Latent Effects for Reusable Language Components Extended Version

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Abstract. The development of programming languages can be quite complicated and costly. Hence, much effort has been devoted to the modular definition of language features that can be reused in various combinations to define new languages and experiment with their semantics. A notable outcome of these efforts is the algebra-based "datatypes à la carte" (DTC) approach. When combined with algebraic effects, DTC can model a wide range of common language features. Unfortunately, the current state of the art does not cover modular definitions of advanced control-flow mechanisms that defer execution to an appropriate point, such as call-by-name and call-by-need evaluation, as well as (multi-)staging. This paper defines *latent effects*, a generic class of such control-flow mechanisms. We demonstrate how function abstractions, lazy computations and a MetaML-like staging can all be expressed in a modular fashion using latent effects, and how they can be combined in various ways to obtain complex semantics. We provide a full Haskell implementation of our effects and handlers with a range of examples.

Keywords: effect handlers, effects, monads, modularity, staging

1 Introduction

Modern programming languages, be they general purpose or domain-specific, can be built in a flexible manner by composing simple, off-the-shelf language components. It is attractive to build languages in this way as it is useful to study language components in isolation. Furthermore, it reduces the cost of developing new and improved programming languages. Indeed, reducing the effort of building languages to the effort of composing off-the-shelf language components for features such as function abstraction, exceptions or mutable state, is likely to enable language designers with limited resources or expertise—e.g., domain experts—to build their own languages. Providing a modular definition for these advanced language features enables a more widespread use of them, especially in the development of domain-specific languages (DSLs).

To build programming languages from reusable components, we need a framework for defining those components. Two promising techniques for such a framework are *Data Types à la Carte* [23] (DTC) and *Algebraic Effects & Handlers* [18] (AE&H). Using these, language interpreters can be implemented in two steps:

$$Syntax \xrightarrow{denote} Effects \xrightarrow{handle} Result$$

The first step defines a *denote* function that maps the syntax of an object language onto effectful (*monadic* [13]) operations. By implementing *denote* using DTC, we can seamlessly compose *denote* functions from isolated cases for different *Syntax* constructors.

In the second step, handle defines the semantics of effectful operations and their effect interactions. Using AE&H allows cases of denote to be defined in an effect polymorphic way; i.e., a case maps to a monad that has at least the relevant operations, and possibly more. Furthermore, we can define handle functions for different effects in isolation, and seamlessly compose them by nesting handlers. Thus, DTC+AE&H provides a powerful framework for composing programming languages from isolated components.

However, not all language fragments have an obvious modular definition in terms of AE&H. Traditional algebraic effects and scoped effects [30] are baking in assumptions about control-flow and data-flow. These assumptions get in the way of expressing a modular semantics for some language features. In particular, control-flow mechanisms that defer execution, such as lambda abstractions with effectful bodies, lazy evaluation strategies, or (multi-)staging, are neither algebraic [17] nor scoped. This implies that the only way to implement these mechanisms is by means of sophisticated encodings that are often relatively low-level and non-modular. As such, these control-flow mechanisms present a severe challenge for allowing non-experts to build programming languages by composing off-the-shelf components.

This paper presents a novel form of effects and handlers that addresses this challenge: *latent effects and handlers*. Latent effect handlers defer the running of side effects, such that they are "latent" until handlers for all effects have been applied. This notion of effects can be used to directly define handlers for language components that defer execution, such as lambda abstractions with effectful bodies, lazy evaluation, and (multi-)staging.

After introducing the background (Section 2), our main contributions are:

- We introduce *latent effect trees*, implemented using Haskell (Section 3).
- We show how to encode lambda abstraction and application in terms of latent effects. We illustrate how this allows us to define lambda abstraction with effectful bodies as a composable language component (Section 3).
- We illustrate how to compose languages from reusable components using latent effects by showing how to encode lazy evaluation (call-by-need and call-by-name), and MetaML-like multi-staging (Section 4).

 We provide an effect library⁴ with syntax and semantics in Haskell for various simple and advanced language features. These features can be used as isolated components to construct languages (Section A).

Finally, we discuss related work (Section 5) and conclude (Section 6).

2 Background and Motivation

This section summarizes the state of the art in modular language definition by means of DTC and AE&H, discusses the challenges and problems associated with treating lambdas in this setting, and sketches how our new latent effects enable the integration of modular forms of advanced control-flow effects.

2.1 Modular Syntax and Semantics with Data Types à la Carte

Data types à la Carte [23] (DTC) solves the core problem of assembling language definitions from isolated components. It solves this for both syntax and semantics to be composed in a modular way. For an accessible introduction to DTC, we refer to [23]; we summarize the key points here.

Firstly, the abstract syntax of a language is modularized into the syntax of each isolated feature. This is achieved by defining a recursive type Syntax S of ASTs in terms of the shape S of its nodes. This shape S can be composed out of shapes S_1, \ldots, S_n of the individual features: $S = S_1 + \ldots + S_n$.

Secondly, the semantic function $denote :: Syntax \ S \to M$ that maps ASTs $Syntax \ S$ to their meaning M is modularized into the separate syntactic maps of the individual features. This is done by parameterizing the recursive semantic map of the AST with the semantic mapping of the individual nodes, $denote = fold \ denote_S$ where $denote_S :: S \ M \to M$ is composed out of the semantic maps:

```
denote_{S_1} :: S_1 M \to M denote_{S_n} :: S_n M \to M denote_{S_n} :: S_n M \to M
```

This modular approach affords great flexibility to quickly assemble a range of different languages and to explore language design: features can be added or removed, and their semantic maps can be changed. We use the DTC approach extensively when defining a library of effects in Section 4.3.

Unfortunately, this approach comes with a serious limitation: to be able to combine the semantic maps of different features, they must agree on the semantic domain M. This often prohibits the unanticipated combinations of features, even due to small differences in their semantic domain, such as one feature requiring access to an environment or store that the other feature does not expect.

Moggi [13] observed that many of the differences in semantic domains of common language features, usually called (side-)effects, can be captured in the category-theoretical pattern of *monads*. Although not all monads compose, the

⁴ https://github.com/birthevdb/Latent-Effect-and-Handlers.git.

state of the art in modular monads, algebraic effects and handlers (AE&H) [18], is well-aligned with the DTC approach of modular language features.

In fact, AE&H tackles the problem of modularly defining a monad in much the same way DTC tackles the problem of modularly defining a language. Indeed, the API of the monad (its "syntax") is modularized into separate effects; the category-theoretical concept of a free monad plays the role of an abstract syntax of API calls. A modular meaning is assigned to the free monad by means of semantic maps for the separate effects. The key difference with DTC is that the different effects do not all have to be mapped to the same semantics. Instead, by means of "modular carriers" their semantics can be layered [20].

Whereas AE&H is used to define (simpler) language features, we use latent effects and handlers, which we introduce in Section 3, to modularly define more complex language features.

2.2 Advanced, Non-Modular Control-Flow Effects

The AE&H literature shows how to express a range of different control-flow effects such as exceptions, non-determinism and even call-with-current-continuation. However, as traditional effects and handlers rely on assumptions about data-flow and control-flow, more advanced and complicated control-flow effects are missing, such as call-by-name and call-by-need evaluation or multi-staging. These features typically defer execution and are *non-algebraic*.

An operation is *algebraic* when it meets the following requirements:

- 1. It can be expressed as $op:D \to \forall a.M\ a \to ... \to M\ a$ where D represents the non-computational parameters that the operation has, M is a monad with algebraic operations, and each M a typed parameter represents a possible continuation of the operation.
- 2. It satisfies the algebraicity property, which states that, no matter which possible continuation m_i we take, the continuation immediately transfers control to the current continuation k.

$$(op \ d \ m_1 \ \dots \ m_n) \gg k \equiv op \ d \ (m_1 \gg k) \ \dots \ (m_n \gg k)$$

Although many operations are algebraic, there are many common controlflow operations that are not. For instance, $catch :: \forall a.M \ a \rightarrow M \ a \rightarrow M \ a$ is an operation that executes the computation in the first argument, and if that computation raises an exception, proceeds by executing the second argument. The catch operation is not algebraic; i.e.,

$$(catch \ m_1 \ m_2) \gg k \not\equiv catch \ (m_1 \gg k) \ (m_2 \gg k)$$

since exceptions raised during evaluation of k should not be handled by m_2 .

The lack of support for control-flow effects such as exception handling, motivated the development of *scoped effects* [30]. An operation is *scoped* when it can be expressed as having the following type:

$$op::D\to \forall a.M\ a\to \ldots\to M\ a\to \forall b.(a\to M\ b)\to \ldots\to (a\to M\ b)\to M\ b$$

The universal quantification over a restricts data flow: for a given operation op d $m_1 \dots m_n$ $k_1 \dots k_m$, it is only the possible continuations $k_1 \dots k_m$ that can inspect values yielded by computations m_i . The allowable destinations of data produced by the computation are restricted to those determined by the operation. The catch operation is compatible with this restriction: it can be implemented as a scoped operation $catch' :: \forall a.M$ $a \to M$ $a \to \forall b.(a \to M$ $b) \to M$ b.

However, this pattern does not apply to more advanced control-flow effects for which the data produced by a computation can be used outside of the operation. For example, lambda abstractions delay the execution of computations in the body of a function until the function is applied (or not, depending on dynamic control- and data flow). To support such deferred execution, the return type V in the body of a lambda abstraction $abstr: String \rightarrow M \ V \rightarrow M \ V$ is not universally quantified. Thus, abstr is not a scoped operation. It is also not algebraic, as the equation $abstr \ x \ m \gg k \equiv abstr \ x \ (m \gg k)$ would cause k to be (wrongly) deferred, and could cause free variables in k to be captured. Other control-flow effects, such as call-by-need and call-by-name evaluation and multi-staging annotations for (dynamically) staging and unstaging code, defer execution in a similar way, and are similarly neither scoped nor algebraic.

It is theoretically possible to define the control-flow effects discussed above, by making the control flow of all operations *explicit*; e.g., by writing interpreters in continuation-passing style (CPS). However, the relatively low-level nature of CPS and its non-modular style, make this approach fall short of our goal of composing languages from simple, off-the-shelf components.

2.3 Our Approach: Latent Effects

We provide modular support for advanced control-flow effects such as function abstraction, with its different evaluation semantics, and staging. Our solution generalizes the approach of DTC and AE&H outlined above. In fact, it does not require any changes to the DTC approach for modular abstract syntax trees and modular semantic mapping. It only impacts the second part of the pipeline, replacing AE&H with a more general notion of *latent effects and handlers* (LE&H).

LE&H is based on a different, more sophisticated structure than AE&H's free monad. This structure supports non-atomic operations (e.g., function abstraction, thunking, quoting) that contain or delimit computations whose execution may be deferred. Also, the layered handling is different. The idea is still the same, to replace bit by bit the structure of the tree by its meaning. Yet, while AE&H grows the meaning around the shrinking tree, LE&H grows little "pockets of meaning" around the individual nodes remaining in the tree, and not just around the root. The latter supports deferred effects because later handlers can still re-arrange the semantic pockets created by earlier handlers.

LE&H are the first to modularly express advanced control-flow effects, such as staging and lambda abstractions, and provide different handlers, e.g., for

call-by-name and call-by-need evaluation. Moreover, they combine with existing algebraic effects to express varying semantics for a large range of languages.

3 Latent Effects

This section presents *latent effects*. Latent effects generalize algebraic effects to include control-flow mechanisms that defer computation. As our running example we use lambda abstraction, which—as discussed in Section 2.2—is neither an algebraic nor a scoped operation. We show that it can be defined as a latent effect. We start from a non-modular version that we refine in two steps. First we add support for modular signatures, and then support for modular handlers.

3.1 Non-Modular Definition of Lambda Abstraction

First we provide a non-modular definition of the lambda abstraction effect.

Monadic Syntax Tree The type $LC\ v\ a$ is a non-modular monadic syntax tree that supports three primitive operations for a de Bruijn indexed λ -calculus.

```
data LC\ v\ a where Return:: a \to LC\ v\ a Var :: Int \to (v \to LC\ v\ a) \to LC\ v\ a App :: v \to v \to (v \to LC\ v\ a) \to LC\ v\ a Abs :: LC\ v\ v \to (v \to LC\ v\ a) \to LC\ v\ a
```

Here, the v of LC v a is a value type parameter, and a represents the return type of the computation. Thus $Return\ x$ is a trivial computation that returns x. $Var\ i\ k$ retrieves the value of type v associated with the ith variable and passes it to the continuation k. The application $App\ v_1\ v_2\ k$ applies the value v_1 to the value v_2 and passes the result to the continuation. Finally, $Abs\ e\ k$ builds a closure value out of the function body e and passes it to the continuation.

For example, we can represent the lambda expression $(\lambda x \to x)$ 1 as the LC expression Abs (Var 0 Return) ($\lambda v \to App$ v 1 Return). This computation constructs an abstraction and passes it to the continuation as a (closure) value v. The continuation applies v to 1 and passes the result to Return. The closure retrieves the the value of variable with index 0 (i.e., x) and passes it to Return.

Handler The idea of a handler is to map the syntax tree onto its meaning. We illustrate this on the $LC\ v\ a$ type, where we use the type v=Closure for values.

```
data Closure_1 where Clos_1 :: FunPtr_1 \rightarrow Env_1 \rightarrow Closure_1 type Env_1 = [Closure_1] type Env_1 = [Closure_1]
```

A closure contains a function pointer and an environment. The function pointer is an index into a list of deferred computations (i.e., function bodies) that we

call the *(resumption) store*. The environment is a list that maps the closure's parameters (which are indexes) onto their values.

Now we are ready to define the handler handleAbs as a function that, given an initial environment and store, maps an LC Closure a computation onto its meaning, which is a tuple of the result value of type a and the final store.

```
\begin{array}{lll} handleAbs :: Env_1 \rightarrow Store_1 \rightarrow LC \ Closure_1 \ Closure_1 \rightarrow (Store_1, Closure_1) \\ handleAbs \_ r \ (Return \ x) &= (r, x) \\ handleAbs \ env \ r \ (Var \ n = k) = handleAbs \ env \ r \ (k \ (env \ !! \ n)) \\ handleAbs \ env \ r \ (App \ v_1 \ v_2 \ k) = handleAbs \ env \ r' \ (k \ v) \\ \textbf{where} \ (Clos_1 \ fp \ env') = v_1 \\ (r', v) &= handleAbs \ (v_2 : env') \ r \ (r \ !! \ fp) \\ handleAbs \ env \ r \ (Abs \ e = k) = handleAbs \ env \ r' \ (k \ v) \\ \textbf{where} \ v = Clos_1 \ (length \ r) \ env \\ r' = r + [e] \end{array}
```

First, the leaf case of the handler returns the value in that leaf, supplemented with the resumption store. Next, the variable case looks up the variable in the environment and passes it to the continuation. Then, the application case unpacks the closure, retrieves the corresponding function body from the resumption store and applies it to the extended environment and store. The resulting value is passed to the continuation. Finally, the abstraction case adds the function body to the resumption store, creates a closure that indexes this new entry, and calls the continuation on this closure value.

3.2 Modular Latent Effect Signatures and Trees, Version 1

We now modularize the definition of LC by separating the recursive structure of the monadic syntax from the node shapes of the Var, App and Abs operations.

Latent Effect Signature We call the node shapes the latent effect signature. In this case, it is called Abstracting v with v the type of values.

```
data Abstracting v :: * \to (* \to *) \to *where

Var' :: Int \to Abstracting v v NoSub

App' :: v \to v \to Abstracting v v NoSub

Abs' :: Abstracting v v (OneSub v)
```

Besides its first parameter v, the type $Abstracting \ v \ p \ c$ is indexed by two further parameters: The parameter p is the return type of the primitive operations; this is the type of value they pass to their continuation. As all three primitive operations return a value of type v, we have that p = v. The parameter $c::* \to *$ captures the number and result type of the subcomputations. As Var' and App' have no subcomputations, they use the type c = NoSub to indicate that. However, Abs' has a subcomputation and it indicates this with $c = OneSub \ v$. This subcomputation is the body of the function abstraction, whose return type is v. Hence, $OneSub \ v$

has one constructor $One :: OneSub \ v \ v$.

```
data NoSub :: * \rightarrow * where
```

```
data OneSub\ v :: * \rightarrow * where One :: OneSub\ v\ v
```

Latent Effect Tree, Version 1 The type $Tree_1 \sigma a$ extends a latent effect signature σ into a recursive syntactic structure that is a monad in a.

```
data Tree_1 (\sigma :: * \to (* \to *) \to *) a where Leaf_1 :: a \to Tree_1 \sigma a Node_1 :: \sigma p \ c \to (\forall x.c \ x \to Tree_1 \sigma \ x) \to (p \to Tree_1 \sigma \ a) \to Tree_1 \sigma \ a
```

The $Leaf_1$ constructor is trivial; $Leaf_1$ x returns a pure computation with result x. The internal nodes are of the form Node op sub k where the fields have the following meaning. The first, op, identifies what primitive operation the node represents. Next, sub is a function that, in case of a non-atomic primitive, selects the subcomputations of the node. Finally, k is the continuation of further operations to perform after the current one.

Some notable characteristics are as follows:

- Every operation has a result type p that is made available to its continuation, and a number of subcomputations c. To model these two, the signature of an operation $op :: \sigma p c$ is parameterized by p and c.
- The function sub has type $\forall x.c \ x \rightarrow Tree_1 \ \sigma \ x$. The input of type $c \ x$ determines what subcomputation to select; the parameter x indicates the result type of that subcomputation.
- Likewise, continuations take as input the operation's output value (p).
- The Tree₁ data type is monadic, with a similar notion of return and bind as the free monad [23]:

```
instance Monad (Tree_1 \sigma) where return = Leaf_1 (Leaf_1 x) \gg f = f x (Node_1 \ op \ sub \ k) \gg f = Node_1 \ op \ sub \ (\lambda x \to k \ x \gg f)
```

A monadic binding $t \gg f$ thus "concatenates" the tree in f to the leaf positions in the continuation (only) of t.

We can emulate the non-modular type $LC \ v \ a$ with $LC' \ v \ a$.

```
type LC' \ v \ a = Tree_1 \ (Abstracting \ v) \ a
```

The corresponding representation for LC's Return constructor is $Leaf_1$. The Var constructor is represented with a $Node_1$.

```
var_1 :: Int \rightarrow (v \rightarrow LC' \ v \ a) \rightarrow LC' \ v \ a

var_1 \ i \ k = Node_1 \ (Var' \ i) \ (\lambda x \rightarrow \mathbf{case} \ x \ \mathbf{of} \ ) \ k
```

```
\begin{array}{l} var :: (Abstracting \ v < \sigma) \Rightarrow Int \rightarrow Tree_1 \ \sigma \ v \\ var \ n = Node_1 \ (injSig \ (Var' \ n)) \ (\lambda x \rightarrow \mathbf{case} \ x \ \mathbf{of} \ ) \ Leaf_1 \\ app :: (Abstracting \ v < \sigma) \Rightarrow v \rightarrow v \rightarrow Tree_1 \ \sigma \ v \\ app \ v_1 \ v_2 = Node_1 \ (injSig \ (App' \ v_1 \ v_2)) \ (\lambda x \rightarrow \mathbf{case} \ x \ \mathbf{of} \ ) \ Leaf_1 \\ abs :: (Abstracting \ v < \sigma) \Rightarrow Tree_1 \ \sigma \ v \rightarrow Tree_1 \ \sigma \ v \\ abs \ t = Node_1 \ (injSig \ Abs') \ (\lambda One \rightarrow t) \ Leaf_1 \end{array}
```

Fig. 1: The modular constructor functions of the *Abstracting* effect. These functions all fix the continuation to $Leaf_1$, which can easily be replaced by an arbitrary continuation k using the \gg operator of $Tree_1$'s monad instance.

This is a Var' i node. As there are no subcomputations, there are no branches in the pattern match in the selection function on the value x of the empty type NoSub. Lastly, the continuation k receives the value produced by the operation.

The encodings of the two other operations are similar. One notable aspect of abs_1 is that it does have one subcomputation. Hence, the selection function matches on the One constructor and returns t:

```
\begin{array}{l} app_1 :: v \rightarrow v \rightarrow (v \rightarrow LC' \ v \ a) \rightarrow LC' \ v \ a \\ app_1 \ v_1 \ v_2 \ k = Node_1 \ (App' \ v_1 \ v_2) \ (\lambda x \rightarrow \mathbf{case} \ x \ \mathbf{of} \ ) \ k \\ abs_1 :: LC' \ v \ v \rightarrow (v \rightarrow LC' \ v \ a) \rightarrow LC' \ v \ a \\ abs_1 \ t \ k = Node_1 \ Abs' \ (\lambda One \rightarrow t) \ k \end{array}
```

Modular Tree Constructors We can create modular constructors for latent effect operations, similarly to how DTC admits modular syntax constructors [23]. To this end, we use a co-product operator + that combines latent effect signatures, and a subtyping relation $\sigma_1 < \sigma_2$ with an associated injection function, $injSig::\sigma_1 \ p \ c \rightarrow \sigma_2 \ p \ c$. Using these, we can implement the modular constructor functions in Figure 1, that allow combining Abstracting with other latent and algebraic effects. The subtyping requirements in the type signatures are automatically inferrable by type class instance resolution in Haskell. The implementation details of + and < are given in Section A.

We can now use these modular constructors to implement denotation function cases. We can also write programs using the constructors directly. For example, the following program with a lambda abstraction with an effectful body:

```
prog :: \forall v.Num \ v \Rightarrow Tree_1 \ (Mutating \ v + Abstracting \ v + Ending) \ v
prog = \mathbf{do}
put \ (1 :: v)
f \leftarrow abs \ (\mathbf{do} \ m \leftarrow var \ 0; (n :: v) \leftarrow get; return \ (m + n))
put \ (2 :: v)
app \ f \ (3 :: v)
```

The body of the function abstraction increments the function argument value (de Bruijn index 0) by the n yielded by the get operation. The signature of

the program tree is the co-product of three signatures: (1) Mutating V for mutable state; (2) function abstractions Abstracting V; and (3) the empty signature Ending, which provides no operations and serves as the base case. The Mutating V effect recasts the traditional algebraic state effect as a latent effect, and has two operations, get and put. Observe that prog is essentially an AST, with multiple possible interpretations. If Mutating V is dynamic (runtime) state, then prog evaluates to 3+2=5. However, if it is for macro bindings that are expanded statically, then the get in the body of the abstraction is evaluated under the state at the "definition site" of the lambda, and prog evaluates to 3+1=4. Next, we show how handlers can map the prog syntax tree to different semantics.

3.3 Trees with Support for Modular Handlers

In the case of a modularly composed signature $\sigma = \sigma_1 + ... + \sigma_n + Ending$, the idea is to compose the handler function from individual handlers for the different components of the signature $h = hEnd \circ h_n \circ ... \circ h_1$. The type of each handler would be $h_i :: \forall \sigma. Tree_1 \ (\sigma_i + \sigma) \ a \to Tree_1 \ \sigma \ (L_i \ a)$. Hence, it is polymorphic in the remaining part of the signature and preserves those nodes in the resulting tree. It only replaces the nodes of σ_i with their meaning, which is given in the form of a functor L_i that decorates the result type a.

Unfortunately, our $Tree_1$ type and, in particular, the type of its $Node_1$ constructor, needs some further refinement to fully support this idea. Indeed, if the signature is $\sigma_i + \sigma$, and we apply h_i to all the recursive occurrences of $Tree\ (\sigma_i + \sigma)$ in a σ -node $Node_1\ (Inr'\ op)\ sub\ k$, we get:

```
Before applying h_i

Int' op :: (\sigma_i + \sigma) p c
sub :: \forall x.c \ x \to Tree_1 \ (\sigma_i + \sigma) x
k :: p \to Tree_1 \ (\sigma_i + \sigma) a

After applying h_i

op :: \sigma p c
h_i \circ sub :: \forall x.c \ x \to Tree_1 \ \sigma \ (L_i \ x)
h_i \circ k :: p \to Tree_1 \ \sigma \ (L_i \ a)
```

The resulting fields do not together form a node of type $Tree_1 \sigma(L_i a)$ because the highlighted result type of the subcomputations is $(L_i x)$ rather than x which the $Node_1$ constructor requires sub to have as return type.

The problem is that the $Tree_1$ type is oblivious to the *effect functor* that the return type of subcomputations in the tree are decorated by. To solve this problem, we can expose the effect functor decoration in the tree type itself; e.g.,

```
data Tree_2 (\sigma :: * \rightarrow (* \rightarrow *) \rightarrow *) (l :: * \rightarrow *) a where Leaf_2 :: a \rightarrow Tree_2 \sigma l a Node_2 :: \sigma p \ c \rightarrow (\forall x.c \ x \rightarrow Tree_2 \sigma l \ (l \ x)) \rightarrow (l \ p \rightarrow Tree_2 \sigma l \ a) \rightarrow Tree_2 \sigma l \ a
```

But the *Tree*₂ type requires effect handlers to be applied to subcomputations *immediately*. Motivated by modeling constructs that defer computation, we generalize the type further by parameterizing subcomputations by the effect

functor state, and making each node "remember" the effect state (the *latent* effects):

```
data Tree\ (\sigma :: * \to (* \to *) \to *)\ (l :: * \to *)\ a\ \mathbf{where}
Leaf\ :: a \to Tree\ \sigma\ l\ a
Node :: \sigma\ p\ c \to \boxed{l\ ()} \to (\forall x.c\ x \to \boxed{l\ ()} \to Tree\ \sigma\ l\ (l\ x))
\to (l\ p \to Tree\ \sigma\ l\ a) \to Tree\ \sigma\ l\ a
```

In section 3.4 we discuss how Tree supports deferring computation. Tree is a monad with a return and bind defined similarly to the ones for $Tree_1$ in section 3.2. We can also define modular tree constructors using similar techniques as in section 3.2. For instance, using Tree instead of $Tree_1$, the type of prog from Section 3.2 becomes:

```
prog :: Tree (Mutating V + Abstracting V + Ending) Id V
```

Here, the Id functor models the absence of latent effects in the tree. The type V represents a concrete value type.

Example Figure 2 shows how the type of the prog tree evolves when applying successive handlers for the three parts of the signature:

```
(hEnd \circ hAbs \ [\ ]\ [\ ] \circ hMut \ 0) \ prog
```

First, we run the modular handler for mutable state *Mutating s*, which has type:

```
hMut :: Functor \ l \Rightarrow s \rightarrow Tree \ (Mutating \ s + \sigma) \ l \ a \rightarrow Tree \ \sigma \ (StateL \ s \ l) \ (s, a)
```

Given an initial state of type s, this handler transforms a tree into another tree. The signature of the tree evolves from $Mutating\ s+\sigma$ to σ because the handler interprets the mutable state, but not the other effects. Also, the latent effect functor evolves from l (the latent effects already present) to $StateL\ s\ l$, which augments l with the value of the intermediate state.

```
newtype StateL \ s \ l \ a = StateL \{ unStateL :: (s, l \ a) \}
```

The result type evolves from a to (s,a), which augments it with the final state. Second, the handler for $Abstracting\ V$ behaves similarly to hMut, removing itself from the signature and growing the latent effect functor. Finally, hEnd handles the Ending base case. It takes a tree with an empty signature, which thus necessarily only contains a leaf, and extracts the final result out of it.

```
hEnd :: Tree \ Ending \ l \ a \rightarrow a
hEnd \ (Leaf \ x) = x
```

The remainder of this section illustrates how modular handlers are implemented.

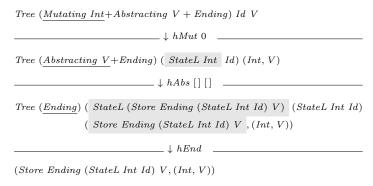


Fig. 2: The type of prog after successive handling steps.

3.4 Example: Two Modular Handlers for Function Abstractions

We implement two different modular handlers for the operations in *Abstracting*, which illustrate (1) how latent effects let us write handlers for function abstraction, and (2) the kind of fine-grained control the handlers provide. The first handler we consider evaluates the body of a function abstraction using the latent effects of its *call site*. Hence, the evaluation of side-effectful operations is postponed until the function is applied. The second handler evaluates the body of the function abstraction using the latent effects of its *definition site*. This immediately enacts the latent effects introduced by previously-applied handlers.

Modular Closure Values A concern that arises when we step away from the earlier non-modular handler for Abstracting is reuse. Notably, in a modular setting we want to allow reuse of both handlers with different notions of values. For that reason, they are parameterized in the type of values v. This type may comprise various shapes of values; all the function abstraction handlers require is that closures are one possible shape of value.

To express this requirement, we introduce another type class $v_1 <: v_2$ for subtyping, this time at kind *, which witnesses (1) that any v_1 can be "upcast" to the type v_2 ; and (2) that a v_2 can be "downcast" to type v_1 . The latter may fail, but the former does not. The minimal requirement for lambda abstractions is that the value type includes closure values.

```
\begin{array}{ll} \textbf{class} \ v_1 <: v_2 \ \textbf{where} \\ inj_v :: v_1 \rightarrow v_2 \\ proj_v :: v_2 \rightarrow Maybe \ v_1 \end{array} \qquad \begin{array}{ll} \textbf{data} \ Closure \ v \ \textbf{where} \\ Clos :: FunPtr \rightarrow Env \ v \rightarrow Closure \ v \\ \textbf{type} \ Env \ v = [v] \end{array}
```

In the modular setting, the types $Closure\ v$ and $Env\ v$, of respectively closures and value environments, are parameterized in the type of values used.

Modular Resumption Store Recall that the resumption store keeps track of the function bodies whose execution has been deferred; i.e., it is a list of

resumptions. In the modular setting, the type of resumptions is parametric in the specific type of signature, latent effect functor and value type.

type Store
$$\sigma$$
 l $v = [R \sigma l v]$

Moreover, depending on whether we want to handle latent effects on the call site or definition site, the definition of a resumption differs.

A resumption of a call-site effect is a function that takes an l () input, which is the latent effect context of the call site where the resumption is evaluated.

type
$$R_{CS} \sigma l v = l () \rightarrow Tree (Abstracting v + \sigma) l (l v)$$

The resumptions of a definition-site effect store are trees instead of functions that produce trees. Indeed, they have no dependency on the latent effects of the call site. Instead, they have been fully determined by the definition site.

type
$$R_{DS} \sigma l v = Tree (Abstracting v + \sigma) l (l v)$$

Although the resumption store makes the handlers verbose, it is a more modular solution than storing *Trees* in values.

Modular Handlers Figure 3 shows the modular handler $hAbs_{CS}$ that uses the call-site latent effects when executing a function body⁵.

Compared to the non-modular definition, there are several differences. Firstly, the handler only interprets part of the work and thus returns the remaining tree instead. Hence, the Leaf case now returns a new leaf, and the other cases use monadic ${\bf do}$ -notation to build a new tree. Secondly, because the signature is a composition and the resulting value type is too, the pattern matching on operations involves the Inl' and Inr' tags of the + co-product. The pattern matching and construction of values involves inj_v and $proj_v$ calls for the same reason. Thirdly, the latent effects now matter and need to be properly threaded in all the operation cases. Finally, there is an additional operation case, to handle unknown operations from the remaining part of the signature by "forwarding" them, i.e., copying them to the resulting tree for later handling.

As discussed, in a modular setting a second handler is possible: one that uses the latent effects of the definition site for function bodies rather than their call site. This definition-site handler, $hAbs_{DS}$, looks much like its sibling. The key difference is the type of resumptions, which affects the code in two places (highlighted in gray). Firstly, the abstraction case applies the subtree function to the latent effect of the definition site instead of deferring the application (st One l instead of st One). Dually, the application case does not have to apply the resumption to the call-site latent effect (r!!p) instead of (r!!p) l).

⁵ The function (<\$) is short for $fmap \circ const$

```
 \begin{array}{l} hAbs_{CS} :: (Closure \ v <: v, Functor \ l) \\ \Rightarrow Env \ v \to Store \ \sigma \ l \ v \to Tree \ (Abstracting \ v + \sigma) \ l \ a \\ \to Tree \ \sigma \ (StateL \ (Store \ \sigma \ l \ v) \ l) \ (Store \ \sigma \ l \ v, a) \\ hAbs_{CS} \ - \ r \ (Leaf \ x) &= Leaf \ (r, x) \\ hAbs_{CS} \ nv \ r \ (Node \ (Inl' \ Abs') \ l \ st \ k) = \mathbf{do} \ \mathbf{let} \ v = inj_v \ (Clos \ (length \ r) \ nv) \\ \mathbf{let} \ r' = r + [ \ st \ One \ ] \\ hAbs_{CS} \ nv \ r' \ (k \ (v <\$ \ l)) \\ hAbs_{CS} \ nv \ r' \ (k \ (v <\$ \ l)) \\ hAbs_{CS} \ nv \ r' \ (k \ v) \to \mathbf{do} \ (r', v) \leftarrow hAbs_{CS} \ (v_2 : nv') \ r \ (r \ (r \ !! \ fp) \ l) \\ hAbs_{CS} \ nv \ r' \ (k \ v) \\ Nothing \to error \ "application \ error" \\ hAbs_{CS} \ nv \ r \ (Node \ (Inl' \ (Var' \ n)) \ l \ - k) = hAbs_{CS} \ nv \ r \ (k \ ((nv \ !! \ n) <\$ \ l)) \\ hAbs_{CS} \ nv \ r \ (Node \ (Inr' \ op) \ l \ st \ k) = Node \ op \ (StateL \ (r, l)) \\ (\lambda c \ (StateL \ (r', lv)) \to hAbs_{CS} \ nv \ r' \ (k \ lv)) \end{array}
```

Fig. 3: Modular call-site abstraction handler. The gray highlights indicate the places where it differs from a definition-site handler.

Example With the abstraction handlers in place, let us revisit the *prog* example. We run the handlers with default initial values, i.e., 0 for the state, the empty variable environment and the empty resumption store. When using the call-site abstraction handler after the state handler, the function body uses the value of the state that was written right before its invocation.

```
> example_{CS} = inspect \ hEnd \ hAbs_{CS} \ [] \ [] \ hMut \ 0 \ prog \ 5
```

If we use the definition-site handler instead, the function body uses the state value that was written right before the abstraction was created.

```
> example_{DS} = inspect \ hEnd \ hAbs_{DS} \ [\ ] \ [\ ] \ hMut \ 0 \ prog \ 4
```

4 Case Study

This section reports on a case study implementation of a library with a range of modular effects (Section A), and two advanced control-flow features implemented using this library: call-by-need lambdas (section 4.1) and multi-staging (section 4.2). For the source code of these case studies, we refer to the implementation available at https://github.com/birthevdb/Latent-Effect-and-Handlers.git.

4.1 Call-by-Need Evaluation

We have implemented two different evaluation strategies, call-by-need (lazy) and call-by-value (CBV), for lambdas by using different latent effect handlers. Our approach is inspired by Levy's call-by-push-value [10], which can express both strategies. We summarize here; Appendix B has all the details.

Call-by-need evaluation lazily delays the evaluation of argument expressions of function applications, and uses *memoization* to ensure that evaluation only happens once for delayed expressions. We build a lazy semantics for function abstractions out of three primitive effects:

- 1. The *Reading* effect corresponds to the well-known *reader monad* from the Haskell monad transformer library [11].
- 2. The Suspending effect delays the evaluation of function bodies, without memoizing the result of the evaluation of the delayed subtrees.
- 3. The *Thunking* effect delays the evaluation of argument expressions of function applications, memoizing the result of forcing a thunked computation.

The definition of these effects and their handlers can be found in the effect library of Appendix A. Using these effects, Appendix B.2 defines three operations for lazy evaluation (abs_{lazy} , var_{lazy} , and app_{lazy}). Lambda abstraction suspends the body of a lambda, and pairs a pointer to the suspension with the environment that the thunk should be evaluated under. The var_{lazy} and app_{lazy} functions memoize and recall argument values (possibly by forcing the evaluation of a thunked computation), and evaluate the body of a lambda. Application evaluates the first argument to a function value, and memoizes the second argument, which is placed in the current environment. Then, the function body is executed.

The following example program evaluates to 0 when using lazy evaluation:

```
\begin{split} prog_{lazy} &:: Tree \quad ( \  \, Mutating \ V + Reading \ [ \, V \,] + Suspending \ V \\ &\quad + Thunking \ V + Ending ) \ Id \ V \\ prog_{lazy} &= app_{lazy} \ (abs_{lazy} \ get) \ (\mathbf{do} \ put \ 42; get) \end{split}
```

Function application delays the evaluation of put in the argument, and is never executed because the function body does not reference its parameter. We can run the program with call-by-need by applying its handlers:

```
>inspect \ hEnd \ hThunk \ [\,] \ hSuspend \ [\,] \ hRead \ [\,] \ hMut \ 0 \ prog_{lazy} \ 0
```

The inspect function extracts the final value out of the result that is decorated with the latent effect functor (in this case nested *StateL's*).

We can also recover a CBV semantics from app_{lazy} , abs_{lazy} , and app_{lazy} by implementing an alternative handler for the *Thunking* effect. This handler eagerly evaluates subtrees and stores their value in a store.

```
>inspect \ hEnd \ hEager \ [] \ hSuspend \ [] \ hRead \ [] \ hMut \ 0 \ prog_{lazy} \ 42
```

```
run <print "bar";
1 + ~(print "foo"; <2>)>

letbind (seq (print "foo") (quote (num 2)))
(unquote (quote (seq (print "bar")
(add (num 1) (splice (var 0)))))))
```

Fig. 4: A MetaML program (left) and its latent effects implementation (right).

This case study demonstrates that modular call-by-need can be implemented by decomposing it into modular, primitive latent effects and handlers. It also shows how *overloading* the handler of the *Thunking* effect provides a means of changing the semantics of a program without touching the program itself.

4.2 Staging

Another advanced control-flow feature that we have implemented with latent effects is multi-staging. By applying effect handlers *before* the handler for the staging effect, we can control which effects should be staged, and which not. The implementation of these staging constructs can be found in Appendix C.

Our inspiration are the three constructs of MetaML [26]: (1) bracket expressions (<_>) delay execution to a later stage; (2) escape expressions (~_) splice a staged expression into another; and (3) run expressions (run _) run an expression that has been dynamically generated by bracket and escape expressions.

A key feature of MetaML is that staged code is *statically typed* and *lexically scoped*. The staging constructs that we implement differ in two ways: our staging constructs are untyped, and we provide *two* constructs for splicing code (*push* and *splice*) instead of the single *escape* expression found in MetaML.

We use *push* for writing programs with escape expressions under binders in staged code. The dynamic semantics of *push* creates an environment with "holes" that represent unknown bindings, and *splice* automatically fills in these holes with bindings from the dynamic context of the *splice* expression.

The four staging constructs we implement are thus: (1) quote, corresponding to brackets in MetaML; (2) unquote, corresponding to run _ in MetaML; and (3+4) splice and push for code splicing. The programs in section 4 illustrate the difference in how splicing works. The MetaML program on the left prints the string "foobar" and returns the value 3. The program on the right desugars into latent effects. With the appropriate handlers, it gives the same output.

Yet, by switching the order of handlers for the *print* effect and staging, we obtain a different semantics that eagerly handles *print* operations inside *quoted* code. This makes the program on the right print "barfoo" instead.

4.3 Library Summary

We have given two examples where latent effects can be modularly composed to form language features. Figure 5 gives an overview of our effect library in Appendix A and how the primitive effects are combined into language features.

The left part shows the general framework code for implementing latent effects, consisting of *Trees*, the DTC approach and helper definitions (e.g. *Id*,

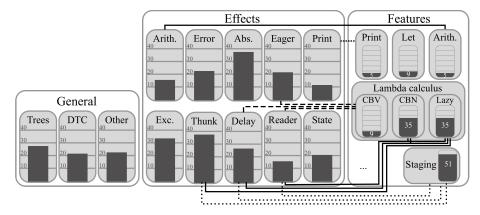


Fig. 5: Effect Library with Lines of Code (LoC) per Effect.

Void); the figure also indicates the associated lines of code (LoC). The middle part shows ten different effects and their LoC. Each effect comes with an effect signature, a handler, and smart constructors for their operations. For the detailed implementation of these effects, we refer to the effect library (Section A). The right part contains several language features that can be implemented using these effects, with their associated LoC. Each feature comes with its object language syntax and a mapping onto the effects. Each language requires an additional two LoC, to indicate the effects and handlers used and their order. A different order of effects and handlers may give different semantics.

Figure 5 only includes a few language features covered in the paper. However, as we provide ten effects and handlers, they can be modularly composed in different order, using different combinations. In theory, when algebras are fixed, we can define $(10! + 9! + \ldots + 2! + 1!) = 4,037,913$ semantics, although some compositions may result in the same behaviour. Even more variations are possible, varying the algebra that maps the object language syntax to the effects.

5 Related Work

Modular Semantics and Effects Modularity has received much attention both at the level of language definitions and of the effects used by those languages. A landmark is the formulation of the expression problem [27], the challenge to modularly extend languages with new features and new interpretation functions. As different language features use different effects, this also requires the modular composition of those effects. Monad transformers [11] are the most prominent approach, including alternative implementations such as Filinksi's layered monads [3] and Jaskelioff's Monatron [7].

Algebraic Effects Algebraic effects [16] have been proposed as a more structured approach to monads that can also be composed [6]. The subsequent introduction

of handlers [18] to deal with exceptions has enabled practical language and library implementations, e.g., [8, 12, 9]. Schrijvers et al. [20] identified when algebraic effect handlers are modular and related this to a subclass of monad transformers, using the notion of modules [15]. Wu et al. [30] have identified a class of what they call *scoped* effects, which cannot be expressed as algebraic operations. To remedy the situation, they have proposed a practical generalization of algebraic effects. Piróg et al. [14] have put this ad-hoc approach for scoped effects on formal footing in terms of a free monad on a level-indexed category.

Latent Effects There are many works on specific types of latent effects. For instance, staging is a widely studied area [19, 25, 22]. Some works have also combined algebraic effects with staging mechanisms [31, 21, 28]. Yet, we are, to the best of our knowledge, the first to consider staging using effect handlers.

The call-by-push-value calculus of Levy [10] provides primitives for expressing both call-by-name and call-by-value. These have been an inspiration for our modular thunking handler. A more generic work is that of Atkey and Johann [1] on interleaving data and effects to model the incremental production of the data, and on interpreting these with f-and-m algebras.

Various forms of delimited control have been used in the literature to realize sophisticated control mechanisms, such as the simulation of call-by-need by Garcia et al. [5]. Moreover, several works [4] show the interdefinability of conventional algebraic effects and delimited control. A further investigation into the relative expressiveness of latent effects would be interesting.

In future work we would like to demonstrate the performance of latent effects, using the techniques of fusion by Wu and Schrijvers [29].

6 Conclusion

This paper has introduced the notion of latent effects. These extend algebraic effects with the ability to modularly model advanced control-flow mechanisms that can postpone the execution of certain computations and require fine-grained control over the effects inside them. Lambda abstraction, lazy evaluation, and staging were shown to be three prominent instances.

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A An Effect Library

This Appendix contains a library of effects that can be modularly composed. Each effect can be implemented with a limited number of lines of code.

A.1 General

Trees and Their Instances The *Tree* datatype for representing latent effects with its corresponding instances.

```
data Tree (\sigma :: * \rightarrow (* \rightarrow *) \rightarrow *) (l :: * \rightarrow *) a where
   Leaf :: a \rightarrow Tree \ \sigma \ l \ a
   Node :: \sigma p c
            \rightarrow (\forall x.c \ x \rightarrow l \ () \rightarrow Tree \ \sigma \ l \ (l \ x))
            \rightarrow (l \ p \rightarrow Tree \ \sigma \ l \ a)
            \rightarrow Tree \sigma l a
instance Functor (Tree \sigma l) where
   fmap = liftM
instance Applicative (Tree \sigma l) where
   pure = return
   (<*>) = ap
instance Monad (Tree \sigma l) where
   return
                                = Leaf
                       \gg f = f x
   Node c \ l \ st \ k \gg f = Node \ c \ l \ st \ (\lambda x \to k \ x \gg f)
```

Datatypes à la Carte We use a DTC approach to compose and project values and signatures.

```
class v_1 <: v_2 where inj_v :: v_1 \rightarrow v_2 proj_v :: v_2 \rightarrow Maybe \ v_1 infixr +  \mathbf{data} \ (\sigma_1 + \sigma_2) :: * \rightarrow (* \rightarrow *) \rightarrow * \mathbf{where} Inl' :: \sigma_1 \ p \ c \rightarrow (\sigma_1 + \sigma_2) \ p \ c Inr' :: \sigma_2 \ p \ c \rightarrow (\sigma_1 + \sigma_2) \ p \ c class (\sigma_1 :: * \rightarrow (* \rightarrow *) \rightarrow *) < \sigma_2 where injSig :: \sigma_1 \ p \ c \rightarrow \sigma_2 \ p \ c instance \sigma < \sigma where injSig = id instance \sigma_1 < (\sigma_1 + \sigma_2) where injSig = Inl'
```

```
instance (\sigma_1 < \sigma_3) \Rightarrow \sigma_1 < (\sigma_2 + \sigma_3) where injSig = Inr' \circ injSig
```

Other Datatypes and Helper Functions Other straightforward datatypes such as the identity functor and *Void*.

```
data Void

newtype Id\ a = Id\ \{unId :: a\} deriving (Functor)

instance Applicative Id\ where

pure\ a = Id\ a

Id\ f <*>Id\ x = Id\ (f\ x)
```

The $StateL\ s\ l\ a$ type is a wrapper for the state type (s,a), where the value is wrapped in a latent effect functor l. Similarly, $EitherL\ left\ l\ a$ is a wrapper for the $Either\ left\ a$ type, where the right value is wrapped in a latent effect functor l

```
newtype StateL \ s \ l \ a = StateL \ \{unStateL :: (s, l \ a)\}
deriving Show
instance Functor \ l \Rightarrow Functor \ (StateL \ s \ l) \ where
fmap \ f \ (StateL \ (s, la)) = StateL \ (s, fmap \ f \ la)
newtype EitherL \ left \ l \ a = EitherL \ \{unEitherL :: Either \ left \ (l \ a)\}
```

Extracting the Value from the Effect Tree Ending has an empty signature, so the tree can only consist of a leave. The handler extracts the value from that leave, possibly decorated with latent effects.

```
data Ending :: * \rightarrow (* \rightarrow *) \rightarrow *
hEnd :: Tree Ending l \ a \rightarrow a
hEnd (Leaf x) = x
```

Subtrees of the *Tree* type data NoSub :: * \rightarrow *where

```
data OneSub\ v :: * \to * where
One :: OneSub\ v\ v
noSubNode :: \forall v\ \sigma_1\ \sigma_2.\sigma_1 < \sigma_2
\Rightarrow \sigma_1\ v\ NoSub \to Tree\ \sigma_2\ Id\ v
noSubNode\ n = Node\ (injSig\ n)\ (Id\ ())\ (\lambda x \to \mathbf{case}\ x\ \mathbf{of}\ ) \qquad (Leaf\ \circ unId)
oneSubNode :: \forall v\ w\ \sigma_1\ \sigma_2.\sigma_1 < \sigma_2
\Rightarrow \sigma_1\ w\ (OneSub\ v) \to Tree\ \sigma_2\ Id\ v \to Tree\ \sigma_2\ Id\ w
oneSubNode\ n\ t = Node\ (injSig\ n)\ (Id\ ())\ (\lambda One\ \_ \to Id\ <\$>\ t)\ (Leaf\ \circ unId)
```

A.2 Effects

In this section, we present small standalone implementations of a single effect, that can be combined in a modular fashion into languages with different effect features. These implementations come with a definition of the effect, a handler for interpreting the effect, and smart constructors.

Reader The *Reading* effect provides two operations: *ask* retrieves the current environment, and *local* passes down an updated environment to a subtree.

```
data Reading r :: * \rightarrow (* \rightarrow *) \rightarrow *where
   Local :: (r \to r) \to Reading \ r \ a \ (OneSub \ a)
   Ask ::
                              Reading \ r \ r \ NoSub
hRead :: Functor l
         \rightarrow Tree (Reading r + \sigma) l (l a)
         \rightarrow Tree \sigma l (l \ a)
hRead\ r\ (Leaf\ x)
                                                      = Leaf x
hRead\ r\ (Node\ (Inl'\ (Local\ f))\ l\ st\ k) = hRead\ (f\ r)\ (st\ One\ l) \gg hRead\ r\circ k
                                           l = k) = hRead \ r \ (k \ (r < l))
hRead \ r \ (Node \ (Inl' \ Ask)
hRead \ r \ (Node \ (Inr' \ op))
                                            l \ st \ k) =
   Node op l (\lambda c_2 \rightarrow hRead \ r \circ st \ c_2) (hRead \ r \circ k)
            :: \forall r \ \sigma \ a. (Reading \ r < \sigma) \Rightarrow (r \rightarrow r) \rightarrow Tree \ \sigma \ Id \ a \rightarrow Tree \ \sigma \ Id \ a
local f t = oneSubNode (Local f) t
ask
            :: (Reading \ r < \sigma) \Rightarrow Tree \ \sigma \ Id \ r
            = noSubNode \ Ask
ask
```

State The Mutating effect provides the two standard operations for state: get and put.

```
data Mutating v :: * \rightarrow (* \rightarrow *) \rightarrow *where
                   Mutating \ v \ v \ NoSub
   Put :: v \to Mutating \ v \ () \ NoSub
hMut :: Functor l
         \Rightarrow s
         \rightarrow Tree (Mutating s + \sigma) l a
         \rightarrow Tree \sigma (StateL s l) (StateL s Id a)
                                                     = Leaf \$ StateL (s, Id x)
hMut\ s\ (Leaf\ x)
hMut\ s\ (Node\ (Inl'\ Get)
                                       l - k = hMut \ s \ (k \ (fmap \ (\lambda \rightarrow s) \ l))
hMut = (Node (Inl' (Put s)) l = k) = hMut s (k l)
hMut\ s\ (Node\ (Inr'\ c)
                                       l \ st \ k) =
   Node\ \ \overset{.}{c}\ (\mathit{StateL}\ (s,\overset{.}{l}))\ (\lambda c\ (\mathit{StateL}\ (s',l'))\ \ \rightarrow \mathit{transS} < \$ > \mathit{hMut}\ s'\ (\mathit{st}\ c\ l'))
                                    (\lambda \quad (StateL\ (s', lv')) \rightarrow hMut\ s'\ (k\ lv'))
transS :: StateL \ s \ Id \ (l \ a) \rightarrow StateL \ s \ l \ a
```

```
\begin{array}{ll} transS \ (StateL \ (s,x)) = StateL \ (s,unId \ x) \\ get & :: (Mutating \ s < \sigma) \Rightarrow Tree \ \sigma \ Id \ s \\ get & = noSubNode \ Get \\ put & :: (Mutating \ s < \sigma) \Rightarrow s \rightarrow Tree \ \sigma \ Id \ () \\ put \ s & = noSubNode \ (Put \ s) \end{array}
```

Exceptions The *Throwing* effect provides the two well-known operations for exceptions: *throw* and *catch*.

```
data Throwing x \ v :: * \rightarrow (* \rightarrow *) \rightarrow *where
   Throw :: x \to \ Throwing \ x \ v \ Void \ NoSub
                       Throwing x v v
                                                   (MaybeSub \ x \ v)
   Catch ::
hExc :: (Functor \ l)
        \Rightarrow Tree (Throwing x \ v + \sigma) l \ a
        \rightarrow Tree \sigma (EitherL (l (), x) l) (EitherL (l (), x) Id a)
hExc (Leaf x)
                                                    = Leaf $ EitherL $ Right $ Id x
hExc\ (Node\ (Inl'\ (Throw\ x))\ l\ \_\ \_) = Leaf\ \$\ EitherL\ \$\ Left\ (l,x)
hExc (Node (Inl' Catch)
                                         l \ st \ k) = hExc \ (st \ NothingS \ l)
    \gg either (\lambda(l', x) \to hExc (st (JustS x) l'))
                       \gg either (Leaf \circ EitherL \circ Left) (hExc \circ k \circ unId))
                   (hExc \circ k \circ unId)
hExc (Node (Inr' c)
                                          l \ st \ k) =
   Node c (EitherL \$ Right l)
              (\lambda z \rightarrow either (Leaf \circ EitherL \circ Left))
                                  (\lambda lv \rightarrow transE < \$ > hExc (st z lv)))
              (either\ (Leaf \circ EitherL \circ Left)\ (hExc \circ k))
throw
             :: \forall x \ v \ \sigma. (Throwing \ x \ v < \sigma) \Rightarrow x \rightarrow Tree \ \sigma \ Id \ v
throw e
             = Node (injSig (Throw e :: Throwing x v Void NoSub))
                          (Id())
                          (\lambda x \_ \rightarrow \mathbf{case} \ x \ \mathbf{of})
                          (\lambda x \to \mathbf{case} \ x \ \mathbf{of} \ )
             :: \forall x \ v \ \sigma. (Throwing \ x \ v < \sigma)
catch
              \Rightarrow Tree \sigma Id v \rightarrow (x \rightarrow Tree \ \sigma \ Id \ v) \rightarrow Tree \ \sigma \ Id \ v
catch \ p \ h = Node \ (injSig \ (Catch :: Throwing \ x \ v \ (MaybeSub \ x \ v)))
                          (Id())
                          (\lambda z \rightarrow \mathbf{case} \ z \ \mathbf{of}
                             JustS \ x \rightarrow Id < \$ > (h \ x)
                             NothingS \rightarrow Id \ll p
                          (Leaf \circ unId)
```

Errors The *Failing* effect has a single operation *err* for constructing an error.

```
data Failing v :: * \rightarrow (* \rightarrow *) \rightarrow *where Err :: v \rightarrow Failing \ v \ Void \ NoSub
```

```
hErr :: Functor l
       \Rightarrow Tree (Failing v + \sigma) l a
       \rightarrow Tree \sigma (EitherL v l) (EitherL v Id a)
hErr(Leaf x)
                                                 = Leaf $ EitherL $ Right $ Id x
hErr\ (Node\ (Inl'\ (Err\ x))\ \_\ \_\ )\ = Leaf\ \$\ EitherL\ \$\ Left\ x
hErr(Node(Inr'c) \quad l \ st \ k) =
   Node\ c\ (EitherL\ \$\ Right\ l)
              (\lambda z \rightarrow either (Leaf \circ EitherL \circ Left) (\lambda lv \rightarrow transE < \$ > hErr (st z lv)))
              (either\ (Leaf\circ EitherL\circ Left)
                                                                  (hErr \circ k)
either :: (v \rightarrow c) \rightarrow (l \ a \rightarrow c) \rightarrow EitherL \ v \ l \ a \rightarrow c
either\ f = (Either L\ (Left
                                      (x) = f(x)
either \ \_g \ (Either L \ (Right \ y)) = g \ y
transE :: EitherL \ v \ Id \ (l \ a) \rightarrow EitherL \ v \ l \ a
transE\ (EitherL\ (Left\ x)) = EitherL\ Left\ x
transE\ (EitherL\ (Right\ y)) = EitherL\ Right\ unId\ y
err :: (Failing v < \sigma) \Rightarrow v \rightarrow Tree \sigma Id a
err \ x = Node \ (injSig \ (Err \ x)) \ (Id \ ()) \ (\lambda x \rightarrow \mathbf{case} \ x \ \mathbf{of} \ ) \ (\lambda x \rightarrow \mathbf{case} \ x \ \mathbf{of} \ )
```

Arithmetics The *Adding* effect provides operations for adding natural numbers: *plus* and *nat*.

```
data Adding\ v :: * \rightarrow (* \rightarrow *) \rightarrow *where
   Nat :: Integer \rightarrow Adding \ v \ NoSub
   Plus :: v \rightarrow v \rightarrow Adding \ v \ NoSub
hPlus :: (Functor \ l, Integer <: v)
        \Rightarrow Tree (Adding v + \sigma) l a
        \rightarrow Tree \sigma l a
hPlus (Leaf x)
                                                    = Leaf x
hPlus\ (Node\ (Inl'\ (Nat\ n)) \qquad l\ \_\ k) = hPlus\ (k\ (const\ (inj_{v}\ n) < > l))
hPlus (Node (Inl' (Plus v_1 v_2)) l = k) = case (proj_v v_1, proj_v v_2) of
   (Just\ n1, Just\ n2) \rightarrow hPlus\ (k\ (const\ (inj_v\ (n1+n2)) < > l))
hPlus (Node (Inr' c))
                                          l \ st \ k) =
   Node c\ l\ (\lambda s \to hPlus \circ st\ s)\ (hPlus \circ k)
               :: (Adding \ v < \sigma) \Rightarrow v \rightarrow v \rightarrow Tree \ \sigma \ Id \ v
plus \ n1 \ n2 = noSubNode (Plus \ n1 \ n2)
nat
               :: (Adding \ v < \sigma) \Rightarrow Integer \rightarrow Tree \ \sigma \ Id \ v
nat n
               = noSubNode (Nat n)
```

Print The *Printing* effect has a single operation for printing a string: *print*.

```
data Printing\ v :: * \rightarrow (* \rightarrow *) \rightarrow *where Print :: String \rightarrow Printing\ v\ v\ NoSub
```

```
\begin{array}{ll} hPrint :: (Functor\ l, String <: v) \\ \qquad \Rightarrow Tree\ (Printing\ v + \sigma)\ l\ a \rightarrow Tree\ \sigma\ l\ a \\ hPrint\ (Leaf\ x) &= Leaf\ x \\ hPrint\ (Node\ (Inl'\ (Print\ s))\ l\ _k) = hPrint\ (k\ (const\ (inj\ _v\ s) < > l)) \\ hPrint\ (Node\ (Inr'\ c)\ l\ st \qquad k) = Node\ c\ l\ (\lambda s \rightarrow hPrint\ \circ st\ s)\ (hPrint\ \circ k) \\ print\ :: (Printing\ v < \sigma) \Rightarrow String \rightarrow Tree\ \sigma\ Id\ v \\ print\ s = noSubNode\ (Print\ s) \end{array}
```

Delay Evaluation The Suspending effect provides two operations: suspend delays the evaluation of a recursive subtree, and enact forces the evaluation of the delayed subtree without memoizing the result.

```
data Suspending v :: * \rightarrow (* \rightarrow *) \rightarrow *where
   Suspend ::
                         Suspending v Ptr (OneSub v)
   Enact :: Ptr \rightarrow Suspending \ v \ NoSub
type Ptr
                             = Int
type Suspension \sigma l v = l () \rightarrow Tree (Suspending v + \sigma) l (l v)
                             = Suspended Ptr r deriving (Functor)
data Suspended r
hSuspend :: Functor l
            \Rightarrow [Suspension \sigma l v]
            \rightarrow Tree (Suspending v + \sigma) l(l a)
            \rightarrow Tree \sigma (StateL [Suspension \sigma l v l l) (StateL [Suspension \sigma l v l l a)
hSuspend \ rs \ (Leaf \ x)
                                                       = Leaf (StateL (rs, x))
hSuspend \ rs \ (Node \ (Inl' \ Suspend) \ l \ st \ k) =
   hSuspend (rs + [st One]) (k (length rs < l))
hSuspend \ rs \ (Node \ (Inl' \ (Enact \ p)) \ l \ \_ \ k) = \mathbf{do}
   StateL(rs', lv) \leftarrow hSuspend\ rs\ ((rs !!\ p)\ l); hSuspend\ rs'\ (k\ lv)
hSuspend rs (Node (Inr' op)
                                             l st k) =
   Node op (StateL (rs, l)) (\lambda c_2 (StateL (rs', l')) \rightarrow hSuspend rs' (st c_2 l'))
                                  (\lambda \quad (StateL (rs', lv')) \rightarrow hSuspend rs' (k lv'))
             :: \forall v \ \sigma.(Suspending \ v < \sigma) \Rightarrow Tree \ \sigma \ Id \ v \rightarrow Tree \ \sigma \ Id \ Ptr
suspend t = oneSubNode (Suspend) t
              :: (Suspending \ v < \sigma) \Rightarrow Ptr \rightarrow Tree \ \sigma \ Id \ v
enact
enact p
              = noSubNode (Enact p)
```

Memoization The *Thunking* effect is analogous to *Suspending*, but its handler *memoizes* the result of forcing a thunked computation.

```
data Thunking v :: * \rightarrow (* \rightarrow *) \rightarrow *where

Thunk :: Thunking v Ptr (OneSub v)

Force :: Ptr \rightarrow Thunking v v NoSub

type Ptr = Int
```

```
type Thunk \sigma l v = Either (l () \rightarrow Tree (Thunking v + \sigma) l (l v)) v data Tree With f \sigma (l :: * \rightarrow *) a =

Tree With \{ unTree With :: Tree \sigma (f l) (f l a)\} data Thunked r = Thunked Ptr r class Modular (m_1 :: * \rightarrow *) (m_2 :: (* \rightarrow *) \rightarrow * \rightarrow *) where fwd :: m_1 (m_2 m_1 a) \rightarrow m_2 m_1 a

replace :: Int \rightarrow a \rightarrow [a] \rightarrow [a]

replace 0 x (\_: xs) = x : xs

replace i x (y : xs) | i > 0 = y : replace <math>(i - 1) x xs

replace \_\_\_ = error "bad index"
```

For this effect, we can provide two handlers, one for lazy evaluation and another one for eager evaluation.

```
hThunk :: (Functor \ l, Modular \ l \ (TreeWith \ (StateL \ ([Thunk \ \sigma \ l \ v])) \ \sigma))
           \Rightarrow [Thunk \sigma l v]
           \rightarrow Tree (Thunking v + \sigma) l(l a)
           \rightarrow Tree \sigma (StateL [Thunk \sigma l v] l) (StateL [Thunk \sigma l v] l a)
hThunk \ rs \ (Leaf \ x)
                                                      = Leaf (StateL (rs, x))
hThunk rs (Node (Inl' Thunk)
                                           l \ st \ k) =
  hThunk (rs + [Left (st One)]) (k (length rs < l))
hThunk \ rs \ (Node \ (Inl' \ (Force \ p)) \ l \ \_ \ k) = \mathbf{case} \ (rs \ !! \ p) \ \mathbf{of}
   Left t \to \mathbf{do}
      StateL(rs', lv) \leftarrow hThunk rs(t l)
      un Tree With \$ fwd \$
         (\lambda v \rightarrow Tree With \$ h Thunk (replace p (Right v) rs') (k lv)) < \$ > lv
   Right v \to hThunk \ rs \ (k \ (v < l))
hThunk rs (Node (Inr' op)
                                            l \ st \ k) =
   Node op (StateL (rs, l)) (\lambda c_2 (StateL (rs', l')) \rightarrow hThunk \ rs' \ (st \ c_2 \ l'))
                                   (\lambda \quad (StateL\ (rs', lv')) \rightarrow hThunk\ rs'\ (k\ lv'))
hEager :: (Functor \ l, Modular \ l \ (TreeWith \ (StateL \ [v]) \ \sigma))
           \Rightarrow [v]
           \rightarrow Tree (Thunking v + \sigma) l(l a)
           \rightarrow Tree \ \sigma \ (StateL \ [v] \ l) \ (StateL \ [v] \ l \ a)
                                                      = Leaf (StateL (vs, x))
hEager\ vs\ (Leaf\ x)
hEager\ vs\ (Node\ (Inl'\ Thunk) \ l\ st\ k) = \mathbf{do}
   StateL(vs', lv) \leftarrow hEager\ vs\ (st\ One\ l)
   un Tree With $ fwd $
     (\lambda v \rightarrow Tree With \$ hEager (vs' + [v]) (k (length vs' < \$ l))) < \$ > lv
hEager\ vs\ (Node\ (Inl'\ (Force\ p))\ l\ \_\ k) = \mathbf{do}
   hEager\ vs\ (k\ ((vs\ !!\ p) < \$\ l))
hEager\ vs\ (Node\ (Inr'\ op)
                                             l \ st \ k) =
   Node op (StateL (vs, l)) (\lambda c_2 (StateL (vs', l')) \rightarrow hEager vs' (st c_2 l'))
                                   (\lambda \quad (StateL (vs', lv')) \rightarrow hEager vs' (k lv'))
```

```
thunk :: \forall v \ \sigma. (Thunking v < \sigma) \Rightarrow Tree \sigma Id v \to Tree \sigma Id Ptr thunk t = oneSubNode Thunk t force :: (Thunking v < \sigma) \Rightarrow Ptr \to Tree \sigma Id v force p = noSubNode (Force p)
```

Function Abstraction The *Abstracting* effect has three operators: var for representing a variable, abs for function abstraction, and app for function application.

```
data Abstracting v :: * \to (* \to *) \to *\mathbf{where}
Var :: Int \to Abstracting v v NoSub
App :: v \to v \to Abstracting v v NoSub
Abs :: Abstracting v v (OneSub v)
data Closure \ r \ \mathbf{where}
Clos :: Ptr \to Env \ r \to Closure \ r
\mathbf{type} \ Env \ v = [v]
\mathbf{type} \ Ptr = Int
```

For this effect, we can define two handlers: one for call site handling of effects an another for definition site handling of effects

```
type Store_{CS} \sigma l v
                                 = [R_{CS} \sigma l v]
                              = l() \rightarrow Tree (Abstracting v + \sigma) l(lv)
                 \sigma l v
type R_{CS}
type Store_{DS} \sigma l v = [R_{DS} \sigma l v]
type R_{DS}
                 \sigma l v = Tree (Abstracting v + \sigma) l (l v)
hAbs_{CS} :: (Closure \ v <: v, Functor \ l)
            \Rightarrow [v]
            \rightarrow Store_{CS} \sigma l v
            \rightarrow Tree (Abstracting v + \sigma) l a
            \rightarrow Tree \ \sigma \ (StateL \ (Store_{CS} \ \sigma \ l \ v) \ l) \ (Store_{CS} \ \sigma \ l \ v, a)
hAbs_{CS} - r (Leaf x) = Leaf (r, x)
hAbs_{CS} \ nv \ r \ (Node \ (Inl' \ Abs) \ l \ st \ k) = \mathbf{do}
   let v = inj_v (Clos (length r) nv)
   let r' = r + [st \ One]
                    nv r' (k (v < l))
   hAbs_{CS}
hAbs_{CS} \ nv \ r \ (Node \ (Inl' \ (App \ v_1 \ v_2)) \ l \ \_k) = \mathbf{case} \ proj_{\mathbf{v}} \ v_1 \ \mathbf{of}
      Just (Clos fp nv') \rightarrow do
          (r', v) \leftarrow hAbs_{CS}(v_2 : nv') r((r !! fp) l)
          hAbs_{CS} \ nv \ r' \ (k \ v)

ightarrow error "application error"
hAbs_{CS} \ nv \ r \ (Node \ (Inl' \ (Var \ n)) \ l \ \underline{\hspace{0.1cm}} \ k) = hAbs_{CS} \ nv \ r \ (k \ ((nv \ !! \ n) < \ l))
hAbs_{CS} \ nv \ r \ (Node \ (Inr' \ op) \ l \ st \ k) = Node \ op \ (StateL \ (r, l))
   (\lambda c \ (StateL \ (r', l))) \rightarrow StateL < \$ > hAbs_{CS} \ nv \ r' \ (st \ c \ l))
```

```
(\lambda \ (StateL\ (r', lv\ )) \rightarrow hAbs_{CS}\ nv\ r'\ (k\ lv))
hAbs_{DS} :: (Closure \ v <: v, Functor \ l)
    \Rightarrow [v]
    \rightarrow Store_{DS} \sigma l v
    \rightarrow Tree (Abstracting v + \sigma) l a
    \rightarrow Tree \sigma (StateL (Store<sub>DS</sub> \sigma l v) l)
                   (Store_{DS} \sigma l v, a)
hAbs_{DS} _ r (Leaf x) = Leaf (r, x)
hAbs_{DS} nv \ r \ (Node \ (Inl' \ Abs) \ l \ st \ k) = \mathbf{do}
   let v = inj_{v} (Clos (length r) nv)
   let r' = r + [st \ One \ l]
   hAbs_{DS}
                   nv r' (k (v < l))
hAbs_{DS} \ nv \ r \ (Node \ (Inl' \ (App \ v_1 \ v_2)) \ \_ \ \_ \ k) =
   case proj_{v} v_1 of
      Just (Clos fp nv') \rightarrow do
          (r', v) \leftarrow hAbs_{DS}(v_2 : nv') \ r \ (r !! fp)
          hAbs_{DS} \ nv \ r' \ (k \ v)
      Nothing \rightarrow error "application error"
hAbs_{DS} \ nv \ r \ (Node \ (Inl' \ (Var \ n)) \ l \ \_k) =
   hAbs_{DS} nv r (k (nv !! n < l))
hAbs_{DS} \ nv \ r \ (Node \ (Inr' \ op) \ l \ st \ k) =
   Node op (StateL(r, l))
         (\lambda c (StateL(r', l)) \rightarrow
             StateL \ll hAbs_{DS} nv r' (st c l)
         (\lambda(StateL\ (r', lv)) \rightarrow hAbs_{DS}\ nv\ r'\ (k\ lv))
              :: (Abstracting \ v < \sigma) \Rightarrow Int \rightarrow Tree \ \sigma \ Id \ v
var
var n
              = noSubNode (Var n)
              :: (Abstracting \ v < \sigma) \Rightarrow v \rightarrow v \rightarrow Tree \ \sigma \ Id \ v
app
app \ v_1 \ v_2 = noSubNode \ (App \ v_1 \ v_2)
              :: (Abstracting \ v < \sigma) \Rightarrow \mathit{Tree} \ \sigma \ \mathit{Id} \ v \rightarrow \mathit{Tree} \ \sigma \ \mathit{Id} \ v
abs
abs t
              = oneSubNode \ Abs \ t
```

A.3 Language Features

This section introduces language features that can be used as individual components to modularly construct languages. We provide a syntax for these features, as well as a mapping to latent effects, by means of an algebra.

Arithmetics We make a simple arithmetic language with numbers and addition.

```
data Arith n = Num \ Int \mid Add \ n \ n
 arithAlg :: (Adding \ v < \sigma) \Rightarrow Arith \ v \rightarrow Tree \ \sigma \ Id \ v
```

```
arithAlg (Num \ n) = nat (toInteger \ n)
arithAlg (Add \ x \ y) = plus \ x \ y
```

Lambda calculus The lambda calculus has constructs for variables, abstraction and application.

```
data \ LamExpr \ e = VarExpr \ Int \mid AbsExpr \ e \mid AppExpr \ e \ e
```

We can define different algebras, depending on the evaluation strategy:

```
\begin{array}{ll} lamAlg::(Abstracting \ v<\sigma)\Rightarrow LamExpr\ (Tree\ \sigma\ Id\ v)\rightarrow Tree\ \sigma\ Id\ v\\ lamAlg\ (VarExpr\ n) &= var\ n\\ lamAlg\ (AbsExpr\ e) &= abs\ e\\ lamAlg\ (AppExpr\ e_1\ e_2) = \mathbf{do}\\ v_1\leftarrow e_1\\ v_2\leftarrow e_2\\ app\ v_1\ v_2 \end{array}
```

Call-by-need evaluation can be expressed through the following algebra, where abs, app and var are defined in Section B.2.

```
\begin{aligned} &lazyAlg :: (Reading \ [v\ ] < \sigma, Thunking \ v < \sigma, Suspending \ v < \sigma, \\ & \quad Thunked \ [v\ ] <: v, Closure \ v <: v) \\ &\Rightarrow LamExpr \ (Tree \ \sigma \ Id \ v) \rightarrow Tree \ \sigma \ Id \ v \\ &lazyAlg \ (VarExpr \ n) &= var_{lazy} \ n \\ &lazyAlg \ (AbsExpr \ e) &= abs_{lazy} \ e \\ &lazyAlg \ (AppExpr \ e_1 \ e_2) = app_{lazy} \ e_1 \ e_2 \end{aligned}
```

Call-by-name evaluation is similar, but without memoizing (intermediate) results.

```
\begin{array}{l} cbnAlg:: (Reading \ [v] < \sigma, Suspending \ v < \sigma, \\ Suspended \ [v] <: v, Closure \ v <: v) \\ \Rightarrow LamExpr \ (Tree \ \sigma \ Id \ v) \rightarrow Tree \ \sigma \ Id \ v \\ cbnAlg \ (VarExpr \ n) = var_{\text{CBN}} \ n \\ cbnAlg \ (AbsExpr \ e) = abs_{\text{CBN}} \ e \\ cbnAlg \ (AppExpr \ e_1 \ e_2) = app_{\text{CBN}} \ e_1 \ e_2 \end{array}
```

Staging For the staging expressions, we refer to Section C for the implementations of *splice*, *quote*, *shift* and *unquote*.

```
data StageExpr\ s = Splice\ s \mid Quote\ s \mid Unquote\ s \mid Shift\ Int\ s stageAlg:: \forall v\ \sigma\ r_1\ r_2. (Suspending v < \sigma, Reading r_1 < \sigma, Reading r_2 < \sigma, Reading (Env_s\ v) < \sigma, Suspended r_1 <: v,
```

```
Suspended r_2 <: v, Suspended (Env_s \ v) <: v)

\Rightarrow StageExpr \ (Tree \ \sigma \ Id \ v) \rightarrow Tree \ \sigma \ Id \ v

stageAlg \ (Splice \ s) = splice \ s

stageAlg \ (Quote \ s) = quote@r_1 \ s

stageAlg \ (Unquote \ s) = unquote@r_2 \ s

stageAlg \ (Shift \ n \ s) = shift \ n \ s
```

Other We can define other simple language features such as printing and let-expressions.

```
\begin{array}{l} \mathbf{data} \; PrintExpr \; e = Pr \; String \\ printAlg :: (Printing \; v < \sigma) \Rightarrow PrintExpr \; v \rightarrow Tree \; \sigma \; Id \; v \\ printAlg \; (Pr \; s) = print \; s \\ \\ \mathbf{data} \; LetExpr \; e = Let \; e \; e \; | \; Seq \; e \; e \; | \; LetVar \; Int \\ letAlg :: (Abstracting \; v < \sigma) \Rightarrow LetExpr \; (Tree \; \sigma \; Id \; v) \rightarrow Tree \; \sigma \; Id \; v \\ letAlg \; (Let \; e_1 \; e_2) = \mathbf{do} \\ v_1 \leftarrow abs \; e_1 \\ v_2 \leftarrow e_2 \\ app \; v_1 \; v_2 \\ letAlg \; (Seq \; e_1 \; e_2) = \mathbf{do} \; e_1; e_2 \\ letAlg \; (LetVar \; n) = var \; n \end{array}
```

B Modular Carriers for Call-by-Need Lambdas

This Appendix shows how we can use an adapted form of the modular carriers of Schrijvers et al. [20] to define call-by-need and call-by-value evaluation for function abstractions. We give semantics to call-by-need lambdas using three different effects: *Reading*, *Suspending* and *Thunking*. For the definition and the handlers of these effects, we refer to Appendix A.

B.1 Modular Carriers

The *hThunk* handler uses a technique that lets us access values wrapped in a latent effect functor: we use a *forwarding algebra*, inspired by the work of Schrijvers et al. [20]. They propose a modular approach to composing algebraic effects so that each handler only has to know about the part of the syntax its effect is handling. This is achieved through the definition of a modular carrier *Modular* that features a forwarding algebra *fwd*.

```
class Modular\ (m_1 :: * \to *)\ (m_2 :: (* \to *) \to * \to *) where fwd :: m_1\ (m_2\ m_1\ a) \to m_2\ m_1\ a
```

However, we use this modular carrier in a different context in the sense that it is now modular with respect to its subcomputations, for which the syntax and structure is known. On the contrary, the original definition is modular with respect to the outer computation.

B.2 Call-by-Need Evaluation

The three operations for call-by-need evaluation are defined below.

```
abs_{lazy} :: \forall v \ \sigma. (Reading \ [v] < \sigma, Suspending \ v < \sigma, Closure \ v <: v)
    \Rightarrow Tree \sigma Id v \rightarrow Tree \sigma Id v
abs_{lazy}\ body = \mathbf{do}
   nv \leftarrow \mathit{ask}
   p \leftarrow suspend\ body
   return (inj_{v} (Clos \ p \ (nv :: [v])))
var_{lazy} :: \forall v \ \sigma. (Reading \ [v] < \sigma, Thunking \ v < \sigma, Thunked \ [v] <: v)
    \Rightarrow Int \rightarrow Tree \sigma Id v
var_{lazy} x = \mathbf{do}
   nv \leftarrow ask
   let v = (nv :: [v]) !! x
   case proj<sub>v</sub> v of
      Just (Thunked p \ nv') \rightarrow local (const (nv' :: [v])) (force p)
      Nothing
                                        \rightarrow return v
app_{lazy} :: \forall v \ \sigma. (Suspending \ v < \sigma, Thunking \ v < \sigma, Reading \ [v] < \sigma,
              Closure v <: v, Thunked [v] <: v)
    \Rightarrow Tree \sigma Id v \rightarrow Tree \sigma Id v \rightarrow Tree \sigma Id v
app_{lazy} e_1 e_2 = \mathbf{do}
   vf \leftarrow e_1; nv \leftarrow ask; p \leftarrow thunk \ e_2
   let th :: v = inj_v $ Thunked p(nv :: [v])
  case proj<sub>v</sub> vf of
      Just (Clos p' nv') \rightarrow do
          local\ (const\ (th:(nv'::[v])))\ (enact\ p')
```

B.3 Call-by-Name

The three operations for call-by-name evaluation are similar to those for call-by-need, but without memoizing the result.

```
abs_{cbn} :: \forall v \ \sigma.(Reading \ [v] < \sigma, Suspending \ v < \sigma, Closure \ v <: v)

\Rightarrow Tree \ \sigma \ Id \ v \rightarrow Tree \ \sigma \ Id \ v

abs_{cbn} \ body = \mathbf{do}

nv \leftarrow ask
```

```
p \leftarrow suspend\ body
   return (inj_{v} (Clos \ p \ (nv :: [v])))
var_{cbn} :: \forall v \ \sigma. (Reading \ [v] < \sigma, Suspending \ v < \sigma, Suspended \ [v] <: v)
    \Rightarrow Int \rightarrow Tree \sigma Id v
var_{cbn} x = \mathbf{do}
   nv \leftarrow \mathit{ask}
  let v = (nv :: [v]) !! x
   case proj<sub>v</sub> v of
      Just (Suspended p nv') \rightarrow local (const (nv' :: [v])) (enact p)
      Nothing
                                          \rightarrow return v
app_{cbn} :: \forall v \ \sigma. (Suspending \ v < \sigma, Reading \ [v] < \sigma,
                     Closure v <: v, Suspended[v] <: v)
    \Rightarrow Tree \sigma Id v \rightarrow Tree \sigma Id v \rightarrow Tree \sigma Id v
app_{cbn} e_1 e_2 = \mathbf{do}
   vf \leftarrow e_1; nv \leftarrow ask; p \leftarrow suspend \ e_2
  let th :: v = inj_v $ Suspended p (nv :: [v])
   case proj<sub>v</sub> vf of
      Just (Clos \ p' \ nv') \rightarrow \mathbf{do}
          local (const (th : (nv' :: [v]))) (enact p')
```

C A Latent Effect Semantics of Staging

In this appendix we show how to model three staging operations, inspired by MetaML [26]: quote for staging a piece of code; unquote for unstaging; and push for defining code with one or more variables whose bindings only become known when the code is unquoted. First, we define a small library of latent effects to work with.

C.1 Reading, Suspending, and Revisiting Lambdas

Appendix A defines the two necessary effect interfaces: *Reading* and *Suspending*. Furthermore, the following *Bindable* type class has operations for binding a value in an environment, and looking up a value in an environment.

```
class Bindable\ r\ v\ where bindVal\ :: v \to r \to r lookupVal :: r \to Int \to v
```

For plain, de Bruijn indexed environments, these functions are straightforwardly defined:

```
instance Bindable [v] v where bindVal = (:) lookupVal = (!!)
```

The challenge with code splicing is to allow spliced code to reference variables from the context that code is being spliced with. We address this challenge by using a refined type of environment and *Bindable*.

C.2 Quote and Unquote

The quote operation is defined in Suspending and Reading. It thunks a subcomputation, remembering the environment that the code is supposed to be evaluated under, reusing the Suspended value from Section ?? as the representation of a quoted piece of code:

```
\begin{array}{l} \textit{quote} :: \forall r \ \textit{v} \ \textit{l} \ \textit{\sigma}.(\textit{Reading} \ r < \sigma, \textit{Suspending} \ \textit{v} < \sigma, \textit{Suspended} \ r <: \textit{v}) \\ \Rightarrow \textit{Tree} \ \sigma \ \textit{Id} \ \textit{v} \rightarrow \textit{Tree} \ \sigma \ \textit{Id} \ \textit{v} \\ \textit{quote} \ m = \mathbf{do} \\ p \ \leftarrow \textit{suspend} \ m \\ \textit{nv} \leftarrow \textit{ask} \\ \textit{return} \ (\textit{inj}_{_{\textbf{v}}} \$ \ \textit{Suspended} \ p \ (\textit{nv} :: r)) \end{array}
```

The unquote operation forces a quoted (thunked) computation.

```
\begin{array}{ll} unquote :: \forall r \ v \ l \ \sigma. (Reading \ r < \sigma, Suspending \ v < \sigma, Suspended \ r <: v) \\ &\Rightarrow Tree \ \sigma \ Id \ v \rightarrow Tree \ \sigma \ Id \ v \\ unquote \ m = \mathbf{do} \\ v \leftarrow m \\ \mathbf{case} \ proj_{\mathbf{v}} \ v \ \mathbf{of} \\ Just \ (Suspended \ p \ nv) \rightarrow local \ (const \ (nv :: r)) \ (enact \ p) \\ Nothing \qquad \rightarrow error \ "bad \ unquote" \end{array}
```

C.3 Pushing and Splicing

We will introduce an operation for defining code with variables whose bindings are not yet known. This operation relies on the following type of *staged environment*:

```
newtype Env_s \ v = Env_s \ \{unEnv_s :: [Maybe \ (Either \ v \ Int)]\}
```

The Maybe type is used to distinguish positions in the environment whose bindings are not yet known. An unknown binding is represented as Nothing. A known binding is Either an actual value v, or an integer representing a forward reference to a value that occurs n positions further down the environment. We use forward references because our staged environments behave in a way that is reminiscent of a telescope [2]: for a bound value v occurring in a staged environment such as Env_s (Just (Left v): nv), the value v is closed under the substitutions in the

environment nv, since nv contains "contextual" bindings that potential unknown bindings in v must be instantiated with.

The implementation of the *Bindable* interface below illustrates how this "closing" of values of type *Val* works:

```
data Val = CloV (Closure (Env_s Val))
          \mid CodV (Suspended (Env_s Val))
          | IntV Int
instance Bindable (Env<sub>s</sub> Val) Val where
                        (Env_s \ nv) = Env_s \$ Just
  bindVal v
            (Right (length nv)) : nv ++ [Just (Left v)]
  lookup Val (Env_s nv) x = go nv x
    where qo (Nothing
                              : nv) \ 0 = error "quote error"
            qo(Just(Right y): nv) 0 = lookup Val(Env_s nv) y
            go (Just (Left v) : nv) 0 = cover v (Env_s nv)
            go (_
                               (x - inv) x \mid x \ge 0 = lookup Val (Env_s nv) (x - 1)
                                     x = error "bad index"
            go nv
```

The bindVal function prepends an environment by a forward reference to a value that is appended to the back of the environment. The lookupVal function follows forward references, and when it finds a value, it uses the cover function to "close" a value over the bindings in the tail of the environment (i.e., the bindings from the context of the bound value):

```
\begin{array}{ll} cover:: Val \rightarrow Env_s \ Val \rightarrow Val \\ cover \ (CloV \ (Clos \ p \ nv')) & nv = CloV \ (Clos \ p \ (combine \ nv' \ nv)) \\ cover \ (CodV \ (Suspended \ p \ nv')) \ nv = CodV \ (Suspended \ p \ (combine \ nv' \ nv)) \\ cover \ (IntV \ i) & nv = IntV \ i \end{array}
```

The *combine* function used by this *cover* function instantiates unknown bindings by bindings coming from the context:

The *push* operation introduces unknown bindings in the context of a subcomputation.

Finally, the *splice* operation is like *unquote*, but it *combines* the environment in a *Suspended* value with the contextual environment of the *splice* operation, thereby instantiating variables whose bindings were unknown when the thunk was created.

```
splice :: \forall r \ v \ l \ \sigma. (Reading \ (Env_s \ v) < \sigma, Suspending \ v < \sigma, \\ Suspended \ (Env_s \ v) <: v) \\ \Rightarrow Tree \ \sigma \ Id \ v \rightarrow Tree \ \sigma \ Id \ v \\ splice \ m = \mathbf{do} \\ v \leftarrow m \\ \mathbf{case} \ proj_v \ v \ \mathbf{of} \\ Just \ (Suspended \ p \ nv) \rightarrow local \ (combine \ (nv :: Env_s \ v)) \ (enact \ p) \\ Nothing \qquad \rightarrow error \ "bad \ unquote"
```

C.4 The Difference between splice and push

While it is straightforward to define latent effect handlers that delay the evaluation of subtrees (which is exactly what Suspending and Thunking do), it is problematic to define handlers that "hoist" code that occurs in the computational subtree or the continuation of a tree. The crux of the problem is that subtrees and continuations require a value and a latent effect context before we can inspect their Trees. Escape expressions in MetaML represent code that must be run before returning a code value, and if escape expressions were to occur in computational subtrees of a quote operation, we would have to hoist the code out of the subtree. Our push and splice operations work around this problem. They allow us to write splicing code so that the effectful operations that compute code to be spliced occur in the "right" place in the latent effect tree.

The program in Figre 6 justifies the need for the two different constructs to mimic escape operations.

The program in Figure 4 of Section 4.1 only uses the *splice* construct. In order to write programs that behave similarly to MetaML programs with escape expressions that occur under a binder inside staged code, we also need *push*. In Figure 6, the MetaML program on the left (due to Taha [24]) contains two escape expressions that both occur under function binders inside staged code. The result of running the program is a value that is alpha-equivalent to the code value $\{\mathbf{fn} \ \mathbf{x1} = \mathbf{x2} = \mathbf{x2} + \mathbf{x1}\}$. The program on the right in Figure 6 uses *push* to indicate how many quoted binders a given computation is scoped by.

⁶ We have also added variable names in {-comments -} in Figure 6 to increase the readability of the de Bruijn indexed program.

```
letbind {-T -} (abs' {-e -} (letbind {-d -} })

1 let T = fn e => (push {-x -} 1 (var {-e -} 1))

2 <fn x => x + ~e > (add (var {-x -} 2 0) (splice (var {-d -} 1)))))))

3 in <fn x => ~(T <x>) > (letbind {-d' -} (push {-x -} 1 (app (var {-T -} 1) (quote (var {-x -} 0)))))

(quote (abs' {-x -} (splice (var {-d' -} 1)))))
```

Fig. 6: A MetaML program and its counterpart with quote, unquote, push, and splice.

The push on line 3 on the right in Figure 6 corresponds to the escape expression in line 2 on the left. Likewise, the push on line 6 on the right corresponds to the escape expression in line 3 on the left. The dynamic semantics of push creates an environment with "holes" that represent unknown bindings. The dynamic semantics of splice automatically fills in the holes of environments in pushed code values, with bindings from the dynamic context of the splice expression. This pushing and filling of holes makes the splicing in the program on the right behave similar to the (lexically-scoped) escape expressions in the program on the left. This is to yield a code value similar to the code value returned by the MetaML program.