Quantum, a look through nonlocality

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Defining quantumness

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I will try to give all the basic notation and ideas necessary to understand most quantum computing talks.
And talk about my own research.

Defining quantumness: the basic idea

Quantum computing is what you get when you move from the usual theory of probability to a similar theory based on the ℓ_2-norm.

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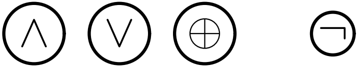
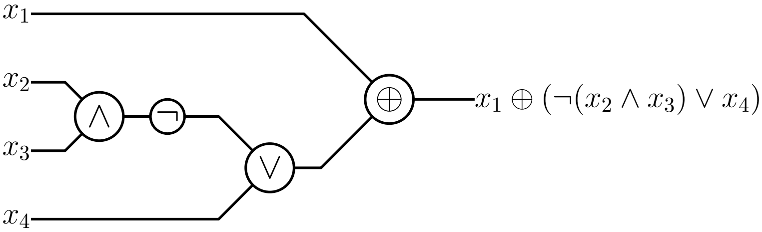
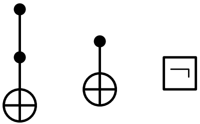
ok ... but precisely ?

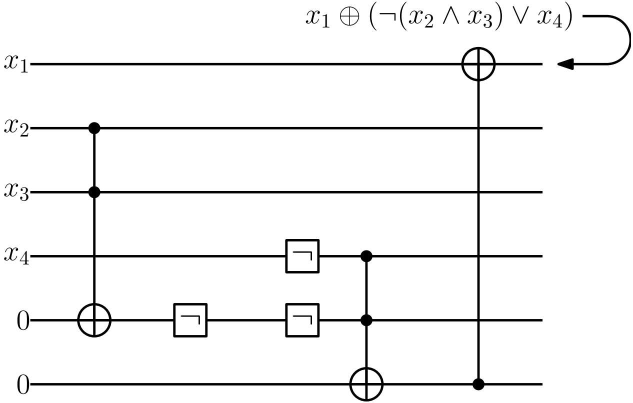
Classical circuits



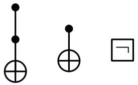


Reversibility





Matrix form



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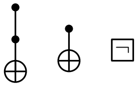

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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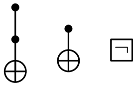

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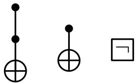

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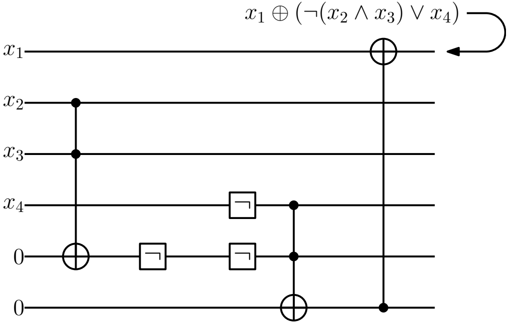
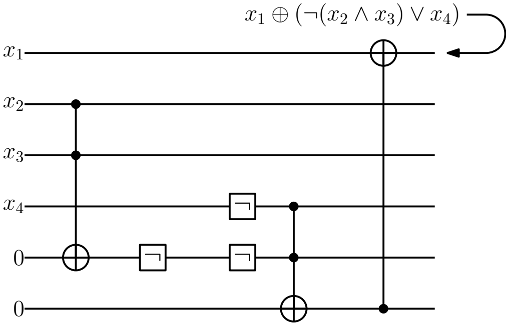
Combining parts

If S_1 and S_2 are the sets of states of two circuits, the states of the two circuits combined are $S_1 \times S_2$.

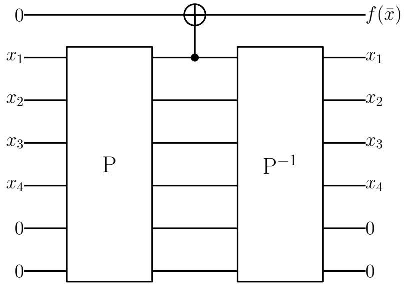


Combining parts

If S_1 and S_2 are the sets of states of two circuits, the states of the two circuits combined are $S_1 \times S_2$.
The matrix representing how a small-dimensional operator is applied to the whole space is obtained with a tensor product $\mathbf{1} \otimes M$



Uncomputation



Reversible circuit correspond to a *permutation matrix* over the 2^n states of the system.
As every gate is its own inverse, taking the circuit from the other side *reverses the computation*.

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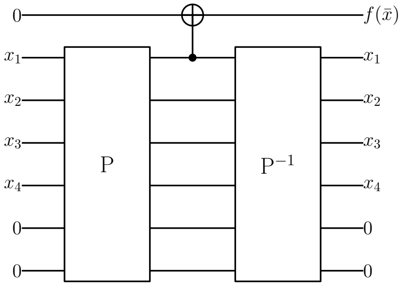

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Every permutation is achievable with the previous gates.

Moving to probabilities

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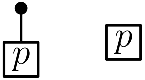
Can we do the same with *probability distributions* over such states, that is *convex combinations* of the previous states ?

What can we add ?



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 1-p \\ 0 & 0 & 1-p & p \end{pmatrix} \quad : \quad \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

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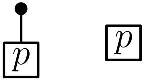

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... it is *very different* !

The sets of ℓ_1-norm-preserving matrices, the *stochastic* matrices, are not necessarily inversible.

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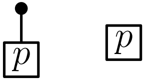

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And their inverse are not necessarily norm-preserving !

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Superposition and observation

In the presence of a state $\sum_i \alpha_i \,|i\rangle$, the probability of observing $|i\rangle$ is $|\alpha_i|^2$

Quantum gates

$\boxed{\mathrm{X}}\ \boxed{\mathrm{Y}}\ \boxed{\mathrm{Z}}\quad \boxed{\mathrm{H}}\quad \boxed{\mathrm{R}_\theta}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;:\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \;:\; \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

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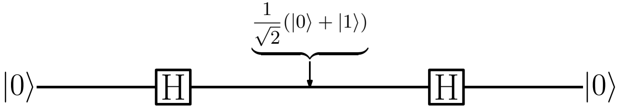
This makes all quantum circuits invertible in the same sense than classical reversible circuits.

Example



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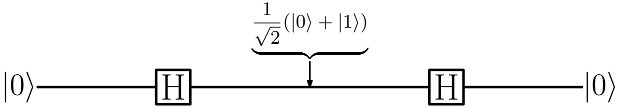

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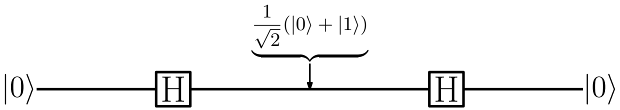

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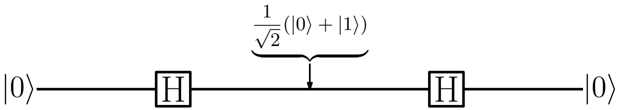

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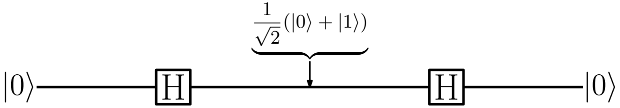

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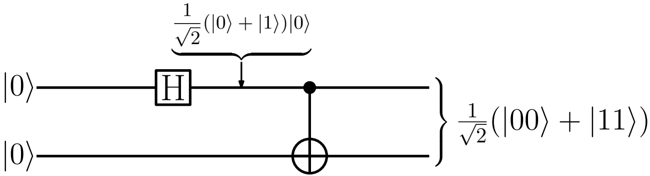
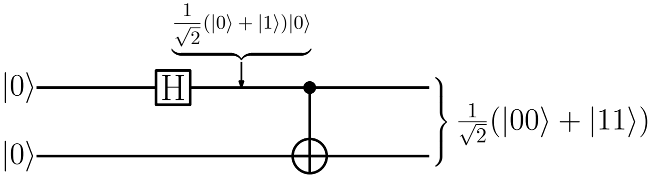
This is definitely *not possible* with probabilistic gates as described before.

EPR pairs


$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$
$|0\rangle$
H
$|0\rangle$
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

EPR pairs



In this example, the two wires are *entangled*, because the state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ *cannot* be factored as $|a\rangle \otimes |b\rangle$

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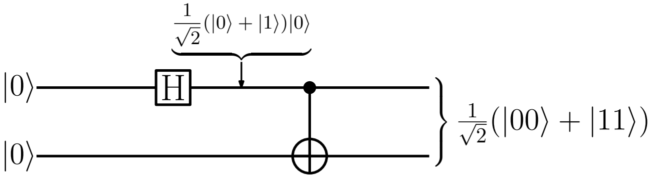

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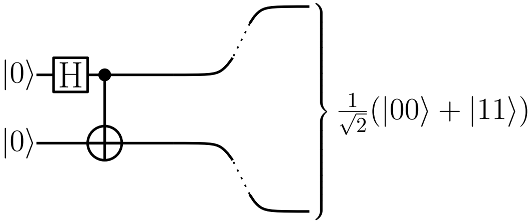

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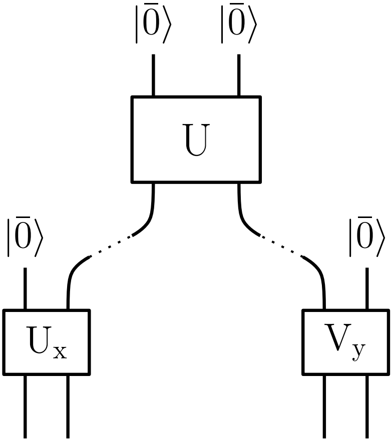
This is even true if the wires are far away from each other.

Moving to the Bell setup



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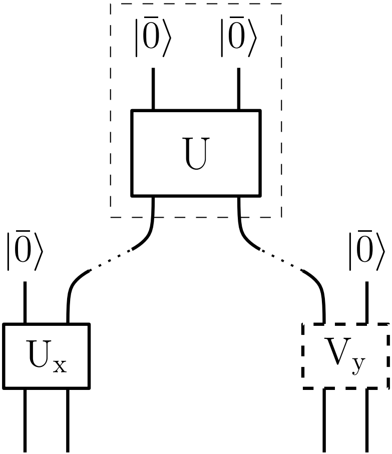

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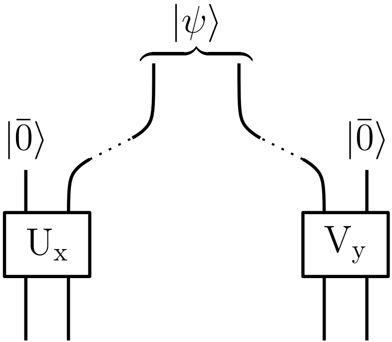

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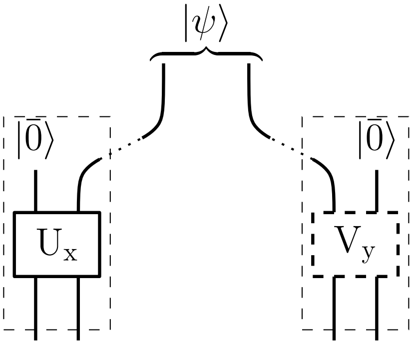

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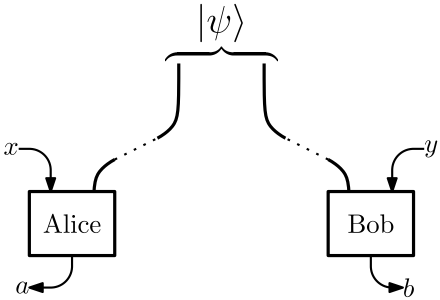
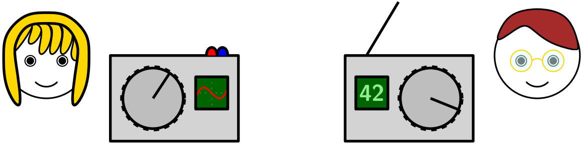
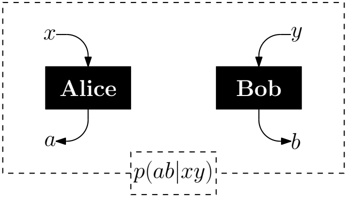
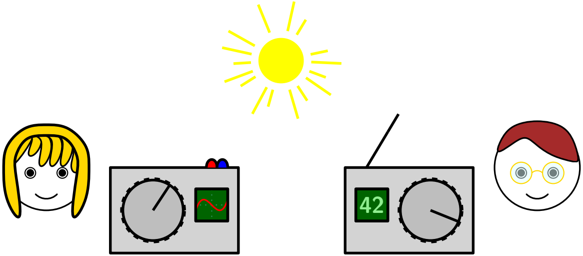


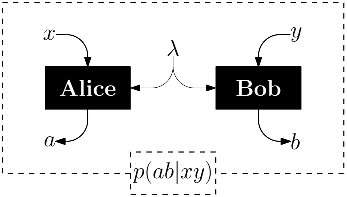
Bell tests : The basic setup

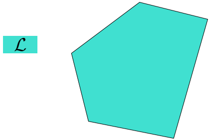




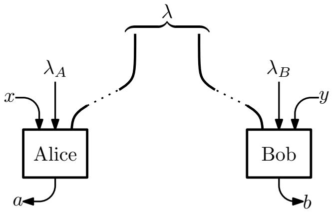
Bell tests : Players using only local hidden variables







Local hidden variables ? Probabilistic circuits !



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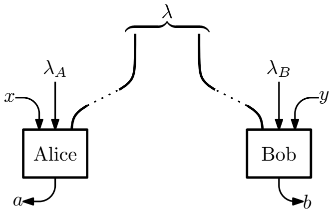

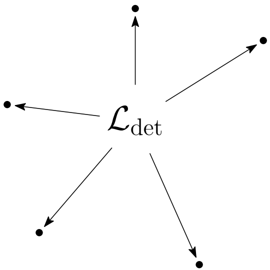

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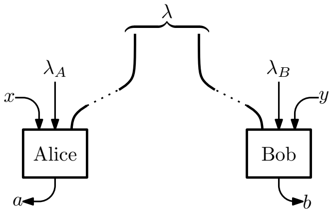
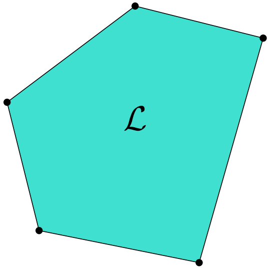
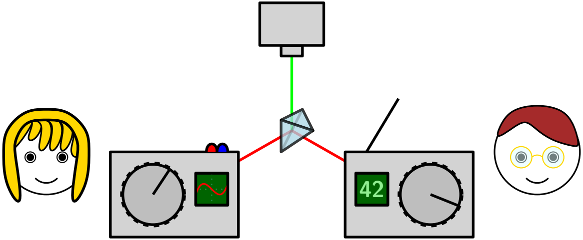
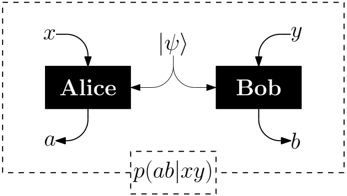
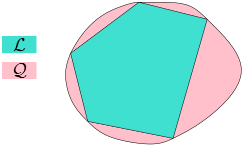




Bell tests : Players with access to a quantum state



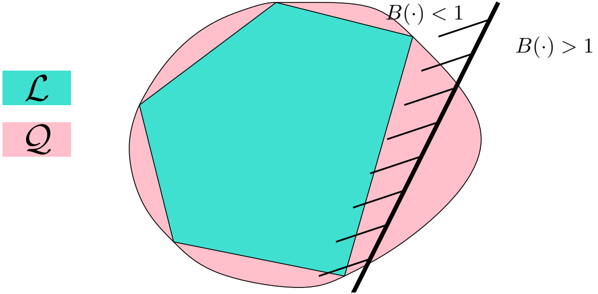




Bell tests : Bell inequalities

Bell functionals $\Leftrightarrow$ linear forms over $\mathbb{R}^{\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}}$

Bell inequalities $\Leftrightarrow$ Inequalities involving a Bell functional that are satisfied by all local distributions



$B : (B_{abxy})_{\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}}$ $\qquad$ $B(\mathbf{p}) = \sum_{abxy} B_{abxy}\, p(ab|xy)$

An example : CHSH

The CHSH game

Alice gets an $x \in \{0, 1\}$, Bob gets an $y \in \{0, 1\}$ (u.a.r).

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Claim

It is impossible to win this game with probability higher than 3/4 classically.

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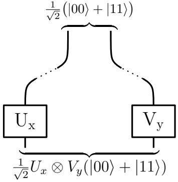
Goal : output a and b s.t. $a \oplus b = x \wedge y$

Claim

It is impossible to win this game with probability higher than 3/4 classically.
However, with quantum ressources, it is possible to achieve $\frac{1+\sqrt{2}}{2\sqrt{2}} \approx 0.85\ldots$

CHSH : winning with $p = 0.85\ldots$



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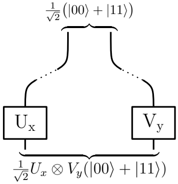

$$U_x = R_{\alpha_x} \qquad V_y = R_{\beta_y}$$
$$\alpha_0 = 0, \alpha_1 = \pi/4 \qquad \beta_0 = \pi/8, \beta_1 = -\pi/8$$

$$\textit{where } R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

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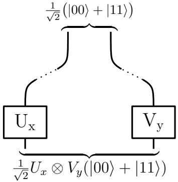

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$$\begin{aligned}
& U_x \otimes V_y \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
= \quad & R_{\alpha_x} \otimes R_{\beta_y} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
= \quad & \frac{1}{\sqrt{2}} \mathbb{1} \otimes R_{\beta_y} (\cos\alpha_x |00\rangle - \sin\alpha_x |10\rangle + \sin\alpha_x |01\rangle + \cos\alpha_x |11\rangle) \\
= \quad & \frac{1}{\sqrt{2}} \Big[(\cos\alpha_x \cos\beta_y + \sin\alpha_x \sin\beta_y)(|00\rangle + |11\rangle) \\
& +(\sin\alpha_x \cos\beta_y - \cos\alpha_x \sin\beta_y)(|01\rangle - |10\rangle) \Big] \\
= \quad & \cos(\alpha_x - \beta_y) \frac{|00\rangle + |11\rangle}{\sqrt{2}} + \sin(\alpha_x - \beta_y) \frac{|01\rangle - |10\rangle}{\sqrt{2}}
\end{aligned}$$

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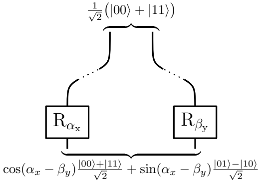

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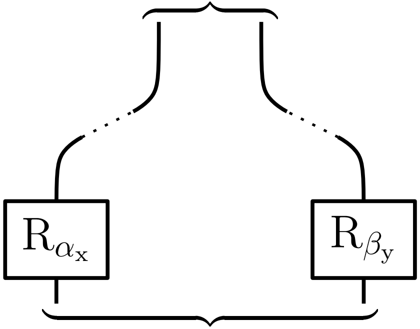
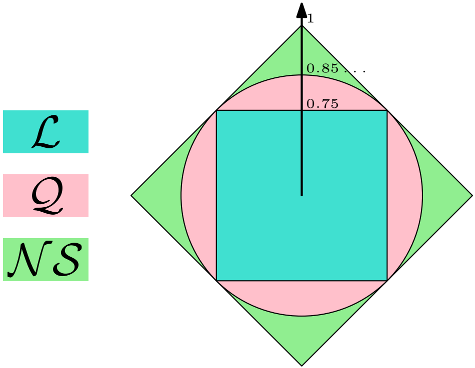
CHSH : winning with $p = 0.85\ldots$

$$\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

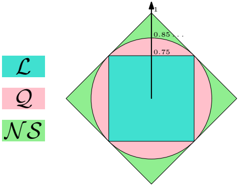


$$\cos(\alpha_x - \beta_y)\frac{|00\rangle+|11\rangle}{\sqrt{2}} + \sin(\alpha_x - \beta_y)\frac{|01\rangle-|10\rangle}{\sqrt{2}}$$

CHSH as bipartite distributions and a Bell inequality

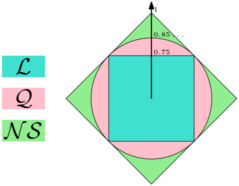


CHSH as bipartite distributions and a Bell inequality



$$B_{abxy} = \mathbb{1}_{\{a \oplus b = x \wedge y\}}$$

CHSH as bipartite distributions and a Bell inequality



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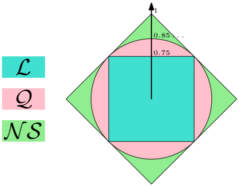
$$B(\ell) \leq 0.75, \forall \ell \in \mathcal{L}$$

CHSH as bipartite distributions and a Bell inequality



$$B_{abxy} = \mathbb{1}_{\{a \oplus b = x \wedge y\}}$$

$$B(\ell) \leq 0.75, \forall \ell \in \mathcal{L}$$

$$\exists \mathbf{q}, B(\mathbf{q}) = \cos^2 \frac{\pi}{8} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

Fun related stuff

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- Non-local boxes (NLBs) of success probability $p > \frac{1}{2} + \frac{1}{\sqrt{6}} \approx 0.91$ trivializes communication complexity [van Dam'00,BBL+'06]

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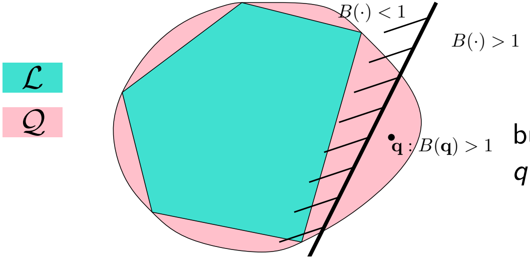
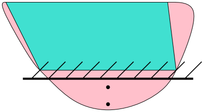
- Non-local boxes (NLBs) of success probability $p > \frac{1}{2} + \frac{1}{\sqrt{6}} \approx 0.91$ trivializes communication complexity [van Dam'00,BBL+'06]
- Because quantum is inherently random, and because deterministic strategies cannot violate Bell inequalities, one can *certify* random bits using Bell inequalities.[PAM'10]
- Such proofs can be *device independent*.

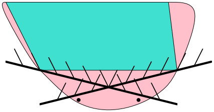
Bell tests : Bell inequality violations

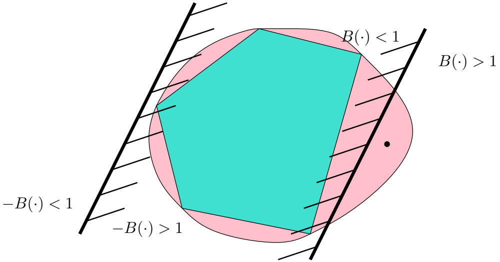


but can we
quantify nonlocality ?

Bell inequality violations : Quantifying nonlocality



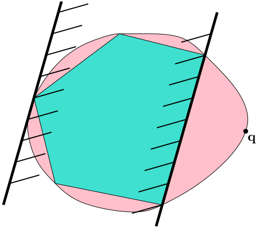




Normalisation constraint : $|B(\cdot)| \leq 1$ on $\mathcal{L}$

$$\nu(\mathbf{p}) := \max B(\mathbf{p}), \qquad |B(\ell)| \leq 1, \; \forall \ell \in \mathcal{L}$$

Maximal violations : Known bounds

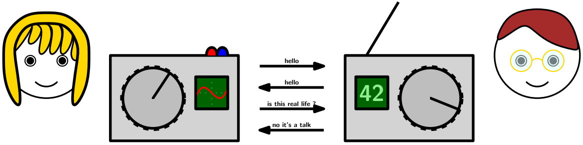


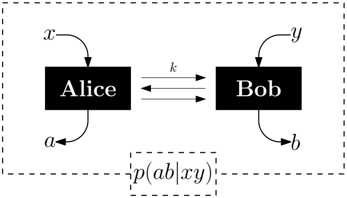
max $B(\mathbf{q})$ over $\mathbf{q} \in \mathcal{Q}$ and
B s.t $|B(\ell)| \leq 1$, $\forall \ell \in \mathcal{L}$?

Parameter	Upper bound	Ad hoc lower bounds	
# inputs N	$2^c \leq N$ [234]	$\frac{\sqrt{N}}{\log(N)}$ [5]	
# outputs K	$O(K)$ [5]	$\Omega\left(\frac{K}{(\log(K))^2}\right)$ [6]	
Dimension d	$O(d)$ [4]	$\Omega\left(\frac{d}{(\log(d))^2}\right)$ [6]	

[1] BCG+'16 [2] LS'09 [3] DKLR'11 [4] JPP+'10 [5] JP'11 [6] BRSdW'12

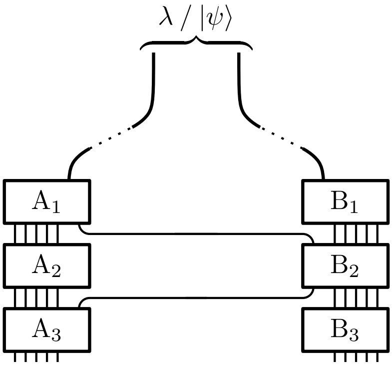
Communication complexity : Definition



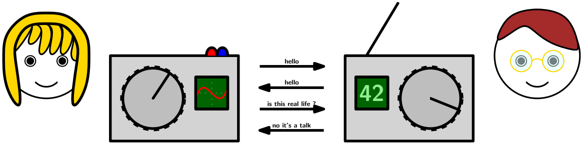


Communication complexity : min k
Classical messages : $\mathrm{CC}(\mathbf{p})$
Quantum messages : $\mathrm{QCC}(\mathbf{p})$

Communication complexity : traveling wires



Communication complexity : Upper and lower bounds



Upper bounds	$\leftrightarrow$	protocols
Lower bounds	$\leftrightarrow$	combinatorial or algebraic measures (ν, γ_2, $\mathrm{disc}\ldots$)

Theorem (DKLR'11)

ν (LS'08) is a lower bound on simulating probability distributions : $\log \nu(\mathbf{p}) \leq \mathrm{CC}(\mathbf{p})$.

Communication complexity : Recent general link to Bell inequalities

Theorem (BCG+'16)

For any communication problem f there exists a normalized Bell inequality B and a quantum distribution $\mathbf{q}$ such that :

$$B(\mathbf{q}) = \Omega\left(\frac{\sqrt{\mathrm{CC}(f)}}{\mathrm{QCC}(f)}\right)$$

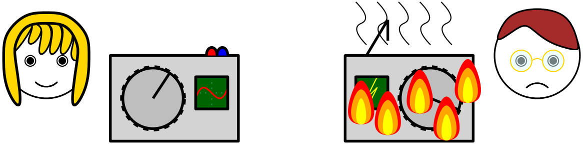
Techniques : protocol tree, port-based teleportation, bias amplification (+Chernoff bound) . . .

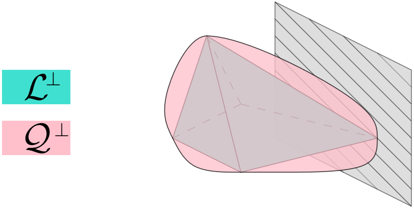
Maximal violations : Known bounds

Parameter	Upper bound	Ad hoc lower bounds	Best possible lower bound from [1]
# inputs N	$2^c \leq N$ [234]	$\frac{\sqrt{N}}{\log(N)}$ [5]	$\frac{\sqrt{c}}{q} \leq \log(N)$
# outputs K	$O(K)$ [5]	$\Omega\left(\frac{K}{(\log(K))^2}\right)$ [6]	$\leq \log(K)$
Dimension d	$O(d)$ [4]	$\Omega\left(\frac{d}{(\log(d))^2}\right)$ [6]	$\leq \log\log(d)$

[1] BCG+'16 [2] LS'09 [3] DKLR'11 [4] JPP+'10 [5] JP'11 [6] BRSdW'12

Efficiency and distributions with abort events

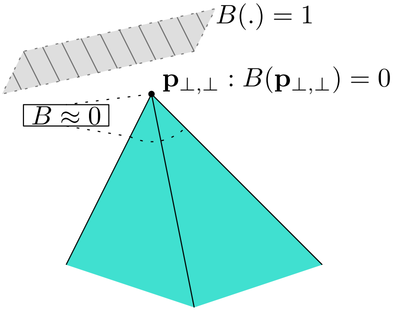




We add a new event $\perp$ (abort) to the outputs
We get new sets $\mathcal{L}^{\perp}$ and $\mathcal{Q}^{\perp}$

Inefficiency-resistant Bell inequalities

Bell inequalities over the larger space $\{\mathcal{A} \cup \perp\} \times \{\mathcal{B} \cup \perp\} \times \mathcal{X} \times \mathcal{Y}$:



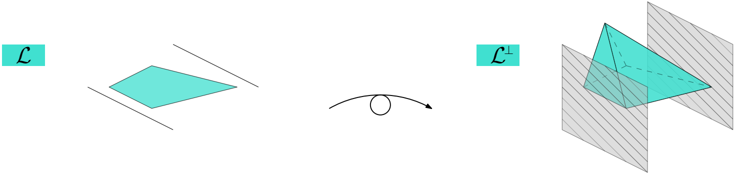
No weights on abort events.
Normalized over a larger set $\mathcal{L}^{\perp}$.
Weaker constraints :

$$B(\ell) \leq 1, \ \forall \ell \in \mathcal{L}^{\perp}.$$

eff$(\mathbf{p}) := \max B(\mathbf{p}), B(\ell) \leq 1, \forall \ell \in \mathcal{L}^{\perp}$ (LLR'12) still well-defined.
It is equivalent to the partition bound of (JK'10).

Theorem (1, this work)

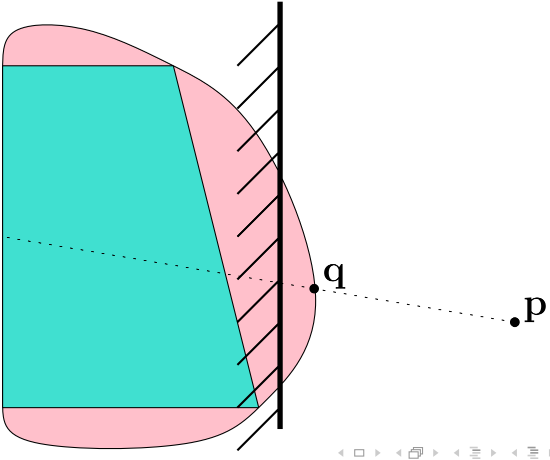
For B such that $|B(\ell)| \leq 1$ over $\mathcal{L}$ There exists B^* such that for all $\mathbf{p} \in \mathcal{C}$ such that $B(\mathbf{p}) \geq 1$, $B^*(\mathbf{p}) \geq \frac{1}{3}B(\mathbf{p}) - \frac{2}{3}$ and $|B(\ell)| \leq 1$ over $\mathcal{L}^\perp$



Ideas : replace aborts by local strategies (preserving locality), make corresponding Bell inequality with weights on aborts, remove aborts from said Bell inequality (MPRG'02)

Theorem (2, this work)

If a distribution $\mathbf{p} \in \mathcal{P}$ *is such that* $CC(\mathbf{p}) \geq \log(B(\mathbf{p})) = C$ *and* $QCC(\mathbf{p}) \leq Q$. *Then there exists* $\mathbf{q} \in \mathcal{Q}$ *with one additional output per player compared to the distribution* $\mathbf{p}$ *such that* $B(\mathbf{q}) \geq 2^{C-2Q}$



Application to some communication problems

Problem	Normalized Bell violations [1]	Inefficiency-resistant Bell violations (this work)
VSP [23]	$\Omega\left(\sqrt[6]{n}/\sqrt{\log n}\right)$	$2^{\Omega(\sqrt[3]{n})-O(\log n)}$
DISJ [456]	N/A	$2^{\Omega(n)-O(\sqrt{n})}$
TRIBES [78]	N/A	$2^{\Omega(n)-O(\sqrt{n}\log^2 n)}$
ORT [98]	N/A	$2^{\Omega(n)-O(\sqrt{n}\log n)}$

[1] BCG'16 [2] Raz'99 [3] KR'11 [4] Razb'92 [5] Razb'03 [6] AA'05 [7] JKS'03 [8] BCW'98 [9] She'12

Maximal violations for inefficiency-resistant Bell inequalities

Parameter	Upper bound	Lower bounds
# inputs N	N	$\Omega(N)$
# outputs K	$2^{O(\frac{K^4}{\epsilon^2}\log^2(K))}$	$\infty(\epsilon = 0)$
Dimension d	$2^{O((\frac{Kd}{\epsilon})^2\log^2(K))}$	?

Very sensitive to arbitrary noise, less so to other noise models.
Uses much less entanglement than (BCG'+16).
Results are quantitatively incomparable.

Thank you !

Why inefficiency-resistant Bell inequalities are normalized

How to get arbitrary violations with normalized Bell inequalities

$\tilde{B} = M \cdot B + \frac{M-1}{\#\mathcal{X}\mathcal{Y}}$

A simple example where this fails for inefficiency-resistant Bell inequalities

- $B_{0,0,0,0} = 1$
- $B_{a,b,x,y} = 0$ for $(a, b, x, y) \neq (0, 0, 0, 0)$

$\forall \mathbf{p} \in \mathcal{P},\ |B(\mathbf{p})| \leq 1$ $\tilde{B}(\ell_{(0,0)\to(0,0)}) = M - \frac{M-1}{\#\mathcal{X}\mathcal{Y}}$