LECTURE_03

MODULE_07

Exercise 1.1

Q.4. Use Euclid's division lemma to show that the square of any positive integer is either of the form $3m$ or $3m + 1$ for some integer m.

Sol. Let x be any positive integer and $b = 3$

$\therefore$ Applying Euclid's Division Algorithm

we get $x = 3q + r$ where $0 \leq r < 3$

$\therefore$ The possible remainders are 0, 1, 2

$\therefore x = 3q$ or $3q + 1$ or $3q + 2$

+ve integer be denoted as 'x'

$(\text{+ve integer})^2 = 3m$

$= 3m + 1$

Here, divisor b is equal to 3

Apply, $(a + b)^2 = a^2 + 2ab + b^2$

$a = bq + r$

i) If $x = 3q$

$\Rightarrow x^2 = (3q)^2$

$= 9q^2$

$= 3(3q^2)$

$= 3m$

Replace $3q^2$ by m

Replace $(3q^2 + 4q + 1)$ by m

Now, for $x = 3q + 1$

Now, for $x = 3q + 2$

$0 \leq r < b$

But, $b = 3$

for some integer m, where $m = 3q^2$

Possible values Of 'r' are 0, 1, 2

We want square of '+ve' integer

ii) If $x = 3q + 1$

$\Rightarrow x^2 = (3q + 1)^2$

$= 9q^2 + 6q + 1$

$= 3(3q^2 + 2q) + 1$

$= 3m + 1$

for some integer m, where $m = (3q^2 + 2q)$

iii) If $x = 3q + 2$

$\Rightarrow x^2 = (3q + 2)^2$

$= 9q^2 + 12q + 4$

$= 9q^2 + 12q + 3 + 1$

$= 3(3q^2 + 4q + 1) + 1$

$= 3m + 1$

for some integer m, where $m = 3q^2 + 4q + 1$

$\therefore$ *Square of any positive odd integer is either of the form $3m$ or $3m + 1$ for some integer m.*

MODULE_08

Exercise 1.1

Q.5. Use Euclid's division lemma to show that the cube of any positive integer is of the form $9m$, $9m + 1$ or $9m + 8$ for some integer m.

Soln. Let x be any positive integer and $b = 3$

$\therefore$ Applying Euclid's Division Algorithm, we get

$x = 3q$, $x = 3q + 1$ or $x = 3q + 2$

i) If $x = 3q$

$\Rightarrow x^3 = (3q)^3$

$= 27q^3$

$= 9\boxed{(3q^3)}$

$= 9m$

for some integer m, where $m = 3q^3$

ii) If $x = 3q + 1$

Apply, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\Rightarrow x^3 = (3q + 1)^3$

$= (3q)^3 + 3(3q)^2(1) + 3(3q)(1)^2 + (1)^3$

$= 27q^3 + 27q^2 + 9q + 1$

$= 9\boxed{(3q^3 + 3q^2 + q)} + 1$

$= 9m + 1$

for some integer m, where $m = 3q^3 + 3q^2 + q$

iii) If $x = 3q + 2$

Apply, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\Rightarrow x^3 = (3q + 2)^3$

$= (3q)^3 + 3(3q)^2(2) + 3(3q)(2)^2 + (2)^3$

$= 27q^3 + 54q^2 + 36q + 8$

$= 9\boxed{(3q^3 + 6q^2 + 4q)} + 8$

$= 9m + 8$

for some integer m, where $m = 3q^3 + 6q^2 + 4q$

$\therefore$ Cube of any positive integer is of the form $9m$, $9m + 1$ or $9m + 8$ for some integer m.

$(x)^3 = 9m$

$= 9m + 1$

$= 9m + 8$

Lets consider divisor as 3

$\therefore$ Possible remainders are

7, 8

i.e. 3q

3q + 1

3q + 2

9q + 4

9q + 5

9q + 6

9q + 7

9q + 8

MODULE_09

Q. Prove that product of three consecutive positive integers is divisible by 6.

Sol. Let three consecutive positive integer's be $n - 1$, n and $n + 1$.

By Euclid's Division Algorithm,

$\boxed{a = 6q + r}$ where $0 \leq r < 6$

$\therefore$ The possible remainders are 0, 1, 2, 3, 4, 5

$\therefore$ $\boxed{a = 6q}$ or $6q + 1$ or $6q + 2$ or $6q + 3$ or $6q + 4$ or $6q + 5$, where q is some integer

Case 1 : If, $n = 6q$

$$n - 1 = 6q - 1$$

$$n + 1 = 6q + 1$$

So, $(n - 1)(n)(n + 1) = (6q - 1)(6q)(6q + 1)$

$= 6\,[(6q - 1)(q)(6q + 1)]$

$= 6\,m$ $\quad$ [where $m = (6q - 1)(q)(6q + 1)$]

Here, the above result is multiple of 6.

Hence, it is divisible by 6.

Q. Prove that product of three consecutive positive integers is divisible by 6.

Sol. Let three consecutive positive integer's be $n - 1$, n and $n + 1$.

By Euclid's Division Algorithm,

$\boxed{a = 6q + r}$ where $0 \leq r < 6$

$\therefore$ The possible remainders are 0, 1, 2, 3, 4, 5

$\therefore$ $a = 6q$ or $\boxed{6q + 1}$ or $6q + 2$ or $6q + 3$ or $6q + 4$ or $6q + 5$, where q is some integer

Case 2 : If, $n = 6q + 1$

$$n - 1 = 6q \cancel{+ 1} \cancel{- 1} = 6q$$

$$n + 1 = 6q + 1 + 1 = 6q + 2$$

$$\text{So, } (n - 1)(n)(n + 1) = (6q)(6q + 1)(6q + 2)$$

$$= 6[q(6q + 1)(6q + 2)]$$

$$= 6m$$

where $m = [q(6q + 1)(6q + 2)]$

Here, the above result is multiple of 6.

Hence, it is divisible by 6.

Q. **Prove that product of three consecutive positive integers is divisible by 6.**

Sol. Let three consecutive positive integer's be $n - 1$, n and $n + 1$.

By Euclid's Division Algorithm,

$\boxed{a = 6q + r}$ where $0 \leq r < 6$

$\therefore$ The possible remainders are 0, 1, 2, 3, 4, 5

$\therefore$ $a = 6q$ or $6q + 1$ or $\boxed{6q + 2}$ or $6q + 3$ or $6q + 4$ or $6q + 5$, where q is some integer

Case 3 : If, $n = 6q + 2$

$$n - 1 = 6q + 2 - 1 = 6q + 1$$

$$n + 1 = 6q + 2 + 1 = 6q + 3$$

So, $(n - 1)(n)(n + 1) = (6q + 1)(6q + 2)(6q + 3)$

$$= (6q + 1)(2)(3q + 1)(3)(2q + 1)$$

$$= 6[(6q + 1)(3q + 1)(2q + 1)]$$

$$= 6m$$

where $m = [(6q + 1)(3q + 1)(2q + 1)]$

Here, the above result is multiple of 6. Hence, it is divisible by 6.

Q. Prove that product of three consecutive positive integers is divisible by 6.

Sol. Let three consecutive positive integer's be
$n - 1$, n and $n + 1$.
By Euclid's Division Algorithm,
$\boxed{a = 6q + r}$ where $0 \leq r < 6$

$\therefore$ The possible remainders are 0, 1, 2, 3, 4, 5

$\therefore$ $a = 6q$ or $6q + 1$ or $6q + 2$ or $\boxed{6q + 3}$ or
$6q + 4$ or $6q + 5$, where q is some integer

Case 4 : If, $n = 6q + 3$

$$n - 1 = 6q + 3 - 1 = 6q + 2$$

$$n + 1 = 6q + 3 + 1 = 6q + 4$$

So, $(n - 1)(n)(n + 1) = (6q + 2)(6q + 3)(6q + 4)$

$$= (2)(3q + 1)(3)(2q + 1)(6q + 4)$$

$$= 6[(3q + 1)(2q + 1)(6q + 4)]$$

$$= 6m$$

where $m = [(3q + 1)(2q + 1)(6q + 4)]$

Here, the above result is multiple of 6. Hence, it is divisible by 6.

Q. Prove that product of three consecutive positive integers is divisible by 6.

Sol. Let three consecutive positive integer's be $n - 1$, n and $n + 1$.

By Euclid's Division Algorithm,

$\boxed{a = 6q + r}$ where $0 \leq r < 6$

$\therefore$ The possible remainders are 0, 1, 2, 3, 4, 5

$\therefore$ $a = 6q$ or $6q + 1$ or $6q + 2$ or $6q + 3$ or $\boxed{6q + 4}$ or $6q + 5$, where q is some integer

Case 5 : If, $n = 6q + 4$

$$n - 1 = 6q + 4 - 1 = 6q + 3$$

$$n + 1 = 6q + 4 + 1 = 6q + 5$$

$$\text{So, } (n - 1)(n)(n + 1) = \underline{(6q + 3)}\,\underline{(6q + 4)}(6q + 5)$$

$$= (\underline{3})(2q + 1)(\underline{2})(3q + 2)(6q + 5)$$

$$= 6[\underline{(2q + 1)(3q + 2)(6q + 5)}]$$

$$= 6\,m$$

$$\text{where } m = [(2q + 1)(3q + 2)(6q + 5)]$$

Here, the above result is multiple of 6. Hence, it is divisible by 6.

Q. **Prove that product of three consecutive positive integers is divisible by 6.**

Sol. Let three consecutive positive integer's be $n - 1$, n and $n + 1$.

By Euclid's Division Algorithm,

$\boxed{a = 6q + r}$ where $0 \leq r < 6$

$\therefore$ The possible remainders are 0, 1, 2, 3, 4, 5

$\therefore$ $a = 6q$ or $6q + 1$ or $6q + 2$ or $6q + 3$ or $6q + 4$ or $\boxed{6q + 5}$, where q is some integer

Case 6 : If, $n = 6q + 5$

$$n - 1 = 6q + 5 - 1 = 6q + 4$$

$$n + 1 = 6q + 5 + 1 = 6q + 6$$

So, $(n - 1)(n)(n + 1) = (6q + 4)(6q + 5)(6q + 6)$

$$= (6q + 4)(6q + 5)(6)(q + 1)$$

$$= 6[(6q + 4)(6q + 5)(q + 1)]$$

$$= 6m$$

where $m = [(6q + 4)(6q + 5)(q + 1)]$

Here, the above result is multiple of 6. Hence, it is divisible by 6.

MODULE_10

Q. Prove that product of three consecutive positive integers is divisible by 6.

Sol.

- The above Question can also be solved in the following way

Q. For any positive integer n, prove that $n^3 - n$ is divisible by 6.

Hint : $n^3 - n = n(n^2 - 1)$

$= n(n - 1)(n + 1)$

........Since, $(a^2 - b^2) = (a - b)(a + b)$

$= (n - 1)(n)(n + 1)$