See the MRF development

The section describes the development for the incompressible Navier-Stokes formulation in the rotating frame.

Contents

1 Accelerations

To start, we will look at the acceleration term for a rotating frame $(\vec{\Omega})$.

Notation: I: inertial, R: rotating

For a general vector:

$$\left[\frac{d\vec{A}}{dt}\right]_I = \left[\frac{d\vec{A}}{dt}\right]_R + \vec{\Omega} \times \vec{A}$$

For the position vector:

$$\left[\frac{d\vec{r}}{dt}\right]_I = \left[\frac{d\vec{r}}{dt}\right]_R + \vec{\Omega} \times \vec{r}$$

$$\vec{u}_I = \vec{u}_R + \vec{\Omega} \times \vec{r}$$

The acceleration is expressed as:

$$\left[\frac{d\vec{u}_I}{dt}\right]_I = \left[\frac{d\vec{u}_I}{dt}\right]_R + \vec{\Omega} \times \vec{u}_I$$

$$\left[\frac{d\vec{u}_I}{dt}\right]_I = \left[\frac{d\left[\vec{u}_R + \vec{\Omega} \times \vec{r}\right]}{dt}\right]_R + \vec{\Omega} \times \left[\vec{u}_R + \vec{\Omega} \times \vec{r}\right]$$

$$\left[\frac{d\vec{u}_I}{dt}\right]_I = \left[\frac{d\vec{u}_R}{dt}\right]_R + \frac{d\vec{\Omega}}{dt} \times \vec{r} + \vec{\Omega} \times \underbrace{\left[\frac{d\vec{r}}{dt}\right]_R}_{\vec{u}_R} + \vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

$$\left[\frac{d\vec{u}_I}{dt}\right]_I = \left[\frac{d\vec{u}_R}{dt}\right]_R + \frac{d\vec{\Omega}}{dt} \times \vec{r} + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$ **Eqn [1]**

2 Navier-Stokes equations in the inertial frame with absolute velocity

The incompressible Navier-Stokes equations in the inertial frame with constant molecular viscosity are:

$$\begin{cases} \frac{D\vec{u}_I}{Dt} = -\nabla(p/\rho) + \nu\nabla \cdot \nabla(\vec{u}_I) \\ \nabla \cdot \vec{u}_I = 0 \end{cases}$$ **Eqn [2]**

$$\begin{cases} \frac{\partial \vec{u}_I}{\partial t} + \vec{u}_I \cdot \nabla \vec{u}_I = -\nabla(p/\rho) + \nu \nabla \cdot \nabla(\vec{u}_I) \\ \nabla \cdot \vec{u}_I = 0 \end{cases}$$

$$\begin{cases} \frac{\partial \vec{u}_I}{\partial t} + \nabla \cdot (\vec{u}_I \otimes \vec{u}_I) - \underbrace{(\nabla \cdot \vec{u}_I)}_{0} \vec{u}_I = -\nabla(p/\rho) + \nu \nabla \cdot \nabla(\vec{u}_I) \\ \nabla \cdot \vec{u}_I = 0 \end{cases}$$

$$\begin{cases} \frac{\partial \vec{u}_I}{\partial t} + \nabla \cdot (\vec{u}_I \otimes \vec{u}_I) = -\nabla(p/\rho) + \nu \nabla \cdot \nabla(\vec{u}_I) \\ \nabla \cdot \vec{u}_I = 0 \end{cases}$$ **Eqn [3]**

3 Navier-Stokes equations in the relative frame with relative velocity

Let's look at the left-hand side of the momentum equation of Eqn [2], by taking into account Eqn [1] for the acceleration term:

$$\frac{D\vec{u}_I}{Dt} = \frac{D\vec{u}_R}{Dt} + \frac{d\vec{\Omega}}{dt} \times \vec{r} + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

$$\frac{D\vec{u}_I}{Dt} = \frac{\partial \vec{u}_R}{\partial t} + \vec{u}_R \cdot \nabla \vec{u}_R + \frac{d\vec{\Omega}}{dt} \times \vec{r} + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

$$\frac{D\vec{u}_I}{Dt} = \frac{\partial \vec{u}_R}{\partial t} + \nabla \cdot (\vec{u}_R \otimes \vec{u}_R) + \frac{d\vec{\Omega}}{dt} \times \vec{r} + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$ **Eqn [4]**

since $\nabla \cdot \vec{u}_R = \nabla \cdot \vec{u}_I = 0$

$$\begin{aligned} \nabla \cdot \vec{u}_I &= \nabla \cdot \left[\vec{u}_R + \vec{\Omega} \times \vec{r}\right] = 0 \\ &= \nabla \cdot \vec{u}_R + \underbrace{\nabla \cdot \left[\vec{\Omega} \times \vec{r}\right]}_{0} = 0 \\ &= \nabla \cdot \vec{u}_R = 0 \end{aligned}$$

Also, it can be noted that

$$\begin{aligned} \nabla \cdot \nabla(\vec{u}_I) &= \nabla \cdot \nabla \left[\vec{u}_R + \vec{\Omega} \times \vec{r}\right] \\ &= \nabla \cdot \nabla(\vec{u}_R) + \nabla \cdot \underbrace{\nabla(\Omega \times \vec{r})}_{0} \\ &= \nabla \cdot \nabla(\vec{u}_R) \end{aligned}$$

Eqn [3] can be written as

$$\begin{cases} \frac{\partial \vec{u}_R}{\partial t} + \frac{d\vec{\Omega}}{dt} \times \vec{r} + \nabla \cdot (\vec{u}_R \otimes \vec{u}_R) + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\nabla(p/\rho) + \nu \nabla \cdot \nabla(\vec{u}_R) \\ \nabla \cdot \vec{u}_R = 0 \end{cases}$$ **Eqn [5]**

Eqn [5] represents the incompressible Navier-Stokes equations in the rotating frame, in terms of rotating velocities (convection velocity and convected velocity).

4 Navier-Stokes equations in the relative frame with absolute velocity

Eqn [5] can be further developed so the convected velocity is the velocity in the inertial frame.

The term $\nabla \cdot (\vec{u}_R \otimes \vec{u}_R)$ can be developed as:

$$\begin{aligned}
\nabla \cdot (\vec{u}_R \otimes \vec{u}_R) &= \nabla \cdot (\vec{u}_R \otimes \left[\vec{u}_I - \vec{\Omega} \times \vec{r}\right]) \\
&= \nabla \cdot (\vec{u}_R \otimes \vec{u}_I) - \underbrace{\nabla \cdot \vec{u}_R}_{0}(\vec{\Omega} \times \vec{r}) - \underbrace{\vec{u}_R \cdot \nabla(\vec{\Omega} \times \vec{r})}_{\vec{\Omega} \times \vec{u}_R} \\
&= \nabla \cdot (\vec{u}_R \otimes \vec{u}_I) - \vec{\Omega} \times \vec{u}_R
\end{aligned}$$

So, the steady term of left-hand side of Eqn [5] can be written as

$$\begin{aligned}
\nabla \cdot (\vec{u}_R \otimes \vec{u}_R) + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} &= \nabla \cdot (\vec{u}_R \otimes \vec{u}_I) - \vec{\Omega} \times \vec{u}_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} \\
&= \nabla \cdot (\vec{u}_R \otimes \vec{u}_I) + \vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} \\
&= \nabla \cdot (\vec{u}_R \otimes \vec{u}_I) + \vec{\Omega} \times (\vec{u}_R + \vec{\Omega} \times \vec{r}) \\
&= \nabla \cdot (\vec{u}_R \otimes \vec{u}_I) + \vec{\Omega} \times \vec{u}_I
\end{aligned}$$

Eqn [5] can be written in terms of the absolute velocity:

$$\begin{cases} \frac{\partial \vec{u}_R}{\partial t} + \frac{d\vec{\Omega}}{dt} \times \vec{r} + \nabla \cdot (\vec{u}_R \otimes \vec{u}_I) + \vec{\Omega} \times \vec{u}_I = -\nabla(p/\rho) + \nu \nabla \cdot \nabla(\vec{u}_I) \\ \nabla \cdot \vec{u}_I = 0 \end{cases}$$ **Eqn [6]**

5 Summary

In summary, for multiple frames of reference, the incompressible Navier-Stokes equations for steady flow can be written

Frame	Convected velocity	Steady incompressible Navier-Stokes equations
Inertial	absolute velocity	$\begin{cases} \nabla \cdot (\vec{u}_I \otimes \vec{u}_I) = -\nabla(p/\rho) + \nu \nabla \cdot \nabla(\vec{u}_I) \\ \nabla \cdot \vec{u}_I = 0 \end{cases}$
Rotating	relative velocity	$\begin{cases} \nabla \cdot (\vec{u}_R \otimes \vec{u}_R) + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\nabla(p/\rho) + \nu \nabla \cdot \nabla(\vec{u}_R) \\ \nabla \cdot \vec{u}_R = 0 \end{cases}$
Rotating	absolute velocity	$\begin{cases} \nabla \cdot (\vec{u}_R \otimes \vec{u}_I) + \vec{\Omega} \times \vec{u}_I = -\nabla(p/\rho) + \nu \nabla \cdot \nabla(\vec{u}_I) \\ \nabla \cdot \vec{u}_I = 0 \end{cases}$

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