Lecture_10

No. 63

Arithmetic Progressions

- **Additional sums based on concepts of AP**

Q.5] The houses in a row are numbered consecutively from 1 to 49. Such that there is a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. Find the value of x.

Sol:

Sum of the numbers of these houses

S_{x-1}

=

Sum of the numbers of these houses

$S_{49} - S_x$

We need to solve:

$S_{x-1} = S_{49} - S_x$

Q.5] The houses in a row are numbered consecutively from 1 to 49. Such that there is a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. Find the value of x.

We need to solve:

$$S_{x-1} = S_{49} - S_x$$

Sol: The number on houses are as follows: 1, 2, 3, …, 49

These numbers form an A.P. with a = 1 and d = 2 – 1 = 1

We know that, $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{x-1} = S_{49} - S_x$$

$$\frac{x-1}{2}[2(1) + (x-1-1)(1)] = \frac{49}{2}[2(1) + (49-1)(1)] - \frac{x}{2}[2(1) + (x-1)(1)]$$

Multiplying throughout by 2,

$$(x-1)(\cancel{2} + x - \cancel{2}) = 49(2 + 48) - x(2 + x - 1)$$

$$(x-1)(x) = 49(50) - x(x+1)$$

$$x^2 - \cancel{x} = 2450 - x^2 - \cancel{x}$$

$$2x^2 = 2450$$

$$x^2 = 1225$$

$x = \pm 35$ [Taking square root]

But x cannot be negative

Value of x is 35

No. 64

Q. Which term of the AP : 121, 117, 113, . . . is its first negative term?

[Hint : Find n for an < 0]

Sol. For given A.P.

$$\begin{array}{cccc} t_1 & t_2 & t_3 & t_n \\ \boxed{121}, & 117, & 113, \ldots & \boxed{-\text{ve}}, \ldots \end{array}$$

$a = t_1 = 121$

$\therefore \quad d = t_2 - t_1 = 117 - 121 = -4$

$\therefore \quad \boxed{d = -4}$

$\therefore \quad t_n = -\text{ve}$

i.e. $\boxed{t_n < 0}$

We know that, $\boxed{t_n = a + (n-1)d}$

$\therefore \quad a + (n-1)d < 0$

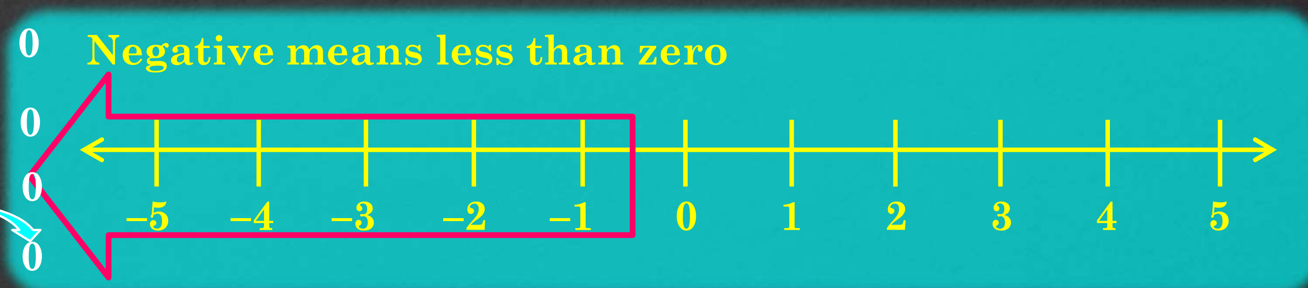
$\therefore \quad 121 + \boxed{(n-1)(-4)} < 0$

$\therefore \quad 121 + \underline{(n)(-4)} - \underline{(1)(-4)} < 0$

$\therefore \quad 121 - 4n + 4 < 0$

$\therefore \quad 125 - 4n < 0$

$\therefore \quad 125 \lessgtr 4n$



Q. Which term of the AP : 121, 117, 113, . . . is its first negative term?

[Hint : Find n for an < 0]

Sol. $\boxed{a = 121,}$ $\boxed{d = -4}$

We know that, $t_n = a + (n - 1)d$

$\therefore \quad 4n > 125$

Dividing both sides by 4

$\therefore \quad \frac{\cancel{4}n}{\cancel{4}} > \frac{125}{4}$

$\therefore \quad n > 31.25$

But 'n' is term number which is a natural number

$\therefore$ The first natural number greater than 31.25 is '32'

When $\boxed{n = 32}$

$t_n = a + \boxed{(n - 1)}\, d$

$\therefore \quad t_{32} = 121 + (32 - 1)(-4)$

$\therefore \quad t_{32} = 121 + 31(-4)$

$\therefore \quad t_{32} = \boxed{121 - 124}$

$\therefore \quad t_{32} = -3$

31.25
4) 125
– 12
5
– 4
10
– 8
20
– 20
00

$\therefore$ $\boxed{\text{First negative term of A.P. is } - 3.}$

Q.2) Find first negative term from following A.P. 122, 116, 110, …

Sol: For given A.P.

t_1 t_2 t_3 t_n

122, 116, 110, … – ve , …

To Find

Lets assume first –ve term as t_n

$a = t_1 = 122$

$d = … – 122 = – 6$

–ve term (t_n) is less than 0

$t_n = –\text{ve}$

Replacing $t_n = a + (n – 1)d$

$t_n < 0$

$\therefore$ a + (n – 1) d

$\therefore$ 122 + (n – 1) (– 6)

$\therefore$ 122 – 6n + 6

$\therefore$ 128 – 6n

$\therefore$ 128 < 6n

Dividing both sides by 6

…ll get … s

$\frac{…8}{…} < \frac{6n}{6}$

…3… < n

$\therefore$ n > 21.33…

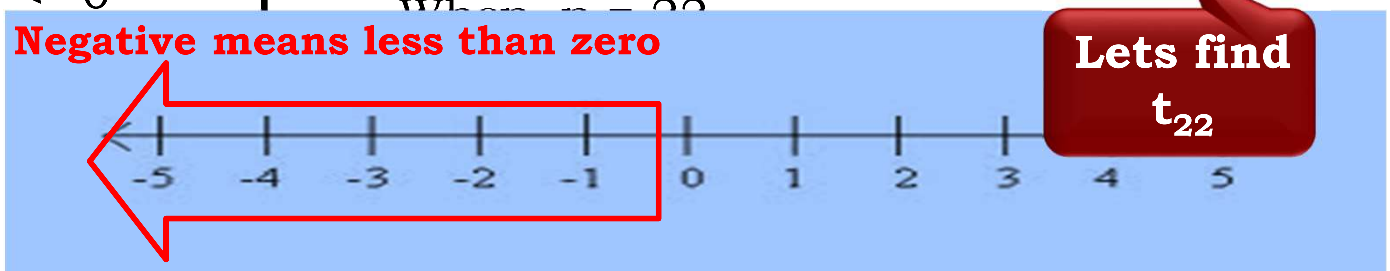
t_{22} , t_{23} , t_{24} ,… are all –ve terms

t_{22} is first –ve term

But 'n' is term number which is a natural number

$\therefore$ The first natural number greater than 21.33… is '22'

When n = 22



Lets find t_{22}

$\therefore$ $t_{22} = 122 – 126$

$\therefore$ $t_{22} = – 4$

$\therefore$ **First negative term of A.P. is – 4.**

No. 65

Q. A ladder has rungs 25 cm apart. (see Diagram). The rungs decrease uniformly in length from 45 cm at the bottom to 25 cm at the top. If the top and the bottom rungs are 2.5 metre apart, what is the length of the wood required for the rungs?

[Hint : Number of rungs = $\frac{250}{25} + 1$]

Sol. It is given that the gap between two consecutive rungs is 25 cm and top and bottom rungs are 2.5 metre i.e., 250 cm apart.

$\therefore$ Number of rungs $= \dfrac{250}{25} + \dfrac{1}{1}$

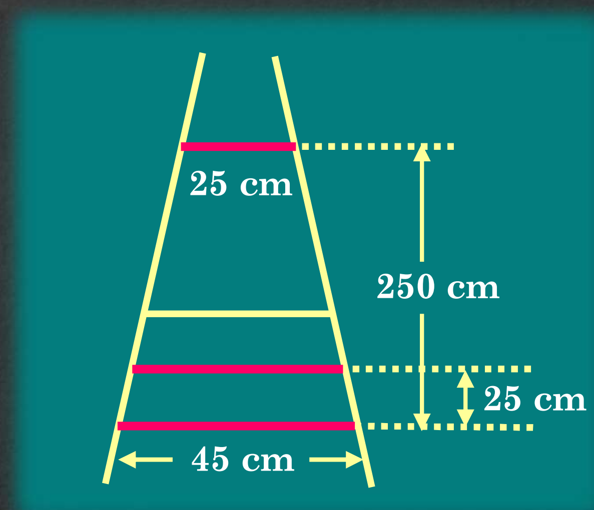
$\therefore$ $\boxed{\text{Number of rungs}} = \dfrac{250 + 25}{25} = \dfrac{\cancel{275}^{11}}{\cancel{25}} \boxed{= 11}$

It is given that the rungs are decreasing uniformly in length from 45 cm at the bottom to 25 cm at the top.

$\therefore$ Lengths of the rungs form an A.P. with first term $(a) = 45$ and 11^{th} term $(t_{11}) = 25$ cm.

$\therefore$ Length of the wood required for rungs

= Sum of 11 term of an A.P. with first term $(a) = 45$ and 11^{th} term $(t_{11}) = 25$ cm.



Q. A ladder has rungs 25 cm apart. (see Diagram). The rungs decrease uniformly in length from 45 cm at the bottom to 25 cm at the top. If the top and the bottom rungs are 2.5 metre apart, what is the length of the wood required for the rungs?

[Hint : Number of rungs = $\frac{250}{25} + 1$]

Sol. $\therefore$ Length of the wood required for rungs

= Sum of 11 term of an A.P. with first term (a) = 45 and 11th term (l) = 25 cm.

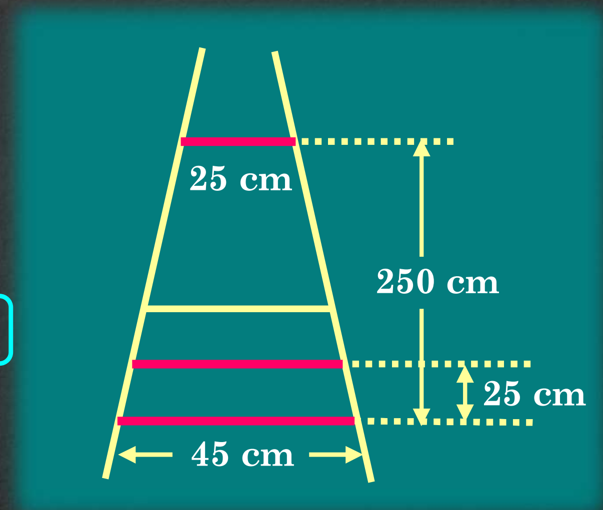
$= \frac{11}{2}(45 + 25)\text{cm}$ $\left[\because S_n = \frac{n}{2}(a + l)\right]$

$= \frac{11}{\cancel{2}}(\cancel{70}^{35})\text{cm}$

= 385 cm

= 3.85 meters [$\because$ 1 meter = 100 cm]

$\therefore$ $\boxed{\text{Length of the wood required for rungs} = 3.85 \text{ meters}}$



No. 66

Q. A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete. Each step has a rise of $\frac{1}{4}$ m and a tread of $\frac{1}{2}$ m. Calculate the total volume of concrete required to build the terrace.

Sol. We observe that the length and width of each step are 50m and $\frac{1}{2}$ m respectively.

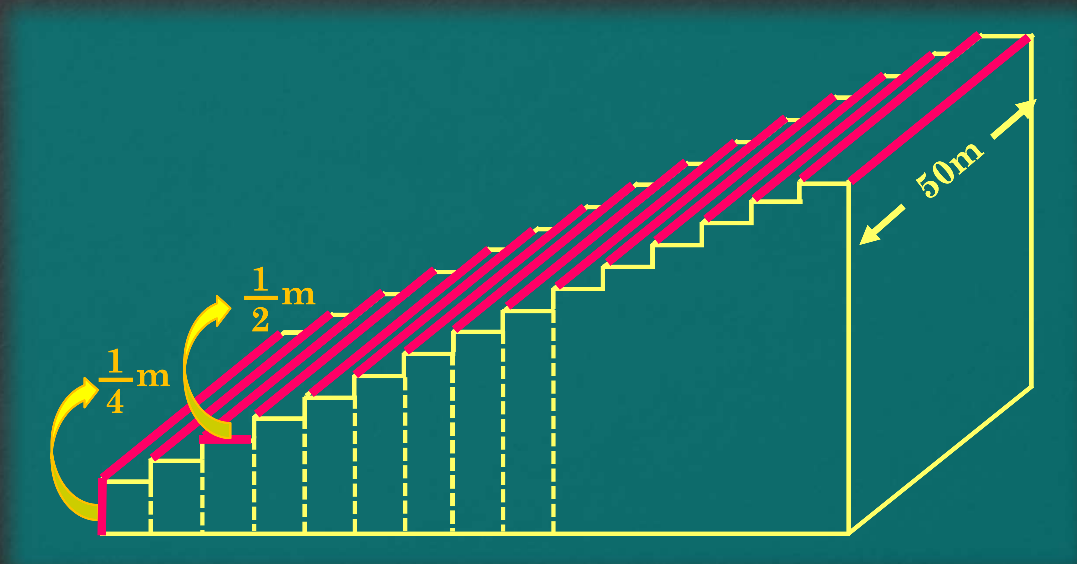
And, Height of 1^{st} step $= \frac{1}{4}$ m.

$\therefore$ Height of 2^{nd} step $= \left(\frac{1}{4} + \frac{1}{4}\right)$ m $= \frac{2}{4}$ m

Similarly,

Height of 3^{rd} step $= \frac{3}{4}$ m and so on.

Let, $V_1, V_2, V_3, \ldots, V_{15}$ respectively denote the volumes of the concrete required to build the first, second, third, …, fifteenth step.



Q. A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete. Each step has a rise of $\frac{1}{4}$m and a tread of $\frac{1}{2}$m. Calculate the total volume of concrete required to build the terrace.

Sol. Height of 1^{st} step $= \frac{1}{4}$ m.

Height of 2^{nd} step $= \left(\frac{1}{4} + \frac{1}{4}\right)$ m $= \frac{2}{4}$ m

Height of 3^{rd} step $= \frac{3}{4}$ m

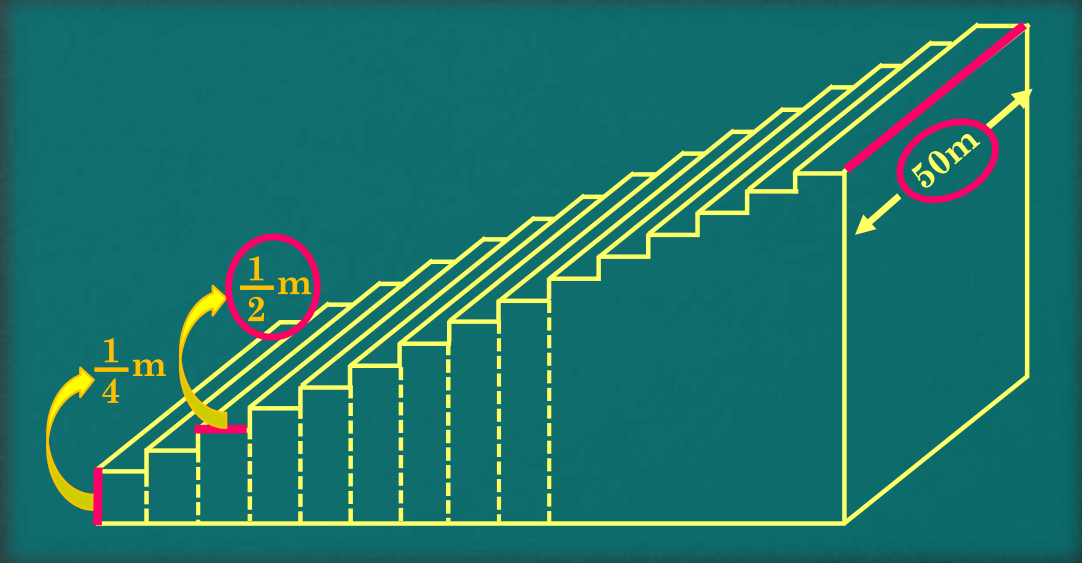
Let, $V_1, V_2, V_3, \ldots, V_{15}$ respectively denote the volumes of the concrete required to build the first, second, third, ..., fifteenth step.

We know, Volume $= l \times b \times h$

$$V_1 = \left(50 \times \frac{1}{2} \times \frac{1}{4}\right) m^3,$$

$$V_2 = \left(50 \times \frac{1}{2} \times \frac{2}{4}\right) m^3,$$

$$V_3 = \left(50 \times \frac{1}{2} \times \frac{3}{4}\right) m^3,$$ And so on



Q. A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete. Each step has a rise of $\frac{1}{4}$ m and a tread of $\frac{1}{2}$ m. Calculate the total volume of concrete required to build the terrace.

Sol. $\boxed{V_1 = \left(50 \times \frac{1}{2} \times \frac{1}{4}\right) m^3}$,

$\boxed{V_2 = \left(50 \times \frac{1}{2} \times \frac{2}{4}\right) m^3}$,

$\boxed{V_3 = \left(50 \times \frac{1}{2} \times \frac{3}{4}\right) m^3}$

Similarly, $\boxed{V_{15} = \left(50 \times \frac{1}{2} \times \frac{15}{4}\right) m^3}$,

Total volume of the concrete

$= V_1 + V_2 + V_3 + \ldots + V_{15}$

$= \left(\boxed{50 \times \frac{1}{2}} \times \frac{1}{4}\right) + \left(\boxed{50 \times \frac{1}{2}} \times \frac{2}{4}\right)$

$+ \left(\boxed{50 \times \frac{1}{2}} \times \frac{3}{4}\right) + \ldots + \left(\boxed{50 \times \frac{1}{2}} \times \frac{15}{4}\right)$

$= \left(\overset{25}{\cancel{50}} \times \frac{1}{\cancel{2}}\right)\left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \ldots + \frac{15}{4}\right) m^3$

$= 25\left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \ldots + \frac{15}{4}\right) m^3$

$= \frac{25}{4}\left[\boxed{1 + 2 + 3 + \ldots + 15}\right] m^3$

$= \frac{25}{\cancel{4}} \times \overset{30}{\cancel{120}}\, m^3$

$= 750\ m^3$

$\therefore$ $\boxed{\text{The total volume of concrete required to build the terrace is } 750\ m^3.}$

No. 67

Q. Divide 32 into four parts which are in A.P . Such that product of extremes is to the product of means is 7:15.

Sol. Let the four parts be $(a - 3d), (a - d), (a + d)$, and $(a + 3d)$

Sum of the numbers = 32

$\therefore \quad (a - 3d) + (a - d) + (a + d) + (a + 3d) = 32$

$\therefore \quad 4a = 32$

$\therefore \quad a = \dfrac{\cancel{32}^{\,8}}{\cancel{4}}$

$\therefore \quad \boxed{a = 8}$

It is given that

$$\frac{\boxed{(a - 3d)(a + 3d)}}{\boxed{(a - d)(a + d)}} = \frac{7}{15}$$

$$\therefore \quad \frac{a^2 - 9d^2}{a^2 - d^2} = \frac{7}{15}$$

$$\therefore \quad \frac{8^2 - 9d^2}{8^2 - d^2} = \frac{7}{15}$$

We know, $(a - b)(a + b) = a^2 - b^2$

Q. **Divide 32 into four parts which are in A.P . Such that product of extremes is to the product of means is 7:15.**

Sol. Let the four parts be $(a - 3d), (a - d), (a + d),$ and $(a + 3d)$

$\therefore \quad a = 8$

$\therefore \quad \dfrac{8^2 - 9d^2}{8^2 - d^2} = \dfrac{7}{15}$

$\therefore \quad \dfrac{64 - 9d^2}{64 - d^2} = \dfrac{7}{15}$

$\therefore \quad 15(64 - 9d^2) = 7(64 - d^2)$

$\therefore \quad 960 - 135d^2 = 448 - 7d^2$

$\therefore \quad 960 - 448 = 135d^2 - 7d^2$

$\therefore \quad 512 = 128d^2$

$\therefore \quad d^2 = \dfrac{\cancel{512}^{\,4}}{\cancel{128}}$

$\therefore \quad d^2 = 4$

$\therefore \quad d = \pm 2$ [Taking square root on both sides]

$\therefore$ **Thus, the four parts $a - 3d$, $a - d$, $a + d$ and $a + 3d$ are 2, 6, 10, 14.**

No. 68

Q. If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn term is is $\frac{1}{2}(mn + 1)$.

Sol. Let a be the first term and d be the common difference of the given A.P.

So, $t_m = \frac{1}{n}$

$\therefore\ a + (m-1)d = \frac{1}{n}$ ...(i)

and $t_n = \frac{1}{m}$

$\therefore\ a + (n-1)d = \frac{1}{m}$ ...(ii)

Subtracting equation (ii) from equation (i), we get

$\therefore\ a + (m-1)d - [a + (n-1)d] = \frac{1}{n} - \frac{1}{m}$

$\therefore\ \cancel{a} + (m-1)d - \cancel{a} - (n-1)d = \frac{m-n}{mn}$

$\therefore\ [(m-1)d - (n-1)d] = \frac{m-n}{mn}$

Q. If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn term is is $\frac{1}{2}(mn+1)$.

Sol. Let a be the first term and d be the common difference of the given A.P.

$$\therefore \quad a + (m-1)d = \frac{1}{n} \qquad \text{...(i)}$$

$$\therefore \quad a + (n-1)d = \frac{1}{m} \qquad \text{...(ii)}$$

$$\therefore \quad [(m-1)d - (n-1)d] = \frac{m-n}{mn}$$

$$\therefore \quad [(m-1) - (n-1)]d = \frac{m-n}{mn}$$

$$\therefore \quad [m \cancel{-1} - n \cancel{+1}]d = \frac{m-n}{mn}$$

$$\therefore \quad (m-n)d = \frac{m-n}{mn}$$

$$\therefore \quad d = \frac{\cancel{m-n}}{mn\cancel{(m-n)}}$$

$$\therefore \quad d = \frac{1}{mn}$$

Q. If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn term is is $\frac{1}{2}(mn + 1)$.

Sol. Let a be the first term and d be the common difference of the given A.P.

$$\therefore \boxed{a + (m-1)d = \frac{1}{n}} \quad \text{...(i)}$$

$$\therefore a + (n-1)d = \frac{1}{m} \quad \text{...(ii)}$$

$$\therefore d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in equation (i), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n}$$

$$a + (\cancel{m})\frac{1}{\cancel{m}n} - \frac{1}{mn} = \frac{1}{n}$$

$$\therefore a + \cancel{\frac{1}{n}} - \frac{1}{mn} = \cancel{\frac{1}{n}}$$

$$\therefore a = \frac{1}{mn}$$

Q. If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn term is is $\frac{1}{2}(mn+1)$.

Sol. Let a be the first term and d be the common difference of the given A.P.

$$\therefore \quad a + (m-1)d = \frac{1}{n} \qquad \text{...(i)}$$

$$\therefore \quad a + (n-1)d = \frac{1}{m} \qquad \text{...(ii)}$$

$$\therefore \quad \boxed{d = \frac{1}{mn}} \text{ and } \boxed{a = \frac{1}{mn}}$$

$$\therefore \quad S_{mn} = \frac{mn}{2}[2a + (mn-1)d]$$

We know, $S_n = \frac{n}{2}[2a + (n-1)d]$

$$\therefore \quad S_{mn} = \boxed{\frac{mn}{2}}\left\{\boxed{\frac{2}{mn}} + \boxed{(mn-1) \times \frac{1}{mn}}\right\}$$

$$\therefore \quad S_{mn} = \left(\frac{\cancel{mn}}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{mn}}\right)^{1} + (mn-1) \times \frac{1}{\cancel{mn}}\left(\frac{\cancel{mn}}{2}\right)$$

$$\therefore \quad S_{mn} = \frac{1}{1} + \frac{mn-1}{2}$$

$$\therefore \quad S_{mn} = \frac{2 + mn - 1}{2}$$

$$\therefore \quad \boxed{S_{mn} = \frac{1}{2}(mn+1)}$$

Hence Proved.

Thank You