LECTURE_04

MODULE_11

Q. Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where q is some integer.

Sol. Let us start with taking a, where a is a positive odd integer.

We apply the division algorithm with a and $b = 4$.

By Euclid's Division Algorithm,

$\boxed{a = 4q + r}$ where $0 \leq r < 4$

Since $0 \leq r < 4$,

$\therefore$ the possible remainders are 0, 1, 2, 3.

That is, a can be $4q$, or $4q + 1$, or $4q + 2$, or $4q + 3$, where q is the quotient.

However, since a is odd, a cannot be $4q$ or $4q + 2$ ....(since they are both divisible by 2).

Therefore, any odd integer is of the form $4q + 1$ or $4q + 3$.

MODULE_12

Q. Show that $n^2 - 1$ is divisible by 8, if n is an odd positive integer.

Sol. Let a be any positive integer
and $b = 4$

$\therefore$ By Euclid's Division Algorithm,

$\boxed{a = 4q + r}$ where $0 \leq r < 4$

$\therefore$ The possible remainders are 0, 1, 2, 3

$\therefore$ $a = 4q$ or $4q + 1$ or $4q + 2$ or $4q + 3$,

where q is some integer

Any odd positive integer n can be written in form of $\boxed{4q + 1}$ or $4q + 3$.

Case 1 : If, $n = 4q + 1$

$$n^2 - 1 = (4q + 1)^2 - 1$$
$$= 16q^2 + 8q \cancel{+ 1} \cancel{- 1}$$
$$= \underline{16q^2 + 8q}$$
$$= (8q)(2q + 1)$$

Here, the above result is multiple of 8.

Hence, it is divisible by 8.

We know, $(a + b)^2 = (a^2 + 2ab + b^2)$

Q. **Show that $n^2 - 1$ is divisible by 8, if n is an odd positive integer.**

Sol. Let a be any positive integer

and $b = 4$

$\therefore$ By Euclid's Division Algorithm,

$a = 4q + r$ where $0 \leq r < 4$

$\therefore$ The possible remainders are 0, 1, 2, 3

$\therefore$ $a = 4q$ or $4q + 1$ or $4q + 2$ or $4q + 3$,

where q is some integer

Any odd positive integer n can be written in form of $4q + 1$ or $\boxed{4q + 3.}$

Case 2 : If, $n = 4q + 3$

$$n^2 - 1 = (4q + 3)^2 - 1$$
$$= 16q^2 + 24q + 9 - 1$$
$$= \underline{16q^2 + 24q + 8}$$
$$= (8)(2q^2 + 3q + 1)$$

Here, the above result is multiple of 8.

Hence, it is divisible by 8.

We know, $(a + b)^2 = (a^2 + 2ab + b^2)$

MODULE_13

Q. Prove that if x and y are odd positive integers, then $x^2 + y^2$ is even but not divisible by 4.

Sol. Let $\boxed{x = 2m + 1}$

$\boxed{y = 2n + 1}$

$x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2$

[where m and n are positive integers]

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$

$= 4m^2 + 4n^2 + 4m + 4n + 2$

$= 4(m^2 + n^2 + m + n) + 2$

$= 4q + 2$ [where $q = m^2 + n^2 + m + n$]

Here, in the above result $4q$ is even,
therefore $4q + 2$ is also even .

And comparing with $a = bq + r$, here
when divisor is 4, remainder is 2.

Hence it is not divisible by 4.

Therefore, $x^2 + y^2$ is even but not divisible by 4.

We know, $(a + b)^2 = (a^2 + 2ab + b^2)$

MODULE_14

Q. Show that one and only one out of n, $n + 2$ or $n + 4$ is divisible by 3, where n is any positive integer.

Sol. Let a be any positive integer for divisor 3

By Euclid's Division Lemma,

$$\boxed{a = 3q + r;} \quad \text{where} \quad r = 0, 1, 2;$$

$\therefore \; a = 3q$ or $3q + 1$ or $3q + 2$;

Case 1 : When $n = 3q$,

$$n + 2 = 3q + 2$$

and $n + 4 = 3q + 4$

$$= 3q + 3 + 1$$

$$= 3(q + 1) + 1$$

Comparing with $3q + r$, here only n is divisible by 3.

Q. Show that one and only one out of n, $n + 2$ or $n + 4$ is divisible by 3, where n is any positive integer.

Sol. Let a be any positive integer for divisor 3

By Euclid's Division Lemma,

$$a = 3q + r \text{ ; } \quad \text{where } r = 0, 1, 2 \text{ ;}$$

$$\therefore \; a = 3q \text{ or } 3q + 1 \text{ or } 3q + 2 \text{ ;}$$

Case 2 : When $n = 3q + 1,$

$$n + 2 = 3q + 1 + 2$$
$$= \underline{3q + 3}$$
$$= 3(q + 1)$$
$$= 3m \qquad \text{...... where } m = (q + 1)$$

$$\text{and } n + 4 = 3q + 1 + 4$$
$$= 3q + \underline{5}$$
$$= \underline{3q + 3} + 2$$
$$= 3(\underline{q + 1}) + 2$$
$$= 3m + 2 \qquad \text{...... where } m = (q + 1)$$

Comparing with $3q + r$, here only $n + 2$ is divisible by 3.

Q. Show that one and only one out of n, $n + 2$ or $n + 4$ is divisible by 3, where n is any positive integer.

Sol. Let a be any positive integer for divisor 3

By Euclid's Division Lemma,

$$a = 3q + r \,; \quad \text{where } r = 0, 1, 2 \,;$$

$$\therefore \; a = 3q \text{ or } 3q + 1 \text{ or } 3q + 2 \,;$$

Case 3 : When $n = 3q + 2$,

$$\begin{aligned}
n + 2 &= 3q + 2 + 2 \\
&= 3q + \underline{4} \\
&= \underline{3q + 3} + 1 \\
&= 3(q + 1) + 1 \\
&= 3m + 1 && \text{......where } m = (q + 1) \\
\text{and } n + 4 &= 3q + 2 + 4 \\
&= 3q + 6 \\
&= 3(\underline{q + 2}) \\
&= 3m && \text{......where } m = (q + 2)
\end{aligned}$$

Comparing with $3q + r$, here only $n + 2$ is divisible by 3.