(Q1) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

$$A^{16} = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$$

$$A^{18} = A^{16} \times A^2 = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 512 & 0 \\ 0 & 512 \end{bmatrix}$$

$$A^{19} = A^{18} \times A = \begin{bmatrix} 512 & 0 \\ 0 & 512 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 512 & 512 \\ 512 & -512 \end{bmatrix}$$

Q2. $\det(A - \lambda I) = 0$

ie $\det\left(\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

~~If~~ $\lambda = 4$ and $\lambda = 8$ are the eigen values of A.

let $\lambda = 4$

$$\det\left(\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -2 & 3 \\ x & y-4 \end{bmatrix}\right) = 0$$

$$-2(y-4) - 3x = 0$$

$$-2y + 8 - 3x = 0$$

$$\boxed{y = \frac{8-3x}{2}} \quad (1)$$

Similarly, let $\lambda = 8$:

$$\det\left(\begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} -6 & 3 \\ x & y-8 \end{bmatrix}\right) = 0$$

$$-6(y-8) - 3x = 0$$

$$-6y + 48 - 3x = 0$$

$$y = \frac{48-3x}{6}$$

$$\boxed{y = 8 - \frac{x}{2}} \quad (2)$$

Set ① = ②

ie $\frac{8-3x}{2} = 8 - \frac{x}{2}$

$4 - \frac{3}{2}x = 8 - \frac{x}{2}$

$-4 = -\frac{x}{2} + \frac{3}{2}x$

$\boxed{-4 = x}$

Subst $x = -4$ into ① :

$\boxed{y = \frac{8-3(-4)}{2} = \frac{8+12}{2} = 10.}$

$\boxed{\therefore \quad x = -4 \quad \text{and} \quad y = 10.}$

LA Q 3

$$-x + 5y = -1 \qquad y = \frac{-1 + x}{5}$$

$$x - y = 2 \qquad y = x - 2$$

$$x + 3y = 3 \qquad y = \frac{3 - x}{3}$$

$$\left[\begin{array}{cc|c} -1 & +5 & -1 \\ 1 & -1 & 2 \\ 1 & 3 & 3 \end{array}\right] \quad \begin{array}{c} -1\times 1 \\ -1\times 1 - 2 \\ -1\times 1 - 3 \end{array} \left[\begin{array}{cc|c} 1 & -5 & 1 \\ 0 & -4 & -1 \\ 0 & -8 & -2 \end{array}\right] \rightarrow$$

$$\begin{array}{c} \\ 2 \times -\frac{1}{4} \\ 3 \times -\frac{1}{8} \end{array} \left[\begin{array}{cc|c} 1 & -5 & 1 \\ 0 & 1 & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} \end{array}\right] \xrightarrow{1 + 5(2)} \left[\begin{array}{cc|c} 1 & 0 & 2\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{array}\right]$$

Let 1 = 2

$$\frac{-1 + x}{5} = x - 2$$

$$-1 + x = 5x - 10$$

$$4x = 9$$

$$\boxed{x = \frac{9}{4}}$$

When $x = \frac{9}{4}$, $y = \frac{9}{4} - 2 = \frac{1}{4}$

Subst $x = \frac{9}{4}$ into 3: $y = \dfrac{3 - \frac{9}{4}}{3} = \dfrac{\frac{3}{4}}{3} = \frac{3}{12} = \frac{1}{4}$

$\therefore$ This system of linear equations has exactly one solution:

$$\boxed{x = \frac{9}{4},\ y = \frac{1}{4}}$$

Q4 $(A - \lambda I)\vec{x} = \vec{0}$, where $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ and $\lambda = 1$

$$\left(\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array} \right]$$

$$\begin{array}{l} 1 + 3 \\ 2 + 3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array} \right]$$

$$\begin{array}{l} \\ -1 \times (1 - 2) \\ -1 \times (1 - 3) \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & +1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right]$$

$$\begin{array}{l} 1 - 2 \\ \\ 2 - 3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_1 + 4x_3 = 0$$
$$x_2 - 2x_3 = 0$$

$\therefore$ $x_1 = -4x_3$

$x_2 = 2x_3$

$x_3 \in \mathbb{R}$.

$\therefore$ $\vec{x} = x_3 \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$ where $x_3 \in \mathbb{R}$

Q5

For A, $\lambda = 1, 2, 4$.

$$\det(A - \lambda I)$$
$$= \det(A - \lambda I)^T$$
$$= \det(A^T - \lambda I)$$

$$\det(A_T^{-1})$$

$$\det(A A^{-1}) = \det(I)$$
$$\det(A)\det(A^{-1}) = \det(I)$$
$$\det(A^{-1}) = \frac{1}{\det(A)}$$
$$\det(A^{-1}) = \det((A_T)^{-1}) = \frac{1}{\det(A)}$$

The product of the eigenvalues of A is the same as the determinant of A.

ie
$$\det(A) = \lambda_1 \times \lambda_2 \times \lambda_3$$
$$= 1 \times 2 \times 4$$
$$= 8.$$

$$\therefore \det((A_T)^{-1}) = \frac{1}{\det(A)}$$
$$= \frac{1}{8}.$$

Q6.

$A = LU$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU \quad \text{where} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

$$\text{and } U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

$\Rightarrow$ $\boxed{U_{11} = 1}$ $\quad \boxed{U_{12} = 2}$ $\quad \boxed{U_{13} = 4}$

$L_{21} \times U_{11} = 3$

$L_{21} \times 1 = 3$

$\boxed{L_{21} = 3}$

$L_{31} \times U_{11} = 2$

$L_{31} \times 1 = 2$

$\therefore \boxed{L_{31} = 2}$

$L_{21} \times U_{12} + U_{22} = 8$

$3 \times 2 + U_{22} = 8$

$\boxed{U_{22} = 2}$

$L_{21}U_{13} + U_{23} = 14$

$3 \times 4 + U_{23} = 14$

$\therefore \boxed{U_{23} = 2}$

$L_{31}U_{12} + L_{32}U_{22} = 6$

$\boxed{2 \times 2 + L_{32}2 = 6}$

$\boxed{L_{32} = 1}$

$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 13$

$$\boxed{\cancel{L_{31}}^{2} \times 4 + \cancel{L_{32}}^{1} \times 2 + U_{33} = 13} \quad (4)$$

~~From eqn (1):~~

$\cancel{L_{31} = \frac{6 - 2 \times L_{32}}{2}}$

$\cancel{L_{31} = 3 - L_{32}}$

~~Subst into eqn (2):~~

$\cancel{(3 - L_{32}) \times 4 + 2L_{32} + U_{33} = 13}$

$\cancel{12 - 4L_{32} + 2L_{32} + U_{33} = 13}$

$\cancel{-2L_{32} + U_{33} = 1}$

$10 + U_{33} = 13$

$$\boxed{U_{33} = 3.}$$

$$\therefore A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}}$$