6 Quadratic Equations

6.1 QUADRATIC EQUATIONS

Quadratic Expression. *An expression of the form* $ax^2 + bx + c$, $a \neq 0$, *is called a* ***quadratic*** *(or* ***second degree****)* ***expression*** *in the variable x.*

Quadratic Equation. *An equation of the type* $ax^2 + bx + c = 0$, $a \neq 0$, *is called a* ***quadratic equation*** *in the variable x.*

The equation $ax^2 + bx + c = 0$, $a \neq 0$, is called the **general** (or **standard**) **form.**

For example : $x^2 - 7x + 12 = 0$ and $3x^2 - 4x - 4 = 0$ are quadratic equations in the variable x.

A number α is a **root** (or **solution**) of the quadratic equation

$ax^2 + bx + c = 0$ if and only if $a\alpha^2 + b\alpha + c = 0$.

For example :

When we substitute $x = 3$ in the quadratic equation $x^2 - 7x + 12 = 0$, we get

$(3)^2 - 7.3 + 12 = 0$ *i.e.* $9 - 21 + 12 = 0$ *i.e.* $0 = 0$, which is true, therefore, 3 is a root of the quadratic equation $x^2 - 7x + 12 = 0$.

When we substitute $x = 2$ in the quadratic equation $x^2 - 7x + 12 = 0$, we get

$(2)^2 - 7.2 + 12 = 0$ *i.e.* $4 - 14 + 12 = 0$ *i.e.* $2 = 0$,

which is wrong, therefore, 2 is not a root of the quadratic equation $x^2 - 7x + 12 = 0$.

6.2 SOLVING QUADRATIC EQUATIONS BY FACTORISATION

Factorisation can be used to solve a quadratic equation. The equation $x^2 - 7x + 12 = 0$ can be written as $(x - 3)(x - 4) = 0$. This equation can be solved by using a property of real numbers called *zero-product rule.*

Zero-Product Rule

If a and b are two numbers or expressions and if $ab = 0$, *then either* $a = 0$ *or* $b = 0$ *or both* $a = 0$ *and* $b = 0$.

Using the above rule, the solutions of the equation $(x - 3)(x - 4) = 0$ can be found by putting each factor equal to zero and then solving for x. Thus we get,

$x - 3 = 0$ or $x - 4 = 0$

$\Rightarrow \quad x = 3$ or $x = 4$.

Hence, the solutions of the equation $x^2 - 7x + 12 = 0$ are 3, 4.

6.2.1 To solve a quadratic equation by factorisation

Proceed as under :

(*i*) Clear all fractions and square roots on variable (if any).

(*ii*) Write the equation in the form $ax^2 + bx + c = 0$.

(*iii*) Factorise the left side into a product of two factors.

(*iv*) Use the zero-product rule to solve the resulting equation.

Remark

The solutions (roots) may be checked by substituting in the original equation.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following equations :*

(*i*) $2x^2 = 3x$ (*ii*) $(x + 3)(x - 3) = 40$.

Solution. (*i*) Given $2x^2 = 3x$

$\Rightarrow \quad 2x^2 - 3x = 0$ (Writing as $ax^2 + bx + c = 0$)

$\Rightarrow \quad x(2x - 3) = 0$ (Factorising left side)

$\Rightarrow \quad x = 0$ or $2x - 3 = 0$ (Zero-product rule)

$\Rightarrow \quad x = 0$ or $2x = 3$

$\Rightarrow \quad x = 0$ or $x = \frac{3}{2}$.

Hence, the roots of the given equation are $0, \frac{3}{2}$.

(*ii*) Given $(x + 3)(x - 3) = 40$

$\Rightarrow \quad x^2 - 9 = 40 \Rightarrow x^2 - 9 - 40 = 0$

$\Rightarrow \quad x^2 - 49 = 0$ (Writing as $ax^2 + bx + c = 0$)

$\Rightarrow \quad (x + 7)(x - 7) = 0$ (Factorising left side)

$\Rightarrow \quad x + 7 = 0$ or $x - 7 = 0$ (Zero-product rule)

$\Rightarrow \quad x = -7$ or $x = 7$.

Hence, the roots of the given equation are $-7, 7$.

Example 2. *Solve the following equations for x :*

(*i*) $8x^2 + 15 = 26x$ (*ii*) $x(2x + 5) = 25$.

Solution. (*i*) Given $8x^2 + 15 = 26x$

$\Rightarrow \quad 8x^2 - 26x + 15 = 0$ (Writing as $ax^2 + bx + c = 0$)

$\Rightarrow \quad 8x^2 - 20x - 6x + 15 = 0$

$\Rightarrow \quad 4x(2x - 5) - 3(2x - 5) = 0$

$\Rightarrow \quad (2x - 5)(4x - 3) = 0$ (Factorising left side)

$\Rightarrow \quad 2x - 5 = 0$ or $4x - 3 = 0$ (Zero-product rule)

$\Rightarrow \quad x = \frac{5}{2}$ or $x = \frac{3}{4}$.

Hence, the roots of the given equation are $\frac{5}{2}, \frac{3}{4}$.

(*ii*) Given $x(2x + 5) = 25$

$\Rightarrow \quad 2x^2 + 5x - 25 = 0$ (Writing as $ax^2 + bx + c = 0$)

$\Rightarrow \quad 2x^2 + 10x - 5x - 25 = 0$

$\Rightarrow \quad 2x(x + 5) - 5(x + 5) = 0$

$\Rightarrow \quad (x + 5)(2x - 5) = 0$ (Factorising left side)

$\Rightarrow \quad x + 5 = 0$ or $2x - 5 = 0$ (Zero-product rule)

$\Rightarrow \quad x = -5$ or $x = \frac{5}{2}$.

Hence, the roots of the given equation are $-5, \frac{5}{2}$.

Example 3. *Solve the equation $3x^2 - 14x + 8 = 0$ when*

(i) $x \in N$ (ii) $x \in Q$.

Solution. Given $3x^2 - 14x + 8 = 0$

$\Rightarrow \quad 3x^2 - 12x - 2x + 8 = 0$

$\Rightarrow \quad 3x(x - 4) - 2(x - 4) = 0$

$\Rightarrow \quad (x - 4)(3x - 2) = 0$ (Factorising left side)

$\Rightarrow \quad x - 4 = 0$ or $3x - 2 = 0$ (Zero-product rule)

$\Rightarrow \quad x = 4$ or $x = \frac{2}{3}$.

(*i*) When $x \in \mathbf{N}$:

As $4 \in \mathbf{N}$ and $\frac{2}{3} \notin \mathbf{N}$, therefore, the given equation has 4 as its root.

(*ii*) When $x \in \mathbf{Q}$:

As $4, \frac{2}{3} \in \mathbf{Q}$, therefore, the given equation has $4, \frac{2}{3}$ as its roots.

Example 4. *Solve for $x : x^2 + (3 - 2a)x - 6a = 0$.*

Solution. Given $x^2 + (3 - 2a)x - 6a = 0$

$\Rightarrow \quad x^2 + 3x - 2ax - 6a = 0$

$\Rightarrow \quad x(x + 3) - 2a(x + 3) = 0$

$\Rightarrow \quad (x + 3)(x - 2a) = 0$ (Factorising left side)

$\Rightarrow \quad x + 3 = 0$ or $x - 2a = 0$ (Zero-product rule)

$\Rightarrow \quad x = -3$ or $x = 2a$.

Hence, the roots of the given equation are $-3, 2a$.

Example 5. *Solve the equation $3a^2x^2 + 8abx + 4b^2 = 0, a \neq 0$.*

Solution. Given $3a^2x^2 + 8abx + 4b^2 = 0$

$\Rightarrow \quad 3a^2x^2 + 6abx + 2abx + 4b^2 = 0$

$\Rightarrow \quad 3ax(ax + 2b) + 2b(ax + 2b) = 0$

$\Rightarrow \quad (ax + 2b)(3ax + 2b) = 0$ (Factorising left side)

$\Rightarrow \quad ax + 2b = 0$ or $3ax + 2b = 0$ (Zero-product rule)

$\Rightarrow \quad ax = -2b$ or $3ax = -2b$

$\Rightarrow \quad x = -\frac{2b}{a}$ or $x = -\frac{2b}{3a}$.

Hence, the roots of the given equation are $-\frac{2b}{a}, -\frac{2b}{3a}$.

Example 6. *Solve for $x : \sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$.*

Solution. Given $\sqrt{3}x^2 + 11x + 6\sqrt{3} = 0$

$\Rightarrow \quad \sqrt{3}x^2 + 9x + 2x + 6\sqrt{3} = 0$ ($\because 9 + 2 = 11$ and $9.2 = \sqrt{3}.6\sqrt{3}$)

$\Rightarrow \quad \sqrt{3}x(x + 3\sqrt{3}) + 2(x + 3\sqrt{3}) = 0$

$\Rightarrow \quad (x + 3\sqrt{3})(\sqrt{3}x + 2) = 0$ (Factorising left side)

$\Rightarrow \quad x + 3\sqrt{3} = 0$ or $\sqrt{3}\,x + 2 = 0$ (Zero-product rule)

$\Rightarrow \quad x = -3\sqrt{3}$ or $x = -\dfrac{2}{\sqrt{3}}$.

Hence, the roots of the given equation are $-3\sqrt{3}, -\dfrac{2}{\sqrt{3}}$.

Example 7. *Solve* $\dfrac{7x+1}{7x+5} = \dfrac{3x+1}{5x+1}$.

Solution. Given $\dfrac{7x+1}{7x+5} = \dfrac{3x+1}{5x+1}$

$\Rightarrow \quad (5x + 1)(7x + 1) = (7x + 5)(3x + 1)$ (Clearing fractions)

$\Rightarrow \quad 35x^2 + 5x + 7x + 1 = 21x^2 + 15x + 7x + 5$

$\Rightarrow \quad 35x^2 + 12x + 1 = 21x^2 + 22x + 5$

$\Rightarrow \quad 35x^2 - 21x^2 + 12x - 22x + 1 - 5 = 0$

$\Rightarrow \quad 14x^2 - 10x - 4 = 0$

$\Rightarrow \quad 7x^2 - 5x - 2 = 0$ (Writing as $ax^2 + bx + c = 0$)

$\Rightarrow \quad 7x^2 - 7x + 2x - 2 = 0$

$\Rightarrow \quad 7x(x - 1) + 2(x - 1) = 0$

$\Rightarrow \quad (x - 1)(7x + 2) = 0$ (Factorising left side)

$\Rightarrow \quad x - 1 = 0$ or $7x + 2 = 0$ (Zero-product rule)

$\Rightarrow \quad x = 1$ or $x = -\dfrac{2}{7}$.

Hence, the roots of the given equation are $1, -\dfrac{2}{7}$.

Note

The given equation is valid only when $5x + 1 \neq 0$ and $7x + 5 \neq 0$ *i.e.* when $x \neq -\dfrac{1}{5}$ and $x \neq -\dfrac{5}{7}$.

Example 8. *Solve* $\dfrac{x-3}{x+3} + \dfrac{x+3}{x-3} = 2\dfrac{1}{2}$.

Solution. Given $\dfrac{x-3}{x+3} + \dfrac{x+3}{x-3} = \dfrac{5}{2}$.

To clear fractions, multiplying both sides by $2(x + 3)(x - 3)$, we get

$\quad 2(x - 3)^2 + 2(x + 3)^2 = 5(x + 3)(x - 3)$

$\Rightarrow \quad 2(x^2 - 6x + 9) + 2(x^2 + 6x + 9) = 5(x^2 - 9)$

$\Rightarrow \quad 2x^2 - 12x + 18 + 2x^2 + 12x + 18 = 5x^2 - 45$

$\Rightarrow \quad 4x^2 + 36 = 5x^2 - 45$

$\Rightarrow \quad 4x^2 - 5x^2 + 36 + 45 = 0 \Rightarrow 81 - x^2 = 0$

$\Rightarrow \quad x^2 - 81 = 0$ (Writing as $ax^2 + bx + c = 0$)

$\Rightarrow \quad (x - 9)(x + 9) = 0$ (Factorising left side)

$\Rightarrow \quad x - 9 = 0$ or $x + 9 = 0$ (Zero-product rule)

$\Rightarrow \quad x = 9$ or $x = -9$.

Hence, the roots of the given equation are $9, -9$.

Note

The given equation is valid only when $x + 3 \neq 0$ and $x - 3 \neq 0$ *i.e.* only when $x \neq -3$ and $x \neq 3$.

Example 9. *Solve the equation* $\frac{a}{x-b} + \frac{b}{x-a} = 2.$

Solution. Given $\frac{a}{x-b} + \frac{b}{x-a} = 2$

$\Rightarrow \quad \left(\frac{a}{x-b} - 1\right) + \left(\frac{b}{x-a} - 1\right) = 0$

$\Rightarrow \quad \frac{a-x+b}{x-b} + \frac{b-x+a}{x-a} = 0$

$\Rightarrow \quad (a + b - x)\left(\frac{1}{x-b} + \frac{1}{x-a}\right) = 0$ (Factorising left side)

$\Rightarrow \quad a + b - x = 0$ or $\frac{1}{x-b} + \frac{1}{x-a} = 0$ (Zero-product rule)

$\Rightarrow \quad x = a + b$ | $\Rightarrow \quad \frac{x-a+x-b}{(x-b)(x-a)} = 0$

$\Rightarrow \quad 2x - a - b = 0$

$\Rightarrow \quad x = \frac{a+b}{2}.$

Hence, the roots of the given equation are $a + b$, $\frac{a+b}{2}$.

Example 10. *Solve* $\sqrt{2x+9} = 13 - x.$

Solution. Given $\sqrt{2x+9} = 13 - x.$

Squaring both sides, we get

$2x + 9 = (13 - x)^2$

$\Rightarrow \quad 2x + 9 = 169 - 26x + x^2$

$\Rightarrow \quad -x^2 + 2x + 26x + 9 - 169 = 0$

$\Rightarrow \quad -x^2 + 28x - 160 = 0$

$\Rightarrow \quad x^2 - 28x + 160 = 0$

$\Rightarrow \quad (x - 8)(x - 20) = 0$ (Factorising left side)

$\Rightarrow \quad x - 8 = 0$ or $x - 20 = 0$ (Zero-product rule)

$\Rightarrow \quad x = 8$ or $x = 20.$

But $x = 20$ does not satisfy the given equation, so $x = 20$ is rejected.

Hence, the root of the given equation is 8.

Caution

When the squaring of both sides of the equation is done, the roots of the final equation must be checked to determine whether they are roots of the original equation or not. Although no root of the original equation will be lost by squaring but certain values may be introduced which are roots of the new equation but not of the original equation. It may be noted that on putting $x = 20$ in the given equation, we get $\sqrt{2 \times 20 + 9} = 13 - 20$

i.e. $\sqrt{49} = -7$ *i.e.* $7 = -7$, which is wrong.

Example 11. *Solve* $2x - 3 = \sqrt{2x^2 - 2x + 21}.$

Solution. Given $2x - 3 = \sqrt{2x^2 - 2x + 21}.$

On squaring both sides, we get

$(2x - 3)^2 = 2x^2 - 2x + 21$

$\Rightarrow \quad 4x^2 - 12x + 9 = 2x^2 - 2x + 21$

$\Rightarrow \quad 4x^2 - 2x^2 - 12x + 2x + 9 - 21 = 0$

$\Rightarrow \quad 2x^2 - 10x - 12 = 0$

$\Rightarrow \quad x^2 - 5x - 6 = 0$

$\Rightarrow \quad (x-6)(x+1) = 0$ (Factorising left side)

$\Rightarrow \quad x-6=0$ or $x+1=0$ (Zero-product rule)

$\Rightarrow \quad x = 6$ or $x = -1$.

But $x = -1$ does not satisfy the given equation, so -1 is rejected.

Hence, the root of the given equation is 6.

Example 12. *Use the substitution $y = 2x - 1$ to solve for x : $3(2x-1)^2 + 4(2x-1) - 4 = 0$.*

Solution. Given $3(2x-1)^2 + 4(2x-1) - 4 = 0$

$\Rightarrow \quad 3y^2 + 4y - 4 = 0$ (Putting $2x - 1 = y$)

$\Rightarrow \quad 3y^2 + 6y - 2y - 4 = 0$

$\Rightarrow \quad 3y(y+2) - 2(y+2) = 0$

$\Rightarrow \quad (y+2)(3y-2) = 0$ (Factorizing left side)

$\Rightarrow \quad y+2=0$ or $3y-2=0$ (Zero-product rule)

$\Rightarrow \quad y = -2$ or $y = \frac{2}{3}$.

When $y = -2$, $2x - 1 = -2 \Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$;

when $y = \frac{2}{3}$, $2x - 1 = \frac{2}{3} \Rightarrow 2x = \frac{2}{3} + 1 = \frac{5}{3} \Rightarrow x = \frac{5}{6}$.

Hence, the roots of the given equation are $-\frac{1}{2}, \frac{5}{6}$.

Exercise 6.1

1. Determine whether $x = \frac{1}{2}$ and $x = \frac{3}{2}$ are the solutions of the equation $2x^2 - 5x + 3 = 0$ or not.

Solve the following equations (2 to 25) by factorisation :

2. (*i*) $4x^2 = 3x$ (*ii*) $\frac{x^2 - 5x}{2} = 0$.
3. (*i*) $(x-3)(2x+5) = 0$ (*ii*) $x(2x+1) = 6$.
4. (*i*) $x^2 - 3x - 10 = 0$ (*ii*) $x(2x+5) = 3$.
5. (*i*) $3x^2 - 5x - 12 = 0$ (*ii*) $21x^2 - 8x - 4 = 0$.
6. (*i*) $3x^2 = x + 4$ (*ii*) $x(6x-1) = 35$.
7. (*i*) $6p^2 + 11p - 10 = 0$ (*ii*) $\frac{2}{3}x^2 - \frac{1}{3}x = 1$.
8. (*i*) $(x-4)^2 + 5^2 = 13^2$ (*ii*) $3(x-2)^2 = 147$.
9. (*i*) $\frac{1}{7}(3x-5)^2 = 28$ (*ii*) $3(y^2 - 6) = y(y+7) - 3$.
10. $x^2 - 4x - 12 = 0$ when $x \in \mathbf{N}$.
11. $2x^2 - 8x - 24 = 0$ when $x \in \mathbf{I}$.
12. $5x^2 - 8x - 4 = 0$ when $x \in \mathbf{Q}$.
13. $2x^2 - 9x + 10 = 0$ when (*i*) $x \in \mathbf{N}$ (*ii*) $x \in \mathbf{Q}$.
14. (*i*) $a^2x^2 + 2ax + 1 = 0$, $a \neq 0$ (*ii*) $x^2 - (p+q)x + pq = 0$.
15. $a^2x^2 + (a^2 + b^2)x + b^2 = 0$, $a \neq 0$.
16. (*i*) $\sqrt{3}x^2 + 10x + 7\sqrt{3} = 0$ (*ii*) $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 0$.
17. (*i*) $x^2 - (1 + \sqrt{2})x + \sqrt{2} = 0$ (*ii*) $x + \frac{1}{x} = 2\frac{1}{20}$.

18. (i) $3x - \frac{8}{x} = 2$ (ii) $\frac{x+2}{x+3} = \frac{2x-3}{3x-7}$.

19. (i) $\frac{8}{x+3} - \frac{3}{2-x} = 2$ (ii) $\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}$.

20. (i) $\frac{x}{x+1} + \frac{x+1}{x} = \frac{34}{15}$ (ii) $\frac{x+1}{x-1} + \frac{x-2}{x+2} = 3$.

21. (i) $\frac{1}{x-3} - \frac{1}{x+5} = \frac{1}{6}$ (ii) $\frac{x-1}{x-2} + \frac{x-3}{x-4} = 3\frac{1}{3}$.

22. $\frac{a}{ax-1} + \frac{b}{bx-1} = a + b,\ a + b \neq 0,\ ab \neq 0.$

Hint

$\left(\frac{a}{ax-1} - b\right) + \left(\frac{b}{bx-1} - a\right) = 0$.

23. $\frac{1}{p} + \frac{1}{q} + \frac{1}{x} = \frac{1}{p+q+x},\ p + q \neq 0,\ p \neq 0,\ q \neq 0.$

24. $\frac{1}{x+6} + \frac{1}{x-10} = \frac{3}{x-4}$.

25. (i) $\sqrt{3x+4} = x$ (ii) $\sqrt{x(x-7)} = 3\sqrt{2}$.

26. Use the substitution $y = 3x + 1$ to solve for x :
$5(3x+1)^2 + 6(3x+1) - 8 = 0.$

27. Find the values of x if $p + 1 = 0$ and $x^2 + px - 6 = 0$.

28. Find the values of x if $p + 7 = 0$, $q - 12 = 0$ and $x^2 + px + q = 0$.

29. If $x = p$ is a solution of the equation $x(2x + 5) = 3$, then find the values of p.

6.3 SOLVING QUADRATIC EQUATIONS BY FORMULA

A general method of solving a quadratic equation :

Let the general quadratic equation be $ax^2 + bx + c = 0,\ a \neq 0$

$\Rightarrow \quad ax^2 + bx = -c$ (Transposing c)

$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$ (Dividing both sides by a)

Adding $\left(\frac{1}{2}\text{ coeff. of } x\right)^2$ *i.e.* $\left(\frac{b}{2a}\right)^2$ to both sides to make L.H.S. a perfect square, we get

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{(Taking square root)}$$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Hence, the roots of the given equation are $\frac{-b+\sqrt{b^2-4ac}}{2a}$, $\frac{-b-\sqrt{b^2-4ac}}{2a}$.

Alternative method (Sridhara Acharyya's method) :

Let the general quadratic equation be $ax^2 + bx + c = 0,\ a \neq 0$.

Multiplying both sides by $4a$, we get

$$4a^2x^2 + 4abx + 4ac = 0$$

$$\Rightarrow \quad (2ax)^2 + 2.2ax.b = -4ac \qquad \text{(Transposing } 4ac\text{)}$$

Adding b^2 to both sides to make L.H.S. a perfect square, we get

$$(2ax)^2 + 2.2ax.b + b^2 = b^2 - 4ac$$

$$\Rightarrow \quad (2ax + b)^2 = b^2 - 4ac$$

$$\Rightarrow \quad 2ax + b = \pm\sqrt{b^2 - 4ac} \qquad \text{(Taking square root)}$$

$$\Rightarrow \quad 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

6.3.1 To solve a quadratic equation by use of formula

Proceed as under :

(*i*) Clear all fractions and square roots on variable (if any).

(*ii*) Write the equation in the form $ax^2 + bx + c = 0$, $a \neq 0$.

(*iii*) Compare the equation obtained in step (*ii*) with the equation
$ax^2 + bx + c = 0$.

(*iv*) Use the formula : $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve $x^2 - 4x + 1 = 0$, by using formula.* (2003)

Solution. The given equation is $x^2 - 4x + 1 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$$a = 1, b = -4, c = 1.$$

By using the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.1}}{2.1} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}.$$

Hence, the roots of the given equation are $2 + \sqrt{3}$, $2 - \sqrt{3}$.

Example 2. *Solve the equation $3x^2 - 4x - 4 = 0$ by using formula.*

Solution. The given equation is $3x^2 - 4x - 4 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$$a = 3, b = -4, c = -4.$$

By using the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.3.(-4)}}{2.3} = \frac{4 \pm \sqrt{16 + 48}}{6} = \frac{4 \pm \sqrt{64}}{6}$$

$$= \frac{4 \pm 8}{6} = \frac{4+8}{6}, \frac{4-8}{6} = \frac{12}{6}, \frac{-4}{6} = 2, -\frac{2}{3}.$$

Hence, the roots of the given equation are $2, -\dfrac{2}{3}$.

Example 3. *Solve $2x^2 + \sqrt{7}x - 7 = 0$ by using formula.*

Solution. The given equation is $2x^2 + \sqrt{7}x - 7 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$$a = 2, b = \sqrt{7}, c = -7.$$

By using the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain

$$x = \frac{-\sqrt{7} \pm \sqrt{(\sqrt{7})^2 - 4.2.(-7)}}{2.2} = \frac{-\sqrt{7} \pm \sqrt{63}}{4}$$

$$= \frac{-\sqrt{7} \pm 3\sqrt{7}}{4} = \frac{2\sqrt{7}}{4}, \frac{-4\sqrt{7}}{4} = \frac{\sqrt{7}}{2}, -\sqrt{7}.$$

Hence, the roots of the given equation are $\dfrac{\sqrt{7}}{2}, -\sqrt{7}$.

Example 4. *Solve the equation $4x^2 - 2x + \frac{1}{4} = 0$.*

Solution. The given equation is $4x^2 - 2x + \dfrac{1}{4} = 0$

$\Rightarrow\ 16x^2 - 8x + 1 = 0$ (Multiplying by 4)

Comparing it with $ax^2 + bx + c = 0$, we get $a = 16, b = -8, c = 1$.

By using the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain

$$\therefore \quad x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4.16.1}}{2.16} = \frac{8 \pm \sqrt{64 - 64}}{32}$$

$$= \frac{8 \pm 0}{32} = \frac{8}{32}, \frac{8}{32} = \frac{1}{4}, \frac{1}{4}.$$

Hence, the roots of the given equation are $\dfrac{1}{4}, \dfrac{1}{4}$.

Example 5. *Solve the equation $3x^2 - x - 7 = 0$ and give your answer correct to two decimal places.* *(2004)*

Solution. The given equation is $3x^2 - x - 7 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$$a = 3, b = -1, c = -7.$$

By using formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.3.(-7)}}{2.3} = \frac{1 \pm \sqrt{85}}{6}.$$

Also $\sqrt{85} = 9{\cdot}219$ (obtain it),

$$\therefore \quad x = \frac{1 \pm 9{\cdot}219}{6} = \frac{10{\cdot}219}{6}, -\frac{8{\cdot}219}{6}$$

$= 1{\cdot}70, -1{\cdot}37$ (correct to two decimal places).

Hence, the roots of the given equations are $1{\cdot}70, -1{\cdot}37$ (correct to two decimal places).

Example 6. *Solve the following quadratic equation and give the answer correct to two significant figures :*

$$4x^2 - 7x + 2 = 0.$$ *(2009)*

Solution. The given equation is $4x^2 - 7x + 2 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$$a = 4, \quad b = -7, \quad c = 2.$$

By using formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4.4.2}}{2.4} = \frac{7 \pm \sqrt{49 - 32}}{8} = \frac{7 \pm \sqrt{17}}{8}.$$

Also $\sqrt{17} = 4{\cdot}123$ (obtain it)

$$\therefore \quad x = \frac{7 \pm 4{\cdot}123}{8} = \frac{11{\cdot}123}{8}, \frac{2{\cdot}877}{8} = 1{\cdot}39,\ 0{\cdot}359$$

$$= 1{\cdot}4,\ 0{\cdot}36 \text{ (correct to two significant figures).}$$

Hence, the roots of the given equation are 1·4, 0·36 (correct to two significant figures).

Example 7. *Solve the equation* $\frac{1}{x+1} + \frac{2}{x+2} = \frac{4}{x+4}$.

Solution. The given equation is $\frac{1}{x+1} + \frac{2}{x+2} = \frac{4}{x+4}$

$$\Rightarrow \quad \frac{1.(x+2) + 2.(x+1)}{(x+1)(x+2)} = \frac{4}{x+4}$$

$$\Rightarrow \quad \frac{3x+4}{(x+1)(x+2)} = \frac{4}{x+4}.$$

To clear fractions, multiplying both sides by $(x + 1)(x + 2)(x + 4)$, we get

$$(3x + 4)(x + 4) = 4(x + 1)(x + 2)$$

$$\Rightarrow \quad 3x^2 + 12x + 4x + 16 = 4(x^2 + 2x + x + 2)$$

$$\Rightarrow \quad 3x^2 + 16x + 16 = 4x^2 + 12x + 8$$

$$\Rightarrow \quad -x^2 + 4x + 8 = 0$$

$$\Rightarrow \quad x^2 - 4x - 8 = 0.$$

Comparing it with $ax^2 + bx + c = 0$, we get $a = 1$, $b = -4$, $c = -8$.

$$\therefore \quad x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.(-8)}}{2.1} \qquad \left|\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right.$$

$$= \frac{4 \pm \sqrt{16 + 32}}{2} = \frac{4 \pm \sqrt{48}}{2}$$

$$= \frac{4 \pm 4\sqrt{3}}{2} = 2 \pm 2\sqrt{3}.$$

Hence, the roots of the given equation are $2 + 2\sqrt{3}$, $2 - 2\sqrt{3}$.

Example 8. *Find the roots of the equation* $x^2 + x - (a + 2)(a + 1) = 0$.

Solution. The given equation is $x^2 + x - (a + 2)(a + 1) = 0$.

Comparing it with $Ax^2 + Bx + C = 0$, we get

$$A = 1, B = 1, C = -(a + 2)(a + 1) = -(a^2 + 3a + 2).$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{1^2 - 4.1.(-(a^2 + 3a + 2))}}{2.1} \qquad \left|\ \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\right.$$

$$= \frac{-1 \pm \sqrt{1 + 4(a^2 + 3a + 2)}}{2} = \frac{-1 \pm \sqrt{4a^2 + 12a + 9}}{2}$$

$$= \frac{-1 \pm \sqrt{(2a + 3)^2}}{2} = \frac{-1 \pm (2a + 3)}{2}$$

$$= \frac{-1+2a+3}{2}, \frac{-1-2a-3}{2} = \frac{2a+2}{2}, \frac{-(2a+4)}{2} = a+1, -(a+2).$$

Hence, the roots of the given equation are $(a+1), -(a+2)$.

Exercise 6.2

Solve the following (1 to 7) equations by using formula :

1. (i) $2x^2 - 7x + 6 = 0$ (ii) $2x^2 - 6x + 3 = 0$.
2. (i) $x^2 + 7x - 7 = 0$ (ii) $(2x+3)(3x-2) + 2 = 0$.
3. (i) $256x^2 - 32x + 1 = 0$ (ii) $25x^2 + 30x + 7 = 0$.
4. (i) $2x^2 + \sqrt{5}\,x - 5 = 0$ (ii) $\sqrt{3}\,x^2 + 10x - 8\sqrt{3} = 0$.
5. (i) $\frac{x-2}{x+2} + \frac{x+2}{x-2} = 4$ (ii) $\frac{x+1}{x+3} = \frac{3x+2}{2x+3}$.
6. (i) $a(x^2+1) = (a^2+1)x,\ a \neq 0$ (ii) $4x^2 - 4ax + (a^2 - b^2) = 0$.
7. $\frac{1}{x-2} + \frac{1}{x-3} + \frac{1}{x-4} = 0$.
8. Solve the following quadratic equations for x and give your answer correct to 2 decimal places :
   (i) $x^2 - 5x - 10 = 0$ *(2013)* (ii) $5x(x+2) = 3$. *(2008)*
9. Solve the following equations by using quadratic formula and give your answer correct to 2 decimal places :
   (i) $x^2 - 3x - 9 = 0$ *(2007)* (ii) $2x - \frac{1}{x} = 7$. *(2006)*
10. Solve the equation : $x - \frac{18}{x} = 6$. Give your answer correct to two significant figures. *(2011)*
11. Solve the equation $5x^2 - 3x - 4 = 0$ and give your answer correct to 3 significant figures. *(2012)*

6.4 NATURE OF ROOTS OF A QUADRATIC EQUATION

In the previous section, we have seen that the roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a, b, c \in \mathbf{R}.$$

Thus, the nature of the roots depends on the quantity under the square root sign *i.e.* $b^2 - 4ac$. This quantity is called **discriminant** of the equation and is usually denoted by D.

$\therefore$ **Discriminant D = $b^2 - 4ac$.**

The following three cases arise :

Case I. If $b^2 - 4ac > 0$, we get two distinct real roots

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Case II. If $b^2 - 4ac = 0$, then $x = \frac{-b \pm 0}{2a} = -\frac{b}{2a}, -\frac{b}{2a}$.

So, the roots of the given equation $ax^2 + bx + c = 0$ are both $-\frac{b}{2a}$. Therefore, we say that the equation has two equal real roots.

Case III. If $b^2 - 4ac < 0$, then there is no real number whose square is $b^2 - 4ac$. Therefore, we say that the given equation has no real roots.

Hence, a quadratic equation $ax^2 + bx + c = 0, a \neq 0$ has

(*i*) **two distinct (unequal) real roots if $b^2 - 4ac > 0$.**

(*ii*) **two equal real roots if $b^2 - 4ac = 0$.**

(*iii*) **no real roots if $b^2 - 4ac < 0$.**

Remember the following :

Table pdf

Discriminant of $ax^2 + bx + c = 0$	*Nature of roots of* $ax^2 + bx + c = 0$	*The roots of* $ax^2 + bx + c = 0$
$b^2 - 4ac > 0$	*two distinct real roots*	$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$b^2 - 4ac = 0$	*two equal real roots*	$-\dfrac{b}{2a}, -\dfrac{b}{2a}$
$b^2 - 4ac < 0$	*not real*	*no real roots*

Note

The quadratic equation $ax^2 + bx + c = 0, a \neq 0$, has real roots if $b^2 - 4ac \geq 0$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the discriminant of the equation $4x^2 - 12x + 9 = 0$, and hence find the nature of the roots.*

Solution. The given equation is $4x^2 - 12x + 9 = 0$

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 4, b = -12, c = 9.$

$\therefore$ Discriminant $= b^2 - 4ac = (-12)^2 - 4.4.9 = 144 - 144 = 0.$

Hence, the given equation has two equal real roots.

Example 2. *Discuss the nature of the roots of the following equations:*

(*i*) $2x^2 - 4x + 3 = 0$ (*ii*) $4x^2 - 5x - 3 = 0$.

Solution. (*i*) The given equation is $2x^2 - 4x + 3 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 2, b = -4, c = 3.$

$\therefore$ Discriminant $= b^2 - 4ac = (-4)^2 - 4.2.3$

$= 16 - 24 = -8 < 0.$

Hence, the given equation has no real roots.

(*ii*) The given equation is $4x^2 - 5x - 3 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 4, b = -5, c = -3.$

$\therefore$ Discriminant $= (-5)^2 - 4.4.(-3)$

$= 25 + 48 = 73 > 0.$

Hence, the given equation has two distinct real roots.

Example 3. *Discuss the nature of the roots of the following equations:*

(*i*) $x^2 + 2x + \frac{1}{3} = 0$ (*ii*) $\sqrt{3}\,x^2 - 5x + 7\sqrt{3} = 0$.

Solution. (*i*) The given equation is $x^2 + 2x + \frac{1}{3} = 0$ *i.e.* $3x^2 + 6x + 1 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 3, b = 6, c = 1.$

$\therefore$ Discriminant $= b^2 - 4ac = (6)^2 - 4.3.1$

$= 36 - 12 = 24 > 0.$

Hence, the given equation has two distinct real roots.

(*ii*) The given equation is $\sqrt{3}x^2 - 5x + 7\sqrt{3} = 0.$

Comparing it with $ax^2 + bx + c = 0$, we get

$a = \sqrt{3}, b = -5, c = 7\sqrt{3}.$

$\therefore$ Discriminant $= b^2 - 4ac = (-5)^2 - 4.\sqrt{3}.7\sqrt{3} = 25 - 84$

$= -59 < 0.$

Hence, the given equation has no real roots.

Example 4. *Find the nature of the roots of the equation $3x^2 - 7x + \frac{1}{2} = 0$. If real roots exist, find them.*

Solution. The given equation is $3x^2 - 7x + \frac{1}{2} = 0$ *i.e.* $6x^2 - 14x + 1 = 0.$

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 6, b = -14, c = 1.$

$\therefore$ Discriminant $= b^2 - 4ac = (-14)^2 - 4 . 6. 1$

$= 196 - 24 = 172 > 0.$

Hence, the given equations has two distinct real roots.

The roots of the equation are given by

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4.6.1}}{2.6} \qquad \left| \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right.$$

$$= \frac{14 \pm \sqrt{172}}{12} = \frac{14 \pm 2\sqrt{43}}{12} = \frac{7 \pm \sqrt{43}}{6}.$$

The roots of the given equation are $\frac{7 + \sqrt{43}}{6}, \frac{7 - \sqrt{43}}{6}.$

Example 5. *Find the value of k so that the equation $2x^2 - 5x + k = 0$ has two equal roots.*

Solution. The given equation is $2x^2 - 5x + k = 0.$

It will have two equal roots if its discriminant = 0

$\Rightarrow \quad (-5)^2 - 4.2.k = 0$

$\Rightarrow \quad 25 - 8k = 0 \Rightarrow k = \frac{25}{8}.$

Example 6. *Find the values of p for which the equation $px^2 + 3x + 2 = 0$ has real roots.*

Solution. The given equation is $px^2 + 3x + 2 = 0.$

It will have real roots if its discriminant ≥ 0

$\Rightarrow \quad 3^2 - 4.p.2 \geq 0 \Rightarrow 9 - 8p \geq 0$

$\Rightarrow \quad 9 \geq 8p \qquad \Rightarrow \frac{9}{8} \geq p$

$\Rightarrow \quad p \leq \frac{9}{8}.$

Example 7. *Find the values of m so that the equation $(4 + m)x^2 + (m + 1)x + 1 = 0$ has equal roots.*

Solution. The given equation is $(4 + m)x^2 + (m + 1)x + 1 = 0.$

It will have equal roots if its discriminant = 0

$\Rightarrow \quad (m + 1)^2 - 4.(4 + m).1 = 0$

$\Rightarrow \quad m^2 + 2m + 1 - 16 - 4m = 0$

$\Rightarrow \quad m^2 - 2m - 15 = 0$

$\Rightarrow \quad m^2 - 5m + 3m - 15 = 0$

$\Rightarrow \quad m(m-5) + 3(m-5) = 0$

$\Rightarrow \quad (m-5)(m+3) = 0$

$\Rightarrow \quad m - 5 = 0$ or $m + 3 = 0$

$\Rightarrow \quad m = 5, -3.$

Hence, the values of m are 5, –3.

Example 8. *Find the values of k so that the equation $3x^2 + kx + 2 = 0$ has equal roots. Also, find the roots in each case.*

Solution. The given equation is $3x^2 + kx + 2 = 0$ ...(*i*)

The equation (*i*) will have equal roots if its discriminant = 0

$\Rightarrow \quad (k)^2 - 4 . 3 . 2 = 0 \Rightarrow k^2 = 24 \Rightarrow k = \pm 2\sqrt{6}.$

$\therefore$ The given equation has equal roots if $k = 2\sqrt{6}$ or $-2\sqrt{6}$.

When k = $2\sqrt{6}$, the equation (*i*) becomes $3x^2 + 2\sqrt{6}x + 2 = 0$.

Its roots are given by

$$x = \frac{-2\sqrt{6} \pm \sqrt{(2\sqrt{6})^2 - 4.3.2}}{2.3} = \frac{-2\sqrt{6} \pm 0}{6} = -\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}.$$

$\therefore$ The roots are $-\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}$.

When k = $-2\sqrt{6}$, the equation (*i*) becomes $3x^2 - 2\sqrt{6}x + 2 = 0$.

Its roots are given by

$$x = \frac{2\sqrt{6} \pm \sqrt{(-2\sqrt{6})^2 - 4.3.2}}{2.3} = \frac{2\sqrt{6} \pm 0}{6} = \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}.$$

$\therefore$ The roots are $\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3}$.

Exercise 6.3

1. Find the discriminant of the following quadratic equations and hence find the nature of roots :
   (*i*) $3x^2 - 5x - 2 = 0$ (*ii*) $2x^2 - 3x + 5 = 0$
   (*iii*) $7x^2 + 8x + 2 = 0$ (*iv*) $3x^2 + 2x - 1 = 0$
   (*v*) $16x^2 - 40x + 25 = 0$ (*vi*) $2x^2 + 15x + 30 = 0$.
2. Discuss the nature of the roots of the following quadratic equations :
   (*i*) $x^2 - 4x - 1 = 0$ (*ii*) $3x^2 - 2x + \frac{1}{3} = 0$
   (*iii*) $3x^2 - 4\sqrt{3}x + 4 = 0$ (*iv*) $x^2 - \frac{1}{2}x + 4 = 0$
   (*v*) $-2x^2 + x + 1 = 0$ (*vi*) $2\sqrt{3}x^2 - 5x + \sqrt{3} = 0$.
3. Find the nature of the roots of the following quadratic equations :
   (*i*) $x^2 - \frac{1}{2}x - \frac{1}{2} = 0$ (*ii*) $x^2 - 2\sqrt{3}x - 1 = 0$.
   If real roots exist, find them.
4. Without solving the following quadratic equations, find the value of 'p' for which the given equations have real and equal roots:
   (*i*) $px^2 - 4x + 3 = 0$ *(2010)* (*ii*) $x^2 + (p-3)x + p = 0$. *(2013)*
5. Find the value(s) of k for which each of the following quadratic equation has equal roots :
   (*i*) $kx^2 - 4x - 5 = 0$ (*ii*) $2x^2 + kx + 3 = 0$
   (*iii*) $(k-4)x^2 + 2(k-4)x + 4 = 0$ (*iv*) $kx(x-2) + 6 = 0$.

6. Find the value(s) of m for which each of the following quadratic equation has real and equal roots :
   (*i*) $(3m + 1)x^2 + 2(m + 1)x + m = 0$ (*ii*) $x^2 + 2(m - 1)x + (m + 5) = 0$. *(2012)*
7. Find the value(s) of p for which each of the following quadratic equation has real roots :
   (*i*) $px^2 + 4x + 1 = 0$ (*ii*) $4x^2 + 8x - p = 0$.
8. Find the values of p for which the equation $3x^2 - px + 5 = 0$ has real roots.
9. Find the values of k for which each of the following quadratic equation has equal roots :
   (*i*) $9x^2 + kx + 1 = 0$ (*ii*) $x^2 - 2kx + 7k - 12 = 0$.
   Also, find the roots for those values of k in each case.

6.5 PROBLEMS ON QUADRATIC EQUATIONS

In chapter 9 (Understanding I.C.S.E. Mathematics Class IX), we explained the term *'applied (or word) problem'*. Solving word (or applied) problem involves two steps. First, translating the words of the problem into an algebraic equation. Second, solving the resulting equation.

6.5.1 Solving applied problems

Due to the wide variety of applied problems, there is no single solving technique that works in all cases. However, the following general suggestions should prove helpful :

(*i*) Read the statement of the problem carefully, and determine what quantity (or quantities) must be found.

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(*ii*) Represent the unknown quantity (or quantities) by a letter (or letters).

(*iii*) Determine which expressions are equal and write an equation (or equations).

(*iv*) Solve the resulting equation (or equations).

Remark

Check the answer (or answers) obtained by determining whether or not they fulfil the condition (or conditions) of the original problem.

ILLUSTRATIVE EXAMPLES

Example 1. *Nine times a certain whole number is equal to five less than twice the square of the number. Find the number.*

Solution. Let the required whole number be x.

Given, nine times the number = 5 less than twice the square of the number

$\Rightarrow \quad 9x = 2x^2 - 5 \Rightarrow 2x^2 - 9x - 5 = 0$

$\Rightarrow \quad (x - 5)(2x + 1) = 0$

$\Rightarrow \quad x - 5 = 0$ or $2x + 1 = 0$

$\Rightarrow \quad x = 5$ or $x = -\frac{1}{2}$.

But x is a whole number, so we reject $x = -\frac{1}{2}$.

Hence, the required whole number is 5.

Example 2. *The sum of two natural numbers is 8. Determine the numbers, if the sum of their reciprocals is* $\frac{8}{15}$.

Solution. As the sum of two natural numbers is 8, let the numbers be x, $8 - x$ where $x \in \mathbf{N}$ and $x < 8$.

According to given, $\frac{1}{x} + \frac{1}{8-x} = \frac{8}{15}$

$\Rightarrow \quad \frac{8-x+x}{x(8-x)} = \frac{8}{15} \quad \Rightarrow \quad \frac{8}{x(8-x)} = \frac{8}{15}$

$\Rightarrow \quad 15 = x(8-x) \quad \Rightarrow \quad 15 = 8x - x^2$

$\Rightarrow \quad x^2 - 8x + 15 = 0$

$\Rightarrow \quad (x-3)(x-5) = 0$

$\Rightarrow \quad x - 3 = 0$ or $x - 5 = 0$

$\Rightarrow \quad x = 3$ or $x = 5$.

When $x = 3$, then the numbers are 3, 8 – 3 *i.e.* 3, 5.

When $x = 5$, then the numbers ae 5, 8 – 5 *i.e.* 5, 3.

Hence, the required numbers are 3, 5. $\left[\textit{Check. } 3 + 5 = 8 \textit{ and } \frac{1}{3} + \frac{1}{5} = \frac{8}{15}\right]$

Example 3. *The difference of two natural numbers is 4 and the difference of their reciprocals is* $\frac{1}{8}$. *Find the numbers.*

Solution. As the difference of two natural numbers is 4, let the numbers be x and $x + 4$, where $x \in \mathbf{N}$.

We note that $x + 4 > x \Rightarrow \frac{1}{x+4} < \frac{1}{x} \Rightarrow \frac{1}{x} - \frac{1}{x+4} > 0$.

According to given, $\frac{1}{x} - \frac{1}{x+4} = \frac{1}{8}$

$\Rightarrow \quad \frac{(x+4)-x}{x(x+4)} = \frac{1}{8} \quad \Rightarrow \quad \frac{4}{x(x+4)} = \frac{1}{8}$

$\Rightarrow \quad x(x+4) = 32 \quad \Rightarrow \quad x^2 + 4x - 32 = 0$

$\Rightarrow \quad (x-4)(x+8) = 0 \quad \Rightarrow x - 4 = 0$ or $x + 8 = 0$

$\Rightarrow \quad x = 4$ or $x = -8$ but $x \in \mathbf{N}$

$\Rightarrow \quad x = 4$.

When $x = 4$, $x + 4 = 4 + 4 = 8$.

Hence, the required numbers are 4, 8.

Example 4. *Find two consecutive odd integers whose product is 195.*

Solution. Let the smaller odd integer be x, then the other consecutive odd integer is $x + 2$.

According to given, $x(x+2) = 195$

$\Rightarrow \quad x^2 + 2x - 195 = 0$

$\Rightarrow \quad (x-13)(x+15) = 0$

$\Rightarrow \quad x - 13 = 0$ or $x + 15 = 0$

$\Rightarrow \quad x = 13$ or $x = -15$.

When $x = 13$, $x + 2 = 15$ and when $x = -15$, $x + 2 = -13$.

Hence, the required integers are 13, 15 or – 15, – 13.

Note

If two consecutive odd natural numbers are required, then $x = -15$ is rejected and the answer would be 13, 15.

Example 5. *The sum of squares of two positive integers is 208. If the square of the larger number is 18 times the smaller number, find the numbers.*

Solution. Let the smaller number be x, then the square of the larger number = $18x$.

Given, sum of squares of numbers is 208

$\Rightarrow \quad x^2 + 18x = 208$

$\Rightarrow \quad x^2 + 18x - 208 = 0$

$\Rightarrow \quad (x - 8)(x + 26) = 0$

$\Rightarrow \quad x - 8 = 0$ or $x + 26 = 0$

$\Rightarrow \quad x = 8$ or $x = -26$.

Since the numbers are positive integers, we reject $x = -26$ and take $x = 8$.

$\therefore$ The square of the larger number = $18 \times 8 = 144$.

$\therefore$ The larger number = $\sqrt{144} = 12$.

Hence, the required numbers are 8, 12. *[Check. $8^2 + 12^2 = 208$ and $12^2 = 18 \times 8$]*

Example 6. *Three consecutive natural numbers are such that the square of the middle number exceeds the difference of the squares of the other two by 60. Assume the middle number to be x and form a quadratic equation satisfying the above statement. Hence, find the three numbers.*

Solution. Since the middle number of the three consecutive numbers is x, therefore, the other two numbers are $x - 1$ and $x + 1$. According to the given condition, we have

$x^2 = [(x + 1)^2 - (x - 1)^2] + 60$

$\Rightarrow \quad x^2 = (x^2 + 2x + 1) - (x^2 - 2x + 1) + 60$

$\Rightarrow \quad x^2 = 4x + 60$

$\Rightarrow \quad x^2 - 4x - 60 = 0$

$\Rightarrow \quad x^2 - 10x + 6x - 60 = 0$

$\Rightarrow \quad x(x - 10) + 6(x - 10) = 0$

$\Rightarrow \quad (x - 10)(x + 6) = 0$

$\Rightarrow \quad x - 10 = 0$ or $x + 6 = 0$

$\Rightarrow \quad x = 10$ or $x = -6$ but x cannot be negative,

$\therefore \quad x = 10$.

Then, $x - 1 = 9$ and $x + 1 = 11$.

Hence, the required numbers are 9, 10, 11. *[Check. $10^2 = (11^2 - 9^2) + 60$]*

Example 7. *The sum of the squares of three consecutive even integers is 200. Find the integers.*

Solution. Let the smallest even integer be x, then the other two consecutive even integers are $x + 2$ and $x + 4$.

According to given, $x^2 + (x + 2)^2 + (x + 4)^2 = 200$

$\Rightarrow \quad x^2 + (x^2 + 4x + 4) + (x^2 + 8x + 16) = 200$

$\Rightarrow \quad 3x^2 + 12x + 20 - 200 = 0$

$\Rightarrow \quad 3x^2 + 12x - 180 = 0$

$\Rightarrow \quad x^2 + 4x - 60 = 0$

$\Rightarrow \quad (x - 6)(x + 10) = 0$

$\Rightarrow \quad x - 6 = 0$ or $x + 10 = 0$

$\Rightarrow \quad x = 6$ or $x = -10$.

When $x = 6$, $x + 2 = 8$ and $x + 4 = 10$;

when $x = -10$, $x + 2 = -8$ and $x + 4 = -6$.

Hence, the required integers are 6, 8, 10 or $-10, -8, -6$.

Note

If three consecutive even natural numbers are required, then $x = -10$ is rejected and the answer would be 6, 8, 10.

Example 8. *The denominator of a fraction is one more than twice the numerator. If the sum of fraction and its reciprocal is $2\frac{16}{21}$, find the fraction.*

Solution. Let the numerator of the fraction be x ($x \in$ **I**), then its denominator $= 2x + 1$; so the fraction is $\frac{x}{2x+1}$.

According to given, $\frac{x}{2x+1} + \frac{2x+1}{x} = 2\frac{16}{21}$

$\Rightarrow \quad \frac{x^2 + (2x+1)^2}{x(2x+1)} = \frac{58}{21}$

$\Rightarrow \quad 58x\,(2x + 1) = 21\,[x^2 + 4x^2 + 4x + 1]$

$\Rightarrow \quad 116x^2 + 58x = 105x^2 + 84x + 21$

$\Rightarrow \quad 11x^2 - 26x - 21 = 0$

$\Rightarrow \quad (x - 3)\,(11x + 7) = 0$

$\Rightarrow \quad x - 3 = 0$ or $11x + 7 = 0$

$\Rightarrow \quad x = 3$ or $x = -\frac{7}{11}$ but x is an integer

$\Rightarrow \quad x = 3.$

$\therefore$ Required fraction $= \frac{3}{2 \times 3 + 1} = \frac{3}{7}$.

Example 9. *The sum S of first n natural numbers is given by the formula* $S = \frac{n(n+1)}{2}$. *If* $S = 231$, *find n.*

Solution. Given, $S = \frac{n(n+1)}{2} = 231$

$\Rightarrow \quad n(n + 1) = 462$

$\Rightarrow \quad n^2 + n - 462 = 0$

$\Rightarrow \quad (n - 21)\,(n + 22) = 0$

$\Rightarrow \quad n - 21 = 0$ or $n + 22 = 0$

$\Rightarrow \quad n = 21$ or $n = -22$ but n is a natural number

$\Rightarrow \quad n = 21.$

Example 10. *A two digit number contains the smaller of two digits in the unit's place. The product of the digits is 24 and the difference between the digits is 5. Find the number.*

Solution. Let the digit in the unit's place be x.

Since the digit at unit's place is smaller and the difference between the two digits is 5, therefore, the digit at ten's place $= x + 5$.

As the product of two digits is 24,

$\therefore \quad (x + 5)\,x = 24$

$\Rightarrow \quad x^2 + 5x - 24 = 0$

$\Rightarrow \quad (x + 8)\,(x - 3) = 0$

$\Rightarrow \quad x + 8 = 0$ or $x - 3 = 0$

$\Rightarrow \quad x = -8$ or $x = 3$ but x cannot be negative,

$\therefore \quad x = 3.$

$\therefore$ The digit at unit's place $= 3$ and the digit at ten's place $= 3 + 5 = 8$.

$\therefore$ The required number $= 83$.

Example 11. *A two digit number is such that the product of its digits is 12. When 36 is added to this number, the digits interchange their places. Find the number.*

Solution. Let the unit's digit of the two digit number be x.

Since the product of its digits is 12, its ten's digit $= \frac{12}{x}$.

$\therefore$ The number $= 10 \times \frac{12}{x} + x.$

On interchanging the digits, the number $= 10 \times x + \frac{12}{x}.$

According to given,

$$10 \times x + \frac{12}{x} = \left(10 \times \frac{12}{x} + x\right) + 36$$

$$\Rightarrow \quad 10x + \frac{12}{x} = \frac{120}{x} + x + 36$$

$$\Rightarrow \quad 10x^2 + 12 = 120 + x^2 + 36x$$

$$\Rightarrow \quad 9x^2 - 36x - 108 = 0$$

$$\Rightarrow \quad x^2 - 4x - 12 = 0$$

$$\Rightarrow \quad (x - 6)(x + 2) = 0$$

$$\Rightarrow \quad x - 6 = 0 \text{ or } x + 2 = 0$$

$\Rightarrow \quad x = 6$ or $x = -2$ but x being a digit of a number cannot be negative

$$\Rightarrow \quad x = 6.$$

$\therefore$ Unit's digit = 6 and ten's digit $= \frac{12}{6} = 2.$

Hence, the required number is 26.

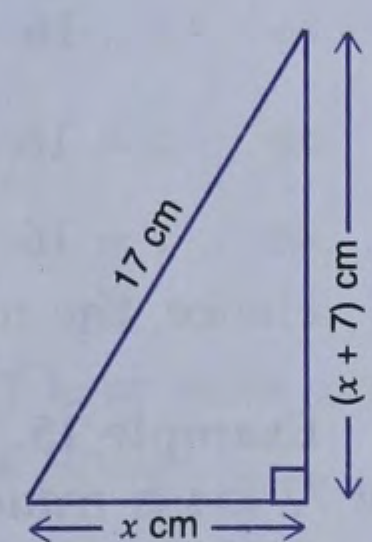
Example 12. *The hypotenuse of a right-angled triangle is 17 cm and the difference between other two sides is 7 cm. Find the other two unknown sides.*

Solution. Let the shorter side be x cm.

Since the difference between the two sides is 7 cm,

$\therefore$ longer side $= (x + 7)$ cm.

As the given triangle is right-angled with hypotenuse = 17 cm, by using Pythagoras theorem, we get



$$x^2 + (x + 7)^2 = (17)^2$$

$$\Rightarrow \quad x^2 + x^2 + 14x + 49 = 289$$

$$\Rightarrow \quad 2x^2 + 14x - 240 = 0$$

$$\Rightarrow \quad x^2 + 7x - 120 = 0$$

$$\Rightarrow \quad x^2 + 15x - 8x - 120 = 0$$

$$\Rightarrow \quad x(x + 15) - 8(x + 15) = 0$$

$$\Rightarrow \quad (x + 15)(x - 8) = 0$$

$$\Rightarrow \quad x + 15 = 0 \text{ or } x - 8 = 0$$

$\Rightarrow \quad x = -15$ or $x = 8$ but x cannot be negative,

$$\therefore \quad x = 8.$$

Hence, the two sides of the triangle are 8 cm and (8 + 7) cm *i.e.* 15 cm.

Example 13. *In an auditorium, seats are arranged in rows and columns. The number of rows was equal to the number of seats in each row. When the number of rows was doubled and the number of seats in each row was reduced by 10, the total number of seats increased by 300. Find :*

(i) the number of rows in the original arrangement.

(ii) the number of seats in the auditorium after re-arrangement. (2003)

Solution. (*i*) Let the number of rows in the original arrangement be x.

Since the number of rows was equal to the number of seats in each row, so the number of seats in the auditorium $= x^2$.

In new arrangement,

number of rows $= 2x$,

number of seats in each row $= x - 10$.

$\therefore$ The number of seats after new arrangement $= 2x(x - 10)$.

According to given, $2x(x-10) = x^2 + 300$

$\Rightarrow \quad 2x^2 - 20x = x^2 + 300$

$\Rightarrow \quad x^2 - 20x - 300 = 0$

$\Rightarrow \quad (x - 30)(x + 10) = 0$

$\Rightarrow \quad x - 30 = 0$ or $x + 10 = 0$

$\Rightarrow \quad x = 30$ or $x = -10$ but number of rows cannot be negative

$\Rightarrow \quad x = 30.$

$\therefore$ The number of rows in the original arrangement was 30.

(*ii*) The number of seats in the auditorium after re-arrangement

$= 30^2 + 300 = 900 + 300 = 1200.$

Example 14. *O Girl ! Out of a group of swans, $\frac{7}{2}$ times the square root of the number are playing on the shore of a tank. The two remaining ones are playing in the water. What is the total number of swans ?*

Solution. Let the total number of swans be x.

Then, the number of swans playing on the shore of the tank $= \frac{7}{2}\sqrt{x}$.

There are two remaining swans playing in the water.

Thus, we have $x = \frac{7}{2}\sqrt{x} + 2$

$\Rightarrow \quad x - 2 = \frac{7}{2}\sqrt{x} \qquad \Rightarrow \quad 2(x-2) = 7\sqrt{x}$

$\Rightarrow \quad 4(x-2)^2 = 49x \qquad \Rightarrow \quad 4(x^2 - 4x + 4) = 49x$

$\Rightarrow \quad 4x^2 - 65x + 16 = 0$

$\Rightarrow \quad (x - 16)(4x - 1) = 0$

$\Rightarrow \quad x - 16 = 0$ or $4x - 1 = 0$

$\Rightarrow \quad x = 16$ or $x = \frac{1}{4}$ but the number of swans cannot be $\frac{1}{4}$

$\Rightarrow \quad x = 16.$

Hence, the total number of swans = 16.

Example 15. *By increasing the speed of a car by 10 km/hr, the time of journey for a distance of 72 km is reduced by 36 minutes. Find the original speed of the car.* (2005)

Solution. Let the original speed of the car be x km/hr.

Time taken to cover a distance of 72 km $= \frac{72}{x}$ hours.

The new speed of the car $= (x + 10)$ km/hr.

Time taken to cover 72 km at new speed $= \frac{72}{x+10}$ hours.

Since the time taken by the car is reduced by 36 minutes *i.e.* $\frac{3}{5}$ hour,

$\therefore \quad \frac{72}{x} - \frac{72}{x+10} = \frac{3}{5}$

$\Rightarrow \quad \frac{72(x+10) - 72x}{x(x+10)} = \frac{3}{5} \qquad \Rightarrow \quad \frac{72 \times 10}{x(x+10)} = \frac{3}{5}$

$\Rightarrow \quad \frac{24 \times 10}{x^2 + 10x} = \frac{1}{5} \qquad \Rightarrow \quad x^2 + 10x = 1200$

$\Rightarrow \quad x^2 + 10x - 1200 = 0$

$\Rightarrow \quad x^2 + 40x - 30x - 1200 = 0$

$\Rightarrow \quad x(x + 40) - 30(x + 40) = 0$

$\Rightarrow \quad (x + 40)(x - 30) = 0$

$\Rightarrow \quad x + 40 = 0$ or $x - 30 = 0$

$\Rightarrow \quad x = -40$ or $x = 30$ but x cannot be negative

$\Rightarrow \quad x = 30.$

Hence, the original speed of the car = 30 km/hr.

Example 16. *An aeroplane left 30 minutes later than its scheduled time, and in order to reach its destination 1500 km away in time, it has to increase its speed by 250 km/hr from its usual speed. Determine its usual speed.*

Solution. Let the usual speed of the aeroplane be x km/hr.

Time taken to cover the journey at usual speed = $\frac{1500}{x}$ hours.

New speed of the aeroplane = $(x + 250)$ km/hr.

Time taken to cover the journey at new speed = $\frac{1500}{x+250}$ hours.

According to given information,

difference in two times = 30 minutes = $\frac{1}{2}$ hours

$\Rightarrow \quad \frac{1500}{x} - \frac{1500}{x+250} = \frac{1}{2}$

$\Rightarrow \quad 1500(x + 250) - 1500x = \frac{x(x+250)}{2}$

$\Rightarrow \quad 1500 \times 250 \times 2 = x^2 + 250x$

$\Rightarrow \quad x^2 + 250x - 750000 = 0$

$\Rightarrow \quad (x - 750)(x + 1000) = 0$

$\Rightarrow \quad x - 750 = 0$ or $x + 1000 = 0$

$\Rightarrow \quad x = 750$ or $x = -1000$ but the speed cannot be negative

$\Rightarrow \quad x = 750.$

Hence, the usual speed of the aeroplane = 750 km/hr.

Example 17. *Sonal can row a boat at a speed of 5 km/hr. If it takes her 1 hour more to row the boat 5·25 km upstream than to return downstream, find the speed of the stream.*

Solution. Let the speed of the stream be x km/hr, $0 < x < 5$.

Speed of the boat upstream = $(5 - x)$ km/hr.

Speed of the boat downstream = $(5 + x)$ km/hr.

Time taken for going 5·25 km upstream = $\frac{5{\cdot}25}{5-x}$ hours.

Time taken for returing 5·25 km downstream = $\frac{5{\cdot}25}{5+x}$ hours.

According to given information,

$\frac{5{\cdot}25}{5-x} - \frac{5{\cdot}25}{5+x} = 1$

$\Rightarrow \quad \frac{21}{4}\left(\frac{1}{5-x} - \frac{1}{5+x}\right) = 1$

$\Rightarrow \quad 21[(5 + x) - (5 - x)] = 4(5 - x)(5 + x)$

$\Rightarrow \quad 42x = 4(25 - x^2)$

$\Rightarrow \quad 4x^2 + 42x - 100 = 0$

$\Rightarrow \quad 2x^2 + 21x - 50 = 0$

$\Rightarrow \quad (x - 2)(2x + 25) = 0$

$\Rightarrow \quad x - 2 = 0$ or $2x + 25 = 0$

$\Rightarrow \quad x = 2$ or $x = -\frac{25}{2}$ but $0 < x < 5$

$\Rightarrow \quad x = 2.$

Hence, the speed of the stream = 2 km/hr.

Example 18. *B takes 16 days less than A to do a piece of work. If both working together can do it in 15 days, in how many days will B alone complete the work ?*

Solution. Let A complete the work in x days, then B will complete it in $(x - 16)$ days.

A's one day work = $\frac{1}{x}$,

B's one day work = $\frac{1}{x - 16}$.

Since A and B together can complete the work in 15 days, therefore,

A's one day work + B's one day work = $\frac{1}{15}$

$\Rightarrow \quad \frac{1}{x} + \frac{1}{x-16} = \frac{1}{15} \Rightarrow \frac{x - 16 + x}{x(x-16)} = \frac{1}{15}$

$\Rightarrow \quad x(x - 16) = 15(2x - 16)$

$\Rightarrow \quad x^2 - 16x = 30x - 240$

$\Rightarrow \quad x^2 - 46x + 240 = 0 \Rightarrow (x - 6)(x - 40) = 0$

$\Rightarrow \quad x - 6 = 0$ or $x - 40 = 0$

$\Rightarrow \quad x = 6$ or $x = 40.$

If $\quad x = 6$, then $x - 16 = 6 - 16 = -10$, which is not possible.

$\therefore \quad x = 40.$

Hence, B alone will complete the work in $(40 - 16)$ days *i.e.* 24 days.

Example 19. *Some students planned a picnic. The budget for the food was ₹ 480. As eight of them failed to join the party, the cost of the food for each member increased by ₹ 10. Find how many students went for the picnic.* *(2008)*

Solution. Let the number of students who planned the picnic be x.

As the budget for the food was ₹ 480, the contribution of each member = ₹ $\frac{480}{x}$.

Since 8 students failed to join the party, the number of students who joined the party $= x - 8$.

$\therefore$ The contribution of each member who joined the party = ₹ $\frac{480}{x-8}$.

As the cost of the food for each member has increased by ₹ 10,

$\therefore \quad \frac{480}{x-8} = \frac{480}{x} + 10 \Rightarrow 480\left(\frac{1}{x-8} - \frac{1}{x}\right) = 10$

$\Rightarrow \quad 48 \cdot \frac{x - (x-8)}{(x-8)x} = 1 \Rightarrow 48 \times 8 = (x - 8)x$

$\Rightarrow \quad x^2 - 8x - 384 = 0 \Rightarrow (x - 24)(x + 16) = 0$

$\Rightarrow \quad x = 24, -16$ but x can not be negative

$\therefore \quad x = 24.$

$\therefore$ The number of students who joined the party = 24 – 8 = 16.

Example 20. *A trader bought a number of articles for ₹ 1200. Ten were damaged and he sold each of the rest at ₹ 2 more than what he paid for it, thus clearing a profit of ₹ 60 on the whole transaction. Taking the number of articles he bought as x, form an equation in x and solve it.*

Solution. As the C.P. of x articles is ₹ 1200,

$\therefore$ C.P. of one article = ₹ $\frac{1200}{x}$.

As the selling price of each article is ₹ 2 more than its C.P.,

$\therefore$ S.P. of each article = ₹ $\left(\frac{1200}{x}+2\right)$.

Since 10 articles were damaged, therefore, the number of articles left for selling = $x - 10$.

$\therefore$ S.P. of all articles (worth selling) = ₹ $(x - 10)\left(\frac{1200}{x}+2\right)$.

As the trader earns a net profit of ₹ 60,

$$\therefore \quad (x-10)\left(\frac{1200}{x}+2\right) = 1200 + 60$$
$$\Rightarrow \quad (x-10) \cdot \frac{1200+2x}{x} = 1260$$
$$\Rightarrow \quad (x-10)(1200+2x) = 1260x$$
$$\Rightarrow \quad 1200x + 2x^2 - 12000 - 20x - 1260x = 0$$
$$\Rightarrow \quad 2x^2 - 80x - 12000 = 0$$
$$\Rightarrow \quad x^2 - 40x - 6000 = 0$$
$$\Rightarrow \quad x^2 - 100x + 60x - 6000 = 0$$
$$\Rightarrow \quad x(x-100) + 60(x-100) = 0$$
$$\Rightarrow \quad (x-100)(x+60) = 0$$
$\Rightarrow \quad x = 100$ or $x = -60$ but x cannot be negative,

$\therefore \quad x = 100.$

Example 21. *Rahul sold an article for ₹ 56 which cost him ₹ x. He finds that he has gained x% on his outlay. Find x.*

Solution. C.P. of the article is ₹ x. Since Rahul gains x% on his outlay,

$\therefore$ S.P. of the article = ₹ $x\left(1+\frac{x}{100}\right)$.

But the S.P. of the article is ₹ 56 (given),

$$\therefore \quad x\left(1+\frac{x}{100}\right) = 56 \quad \Rightarrow \quad x \cdot \frac{100+x}{100} = 56$$
$$\Rightarrow \quad 100x + x^2 = 5600$$
$$\Rightarrow \quad x^2 + 100x - 5600 = 0$$
$$\Rightarrow \quad (x-40)(x+140) = 0$$
$$\Rightarrow \quad x - 40 = 0 \text{ or } x + 140 = 0$$
$\Rightarrow \quad x = 40$ or $x = -140$ but x cannot be negative,

$\therefore \quad x = 40.$

Example 22. *Five years ago, a woman's age was the square of her son's age. Ten years hence, her age will be twice that of her son's age. Find :*

(i) the age of the son five years ago.

(ii) the present age of the woman. (2007)

Solution. Let the age of the son 5 years ago be x years, then the age of the woman 5 years ago = x^2 years.

$\therefore$ The present age of the son = $(x + 5)$ years

and the present age of the woman = $(x^2 + 5)$ years.

Ten years hence *i.e.* after 10 years from now,

the age of the son $= ((x + 5) + 10)$ years $= (x + 15)$ years.

and the age of the woman $= ((x^2 + 5) + 10)$ years $= (x^2 + 15)$ years.

According to given, $x^2 + 15 = 2(x + 15)$

$\Rightarrow \quad x^2 + 15 = 2x + 30 \Rightarrow x^2 - 2x - 15 = 0$

$\Rightarrow \quad (x - 5)(x + 3) = 0 \Rightarrow x - 5 = 0$ or $x + 3 = 0$

$\Rightarrow \quad x = 5$ or $x = -3$ but x being age cannot be negative

$\Rightarrow \quad x = 5.$

(*i*) The age of the son 5 years ago = 5 years.

(*ii*) The present age of the woman $= (x^2 + 5)$ years

$= (5^2 + 5)$ years $= 30$ years.

Example 23. *The sum of the areas of two squares is 640 m². If the difference in their perimeters is 64 m, find the sides of the two squares.*

Solution. Let side of 1st sqaure be x metres and side of 2nd square be y metres. Then

area of 1st square $= x^2$ m²,

area of 2nd square $= y^2$ m²,

perimeter of 1st square $= 4x$ m

and perimeter of 2nd square $= 4y$ m.

Let 1st square be bigger than 2nd square, then $x > y$

$\Rightarrow \quad 4x > 4y.$

According to given,

$$x^2 + y^2 = 640 \qquad \ldots(i)$$

and $4x - 4y = 64 \Rightarrow x - y = 16$

$$\Rightarrow \quad x = y + 16 \qquad \ldots(ii)$$

Substituting the value of x from (*ii*) in (*i*), we get

$(y + 16)^2 + y^2 = 640$

$\Rightarrow \quad y^2 + 32y + 256 + y^2 = 640$

$\Rightarrow \quad 2y^2 + 32y - 384 = 0 \Rightarrow y^2 + 16y - 192 = 0$

$\Rightarrow \quad (y - 8)(y + 24) = 0 \Rightarrow y - 8 = 0$ or $y + 24 = 0$

$\Rightarrow \quad y = 8$ or $y = -24$ but y being side of a square cannot be negative

$\Rightarrow \quad y = 8.$

From (*ii*), when $y = 8$, $x = 8 + 16 = 24$.

Hence, the sides of the two squares are 24 m and 8 m.

Exercise 6.4

1. (*i*) Find two consecutive natural numbers such that the sum of their squares is 61.
   (*ii*) Find two consecutive integers such that the sum of their squares is 61.
2. (*i*) If the product of two positive consecutive even integers is 288, find the integers.
   (*ii*) If the product of two consecutive even integers is 224, find the integers.
   (*iii*) Find two consecutive even natural numbers such that the sum of their squares is 340.
   (*iv*) Find two consecutive odd integers such that the sum of their squares is 394.
3. The sum of two numbers is 9 and the sum of their squares is 41. Taking one number as x, form an equation in x and solve it to find the numbers.

4. Five times a certain whole number is equal to three less than twice the square of the number. Find the number.

5. Divide 15 into two parts such that the sum of their reciprocals is $\frac{3}{10}$.

6. The difference of the squares of two numbers is 45. The square of the smaller number is 4 times the larger number. Determine the numbers.

Hint

Let the larger number be x, then the square of the smaller number = $4x$. According to given, $x^2 - 4x = 45$.

7. There are three consecutive positive integers such that the sum of the square of the first and the product of other two is 154. What are the integers ?

Hint

Let the integers be x, $x + 1$, $x + 2$. Then $x^2 + (x + 1)(x + 2) = 154$.

8. (*i*) Find three successive even natural numbers, the sum of whose squares is 308.

(*ii*) Find three consecutive odd integers, the sum of whose squares is 83.

Hint

(*i*) Let the numbers be x, $x + 2$, $x + 4$.

9. In a certain positive fraction, the denominator is greater than the numerator by 3. If 1 is subtracted from both the numerator and denominator, the fraction is decreased by $\frac{1}{14}$. Find the fraction.

10. The sum of the numerator and denominator of a certain positive fraction is 8. If 2 is added to both the numerator and denominator, the fraction is increased by $\frac{4}{35}$. Find the fraction.

11. A two digit number contains the bigger at ten's place. The product of the digits is 27 and the difference between two digits is 6. Find the number.

12. A two digit number is such that the product of its digits is 24. When 18 is subtracted from this number, the digits interchange their places. Find the number.

13. A rectangle of area 105 cm^2 has its length equal to x cm. Write down its breadth in terms of x. Given that the perimeter is 44 cm, write down an equation in x and solve it to determine the dimensions of the rectangle.

14. A rectangular garden 10 m by 16 m is to be surrounded by a concrete walk of uniform width. Given that the area of the walk is 120 square metres, assuming the width of the walk to be x, form an equation in x and solve it to find the value of x.

15. (*i*) Harish made a rectangular garden, with its length 5 metres more than its width. The next year, he increased the length by 3 metres and decreased the width by 2 metres. If the area of the second garden was 119 sq m, was the second garden larger or smaller ?

(*ii*) The length of a rectangle exceeds its breadth by 5 m. If the breadth were doubled and the length reduced by 9 m, the area of the rectangle would have increased by 140 m^2. Find its dimensions.

16. The perimeter of a rectangular plot is 180 m and its area is 1800 m^2. Take the length of the plot as x m. Use the perimeter 180 m to write the value of the breadth in terms of x. Use the values of length, breadth and the area to write an equation in x. Solve the equation to calculate the length and breadth of the plot.

17. The lengths of the parallel sides of a trapezium are $(x + 9)$ cm and $(2x - 3)$ cm, and the distance between them is $(x + 4)$ cm. If its area is 540 cm^2, find x.

18. If the perimeter of a rectangular plot is 68 m and length of its diagonal is 26 m, find its area.

19. If ths sum of two smaller sides of a right-angled triangle is 17 cm and the perimeter is 30 cm, then find the area of the triangle.

Hint

Let one of the two smaller sides be x cm, then the other side is $(17 - x)$ cm.

Length of hypotenuse = perimeter − sum of other two sides

= 30 cm − 17 cm = 13 cm.

$\therefore\ x^2 + (17 - x)^2 = 13^2 \Rightarrow x = 12, 5.$

20. The hypotenuse of a grassy land in the shape of a right triangle is 1 metre more than twice the shortest side. If the third side is 7 metres more than the shortest side, find the sides of the grassy land.

21. Mohini wishes to fit three rods together in the shape of a right triangle. If the hypotenuse is 2 cm longer than the base and 4 cm longer than the shortest side, find the lengths of the rods.

Hint

Let length of hypotenuse be x cm, then longer side is $(x - 2)$ cm and shortest side = $(x - 4)$ cm.

22. In a P.T. display, 480 students are arranged in rows and columns. If there are 4 more students in each row than the number of rows, find the number of students in each row.

23. In an auditorium, the number of rows was equal to the number of seats in each row. If the number of rows is doubled and the number of seats in each row is reduced by 5, then the total number of seats is increased by 375. How many rows were there ?

24. At an annual function of a school, each student gives gift to every other student. If the number of gifts is 1980, find the number of students.

25. The speed of an express train is x km/hr and the speed of an ordinary train is 12 km/hr less than that of the express train. If the ordinary train takes one hour longer than the express train to cover a distance of 240 km, find the speed of the express train. *(2009)*

26. A car covers a distance of 400 km at a certain speed. Had the speed been 12 km/hr more, the time taken for the journey would have been 1 hour 40 minutes less. Find the original speed of the car. *(2012)*

27. An aeroplane travelled a distance of 400 km at an average speed of x km/hr. On the return journey, the speed was increased by 40 km/hr. Write down an expression for the time taken for :

    (*i*) the onward journey, (*ii*) the return journey.

    If the return journey took 30 minutes less than the onward journey, write down an equation in x and find its value. (2002)

28. The distance by road between two towns A and B, is 216 km, and by rail it is 208 km. A car travels at a speed of x km/hr, and the train travels at a speed which is 16 km/hr faster than the car. Calculate :

(*i*) The time taken by the car, to reach town B from A, in terms of x.

(*ii*) The time taken by the train, to reach town B from A, in terms of x.

(*iii*) If the train takes 2 hours less than the car, to reach town B, obtain an equation in x, and solve it.

(*iv*) Hence find the speed of the train.

29. An aeroplane flying with a wind of 30 km/hr takes 40 minutes less to fly 3600 km, than what it would have taken to fly against the same wind. Find the plane's speed of flying in still air.

Hint

Let the speed of the plane in still air be x km/hr, then $\frac{3600}{x-30} - \frac{3600}{x+30} = \frac{2}{3}$.

30. A school bus transported an excursion party to a picnic spot 150 km away. While returning, it was raining and the bus had to reduce its speed by 5 km/hr, and it took one hour longer to make the return trip. Find the time taken to return.

31. A boat can cover 10 km up the stream and 5 km down the stream in 6 hours. If the speed of the stream is 1·5 km/hr, find the speed of the boat in still water.

32. Two pipes running together can fill a tank in $11\frac{1}{9}$ minutes. If one pipe takes 5 minutes more than the other to fill the tank, find the time in which each pipe would fill the tank.

Hint

Let the time taken by the two pipes to fill the tank be x minutes and $(x + 5)$ minutes, then $\frac{1}{x} + \frac{1}{x+5} = \frac{9}{100}$.

33. (*i*) ₹ 480 is divided equally among 'x' children. If the number of children were 20 more, then each would have got ₹ 12 less. Find 'x'. *(2011)*

(*ii*) ₹ 6500 is divided equally among a certain number of persons. Had there been 15 more persons, each would have got ₹ 30 less. Find the original number of persons.

34. $2x$ articles cost ₹ $(5x + 54)$ and $(x + 2)$ similar articles cost ₹ $(10x - 4)$; find x.

35. A trader buys x articles for a total cost of ₹ 600.

(*i*) Write down the cost of one article in terms of x.

If the cost per article were ₹ 5 more, the number of articles that can be bought for ₹ 600 would be four less.

(*ii*) Write down the equation in x for the above situation and solve it to find x.

36. A shopkeeper buys a certain number of books for ₹ 960. If the cost per book was ₹ 8 less, the number of books that could be bought for ₹ 960 would be 4 more. Taking the original cost of each book to be ₹ x, write an equation in x and solve it to find the original cost of each book. *(2013)*

37. A piece of cloth costs ₹ 300. If the piece was 5 metre longer and each metre of cloth costs ₹ 2 less, the cost of the piece would have remained unchanged. How long is the original piece of cloth and what is the rate per metre?

38. A cottage industry produces a certain number of pottery articles a day. One day, it was observed that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was ₹ 90, find the number of articles produced and the cost of each article.

39. The hotel bill for a number of people for overnight stay is ₹ 4800. If there were 4 more, the bill each person had to pay would have reduced by ₹ 200. Find the number of people staying overnight. *(2000)*

40. A person was given ₹ 3000 for a tour. If he extends his tour programme by 5 days, he must cut down his daily expenses by ₹ 20. Find the number of days of his tour programme.

41. Ritu bought a saree for ₹ 60 x and sold it for ₹ $(500 + 4x)$ at a loss of $x\%$. Find the cost price.

42. Paul is x years old and his father's age is twice the square of Paul's age. Ten years hence, father's age will be four times Paul's age. Find their present ages.

43. The age of a man is twice the square of the age of his son. Eight years hence, the age of the man will be 4 years more than three times the age of his son. Find their present age.

Hint

Let the present age of the son be x years, then the present age of the man $= 2x^2$ years.
After 8 years, the age of the son $= (x + 8)$ years and the age of the man $= (2x^2 + 8$ years).
According to question, $2x^2 + 8 = 3(x + 8) + 4$.

44. Two years ago, a man's age was three times the square of his daughter's age. Three years hence, his age will be four times his daughter's age. Find their present ages.

45. The length (in cm) of the hypotenuse of a right-angled triangle exceeds the length of one side by 2 cm and exceeds twice the length of other side by 1 cm. Find the length of each side. Also find the perimeter and the area of the triangle.

Hint

Let the lengths of other two sides be x cm and y cm, then $x + 2 = 2y + 1$. Also $x^2 + y^2 = (2y + 1)^2$.

46. If twice the area of a smaller square is subtracted from the area of a larger square, the result is 14 cm^2. However, if twice the area of the larger square is added to three times the area of the smaller square, the result is 203 cm^2. Determine the sides of the two squares.

Hint

Let the sides of the smaller and bigger square be x cm and y cm respectively. Then $y^2 - 2x^2 = 14$, $2y^2 + 3x^2 = 203$. Eliminate y^2 and solve the resulting equation.

CHAPTER TEST

Solve the following equations (1 to 4) by factorisation :

1. (*i*) $x^2 + 6x - 16 = 0$ (*ii*) $3x^2 + 11x + 10 = 0$.

2. (*i*) $2x^2 + ax - a^2 = 0$ (*ii*) $\sqrt{3}\,x^2 + 10x + 7\sqrt{3} = 0$.

3. (*i*) $x(x + 1) + (x + 2)(x + 3) = 42$ (*ii*) $\frac{6}{x} - \frac{2}{x-1} = \frac{1}{x-2}$.

4. (*i*) $\sqrt{x+15} = x + 3$ (*ii*) $\sqrt{3x^2 - 2x - 1} = 2x - 2$.

Solve the following equations (5 to 8) by using formula :

5. (*i*) $2x^2 - 3x - 1 = 0$ (*ii*) $x\left(3x + \frac{1}{2}\right) = 6$.

6. (*i*) $\frac{2x+5}{3x+4} = \frac{x+1}{x+3}$ (*ii*) $\frac{2}{x+2} - \frac{1}{x+1} = \frac{4}{x+4} - \frac{3}{x+3}$.

7. $x^2 + (4 - 3a)x - 12a = 0$.

8. $10ax^2 - 6x + 15ax - 9 = 0, a \neq 0$.

9. Solve the following quadratic equation for x and give your answer correct to 2 decimal places : $2x^2 + 2x = 3$.

10. Discuss the nature of the roots of the following equations :

(*i*) $3x^2 - 7x + 8 = 0$ (*ii*) $x^2 - \frac{1}{2}x - 4 = 0$

(*iii*) $5x^2 - 6\sqrt{5}\,x + 9 = 0$ (*iv*) $\sqrt{3}\,x^2 - 2x - \sqrt{3} = 0$.

In case the real roots exist, then find them.

11. Find the values of k so that the quadratic equation

$(4 - k)x^2 + 2(k + 2)x + (8k + 1) = 0$ has equal roots.

12. Find the values of m so that the quadratic equation $3x^2 - 5x - 2m = 0$ has two distinct real roots.

13. Find the value(s) of k for which each of the following quadratic equation has equal roots :

(*i*) $3kx^2 = 4(kx - 1)$ (*ii*) $(k + 4)x^2 + (k + 1)x + 1 = 0$.

Also, find the roots for that value(s) of k in each case.

14. Find two natural numbers which differ by 3 and whose squares have the sum 117.

15. Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.

16. Two natural numbers are in the ratio 3 : 4. Find the numbers if the difference between their squares is 175.

17. Two squares have sides x cm and $(x + 4)$ cm. The sum of their areas is 656 sq. cm. Express this as an algebraic equation and solve it to find the sides of the squares.

18. The length of a rectangular garden is 12 m more than its breadth. The numerical value of its area is equal to 4 times the numerical value of its perimeter. Find the dimensions of the garden.

19. A farmer wishes to grow a 100 m^2 rectangular vegetable garden. Since he has with him only 30 m barbed wire, he fences three sides of the rectangular garden letting compound wall of his house act as the fourth side fence. Find the dimensions of his garden.

Hint

Let x metres be the length of the side opposite to unfenced side, then length of each of two other sides = $\frac{1}{2}(30 - x)$. Therefore $x . \frac{1}{2}\ (30 - x) = 100$.

20. The hypotenuse of a right angled triangle is 1 m less than twice the shortest side. If the third side is 1 m more than the shortest side, find the sides of the triangle.

21. A wire, 112 cm long, is bent to form a right angled triangle. If the hypotenuse is 50 cm long, find the area of the triangle.

22. Car A travels x km for every litre of petrol, while car B travels $(x + 5)$ km for every litre of petrol.

(*i*) Write down the number of litres of petrol used by car A and car B in covering a distance of 400 km.

(*ii*) If car A uses 4 litres of petrol more than car B in covering 400 km, write down an equation in x and solve it to determine the number of litres of petrol used by car B for the journey.

23. The speed of a boat in still water is 11 km/hr. It can go 12 km upstream and return downstream to the original point in 2 hours 45 minutes. Find the speed of the stream.

24. By selling an article for ₹21, a trader loses as much percent as the cost price of the article. Find the cost price.

25. A man spent ₹2800 on buying a number of plants priced at ₹x each. Because of the number involved, the supplier reduced the price of each plant by one rupee. The man finally paid ₹2730 and received 10 more plants. Find x.

26. Forty years hence, Mr. Pratap's age will be the square of what it was 32 years ago. Find his present age.