1 MATHEMATICS

MATHEMATICS–1

The set A consists of elements $\{1, 3, 6\}$, and the set B consists of elements $\{1, 2, 6, 7\}$. Both sets come from the universe of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. What is the intersection, $\bar{A} \cap B$?

(A) $\{2, 7\}$ (B) $\{2, 3, 7\}$ (C) $\{2, 4, 5, 7, 8\}$ (D) $\{4, 5, 8\}$

The set of "not A" consists of all universe elements not in set A: $\{2, 4, 5, 7, 8\}$.

The intersection of $\{2, 4, 5, 7, 8\}$ and $\{1, 2, 6, 7\}$ is the set of all elements appearing in both.

Thus $\bar{A} \cap B$ is $\{2, 7\}$.

The answer is (A).

MATHEMATICS–2

For a given function, $f(t) = f(-t)$. What type of symmetry does $f(t)$ have?

(A) odd symmetry
(B) even symmetry
(C) rotational symmetry
(D) quarter-wave symmetry

When $f(t) = f(-t)$, the function is "mirrored" on either side of the vertical axis. This is known as even symmetry.

The answer is (B).

MATHEMATICS–3

What is the value of each interior angle of a regular pentagon?

(A) $\frac{\pi}{5}$ (B) $\frac{2\pi}{5}$ (C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{5}$

For a regular polygon, the value of each interior angle, θ, is

$$\theta = \frac{\pi(\text{number of sides} - 2)}{\text{number of sides}}$$

For a regular pentagon,

$$\begin{aligned}\theta &= \frac{\pi(5-2)}{5} \\ &= \frac{3\pi}{5}\end{aligned}$$

The answer is (D).

MATHEMATICS–4

A cubical container that measures 2 m on a side is tightly packed with eight balls and is filled with water. All eight balls are in contact with the walls of the container and the adjacent balls. All of the balls are the same size. What is the volume of water in the container?

(A) 0.38 m^3 (B) 2.5 m^3 (C) 3.8 m^3 (D) 4.2 m^3

Since the balls are tightly packed, $r_{ball} = 0.5$ m.

$$\begin{aligned}V_{water} &= V_{box} - 8V_{ball} \\ &= (2\text{ m})^3 - (8)\left(\tfrac{4}{3}\pi(0.5\text{ m})^3\right) \\ &= 3.8\text{ m}^3\end{aligned}$$

The answer is (C).

MATHEMATICS–5

Which number has four significant figures?

(A) 0.0014 (B) 0.01414 (C) 0.141 (D) 1.4140

The number of significant figures, or digits, for each choice is (A) 2, (B) 4, (C) 3, and (D) 5.

Only option (B) has four significant figures.

The answer is (B).

MATHEMATICS–6

What is the solution of the equation $50x^2 + 5(x - 2)^2 = -1$, where x is a real-valued variable?

(A) -6.12 and -3.88 (B) -0.52 and 0.700
(C) 7.55 (D) no solution

For real-valued x, the left-hand side of the equation must always be greater than or equal to zero, since all terms containing x are squared. There is no solution to this equation for real values of x.

The answer is (D).

MATHEMATICS–7

What are the roots of the cubic equation $x^3 - 8x - 3 = 0$?

(A) $x = -7.90, -3, -0.38$
(B) $x = -3, -2, 2$
(C) $x = -3, -0.38, 2$
(D) $x = -2.62, -0.38, 3$

By inspection, $+3$ is a root, and $(x-3)$ is a factor. Factor out $(x-3)$.

$$\frac{x^3 - 8x - 3}{x - 3} = x^2 + 3x + 1$$

Use the quadratic equation to solve $x^2 + 3x + 1 = 0$.

$$x = 3, \frac{-3 \pm \sqrt{9-4}}{2}$$
$$= -2.62,\ -0.38,\ 3$$

The answer is (D).

MATHEMATICS–8

Naperian logarithms have a base closest to which number?

(A) 2.17 (B) 2.72 (C) 3.14 (D) 10.0

The base of Naperian logarithms is the number $e \approx 2.7183$. Of the choices given, 2.72 is the closest to e.

The answer is (B).

MATHEMATICS–9

What is the radius of the circle defined by $x^2 + y^2 - 4x + 8y = 7$?

(A) $\sqrt{3}$ (B) $2\sqrt{5}$ (C) $3\sqrt{3}$ (D) $4\sqrt{3}$

Since the general equation for a circle is $(x - a)^2 + (y - b)^2 = r^2$, rearrange the equation given to fit the general equation.

$x^2 - 4x + y^2 + 8y = 7$

Complete the binomial forms.

$x^2 - 4x + 4 + y^2 + 8y + 16 = 7 + 4 + 16$

$(x - 2)^2 + (y + 4)^2 = 27$

$r^2 = 27$

$r = 3\sqrt{3}$

The answer is (C).

MATHEMATICS–10

What is the natural logarithm of e^{xy}?

(A) $\frac{1}{xy}$ (B) xy (C) $2.718xy$ (D) $\frac{2.718}{xy}$

By definition, the natural logarithm of a number is

$$\ln e^G = G$$
$$\ln e^{xy} = xy$$

The answer is (B).

MATHEMATICS–11

What is the value of $(0.001)^{2/3}$?

(A) antilog $\left(\frac{3}{2}\log 0.001\right)$

(B) $\frac{2}{3}$ antilog $(\log 0.001)$

(C) antilog $\left(\log \frac{0.001}{\frac{2}{3}}\right)$

(D) antilog $\left(\frac{2}{3}\log 0.001\right)$

$$\log x^a = a \log x$$
$$\log(0.001)^{2/3} = \tfrac{2}{3}\log 0.001$$
$$(0.001)^{2/3} = \text{antilog}\left(\tfrac{2}{3}\log 0.001\right)$$

The answer is (D).

MATHEMATICS–12

The salary of an employee's job has five levels, each one 5% greater than the one below it. Due to circumstances, the salary of the employee must be reduced from the top (fifth) level to the second level, which results in a reduction of $122.00 per month. What is the employee's present salary per month?

(A) $440/mo (B) $570/mo (C) $680/mo (D) $900/mo

The salary levels represent a geometric sequence. Let S_i be the salary at level i.

$$\begin{aligned} S_3 &= 1.05S_2 \\ S_4 &= 1.05S_3 \\ S_5 &= 1.05S_4 \\ &= (1.05)^3 S_2 \\ S_5 - 122 &= S_2 \\ S_5 &= (1.05)^3 \left(S_5 - 122\ \frac{\$}{\text{mo}}\right) \\ &= \$896/\text{mo} \quad (\$900/\text{mo}) \end{aligned}$$

The answer is (D).

MATHEMATICS–13

Which of the following statements regarding matrices is FALSE?

(A) $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$

(B) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

(C) $\begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$

(D) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

The inverse of a product of two matrices is the product of the inverses, in reverse order.

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

The answer is (D).

MATHEMATICS–14

What is the determinant of the following 2 × 2 matrix?

$$\begin{pmatrix} 5 & 9 \\ 7 & 6 \end{pmatrix}$$

(A) −33 (B) −27 (C) 27 (D) 33

The determinant, **D**, is calculated as follows.

$$\begin{aligned}\mathbf{D} &= \begin{vmatrix} 5 & 9 \\ 7 & 6 \end{vmatrix} \\ &= (5)(6) - (7)(9) \\ &= -33\end{aligned}$$

The answer is (A).

MATHEMATICS–15

What is the determinant of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

(A) 0 (B) 1 (C) 5 (D) 7

To find the determinant, expand by minors across the top row.

$$\begin{aligned}\mathbf{D} &= 1\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \\ &= \big((-1)(-1) - (2)(1)\big) - \big((2)(-1) - (1)(1)\big) + \big((2)(2) - (1)(-1)\big) \\ &= 7\end{aligned}$$

The answer is (D).

MATHEMATICS–16

What is the inverse of the matrix **A**?

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(A) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(B) $\begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(C) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

(D) $\begin{pmatrix} \cos\theta\sin\theta & 0 \\ 0 & \sin\theta\cos\theta \end{pmatrix}$

For a 2×2 matrix, $\mathbf{X}$,

$$\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The inverse, $\mathbf{X}^{-1}$, is

$$\mathbf{X}^{-1} = \frac{1}{\mathbf{D}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\mathbf{D}$ is the determinant of $\mathbf{X}$. For matrix $\mathbf{A}$,

$$\begin{aligned} \mathbf{D} &= \cos^2\theta - (\sin\theta)(-\sin\theta) \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

The answer is (C).

MATHEMATICS–17

What is the rank of the matrix $\mathbf{A}$?

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 2 & 0 & 1 & -2 \end{pmatrix}$$

(A) 0 (B) 1 (C) 2 (D) 3

The rank of a matrix is the number of independent vectors (rows). The rank can be found by row-reducing (diagonalizing) the matrix and counting the number of pivots in the row-reduced form of the matrix.

$$\text{Row } 2 = (-2)(\text{Row } 2) + (3)(\text{Row } 4)$$
$$\text{Row } 4 = \text{Row } 4 - \text{Row } 2$$
$$\text{Row } 3 = (2)(\text{Row } 3) + \text{Row } 2$$

The row-reduced form of **A** is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix cannot be further row-reduced. There are three pivots and, therefore, three independent rows. The rank of matrix **A** is 3.

The answer is (D).

MATHEMATICS–18

Determine the values of x_1 and x_2 that satisfy the following linear system.

$$\begin{pmatrix} 3 & 7 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

(A) $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ (B) $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ (C) $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ (D) $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$

The linear equations represented by this system are

$$3x_1 + 7x_2 = 2$$
$$2x_1 + 6x_2 = 4$$

Use Cramer's rule to solve the system of equations.

$$x_1 = \frac{\begin{vmatrix} 2 & 7 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 7 \\ 2 & 6 \end{vmatrix}}$$

$$= \frac{-16}{4}$$

$$x_1 = -4$$

$$x_2 = \frac{\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 7 \\ 2 & 6 \end{vmatrix}}$$

$$= \frac{8}{4}$$

$$x_2 = 2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

The answer is (D).

MATHEMATICS–19

If $\sin\alpha = x$, what is $\sec\alpha$?

(A) $\sqrt{1-x^2}$ (B) $\dfrac{x}{\sqrt{1-x^2}}$ (C) $\dfrac{1}{\sqrt{1-x^2}}$ (D) $\dfrac{x}{\sqrt{1+x^2}}$

Since $\sin\alpha$ is the side facing angle α divided by the hypotenuse, the hypotenuse = 1. Therefore,

$$\text{side adjacent to angle } \alpha = \sqrt{1-x^2}$$

$$\sec\alpha = \frac{\text{hypotenuse}}{\text{side adjacent to angle } \alpha} = \frac{1}{\sqrt{1-x^2}}$$

The answer is (C).

MATHEMATICS–20

Experimental data show that a body's temperature declines exponentially in time according to the expression $T(t) = 50e^{-0.04t}$ (where 50 is a constant expressed in °C, 0.04 is the cooling rate in min^{-1}, and t is the cooling time expressed in minutes). How long would it take the body to reach 25°C?

(A) 12.4 min (B) 15.6 min (C) 16.5 min (D) 17.3 min

$$T(t) = 50e^{-0.04t}$$

$$25°\text{C} = (50°\text{C})e^{-0.04\ \text{min}^{-1}t}$$

$$e^{-0.04\ \text{min}^{-1}t} = \frac{25°\text{C}}{50°\text{C}} = 0.5$$

Take the natural logarithm of both sides.

$$-0.04\ \text{min}^{-1}t = \ln 0.5$$

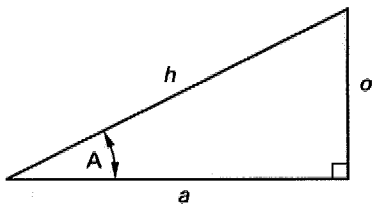
$$t = \frac{\ln 0.5}{-0.04\ \text{min}^{-1}}$$

$$= 17.3\ \text{min}$$

The answer is (D).

MATHEMATICS–21

If the sine of angle A is given as K, what is the tangent of angle A?



(A) $\frac{hK}{o}$ (B) $\frac{aK}{h}$ (C) $\frac{ha}{K}$ (D) $\frac{hK}{a}$

$$\begin{aligned}\sin A &= \frac{o}{h} \\ &= K \\ \tan A &= \frac{o}{a} \\ &= \left(\frac{h}{a}\right)\left(\frac{o}{h}\right) \\ &= \frac{hK}{a}\end{aligned}$$

The answer is (D).

MATHEMATICS–22

Which is true regarding the signs of the natural functions for angles between 90° and 180°?

(A) The tangent is positive.
(B) The cotangent is positive.
(C) The cosine is negative.
(D) The sine is negative.

In the second quadrant, the natural functions and their signs are as follows.

sin	positive
cos	negative
tan	negative
cot	negative
sec	negative
csc	positive

The answer is (C).

MATHEMATICS–23

What is the inverse natural function of the cosecant?

(A) secant (B) sine (C) cosine (D) tangent

In a right triangle, the cosecant is the hypotenuse divided by the opposite side. The sine is the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{1}{\csc\theta}$$

The answer is (B).

MATHEMATICS–24

What is the sum of the squares of the sine and cosine of an angle?

(A) 0 (B) 1 (C) $\sqrt{3}$ (D) 2

For any angle,

$$\cos^2 x + \sin^2 x = 1$$

The answer is (B).

MATHEMATICS–25

What is an equivalent expression for $\sin 2x$?

(A) $\frac{1}{2}\sin x \cos x$ (B) $2\sin x \cos \frac{1}{2}x$ (C) $-2\sin x \cos x$ (D) $\dfrac{2\sin x}{\sec x}$

The double angle formula for the sine function is

$$\begin{aligned}\sin 2x &= 2\sin x \cos x \\ &= \frac{2\sin x}{\sec x}\end{aligned}$$

The answer is (D).

MATHEMATICS–26

The Taylor series expansion for $\cos x$ contains which powers of x?

(A) 0, 2, 4, 6, 8, ...
(B) 1, 3, 5, 9, ...
(C) 1, 2, 3, 4, 5, ...
(D) 1/2, 3/2, 5/2, 7/2, ...

The Taylor series expansion for $\cos x$ is as follows.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Only the positive even powers of x are contained in the expansion of $\cos x$.

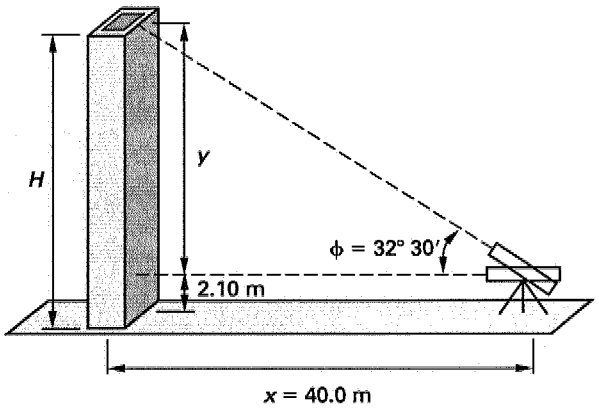
The answer is (A).

MATHEMATICS–27

A transit set up 40 m from the base of a vertical chimney reads 32°30′ with the crosshairs set on the top of the chimney. With the telescope level, the vertical rod at the base of the chimney is 2.1 m. Approximately how tall is the chimney?

(A) 15 m (B) 26 m (C) 28 m (D) 38 m

To find the height, H, of the chimney, refer to the following figure.



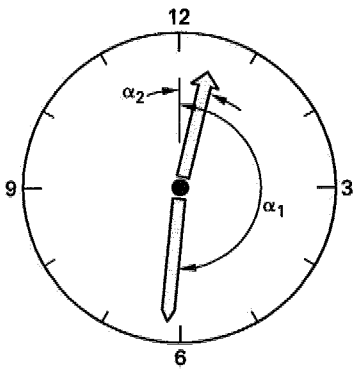
$$\tan\phi = \frac{y}{x}$$
$$y = (40.0\text{ m})\tan 32.5^\circ$$
$$= 25.5\text{ m}$$
$$H = 2.10\text{ m} + y$$
$$= 2.10\text{ m} + 25.5\text{ m}$$
$$= 27.60\text{ m} \quad (28\text{ m})$$

The answer is (C).

MATHEMATICS–28

At approximately what time between the hours of 12:00 noon and 1:00 p.m. would the angle between the hour hand and the minute hand of a continuously driven clock be exactly 180°?

(A) 12:28 p.m. (B) 12:30 p.m. (C) 12:33 p.m. (D) 12:37 p.m.



The change in the angle of the minute hand between 12:00 p.m. and 1:00 p.m., α_1, is

$$\alpha_1 = \frac{360^\circ}{60\text{ min}}t$$
$$= (6t)^\circ$$

The change in the angle of the hour hand between 12:00 noon and 1:00 p.m., α_2, is

$$\alpha_2 = \frac{360^\circ}{(12)(60\text{ min})}t$$
$$= (0.5t)^\circ$$

In the preceding equations, t is in minutes past 12:00 noon. The angle between the two hands is $\alpha_1 - \alpha_2$.

$$\begin{aligned} \alpha_1 - \alpha_2 &= 180^\circ \\ (6t)^\circ - (0.5t)^\circ &= 180^\circ \\ (5.5t)^\circ &= 180^\circ \\ t &= 32.7 \text{ min} \end{aligned}$$

The time is approximately 12:33 p.m.

The answer is (C).

MATHEMATICS–29

In finding the distance, d, between two points, which equation is the appropriate one to use?

(A) $d = \sqrt{(x_1 - x_2)^2 - (y_2 - y_1)^2}$
(B) $d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
(C) $d = \sqrt{(x_1^2 - x_2^2) + (y_1^2 - y_2^2)}$
(D) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The distance formula is defined as follows.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The answer is (D).

MATHEMATICS–30

The equation $y = a_1 + a_2 x$ is an algebraic expression for which of the following?

(A) a cosine expansion series
(B) projectile motion
(C) a circle in polar form
(D) a straight line

$y = mx + b$ is the slope-intercept form of the equation of a straight line. Thus, $y = a_1 + a_2x$ describes a straight line.

The answer is (D).

MATHEMATICS–31

Find the slope of the line defined by $y - x = 5$.

(A) $5 + x$ (B) $-1/2$ (C) $1/4$ (D) 1

The slope-intercept form of the equation of a straight line is $y = mx + b$, where m is the slope and b is the y-intercept.

$$y - x = 5$$
$$y = x + 5$$

The coefficient of x, m, is

$$m = 1$$

The answer is (D).

MATHEMATICS–32

Find the equation of a line with slope = 2 and y-intercept = -3.

(A) $y = -3x + 2$
(B) $y = 2x - 3$
(C) $y = \frac{2}{3}x + 1$
(D) $y = 2x + 3$

The slope-intercept form of the given equation is

$$y = 2x - 3$$

The answer is (B).

MATHEMATICS–33

Find the equation of the line that passes through the points $(0, 0)$ and $(2, -2)$.

(A) $y = x$ (B) $y = -2x + 2$ (C) $y = -2x$ (D) $y = -x$

Since the line passes through the origin, the y-intercept is 0. Thus, the equation simplifies to $y = mx$. Substituting for the known points,

$$y = \left(\frac{-2-0}{2-0}\right) x$$
$$= -x$$

The answer is (D).

MATHEMATICS–34

What is the name for a vector that represents the sum of two vectors?

(A) scalar (B) resultant (C) tensor (D) moment

By definition, the sum of two vectors is known as the resultant.

The answer is (B).

MATHEMATICS–35

What is the resultant, $\mathbf{R}$, of the vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$?

$$\mathbf{F}_1 = 4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$
$$\mathbf{F}_2 = 9\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$$
$$\mathbf{F}_3 = 5\mathbf{i} - 3\mathbf{j} - 8\mathbf{k}$$

(A) $\mathbf{R} = 18\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$
(B) $\mathbf{R} = -18\mathbf{i} - 6\mathbf{j} - 9\mathbf{k}$
(C) $\mathbf{R} = 18\mathbf{i} + 12\mathbf{j} + 25\mathbf{k}$
(D) $\mathbf{R} = 21\mathbf{i}$

The resultant of vectors given in unit-vector form is the sum of the components.

$$\begin{aligned}\mathbf{R} &= (4+9+5)\mathbf{i} + (7+2-3)\mathbf{j} + (6+11-8)\mathbf{k} \\ &= 18\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}\end{aligned}$$

The answer is (A).

MATHEMATICS–36

Simplify the expression $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$, given

$$\begin{aligned}\mathbf{A} &= 3\mathbf{i} + 2\mathbf{j} \\ \mathbf{B} &= 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \\ \mathbf{C} &= 5\mathbf{i} + 2\mathbf{k}\end{aligned}$$

(A) 0 (B) 20 (C) $60\mathbf{i} + 24\mathbf{k}$ (D) $5\mathbf{i} + 2\mathbf{k}$

First find $\mathbf{A} \times \mathbf{B}$.

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (3\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 0 \\ 2 & 3 & 1 \end{vmatrix} \\ &= \mathbf{i}(2-0) - \mathbf{j}(3-0) + \mathbf{k}(9-4) \\ &= 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} \\ (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} &= (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) \cdot (5\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}) \\ &= (2)(5) + (-3)(0) + (5)(2) \\ &= 20\end{aligned}$$

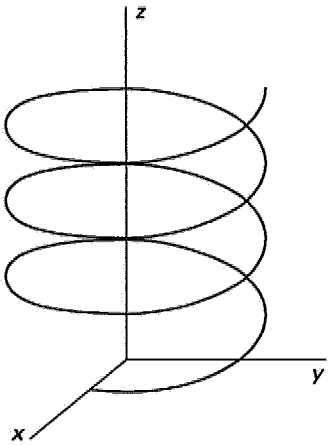
The answer is (B).

MATHEMATICS–37

What type of curve is generated by a point that moves in uniform circular motion about an axis, while travelling with a constant speed, v, parallel to the axis?

(A) a cycloid (B) an epicycloid (C) a hypocycloid (D) a helix

A curve generated by the method described is called a helix and is illustrated in the following figure.



The answer is (D).

MATHEMATICS–38

What is the term that describes a possible outcome of an experiment?

(A) a sample space (B) a random point
(C) an event (D) a finite set

By definition, an event is a possible outcome of a trial or experiment.

The answer is (C).

MATHEMATICS–39

In probability theory, what is the term that describes the set of all possible outcomes of an experiment?

(A) a set of random events
(B) a fuzzy set
(C) a cumulative distribution
(D) a sample space

By definition, the sample space is the set of all possible outcomes of an experiment.

The answer is (D).

MATHEMATICS–40

How can the values of a random variable defined over a sample space be described?

(A) always continuous
(B) always numerical
(C) strictly nonzero
(D) defined only over a finite horizon

The values of a random variable can be continuous or discrete over a finite or infinite domain. The values in the sample space can be shared by other sample spaces. However, the values of a random variable must be numerical.

The answer is (B).

MATHEMATICS–41

If two random variables are independently distributed, what is their relationship?

(A) They are not identically distributed.
(B) They are uncorrelated.
(C) They are mutually exclusive.
(D) Either option (A) or option (B) is true.

By definition, two independently distributed random variables are uncorrelated. Any two random variables may or may not be identically distributed. Independent events cannot be mutually exclusive.

The answer is (B).

MATHEMATICS–42

Which of the following properties of probability is NOT valid?

(A) The probability of an event is always positive and less than or equal to one.

(B) If E_0 is an event which cannot occur in the sample space, the probability of E_0 is zero.

(C) If events E_1 and E_2 are mutually exclusive, then the probability of both events occurring is zero.

(D) If events E_1 and E_2 are events from the same sample space, then $P(E_1 + E_2) = P(E_1) + P(E_2) - P(E_1E_2)$.

The probability law given in option (D) is valid for events from two sample spaces, not events from a single sample space. The correct rule for events from a single sample space is

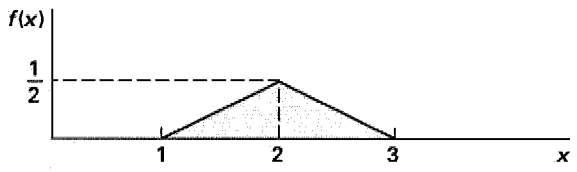
$$P(E_1 + E_2) = P(E_1) + P(E_2)$$

The answer is (D).

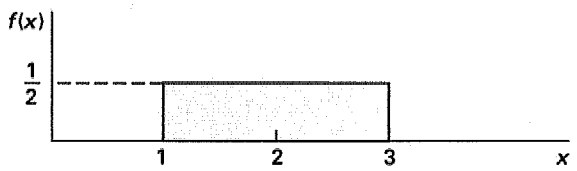
MATHEMATICS–43

Which one of the following functions cannot be a probability density function for the variable x?

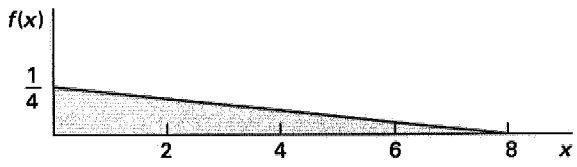
(A)



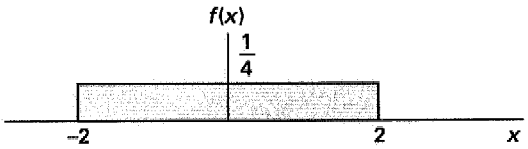
(B)



(C)



(D)



To be a probability density function, the area under the curve must equal 1. That is, the cumulative density function must sum to 1. The area under the curve for option (A) is $^1/_2$. Therefore, it cannot be a probability density function.

The answer is (A).

MATHEMATICS–44

If n is the number of trials, and m is the number of successes, what is the frequency based interpretation of the probability of event E?

(A) $P(E) = \lim_{n \to \infty} \frac{n - m}{n}$

(B) $P(E) = \lim_{n \to \infty} \frac{n}{m}$

(C) $P(E) = \lim_{n \to \infty} \frac{m}{m - n}$

(D) $P(E) = \lim_{n \to \infty} \frac{m}{n}$

The probability of an event can be interpreted as the fraction of successful outcomes when the experiment is performed an infinite number of times. Thus,

$$P(E) = \lim_{n \to \infty} \frac{m}{n}$$

The answer is (D).

MATHEMATICS–45

For a continuous random variable X with probability density function $f(x)$, what is the expected value of X?

(A) $E(X) = \int_0^\infty xf(x)\,dx$

(B) $E(X) = \int_{-\infty}^\infty xf(x)\,dx$

(C) $E(X) = \int_0^\infty f(x)\,dx$

(D) $E(X) = \int_0^\infty x dx$

The expected value or average of X can be defined mathematically as follows.

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

The answer is (B).

MATHEMATICS–46

If $P(B) \neq 1$, and A and B are not independent events, what is $P(A|B)$?

(A) $\big(P(A)\big)\big(P(B)\big)$ (B) $\big(P(B|A)\big)\left(\dfrac{P(A)}{P(B)}\right)$

(C) $P(A)$ (D) $\big(P(A|B)\big)\left(\dfrac{P(B)}{P(A)}\right)$

The probability of event A occurring, given that the dependent event B has occurred, is predicted by the conditional probability law, commonly known as Bayes theorem.

$$P(B|A) = \frac{P(AB)}{P(A)}$$

Similarly,

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Therefore,

$$P(A|B) = \frac{\big(P(B|A)\big)\big(P(A)\big)}{P(B)}$$

The answer is (B).

MATHEMATICS–47

If the discrete random variable X has a geometric distribution parameter P and smallest mass point 0, what is the expected value of X?

(A) P (B) P^{-1} (C) P^{1-P} (D) $\frac{1-P}{P}$

The geometric distribution is a special case of the negative binomial distribution. The mean is $(1-P)/P$, and the variance is $(1-P)/P^2$. Note: Some authors define the geometric distribution with the smallest mass point being 1 (instead of 0). In that case, the mean is $1/P$ and the variance is the same as before.

The answer is (D).

MATHEMATICS–48

If the variable X has a Poisson distribution with parameter λ, what is the expected value of X?

(A) λ^2 (B) $\lambda(1-\lambda)$ (C) λ^{-1} (D) λ

For the Poisson distribution, both the mean and variance are equal to λ.

The answer is (D).

MATHEMATICS–49

If X is a binomial random variable with parameters n and p, what is the expected value of X?

(A) $n(1-p)$ (B) $np(1-p)$ (C) p^{-1} (D) np

For a binomial distribution, the mean is np, and the variance is $np(1-p)$.

The answer is (D).

MATHEMATICS–50

For a discrete random variable X with probability mass function $P(X)$, what is the expected value of X?

(A) $E(X) = \sum_{\text{all } x_i} x_i P(x_i)$

(B) $E(X) = \sum_{\text{all } x_i} x_i^2 P(x_i)$

(C) $E(X) = \sum_{\text{all } x_i} P(x_i)$

(D) $E(X) = \sum_{\text{all } x_i} (x_i - \bar{x}) P(x_i)$

The expected value of a discrete function is given by the following.

$$\begin{aligned} E(X) &= x_1 P(x_1) + x_2 P(x_2) + \cdots \\ &= \sum_{\text{all } x_i} x_i P(x_i) \end{aligned}$$

The answer is (A).

MATHEMATICS–51

An item's cost distribution is given. What is the approximate expected cost?

cost ($)	probability
1	0.07
2	0.23
3	0.46
4	0.17
5	0.04
6	0.03

(A) \$2.5 (B) \$2.9 (C) \$3.0 (D) \$3.1

The expected value is the sum of the products of the individual values and their respective probabilities.

$$\begin{aligned} E(\text{cost}) &= (\$1)(0.07) + (\$2)(0.23) + (\$3)(0.46) + (\$4)(0.17) \\ &\quad + (\$5)(0.04) + (\$6)(0.03) \\ &= \$2.97 \quad (\$3.0) \end{aligned}$$

The answer is (C).

MATHEMATICS–52

One fair die is used in a dice game. The player wins \$10 if he rolls either a 1 or a 6. He loses \$5 if he turns up any other face. What is the expected winning for one roll of the die?

(A) \$0.00 (B) \$3.33 (C) \$5.00 (D) \$6.67

For a fair die, the probability of any face turning up is $1/6$. Therefore, the expected value is

$$E_{\text{win}} = (\$10)\left((2)\left(\frac{1}{6}\right)\right) - (\$5)\left((4)\left(\frac{1}{6}\right)\right) = \$0.00$$

The answer is (A).

MATHEMATICS–53

An urn contains four black balls and six white balls. What is the probability of getting one black ball and one white ball in two consecutive draws from the urn without replacement?

(A) 0.040 (B) 0.24 (C) 0.27 (D) 0.53

$$\begin{aligned} P(\text{black and white}) &= P(\text{black then white}) + \text{P(white then black)} \\ &= \left(\frac{4}{10}\right)\left(\frac{6}{9}\right) + \left(\frac{6}{10}\right)\left(\frac{4}{9}\right) \\ &= 0.53 \end{aligned}$$

The answer is (D).

MATHEMATICS–54

The probability that both stages of a two-stage rocket will function correctly is 0.95. The reliability of the first stage is 0.98. What is the reliability of the second stage?

(A) 0.95 (B) 0.96 (C) 0.97 (D) 0.98

In a serial system consisting of two units,

$$R_t = R_1 R_2$$

In the preceding question, R_2 is the reliability of stage 2, and R_t is the total reliability of all stages. For the second stage,

$$\begin{aligned} R_2 &= \frac{R_t}{R_1} \\ &= \frac{0.95}{0.98} \\ &= 0.97 \end{aligned}$$

The answer is (C).

MATHEMATICS–55

What is the exponential form of the complex number $3 + 4i$?

(A) $e^{i53.1^\circ}$ (B) $5e^{i53.1^\circ}$ (C) $5e^{i126.9^\circ}$ (D) $7e^{i53.1^\circ}$

Any complex number $a + bi$ can be converted to its equivalent exponential form as follows.

$$a + bi = \sqrt{a^2 + b^2}e^{i \arctan b/a}$$

Therefore,

$$\begin{aligned} 3 + 4i &= \sqrt{3^2 + 4^2}e^{i \arctan 4/3} \\ \arctan \frac{4}{3} &= 53.1^\circ \\ 3 + 4i &= 5e^{i53.1^\circ} \end{aligned}$$

The answer is (B).

MATHEMATICS–56

What is the product of the complex numbers $3 + 4i$ and $7 - 2i$?

(A) $10 + 2i$ (B) $13 + 22i$ (C) $13 + 34i$ (D) $29 + 22i$

$$\begin{aligned}(3 + 4i)(7 - 2i) &= 21 - 8i^2 + 28i - 6i \\ &= 21 + 8 + 28i - 6i \\ &= 29 + 22i\end{aligned}$$

The answer is (D).

MATHEMATICS–57

What is the rectangular form of the complex number $7.2e^{i7\pi/13}$?

(A) $7.15 + 0.87i$ (B) $7.15 - 0.87i$
(C) $-0.87 + 7.15i$ (D) $-0.87 - 7.15i$

A complex number of the form ce^{id} can be converted to rectangular form as follows.

$$ce^{id} = c\cos d + (c\sin d)i$$

$$\begin{aligned}7.2e^{i(7\pi/13)} &= (7.2)\left(\cos\frac{7\pi}{13} + \left(\sin\frac{7\pi}{13}\right)i\right) \\ &= -0.87 + 7.15i\end{aligned}$$

The answer is (C).

MATHEMATICS–58

What is the product of the complex numbers $2 - 2i$ and $\sqrt{32}e^{i\pi/4}$?

(A) 16 (B) $16i$ (C) $16e^{i\pi/4}$ (D) $16(1 - i)$

$$\begin{aligned}2 - 2i &= \sqrt{2^2 + 2^2}e^{i \arctan -2/2} \\ &= \sqrt{8}e^{-i\pi/4}\end{aligned}$$

Therefore,

$$\begin{aligned}(2-2i)\sqrt{32}e^{i\pi/4} &= \sqrt{8}e^{-i\pi/4}\sqrt{32}e^{i\pi/4} \\ &= \sqrt{8}\sqrt{32}e^{i(\pi/4-\pi/4)} \\ &= 16\end{aligned}$$

The answer is (A).

MATHEMATICS–59

What is the rationalized value of the following complex quotient?

$$\frac{6+2.5i}{3+4i}$$

(A) $-0.32+0.66i$ (B) $0.32-0.66i$ (C) $1.1-0.66i$ (D) $-1.7+1.1$

In order to rationalize a complex number, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\begin{aligned}\frac{6+2.5i}{3+4i} &= \frac{(6+2.5i)(3-4i)}{(3+4i)(3-4i)} \\ &= \frac{28-16.5i}{25} \\ &= 1.1-0.66i\end{aligned}$$

The answer is (C).

MATHEMATICS–60

What is the first derivative with respect to x of the function $g(x) = 4\sqrt{9}$?

(A) 0 (B) 4/9 (C) 4 (D) 12

The derivative of a constant is zero. Therefore $g'(x) = 0$.

The answer is (A).

MATHEMATICS–61

If a is a simple constant, what is the derivative of $y = x^a$?

(A) ax (B) x^{a-1} (C) ax^{a-1} (D) $(a-1)x$

$$y = x^a$$
$$y' = ax^{a-1}$$

The answer is (C).

MATHEMATICS–62

What is the derivative of $f(x) = \left(x^3 - (x-1)^3\right)^3$?

(A) $3x^2 - 3(x-1)^2$

(B) $3\left(x^3 - (x-1)^3\right)^2$

(C) $9\left(x^3 - (x-1)^3\right)\left(x^2 - (x-1)^2\right)$

(D) $9\left(x^3 - (x-1)^3\right)^2\left(x^2 - (x-1)^2\right)$

$$\begin{aligned} f(x) &= \left(x^3 - (x-1)^3\right)^3 \\ f'(x) &= 3\left(x^3 - (x-1)^3\right)^2 \frac{d}{dx}\left(x^3 - (x-1)^3\right) \\ &= 3\left(x^3 - (x-1)^3\right)^2 \left(3x^2 - 3(x-1)^2(1)\right) \\ &= 9\left(x^3 - (x-1)^3\right)^2 \left(x^2 - (x-1)^2\right) \end{aligned}$$

The answer is (D).

MATHEMATICS–63

Differentiate $f(x) = \sqrt{2x^2 + 4x + 1}$.

(A) $2x + 2$

(B) $\frac{1}{2}\sqrt{2x^2 + 4x + 1}$

(C) $\frac{2x + 2}{\sqrt{2x^2 + 4x + 1}}$

(D) $\frac{4x + 4}{\sqrt{2x^2 + 4x + 1}}$

$$\begin{aligned} f(x) &= \sqrt{2x^2 + 4x + 1} \\ &= \left(2x^2 + 4x + 1\right)^{1/2} \\ f'(x) &= \frac{1}{2}\left(2x^2 + 4x + 1\right)^{-1/2} \frac{d}{dx}\left(2x^2 + 4x + 1\right) \\ &= \frac{1}{2}\left(2x^2 + 4x + 1\right)^{-1/2}(4x + 4) \\ &= \frac{2x + 2}{\sqrt{2x^2 + 4x + 1}} \end{aligned}$$

The answer is (C).

MATHEMATICS–64

Find the second derivative of $y = \sqrt{x^2} + x^{-2}$.

(A) $1 - 2x^{-3}$ (B) $1 - 6x^{-4}$ (C) 3 (D) $\frac{6}{x^4}$

$$\begin{aligned} y &= \sqrt{x^2} + x^{-2} \\ y' &= \frac{x}{\sqrt{x^2}} - 2x^{-3} \\ &= \pm 1 - 2x^{-3} \\ y'' &= 6x^{-4} \end{aligned}$$

Note: $x/\sqrt{x^2} = \pm 1$ because by definition, $\sqrt{x^2} = |x|$.

The answer is (D).

MATHEMATICS–65

Find dy/dt given the following two simultaneous differential equations.

$$2\frac{dx}{dt} - 3\frac{dy}{dt} + x - y = k$$
$$3\frac{dx}{dt} + 2\frac{dy}{dt} - x = \cos t$$

(A) $\frac{2}{13}\left(\cos t + \frac{5}{2}x - \frac{3}{2}y - \frac{3}{2}k\right)$

(B) $\frac{1}{3}\left(\sin t + \frac{1}{9}x - y^3 - \frac{3}{2}k\right)$

(C) $-\frac{1}{6}\left(\sin t + \frac{1}{9}x + y^2 - \frac{3}{2}k\right)$

(D) $\frac{2}{9}\left(\cos t + \frac{3}{2}x - \frac{5}{2}y - \frac{3}{2}k\right)$

Solve both equations for dx/dt.

$$\frac{dx}{dt} = \frac{1}{2}\left(k + y - x + 3\frac{dy}{dt}\right)$$
$$\frac{dx}{dt} = \frac{1}{3}\left(\cos t + x - 2\frac{dy}{dt}\right)$$

Combine and solve for dy/dt.

$$\frac{1}{2}\left(k + y - x + 3\frac{dy}{dt}\right) = \frac{1}{3}\left(\cos t + x - 2\frac{dy}{dt}\right)$$
$$9\frac{dy}{dt} + 4\frac{dy}{dt} = -3k - 3y + 3x + 2\cos t + 2x$$
$$13\frac{dy}{dt} = 2\cos t + 5x - 3y - 3k$$
$$\frac{dy}{dt} = \frac{2}{13}\left(\cos t + \frac{5}{2}x - \frac{3}{2}y - \frac{3}{2}k\right)$$

The answer is (A).

MATHEMATICS–66

If $y = \cos x$, what is dy/dx?

(A) $\sec x$ (B) $-\sec x$ (C) $\csc x$ (D) $-\sin x$

$$\frac{d}{dx}\cos x = -\sin x$$

The answer is (D).

MATHEMATICS–67

If the second derivative of the equation of a curve is proportional to the negative of the equation of the same curve, what is that curve?

(A) a hyperbola (B) a square wave (C) a sinusoid (D) a cycloid

The only type of function that fits the description is a sinusoidal one.

$$\begin{aligned}\frac{d^2}{dx^2}\sin x &= \frac{d}{dx}\cos x \\ &= -\sin x\end{aligned}$$

The answer is (C).

MATHEMATICS–68

Given $P = 2R^2S^3T^{1/2} + R^{1/3}S\sin 2T$, what is $\partial P/\partial T$?

(A) $R^2S^3T^{3/2} + 2R^{1/3}\cos 2T$

(B) $6RS^2T^{-1/2} + \frac{2}{3}R^{-2/3}\cos 2T$

(C) $2R^2S^3T^{1/2} + R^{1/3}S\cos 2T$

(D) $R^2S^3T^{-1/2} + 2R^{1/3}S\cos 2T$

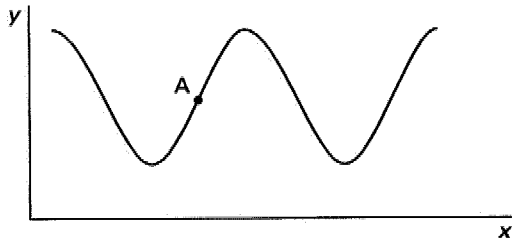
All variables other than T are treated as constants.

$$\begin{aligned}\frac{\partial P}{\partial T} &= 2R^2S^3\left(\frac{1}{2}T^{-1/2}\right) + R^{1/3}S(\cos 2T)(2) \\ &= R^2S^3T^{-1/2} + 2R^{1/3}S\cos 2T\end{aligned}$$

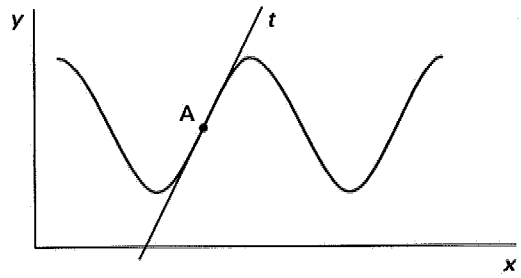
The answer is (D).

MATHEMATICS–69

Which of the following describes the first derivative at point A of the function shown in the figure?



(A) positive only
(B) negative only
(C) zero
(D) positive or negative

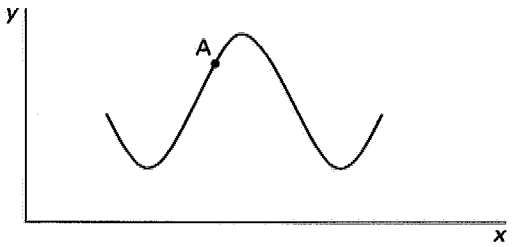


The first derivative corresponds to the slope of a tangent line at the point. The slope of this tangent line is positive. Therefore, the first derivative of the function at point A is also positive.

The answer is (A).

MATHEMATICS–70

Which of the following describes the second derivative at point A of the function shown?



(A) positive only
(B) negative only
(C) zero
(D) positive or negative

The second derivative corresponds to the concavity of the function. Since the curvature at this point is concave down, the second derivative is negative. The second derivative also indicates what is happening to the first derivative, the slope. Since the slope is decreasing at point A, the second derivative must be negative.

The answer is (B).

MATHEMATICS–71

What is the slope of the graph $y = -x^2$ at $x = 2$?

(A) -4 (B) -2 (C) 1 (D) 3

The slope of a curve is given by the first derivative.

$$y(x) = -x^2$$
$$y'(x) = -2x$$

At $x = 2$,

$$y'(2) = (-2)(2)$$
$$= -4$$

The answer is (A).

MATHEMATICS–72

Given the function $f(x) = x^3 - 5x + 2$, find $f'(2)$, the value of the first derivative at $x = 2$.

(A) 2 (B) $3x^2 - 5$ (C) 7 (D) 8

$$\begin{aligned} f(x) &= x^3 - 5x + 2 \\ f'(x) &= 3x^2 - 5 \\ f'(2) &= (3)(2)^2 - 5 \\ &= 7 \end{aligned}$$

The answer is (C).

MATHEMATICS–73

Find the slope of the tangent to a parabola, $y = x^2$, at a point on the curve where $x = {}^1/_2$.

(A) −1/2 (B) 0 (C) 1/4 (D) 1

$$\begin{aligned} y &= x^2 \\ y' &= 2x \\ y'\left(\frac{1}{2}\right) &= (2)\left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

The answer is (D).

MATHEMATICS–74

What is the slope of the curve $y = x^2 - 4x$ at the origin?

(A) −4 (B) −3 (C) 0 (D) 4

$$\begin{aligned} y &= x^2 - 4x \\ \frac{dy}{dx} &= 2x - 4 \\ \left.\frac{dy}{dx}\right|_{x=0} &= (2)(0) - 4 \\ &= -4 \end{aligned}$$

The answer is (A).

MATHEMATICS–75

Find the slope of the line tangent to the curve $y = x^3 - 2x + 1$ at $x = 1$.

(A) 1/4 (B) 1/3 (C) 1/2 (D) 1

$$\begin{aligned} y &= x^3 - 2x + 1 \\ y' &= 3x^2 - 2 \\ y'(1) &= 3 - 2 \\ &= 1 \end{aligned}$$

The answer is (D).

MATHEMATICS–76

Determine the equation of the line tangent to the graph $y = 2x^2 + 1$ at the point (1,3).

(A) $y = 2x + 1$
(B) $y = 4x - 1$
(C) $y = 2x - 1$
(D) $y = 4x + 1$

First, determine the slope of the graph at $x = 1$.

$$\begin{aligned} y &= 2x^2 + 1 \\ y' &= 4x \\ y'(1) &= 4 \end{aligned}$$

Since the tangent line intersects the graph at (1,3), the equation of the tangent line is

$$\begin{aligned} y &= 4x + b \\ 3 &= (4)(1) + b \\ b &= -1 \\ y &= 4x - 1 \end{aligned}$$

The answer is (B).

MATHEMATICS–77

Given $y_1 = 4x + 3$ and $y_2 = x^2 + C$, find C such that y_2 is tangent to y_1.

(A) 2 (B) 4 (C) 5 (D) 7

The slope of $y_1 = 4x + 3$ is 4 everywhere. Therefore, y_2 has a slope of 4 at the tangent point.

$$\begin{aligned} y_2' &= 2x \\ 4 &= 2x \\ x &= 2 \end{aligned}$$

$x = 2$ at the tangent point. Find $y_1 = y_2$ at the tangent point and substitute in to find C.

$$\begin{aligned} y_1 &= (4)(2) + 3 \\ &= 11 \\ y_2 &= 11 \\ 11 &= (2)^2 + C \\ C &= 7 \end{aligned}$$

The answer is (D).

MATHEMATICS–78

Given

$$\frac{dy_1}{dx} = \tfrac{2}{13}\left(1 + \tfrac{5}{2}x - \tfrac{3}{2} - \tfrac{3}{4}k\right)$$

What is the value of k such that y_1 is perpendicular to the curve $y_2 = 2x$ at $x = 1$?

(A) 2 (B) 3 (C) 6 (D) 7

For two lines to be perpendicular, $m_1 m_2 = -1$, where m_n is the slope of line n.

$$\frac{dy_2}{dx} = 2$$

Therefore, at $(1, 1)$,

$$\begin{aligned}
\frac{dy_1}{dx} &= -\frac{1}{2} \\
&= \frac{2}{13}\left(1 + \tfrac{5}{2}x - \frac{3}{2} - \tfrac{3}{4}k\right) \\
-\frac{1}{2} &= \left(\frac{2}{13}\right)\left(1 + \left(\frac{5}{2}\right)(1) - \frac{3}{2} - \tfrac{3}{4}k\right) \\
-\frac{13}{4} &= 2 - \tfrac{3}{4}k \\
\tfrac{3}{4}k &= \frac{21}{4} \\
k &= 7
\end{aligned}$$

The answer is (D).

MATHEMATICS–79

The location of a body as a function of time is $x(t) = 18t + 9t^2$. Find the body's velocity at $t = 2$.

(A) 20 (B) 24 (C) 36 (D) 54

Velocity is the first time derivative of the position function.

$$\begin{aligned} x(t) &= 18t + 9t^2 \\ \mathrm{v}(t) &= x'(t) \\ &= 18 + 18t \\ \mathrm{v}(2) &= 18 + (18)(2) \\ &= 54 \end{aligned}$$

The answer is (D).

MATHEMATICS–80

A particle moves according to the following functions of time.

$$\begin{aligned} x(t) &= 3\sin t \\ y(t) &= 4\cos t \end{aligned}$$

What is the resultant velocity at $t = \pi$?

(A) 0 (B) 3 (C) 4 (D) 9

$$\mathrm{v} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\frac{dx}{dt} = 3\cos t$$

$$\frac{dy}{dt} = -4\sin t$$

$$\begin{aligned} \mathrm{v}(t) &= \sqrt{9\cos^2 t + 16\sin^2 t} \\ \mathrm{v}(\pi) &= \sqrt{(9)(-1)^2 + (16)(0)^2} \\ &= 3 \end{aligned}$$

The answer is (B).

MATHEMATICS–81

Water is pouring at a varying rate into a swimming pool that is initially empty. After t hours, there are $t + \sqrt{t}$ liters in the pool. At what rate is the water pouring into the pool when $t = 9$ h?

(A) 1/6 L/h (B) 1/2 L/h (C) 1 L/h (D) 7/6 L/h

Let V = volume of water in the tank in liters and Q = flow rate in liters per hour.

$$V = t + \sqrt{t}$$

$$Q = \frac{dV}{dt} = 1 + \frac{1}{2\sqrt{t}}$$

At $t = 9$ h,

$$Q = 1 + \frac{1}{2\sqrt{9}}$$

$$= 7/6 \text{ L/h}$$

The answer is (D).

MATHEMATICS–82

If x increases uniformly at the rate of 0.001 per unit time, at what rate is the expression $(1 + x)^3$ increasing when x becomes 9?

(A) 0.001 (B) 0.003 (C) 0.3 (D) 1

$$\frac{dx}{dt} = 0.001$$

$$f(x) = (1 + x)^3$$

$$\frac{df}{dx} = 3(1 + x)^2$$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

$$= 0.003(1 + x)^2$$

$$\left.\frac{df}{dt}\right|_{x=9} = (0.003)(1 + 9)^2$$

$$= 0.3$$

The answer is (C).

MATHEMATICS–83

A spherical balloon is filled with air at a rate of 1 m^3/s. Compute the time rate of change of the surface area of the balloon at the instant the balloon's volume is 113.1 m^3.

(A) 0.67 m^2/s (B) 1.7 m^2/s (C) 3.1 m^2/s (D) 3.7 m^2/s

$$V = \tfrac{4}{3}\pi r^3$$

$$A = 4\pi r^2$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \left(\frac{1}{4\pi r^2}\right)\frac{dV}{dt}$$

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$

$$= 8\pi r \frac{dr}{dt}$$

$$= \left(\frac{8\pi r}{4\pi r^2}\right)\frac{dV}{dt}$$

$$= \left(\frac{2}{r}\right)\frac{dV}{dt}$$

Solve for the radius of the balloon when the volume is 113.1 m^3, and substitute into the equation for the rate of change of the surface area.

$$r = \left(\frac{3V}{4\pi}\right)^{1/3}$$

$$= \left(\frac{(3)(113.1\ \text{m}^3)}{4\pi}\right)^{1/3}$$

$$r = 3\ \text{m}$$

$$\frac{dA}{dt} = \left(\frac{2}{r}\right)\frac{dV}{dt}$$

$$= \left(\frac{2}{3\ \text{m}}\right)\left(1\ \frac{\text{m}^3}{\text{s}}\right)$$

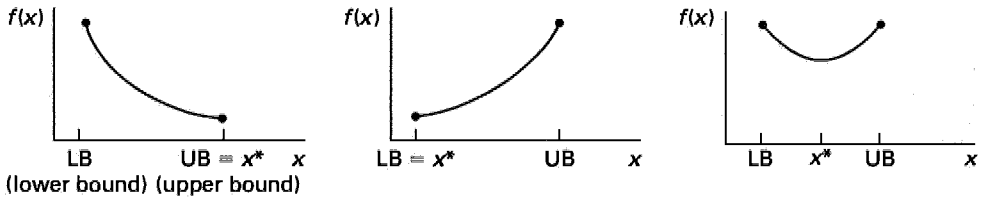
$$= 0.67\ \text{m}^2/\text{s}$$

The answer is (A).

MATHEMATICS–84

Consider a strictly concave up function of one variable, x, with lower and upper bounds on x. At what value(s) of x will the function be minimized?

(A) at the upper bound of x
(B) at the lower bound of x
(C) strictly between the upper and lower bounds of x
(D) at any of the above



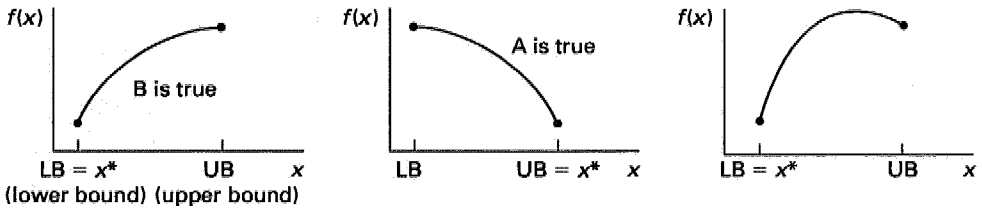
The examples given demonstrate that, for a concave up function, the minimum could occur at the lower bound, the upper bound, or somewhere between. Option (A), (B), or (C) could be correct.

The answer is (D).

MATHEMATICS–85

Consider a strictly concave down function in one variable, x, with lower and upper bounds on x. At what value(s) of x will the function be minimized?

(A) at the upper bound of x
(B) at the lower bound of x
(C) strictly between the upper and lower bounds of x
(D) at either the upper or lower bound of x



The illustrations demonstrate that for a concave down curve, the minimum could occur at either the lower or the upper bound. Therefore, option (D) is correct.

The answer is (D).

MATHEMATICS–86

What is the maximum of the function $y = -x^3 + 3x$, for $x \geq -1$?

(A) -2 (B) -1 (C) 0 (D) 2

The maximum occurs where $y' = 0$ and $y'' < 0$ or at an endpoint.

$$\begin{aligned} y &= -x^3 + 3x \\ y' &= -3x^2 + 3 \\ y'' &= -6x \\ y' &= 0 \\ 0 &= -3x^2 + 3 \\ x^2 &= 1 \\ x &= \pm 1 \quad [-1 \text{ is also an endpoint}] \\ y(-1) &= -(-1)^3 + (3)(-1) \\ &= -2 \\ y(1) &= -(1)^3 + (3)(1) \\ &= 2 \end{aligned}$$

Therefore,

$$y_{\max} = 2$$

The answer is (D).

MATHEMATICS–87

The cost, C, of an item is a function of the quantity, x, of the item: $C(x) = x^2 - 4000x + 50$. Find the quantity for which the cost is minimum.

(A) 1000 (B) 1500 (C) 2000 (D) 3000

$$\begin{aligned} C &= x^2 - 4000x + 50 \\ C' &= 2x - 4000 \\ C'' &= 2 \\ C' &= 0 \\ 2x - 4000 &= 0 \\ x &= 2000 \\ C'' &> 0 \end{aligned}$$

Thus, cost is a minimum when $x = 2000$.

The answer is (C).

MATHEMATICS–88

Compute the following limit.

$$\lim_{x \to \infty} \frac{x+2}{x-2}$$

(A) 0 (B) 1 (C) 2 (D) ∞

Divide both the numerator and denominator by x, and allow x to approach infinity.

$$\begin{aligned} \lim_{x \to \infty} \frac{x+2}{x-2} &= \lim_{x \to \infty} \frac{1 + \frac{2}{x}}{1 - \frac{2}{x}} \\ &= \frac{1+0}{1-0} \\ &= 1 \end{aligned}$$

The answer is (B).

MATHEMATICS–89

Simplify the following expression.

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

(A) 0 (B) 8 (C) 12 (D) 16

Factor the numerator, and simplify the fraction before taking the limit.

$$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 16}{x - 4} &= \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4} \\ &= \lim_{x \to 4} (x + 4) \\ &= 8 \end{aligned}$$

The answer is (B).

MATHEMATICS–90

Compute the following limit.

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

(A) 0 (B) 1/4 (C) 1/2 (D) 1

Since both the numerator and denominator approach zero, use L'Hôpital's rule. L'Hôpital's rule states that the derivative of the numerator divided by the derivative of the denominator has the same limit as the original fraction, provided that both the numerator and denominator of the original fraction approach zero.

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x}$$

Since the numerator and denominator both approach zero, apply L'Hôpital's rule again.

$$\begin{aligned} \lim_{x \to 0} \frac{1 - \cos x}{x^2} &= \lim_{x \to 0} \frac{\cos x}{2} \\ &= \frac{\cos 0}{2} \\ &= 1/2 \end{aligned}$$

The answer is (C).

MATHEMATICS–91

The existence of the two equations, $y' = f(x)$ and $y = \phi(x)$, implies that which of the following equations is true?

(A) $\phi(x) = \int f(x)dx + C$
(B) $\phi(x) = f(x)$
(C) $\phi'(x) = \int f(x)dx + C$
(D) $\phi'(x) = y$

$$y = \phi(x)$$
$$\frac{d\phi}{dx} = y'$$
$$\phi(x) = \int y'dx + C$$

Since

$$y' = f(x)$$
$$\phi(x) = \int f(x)dx + C$$

The answer is (A).

MATHEMATICS–92

Fill in the blank in the following statement.

The integral of a function between certain limits divided by the difference in abscissas between those limits gives the _____ of the function.

(A) average (B) middle (C) intercept (D) asymptote

$$\frac{1}{b-a}\int_a^b f(x) = \text{the average value of the function}$$

The answer is (A).

MATHEMATICS–93

Find the area under the curve $y = 1/x$ between the limits $y = 2$ and $y = 10$.

(A) 1.61 (B) 2.39 (C) 3.71 (D) 3.97

The area under the curve $f(x)$ between x_1 and x_2, A, is given by

$$A = \int_{x_1}^{x_2} f(x)dx$$

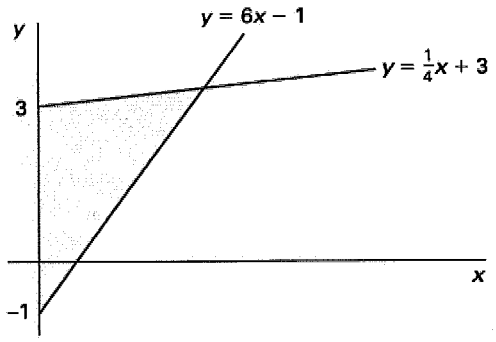
The x limits corresponding to the y limits are $x = 1/2$ and $x = 1/10$.

$$\begin{aligned} A &= \int_{1/10}^{1/2} \frac{1}{x} dx \\ &= \ln x \Big|_{1/10}^{1/2} \\ &= 1.61 \end{aligned}$$

The answer is (A).

MATHEMATICS–94

Find the area of the shaded region between $y = 6x - 1$ and $y = {}^1/_4x + 3$, bounded by $x = 0$ and the intersection point.



(A) 32/529 (B) 16/23 (C) 32/23 (D) 1440/529

The area between curve 1 and curve 2 is equal to the area under curve 1 minus the area under curve 2. The intersection point of the two curves is found by equating both functions.

$$\begin{aligned} 6x - 1 &= \tfrac{1}{4}x + 3 \\ \tfrac{23}{4}x &= 4 \\ x &= 16/23 \end{aligned}$$

The area, A, is

$$\begin{aligned} A &= \int_0^{16/23} \left(\tfrac{1}{4}x + 3 - 6x + 1\right) dx \\ &= \int_0^{16/23} \left(-\tfrac{23}{4}x + 4\right) dx \\ &= \left| \left(-\tfrac{23}{8}x^2 + 4x\right) \right|_0^{16/23} \\ &= \left(-\frac{23}{8}\right)\left(\frac{16}{23}\right)^2 + (4)\left(\frac{16}{23}\right) \\ &= -\frac{32}{23} + \frac{64}{23} \\ &= 32/23 \end{aligned}$$

The answer is (C).

MATHEMATICS–95

If it is known that $y = 1$ when $x = 1$, what is the constant of integration for the following integral?

$$y(x) = \int \left(e^{2x} - 2x\right) dx$$

(A) $C = 2 - e^2$

(B) $C = 3 - e^2$

(C) $C = 4 - e^2$

(D) $C = \frac{1}{2}\left(4 - e^2\right)$

$$\begin{aligned} y(x) &= \int e^{2x}dx - 2\int x dx \\ &= \tfrac{1}{2}e^{2x} - x^2 + C \\ &= \frac{1}{2}\left(e^{2x} - 2x^2\right) + C \end{aligned}$$

However, $y(1) = 1$.

$$\begin{aligned} 1 &= \frac{1}{2}\left(e^{2(1)} - (2)(1)^2\right) + C \\ &= \frac{1}{2}\left(e^2 - 2\right) + C \\ C &= 1 + 1 - \tfrac{1}{2}e^2 \\ &= \frac{1}{2}\left(4 - e^2\right) \end{aligned}$$

The answer is (D).

MATHEMATICS–96

It is known that $y(x)$ passes through the points (0,2) and (1,4). Solve for $y(x)$ if the second derivative is

$$\frac{d^2y}{dx^2} = 1$$

(A) $y = \left(x^2 + 3x\right) + 2$

(B) $y = \frac{1}{2}\left(x^2 + 3x\right) + 2$

(C) $y = \frac{1}{2}\left(x^2 - 3x\right) - 2$

(D) $y = \frac{1}{2}\left(x^2 + 3x\right) - 2$

Integrate twice to get the general form of the equation.

$$\begin{aligned} \frac{d^2y}{dx^2} &= 1 \\ \frac{dy}{dx} &= \int 1dx \\ &= x + C_1 \\ y &= \int (x + C_1) \\ &= \tfrac{1}{2}x^2 + C_1x + C_2 \end{aligned}$$

Now solve for C_1 and C_2 using the given conditions.

$$\begin{aligned} 2 &= \frac{1}{2}(0) + C_1(0) + C_2 \\ C_2 &= 2 \\ 4 &= \frac{1}{2}(1)^2 + C_1(1) + 2 \\ C_1 &= 3/2 \\ y &= \tfrac{1}{2}x^2 + \tfrac{3}{2}x + 2 \end{aligned}$$

The answer is (B).

MATHEMATICS–97

What is a solution of the first-order difference equation $y(k+1) = y(k) + 5$?

(A) $y(k) = 4 - \dfrac{5}{k}$

(B) $y(k) = C - k$, where C is a constant

(C) $y(k) = 5^k + \dfrac{1 - 5^k}{-4}$

(D) $y(k) = 20 + 5k$

Assume the solution has the form

$$y(k) = 20 + 5k$$

Substitute the assumed solution into the difference equation.

$$\begin{aligned} y(k+1) &= 20 + 5(k+1) \\ &= 20 + 5k + 5 \\ &= y(k) + 5 \end{aligned}$$

The answer is (D).

MATHEMATICS–98

What is the solution of the linear difference equation $y(k+1) = 15y(k)$?

(A) $y(k) = \dfrac{15}{1+15k}$

(B) $y(k) = \dfrac{15k}{16}$

(C) $y(k) = C + 15^k$, where C is a constant

(D) $y(k) = 15^k$

Assume the solution has the form

$$y(k) = 15^k$$

Substitute into the difference equation.

$$\begin{aligned} y(k+1) &= 15^{k+1} \\ &= (15)\left(15^k\right) \\ &= 15y(k) \end{aligned}$$

Note: If $y(k) = C + 15^k$, then $y(k+1) = C + 15^{k+1} \neq 15y(k)$.

The answer is (D).

MATHEMATICS–99

What is the solution of the linear difference equation $(k+1)y(k+1) - ky(k) = 1$?

(A) $y(k) = 12 - \dfrac{1}{k}$

(B) $y(k) = 1 - \dfrac{12}{k}$

(C) $y(k) = 12 + 3k$

(D) $y(k) = 3 + \dfrac{1}{k}$

Assume the solution has the form

$$y(k) = 1 - \frac{12}{k}$$

Substitute the solution into the difference equation.

$$\begin{aligned}(k+1)\big(y(k+1)\big) - k\big(y(k)\big) &= 1\\ (k+1)\left(1 - \frac{12}{k+1}\right) - k\left(1 - \frac{12}{k}\right) &= 1\\ (k+1)\left(\frac{k+1-12}{k+1}\right) - k\left(\frac{k-12}{k}\right) &= 1\\ k+1-12-k+12 &= 1\\ 1 &= 1\end{aligned}$$

Thus, $y(k) = 1 - 12/k$ solves the difference equation.

The answer is (B).

MATHEMATICS–100

Which of the following is a differential equation of the first order?

(A) $(y'')^3 + 2y' = -3$

(B) $\dfrac{\partial Q}{\partial x} - \dfrac{\partial Q}{\partial y} = 0$

(C) $\dfrac{dy}{dx} + \dfrac{9-x}{x} = y^3$

(D) $\left(\dfrac{dy}{dx}\right)^2 = -y + x$

A first-order differential equation contains only first derivatives and does not have partial derivatives. The only choice that fulfills this requirement is option (C).

The answer is (C).

MATHEMATICS–101

How can the following differential equation best be described?

$$a\frac{d^2x}{dt^2} + B(t)\frac{dx}{dt} + C = D(t)$$

(A) linear, homogeneous, and first order
(B) homogeneous and first order
(C) linear, second order, and nonhomogeneous
(D) linear, homogeneous, and second order

The differential equation has a second derivative, so it is of second order. The forcing function is nonzero, so the equation is nonhomogeneous. All of the terms on the left-hand side only have coefficients that are either constant or a function of the independent variable. Therefore the equation is also linear.

The answer is (C).

MATHEMATICS–102

The differential equation given is correctly described by which of the following choices?

$$a\frac{d^2y}{dx^2} + bxy\frac{dy}{dx} = f(x)$$

(A) linear, second order, homogeneous
(B) nonlinear, second order, homogeneous
(C) linear, second order, nonhomogeneous
(D) nonlinear, second order, nonhomogeneous

Since there is a second derivative, the differential equation is of second order. Since the coefficient of one of the terms contains the dependent variable, y, the equation is nonlinear. Since the forcing function, $f(x)$, is implied to be nonzero, the differential equation is also nonhomogeneous.

The answer is (D).

MATHEMATICS–103

Determine the solution of the following differential equation.

$$y' + 5y = 0$$

(A) $y = 5x + C$ (B) $y = Ce^{-5x}$ (C) $y = Ce^{5x}$ (D) (A) or (B)

This is a first-order linear equation with characteristic equation $r + 5 = 0$. Therefore, the form of the solution is

$$y = Ce^{-5x}$$

In the preceding equation, the constant, C, could be determined from additional information.

The answer is (B).

MATHEMATICS–104

What is the general solution of the following differential equation?

$$\frac{d^2y}{dx^2} + 4y = 0$$

(A) $y = \sin x + 2\tan x + C$
(B) $y = e^x - 2e^{-x} + C$
(C) $y = 2x^2 - x + C$
(D) $y = \sin 2x + \cos 2x + C$

Examination of the differential equation shows that a multiple of the function and its second derivative must sum to zero. Sines and cosines have the property that their second derivatives are the negatives of the original natural function.

If $y = \sin 2x + \cos 2x$, then

$$\begin{aligned} y' &= 2\cos 2x - 2\sin 2x \\ y'' &= -4\sin 2x - 4\cos 2x \\ y'' + 4y &= -4\sin 2x - 4\cos 2x + 4(\sin 2x + \cos 2x) \\ &= 0 \end{aligned}$$

The function in option (D) solves the differential equation.

The answer is (D).

MATHEMATICS–105

In the following differential equation with the initial condition $x(0) = 12$, what is the value of $x(2)$?

$$\frac{dx}{dt} + 4x = 0$$

(A) 3.35×10^{-4} (B) 4.03×10^{-3} (C) 3.35 (D) 6.04

This is a first-order, linear, homogeneous differential equation with characteristic equation $r + 4 = 0$.

$$\begin{aligned} x' + 4x &= 0 \\ x &= x_0 e^{-4t} \\ x(0) &= x_0 e^{(-4)(0)} \\ &= 12 \\ x_0 &= 12 \\ x &= 12e^{-4t} \\ x(2) &= 12e^{(-4)(2)} \\ &= 12e^{-8} \\ &= 4.03 \times 10^{-3} \end{aligned}$$

The answer is (B).

MATHEMATICS–106

A curve passes through the point (1,1). Determine the absolute value of the slope of the curve at $x = 25$ if the differential equation of the curve is the exact equation $y^2dx + 2xydy = 0$.

(A) 1/250 (B) 1/125 (C) $1/50\sqrt{5}$ (D) $1/\sqrt{125}$

$$\begin{aligned} y^2dx + 2xydy &= 0 \\ 2xydy &= -y^2dx \\ 2\frac{dy}{y} &= \frac{-dx}{x} \end{aligned}$$

Integrating both sides,

$$\begin{aligned} 2\ln y &= -\ln x + \ln C \\ \ln y^2 + \ln x &= \ln C \\ \ln xy^2 &= \ln C \\ xy^2 &= C \end{aligned}$$

Use the fact that the curve passes through the point (1,1) to solve for C, then determine the slope at $x = 25$.

$$\begin{aligned} (1)(1)^2 &= C \\ C &= 1 \\ xy^2 &= 1 \\ y &= \pm\sqrt{\frac{1}{x}} \\ y(25) &= \pm\sqrt{\frac{1}{25}} \\ &= \pm 1/5 \\ y^2dx + 2xydy &= 0 \\ \frac{dy}{dx} &= -\frac{y}{2x} \\ \left.\frac{dy}{dx}\right|_{x=25} &= -\frac{\pm\frac{1}{5}}{(2)(25)} \\ &= \pm 1/250 \end{aligned}$$

The answer is (A).

MATHEMATICS–107

Determine the constant of integration for the separable differential equation $xdx + 6y^5dy = 0$. It is known that $x = 0$ when $y = 2$.

(A) 12 (B) 16 (C) 24 (D) 64

Since this differential equation is already separated, integrate to find the solution.

$$\begin{aligned} \int xdx + \int 6y^5dy &= \int 0 \\ \tfrac{1}{2}x^2 + y^6 &= C \end{aligned}$$

Use the initial conditions to solve for C.

$$\left(\frac{1}{2}\right)(0)^2 + (2)^6 = C$$

$$C = 64$$

The answer is (D).

MATHEMATICS–108

What is the Laplace transform of e^{-6t}?

(A) $\frac{1}{s+6}$ (B) $\frac{1}{s-6}$ (C) e^{-6+s} (D) e^{6+s}

The Laplace transform of a function, $\mathcal{L}(f)$, can be calculated for the definition of a transform. However, it is easier to refer to a table of transforms.

$$\mathcal{L}(e^{-6t}) = \int_0^\infty e^{-(s+6)t}dt$$

$$= -\frac{e^{-(s+6)t}}{s+6}\Big|_0^\infty$$

$$= \frac{1}{s+6}$$

The answer is (A).