MVC. Q1

~~Let $w = x^2 + 2y^2 + 2z^2$~~

~~The level surface $w = 5$ is the surface~~

~~The gradient of w at $(1,1,1)$ is~~

Let $f(x,y,z) = x^2 + 2y^2 + 2z^2$

$f(1,1,1) = 1^2 + 2(1)^2 + 2(1)^2 = 5$ lies on

the surface $x^2 + 2y^2 + 2z^2 = 5$

The gradient of $f(x,y,z)$ is perpendicular to

the tangent plane at $(1,1,1)$

ie $\nabla f(x,y,z) = \begin{bmatrix} 2x \\ 4y \\ 4z \end{bmatrix}$ is $\perp$ to $x^2 + 2y^2 + 2z^2 = 5$

At $(1,1,1)$, $\nabla f(1,1,1) = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$

~~we want~~ The points $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ passing through $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

and perdendicular to $\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$ ~~on the~~ must satisfy the

following: $\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} = 0$

$2(x-1) + 4(y-1) + 4(z-1) = 0$

$2x + 4y + 4z = 10$

$\therefore$ The plane $x + 2y + 2z = 5$ passes through $(1,1,1)$ and is tangent to $f(x,y,z)$

MVC Q 2.

$w = x^2 - xy^3$

$P = (2,1)$

a) $\vec{A} = 3\vec{i} + 4\vec{j}$

$\vec{u} = \dfrac{\vec{A}}{|\vec{A}|} = \dfrac{3\vec{i} + 4\vec{j}}{\sqrt{3^2 + 4^2}} = \dfrac{3}{5}\vec{i} + \dfrac{4}{5}\vec{j}$

$\dfrac{dw}{ds}$ at P in the direction of $\vec{u}$ is

$\nabla w(\vec{P}) \cdot \vec{u}$

$= \begin{bmatrix} \dfrac{\partial w(P)}{\partial x} \\ \dfrac{\partial w(P)}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5} \\ 4/5 \end{bmatrix}$

$= \begin{bmatrix} 2x - y^3 \\ -3xy^2 \end{bmatrix}_P \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$= \begin{bmatrix} 2(2) - 1^3 \\ -3(2)(1)^2 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$= \dfrac{9}{5} - 24/5$

$= -3$

$\therefore$ the directional derivative $\dfrac{dw}{ds}$ at $P = (2,1)$ in the direction of $A = 3\vec{i} + 4\vec{j}$ is -3.

MVC Q2b.

$$\left.\frac{dw}{ds}\right|_{P\vec{u}} = -3.$$

Since $\left.\frac{dw}{ds}\right|_{P,\vec{u}} \approx \left.\frac{\Delta w}{\Delta s}\right|_{P,\vec{u}}$

$$\left.\frac{\Delta w}{\Delta s}\right|_{P,\vec{u}} \approx 3.$$

Since $\Delta s = 0.01$,

$$\frac{\Delta w}{\Delta s} \approx -3$$

$$\Delta w \approx -3 \times 0.01$$
$$= -0.03.$$

ie $\Delta w = -0.03.$

3. We want to solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ where

$f(x,y,z) = 2x + y - z - 6 = 0$ and

$g(x,y,z) = (x^2+y^2+z^2)^{\frac{1}{2}}$ is the distance from the origin to the plane $2x + y - z = 6$

$$\nabla f(x,y,z) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda \nabla g(x,y,z) = \lambda \begin{bmatrix} \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot 2x \\ \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot 2y \\ \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} \cdot 2z \end{bmatrix} = \lambda (x^2+y^2+z^2)^{-\frac{1}{2}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

From $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ we get 3 eqns.

① $2 = \lambda (x^2+y^2+z^2)^{-\frac{1}{2}} x \Rightarrow \lambda = \frac{2}{x}(x^2+y^2+z^2)^{\frac{1}{2}}$

② $1 = \lambda (x^2+y^2+z^2)^{-\frac{1}{2}} y \Rightarrow \lambda = \frac{1}{y}(x^2+y^2+z^2)^{\frac{1}{2}}$

③ $-1 = \lambda (x^2+y^2+z^2)^{-\frac{1}{2}} z \Rightarrow \lambda = -\frac{1}{z}(x^2+y^2+z^2)^{\frac{1}{2}}$.

with the constraint $2x + y - z = 6$

ie ④ $x = \dfrac{6 - y + z}{2}$

3. cont.

Setting ① = ② we get

$$\frac{2}{x}(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{1}{y}(x^2+y^2+z^2)^{\frac{1}{2}}.$$

$$\boxed{⑤ \quad x = 2y.}$$

Setting ① = ③ we get

$$\frac{2}{x}(x^2+y^2+z^2)^{\frac{1}{2}} = -\frac{1}{z}(x^2+y^2+z^2)^{\frac{1}{2}}.$$

$$\boxed{⑥ \quad x = -2z.}$$

Substituting ④ into ⑤ we get.

$$\frac{6-y+z}{2} = 2y$$

$$6-y+z = 4y$$

$$\boxed{⑦ \quad \therefore \; y = \frac{6+z}{5}}$$

Substituting ④ into ⑥ we get.

$$\frac{6-y+z}{2} = -2z.$$

$$6-y+z = -4z.$$

$$\boxed{⑧ \quad y = 6+5z}$$

Setting ⑦ = ⑧ we can solve for z.

$$\frac{6+z}{5} = 6+5z.$$

$$6+z = 30+25z.$$

$$-24 = 24z$$

$$z = -1$$

If $z = -1$, $y = 6+5(-1) = 1$ & $x = -2(-1) = 2$

3 cont.

$\therefore$ The point $P = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ lies on the plane $2x + y - z = 6$ and is closest to the origin.

Q4 MVC.

a) $|A_2| = \det\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix}$

$= 1 \cdot (2 - (-1)) - 0 + 3(-2 - (-1))$

$= 3 - 3$

$= 0.$

b) $A_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ we want $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that

$\cancel{\begin{bmatrix} x + 3z \\ -2x + y - z \\ -x + y + 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is orthogonal to each of the rows of A_2.

Since $|A_2| = 0$, all rows lie in the same plane and therefore if we take the cross product of any two rows, we will get an orthogonal vector.

$\langle 1, 0, 3 \rangle \times \langle -2, 1, -1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 3 \\ -2 & 1 & -1 \end{vmatrix} = \langle -3, -5, 1 \rangle$

Since a scalar multiple of $\langle -3, -5, 1 \rangle$ will also be orthogonal, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a\begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$ where a is any real number.

MVC.

4c) If $a=1$, $A_1 = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

Since $A_1 A_1^{-1} = I$

$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} * & * & * \\ -3 & p & 5 \\ * & * & * \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting some variables into the inverse matrix:

$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ -3 & p & 5 \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From this we can generate equations involving p:

① $b + 0p + 3e = 0 \Rightarrow b = -3e.$

② $-2b + p - e = 1$

③ $-b + p + e = 0$

Subst $\boxed{b = -3e}$ into ② we get

$$-2(-3e) + p - e = 1$$

$$\boxed{p = 1 - 5e} \quad ④$$

Subst. $\boxed{b = -3e}$ into ③ we get

$$-(-3e) + p + e = 0$$

$$\boxed{p = -4e} \quad ⑤$$

Since ④ = ⑤ we can solve for e:

$$1 - 5e = -4e$$

$$e = 1$$

$$\boxed{\therefore \quad p = -4(1) = -4}$$

MVC Q5

a) $f(x,y) = x + 4y + \frac{2}{xy}$

$= x + 4y + 2x^{-1}y^{-1}$

$\frac{\partial f}{\partial x} = 1 - 2x^{-2}y^{-1} = 1 - \frac{2}{x^2 y}$

if $\frac{\partial f}{\partial x} = 0 \Rightarrow 1 = \frac{2}{x^2 y}$

$\therefore y = \frac{2}{x^2}$.

$\frac{\partial f}{\partial y} = 4 - 2x^{-1}y^{-2} = 4 - \frac{2}{xy^2}$

if $\frac{\partial f}{\partial y} = 0 \Rightarrow 4 = \frac{2}{xy^2}$

$y = \sqrt{\frac{1}{2x}}$

Solving for x, $\frac{2}{x^2} = \sqrt{\frac{1}{2x}}$

$\frac{4}{x^4} = \frac{1}{2x}$

$8x = x^4$

$\therefore 8 = x^3$

$\Rightarrow x = 2$.

if $x = 2$, $y = \frac{1}{2}$.

The only critical point is $(2, \frac{1}{2})$

Q5b)

Computing the Second Partial Derivatives:

$$\frac{\partial f}{\partial x} = 1 - \frac{2}{x^2 y} \quad \text{and} \quad \frac{\partial f}{\partial y} = 4 - \frac{2}{xy^2}$$

$$= 1 - 2x^{-2}y^{-1} \qquad\qquad = 4 - 2x^{-1}y^{-2}$$

$$\boxed{\therefore \frac{\partial^2 f}{\partial x^2} = 4x^{-3}y^{-1} = \frac{4}{x^3 y}}$$

$$\boxed{\therefore \frac{\partial^2 f}{\partial y^2} = 4x^{-1}y^{-3} = \frac{4}{xy^3}}$$

$$\boxed{\therefore \frac{\partial^2 f}{\partial y \partial x} = -2x^{-2}(-1)y^{-2} = \frac{2}{x^2 y^2}}$$

$$H = \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\delta f}{\partial y \delta x}\right)^2$$

$$\boxed{\therefore H = \frac{4}{x^3 y} \times \frac{4}{xy^3} - \frac{4}{x^4 y^4}}$$

At $(2, \frac{1}{2})$, $H = \frac{4}{4} \times \frac{4}{(\frac{1}{4})} - \frac{4}{16 \times (\frac{1}{16})}$

$$= 1 \times 16 - 1$$

$$= 15$$

$\therefore$ local Maxima or minima.

as $\frac{\partial^2 f}{\partial x^2} > 0$ and $\frac{\partial^2 f}{\partial y^2} > 0$, this is a local minima

6 a) If $f(x,y,z)$ is a scalar function ~~function~~ such that

$\vec{F} = \nabla f(x,y,z)$, then $\text{curl } \vec{F} = 0$.

$\therefore$ If $\text{curl } \vec{F} = 0 \Rightarrow \vec{F}$ is a gradient field because the gradient of a scalar function is a gradient field.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\vec{F} = \begin{bmatrix} y + y^2 z \\ x - z + 2xyz \\ -y + xy^2 \end{bmatrix}$$

Let $\vec{F} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$,

$$\det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

Then $\text{curl } \vec{F} = \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

$= \vec{i}\left[(-1+2xy) - (-1+2xy)\right] - \vec{j}\left((y^2) - (y^2)\right) + \vec{k}\left((1+2yz) - (1+2yz)\right)$

$= 0$.

$\therefore$ Since $\text{curl } \vec{F} = 0$, $\vec{F}$ is a gradient field.

6) b) If $\vec{F}(x,y,z)$ is a gradient field

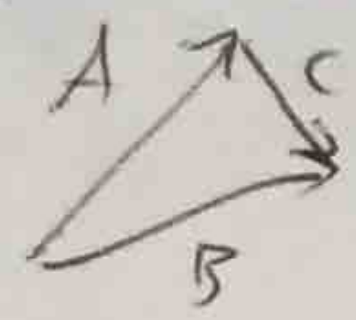
$$f(x,y,z) = \int \vec{i}\,dx\ \vec{i},\ \int \vec{j}\,dy\ \vec{j},\ \int \vec{k}\,dz$$

$$= x(y+y^2z)\ ;\ (xy - zy + xy^2z)\ ;$$

$$z(-y + xy^2)$$

The function $f(x,y,z) = xy - zy + xy^2z + C$ satisfies $\vec{F}(x,y,z)$ because $\nabla f(x,y,z) = \vec{F}(x,y,z)$

~~A~~ At $P = (1,1,1)$, $\vec{F} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$



at $A = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$\vec{c} = -\vec{A} + \vec{B}$$

$$= \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

6a). 7.

The line $c = \vec{A} + d\,\vec{C}$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$$

d is any constant.

6 c) $\int_C \vec{F} \cdot dr = f(1,-1,2) - f(2,2,1)$, where $f(x,y,z)$

$= xy - zy + xy^2z + C$

$= (1)(-1) - (2)(-1) + (1)(-1)^2(2)$

$- \left((2)(2) - (1)(2) + (2)(2)^2(1)\right)$

$= -1 + 2 + 2 - (4 - 2 + 8)$

$= 3 - 10$

$= -7$