Lecture 6

Module 21

Q. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - x - 4$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} - \alpha\beta$.

Sol. $f(x) = 1x^2 - 1x - 4$

Here $a = 1$, $b = -1$, $c = -4$

$\because$ α and β are the zeros of $f(x)$

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-1)}{1} = 1$$

$$\alpha\beta = \frac{c}{a} = \frac{-4}{1} = -4$$

$$\frac{1}{\alpha} + \frac{1}{\beta} - \alpha\beta = \frac{\beta + \alpha}{\alpha\beta} - \alpha\beta$$

$$= \frac{\beta + \beta}{\alpha\beta} - \alpha\beta$$

$$= \frac{1}{-4} - (-4) = \frac{-1}{4} + \frac{4}{1} = \frac{-1 + 16}{4} = \frac{15}{4}$$

$$\therefore \boxed{\frac{1}{\alpha} + \frac{1}{\beta} = \frac{15}{4}}$$

Module 22

Q. If α and β are the zeros of the quadratic polynomial $f(t) = t^2 - 4t + 3$, find the value of $\alpha^4\beta^3 + \alpha^3\beta^4$.

Sol. $f(t) = 1t^2 - 4t + 3$

Here $a = 1$, $b = -4$, $c = 3$

$\because$ α and β are the zeros of $f(t)$

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-4)}{1} = 4$$

$$\text{and} \quad \alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$$

$$\alpha^4\beta^3 + \alpha^3\beta^4 = \alpha^3\beta^3(\alpha + \beta)$$

$$= (\alpha\beta)^3(\alpha + \beta) = (3)^3 \times 4$$

$$= 27 \times 4 = 108$$

$$\therefore \boxed{\alpha^4\beta^3 + \alpha^3\beta^4 = 108}$$

Q. If α and β are the zeros of the quadratic polynomial $\boxed{p(s) = 3s^2 - 6s + 4,}$ find the value of $\boxed{\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} + 2\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right) + 3\alpha\beta.}$

Sol. $p(s) = 3s^2 - 6s + 4$

Here $a = 3,\ b = -6,\ c = 4$

$\because$ α and β are the zeros of p(s)

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(\overset{2}{\cancel{-6}})}{\cancel{3}} = 2$$

$$\text{and} \quad \alpha\beta = \frac{c}{a} = \frac{4}{3}$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta$$

$$= \frac{\alpha^2 + \beta^2}{\beta\alpha} + 2\left[\frac{\beta + \alpha}{\alpha\beta}\right] + 3\alpha\beta$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\beta\beta} + 2\left(\frac{\beta + \beta}{\alpha\beta}\right) + 3\alpha\beta$$

$\boxed{\checkmark}$ $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

Q. If α and β are the zeros of the quadratic polynomial $p(s) = 3s^2 - 6s + 4$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta$.

Sol. $p(s) = 3s^2 - 6s + 4$

$\alpha + \beta = 2$ and $\alpha\beta = \frac{c}{a} = \frac{4}{3}$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} + 2\left[\frac{\alpha+\beta}{\alpha\beta}\right] + 3\alpha\beta$$

$$= \frac{(2)^2 - 2\left(\frac{4}{3}\right)}{\frac{4}{3}} + 2\left[2 \div \frac{4}{3}\right] + \cancel{3}\left[\frac{4}{\cancel{3}}\right]$$

$$= \frac{\frac{4}{1} - \frac{8}{3}}{\frac{4}{3}} + \cancel{2}\left[\cancel{2} \times \frac{3}{\cancel{4}}\right] + 4$$

$$\cancel{2}$$

$$= \left[\frac{12-8}{3} \div \frac{4}{3}\right] + 3 + 4$$

$$= \left[\frac{\cancel{4}^{\,1}}{\cancel{3}} \times \frac{\cancel{3}^{\,1}}{\cancel{4}}\right] + 7$$

$$= 1 + 7$$

$$= 8$$

$$\therefore \boxed{\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta = 8}$$

Module 23

Q. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - px + q$.

Prove that $\dfrac{\alpha^2}{\beta^2} + \dfrac{\beta^2}{\alpha^2} = \dfrac{p^4}{q^2} - \dfrac{4p^4}{q} + 2$

Sol. $f(x) = 1x^2 - px + q$

Here $a = 1$, $b = -p$, $c = q$

$\because$ α and β are the zeros of $f(x)$

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-p)}{1} = p$$

$$\text{and} \quad \alpha\beta = \frac{c}{a} = \frac{q}{1} = q$$

$$\text{L.H.S.} = \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{(\alpha^2)^2 + (\beta^2)^2}{\alpha^2\beta^2}$$

$$= \frac{[(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2]}{\alpha^2\beta^2}$$

$$= \frac{[\{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2(\alpha\beta)^2]}{(\alpha\beta)^2}$$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\therefore (\alpha^2)^2 + (\beta^2)^2 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

Q. If α and β are the zeros of the quadratic polynomial $f(x) = x^2 - px + q$,

Prove that $\dfrac{\alpha^2}{\beta^2} + \dfrac{\beta^2}{\alpha^2} = \dfrac{p^4}{q^2} - \dfrac{4p^4}{q} + 2$

Sol. $f(x) = x^2 - px + q$

$\alpha + \beta = p$ and $\alpha\beta = q$

$$\text{L.H.S.} = \frac{[\{(\alpha+\beta)^2 - 2\alpha\beta\}^2 - 2(\alpha\beta)^2]}{(\alpha\beta)^2}$$

$$= \frac{\{[(p)^2 - 2q]^2 - 2(q)^2\}}{(q)^2}$$

$$= \frac{(p^2)^2 - 2(p)^2(2q) + (2q)^2 - 2(q)^2}{(q)^2}$$

$$= \frac{p^4 - 4p^2q + 4q^2 - 2q^2}{(q)^2}$$

$$= \frac{p^4 - 4p^2q + 2q^2}{q^2}$$

$$= \frac{p^4}{q^2} - \frac{4p^2\cancel{q}}{\cancel{q^2}} + \frac{2\cancel{q^2}}{\cancel{q^2}}$$

$$= \frac{p^4}{q^2} - \frac{4p^2}{q} + 2 = \text{R.H.S.}$$

$$\therefore \boxed{\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2} + \frac{4p^4}{q} + 2}$$

Module 24

Q. If one zeros of the quadratic polynomial $f(x) = 4x^2 - 8kx - 9$ is negative of the other, find the value of k.

Sol. $f(x) = 4x^2 - 8kx - 9$

Here $a = 4$, $b = -8k$, $c = -9$

Let α and β be the zeros of polynomial $f(x)$

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-\overset{2}{\cancel{8}}k)}{\cancel{4}} = 2k$$

$$\alpha\beta = \frac{c}{a} = \frac{-9}{4}$$

$\because$ One zero of the polynomial is negative of the other

$$\therefore \quad \beta = -\alpha$$

$$\alpha + \beta = 2k$$

$$\therefore \quad \alpha + (-\alpha) = 2k$$

$$\therefore \quad \alpha - \alpha = 2k$$

$$\therefore \quad 0 = 2k$$

$$\therefore \quad \boxed{k = 0}$$

Q. If α and β are the zeros of the polynomial $f(x) = x^2 - 5x + k$ such that $\alpha - \beta = 1$, find the value of k.

Sol. Since, α and β are the zeros of polynomial

$f(x) = 1x^2 - 5x + k$

Here $a = 1$, $b = -5$, $c = k$

$\therefore \quad \alpha + \beta = \dfrac{-b}{a} = \dfrac{-(-5)}{1} = 5$

$\alpha\beta = \dfrac{c}{a} = \dfrac{k}{1} = k$

$\alpha - \beta = 1$ [Given]

Squaring both sides, we get

$(\alpha - \beta)^2 = (1)^2$

$\boxed{(\alpha - \beta)^2} = 1$

$\therefore \quad (\alpha + \beta)^2 - 4\alpha\beta = 1$

$\therefore \quad (5)^2 - 4\alpha\beta = 1$

$\therefore \quad 25 - 4\alpha\beta = 1$

$\therefore \quad 25 - 1 = 4\alpha\beta$

$\therefore \quad 4\alpha\beta = 24$

$\therefore \quad \alpha\beta = \dfrac{\cancel{24}^{\,6}}{\cancel{4}}$

$\therefore \quad \boxed{\alpha\beta} = 6$

$\therefore \quad k = 6$

Hence, the value of k is 6.

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

Q. If α and β are the zeros of the quadratic polynomial $\boxed{f(x) = kx^2 + 4x + 4}$ such that $\boxed{\alpha^2 + \beta^2 = 24}$, find the value of k.

Sol. It is given that α and β are the zeros of quadratic polynomial

$$f(x) = kx^2 + 4x + 4$$

Here $a = k$, $b = 4$, $c = 4$

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-4}{k}$$

$$\text{and} \quad \alpha\beta = \frac{c}{a} = \frac{4}{k}$$

We have, $\alpha^2 + \beta^2 = 24$

$$\therefore \quad (\alpha + \beta)^2 - 2\alpha\beta = 24$$

$$\therefore \quad \left(-\frac{4}{k}\right)^2 - 2 \times \frac{4}{k} = 24$$

$$\therefore \quad \frac{16}{k^2} - \frac{8}{k} = 24$$

Multiplying throughout by k^2, we get

$$\therefore \quad \cancel{k^2}\left(\frac{16}{\cancel{k^2}}\right) - \cancel{k^2}\left(\frac{8}{\cancel{k}}\right) = k^2(24)$$

$$\therefore \quad 16 - 8k = 24k^2$$

$$\therefore \quad 24k^2 + 8k - 16 = 0$$

Dividing throughout by 8, we get

$$\therefore \quad 3k^2 + k - 2 = 0$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

Q. If α and β are the zeros of the quadratic polynomial $f(x) = kx^2 + 4x + 4$ such that $\alpha^2 + \beta^2 = 24$, find the value of k.

Sol.

$\therefore \quad 3k^2 + k - 2 = 0$

$\therefore \quad \underline{3k^2 + 3k} - \underline{2k - 2} = 0$

$\therefore \quad 3k\,\underline{(k+1)} - 2\,\underline{(k+1)} = 0$

$\therefore \quad (k + 1)(3k - 2) = 0$

$\therefore \quad k + 1 = 0 \quad \text{or} \quad 3k - 2 = 0$

$\therefore \quad k = -1 \quad \text{or} \quad 3k = 2$

$\therefore \quad k = -1 \quad \text{or} \quad k = \dfrac{2}{3}$

Hence, $\boxed{k = -1 \;\text{ or }\; k = \dfrac{2}{3}}$

Q. If sum of the squares of the quadratic polynomial $\boxed{f(x) = x^2 - 8x + k}$ is 40, find the value of k.

Sol. Let α and β are the zeros of given polynomial

$$f(x) = 1x^2 - 8x + k$$

Here $a = 1$, $b = -8$, $c = k$

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-8)}{1} = 8$$

$$\alpha\beta = \frac{c}{a} = \frac{k}{1} = k$$

$$\boxed{\alpha^2 + \beta^2} = 40 \quad \text{[Given]}$$

$$\therefore \quad (\alpha + \beta)^2 - 2\boxed{\alpha\beta} = 40$$

$$\therefore \quad (8)^2 - 2(k) = 40$$

$$\therefore \quad 64 - 2k = 40$$

$$\therefore \quad 64 - 40 = 2k$$

$$\therefore \quad 24 = 2k$$

$$\therefore \quad k = \frac{\cancel{24}^{\,12}}{\cancel{2}}$$

$$\therefore \quad k = 6$$

$$\therefore \quad k = 6$$

$\boxed{\text{Hence, the value of } k \text{ is 6.}}$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

Thank You