Lecture 7

Module 25

Q. If α and β are the zeros of the quadratic polynomial $f(x) = 2x^2 - 5x + 7$, find a polynomial whose zeros are $2\alpha + 3\beta$ and $3\alpha + 2\beta$.

Sol. It is given that α and β are the zeros of quadratic polynomial

$$f(x) = 2x^2 - 5x + 7$$

Here $a = 2,\ b = -5,\ c = 7$

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-5)}{2} = \frac{5}{2}$$

$$\alpha\beta = \frac{c}{a} = \frac{7}{2}$$

Let S and P denote respectively the sum and product of zeroes of the required polynomial.

Also, $\boxed{\alpha_1 = 2\alpha + 3\beta}$ & $\boxed{\beta_1 = 3\alpha + 2\beta}$ are the roots of the polynomial to be formed

Then, $S = \boxed{\alpha_1} + \boxed{\beta_1}$ and $P = \alpha_1 \beta_1$

$$\therefore \quad S = (2\alpha + 3\beta) + (3\alpha + 2\beta)$$

$$S = \boxed{2\alpha} + \boxed{3\beta} + \boxed{3\alpha} + \boxed{2\beta}$$

$$= 5\alpha + 5\beta = 5(\alpha + \beta) = 5 \times \frac{5}{2} = \frac{25}{2}$$

Q. If α and β are the zeros of the quadratic polynomial $f(x) = 2x^2 - 5x + 7$, find a polynomial whose zeros are $2\alpha + 3\beta$ and $3\alpha + 2\beta$.

Sol. $\alpha + \beta = \frac{5}{2}$ and $\alpha\beta = \frac{7}{2}$

$S = \frac{25}{2}$ and $P = \boxed{\alpha_1}\boxed{\beta_1}$

$\boxed{\alpha_1 = 2\alpha + 3\beta}$ and $\boxed{\beta_1 = 3\alpha + 2\beta}$

$$\begin{aligned}
\therefore \quad P &= (\boxed{2\alpha} + \boxed{3\beta})\boxed{(3\alpha + 2\beta)} \\
&= 2\alpha(3\alpha + 2\beta) + 3\beta(3\alpha + 2\beta) \\
&= 6\alpha^2 + \boxed{4\alpha\beta + 9\alpha\beta} + 6\beta^2 \\
&= \underline{6\alpha^2 + 6\beta^2} + 13\alpha\beta \\
&= 6(\alpha^2 + \beta^2) + 13\alpha\beta \\
&= 6[(\alpha + \beta)^2 - 2\alpha\beta] + 13\alpha\beta \\
&= 6(\alpha + \beta)^2 \underline{- 12\alpha\beta + 13\alpha\beta} \\
&= 6(\alpha + \beta)^2 + \alpha\beta \\
&= 6\left(\frac{5}{2}\right)^2 + \frac{7}{2} \\
&= \overset{3}{\cancel{6}}\left(\frac{25}{\cancel{4}_{2}}\right) + \frac{7}{2}
\end{aligned}$$

$\therefore$ P $= \frac{75}{2} + \frac{7}{2}$ ... 41

$\boxed{\checkmark}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

$\therefore$ P = 41

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

...nomial $g(x)$ is given by,

$g(x) = k(x^2 - Sx + P)$ where k is any non-zero real number.

$$\therefore \quad \boxed{g(x) = k\left(x^2 - \frac{25}{2}x + 41\right)}$$

Q. If α and β are the zeros of the quadratic polynomial $\boxed{f(x) = 3x^2 - 4x + 1}$, find a quadratic polynomial whose zeros are. $\boxed{\dfrac{\alpha^2}{\beta} \text{ and } \dfrac{\beta^2}{\alpha}}$

Sol. It is given that α and β are the zeros of the polynomial

$$f(x) = 3x^2 - 4x + 1$$

Here $a = 3,\ b = -4,\ c = 1$

$$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(-4)}{3} = \frac{4}{3}$$

$$\alpha\beta = \frac{c}{a} = \frac{1}{3} =$$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

Let S and P denote

product of zeroes of

Also, $\alpha_1 = \dfrac{\alpha^2}{\beta}$ and β_1

polynomial to be formed

Then, $S = \dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3 + \beta^3}{\beta\beta}$

$$= \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{\left(\frac{4}{3}\right)^3 - 3\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)}{\frac{1}{3}}$$

$$= \frac{64}{27} - \frac{12}{9}\left(\frac{3}{3}\right) \div \frac{1}{3} = \frac{64}{27} - \frac{36}{27} \times 3$$

L.C.M of 27 and 9 = 27

$$= \frac{64 - 36}{\cancel{27}_{\,9}} \times \cancel{3} = \frac{28}{9}$$

Q. If α and β are the zeros of the quadratic polynomial $f(x) = 3x^2 - 4x + 1$, find a quadratic polynomial whose zeros are. $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$

Sol. $\alpha + \beta = \frac{4}{3}$ and $\boxed{\alpha\beta = \frac{1}{3}}$

$S = \frac{28}{9}$ and $P = \boxed{\alpha_1}\boxed{\beta_1}$

$\boxed{\alpha_1 = \frac{\alpha^2}{\beta}}$ and $\boxed{\beta_1 = \frac{\beta^2}{\alpha}}$

$$\therefore \quad P = \left(\frac{\alpha^2}{\beta}\right)\left(\frac{\beta^2}{\alpha}\right)$$

$$= \frac{\alpha^{\cancel{2}}\beta^{\cancel{2}}}{\cancel{\beta}\cancel{\alpha}}$$

$$= \alpha\beta$$

$$\therefore \quad P = \frac{1}{3}$$

Hence, the required polynomial $g(x)$ is given by,

$g(x) = k(x^2 - Sx + P)$ where k is any non-zero real number.

$$\therefore \quad \boxed{g(x) = k\left(x^2 - \frac{28}{9}x + \frac{1}{3}\right)}$$

Q. If α and β are the zeros of the quadratic polynomial $\boxed{f(x) = kx^2 - 1}$, find a quadratic polynomial whose zeros are $\boxed{\frac{2\alpha}{\beta} \text{ and } \frac{2\beta}{\alpha}}$.

Sol. It is given that α and β are the zeros of the polynomial

$f(x) = kx^2 - 1 = kx^2 + 0x - 1$

Here $a = k$, $b = 0$, $c = -1$

$\therefore \quad \alpha + \beta = \frac{-b}{a} = \frac{-(0)}{k} = \frac{0}{k} = 0$

$\alpha\beta = \frac{c}{a} = \frac{-1}{k}$ ✓

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

$(\alpha - \beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

Let S and P denote r... product of zeroes of...

Also, $\alpha_1 = \frac{2\alpha}{\beta}$ and $\beta_1 = \ldots \alpha$

polynomial to be formed

Then, $S = \frac{2\alpha}{\beta} + \frac{2\beta}{\alpha} = \frac{2\alpha^2 + 2\beta^2}{\beta\beta} = \frac{2(\alpha^2+\beta^2)}{\alpha\beta}$

$= \frac{2\{(\alpha+\beta)^2 - 2\alpha\beta\}}{\alpha\beta} = \frac{2\left\{(0)^2 - 2\left(\frac{-1}{k}\right)\right\}}{\frac{-1}{k}}$

$= 2\left\{0 + \left(\frac{2}{k}\right)\right\} \div \frac{-1}{k} = 2\left(\frac{2}{k}\right) \times \frac{k}{-1}$

$= \frac{4}{\cancel{k}} \times \frac{\cancel{k}}{-1} = \frac{4}{-1} = -4$

Therefore $S = -4$

Q. If α and β are the zeros of the quadratic polynomial $f(x) = kx^2 - 1$, find a quadratic polynomial whose zeros are $\frac{2\alpha}{\beta}$ and $\frac{2\beta}{\alpha}$.

Sol. $\alpha + \beta = 0$ and $\alpha\beta = \frac{-1}{k}$

$S = -4$ and $P = \boxed{\alpha_1}\boxed{\beta_1}$

$\boxed{\alpha_1 = \frac{2\alpha}{\beta}}$ and $\boxed{\beta_1 = \frac{2\beta}{\alpha}}$

$\therefore \quad P = \left(\frac{2\alpha}{\beta}\right)\left(\frac{2\beta}{\alpha}\right)$

$= \frac{4\cancel{\alpha\beta}}{\cancel{\beta\alpha}}$

$\therefore \quad P = 4$

Hence, the required polynomial $g(x)$ is given by,

$g(x) = k(x^2 - Sx + P)$ where k is any non - zero real number.

$\therefore \quad g(x) = k\,(x^2 - (-4)x + 4)$

$\therefore \quad \boxed{g(x) = k\,(x^2 + 4x + 4)}$

Module 26

Relationship between zeroes and coefficients of a Cubic Polynomial

- **If α, β and γ are zeroes of $p(x) = \boxed{a}x^3 + \boxed{b}x^2 + \boxed{c}x + \boxed{d}$, then**

Sum of the zeroes $= \alpha + \beta + \gamma = \dfrac{-(\text{coefficient of } x^2)}{\text{coefficient of } x^3} = \dfrac{-b}{a}$

Sum of product of two zeroes $= \alpha\beta + \beta\gamma + \alpha\gamma = \dfrac{(\text{coefficient of } x)}{\text{coefficient of } x^3} = \dfrac{c}{a}$

Product of the zeroes $= \alpha\beta\gamma = \dfrac{-\text{ Constant term}}{\text{coefficient of } x^3} = \dfrac{-d}{a}$

EXERCISE 2.4

Q. 1 Verify that numbers given alongside of cubic polynomials below are their zeroes. Also verify the relationship between zeroes and the coefficient in each case

i) $2x^3 + x^2 - 5x + 2; \frac{1}{2}, 1, -2$

Sol : $p(x) = 2x^3 + x^2 - 5x + 2$

Zeroes for this polynomial are $\frac{1}{2}$, 1, – 2

$$p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2$$

$$= \frac{\cancel{2}}{4}\left(\frac{1}{\cancel{8}}\right) + \frac{1}{4} - \frac{5}{2} + \frac{2}{1}$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{\boxed{\times 2}}{\boxed{\times 2}} + \frac{\boxed{\times 4}}{\boxed{\times 4}}$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{10}{4} + \frac{8}{4}$$

$$= \frac{1 + 1 - 10 + 8}{4} = \frac{0}{4}$$

$$\therefore \quad p\left(\frac{1}{2}\right) = 0$$

L.C.M of 4, 2 and 1 = 4

EXERCISE 2.4

Q. 1 Verify that numbers given alongside of cubic polynomials below are their zeroes. Also verify the relationship between zeroes and the coefficient in each case

i) $2x^3 + x^2 - 5x + 2;\ \frac{1}{2}, 1, -2$

Sol : $p(x) = 2x^3 + x^2 - 5x + 2$

Zeroes for this polynomial are $\frac{1}{2}$, 1, – 2

$\therefore\ p\left(\frac{1}{2}\right) = 0$

$$p(1) = \underline{2(1)^3} + \underline{(1)^2} - \underline{5(1)} + 2$$
$$= 2 + 1 - 5 + 2$$
$$\therefore\ p(1) = 0$$
$$p(-2) = 2\underline{(-2)^3} + \underline{(-2)^2} - \underline{5(-2)} + 2$$
$$= 2(-8) + 4 + 10 + 2$$
$$= -16 + 16$$
$$\therefore\ p(-2) = 0$$

$\therefore\ p\left(\frac{1}{2}\right) = 0,\ p(-2) = 0$ and $p(1) = 0$

Hence $\frac{1}{2}$, 1 and –2 are zeroes of given polynomial.

EXERCISE 2.4

Q. 1 **Verify that numbers given alongside of cubic polynomials below are their zeroes. Also verify the relationship between zeroes and the coefficient in each case**

i) $2x^3 + x^2 - 5x + 2; \frac{1}{2}, 1, -2$

Sol : $p(x) = 2x^3 + 1x^2 - 5x + 2$

Zeroes for this polynomial are $\frac{1}{2}, 1, -2$

Comparing the given polynomial with

$ax^3 + bx^2 + cx + d,$

we get, $a = 2,\ b = 1,\ c = -5,\ d = 2$

we can take $\alpha = \frac{1}{2}, \beta = 1, \gamma = -2$

$$\therefore \quad \alpha + \beta + \gamma = \frac{1}{2} + 1 + (-2)$$

$$= \frac{1}{2}\ \frac{-1}{1}$$

$$= \frac{1-2}{2}$$

$$\therefore \quad \alpha + \beta + \gamma = \frac{-1}{2} = \frac{-b}{a}$$

Exercise 2.4

Q. 1 Verify that numbers given alongside of cubic polynomials below are their zeroes. Also verify the relationship between zeroes and the coefficient in each case

i) $2x^3 + x^2 - 5x + 2; \frac{1}{2}, 1, -2$

Sol : $p(x) = 2x^3 + x^2 - 5x + 2$

Zeroes for this polynomial are $\frac{1}{2}, 1, -2$

we can take $\alpha = \frac{1}{2}, \beta = 1, \gamma = -2$

$$\therefore \alpha\beta + \beta\gamma + \alpha\gamma = \frac{1}{2}(1) + 1(-2) + \frac{1}{2}(-2)$$

$$= \frac{1}{2} - 2 - \frac{\overset{1}{\cancel{2}}}{\cancel{2}}$$

$$= \frac{1}{2} - 2 - 1 = \frac{1}{2} - \frac{3}{1}$$

$$\therefore \alpha\beta + \beta\gamma + \alpha\gamma = \frac{1-6}{2} = \frac{-5}{2} = \frac{c}{a}$$

$$\therefore \alpha\beta\gamma = \frac{1}{2} \times 1 \times (-2) = \frac{-2}{2} = \frac{-d}{a}$$

Therefore, the relationship between the zeroes and the coefficient is verified.

EXERCISE 2.4

Q. 1 Verify that numbers given alongside of cubic polynomials below are their zeroes. Also verify the relationship between zeroes and the coefficient in each case

ii) $x^3 - 4x^2 + 5x - 2$; 2, 1,1

Sol: $p(x) = x^3 - 4x^2 + 5x - 2$,

Zeroes for this polynomial are 2, 1, 1

$p(2) = \underline{(2)^3} - \underline{4(2)^2} + 5(2) - 2$

$= 8 - 16 - 10 - 2$

$\therefore\ p(2) = 0$

$p(1) = \underline{1^3} - \underline{4(1)^2} + \underline{5(1)} - 2$

$= 1 - 4 + 5 - 2$

$= 6 - 6$

$\therefore\ p(1) = 0$

Therefore, 2, 1, 1 are the zeroes of the given polynomial.

EXERCISE 2.4

Q. 1 Verify that numbers given alongside of cubic polynomials below are their zeroes. Also verify the relationship between zeroes and the coefficient in each case

ii) $x^3 - 4x^2 + 5x - 2;\ 2, 1, 1$

Sol: $p(x) = 1x^3 - 4x^2 + 5x - 2,$

Comparing the given polynomial with

$ax^3 + bx^2 + cx + d.$

we get, $a = 1,\ b = -4,\ c = 5,\ d = -2$

we can take $\boxed{\alpha = 2}, \boxed{\beta = 1,} \boxed{\gamma = 1}$

$\therefore\ \alpha + \beta + \gamma = \boxed{2 + 1 + 1} = 4 = \dfrac{-(-4)}{1} = \dfrac{-b}{a}$

$\therefore\ \alpha\beta + \beta\gamma + \alpha\gamma = (2)(1) + (1)(1) + (2)(1)$

$= \boxed{2 + 1 + 2} = 5 = \dfrac{5}{1} = \dfrac{c}{a}$

$\therefore\ \alpha\beta\gamma = \boxed{2 \times 1 \times 1} = 2 = \dfrac{-(-2)}{1} = \dfrac{-d}{a}$

Therefore, the relationship between the zeroes and the coefficient is verified.

Module 27

Standard form of Cubic Polynomial in terms of a, b and g

$$x^3 - (a + b + g)x^2 + (ab + bg + ag)x - abg$$

EXERCISE 2.4

Q. 2 Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, – 7, – 14 respectively.

Sol: Given that,

Sum of zeroes $(\alpha + \beta + \gamma) = 2$ ....(i)

Sum of product of zeroes taken two at a time $(\alpha\beta + \beta\gamma + \alpha\gamma) = -7$ ....(ii)

Product of zeroes $(\alpha\beta\gamma) = -14$ ....(iii)

We know, a cubic polynomial is of the form,

$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - (\alpha\beta\gamma)$

substituting (i), (ii) and (iii) in the above polynomial

We get, $x^3 - 2x^2 + (-7)x - (-14)$

$\Rightarrow x^3 - 2x^2 - 7x + 14$

Therefore, the required polynomial is $x^3 - 2x^2 - 7x + 14$

Q. Find the zeroes of the polynomial $\boxed{f(x) = x^3 - 12x^2 + 39x - 28,}$ if it is given that the zeroes are in A.P.

Sol. Let $\boxed{\alpha = a - d,}$ $\boxed{\beta = a}$ and $\boxed{\gamma = a + d}$ be the zeros of the polynomial

$f(x) = 1x^3 - 12x^2 + 39x - 28$

$$\therefore \quad \alpha + \beta + \gamma = \frac{-b}{a}$$

$$\therefore \quad \alpha + \beta + \gamma = \frac{-(-12)}{1} = 12 \qquad \text{....(i)}$$

$$\text{and, } \alpha\beta\gamma = \frac{-d}{a}$$

$$\therefore \quad \alpha\beta\gamma = \frac{-(-28)}{1} = 28 \qquad \text{....(ii)}$$

Now, from (i)

$$(a - d) + a + (a + d) = 12$$

$$\therefore \quad a \cancel{- d} + a + a \cancel{+ d} = 12$$

$$\therefore \quad 3a = 12$$

$$\therefore \quad a = \frac{\overset{4}{\cancel{12}}}{\cancel{3}} = 4$$

Q. Find the zeroes of the polynomial $f(x) = x^3 - 12x^2 + 39x - 28$, if it is given that the zeroes are in A.P.

Sol. Let $\boxed{\alpha = a - d,}$ $\boxed{\beta = a}$ and $\boxed{\gamma = a + d}$ be the zeros of the polynomial $f(x) = x^3 - 12x^2 + 39x - 28$.

$$\alpha + \beta + \gamma = 12 \qquad \text{....(i)}$$

$$\alpha\beta\gamma = 28 \qquad \text{....(ii)}$$

$$\therefore \quad \boxed{a = 4}$$

Now, from (ii)

$$(a - d)(a)(a + d) = 28$$

$$\therefore \quad a(a^2 - d^2) = 28$$

$$\therefore \quad 4[(4)^2 - d^2)] = 28$$

$$\therefore \quad 4(16 - d^2) = 28$$

$$\therefore \quad 16 - d^2 = \frac{\cancel{28}^{\,7}}{\cancel{4}}$$

$$16 - d^2 = 7$$

$$\therefore \quad 16 - 7 = d^2$$

$$\therefore \quad d^2 = 9$$

$$\therefore \quad d = \pm 3$$

<u>Case I</u> : When $\boxed{a = 4}$ and $\boxed{d = 3,}$ In this case,

$\alpha = a - d = 4 - 3 = 1,$

$\beta = a = 4$ and $\gamma = a + d = 4 + 3 = 7$

$\therefore$ $\boxed{\text{When } a = 4 \text{ and } d = 3,\ \alpha = 1,\ \beta = 4 \text{ and } \gamma = 7}$

<u>Case II</u> : When $\boxed{a = 4}$ and $\boxed{d = -3,}$ In this case,

$\alpha = a - d = 4 - (-3) = 4 + 3 = 7$

$\beta = a = 4$ and $\gamma = a + d = 4 + (-3) = 4 - 3 = 1$

$\therefore$ $\boxed{\text{When } a = 4 \text{ and } d = -3,\ \alpha = 7,\ \beta = 4 \text{ and } \gamma = 1}$

Thank You