# Graph Theory Exercises

In these exercises, $p$ denotes the number of nodes and $q$ the number of edges of the graph.

1. A graph has 12 edges and 6 nodes, each of which has degree 2 or 5. How many nodes are there of each degree?

2. For each of the following, describe a graph model and then answer the question.

   (a) Must the number of people at a party who know an odd number of other people be even?

   (b) Must the number of people at a party who do not know an odd number of other people be even?

   (c) Must the number of people ever born who had (have) an odd number of brothers and sisters be even?

   (d) Must the number of families in Kerry with an odd number of children be even?

3. There used to be 26 teams in the U.S. National Football League with 13 teams in each of two divisions. A League guideline said that each team's 14-game schedule should include exactly 11 games against teams in its own division and 3 games against teams in the other division. By considering part of a graph model of this scheduling problem, show that this guideline could not be satisfied by all the teams!

4. Either find a graph that models the following or show that none exists: Each of 102 students will be assigned the use of 1 of 35 computers, and each of the 35 computers will be used by exactly 1 or 3 students.

5. (a) Is it possible to have a group of 11 people, each of whom knows exactly 3 others of the group? If it is possible, draw a graph illustrating the situation. If it's impossible, explain why.

   (b) Same as part (a), except 11 is replaced by 8.

6. Show that $K_n$ has $n(n-1)/2$ edges for $n = 1, 2, \ldots$

7. Suppose that there are 7 committees with each pair of committees having a common member and each person is on two committees. How many people are there? Hint: use the previous exercise.

8. A graph is *connected* if there is a path joining each pair of distinct vertices. Otherwise the graph is *disconnected*. A disconnected graph can be written as a union of connected subgraphs called *components* where different components have no vertex in common. Two nodes are in the same component of a graph if and only if there is a path joining them.

   Show that if $G$ is a simple graph with $p$ vertices and each vertex has degree at least $(p-1)/2$, then $G$ must be connected. Hint: how many vertices must each component of $G$ have?

9. Let $G$ be a simple connected plane graph with $p \geq 3$. Prove that $q \leq 3p - 6$. Hint: since $G$ is simple, every region is bounded by at least 3 edges ($p \geq 3$ implies this is true for the infinite region). Add the degrees of all the regions; what can you say about this sum (i) in terms of $r$ (ii) in terms of q? Finally, apply Euler's formula.

10. Prove that $K_n$ is planar for $n \leq 4$ and non-planar for $n \geq 5$. Hint: use the previous exercise for $n = 5$.

11. Suppose $n$ cuts are made across a pizza. Let $p_n$ denote the maximum number of pieces that can result (this happens when no two cuts are parallel or meet outside the pizza, and no three are concurrent). Prove that $p_n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}$.

12. For which values of $n$ does $K_n$ have an eulerian circuit?

13. A road inspector must travel a network of roads to inspect them. She wishes to travel along each road exactly once, returning eventually to her starting point. Under which condition on the road network can this be done?

14. A postman must travel a connected network of roads, delivering post to houses on both sides of each road. He wishes to drive along each road exactly twice (once in each direction). Show that this can always be done.

15. A standard set of dominoes has one piece for each (unordered) pair of integers from 0 to 6 inclusive; that is, the 28 pieces are (0,0), (0,1), ..., (0,6), (1,1), (1,2), ..., (1,6), (2,2), (2,3), ..., (2,6), (3,3), (3,4), ..., (6,6). Here a piece such as (1,0) is not listed because it has already appeared as (0,1).

    Can you arrange the 28 dominoes in a closed pattern (like a large circle), so that each matches with its neighbour in the usual way? (This means that the adjacent halves of every two adjacent dominoes are the same.) Solve this problem by expressing it as a problem in graph theory then applying a theorem about eulerian circuits. Hint: consider the graph with nodes $0, 1, 2, \ldots, 6$ and dominoes as edges — for example the domino (3,5) is the edge incident to nodes 3 and 5.

    If all dominoes with a 6 on them are removed from the set, can you arrange the remaining dominoes in a closed pattern, so that each matches with its neighbour?

    Can you state a general theorem about arranging dominoes with numbers $0, 1, \ldots, n$ on them into a closed pattern?

16. You can arrange the numbers 1,2,3,4,5 in a circle so that each number is adjacent to every other number exactly once by putting them in the order 1,2,3,4,5,3,1,4,2,5. Can you produce a similar arrangement for 1,2,3,4,5,6,7? Use the theorem for eulerian circuits to show that there is a solution for $n$ numbers if and only if $n$ is odd. Can you salvage a similar type of result when $n$ is even?