NCEES

FE Reference Handbook 10.1

EXAMEN DE COMPETENCIAS FUNDAMENTALES

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Preface

About the *Handbook*

The Fundamentals of Engineering (FE) exam is computer-based, and the *FE Reference Handbook* is the only resource material you may use during the exam. Reviewing it before exam day will help you become familiar with the charts, formulas, tables, and other reference information provided. You won't be allowed to bring your personal copy of the *Handbook* into the exam room. Instead, the computer-based exam will include a PDF version of the *Handbook* for your use. No printed copies of the *Handbook* will be allowed in the exam room.

The PDF version of the *FE Reference Handbook* that you use on exam day will be very similar to the printed version. Pages not needed to solve exam questions—such as the cover, introductory material, index, and exam specifications—will not be included in the PDF version. In addition, NCEES will periodically revise and update the *Handbook*, and each FE exam will be administered using the updated version.

The *FE Reference Handbook* does not contain all the information required to answer every question on the exam. Basic theories, conversions, formulas, and definitions examinees are expected to know have not been included. Special material required for the solution of a particular exam question will be included in the question itself.

Updates on exam content and procedures

NCEES.org is our home on the web. Visit us there for updates on everything exam-related, including specifications, exam-day policies, scoring, and practice tests. A PDF version of the *FE Reference Handbook* similar to the one you will use on exam day is also available there.

Errata

To report errata in this book, send your correction using our chat feature or your account on NCEES.org. We will also post errata on the website. Examinees are not penalized for any errors in the *Handbook* that affect an exam question.

CONTENTS

Units and Conversion Factors

Distinguishing pound-force from pound-mass

The FE exam and this handbook use both the metric system of units and the U.S. Customary System (USCS). In the USCS system of units, both force and mass are called pounds. Therefore, one must distinguish the pound-force (lbf) from the pound-mass (lbm).

The pound-force is that force which accelerates one pound-mass at 32.174 ft/sec^2. Thus, 1 lbf = 32.174 lbm-ft/sec^2. The expression 32.174 lbm-ft/(lbf-sec^2) is designated as g_c and is used to resolve expressions involving both mass and force expressed as pounds. For instance, in writing Newton's second law, the equation would be written as $F = ma/g_c$, where F is in lbf, m in lbm, and a is in ft/sec^2.

Similar expressions exist for other quantities: kinetic energy, $KE = mv^2/2g_c$, with KE in (ft-lbf); potential energy, $PE = mgh/g_c$, with PE in (ft-lbf); fluid pressure, $p = \rho gh/g_c$, with p in (lbf/ft^2); specific weight, $SW = \rho g/g_c$, in (lbf/ft^3); shear stress, $\tau = (\mu/g_c)(dv/dy)$, with shear stress in (lbf/ft^2). In all these examples, g_c should be regarded as a force unit conversion factor. It is frequently not written explicitly in engineering equations. However, its use is required to produce a consistent set of units.

Note that the force unit conversion factor g_c [lbm-ft/(lbf-sec^2)] should not be confused with the local acceleration of gravity g, which has different units (m/s^2 or ft/sec^2) and may be either its standard value (9.807 m/s^2 or 32.174 ft/sec^2) or some other local value.

If the problem is presented in USCS units, it may be necessary to use the constant g_c in the equation to have a consistent set of units.

Constants and conversion factors provided are approximate, with sufficient accuracy to solve exam questions.

METRIC PREFIXES		
Multiple	Prefix	Symbol
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	µ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10^{1}	deka	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E

COMMONLY USED EQUIVALENTS	
1 gallon of water weighs	8.34 lbf
1 cubic foot of water weighs	62.4 lbf
1 cubic inch of mercury weighs	0.491 lbf
The mass of 1 cubic meter of water is	1,000 kilograms
1 mg/L is	8.34×10^{-6} lbf/gal

TEMPERATURE CONVERSIONS
ºF = 1.8 (ºC) + 32
ºC = (ºF – 32)/1.8
ºR = ºF + 459.69
K = ºC + 273.15

Significant Figures

Significant figures of numbers in math operations will determine the accuracy of the result. General rules for significant digits are:

Rule 1: Non-zero digits are always significant.

Rule 2: Any zeros between two significant digits are significant.

Rule 3: All zeros in the decimal portion are significant.

Rule 4 (Addition and Subtraction): The number used in the calculation with the least number of significant digits after the decimal point dictates the number of significant figures after the decimal point. The number with the most significant figures to the left of the decimal point dictates the number of significant digits to the left of decimal point.

Rule 5 (Multiplication and Division): The result of the operation has the same number of significant digits as the input number with the least number of significant digits.

Rule 6: In the solution of engineering problems, it is customary to retain 3–4 significant digits in the final result.

Ideal Gas Constants

The universal gas constant, designated as $\overline{R}$ in the table below, relates pressure, volume, temperature, and number of moles of an ideal gas. When that universal constant, $\overline{R}$, is divided by the molecular weight of the gas, the result, often designated as R, has units of energy per degree per unit mass [kJ/(kg·K) or ft-lbf/(lbm-ºR)] and becomes characteristic of the particular gas. Some disciplines, notably chemical engineering, often use the symbol R to refer to the universal gas constant $\overline{R}$.

Fundamental Constants

Quantity		Symbol	Value	Units
electron charge		e	1.6022×10^{-19}	C (coulombs)
Faraday constant		F	96,485	coulombs/(mol)
gas constant	metric	$\overline{R}$	8,314	J/(kmol·K)
gas constant	metric	$\overline{R}$	8.314	kPa·m^3/(kmol·K)
gas constant	USCS	$\overline{R}$	1,545	ft-lbf/(lb mole-ºR)
		$\overline{R}$	0.08206	L·atm/(mole·K)
gravitation–Newtonian constant		G	6.673×10^{-11}	m^3/(kg·s^2)
gravitation–Newtonian constant		G	6.673×10^{-11}	N·m^2/kg^2
gravity acceleration (standard)	metric	g	9.807	m/s^2
gravity acceleration (standard)	USCS	g	32.174	ft/sec^2
molar volume (ideal gas), T = 273.15 K, p = 101.3 kPa		V_m	22,414	L/kmol
speed of light (exact)		c	299,792,458	m/s
Stefan-Boltzmann constant		σ	5.67×10^{-8}	W/(m^2·K^4)

Units and Conversion Factors

Multiply	By	To Obtain
acre	43,560	square feet (ft^2)
ampere-hr (A-hr)	3,600	coulomb (C)
ångström (Å)	1×10^{-10}	meter (m)
atmosphere (atm)	76.0	cm, mercury (Hg)
atm, std	29.92	in., mercury (Hg)
atm, std	14.70	lbf/in^2 abs (psia)
atm, std	33.90	ft, water
atm, std	1.013×10^5	pascal (Pa)
bar	1×10^5	Pa
bar	0.987	atm
barrels–oil	42	gallons–oil
Btu	1,055	joule (J)
Btu	2.928×10^{-4}	kilowatt-hr (kWh)
Btu	778	ft-lbf
Btu/hr	3.930×10^{-4}	horsepower (hp)
Btu/hr	0.293	watt (W)
Btu/hr	0.216	ft-lbf/sec
calorie (g-cal)	3.968×10^{-3}	Btu
cal	1.560×10^{-6}	hp-hr
cal	4.184	joule (J)
cal/sec	4.184	watt (W)
centimeter (cm)	3.281×10^{-2}	foot (ft)
cm	0.394	inch (in)
centipoise (cP)	0.001	pascal•sec (Pa•s)
centipoise (cP)	1	g/(m•s)
centipoise (cP)	2.419	lbm/hr-ft
centistoke (cSt)	1×10^{-6}	m^2/sec (m^2/s)
cubic feet/second (cfs)	0.646317	million gallons/day (MGD)
cubic foot (ft^3)	7.481	gallon
cubic meters (m^3)	1,000	liters
electronvolt (eV)	1.602×10^{-19}	joule (J)
foot (ft)	30.48	cm
ft	0.3048	meter (m)
ft of H_2O	0.4332	psi
ft-pound (ft-lbf)	1.285×10^{-3}	Btu
ft-lbf	3.766×10^{-7}	kilowatt-hr (kWh)
ft-lbf	0.324	calorie (g-cal)
ft-lbf	1.356	joule (J)
ft-lbf/sec	1.818×10^{-3}	horsepower (hp)
gallon (U.S. Liq)	3.785	liter (L)
gallon (U.S. Liq)	0.134	ft^3
gallons of water	8.3453	pounds of water
gamma (γ, Γ)	1×10^{-9}	tesla (T)
gauss	1×10^{-4}	T
gram (g)	2.205×10^{-3}	pound (lbm)
hectare	1×10^4	square meters (m^2)
hectare	2.47104	acres
horsepower (hp)	42.4	Btu/min
hp	745.7	watt (W)
hp	33,000	(ft-lbf)/min
hp	550	(ft-lbf)/sec
hp-hr	2,545	Btu
hp-hr	1.98×10^6	ft-lbf
hp-hr	2.68×10^6	joule (J)
hp-hr	0.746	kWh
inch (in.)	2.540	centimeter (cm)
in. of Hg	0.0334	atm
in. of Hg	13.60	in. of H_2O
in. of H_2O	0.0361	lbf/in^2 (psi)
in. of H_2O	0.002458	atm

Multiply	By	To Obtain
joule (J)	9.478×10^{-4}	Btu
J	0.7376	ft-lbf
J	1	newton•m (N•m)
J/s	1	watt (W)
kilogram (kg)	2.205	pound-mass (lbm)
kgf	9.8066	newton (N)
kilometer (km)	3,281	feet (ft)
km/hr	0.621	mph
kilopascal (kPa)	0.145	lbf/in^2 (psi)
kilowatt (kW)	1.341	horsepower (hp)
kW	3,413	Btu/hr
kW	737.6	(ft-lbf)/sec
kW-hour (kWh)	3,413	Btu
kWh	1.341	hp-hr
kWh	3.6×10^6	joule (J)
kip (K)	1,000	lbf
K	4,448	newton (N)
liter (L)	61.02	in^3
L	0.264	gal (U.S. Liq)
L	10^{-3}	m^3
L/second (L/s)	2.119	ft^3/min (cfm)
L/s	15.85	gal (U.S.)/min (gpm)
meter (m)	3.281	feet (ft)
m	1.094	yard
m/second (m/s)	196.8	feet/min (ft/min)
mile (statute)	5,280	feet (ft)
mile (statute)	1.609	kilometer (km)
mile/hour (mph)	88.0	ft/min (fpm)
mph	1.609	km/h
mm of Hg	1.316×10^{-3}	atm
mm of H_2O	9.678×10^{-5}	atm
newton (N)	0.225	lbf
newton (N)	1	kg•m/s^2
N•m	0.7376	ft-lbf
N•m	1	joule (J)
pascal (Pa)	9.869×10^{-6}	atmosphere (atm)
Pa	1	newton/m^2 (N/m^2)
Pa•sec (Pa•s)	10	poise (P)
pound (lbm, avdp)	0.454	kilogram (kg)
lbf	4.448	N
lbf-ft	1.356	N•m
lbf/in^2 (psi)	0.068	atm
psi	2.307	ft of H_2O
psi	2.036	in. of Hg
psi	6,895	Pa
radian (rad)	$180/\pi$	degree
slug	32.174	pound-mass (lbm)
stokes	1×10^{-4}	m^2/s
tesla	1.0	weber/m^2
therm	1×10^5	Btu
ton (metric)	1,000	kilogram (kg)
ton (short)	2,000	pound-force (lbf)
watt (W)	3.413	Btu/hr
W	1.341×10^{-3}	horsepower (hp)
W	1	joule/s (J/s)
weber/m^2 (Wb/m^2)	10,000	gauss

Mathematics

Discrete Math

Symbols

$x \in X$	x is a member of X
$\{\ \}$, ϕ	The empty (or null) set
$S \subseteq T$	S is a subset of T
$S \subset T$	S is a proper subset of T
(a,b)	Ordered pair
$P^{(s)}$	Power set of S
$(a_1, a_2, ..., a_n)$	n-tuple
$A \times B$	Cartesian product of A and B
$A \cup B$	Union of A and B
$A \cap B$	Intersection of A and B
$\forall$ x	Universal qualification for all x; for any x; for each x
$\exists$ y	Uniqueness qualification there exists y

A binary relation from A to B is a subset of $A \times B$.

Matrix of Relation

If $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$ are finite sets containing m and n elements, respectively, then a relation R from A to B can be represented by the $m \times n$ matrix
$M_R < [m_{ij}]$, which is defined by:

$$m_{ij} = \{\ 1 \text{ if } (a_i, b_j) \in R$$
$$0 \text{ if } (a_i, b_j) \notin R\}$$

Directed Graphs, or Digraphs, of Relation

A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). For edge (a, b), the vertex a is called the initial vertex and vertex b is called the terminal vertex. An edge of form (a, a) is called a loop.

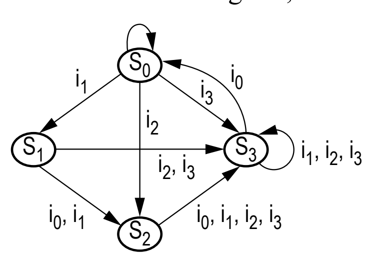
Finite State Machine

A finite state machine consists of a finite set of states
$S_i = \{s_0, s_1, ..., s_n\}$ and a finite set of inputs I; and a transition function f that assigns to each state and input pair a new state.

A state (or truth) table can be used to represent the finite state machine.

	Input			
State	i_0	i_1	i_2	i_3
S_0	S_0	S_1	S_2	S_3
S_1	S_2	S_2	S_3	S_3
S_2	S_3	S_3	S_3	S_3
S_3	S_0	S_3	S_3	S_3

Another way to represent a finite state machine is to use a state diagram, which is a directed graph with labeled edges.



The characteristic of how a function maps one set (X) to another set (Y) may be described in terms of being either injective, surjective, or bijective.

An injective (one-to-one) relationship exists if, and only if,
$\forall\ x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

A surjective (onto) relationship exists when $\forall\ y \in Y, \exists\ x \in X$ such that $f(x) = y$

A bijective relationship is both injective (one-to-one) and surjective (onto).

Straight Line

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is

$$y - y_1 = m(x - x_1)$$

Given two points: slope,

$$m = (y_2 - y_1)/(x_2 - x_1)$$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\left[(m_2 - m_1)/(1 + m_2 \cdot m_1)\right]$$

Two lines are perpendicular if $m_1 = -1/m_2$

The distance between two points is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Quadratic Equation

$$ax^2 + bx + c = 0$$

$$x = \text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadric Surface (SPHERE)

The standard form of the equation is

$$(x - h)^2 + (y - k)^2 + (z - m)^2 = r^2$$

with center at (h, k, m).

In a three-dimensional space, the distance between two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Logarithms

The logarithm of x to the Base b is defined by

$$\log_b(x) = c$$

where $b^c = x$

Special definitions for $b = e$ or $b = 10$ are:

$\ln x$, Base = e

$\log x$, Base = 10

To change from one Base to another:

$$\log_b x = (\log_a x)/(\log_a b)$$

e.g., $\ln x = (\log_{10} x)/(\log_{10} e) = 2.302585\ (\log_{10} x)$

Identities

$\log_b b^n = n$

$\log x^c = c \log x$; $x^c = \text{antilog}(c \log x)$

$\log xy = \log x + \log y$

$\log_b b = 1$; $\log 1 = 0$

$\log x/y = \log x - \log y$

Algebra of Complex Numbers

Complex numbers may be designated in rectangular form or polar form. In rectangular form, a complex number is written in terms of its real and imaginary components.

$z = a + jb$

where

a = real component

b = imaginary component

$j = \sqrt{-1}$ (some disciplines use $i = \sqrt{-1}$)

In polar form $z = c \angle \theta$

where

$c = \sqrt{a^2 + b^2}$

$\theta = \tan^{-1}(b/a)$

$a = c \cos \theta$

$b = c \sin \theta$

Complex numbers can be added and subtracted in rectangular form. If

$z_1 = a_1 + jb_1 = c_1(\cos \theta_1 + j\sin \theta_1) = c_1 \angle \theta_1$ and

$z_2 = a_2 + jb_2 = c_2(\cos \theta_2 + j\sin \theta_2) = c_2 \angle \theta_2$, then

$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2)$ and

$z_1 - z_2 = (a_1 - a_2) + j(b_1 - b_2)$

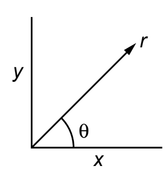
While complex numbers can be multiplied or divided in rectangular form, it is more convenient to perform these operations in polar form.

$z_1 \times z_2 = (c_1 \times c_2) \angle (\theta_1 + \theta_2)$

$z_1/z_2 = (c_1/c_2) \angle (\theta_1 - \theta_2)$

The complex conjugate of a complex number $z_1 = (a_1 + jb_1)$ is defined as $z_1^* = (a_1 - jb_1)$. The product of a complex number and its complex conjugate is $z_1 z_1^* = a_1^2 + b_1^2$.

Polar Coordinate System



$x = r \cos \theta$; $y = r \sin \theta$; $\theta = \arctan(y/x)$

$r = |x + jy| = \sqrt{x^2 + y^2}$

$x + jy = r(\cos \theta + j \sin \theta) = re^{j\theta}$

$[r_1(\cos \theta_1 + j \sin \theta_1)][r_2(\cos \theta_2 + j \sin \theta_2)] = r_1 r_2[\cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)]$

$(x + jy)^n = [r(\cos \theta + j \sin \theta)]^n = r^n(\cos n\theta + j \sin n\theta)$

$$\frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)\right]$$

Euler's Identity

$e^{j\theta} = \cos \theta + j \sin \theta$

$e^{-j\theta} = \cos \theta - j \sin \theta$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \ \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Roots

If k is any positive integer, any complex number (other than zero) has k distinct roots. The k roots of $r (\cos \theta + j \sin \theta)$ can be found by substituting successively $n = 0, 1, 2, ..., (k - 1)$ in the formula

$$w = \sqrt[k]{r}\left[\cos\left(\frac{\theta}{k} + n\frac{360^\circ}{k}\right) + j \sin\left(\frac{\theta}{k} + n\frac{360^\circ}{k}\right)\right]$$

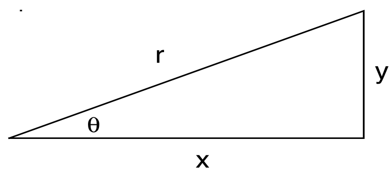
Trigonometry

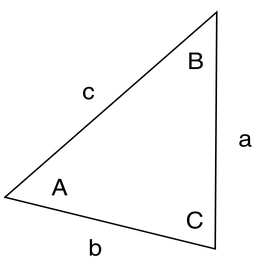
Trigonometric functions are defined using a right triangle.

$\sin \theta = y/r$, $\cos \theta = x/r$

$\tan \theta = y/x$, $\cot \theta = x/y$

$\csc \theta = r/y$, $\sec \theta = r/x$





Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Brink, R.W., *A First Year of College Mathematics*, D. Appleton-Century Co., Inc., Englewood Cliffs, NJ, 1937.

Identities

$\cos\theta = \sin(\theta + \pi/2) = -\sin(\theta - \pi/2)$

$\sin\theta = \cos(\theta - \pi/2) = -\cos(\theta + \pi/2)$

$\csc\theta = 1/\sin\theta$

$\sec\theta = 1/\cos\theta$

$\tan\theta = \sin\theta/\cos\theta$

$\cot\theta = 1/\tan\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = (2\tan\alpha)/(1 - \tan^2\alpha)$

$\cot 2\alpha = (\cot^2\alpha - 1)/(2\cot\alpha)$

$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$

$\cot(\alpha + \beta) = (\cot\alpha\cot\beta - 1)/(\cot\alpha + \cot\beta)$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\tan(\alpha - \beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha\tan\beta)$

$\cot(\alpha - \beta) = (\cot\alpha\cot\beta + 1)/(\cot\beta - \cot\alpha)$

$\sin(\alpha/2) = \pm\sqrt{(1 - \cos\alpha)/2}$

$\cos(\alpha/2) = \pm\sqrt{(1 + \cos\alpha)/2}$

$\tan(\alpha/2) = \pm\sqrt{(1 - \cos\alpha)/(1 + \cos\alpha)}$

$\cot(\alpha/2) = \pm\sqrt{(1 + \cos\alpha)/(1 - \cos\alpha)}$

$\sin\alpha\sin\beta = (1/2)[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

$\cos\alpha\cos\beta = (1/2)[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$

$\sin\alpha\cos\beta = (1/2)[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$\sin\alpha + \sin\beta = 2\sin[(1/2)(\alpha + \beta)]\cos[(1/2)(\alpha - \beta)]$

$\sin\alpha - \sin\beta = 2\cos[(1/2)(\alpha + \beta)]\sin[(1/2)(\alpha - \beta)]$

$\cos\alpha + \cos\beta = 2\cos[(1/2)(\alpha + \beta)]\cos[(1/2)(\alpha - \beta)]$

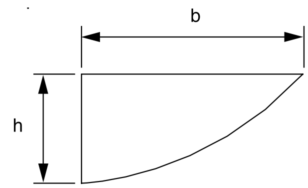
$\cos\alpha - \cos\beta = -2\sin[(1/2)(\alpha + \beta)]\sin[(1/2)(\alpha - \beta)]$

Mensuration of Areas and Volumes

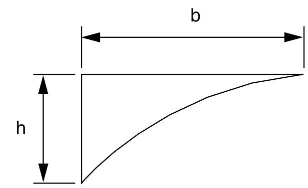
Nomenclature

A = total surface area
P = perimeter
V = volume

Parabola

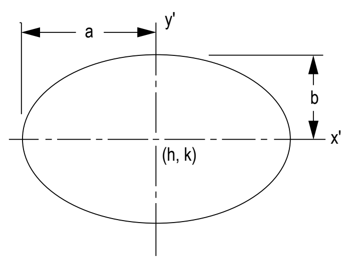


$A = 2bh/3$



$A = bh/3$

Ellipse



$A = \pi ab$

$P_{approx} = 2\pi\sqrt{\left(a^2 + b^2\right)/2}$

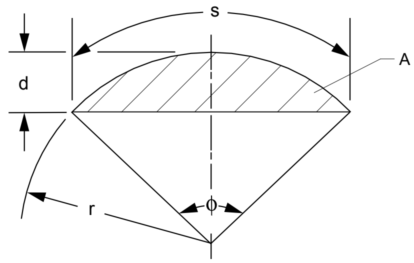
$$P = \pi(a+b)\begin{bmatrix} 1 + (1/2)^2\lambda^2 + (1/2 \times 1/4)^2\lambda^4 \\ +(1/2 \times 1/4 \times 3/6)^2\lambda^6 + (1/2 \times 1/4 \times 3/6 \times 5/8)^2\lambda^8 \\ +(1/2 \times 1/4 \times 3/6 \times 5/8 \times 7/10)^2\lambda^{10} + \ldots \end{bmatrix}$$

where

$\lambda = (a-b)/(a+b)$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

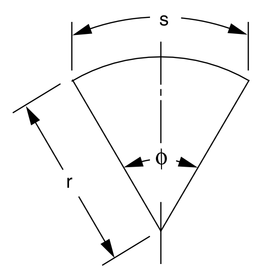
Circular Segment



$$A = \left[r^2\left(\phi - \sin\phi\right)\right]/2$$
$$\phi = s/r = 2\left\{\arccos\left[\left(r - d\right)/r\right]\right\}$$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

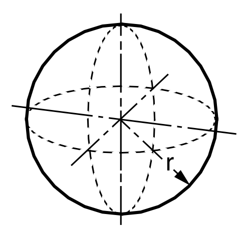
Circular Sector



$$A = \phi r^2/2 = sr/2$$
$$\phi = s/r$$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

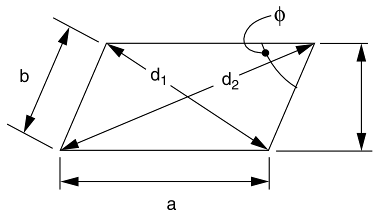
Sphere



$$V = 4\pi r^3/3 = \pi d^3/6$$
$$A = 4\pi r^2 = \pi d^2$$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

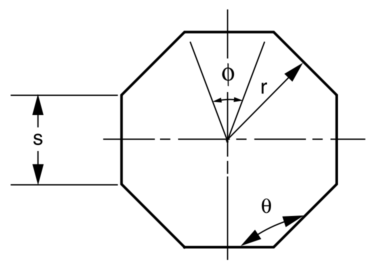
Parallelogram



$$P = 2(a + b)$$
$$d_1 = \sqrt{a^2 + b^2 - 2ab(\cos\phi)}$$
$$d_2 = \sqrt{a^2 + b^2 + 2ab(\cos\phi)}$$
$$d_1^2 + d_2^2 = 2(a^2 + b^2)$$
$$A = ah = ab(\sin\phi)$$

If $a = b$, the parallelogram is a rhombus.

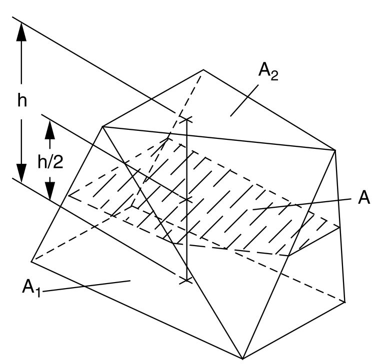
Regular Polygon (*n* equal sides)



$$\phi = 2\pi/n$$
$$\theta = \left[\frac{\pi(n-2)}{n}\right] = \pi\left(1 - \frac{2}{n}\right)$$
$$P = ns$$
$$s = 2r\left[\tan(\phi/2)\right]$$
$$A = (nsr)/2$$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

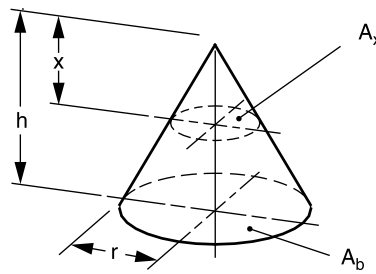
Prismoid



$$V = (h/6)(A_1 + A_2 + 4A)$$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Right Circular Cone



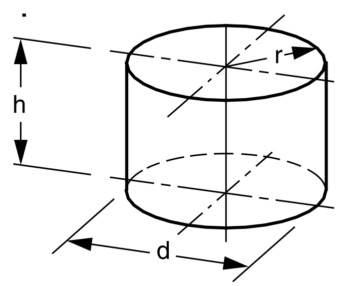
$V = (\pi r^2 h)/3$

$A = \text{side area} + \text{base area} = \pi r\left(r + \sqrt{r^2 + h^2}\right)$

$A_x : A_b = x^2 : h^2$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Right Circular Cylinder

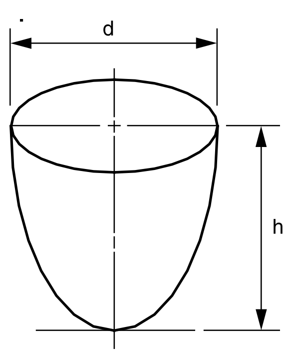


$V = \pi r^2 h = \dfrac{\pi d^2 h}{4}$

$A = \text{side area} + \text{end areas} = 2\pi r(h + r)$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

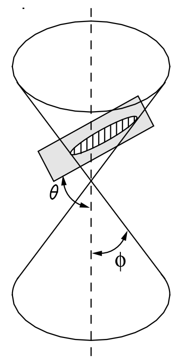
Paraboloid of Revolution



$V = \dfrac{\pi d^2 h}{8}$

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Conic Sections

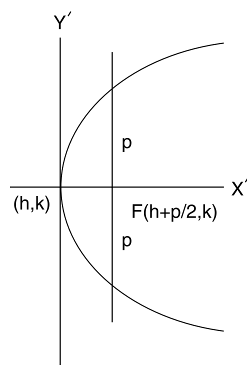


e = eccentricity = $\cos\theta/(\cos\phi)$

[Note: X' and Y', in the following cases, are translated axes.]

Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Case 1. Parabola $e = 1$:

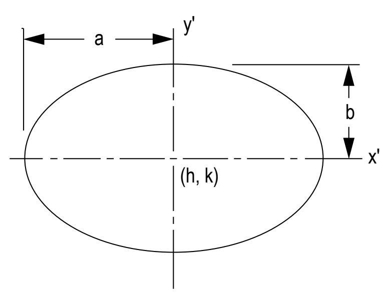


$(y - k)^2 = 2p(x - h)$; Center at (h, k) is the standard form of the equation. When $h = k = 0$,

Focus: $(p/2, 0)$; Directrix: $x = -p/2$

Brink, R.W., *A First Year of College Mathematics,* D. Appleton-Century Company, Inc. (Prentice Hall), 1937.

Case 2. Ellipse $e < 1$:



$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; Center at (h, k) is the standard form of the equation. When $h = k = 0$,

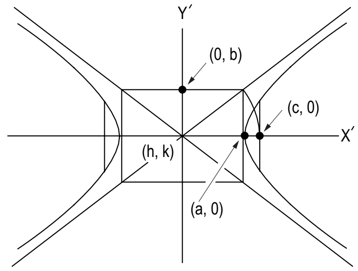
Eccentricity: $e = \sqrt{1 - \left(b^2/a^2\right)} = c/a$

$$b = a\sqrt{1 - e^2};$$

Focus: $(\pm\, ae, 0)$; Directrix: $x = \pm\, a/e$

Brink, R.W., *A First Year of College Mathematics,* D. Appleton-Century Company, Inc. (Prentice Hall), 1937.

Case 3. Hyperbola $e > 1$:



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1;$$

Center at (h, k) is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 + (b^2/a^2)} = c/a$

$$b = a\sqrt{e^2 - 1};$$

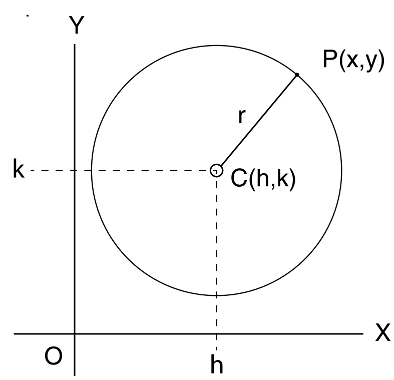
Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Brink, R.W., *A First Year of College Mathematics,* D. Appleton-Century Company, Inc. (Prentice Hall), 1937.

Case 4. Circle $e = 0$:

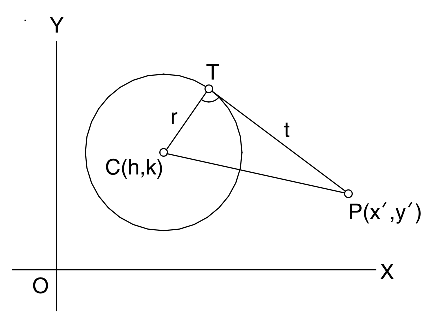
$(x - h)^2 + (y - k)^2 = r^2$; Center at (h, k) is the standard form of the equation with radius

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$



Length of the tangent line from a point on a circle to a point (x', y'):

$$t^2 = (x' - h)^2 + (y' - k)^2 - r^2$$



Brink, R.W., *A First Year of College Mathematics,* D. Appleton-Century Company, Inc. (Prentice Hall), 1937.

Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where not both A and C are zero.

If $B^2 - 4AC < 0$, an ellipse is defined.

If $B^2 - 4AC > 0$, a hyperbola is defined.

If $B^2 - 4AC = 0$, the conic is a parabola.

If $A = C$ and $B = 0$, a circle is defined.

If $A = B = C = 0$, a straight line is defined.

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$$h = -a;\ k = -b$$

$$r = \sqrt{a^2 + b^2 - c}$$

If $a^2 + b^2 - c$ is positive, a circle, center $(-a, -b)$.

If $a^2 + b^2 - c$ equals zero, a point at $(-a, -b)$.

If $a^2 + b^2 - c$ is negative, locus is imaginary.

Differential Calculus

The Derivative

For any function $y = f(x)$, the derivative $= D_x y = dy/dx = y'$

$$y' = \lim_{\Delta x \to 0} \left[(\Delta y)/(\Delta x)\right]$$

$$= \lim_{\Delta x \to 0} \left\{\left[f(x + \Delta x) - f(x)\right]/(\Delta x)\right\}$$

$y' =$ the slope of the curve $f(x)$.

Test for a Maximum

$y = f(x)$ is a maximum for

$x = a$, if $f'(a) = 0$ and $f''(a) < 0$.

Test for a Minimum

$y = f(x)$ is a minimum for

$x = a$, if $f'(a) = 0$ and $f''(a) > 0$.

Test for a Point of Inflection

$y = f(x)$ has a point of inflection at $x = a$,

if $f''(a) = 0$, and

if $f''(x)$ changes sign as x increases through

$x = a$.

The Partial Derivative

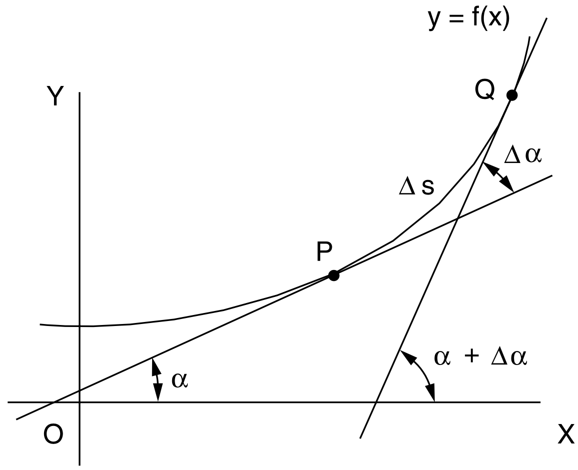
In a function of two independent variables x and y, a derivative with respect to one of the variables may be found if the other variable is *assumed* to remain constant. If *y is kept fixed*, the function

$$z = f(x, y)$$

becomes a function of the *single variable* x, and its derivative (if it exists) can be found. This derivative is called the *partial derivative of z with respect to x*. The partial derivative with respect to x is denoted as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

The Curvature of Any Curve



The curvature K of a curve at P is the limit of its average curvature for the arc PQ as Q approaches P. This is also expressed as: the curvature of a curve at a given point is the rate-of-change of its inclination with respect to its arc length.

$$K = \lim_{\Delta s \to 0} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}$$

Wade, Thomas L., *Calculus*, Boston, Ginn and Company, 1953.

Curvature in Rectangular Coordinates

$$K = \frac{y''}{\left[1 + \left(y'\right)^2\right]^{3/2}}$$

When it may be easier to differentiate the function with respect to y rather than x, the notation x' will be used for the derivative.

$$x' = dx/dy$$

$$K = \frac{-x''}{\left[1 + \left(x'\right)^2\right]^{3/2}}$$

The Radius of Curvature

The *radius of curvature R* at any point on a curve is defined as the absolute value of the reciprocal of the curvature K at that point.

$$R = \frac{1}{|K|} \qquad (K \neq 0)$$

$$R = \left|\frac{\left[1 + \left(y'\right)^2\right]^{3/2}}{|y''|}\right| \qquad (y'' \neq 0)$$

L'Hospital's Rule (L'Hôpital's Rule)

If the fractional function $f(x)/g(x)$ assumes one of the indeterminate forms 0/0 or ∞/∞ (where α is finite or infinite), then

$$\underset{x \to \alpha}{\text{limit}}\, f(x)/g(x)$$

is equal to the first of the expressions

$$\underset{x \to \alpha}{\text{limit}} \frac{f'(x)}{g'(x)}, \underset{x \to \alpha}{\text{limit}} \frac{f''(x)}{g''(x)}, \underset{x \to \alpha}{\text{limit}} \frac{f'''(x)}{g'''(x)}$$

which is not indeterminate, provided such first indicated limit exists.

Integral Calculus

The definite integral is defined as:

$$\underset{n \to \infty}{\text{limit}} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

Also, $\Delta x_i \to 0$ for all i.

A table of derivatives and integrals is available in the Derivatives and Indefinite Integrals sections. The integral equations can be used along with the following methods of integration:

A. Integration by Parts (integral equation #6),

B. Integration by Substitution, and

C. Separation of Rational Fractions into Partial Fractions.

Derivatives

In these formulas, u, v, and w represent functions of x. Also, a, c, and n represent constants. All arguments of the trigonometric functions are in radians. A constant of integration should be added to the integrals. The following definitions are followed: arcsin $u = \sin^{-1} u$, $(\sin u)^{-1} = 1/\sin u$.

1. $dc/dx = 0$
2. $dx/dx = 1$
3. $d(cu)/dx = c\, du/dx$
4. $d(u + v - w)/dx = du/dx + dv/dx - dw/dx$
5. $d(uv)/dx = u\, dv/dx + v\, du/dx$
6. $d(uvw)/dx = uv\, dw/dx + uw\, dv/dx + vw\, du/dx$
7. $\dfrac{d(u/v)}{dx} = \dfrac{v\, du/dx - u\, dv/dx}{v^2}$
8. $d(u^n)/dx = nu^{n-1}\, du/dx$
9. $d[f(u)]/dx = \{d[f(u)]/du\}\, du/dx$
10. $du/dx = 1/(dx/du)$
11. $\dfrac{d(\log_a u)}{dx} = (\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$
12. $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$
13. $\dfrac{d(a^u)}{dx} = (\ln a)a^u\dfrac{du}{dx}$
14. $d(e^u)/dx = e^u\, du/dx$
15. $d(u^v)/dx = vu^{v-1}\, du/dx + (\ln u)\, u^v\, dv/dx$
16. $d(\sin u)/dx = \cos u\, du/dx$
17. $d(\cos u)/dx = -\sin u\, du/dx$
18. $d(\tan u)/dx = \sec^2 u\, du/dx$
19. $d(\cot u)/dx = -\csc^2 u\, du/dx$
20. $d(\sec u)/dx = \sec u \tan u\, du/dx$
21. $d(\csc u)/dx = -\csc u \cot u\, du/dx$
22. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad (-\pi/2 \le \sin^{-1} u \le \pi/2)$
23. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad (0 \le \cos^{-1} u \le \pi)$
24. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad (-\pi/2 < \tan^{-1} u < \pi/2)$
25. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad (0 < \cot^{-1} u < \pi)$
26. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$ $\quad (0 < \sec^{-1} u < \pi/2)(-\pi \le \sec^{-1} u < -\pi/2)$
27. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$ $\quad (0 < \csc^{-1} u \le \pi/2)(-\pi < \csc^{-1} u \le -\pi/2)$

Indefinite Integrals

1. $\int d\,f(x) = f(x)$
2. $\int dx = x$
3. $\int a\,f(x)\,dx = a\int f(x)\,dx$
4. $\int [u(x) \pm v(x)]\,dx = \int u(x)\,dx \pm \int v(x)\,dx$
5. $\int x^m dx = \dfrac{x^{m+1}}{m+1} \qquad (m \neq -1)$
6. $\int u(x)\,dv(x) = u(x)\,v(x) - \int v(x)\,du(x)$
7. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\ln|ax+b|$
8. $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$
9. $\int a^x dx = \dfrac{a^x}{\ln a}$
10. $\int \sin x\,dx = -\cos x$
11. $\int \cos x\,dx = \sin x$
12. $\int \sin^2 x\,dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$
13. $\int \cos^2 x\,dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4}$
14. $\int x\sin x\,dx = \sin x - x\cos x$
15. $\int x\cos x\,dx = \cos x + x\sin x$
16. $\int \sin x\cos x\,dx = (\sin^2 x)/2$
17. $\int \sin ax\cos bx\,dx = -\dfrac{\cos(a-b)x}{2(a-b)} - \dfrac{\cos(a+b)x}{2(a+b)} \qquad (a^2 \neq b^2)$
18. $\int \tan x\,dx = -\ln|\cos x| = \ln|\sec x|$
19. $\int \cot x\,dx = -\ln|\csc x| = \ln|\sin x|$
20. $\int \tan^2 x\,dx = \tan x - x$
21. $\int \cot^2 x\,dx = -\cot x - x$
22. $\int e^{ax} dx = (1/a)\,e^{ax}$
23. $\int xe^{ax} dx = (e^{ax}/a^2)(ax-1)$
24. $\int \ln x\,dx = x\,[\ln(x) - 1] \qquad (x > 0)$
25. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} \qquad (a \neq 0)$
26. $\int \dfrac{dx}{ax^2+c} = \dfrac{1}{\sqrt{ac}}\tan^{-1}\left(x\sqrt{\dfrac{a}{c}}\right) \qquad (a > 0, c > 0)$

27a. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\dfrac{2ax+b}{\sqrt{4ac-b^2}} \qquad (4ac-b^2 > 0)$

27b. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| \qquad (b^2-4ac > 0)$

27c. $\int \dfrac{dx}{ax^2+bx+c} = -\dfrac{2}{2ax+b} \qquad (b^2-4ac = 0)$

Progression and Series

Arithmetic Progression

To determine whether a given finite sequence of numbers is an arithmetic progression, subtract each number from the following number. If the differences are equal, the series is arithmetic.

1. The first term is a.
2. The common difference is d.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = a + (n - 1)d$$

$$S = n(a + l)/2 = n\,[2a + (n - 1)\,d]/2$$

Geometric Progression

To determine whether a given finite sequence is a geometric progression (G.P.), divide each number after the first by the preceding number. If the quotients are equal, the series is geometric:

1. The first term is a.
2. The common ratio is r.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = ar^{n-1}$$

$$S = a\,(1 - r^n)/(1 - r);\ r \neq 1$$

$$S = (a - rl)/(1 - r);\ r \neq 1$$

$$\lim_{n \to \infty} S_n = a/(1 - r);\ r < 1$$

A G.P. converges if $|r| < 1$ and it diverges if $|r| > 1$.

Properties of Series

$$\sum_{i=1}^{n} c = nc; \qquad c = \text{constant}$$

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} \left(x_i + y_i - z_i\right) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i$$

$$\sum_{x=1}^{n} x = \left(n + n^2\right)/2$$

$$\prod_{i=1}^{n} x_i = x_1 x_2 x_3 \ldots x_n$$

Power Series

$$\sum_{i=0}^{\infty} a_i (x - a)^i$$

1. A power series, which is convergent in the interval $-R < x < R$, defines a function of x that is continuous for all values of x within the interval and is said to represent the function in that interval.
2. A power series may be differentiated term by term within its interval of convergence. The resulting series has the same interval of convergence as the original series (except possibly at the end points of the series).
3. A power series may be integrated term by term provided the limits of integration are within the interval of convergence of the series.
4. Two power series may be added, subtracted, or multiplied, and the resulting series in each case is convergent, at least, in the interval common to the two series.
5. Using the process of long division (as for polynomials), two power series may be divided one by the other within their common interval of convergence.

Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

is called *Taylor's series*, and the function $f(x)$ is said to be expanded about the point a in a Taylor's series.

If $a = 0$, the Taylor's series equation becomes a *Maclaurin's series*.

Differential Equations

A common class of ordinary linear differential equations is

$$b_n \frac{d^n y(x)}{dx^n} + \ldots + b_1 \frac{dy(x)}{dx} + b_0 y(x) = f(x)$$

where $b_n, \ldots, b_i, \ldots, b_1, b_0$ are constants.

When the equation is a homogeneous differential equation, $f(x) = 0$, the solution is

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \ldots + C_i e^{r_i x} + \ldots + C_n e^{r_n x}$$

where r_n is the nth distinct root of the characteristic polynomial $P(x)$ with

$$P(r) = b_n r^n + b_{n-1} r^{n-1} + \ldots + b_1 r + b_0$$

If the root $r_1 = r_2$, then $C_2 e^{r_2 x}$ is replaced with $C_2 x e^{r_1 x}$.

Higher orders of multiplicity imply higher powers of x. The complete solution for the differential equation is

$$y(x) = y_h(x) + y_p(x),$$

where $y_p(x)$ is any particular solution with $f(x)$ present. If $f(x)$ has $e^{r_n x}$ terms, then resonance is manifested.

Furthermore, specific $f(x)$ forms result in specific $y_p(x)$ forms, some of which are:

$f(x)$	$y_p(x)$
A	B
$Ae^{\alpha x}$	$Be^{\alpha x}, \alpha \neq r_n$
$A_1 \sin \omega x + A_2 \cos \omega x$	$B_1 \sin \omega x + B_2 \cos \omega x$

If the independent variable is time t, then transient dynamic solutions are implied.

First-Order Linear Homogeneous Differential Equations with Constant Coefficients

$$y' + ay = 0$$

where a is a real constant:

Solution, $y = Ce^{-at}$

where C = a constant that satisfies the initial conditions.

First-Order Linear Nonhomogeneous Differential Equations

$$\tau \frac{dy}{dt} + y = Kx(t) \qquad x(t) = \begin{Bmatrix} A & t < 0 \\ B & t > 0 \end{Bmatrix}$$

$$y(0) = KA$$

τ = time constant

K = gain

The solution is

$$y(t) = KA + (KB - KA)\left(1 - \exp\left(\frac{-t}{\tau}\right)\right) \text{ or}$$

$$\frac{t}{\tau} = 1\text{n}\left[\frac{KB - KA}{KB - y}\right]$$

Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

An equation of the form

$$y'' + ay' + by = 0$$

can be solved by the method of undetermined coefficients where a solution of the form $y = Ce^{rx}$ is sought. Substitution of this solution gives

$$(r^2 + ar + b)\, Ce^{rx} = 0$$

and since Ce^{rx} cannot be zero, the characteristic equation must vanish or

$$r^2 + ar + b = 0$$

The roots of the characteristic equation are

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

and can be real and distinct for $a^2 > 4b$, real and equal for $a^2 = 4b$, and complex for $a^2 < 4b$.

If $a^2 > 4b$, the solution is of the form (overdamped)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $a^2 = 4b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2 x) e^{r_1 x}$$

If $a^2 < 4b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x), \text{ where}$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

Fourier Transform

The Fourier transform pair, one form of which is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt$$

$$f(t) = [1/(2\pi)] \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

can be used to characterize a broad class of signal models in terms of their frequency or spectral content. Some useful transform pairs are:

f(t)	***F*(ω)**
$\delta(t)$	1
$u(t)$	$\pi\delta(\omega) + 1/j\omega$
$u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) = r_{rect}\frac{t}{\tau}$	$\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}$
$e^{j\omega_o t}$	$2\pi\delta(\omega - \omega_o)$

Some mathematical liberties are required to obtain the second and fourth form. Other Fourier transforms are derivable from the Laplace transform by replacing s with $j\omega$ provided

$$f(t) = 0, t < 0$$

$$\int_0^{\infty} |f(t)| dt < \infty$$

Fourier Series

Every periodic function $f(t)$ which has the period $T = 2\pi/\omega_0$ and has certain continuity conditions can be represented by a series plus a constant

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right)\right]$$

The above holds if $f(t)$ has a continuous derivative $f'(t)$ for
all t. It should be noted that the various sinusoids present in the series are orthogonal on the interval 0 to T and as a result the coefficients are given by

$$a_0 = (1/T)\int_0^T f(t)dt$$

$$a_n = (2/T)\int_0^T f(t)\cos(n\omega_0 t)dt \qquad n = 1, 2, \ldots$$

$$b_n = (2/T)\int_0^T f(t)\sin(n\omega_0 t)dt \qquad n = 1, 2, \ldots$$

The constants a_n and b_n are the *Fourier coefficients* of $f(t)$ for the interval 0 to T and the corresponding series is called the *Fourier series of* $f(t)$ over the same interval.

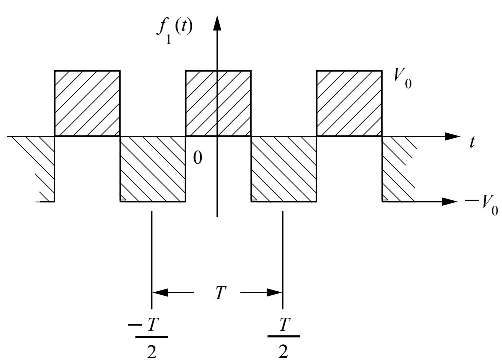
The integrals have the same value when evaluated over any interval of length T.

If a Fourier series representing a periodic function is truncated after term $n = N$, the mean square value F_N^2 of the truncated series is given by Parseval's relation. This relation says that the mean-square value is the sum of the mean-square values of the Fourier components, or

$$F_N^2 = a_0^2 + (1/2)\sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)$$

and the RMS value is then defined to be the square root of this quantity or F_N.

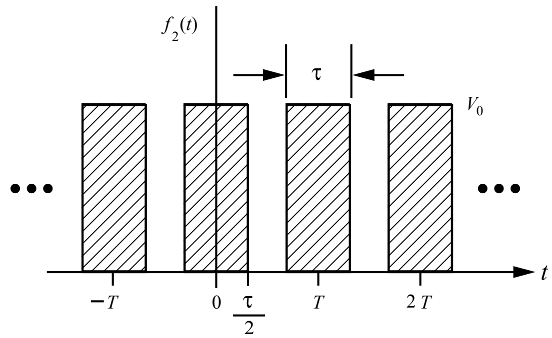
Three useful and common Fourier series forms are defined in terms of the following graphs (with $\omega_0 = 2\pi/T$).
Given:



then

$$f_1(t) = \sum_{\substack{n=1 \\ (\text{n odd})}}^{\infty} (-1)^{(n-1)/2}\left(4V_0/n\pi\right)\cos\left(n\omega_0 t\right)$$

Given:

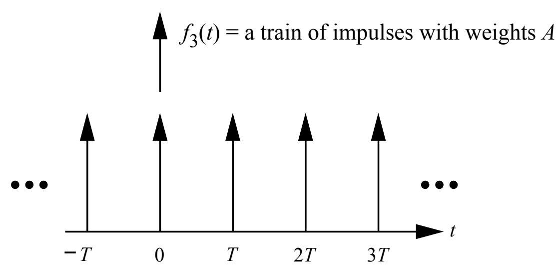


then

$$f_2(t) = \frac{V_0\tau}{T} + \frac{2V_0\tau}{T}\sum_{n=1}^{\infty}\frac{\sin(n\pi\tau/T)}{(n\pi\tau/T)}\cos(n\omega_0 t)$$

$$f_2(t) = \frac{V_0\tau}{T}\sum_{n=-\infty}^{\infty}\frac{\sin(n\pi\tau/T)}{(n\pi\tau/T)}e^{jn\omega_0 t}$$

Given:



then

$$f_3(t) = \sum_{n=-\infty}^{\infty} A\delta(t - nT)$$

$$f_3(t) = (A/T) + (2A/T)\sum_{n=1}^{\infty}\cos(n\omega_0 t)$$

$$f_3(t) = (A/T)\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

The Fourier Transform and its Inverse

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df$$

We say that $x(t)$ and $X(f)$ form a *Fourier transform pair*:

$x(t) \leftrightarrow X(f)$

Fourier Transform Pairs

$x(t)$	$X(f)$
1	$\delta(f)$
$\delta(t)$	1
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\Pi\left(\frac{t}{\tau}\right)$	$\tau\,\mathrm{sinc}\,(\tau f)$
$\mathrm{sinc}\,(Bt)$	$\frac{1}{B}\Pi\left(\frac{f}{B}\right)$
$\Lambda\left(\frac{t}{\tau}\right)$	$\tau\,\mathrm{sinc}^2(\tau f)$
$e^{-at}u(t)$	$\frac{1}{a+j2\pi f} \qquad a>0$
$te^{-at}u(t)$	$\frac{2a}{a^2+(2\pi f)^2} \qquad a>0$
$e^{-a\|t\|}$	$\frac{2a}{a^2+(2\pi f)^2} \qquad a>0$
$e^{-(at)^2}$	$\frac{\sqrt{\pi}}{a}e^{-\left(\frac{\pi f}{a}\right)^2}$
$\cos(2\pi f_0 t+\theta)$	$\frac{1}{2}\left[e^{j\theta}\delta(f-f_0)+e^{-j\theta}\delta(f+f_0)\right]$
$\sin(2\pi f_0 t+\theta)$	$\frac{1}{2j}\left[e^{j\theta}\delta(f-f_0)-e^{-j\theta}\delta(f+f_0)\right]$
$\sum_{n=-\infty}^{n=+\infty}\delta(t-nT_s)$	$f_s\sum_{k=-\infty}^{k=+\infty}\delta(f-kf_s) \qquad f_s=\frac{1}{T_s}$

Fourier Transform Theorems

Linearity	$ax(t)+by(t)$	$aX(f)+bY(f)$
Scale change	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{f}{a}\right)$
Time reversal	$x(-t)$	$X(-f)$
Duality	$X(t)$	$x(-f)$
Time shift	$x(t-t_0)$	$X(f)e^{-j2\pi f t_0}$
Frequency shift	$x(t)e^{j2\pi f_0 t}$	$X(f-f_0)$
Modulation	$x(t)\cos 2\pi f_0 t$	$\frac{1}{2}X(f-f_0)$ $+\frac{1}{2}X(f+f_0)$
Multiplication	$x(t)y(t)$	$X(f)*Y(f)$
Convolution	$x(t)*y(t)$	$X(f)Y(f)$
Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
Integration	$\int_{-\infty}^{t}x(\lambda)d\lambda$	$\frac{1}{j2\pi f}X(f)$ $+\frac{1}{2}X(0)\delta(f)$

where:

$$\mathrm{sinc}(t)=\frac{\sin(\pi t)}{\pi t}$$

$$\Pi(t)=\begin{cases}1, & |t|\le\frac{1}{2}\\ 0, & \text{otherwise}\end{cases}$$

$$\Lambda(t)=\begin{cases}1-|t|, & |t|\le 1\\ 0, & \text{otherwise}\end{cases}$$

Laplace Transforms

The unilateral Laplace transform pair

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

where $s = \sigma + j\omega$

represents a powerful tool for the transient and frequency response of linear time invariant systems. Some useful Laplace transform pairs are:

Laplace Transform Pairs

f(t)	*F(s)*
$\delta(t)$, Impulse at $t = 0$	1
$u(t)$, Step at $t = 0$	$\frac{1}{s}$
$t[u(t)]$, Ramp at $t = 0$	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{(s+a)}$
te^{-at}	$\frac{1}{(s+\alpha)^2}$
$e^{-\alpha t} \sin \beta t$	$\frac{\beta}{\left[(s+\alpha)^2 + \beta^2\right]}$
$e^{-\alpha t} \cos \beta t$	$\frac{(s+\alpha)}{\left[(s+\alpha)^2 + \beta^2\right]}$
$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - \sum_{m=0}^{n-1} s^{n-m-1} \frac{d^m f(0)}{dt^m}$
$\int_0^t f(\tau) d\tau$	$\left(\frac{1}{s}\right) F(s)$
$\int_0^t x(t-\tau) h(\tau) d\tau$	$H(s) X(s)$
$f(t-\tau) u(t-\tau)$	$e^{-\tau s} F(s)$
$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s)$
$\lim_{t \to 0} f(t)$	$\lim_{s \to \infty} sF(s)$

The last two transforms represent the Final Value Theorem (F.V.T.) and Initial Value Theorem (I.V.T.), respectively. It is assumed that the limits exist.

Matrices

A matrix is an ordered rectangular array of numbers with m rows and n columns. The element a_{ij} refers to row i and column j.

The rank of a matrix is equal to the number of rows that are linearly independent.

Multiplication of Two Matrices

$$A = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \quad A_{3,2} \text{ is a 3-row, 2-column matrix}$$

$$B = \begin{bmatrix} H & I \\ J & K \end{bmatrix} \quad B_{2,2} \text{ is a 2-row, 2-column matrix}$$

In order for multiplication to be possible, the number of columns in A must equal the number of rows in B.

Multiplying matrix B by matrix A occurs as follows:

$$C = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \cdot \begin{bmatrix} H & I \\ J & K \end{bmatrix}$$

$$C = \begin{bmatrix} (A \cdot H + B \cdot J) & (A \cdot I + B \cdot K) \\ (C \cdot H + D \cdot J) & (C \cdot I + D \cdot K) \\ (E \cdot H + F \cdot J) & (E \cdot I + F \cdot K) \end{bmatrix}$$

Matrix multiplication is not commutative.

Addition of Two Matrices

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} + \begin{bmatrix} G & H & I \\ J & K & L \end{bmatrix} = \begin{bmatrix} A+G & B+H & C+I \\ D+J & E+K & F+L \end{bmatrix}$$

Identity Matrix

The matrix $\mathbf{I} = (a_{ij})$ is a square $n \times n$ matrix with 1's on the diagonal and 0's everywhere else.

Matrix Transpose

Rows become columns. Columns become rows.

$$A = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \quad A^{T} = \begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

Inverse $[\ ]^{-1}$

The inverse $\boldsymbol{B}$ of a square $n \times n$ matrix $\boldsymbol{A}$ is

$$\boldsymbol{B} = \boldsymbol{A}^{-1} = \frac{\text{adj}(\boldsymbol{A})}{|\boldsymbol{A}|}$$

where

adj($\boldsymbol{A}$) = adjoint of $\boldsymbol{A}$ (obtained by replacing $\boldsymbol{A}^{T}$ elements with their cofactors)

$|\boldsymbol{A}|$= determinant of $\boldsymbol{A}$

$\mathbf{[A][A]^{-1} = [A]^{-1}[A] = [I]}$

where **I** is the identity matrix.

Matrix Properties

Suppose A is N × N over real numbers. Then if one of the following is true, all are true. If one of the following is false, all are false.

1. A is nonsingular.
2. A has an inverse.
3. A*X = 0 has a unique solution.
4. Determinant of A is not equal to zero.
5. Columns of A are linearly independent.
6. Rows of A are linearly independent.
7. Rank of A is N.
8. A is row equivalent to I (identity matrix).
9. Null Space of A = {0}.

Cullen, C., *Matrices and Linear Transformations*. Reading, Massachusetts: Addison-Wesley, 1967.

Determinants

A *determinant of order n* consists of n^2 numbers, called the *elements* of the determinant, arranged in *n* rows and *n* columns and enclosed by two vertical lines.

In any determinant, the *minor* of a given element is the determinant that remains after all of the elements are struck out that lie in the same row and in the same column as the given element. Consider an element which lies in the *j*th column and the *i*th row. The *cofactor* of this element is the value of the minor of the element (if $i + j$ is *even*), and it is the negative of the value of the minor of the element (if $i + j$ is *odd*).

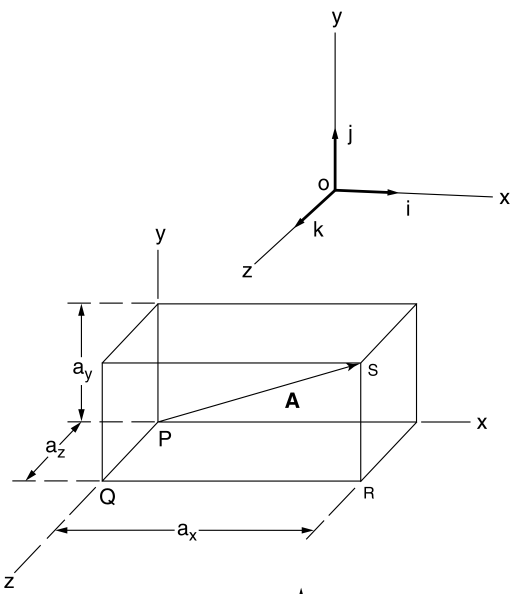
If *n* is greater than 1, the *value* of a determinant of order *n* is the sum of the *n* products formed by multiplying each element of some specified row (or column) by its cofactor. This sum is called the *expansion of the determinant* [according to the elements of the specified row (or column)]. For a second-order determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

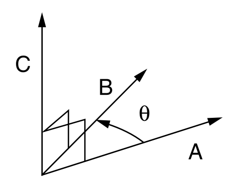
For a third-order determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

Vectors



$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$



Addition and *subtraction:*

$\mathbf{A} + \mathbf{B} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$

$\mathbf{A} - \mathbf{B} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k}$

The *dot product* is a *scalar product* and represents the projection of **B** onto **A** times |**A**|. It is given by

$\mathbf{A}\bullet\mathbf{B} = a_xb_x + a_yb_y + a_zb_z = |\mathrm{A}||\mathbf{B}|\cos\theta = \mathbf{B}\bullet\mathbf{A}$

The *cross product* is a *vector product* of magnitude |**B**| |**A**| sin θ which is perpendicular to the plane containing **A** and **B**. The product is

$$\mathrm{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\ \mathbf{B} \times \mathrm{A}$$

The sense of **A** × **B** is determined by the right-hand rule.

$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\mathbf{n}\ \sin\theta$

where

n = unit vector perpendicular to the plane of **A** and **B**

Gradient, Divergence, and Curl

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi$$

$$\nabla \bullet \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\bullet\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

$$\nabla \times \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times\left(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}\right)$$

The Laplacian of a scalar function ϕ is

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Identities

$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A}$; $\mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$

$\mathbf{A} \bullet \mathbf{A} = |\mathbf{A}|^2$

$\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$

$\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$

If $\mathbf{A} \bullet \mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$

$(\mathbf{B} + \mathbf{C}) \times \mathbf{A} = (\mathbf{B} \times \mathbf{A}) + (\mathbf{C} \times \mathbf{A})$

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

$\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j}$

$\mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k}$

If $\mathbf{A} \times \mathbf{B} = \mathbf{0}$, then either $\mathbf{A} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$, or $\mathbf{A}$ is parallel to $\mathbf{B}$.

$\nabla^2 \phi = \nabla \bullet (\nabla \phi) = (\nabla \bullet \nabla) \phi$

$\nabla \times \nabla \phi = \mathbf{0}$

$\nabla \bullet (\nabla \times \mathbf{A}) = \mathbf{0}$

$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A}$

Numerical Methods

Difference Equations

Any system whose input $v(t)$ and output $y(t)$ are defined only at the equally spaced intervals

$$f(t) = y' = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$$

can be described by a difference equation.

First-Order Linear Difference Equation

$$\Delta t = t_{i+1} - t_i$$

$$y_{i+1} = y_i + y'(\Delta t)$$

Newton's Method for Root Extraction

Given a function $f(x)$ which has a simple root of $f(x) = 0$ at $x = a,$ an important computational task would be to find that root. If $f(x)$ has a continuous first derivative then the $(j + 1)$st estimate of the root is

$$a^{j+1} = a^j - \left. \frac{f(x)}{\dfrac{df(x)}{dx}} \right|_{x = a^j}$$

The initial estimate of the root a^0 must be near enough to the actual root to cause the algorithm to converge to the root.

Newton's Method of Minimization

Given a scalar value function

$h(\boldsymbol{x}) = h(x_1, x_2, \ldots, x_n)$

find a vector $\boldsymbol{x}^* \in R_n$ such that

$h(\boldsymbol{x}^*) \le h(\boldsymbol{x})$ for all $\boldsymbol{x}$

Newton's algorithm is

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \left(\left. \frac{\partial^2 h}{\partial x^2} \right|_{\boldsymbol{x} = \boldsymbol{x}_k} \right)^{-1} \left. \frac{\partial h}{\partial x} \right|_{\boldsymbol{x} = \boldsymbol{x}_k}, \text{ where}$$

$$\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \\ \ldots \\ \ldots \\ \frac{\partial h}{\partial x_n} \end{bmatrix}$$

and

$$\frac{\partial^2 h}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 h}{\partial x_1 \partial x_2} & \ldots & \ldots & \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 h}{\partial x_2^2} & \ldots & \ldots & \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial^2 h}{\partial x_1 \partial x_n} & \frac{\partial^2 h}{\partial x_2 \partial x_n} & \ldots & \ldots & \frac{\partial^2 h}{\partial x_n^2} \end{bmatrix}$$

Numerical Integration

Three of the more common numerical integration algorithms used to evaluate the integral

$$\int_a^b f(x)\,dx$$

are:

Euler's or Forward Rectangular Rule

$$\int_a^b f(x)\,dx \approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x)$$

Trapezoidal Rule

for $n = 1$

$$\int_a^b f(x)\,dx \approx \Delta x \left[\frac{f(a) + f(b)}{2} \right]$$

for $n > 1$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2} \left[f(a) + 2 \sum_{k=1}^{n-1} f(a + k\Delta x) + f(b) \right]$$

Simpson's Rule/Parabolic Rule (n must be an even integer)
for $n = 2$

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

for $n \geq 4$

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\begin{bmatrix} f(a) + 2 \sum\limits_{k=2,4,6,\ldots}^{n-2} f(a + k\Delta x) \\ +4 \sum\limits_{k=1,3,5,\ldots}^{n-1} f(a + k\Delta x) + f(b) \end{bmatrix}$$

with $\Delta x = (b - a)/n$
n = number of intervals between data points

Numerical Solution of Ordinary Differential Equations

Euler's Approximation
Given a differential equation

$dx/dt = f(x, t)$ with $x(0) = x_o$

At some general time $k\Delta t$

$x[(k + 1)\Delta t] \cong x(k\Delta t) + \Delta t f[x(k\Delta t), k\Delta t]$

which can be used with starting condition x_o to solve recursively for $x(\Delta t), x(2\Delta t), \ldots, x(n\Delta t)$.
The method can be extended to nth order differential equations by recasting them as n first-order equations.
In particular, when $dx/dt = f(x)$

$x[(k + 1)\Delta t] \cong x(k\Delta t) + \Delta t f[x(k\Delta t)]$

which can be expressed as the recursive equation

$x_{k+1} = x_k + \Delta t\,(dx_k/dt)$

$x_{k+1} = x + \Delta t\,[f(x(k), t(k))]$

Engineering Probability and Statistics

Dispersion, Mean, Median, and Mode Values

If $X_1, X_2, \ldots, X_n$ represent the values of a random sample of n items or observations, the *arithmetic mean* of these items or observations, denoted $\overline{X}$, is defined as

$$\overline{X} = (1/n)(X_1 + X_2 + \ldots + X_n) = (1/n) \sum_{i=1}^{n} X_i$$

$$\overline{X} \to \mu \text{ for sufficiently large values of } n.$$

The *weighted arithmetic mean* is

$$\overline{X}_w = \frac{\sum w_i X_i}{\sum w_i}$$

where

X_i = the value of the ith observation, and

w_i = the weight applied to X_i.

The *variance* of the population is the *arithmetic mean* of the *squared deviations from the population mean*. If μ is the arithmetic mean of a discrete population of size N, the *population variance* is defined by

$$\sigma^2 = (1/N)\left[(X_1 - \mu)^2 + (X_2 - \mu)^2 + \ldots + (X_N - \mu)^2\right]$$

$$= (1/N) \sum_{i=1}^{N} (X_i - \mu)^2$$

Standard deviation formulas (assuming statistical independence) are

$$\sigma_{\text{population}} = \sqrt{(1/N)\sum\left(X_i - \mu\right)^2}$$

$$\sigma_{\text{sum}} = \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2}$$

$$\sigma_{\text{series}} = \sigma\sqrt{n}$$

$$\sigma_{\text{mean}} = \frac{\sigma}{\sqrt{n}}$$

$$\sigma_{\text{product}} = \sqrt{\mathrm{A}^2\sigma_\mathrm{b}^2 + \mathrm{B}^2\sigma_\mathrm{a}^2}$$

The *sample variance* is

$$s^2 = \left[1/(n-1)\right] \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$$

The *sample standard deviation* is

$$s = \sqrt{\left[1/(n-1)\right] \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2}$$

The *sample coefficient of variation* = $CV = s/\overline{X}$

The *sample geometric mean* = $\sqrt[n]{X_1 X_2 X_3 \ldots X_n}$

The *sample root-mean-square value* = $\sqrt{(1/n)\sum X_i^2}$

When the discrete data are rearranged in increasing order and n is odd, the median is the value of the $\left(\frac{n+1}{2}\right)^{\text{th}}$ item

When n is even, the median is the average of the $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2} + 1\right)^{\text{th}}$ items.

The *mode* of a set of data is the value that occurs with greatest frequency.

The *sample range R* is the largest sample value minus the smallest sample value.

Permutations and Combinations

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of n distinct objects *taken r at a time* is

$$P(n,r) = \frac{n!}{(n-r)!}$$

   nPr is an alternative notation for $P(n,r)$

2. The number of different *combinations* of n distinct objects *taken r at a time* is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{[r!(n-r)!]}$$

   nCr and $\binom{n}{r}$ are alternative notations for $C(n,r)$

3. The number of different *permutations* of n objects *taken n at a time*, given that n_i are of type i, where $i = 1, 2, \ldots, k$ and $\sum n_i = n$, is

$$P(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1! n_2! \ldots n_k!}$$

Sets

De Morgan's Law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Associative Law

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Laws of Probability

Property 1. General Character of Probability

The probability $P(E)$ of an event E is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

Property 2. Law of Total Probability

$$P(A + B) = P(A) + P(B) - P(A, B)$$

where

$P(A + B)$ = the probability that either A or B occur alone or that both occur together

$P(A)$ = the probability that A occurs

$P(B)$ = the probability that B occurs

$P(A, B)$ = the probability that both A and B occur simultaneously

Property 3. Law of Compound or Joint Probability

If neither $P(A)$ nor $P(B)$ is zero,

$$P(A, B) = P(A)P(B \mid A) = P(B)P(A \mid B)$$

where

$P(B \mid A)$ = the probability that B occurs given the fact that A has occurred

$P(A \mid B)$ = the probability that A occurs given the fact that B has occurred

If either $P(A)$ or $P(B)$ is zero, then $P(A, B) = 0$.

Bayes' Theorem

$$P\left(B_j \middle| A\right) = \frac{P\left(B_j\right)P\left(A \middle| B_j\right)}{\sum_{i=1}^{n} P\left(A \middle| B_i\right)P\left(B_i\right)}$$

where

$P\left(A_j\right)$ = the probability of event A_j within the population of A

$P\left(B_j\right)$ = the probability of event B_j within the population of B

Probability Functions, Distributions, and Expected Values

A random variable X has a probability associated with each of its possible values. The probability is termed a discrete probability if X can assume only discrete values, or

$$X = x_1, x_2, x_3, \ldots, x_n$$

The *discrete probability* of any single event, $X = x_i$, occurring is defined as $P(x_i)$ while the *probability mass function* of the random variable X is defined by

$$f(x_k) = P(X = x_k), k = 1, 2, ..., n$$

Probability Density Function

If X is continuous, the *probability density function, f,* is defined such that

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Cumulative Distribution Functions

The *cumulative distribution function, F,* of a discrete random variable X that has a probability distribution described by $P(x_i)$ is defined as

$$F(x_m) = \sum_{k=1}^{m} P(x_k) = P(X \leq x_m), m = 1, 2, \ldots, n$$

If X is continuous, the *cumulative distribution function, F,* is defined by

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

which implies that $F(a)$ is the probability that $X \leq a$.

Expected Values

Let X be a discrete random variable having a probability mass function

$$f(x_k), k = 1, 2,..., n$$

The expected value of X is defined as

$$\mu = E[X] = \sum_{k=1}^{n} x_k f(x_k)$$

The variance of X is defined as

$$\sigma^2 = V[X] = \sum_{k=1}^{n} (x_k - \mu)^2 f(x_k)$$

Let X be a continuous random variable having a density function $f(X)$ and let $Y = g(X)$ be some general function. The expected value of Y is:

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

The mean or expected value of the random variable X is now defined as

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

while the variance is given by

$$\sigma^2 = V[X] = E\left[(X-\mu)^2\right] = E[x^2] - \mu^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

The standard deviation is given by

$$\sigma = \sqrt{V[X]}$$

The coefficient of variation is defined as σ/μ.

Combinations of Random Variables

$$Y = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$$

The expected value of Y is:

$$\mu_y = E(Y) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$$

If the random variables are statistically *independent*, then the variance of Y is:

$$\sigma_y^2 = V(Y) = a_1^2 V(X_1) + a_2^2 V(X_2) + \ldots + a_n^2 V(X_n)$$
$$= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \ldots + a_n^2 \sigma_n^2$$

Also, the standard deviation of Y is:

$$\sigma_y = \sqrt{\sigma_y^2}$$

When $Y = f(X_1, X_2, .., X_n)$ and X_i are independent, the standard deviation of Y is expressed as:

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial X_1}\sigma_{X_1}\right)^2 + \left(\frac{\partial f}{\partial X_2}\sigma_{X_2}\right)^2 + \ldots + \left(\frac{\partial f}{\partial X_n}\sigma_{X_n}\right)^2}$$

Binomial Distribution

$P(x)$ is the probability that x successes will occur in n trials.

If p = probability of success and q = probability of failure = $1 - p$, then

$$P_n(x) = C(n,x)\, p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

where

x = 0, 1, 2, …, n

$C(n, x)$ = number of combinations

n, p = parameters

The variance is given by the form:

$\sigma^2 = npq$

Normal Distribution (Gaussian Distribution)

This is a unimodal distribution, the mode being $x = \mu$, with two points of inflection (each located at a distance σ to either side of the mode). The averages of n observations tend to become normally distributed as n increases. The variate x is said to be normally distributed if its density function $f(x)$ is given by an expression of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where

μ = population mean

σ = standard deviation of the population

$-\infty \le x \le \infty$

When $\mu = 0$ and $\sigma^2 = \sigma = 1$, the distribution is called a *standardized* or *unit normal* distribution. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ where } -\infty \le x \le \infty.$$

A unit normal distribution table is included at the end of this section. In the table, the following notations are utilized:

$F(x)$ = area under the curve from $-\infty$ to x

$R(x)$ = area under the curve from x to ∞

$W(x)$ = area under the curve between $-x$ and x

$F(-x) = 1 - F(x)$

It should be noted that for any normal distribution with mean μ and standard deviation σ, the table for the unit normal distribution can be used by utilizing the following transformation:

$$z = \frac{x - \mu}{\sigma}$$

$f(x)$ then becomes $f(z)$, $F(x)$ becomes $F(z)$, etc.

The Central Limit Theorem

Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. Then for large n, the Central Limit Theorem asserts that the sum

$Y = X_1 + X_2 + ... X_n$ is approximately normal.

$$\mu_{\bar{y}} = \mu$$

and the standard deviation

$$\sigma_{\bar{y}} = \frac{\sigma}{\sqrt{n}}$$

t-Distribution

Student's *t*-distribution has the probability density function given by:

$$f(t)=\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{t^2}{v}\right)^{-\frac{v+1}{2}}$$

where

v = number of degrees of freedom

n = sample size

$v = n - 1$

Γ = gamma function

$t = \dfrac{\bar{x}-\mu}{s/\sqrt{n}}$

$-\infty \le t \le \infty$

A table later in this section gives the values of $t_{\alpha,\,v}$ for values of α and v. Note that, in view of the symmetry of the *t*-distribution, $t_{1-\alpha,v} = -t_{\alpha,v}$

The function for α follows:

$$\alpha = \int_{t_{\alpha,v}}^{\infty} f(t)\,dt$$

χ^2 - Distribution

If $Z_1, Z_2, \ldots, Z_n$ are independent unit normal random variables, then

$$\chi^2 = Z_1^2 + Z_2^2 + \ldots + Z_n^2$$

is said to have a chi-square distribution with n degrees of freedom.

A table at the end of this section gives values of $\chi^2_{\alpha,n}$ for selected values of α and n.

Gamma Function

$$\Gamma(n) = \int_0^{\infty} t^{n-1}e^{-t}dt,\ n > 0$$

Propagation of Error

Measurement Error

Measurement error is defined as: *Measured quantity value minus a reference quantity value*. [Source: ISO JCGM 200:2012 definition 2.16]

Sources of errors in measurements arise from imperfections and disturbances in the measurement process, and added noise. One may model a measurement as:

$$x = x_{\text{ref}} + d_{\text{systematic}} + d_{\text{random}}$$

where x is the measurand (value being measured), x_{ref} is the reference value, $d_{\text{systematic}}$ is a disturbance from the measurement process such as a drift or bias, and d_{random} is a disturbance such as random noise.

Linear Combinations

In mathematics, a linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the results (e.g., if z is a linear combination of x and y, then $z = ax+by$ where a and b are constants).

See the section "Combinations of Random Variables" for how variances and standard deviations of random variables combine.

Measurement Uncertainty

Measurement uncertainty is defined as: *A quantitative estimate of the range of values about the reported or measured value in which the true value is believed to lie.* [Source: ISO JCGM 200:2012, definition 2.26]

Given a desired state or measurement y, which is a function of different measured or available states x_i:

$$y = f(x_1, x_2, \ldots, x_n)$$

Given the individual states x_i and their standard deviations σ_{x_i}, and assuming that the different x_i are uncorrelated, the Kline-McClintock equation can be used to compute the expected standard uncertainty of y (σ_y) is:

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{x_n}^2}$$

Expanded uncertainties are typically given at an approximately 95% level of confidence with a coverage factor of k = 2. This represents 95% of the area under a Normal probability distribution and is often called 2 sigma.

Linear Regression and Goodness of Fit

Least Squares

$$\hat{y} = \hat{a} + \hat{b}x$$

where

$$\hat{b} = S_{xy}/S_{xx}$$

$$\hat{a} = \overline{y} - \hat{b}\overline{x}$$

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - (1/n)\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)$$

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - (1/n)\left(\sum_{i=1}^{n} x_i\right)^2$$

$$\overline{y} = (1/n)\left(\sum_{i=1}^{n} y_i\right)$$

$$\overline{x} = (1/n)\left(\sum_{i=1}^{n} x_i\right)$$

where

n = sample size

S_{xx} = sum of squares of x

S_{yy} = sum of squares of y

S_{xy} = sum of x-y products

Residual

$$e_i = y_i - \hat{y} = y_i - \left(\hat{a} + \hat{b}x_i\right)$$

Standard Error of Estimate $\left(S_e^2\right)$:

$$S_e^2 = \frac{S_{xx}S_{yy} - S_{xy}^2}{S_{xx}(n-2)} = MSE$$

where

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - (1/n)\left(\sum_{i=1}^{n} y_i\right)^2$$

Confidence Interval for Intercept ($\hat{a}$):

$$\hat{a} \pm t_{\alpha/2,n-2}\sqrt{\left(\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}}\right)MSE}$$

Confidence Interval for Slope ($\hat{b}$):

$$\hat{b} \pm t_{\alpha/2,n-2}\sqrt{\frac{MSE}{S_{xx}}}$$

Sample Correlation Coefficient (R) and Coefficient of Determination (R^2):

$$R = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$R^2 = \frac{S_{xy}^2}{S_{xx}S_{yy}}$$

Hypothesis Testing

Let a "dot" subscript indicate summation over the subscript. Thus:

$$y_{i\bullet} = \sum_{j=1}^{n} y_{ij} \qquad \text{and} \qquad y_{\bullet\bullet} = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}$$

One-Way Analysis of Variance (ANOVA)

Given independent random samples of size n_i from k populations, then:

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(y_{ij}-\overline{y}_{\bullet\bullet}\right)^2 = \sum_{i=1}^{k} n_i\left(\overline{y}_{i\bullet}-\overline{y}_{\bullet\bullet}\right)^2 + \sum_{i=1}^{k}\sum_{i=1}^{n_i}\left(y_{ij}-\overline{y}_{i\bullet}\right)^2$$

$$SS_{\text{total}} = SS_{\text{treatments}} + SS_{\text{error}}$$

If N = total number observations

$$N = \sum_{i=1}^{k} n_i, \textit{ then}$$

$$SS_{\text{total}} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{\bullet\bullet}^2}{N}$$

$$SS_{\text{treatments}} = \sum_{i=1}^{k}\frac{y_{i\bullet}^2}{n_i} - \frac{y_{\bullet\bullet}^2}{N}$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Randomized Complete Block Design

For k treatments and b blocks

$$\sum_{i=1}^{k}\sum_{j=1}^{b}\left(y_{ij}-\overline{y}_{\bullet\bullet}\right)^2 = b\sum_{i=1}^{k}\left(\overline{y}_{i\bullet}-\overline{y}_{\bullet\bullet}\right)^2 + k\sum_{j=1}^{b}\left(\overline{y}_{\bullet j}-\overline{y}_{\bullet\bullet}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{b}\left(\overline{y}_{ij}-\overline{y}_{\bullet j}-\overline{y}_{i\bullet}+\overline{y}_{\bullet\bullet}\right)^2$$

$$SS_{\text{total}} = SS_{\text{treatments}} + SS_{\text{blocks}} + SS_{\text{error}}$$

$$SS_{\text{total}} = \sum_{i=1}^{k}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{\bullet\bullet}^2}{kb}$$

$$SS_{\text{treatments}} = \frac{1}{b}\sum_{i=1}^{k} y_{i\bullet}^2 - \frac{y_{\bullet\bullet}^2}{bk}$$

$$SS_{blocks} = \frac{1}{k}\sum_{j=1}^{b} y_{\bullet j}^2 - \frac{y_{\bullet\bullet}^2}{bk}$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}} - SS_{\text{blocks}}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Two-Factor Factorial Designs

For a levels of Factor A, b levels of Factor B, and n repetitions per cell:

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk}-\overline{y}_{\bullet\bullet\bullet}\right)^2 = bn\sum_{i=1}^{a}\left(\overline{y}_{i\bullet\bullet}-\overline{y}_{\bullet\bullet\bullet}\right)^2 + an\sum_{j=1}^{b}\left(\overline{y}_{\bullet j\bullet}-\overline{y}_{\bullet\bullet\bullet}\right)^2$$

$$+\,n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\overline{y}_{ij\bullet}-\overline{y}_{i\bullet\bullet}-\overline{y}_{\bullet j\bullet}+\overline{y}_{\bullet\bullet\bullet}\right)^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk}-\overline{y}_{ij\bullet}\right)^2$$

$$SS_{\text{total}} = SS_{\text{A}} + SS_{\text{B}} + SS_{\text{AB}} + SS_{\text{error}}$$

$$SS_{\text{total}} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$

$$SS_{\text{A}} = \sum_{i=1}^{a}\frac{y_{i\bullet\bullet}^2}{bn} - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$

$$SS_{\text{B}} = \sum_{j=1}^{b}\frac{y_{\bullet j\bullet}^2}{an} - \frac{y_{\bullet\bullet\bullet}^2}{abn}$$

$$SS_{\text{AB}} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{y_{ij\bullet}^2}{n} - \frac{y_{\bullet\bullet\bullet}^2}{abn} - SS_{\text{A}} - SS_{\text{B}}$$

$$SS_{\text{error}} = SS_{\text{T}} - SS_{\text{A}} - SS_{\text{B}} - SS_{\text{AB}}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

One-Way ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Between Treatments	$k-1$	$SS_{\text{treatments}}$	$MST = \dfrac{SS_{\text{treatments}}}{k-1}$	$\dfrac{MST}{MSE}$
Error	$N-k$	SS_{error}	$MSE = \dfrac{SS_{\text{error}}}{N-k}$	
Total	$N-1$	SS_{total}		

Randomized Complete Block ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Between Treatments	$k-1$	$SS_{\text{treatments}}$	$MST = \dfrac{SS_{\text{treatments}}}{k-1}$	$\dfrac{MST}{MSE}$
Between Blocks	$n-1$	SS_{blocks}	$MSB = \dfrac{SS_{\text{blocks}}}{n-1}$	$\dfrac{MSB}{MSE}$
Error	$(k-1)(n-1)$	SS_{error}	$MSE = \dfrac{SS_{\text{error}}}{(k-1)(n-1)}$	
Total	$N-1$	SS_{total}		

Two-Way Factorial ANOVA Table

Source of Variation	**Degrees of Freedom**	**Sum of Squares**	**Mean Square**	***F***
A Treatments	$a-1$	SS_A	$MSA = \dfrac{SS_A}{a-1}$	$\dfrac{MSA}{MSE}$
B Treatments	$b-1$	SS_B	$MSB = \dfrac{SS_B}{b-1}$	$\dfrac{MSB}{MSE}$
AB Interaction	$(a-1)(b-1)$	SS_{AB}	$MSAB = \dfrac{SS_{AB}}{(a-1)(b-1)}$	$\dfrac{MSAB}{MSE}$
Error	$ab(n-1)$	SS_{error}	$MSE = \dfrac{SS_E}{ab(n-1)}$	
Total	$abn-1$	SS_{total}		

Consider an unknown parameter θ of a statistical distribution. Let the null hypothesis be

H_0: $\mu = \mu_0$

and let the alternative hypothesis be

H_1: $\mu \neq \mu_0$

Rejecting H_0 when it is true is known as a Type I error, while accepting H_0 when it is wrong is known as a Type II error. Furthermore, the probabilities of Type I and Type II errors are usually represented by the symbols α and β, respectively:

α = probability (Type I error)

β = probability (Type II error)

The probability of a Type I error is known as the level of significance of the test.

Table A. Tests on Means of Normal Distribution—Variance Known

Hypothesis	*Test Statistic*	*Criteria for Rejection*
H_0: $\mu = \mu_0$ H_1: $\mu \neq \mu_0$		$\lvert Z_0 \rvert > Z_{\alpha/2}$
H_0: $\mu = \mu_0$ H_1: $\mu < \mu_0$	$Z_0 \equiv \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$	$Z_0 < -Z_\alpha$
H_0: $\mu = \mu_0$ H_1: $\mu > \mu_0$		$Z_0 > Z_\alpha$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 \neq \gamma$		$\lvert Z_0 \rvert > Z_{\alpha/2}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 < \gamma$	$Z_0 \equiv \dfrac{\overline{X_1} - \overline{X_2} - \gamma}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$	$Z_0 < -Z_\alpha$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 > \gamma$		$Z_0 > Z_\alpha$

Table B. Tests on Means of Normal Distribution—Variance Unknown

Hypothesis	*Test Statistic*	*Criteria for Rejection*
H_0: $\mu = \mu_0$ H_1: $\mu \neq \mu_0$		$\lvert t_0 \rvert > t_{\alpha/2,\, n-1}$
H_0: $\mu = \mu_0$ H_1: $\mu < \mu_0$	$t_0 = \dfrac{\overline{X} - \mu_0}{s/\sqrt{n}}$	$t_0 < -t_{\alpha,\, n-1}$
H_0: $\mu = \mu_0$ H_1: $\mu > \mu_0$		$t_0 > t_{\alpha,\, n-1}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 \neq \gamma$	Variances equal: $t_0 = \dfrac{\overline{X_1} - \overline{X_2} - \gamma}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ $v = n_1 + n_2 - 2$	$\lvert t_0 \rvert > t_{\alpha/2,\, v}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 < \gamma$	Variances unequal: $t_0 = \dfrac{\overline{X_1} - \overline{X_2} - \gamma}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$	$t_0 < -t_{\alpha,\, v}$
H_0: $\mu_1 - \mu_2 = \gamma$ H_1: $\mu_1 - \mu_2 > \gamma$	$v = \dfrac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \dfrac{\left(s_2^2/n_2\right)^2}{n_2 - 1}}$	$t_0 > t_{\alpha,\, v}$

In Table B, $s_p^2 = [(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2]/v$

Table C. Tests on Variances of Normal Distribution with Unknown Mean

Hypothesis	*Test Statistic*	*Criteria for Rejection*
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 \neq \sigma_0^2$		$\chi_0^2 > \chi^2_{\alpha/2,\, n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2,\, n-1}$
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 < \sigma_0^2$	$\chi_0^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$	$\chi_0^2 < \chi^2_{1-\alpha,\, n-1}$
H_0: $\sigma^2 = \sigma_0^2$ H_1: $\sigma^2 > \sigma_0^2$		$\chi_0^2 > \chi^2_{\alpha,\, n-1}$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 \neq \sigma_2^2$	$F_0 = \dfrac{s_1^2}{s_2^2}$	$F_0 > F_{\alpha/2,\, n_1-1,\, n_2-1}$ $F_0 < F_{1-\alpha/2,\, n_1-1,\, n_2-1}$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 < \sigma_2^2$	$F_0 = \dfrac{s_2^2}{s_1^2}$	$F_0 > F_{\alpha,\, n_2-1,\, n_1-1}$
H_0: $\sigma_1^2 = \sigma_2^2$ H_1: $\sigma_1^2 > \sigma_2^2$	$F_0 = \dfrac{s_1^2}{s_2^2}$	$F_0 > F_{\alpha,\, n_1-1,\, n_2-1}$

Assume that the values of α and β are given. The sample size can be obtained from the following relationships. In (A) and (B), μ_1 is the value assumed to be the true mean.

(A) $H_0\!: \mu = \mu_0;\ H_1\!: \mu \neq \mu_0$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_{a/2}\right) - \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - Z_{a/2}\right)$$

An approximate result is

$$n \simeq \frac{\left(Z_{a/2} + Z_b\right)^2 \sigma^2}{\left(\mu_1 - \mu_0\right)^2}$$

(B) $H_0\!: \mu = \mu_0;\ H_1\!: \mu > \mu_0$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_a\right)$$

$$n = \frac{\left(Z_a + Z_b\right)^2 \sigma^2}{\left(\mu_1 - \mu_0\right)^2}$$

Confidence Intervals, Sample Distributions and Sample Size

Confidence Interval for the Mean μ of a Normal Distribution

(A) Standard deviation σ is known

$$\overline{X} - Z_{a/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + Z_{a/2}\frac{\sigma}{\sqrt{n}}$$

(B) Standard deviation σ is not known

$$\overline{X} - t_{a/2}\frac{s}{\sqrt{n}} \le \mu \le \overline{X} + t_{a/2}\frac{s}{\sqrt{n}}$$

where $t_{a/2}$ corresponds to $n-1$ degrees of freedom.

Confidence Interval for the Difference Between Two Means μ_1 and μ_2

(A) Standard deviations σ_1 and σ_2 known

$$\overline{X}_1 - \overline{X}_2 - Z_{a/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \overline{X}_1 - \overline{X}_2 + Z_{a/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

(B) Standard deviations σ_1 and σ_2 are not known

$$\overline{X}_1 - \overline{X}_2 - t_{a/2}\sqrt{\frac{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right]}{n_1 + n_2 - 2}} \le \mu_1 - \mu_2 \le \overline{X}_1 - \overline{X}_2 + t_{a/2}\sqrt{\frac{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2\right]}{n_1 + n_2 - 2}}$$

where $t_{a/2}$ corresponds to $n_1 + n_2 - 2$ degrees of freedom.

Confidence Intervals for the Variance σ^2 of a Normal Distribution

$$\frac{(n-1)s^2}{x^2_{\alpha/2,\, n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{x^2_{1-\alpha/2,\, n-1}}$$

Sample Size

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \qquad n = \left[\frac{z_{\alpha/2}\sigma}{\overline{x} - \mu}\right]^2$$

Test Statistics

The following definitions apply.

$$Z_{\text{var}} = \frac{\overline{X} - \mu_o}{\frac{\sigma}{\sqrt{n}}}$$

$$t_{\text{var}} = \frac{\overline{X} - \mu_o}{\frac{s}{\sqrt{n}}}$$

where

Z_{var} = standard normal Z score

t_{var} = sample distribution test statistic

σ = standard deviation

μ_o = population mean

$\overline{X}$ = hypothesized mean or sample mean

n = sample size

s = computed sample standard deviation

The Z score is applicable when the standard deviation (s) is known. The test statistic is applicable when the standard deviation (s) is computed at time of sampling.

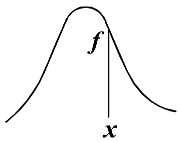
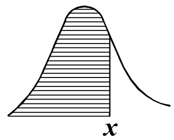
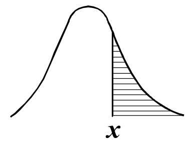
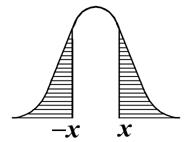
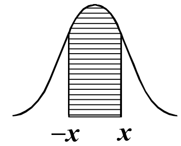
Z_α corresponds to the appropriate probability under the normal probability curve for a given Z_{var}.

$t_{\alpha,\, n-1}$ corresponds to the appropriate probability under the t distribution with $n-1$ degrees of freedom for a given t_{var}.

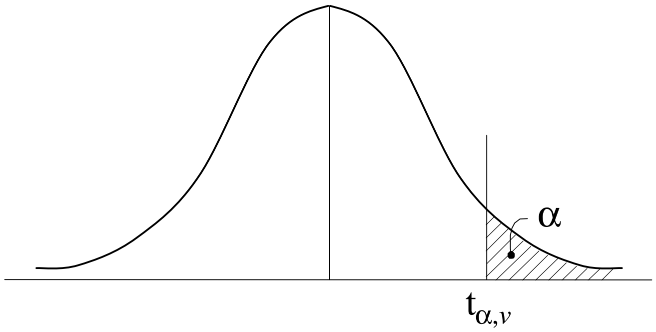
Values of $Z_{\alpha/2}$

Confidence Interval	$Z_{\alpha/2}$
80%	1.2816
90%	1.6449
95%	1.9600
96%	2.0537
98%	2.3263
99%	2.5758

Unit Normal Distribution ($\mu = 0$, $\sigma = 1$)

					
x	***f*(*x*)**	***F*(*x*)**	***R*(*x*)**	**2*R*(*x*)**	***W*(*x*)**
0.0	0.3989	0.5000	0.5000	1.0000	0.0000
0.1	0.3970	0.5398	0.4602	0.9203	0.0797
0.2	0.3910	0.5793	0.4207	0.8415	0.1585
0.3	0.3814	0.6179	0.3821	0.7642	0.2358
0.4	0.3683	0.6554	0.3446	0.6892	0.3108
0.5	0.3521	0.6915	0.3085	0.6171	0.3829
0.6	0.3332	0.7257	0.2743	0.5485	0.4515
0.7	0.3123	0.7580	0.2420	0.4839	0.5161
0.8	0.2897	0.7881	0.2119	0.4237	0.5763
0.9	0.2661	0.8159	0.1841	0.3681	0.6319
1.0	0.2420	0.8413	0.1587	0.3173	0.6827
1.1	0.2179	0.8643	0.1357	0.2713	0.7287
1.2	0.1942	0.8849	0.1151	0.2301	0.7699
1.3	0.1714	0.9032	0.0968	0.1936	0.8064
1.4	0.1497	0.9192	0.0808	0.1615	0.8385
1.5	0.1295	0.9332	0.0668	0.1336	0.8664
1.6	0.1109	0.9452	0.0548	0.1096	0.8904
1.7	0.0940	0.9554	0.0446	0.0891	0.9109
1.8	0.0790	0.9641	0.0359	0.0719	0.9281
1.9	0.0656	0.9713	0.0287	0.0574	0.9426
2.0	0.0540	0.9772	0.0228	0.0455	0.9545
2.1	0.0440	0.9821	0.0179	0.0357	0.9643
2.2	0.0355	0.9861	0.0139	0.0278	0.9722
2.3	0.0283	0.9893	0.0107	0.0214	0.9786
2.4	0.0224	0.9918	0.0082	0.0164	0.9836
2.5	0.0175	0.9938	0.0062	0.0124	0.9876
2.6	0.0136	0.9953	0.0047	0.0093	0.9907
2.7	0.0104	0.9965	0.0035	0.0069	0.9931
2.8	0.0079	0.9974	0.0026	0.0051	0.9949
2.9	0.0060	0.9981	0.0019	0.0037	0.9963
3.0	0.0044	0.9987	0.0013	0.0027	0.9973
Fractiles					
1.2816	0.1755	0.9000	0.1000	0.2000	0.8000
1.6449	0.1031	0.9500	0.0500	0.1000	0.9000
1.9600	0.0584	0.9750	0.0250	0.0500	0.9500
2.0537	0.0484	0.9800	0.0200	0.0400	0.9600
2.3263	0.0267	0.9900	0.0100	0.0200	0.9800
2.5758	0.0145	0.9950	0.0050	0.0100	0.9900

Student's *t*-Distribution

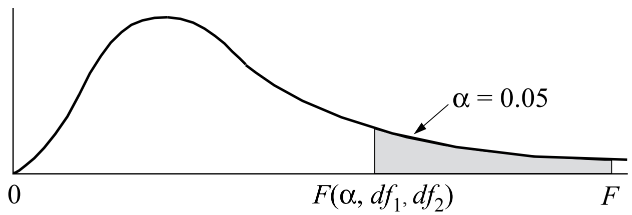


VALUES OF $t_{\alpha,v}$

v	α								v
	0.25	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	1
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	2
3	0.765	0.978	1.350	1.638	2.353	3.182	4.541	5.841	3
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	4
5	**0.727**	**0.920**	**1.156**	**1.476**	**2.015**	**2.571**	**3.365**	**4.032**	**5**
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	6
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	7
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	8
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	9
10	**0.700**	**0.879**	**1.093**	**1.372**	**1.812**	**2.228**	**2.764**	**3.169**	**10**
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	11
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	12
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	13
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	14
15	**0.691**	**0.866**	**1.074**	**1.341**	**1.753**	**2.131**	**2.602**	**2.947**	**15**
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	16
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	17
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	18
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	19
20	**0.687**	**0.860**	**1.064**	**1.325**	**1.725**	**2.086**	**2.528**	**2.845**	**20**
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	21
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	22
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	23
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	24
25	**0.684**	**0.856**	**1.058**	**1.316**	**1.708**	**2.060**	**2.485**	**2.787**	**25**
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	26
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	27
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	28
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	29
30	**0.683**	**0.854**	**1.055**	**1.310**	**1.697**	**2.042**	**2.457**	**2.750**	**30**
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	∞

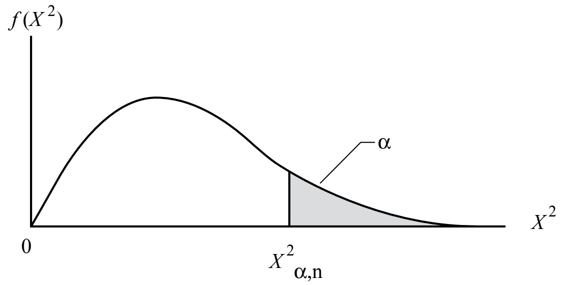
CRITICAL VALUES OF THE F DISTRIBUTION – TABLE

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of F corresponding to a specified upper tail area (α).



Denominator df_2	Numerator df_1																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	**6.61**	**5.79**	**5.41**	**5.19**	**5.05**	**4.95**	**4.88**	**4.82**	**4.77**	**4.74**	**4.68**	**4.62**	**4.56**	**4.53**	**4.50**	**4.46**	**4.43**	**4.40**	**4.36**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	**4.96**	**4.10**	**3.71**	**3.48**	**3.33**	**3.22**	**3.14**	**3.07**	**3.02**	**2.98**	**2.91**	**2.85**	**2.77**	**2.74**	**2.70**	**2.66**	**2.62**	**2.58**	**2.54**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	**4.54**	**3.68**	**3.29**	**3.06**	**2.90**	**2.79**	**2.71**	**2.64**	**2.59**	**2.54**	**2.48**	**2.40**	**2.33**	**2.29**	**2.25**	**2.20**	**2.16**	**2.11**	**2.07**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	**4.35**	**3.49**	**3.10**	**2.87**	**2.71**	**2.60**	**2.51**	**2.45**	**2.39**	**2.35**	**2.28**	**2.20**	**2.12**	**2.08**	**2.04**	**1.99**	**1.95**	**1.90**	**1.84**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	**4.24**	**3.39**	**2.99**	**2.76**	**2.60**	**2.49**	**2.40**	**2.34**	**2.28**	**2.24**	**2.16**	**2.09**	**2.01**	**1.96**	**1.92**	**1.87**	**1.82**	**1.77**	**1.71**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	**4.17**	**3.32**	**2.92**	**2.69**	**2.53**	**2.42**	**2.33**	**2.27**	**2.21**	**2.16**	**2.09**	**2.01**	**1.93**	**1.89**	**1.84**	**1.79**	**1.74**	**1.68**	**1.62**
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

CRITICAL VALUES OF X^2 DISTRIBUTION



Degrees of Freedom	$X^2_{.995}$	$X^2_{.990}$	$X^2_{.975}$	$X^2_{.950}$	$X^2_{.900}$	$X^2_{.100}$	$X^2_{.050}$	$X^2_{.025}$	$X^2_{.010}$	$X^2_{.005}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720	4.60517	5.99147	7.37776	9.21034	10.5966
3	0.0717212	0.114832	0.215795	0.351846	0.584375	6.25139	7.81473	9.34840	11.3449	12.8381
4	0.206990	0.297110	0.484419	0.710721	1.063623	7.77944	9.48773	11.1433	13.2767	14.8602
5	0.411740	0.554300	0.831211	1.145476	1.61031	9.23635	11.0705	12.8325	15.0863	16.7496
6	0.675727	0.872085	1.237347	1.63539	2.20413	10.6446	12.5916	14.4494	16.8119	18.5476
7	0.989265	1.239043	1.68987	2.16735	2.83311	12.0170	14.0671	16.0128	18.4753	20.2777
8	1.344419	1.646482	2.17973	2.73264	3.48954	13.3616	15.5073	17.5346	20.0902	21.9550
9	1.734926	2.087912	2.70039	3.32511	4.16816	14.6837	16.9190	19.0228	21.6660	23.5893
10	2.15585	2.55821	3.24697	3.94030	4.86518	15.9871	18.3070	20.4831	23.2093	25.1882
11	2.60321	3.05347	3.81575	4.57481	5.57779	17.2750	19.6751	21.9200	24.7250	26.7569
12	3.07382	3.57056	4.40379	5.22603	6.30380	18.5494	21.0261	23.3367	26.2170	28.2995
13	3.56503	4.10691	5.00874	5.89186	7.04150	19.8119	22.3621	24.7356	27.6883	29.8194
14	4.07468	4.66043	5.62872	6.57063	7.78953	21.0642	23.6848	26.1190	29.1413	31.3193
15	4.60094	5.22935	6.26214	7.26094	8.54675	22.3072	24.9958	27.4884	30.5779	32.8013
16	5.14224	5.81221	6.90766	7.96164	9.31223	23.5418	26.2962	28.8454	31.9999	34.2672
17	5.69724	6.40776	7.56418	8.67176	10.0852	24.7690	27.5871	30.1910	33.4087	35.7185
18	6.26481	7.01491	8.23075	9.39046	10.8649	25.9894	28.8693	31.5264	34.8053	37.1564
19	6.84398	7.63273	8.90655	10.1170	11.6509	27.2036	30.1435	32.8523	36.1908	38.5822
20	7.43386	8.26040	9.59083	10.8508	12.4426	28.4120	31.4104	34.1696	37.5662	39.9968
21	8.03366	8.89720	10.28293	11.5913	13.2396	29.6151	32.6705	35.4789	38.9321	41.4010
22	8.64272	9.54249	10.9823	12.3380	14.0415	30.8133	33.9244	36.7807	40.2894	42.7956
23	9.26042	10.19567	11.6885	13.0905	14.8479	32.0069	35.1725	38.0757	41.6384	44.1813
24	9.88623	10.8564	12.4011	13.8484	15.6587	33.1963	36.4151	39.3641	42.9798	45.5585
25	10.5197	11.5240	13.1197	14.6114	16.4734	34.3816	37.6525	40.6465	44.3141	46.9278
26	11.1603	12.1981	13.8439	15.3791	17.2919	35.5631	38.8852	41.9232	45.6417	48.2899
27	11.8076	12.8786	14.5733	16.1513	18.1138	36.7412	40.1133	43.1944	46.9630	49.6449
28	12.4613	13.5648	15.3079	16.9279	18.9392	37.9159	41.3372	44.4607	48.2782	50.9933
29	13.1211	14.2565	16.0471	17.7083	19.7677	39.0875	42.5569	45.7222	49.5879	52.3356
30	13.7867	14.9535	16.7908	18.4926	20.5992	40.2560	43.7729	46.9792	50.8922	53.6720
40	20.7065	22.1643	24.4331	26.5093	29.0505	51.8050	55.7585	59.3417	63.6907	66.7659
50	27.9907	29.7067	32.3574	34.7642	37.6886	63.1671	67.5048	71.4202	76.1539	79.4900
60	35.5346	37.4848	40.4817	43.1879	46.4589	74.3970	79.0819	83.2976	88.3794	91.9517
70	43.2752	45.4418	48.7576	51.7393	55.3290	85.5271	90.5312	95.0231	100.425	104.215
80	51.1720	53.5400	57.1532	60.3915	64.2778	96.5782	101.879	106.629	112.329	116.321
90	59.1963	61.7541	65.6466	69.1260	73.2912	107.565	113.145	118.136	124.116	128.299
100	67.3276	70.0648	74.2219	77.9295	82.3581	118.498	124.342	129.561	135.807	140.169

Source: Thompson, C. M., "Tables of the Percentage Points of the X^2–Distribution," *Biometrika,* ©1941, 32, 188-189. Reproduced by permission of Oxford University Press.

Cumulative Binomial Probabilities $P(X \leq x)$

		P										
n	*x*	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
1	0	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	0.0500	0.0100
2	0	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100	0.0025	0.0001
	1	0.9900	0.9600	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900	0.0975	0.0199
3	0	0.7290	0.5120	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010	0.0001	0.0000
	1	0.9720	0.8960	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280	0.0073	0.0003
	2	0.9990	0.9920	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710	0.1426	0.0297
4	0	0.6561	0.4096	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001	0.0000	0.0000
	1	0.9477	0.8192	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037	0.0005	0.0000
	2	0.9963	0.9728	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523	0.0140	0.0006
	3	0.9999	0.9984	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439	0.1855	0.0394
5	0	0.5905	0.3277	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000	0.0000	0.0000
	1	0.9185	0.7373	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005	0.0000	0.0000
	2	0.9914	0.9421	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086	0.0012	0.0000
	3	0.9995	0.9933	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815	0.0226	0.0010
	4	1.0000	0.9997	0.9976	0.9898	0.9688	0.9222	0.8319	0.6723	0.4095	0.2262	0.0490
6	0	0.5314	0.2621	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000	0.0000	0.0000
	1	0.8857	0.6554	0.4202	0.2333	0.1094	0.0410	0.0109	0.0016	0.0001	0.0000	0.0000
	2	0.9842	0.9011	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013	0.0001	0.0000
	3	0.9987	0.9830	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0159	0.0022	0.0000
	4	0.9999	0.9984	0.9891	0.9590	0.8906	0.7667	0.5798	0.3446	0.1143	0.0328	0.0015
	5	1.0000	0.9999	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686	0.2649	0.0585
7	0	0.4783	0.2097	0.0824	0.0280	0.0078	0.0016	0.0002	0.0000	0.0000	0.0000	0.0000
	1	0.8503	0.5767	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000	0.0000	0.0000
	2	0.9743	0.8520	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002	0.0000	0.0000
	3	0.9973	0.9667	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027	0.0002	0.0000
	4	0.9998	0.9953	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257	0.0038	0.0000
	5	1.0000	0.9996	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497	0.0444	0.0020
	6	1.0000	1.0000	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217	0.3017	0.0679
8	0	0.4305	0.1678	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.8131	0.5033	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	0.0000	0.0000	0.0000
	2	0.9619	0.7969	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000	0.0000	0.0000
	3	0.9950	0.9437	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004	0.0000	0.0000
	4	0.9996	0.9896	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050	0.0004	0.0000
	5	1.0000	0.9988	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381	0.0058	0.0001
	6	1.0000	0.9999	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869	0.0572	0.0027
	7	1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695	0.3366	0.0773
9	0	0.3874	0.1342	0.0404	0.0101	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7748	0.4362	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	0.0000	0.0000	0.0000
	2	0.9470	0.7382	0.4628	0.2318	0.0898	0.0250	0.0043	0.0003	0.0000	0.0000	0.0000
	3	0.9917	0.9144	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001	0.0000	0.0000
	4	0.9991	0.9804	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009	0.0000	0.0000
	5	0.9999	0.9969	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083	0.0006	0.0000
	6	1.0000	0.9997	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530	0.0084	0.0001
	7	1.0000	1.0000	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252	0.0712	0.0034
	8	1.0000	1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126	0.3698	0.0865

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Cumulative Binomial Probabilities $P(X \le x)$ (*continued*)

		P										
n	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
10	0	0.3487	0.1074	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7361	0.3758	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	0.0000	0.0000
	2	0.9298	0.6778	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	0.0000	0.0000	0.0000
	3	0.9872	0.8791	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000	0.0000	0.0000
	4	0.9984	0.9672	0.8497	0.6331	0.3770	0.1662	0.0473	0.0064	0.0001	0.0000	0.0000
	5	0.9999	0.9936	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016	0.0001	0.0000
	6	1.0000	0.9991	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128	0.0010	0.0000
	7	1.0000	0.9999	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702	0.0115	0.0001
	8	1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639	0.0861	0.0043
	9	1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513	0.4013	0.0956
15	0	0.2059	0.0352	0.0047	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.5490	0.1671	0.0353	0.0052	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.8159	0.3980	0.1268	0.0271	0.0037	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.9444	0.6482	0.2969	0.0905	0.0176	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000
	4	0.9873	0.8358	0.5155	0.2173	0.0592	0.0093	0.0007	0.0000	0.0000	0.0000	0.0000
	5	0.9978	0.9389	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	0.0000	0.0000	0.0000
	6	0.9997	0.9819	0.8689	0.6098	0.3036	0.0950	0.0152	0.0008	0.0000	0.0000	0.0000
	7	1.0000	0.9958	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000	0.0000	0.0000
	8	1.0000	0.9992	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003	0.0000	0.0000
	9	1.0000	0.9999	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022	0.0001	0.0000
	10	1.0000	1.0000	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127	0.0006	0.0000
	11	1.0000	1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556	0.0055	0.0000
	12	1.0000	1.0000	1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841	0.0362	0.0004
	13	1.0000	1.0000	1.0000	1.0000	0.9995	0.9948	0.9647	0.8329	0.4510	0.1710	0.0096
	14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9953	0.9648	0.7941	0.5367	0.1399
20	0	0.1216	0.0115	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.3917	0.0692	0.0076	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.6769	0.2061	0.0355	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.8670	0.4114	0.1071	0.0160	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.9568	0.6296	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.9887	0.8042	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.9976	0.9133	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000
	7	0.9996	0.9679	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	0.0000	0.0000	0.0000
	8	0.9999	0.9900	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	0.0000	0.0000	0.0000
	9	1.0000	0.9974	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	0.0000	0.0000	0.0000
	10	1.0000	0.9994	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000	0.0000	0.0000
	11	1.0000	0.9999	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001	0.0000	0.0000
	12	1.0000	1.0000	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004	0.0000	0.0000
	13	1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024	0.0000	0.0000
	14	1.0000	1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113	0.0003	0.0000
	15	1.0000	1.0000	1.0000	0.9997	0.9941	0.9490	0.7625	0.3704	0.0432	0.0026	0.0000
	16	1.0000	1.0000	1.0000	1.0000	0.9987	0.9840	0.8929	0.5886	0.1330	0.0159	0.0000
	17	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9645	0.7939	0.3231	0.0755	0.0010
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9924	0.9308	0.6083	0.2642	0.0169
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9885	0.8784	0.6415	0.1821

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Statistical Quality Control

Average and Range Charts

n	A_2	D_3	D_4
2	1.880	0	3.268
3	1.023	0	2.574
4	0.729	0	2.282
5	0.577	0	2.114
6	0.483	0	2.004
7	0.419	0.076	1.924
8	0.373	0.136	1.864
9	0.337	0.184	1.816
10	0.308	0.223	1.777

X_i = an individual observation

n = the sample size of a group

k = the number of groups

R = (range) the difference between the largest and smallest observations in a sample of size n.

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

$$\overline{\overline{X}} = \frac{\overline{X}_1 + \overline{X}_2 + \ldots + \overline{X}_k}{k}$$

$$\overline{R} = \frac{R_1 + R_2 + \ldots + R_k}{k}$$

The R Chart formulas are:

$$CL_R = \overline{R}$$
$$UCL_R = D_4\overline{R}$$
$$LCL_R = D_3\overline{R}$$

The $\overline{X}$ Chart formulas are:

$$CL_X = \overline{\overline{X}}$$
$$UCL_X = \overline{\overline{X}} + A_2\overline{R}$$
$$LCL_X = \overline{\overline{X}} - A_2\overline{R}$$

Standard Deviation Charts

n	A_3	B_3	B_4
2	2.659	0	3.267
3	1.954	0	2.568
4	1.628	0	2.266
5	1.427	0	2.089
6	1.287	0.030	1.970
7	1.182	0.119	1.882
8	1.099	0.185	1.815
9	1.032	0.239	1.761
10	0.975	0.284	1.716

$$UCL_X = \overline{\overline{X}} + A_3\overline{S}$$
$$CL_X = \overline{\overline{X}}$$
$$LCL_X = \overline{\overline{X}} - A_3\overline{S}$$
$$UCL_S = B_4\overline{S}$$
$$CL_S = \overline{S}$$
$$LCL_S = B_3\overline{S}$$

Approximations

The following table and equations may be used to generate initial approximations of the items indicated.

n	c_4	d_2	d_3
2	0.7979	1.128	0.853
3	0.8862	1.693	0.888
4	0.9213	2.059	0.880
5	0.9400	2.326	0.864
6	0.9515	2.534	0.848
7	0.9594	2.704	0.833
8	0.9650	2.847	0.820
9	0.9693	2.970	0.808
10	0.9727	3.078	0.797

$$\hat{\sigma} = \overline{R}/d_2$$
$$\hat{\sigma} = \overline{S}/c_4$$
$$\sigma_R = d_3\hat{\sigma}$$
$$\sigma_S = \hat{\sigma}\sqrt{1 - c_4^2}$$

where

$\hat{\sigma}$ = an estimate of σ

σ_R = an estimate of the standard deviation of the ranges of the samples

σ_S = an estimate of the standard deviation of the standard deviations of the samples

Tests for Out of Control

1. A single point falls outside the (three sigma) control limits.
2. Two out of three successive points fall on the same side of and more than two sigma units from the center line.
3. Four out of five successive points fall on the same side of and more than one sigma unit from the center line.
4. Eight successive points fall on the same side of the center line.

Probability and Density Functions: Means and Variances

Variable	Equation	Mean	Variance
Binomial Coefficient	$\binom{n}{x} = \dfrac{n!}{x!(n-x)!}$		
Binomial	$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Hyper Geometric	$h(x; n, r, N) = \binom{r}{x} \dfrac{\binom{N-r}{n-x}}{\binom{N}{n}}$	$\dfrac{nr}{N}$	$\dfrac{r(N-r)n(N-n)}{N^2(N-1)}$
Poisson	$f(x; \lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}$	λ	λ
Geometric	$g(x; p) = p\,(1-p)^{x-1}$	$1/p$	$(1-p)/p^2$
Negative Binomial	$f(y; r, p) = \binom{y+r-1}{r-1} p^r (1-p)^y$	r/p	$r\,(1-p)/p^2$
Multinomial	$f(x_1, \ldots, x_k) = \dfrac{n!}{x_1!, \ldots, x_k!} p_1^{x_1} \ldots p_k^{x_k}$	np_i	$np_i\,(1-p_i)$
Uniform	$f(x) = 1/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Gamma	$f(x) = \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}; \quad \alpha > 0, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$
Exponential	$f(x) = \dfrac{1}{\beta} e^{-x/\beta}$	β	β^2
Weibull	$f(x) = \dfrac{\alpha}{\beta} x^{\alpha-1} e^{-x^\alpha/\beta}$	$\beta^{1/\alpha}\Gamma[(\alpha+1)/\alpha]$	$\beta^{2/\alpha}\left[\Gamma\left(\dfrac{\alpha+1}{\alpha}\right) - \Gamma^2\left(\dfrac{\alpha+1}{\alpha}\right)\right]$
Normal	$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2
Triangular	$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(m-a)} \text{ if } a \le x \le m \\ \dfrac{2(b-x)}{(b-a)(b-m)} \text{ if } m < x \le b \end{cases}$	$\dfrac{a+b+m}{3}$	$\dfrac{a^2 + b^2 + m^2 - ab - am - bm}{18}$

Chemistry and Biology

Definitions

Avogadro's Number – The number of elementary particles in a mol of a substance.

1 mol = 1 gram mole
1 mol = 6.02×10^{23} particles

Molarity of Solutions – The number of gram moles of a substance dissolved in a liter of solution.

Molality of Solutions – The number of gram moles of a substance per 1,000 grams of solvent.

Normality of Solutions – The product of the molarity of a solution and the number of valence changes taking place in a reaction.

Molar Volume of an Ideal Gas [at 0°C (32°F) and 1 atm (14.7 psia)]; 22.4 L/(g mole) [359 ft^3/(lb mole)].

$$K_{EQ} = \frac{[C]^c[D]^d}{[A]^a[B]^b}$$

where $[x]$ is the thermodynamic activity of x unless otherwise noted

$[x]$ = concentration of x in dilute solution, or
= partial pressure of x, or
= 1 for solids and liquids

Equilibrium Constant of a Chemical Reaction

$$aA + bB \rightleftharpoons cC + dD$$

Heats of Reaction, Solution, Formation, and Combustion – Chemical processes generally involve the absorption or evolution of heat. In an endothermic process, heat is absorbed (enthalpy change is positive). In an exothermic process, heat is evolved (enthalpy change is negative).

Solubility Product of a slightly soluble substance *AB:*

$$A_mB_n \rightarrow mA^{n+} + nB^{m-}$$

Solubility Product Constant = $K_{SP} = [A^+]^m\,[B^-]^n$

Faraday's Equation

$$m = \left(\frac{Q}{F}\right)\left(\frac{M}{z}\right)$$

where

m = mass (grams) of substance liberated at electrode
Q = total electric charge passed through electrolyte (coulomb or ampere•second)
F = 96,485 coulombs/mol
M = molar mass of the substance (g/mol)
z = valence number

A *catalyst* is a substance that alters the rate of a chemical reaction. The catalyst does not affect the position of equilibrium of a reversible reaction.

The *atomic number* is the number of protons in the atomic nucleus.

Boiling Point Elevation – The presence of a nonvolatile solute in a solvent raises the boiling point of the resulting solution.

Freezing Point Depression – The presence of a solute lowers the freezing point of the resulting solution.

Nernst Equation

$$\Delta E = \left(E_2^0 - E_1^0\right) - \frac{RT}{nF}\ln\left[\frac{M_1^{n+}}{M_2^{n+}}\right]$$

where

E_1^0 = half-cell potential (volts)

R = ideal gas constant (J/kmol•K) [Note: 1 J = (1 volt)(1 coulomb)]

n = number of electrons participating in either half-cell reaction (dimensionless)

T = absolute temperature (K)

M_1^{n+} and M_2^{n+} = molar ion concentration (mol/L of solution)

Acids, Bases, and pH (aqueous solutions)

$$\text{pH} = \log_{10}\left(\frac{1}{[H^+]}\right)$$

where

$[H^+]$ = molar concentration of hydrogen ion, in gram moles per liter. Acids have pH < 7. Bases have pH > 7.

$$HA \longleftrightarrow A^- + H^+$$

$$K_a = \frac{[A^-][H^+]}{[HA]}$$

$$pK_a = -\log(K_a)$$

For water $[H^+][OH^-] = 10^{-14}$

$[\ \ ]$ denotes molarity

Bioconversion

Aerobic Biodegradation of Glucose with No Product, Ammonia Nitrogen Source, Cell Production Only, where Respiration Quotent (RQ) = 1.1

$$C_6H_{12}O_6 + aO_2 + bNH_3 \rightarrow cCH_{1.8}O_{0.5}N_{0.2} + dCO_2 + eH_2O$$

Substrate Cells

For the above conditions, one finds that:

a = 1.94

b = 0.77

c = 3.88

d = 2.13

e = 3.68

The *c* coefficient represents a theoretical maximum yield coefficient, which may be reduced by a yield factor.

The respiratory quotient (RQ) is a dimensionless number used in calculations of basal metabolic rate when estimated from the ratio of CO_2 produced to the O_2 consumed. The RQ depends on substrates and organisms involved.

Anaerobic Biodegradation of Organic Wastes, Incomplete Stabilization

$$C_aH_bO_cN_d \rightarrow nC_wH_xO_yN_z + mCH_4 + sCO_2 + rH_2O + (d - nz)NH_3$$

s = a – nw – m

r = c – ny – 2s

Knowledge of product composition, yield coefficient (n) and a methane/CO_2 ratio is needed.

Anaerobic Biodegradation of Organic Wastes, Complete Stabilization

$$C_aH_bO_cN_d + rH_2O \rightarrow mCH_4 + sCO_2 + dNH_3$$

$$r = \frac{4a - b - 2c + 3d}{4}$$

$$s = \frac{4a - b + 2c + 3d}{8}$$

$$m = \frac{4a + b - 2c - 3d}{8}$$

Photosynthesis

Photosynthesis is a most important process form synthesizing glucose from carbon dioxide. It also produces oxygen. The most important photosynthesis reaction is summarized as follows.

$$6CO_2 + 6H_2O + \text{light} \rightarrow C_6H_{12}O_6 + 6O_2$$

The light is required to be in the 400- to 700-nm range (visible light). Chlorophyll is the primary photosynthesis compound and it is found in organisms ranging from tree and plant leaves to single celled algae.

Instrumental Methods of Analysis

Method	Qualitative		Quantitative	
	Elemental	Molecular	Elemental	Molecular
Atomic absorption spectrometry	No	No	Yes	No
Atomic emission spectrometry (AES)	Yes	No	Yes	No
Capillary electrophoresis (CE)	Yes	Yes	Yes	Yes
Electrochemistry	Yes	Yes	Yes	Yes
Gas Chromatography (GC)	No	Yes	No	Yes
ICP-mass spectrometry(ICP MS)	Yes	No	Yes	No
Infrared spectroscopy (IS)	No	Yes	No	Yes
Ion chromatography	Yes	Yes	Yes	Yes
Liquid chromatography (LC)	No	Yes	No	Yes
Mass spectrometry (MS)	Yes	Yes	Yes	Yes
Nuclear Magnetic Resonance (NMR)	No	Yes	No	Yes
Raman spectroscopy	No	Yes	No	Yes
Thermal analysis (TA)	No	Yes	No	Yes
UV and visible (UV/VIS) spectrophotometry	Yes	Yes	Yes	Yes
UV absorption	No	Yes	No	Yes
UV fluorescence	No	Yes	No	Yes
X-ray absorption	Yes	No	Yes	No
X-ray diffraction (XRF)	No	Yes	No	Yes
X-ray fluorescence	Yes	No	Yes	No

Adapted from Robinson, James W., Eileen M. Skelly Frame, George M. Frame II, *Undergraduate Instrumental Analysis*, 6th ed., p. 8.

Periodic Table of Elements

Atomic Number
Symbol
Atomic Weight

I	II											III	IV	V	VI	VII	VIII
1 **H** 1.0079																	2 **He** 4.0026
3 **Li** 6.941	4 **Be** 9.0122											5 **B** 10.811	6 **C** 12.011	7 **N** 14.007	8 **O** 15.999	9 **F** 18.998	10 **Ne** 20.179
11 **Na** 22.990	12 **Mg** 24.305											13 **Al** 26.981	14 **Si** 28.086	15 **P** 30.974	16 **S** 32.066	17 **Cl** 35.453	18 **Ar** 39.948
19 **K** 39.098	20 **Ca** 40.078	21 **Sc** 44.956	22 **Ti** 47.88	23 **V** 50.941	24 **Cr** 51.996	25 **Mn** 54.938	26 **Fe** 55.847	27 **Co** 58.933	28 **Ni** 58.69	29 **Cu** 63.546	30 **Zn** 65.39	31 **Ga** 69.723	32 **Ge** 72.61	33 **As** 74.921	34 **Se** 78.96	35 **Br** 79.904	36 **Kr** 83.80
37 **Rb** 85.468	38 **Sr** 87.62	39 **Y** 88.906	40 **Zr** 91.224	41 **Nb** 92.906	42 **Mo** 95.94	43 **Tc** (98)	44 **Ru** 101.07	45 **Rh** 102.91	46 **Pd** 106.42	47 **Ag** 107.87	48 **Cd** 112.41	49 **In** 114.82	50 **Sn** 118.71	51 **Sb** 121.75	52 **Te** 127.60	53 **I** 126.90	54 **Xe** 131.29
55 **Cs** 132.91	56 **Ba** 137.33	57–71	72 **Hf** 178.49	73 **Ta** 180.95	74 **W** 183.85	75 **Re** 186.21	76 **Os** 190.2	77 **Ir** 192.22	78 **Pt** 195.08	79 **Au** 196.97	80 **Hg** 200.59	81 **Tl** 204.38	82 **Pb** 207.2	83 **Bi** 208.98	84 **Po** (209)	85 **At** (210)	86 **Rn** (222)
87 **Fr** (223)	88 **Ra** 226.02	89–103	104 **Rf** (261)	105 **Db** (262)	106 **Sg** (266)	107 **Bh** (264)	108 **Hs** (269)	109 **Mt** (268)	110 **Ds** (269)	111 **Rg** (272)	112 **Cn** (277)	113 **Uut** unknown	114 **Fl** (289)	115 **Uup** unknown	116 **Lv** (298)	117 **Uus** unknown	118 **Uuo** unknown

Lanthanide Series	57 **La** 138.91	58 **Ce** 140.12	59 **Pr** 140.91	60 **Nd** 144.24	61 **Pm** (145)	62 **Sm** 150.36	63 **Eu** 151.96	64 **Gd** 157.25	65 **Tb** 158.92	66 **Dy** 162.50	67 **Ho** 164.93	68 **Er** 167.26	69 **Tm** 168.93	70 **Yb** 173.04	71 **Lu** 174.97
Actinide Series	89 **Ac** 227.03	90 **Th** 232.04	91 **Pa** 231.04	92 **U** 238.03	93 **Np** 237.05	94 **Pu** (244)	95 **Am** (243)	96 **Cm** (247)	97 **Bk** (247)	98 **Cf** (251)	99 **Es** (252)	100 **Fm** (257)	101 **Md** (258)	102 **No** (259)	103 **Lr** (260)

Selected Rules of Nomenclature in Organic Chemistry

Alcohols

Three systems of nomenclature are in general use. In the first, the alkyl group attached to the hydroxyl group is named and the separate word *alcohol* is added. In the second system, the higher alcohols are considered as derivatives of the first member of the series, which is called *carbinol*. The third method is the modified Geneva system in which (1) the longest carbon chain containing the hydroxyl group determines the surname, (2) the ending *e* of the corresponding saturated hydrocarbon is replaced by *ol*, (3) the carbon chain is numbered from the end that gives the hydroxyl group the smaller number, and (4) the side chains are named and their positions indicated by the proper number. Alcohols in general are divided into three classes. In *primary* alcohols the hydroxyl group is united to a primary carbon atom, that is, a carbon atom united directly to only one other carbon atom. *Secondary* alcohols have the hydroxyl group united to a secondary carbon atom, that is, one united to two other carbon atoms. *Tertiary* alcohols have the hydroxyl group united to a tertiary carbon atom, that is, one united to three other carbon atoms.

Ethers

Ethers are generally designated by naming the alkyl groups and adding the word *ether*. The group RO is known as an *alkoxyl group*. Ethers may also be named as alkoxy derivatives of hydrocarbons.

Carboxylic Acids

The name of each linear carboxylic acid is unique to the number of carbon atoms it contains. 1: (one carbon atom) Formic. 2: Acetic. 3: Propionic. 4: Butyric. 5: Valeric. 6: Caproic. 7: Enanthic. 8: Caprylic. 9: Pelargonic. 10: Capric.

Aldehydes

The common names of aldehydes are derived from the acids that would be formed on oxidation, that is, the acids having the same number of carbon atoms. In general the *ic acid* is dropped and *aldehyde* added.

Ketones

The common names of ketones are derived from the acid which on pyrolysis would yield the ketone. A second method, especially useful for naming mixed ketones, simply names the alkyl groups and adds the word *ketone*. The name is written as three separate words.

Unsaturated Acyclic Hydrocarbons

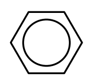
The simplest compounds in this class of hydrocarbon chemicals are olefins or alkenes with a single carbon-carbon double bond, having the general formula of C_nH_{2n}. The simplest example in this category is ethylene, C_2H_4.

Dienes are acyclic hydrocarbons with two carbon-carbon double bonds, having the general formula of C_nH_{2n-2}; butadiene (C_4H_6) is an example of such.

Similarly, trienes have three carbon-carbon double bonds with the general formula of C_nH_{2n-4}; hexatriene (C_6H_8) is such an example.

The simplest alkynes have a single carbon-carbon triple bond with the general formula of C_nH_{2n-2}. This series of compounds begins with acetylene, or C_2H_2.

Important Families of Organic Compounds

	FAMILY											
	Alkane	**Alkene**	**Alkyne**	**Arene**	**Haloalkane**	**Alcohol**	**Ether**	**Amine**	**Aldehyde**	**Ketone**	**Carboxylic Acid**	**Ester**
Specific Example	CH_3CH_3	$H_2C = CH_2$	$HC \equiv CH$		CH_3CH_2Cl	CH_3CH_2OH	CH_3OCH_3	CH_3NH_2	$CH_3\overset{\overset{O}{\Vert}}{C}H$	$CH_3\overset{\overset{O}{\Vert}}{C}CH_3$	$CH_3\overset{\overset{O}{\Vert}}{C}OH$	$CH_3\overset{\overset{O}{\Vert}}{C}OCH_3$
IUPAC Name	Ethane	Ethene or Ethylene	Ethyne or Acetylene	Benzene	Chloroethane	Ethanol	Methoxy-methane	Methan-amine	Ethanal	Acetone	Ethanoic Acid	Methyl ethanoate
Common Name	Ethane	Ethylene	Acetylene	Benzene	Ethyl chloride	Ethyl alcohol	Dimethyl ether	Methyl-amine	Acetal-dehyde	Dimethyl ketone	Acetic Acid	Methyl acetate
General Formula	RH	$RCH = CH_2$ $RCH = CHR$ $R_2C = CHR$ $R_2C = CR_2$	$RC \equiv CH$ $RC \equiv CR$	ArH	RX	ROH	ROR	RNH_2 R_2NH R_3N	$R\overset{\overset{O}{\Vert}}{C}H$	$R_1\overset{\overset{O}{\Vert}}{C}R_2$	$R\overset{\overset{O}{\Vert}}{C}OH$	$R\overset{\overset{O}{\Vert}}{C}OR$
Functional Group	C–H and C–C bonds	$\gt C = C \lt$	$- C \equiv C -$	Aromatic Ring	$-\overset{\vert}{\underset{\vert}{C}}-X$	$-\overset{\vert}{\underset{\vert}{C}}-OH$	$-\overset{\vert}{\underset{\vert}{C}}-O-\overset{\vert}{\underset{\vert}{C}}-$	$-\overset{\vert}{\underset{\vert}{C}}-\underset{\vert}{N}-$	$-\overset{\overset{O}{\Vert}}{C}-H$	$-\overset{\overset{O}{\Vert}}{C}-$	$-\overset{\overset{O}{\Vert}}{C}-OH$	$-\overset{\overset{O}{\Vert}}{C}-O-\overset{\vert}{\underset{\vert}{C}}-$

Common Names and Molecular Formulas of Some Industrial (Inorganic and Organic) Chemicals

Common Name	Chemical Name	Molecular Formula
Muriatic acid	Hydrochloric acid	HCl
Cumene	Isopropyl benzene	$C_6H_5CH(CH_3)_2$
Styrene	Vinyl benzene	$C_6H_5CH{=}CH_2$
—	Hypochlorite ion	OCl^{-1}
—	Chlorite ion	ClO_2^{-1}
—	Chlorate ion	ClO_3^{-1}
—	Perchlorate ion	ClO_4^{-1}
Gypsum	Calcium sulfate	$CaSO_4$
Limestone	Calcium carbonate	$CaCO_3$
Dolomite	Magnesium carbonate	$MgCO_3$
Bauxite	Aluminum oxide	Al_2O_3
Anatase	Titanium dioxide	TiO_2
Rutile	Titanium dioxide	TiO_2
—	Vinyl chloride	$CH_2{=}CHCl$
—	Ethylene oxide	C_2H_4O
Pyrite	Ferrous sulfide	FeS
Epsom salt	Magnesium sulfate	$MgSO_4$
Hydroquinone	p-Dihydroxy benzene	$C_6H_4(OH)_2$
Soda ash	Sodium carbonate	Na_2CO_3
Salt	Sodium chloride	$NaCl$
Potash	Potassium carbonate	K_2CO_3
Baking soda	Sodium bicarbonate	$NaHCO_3$
Lye	Sodium hydroxide	$NaOH$
Caustic soda	Sodium hydroxide	$NaOH$
—	Vinyl alcohol	$CH_2{=}CHOH$
Carbolic acid	Phenol	C_6H_5OH
Aniline	Aminobenzene	$C_6H_5NH_2$
—	Urea	$(NH_2)_2CO$
Toluene	Methyl benzene	$C_6H_5CH_3$
Xylene	Dimethyl benzene	$C_6H_4(CH_3)_2$
—	Silane	SiH_4
—	Ozone	O_3
Neopentane	2,2-Dimethylpropane	$CH_3C(CH_3)_2CH_3$
Magnetite	Ferrous/ferric oxide	Fe_3O_4
Quicksilver	Mercury	Hg
Heavy water	Deuterium oxide	^2H_2O
—	Borane	BH_3
Eyewash	Boric acid (solution)	H_3BO_3
—	Deuterium	^2H
—	Tritium	^3H
Laughing gas	Nitrous oxide	N_2O
—	Phosgene	$COCl_2$
Wolfram	Tungsten	W
—	Permanganate ion	MnO_4^{-1}
—	Dichromate ion	$Cr_2O_7^{-2}$
—	Hydronium ion	H_3O^{+1}
Brine	Sodium chloride (solution)	$NaCl$
Battery acid	Sulfuric acid	H_2SO_4

Electrochemistry

Cathode – The electrode at which reduction occurs.

Anode – The electrode at which oxidation occurs.

Oxidation – The loss of electrons.

Reduction – The gaining of electrons.

Cation – Positive ion

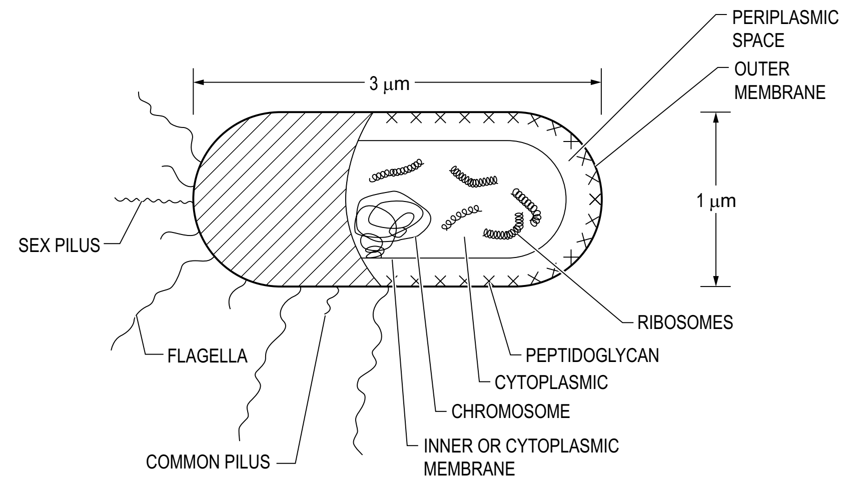
Anion – Negative ion

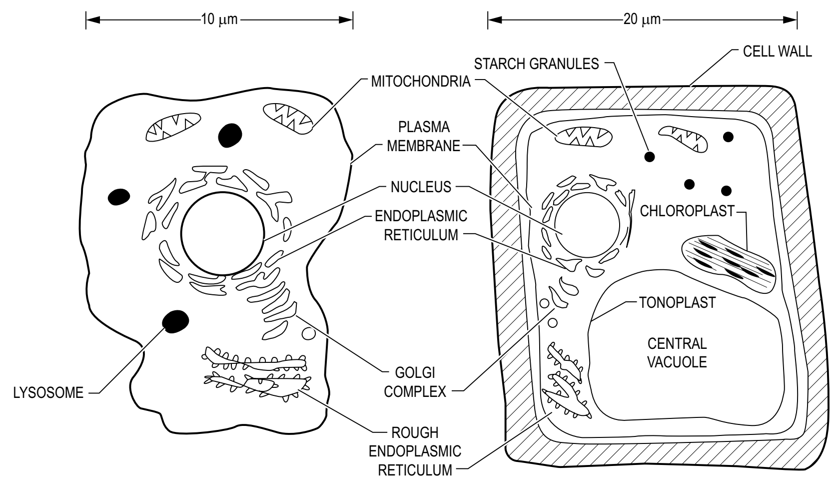
Standard Oxidation Potentials for Corrosion Reactions*	
Corrosion Reaction	**Potential, E_o, Volts vs. Normal Hydrogen Electrode**
$Au \rightarrow Au^{3+} + 3e^-$	–1.498
$2H_2O \rightarrow O_2 + 4H^+ + 4e^-$	–1.229
$Pt \rightarrow Pt^{2+} + 2e^-$	–1.200
$Pd \rightarrow Pd^{2+} + 2e^-$	–0.987
$Ag \rightarrow Ag^+ + e^-$	–0.799
$2Hg \rightarrow Hg_2^{2+} + 2e^-$	–0.788
$Fe^{2+} \rightarrow Fe^{3+} + e^-$	–0.771
$4(OH)^- \rightarrow O_2 + 2H_2O + 4e^-$	–0.401
$Cu \rightarrow Cu^{2+} + 2e^-$	–0.337
$Sn^{2+} \rightarrow Sn^{4+} + 2e^-$	–0.150
$H_2 \rightarrow 2H^+ + 2e^-$	0.000
$Pb \rightarrow Pb^{2+} + 2e^-$	+0.126
$Sn \rightarrow Sn^{2+} + 2e^-$	+0.136
$Ni \rightarrow Ni^{2+} + 2e^-$	+0.250
$Co \rightarrow Co^{2+} + 2e^-$	+0.277
$Cd \rightarrow Cd^{2+} + 2e^-$	+0.403
$Fe \rightarrow Fe^{2+} + 2e^-$	+0.440
$Cr \rightarrow Cr^{3+} + 3e^-$	+0.744
$Zn \rightarrow Zn^{2+} + 2e^-$	+0.763
$Al \rightarrow Al^{3+} + 3e^-$	+1.662
$Mg \rightarrow Mg^{2+} + 2e^-$	+2.363
$Na \rightarrow Na^+ + e^-$	+2.714
$K \rightarrow K^+ + e^-$	+2.925
* Measured at 25°C. Reactions are written as anode half-cells. Arrows are reversed for cathode half-cells.	

Flinn, Richard A., and Paul K. Trojan, *Engineering Materials and Their Applications*, 4th ed., Houghton Mifflin Company, 1990.

NOTE: In some chemistry texts, the reactions and the signs of the values (in this table) are reversed; for example, the half-cell potential of zinc is given as –0.763 volt for the reaction $Zn^{2+} + 2e^- \rightarrow Zn$. When the potential E_o is positive, the reaction proceeds spontaneously as written.

Cellular Biology





Shuler, Michael L., & Fikret Kargi, *Bioprocess Engineering Basic Concepts*, Prentice Hall PTR, New Jersey, 1992.

Materials Science/Structure of Matter

Atomic Bonding

Primary Bonds

Ionic (e.g., salts, metal oxides)
Covalent (e.g., within polymer molecules)
Metallic (e.g., metals)

Corrosion

A table listing the standard electromotive potentials of metals is shown on the previous page.

For corrosion to occur, there must be an anode and a cathode in electrical contact in the presence of an electrolyte.

Anode Reaction (Oxidation) of a Typical Metal, M

$M^o \rightarrow M^{n+} + ne^-$

Possible Cathode Reactions (Reduction)

$\frac{1}{2} O_2 + 2 e^- + H_2O \rightarrow 2 OH^-$

$\frac{1}{2} O_2 + 2 e^- + 2 H_3O^+ \rightarrow 3 H_2O$

$2 e^- + 2 H_3O^+ \rightarrow 2 H_2O + H_2$

When dissimilar metals are in contact, the more electropositive one becomes the anode in a corrosion cell. Different regions of carbon steel can also result in a corrosion reaction: e.g., cold-worked regions are anodic to noncold-worked; different oxygen concentrations can cause oxygen-deficient regions to become cathodic to oxygen-rich regions; grain boundary regions are anodic to bulk grain; in multiphase alloys, various phases may not have the same galvanic potential.

Diffusion

Diffusion Coefficient

$$D = D_o e^{-Q/(RT)}$$

where

D = diffusion coefficient
D_o = proportionality constant
Q = activation energy
R = gas constant [8.314 J/(mol•K)]
T = absolute temperature

Thermal and Mechanical Processing

Cold working (plastically deforming) a metal increases strength and lowers ductility.

Raising temperature causes (1) recovery (stress relief), (2) recrystallization, and (3) grain growth. *Hot working* allows these processes to occur simultaneously with deformation.

Quenching is rapid cooling from elevated temperature, preventing the formation of equilibrium phases.

In steels, quenching austenite [FCC (γ) iron] can result in martensite instead of equilibrium phases—ferrite [BCC (α) iron] and cementite (iron carbide).

Properties of Materials

Electrical

Capacitance: The charge-carrying capacity of an insulating material

Charge held by a capacitor

$$q = CV$$

where

q = charge

C = capacitance

V = voltage

Capacitance of a parallel plate capacitor

$$C = \frac{\varepsilon A}{d}$$

where

C = capacitance

ε = permittivity of material

A = cross-sectional area of the plates

d = distance between the plates

ε is also expressed as the product of the dielectric constant (κ) and the permittivity of free space ($\varepsilon_0 = 8.85 \times 10^{-12}$ F/m)

Resistivity: The material property that determines the resistance of a resistor

Resistivity of a material within a resistor

$$\rho = \frac{RA}{L}$$

where

ρ = resistivity of the material

R = resistance of the resistor

A = cross-sectional area of the resistor

L = length of the resistor

Conductivity is the reciprocal of the resistivity

Photoelectric effect–electrons are emitted from matter (metals and nonmetallic solids, liquids or gases) as a consequence of their absorption of energy from electromagnetic radiation of very short wavelength and high frequency.

Piezoelectric effect–the electromechanical and the electrical state in crystalline materials.

Mechanical

Strain is defined as change in length per unit length; for pure tension the following apply:

Engineering strain

$$\varepsilon = \frac{\Delta L}{L_0}$$

where

ε = engineering strain
ΔL = change in length
L_0 = initial length

True strain

$$\varepsilon_T = \frac{dL}{L}$$

where

ε_T = true strain
dL = differential change in length
L = initial length
ε_T = ln (1 + ε)

Properties of Metals

Metal	Symbol	Atomic Weight	Density ρ (kg/m^3) Water = 1000	Melting Point (°C)	Melting Point (°F)	Specific Heat (J/(kg·K))	Electrical Resistivity (10^{-8} Ω·m) at 0°C (273.2 K)	Heat Conductivity λ (W/(m·K)) at 0°C (273.2 K)
Aluminum	Al	26.98	2,698	660	1,220	895.9	2.5	236
Antimony	Sb	121.75	6,692	630	1,166	209.3	39	25.5
Arsenic	As	74.92	5,776	subl. 613	subl. 1,135	347.5	26	–
Barium	Ba	137.33	3,594	710	1,310	284.7	36	–
Beryllium	Be	9.012	1,846	1,285	2,345	2,051.5	2.8	218
Bismuth	Bi	208.98	9,803	271	519	125.6	107	8.2
Cadmium	Cd	112.41	8,647	321	609	234.5	6.8	97
Caesium	Cs	132.91	1,900	29	84	217.7	18.8	36
Calcium	Ca	40.08	1,530	840	1,544	636.4	3.2	–
Cerium	Ce	140.12	6,711	800	1,472	188.4	7.3	11
Chromium	Cr	52	7,194	1,860	3,380	406.5	12.7	96.5
Cobalt	Co	58.93	8,800	1,494	2,721	431.2	5.6	105
Copper	Cu	63.54	8,933	1,084	1,983	389.4	1.55	403
Gallium	Ga	69.72	5,905	30	86	330.7	13.6	41
Gold	Au	196.97	19,281	1,064	1,947	129.8	2.05	319
Indium	In	114.82	7,290	156	312	238.6	8	84
Iridium	Ir	192.22	22,550	2,447	4,436	138.2	4.7	147
Iron	Fe	55.85	7,873	1,540	2,804	456.4	8.9	83.5
Lead	Pb	207.2	11,343	327	620	129.8	19.2	36
Lithium	Li	6.94	533	180	356	4,576.2	8.55	86
Magnesium	Mg	24.31	1,738	650	1,202	1,046.7	3.94	157
Manganese	Mn	54.94	7,473	1,250	2,282	502.4	138	8
Mercury	Hg	200.59	13,547	–39	–38	142.3	94.1	7.8
Molybendum	Mo	95.94	10,222	2,620	4,748	272.1	5	139
Nickel	Ni	58.69	8,907	1,455	2,651	439.6	6.2	94
Niobium	Nb	92.91	8,578	2,425	4,397	267.9	15.2	53
Osmium	Os	190.2	22,580	3,030	5,486	129.8	8.1	88
Palladium	Pd	106.4	11,995	1,554	2,829	230.3	10	72
Platinum	Pt	195.08	21,450	1,772	3,221	134	9.81	72
Potassium	K	39.09	862	63	145	753.6	6.1	104
Rhodium	Rh	102.91	12,420	1,963	3,565	242.8	4.3	151
Rubidium	Rb	85.47	1,533	38.8	102	330.7	11	58
Ruthenium	Ru	101.07	12,360	2,310	4,190	255.4	7.1	117
Silver	Ag	107.87	10,500	961	1,760	234.5	1.47	428
Sodium	Na	22.989	966	97.8	208	1,235.1	4.2	142
Strontium	Sr	87.62	2,583	770	1,418	–	20	–
Tantalum	Ta	180.95	16,670	3,000	5,432	150.7	12.3	57
Thallium	Tl	204.38	11,871	304	579	138.2	10	10
Thorium	Th	232.04	11,725	1,700	3,092	117.2	14.7	54
Tin	Sn	118.69	7,285	232	449	230.3	11.5	68
Titanium	Ti	47.88	4,508	1,670	3,038	527.5	39	22
Tungsten	W	183.85	19,254	3,387	6,128	142.8	4.9	177
Uranium	U	238.03	19,050	1,135	2,075	117.2	28	27
Vanadium	V	50.94	6,090	1,920	3,488	481.5	18.2	31
Zinc	Zn	65.38	7,135	419	786	393.5	5.5	117
Zirconium	Zr	91.22	6,507	1,850	3,362	284.7	40	23

Some Extrinsic, Elemental Semiconductors

Element	Dopant	Periodic table group of dopant	Maximum solid solubility of dopant (atoms/m^3)
Si	B	III A	600×10^{24}
	AI	III A	20×10^{24}
	Ga	III A	40×10^{24}
	P	V A	$1{,}000 \times 10^{24}$
	As	V A	$2{,}000 \times 10^{24}$
	Sb	V A	70×10^{24}
Ge	Al	III A	400×10^{24}
	Ga	III A	500×10^{24}
	In	III A	4×10^{24}
	As	V A	80×10^{24}
	Sb	V A	10×10^{24}

Impurity Energy Levels for Extrinsic Semiconductors

Semiconductor	Dopant	$E_g - E_d$ (eV)	E_a (eV)
Si	P	0.044	–
	As	0.049	–
	Sb	0.039	–
	Bi	0.069	–
	B	–	0.045
	Al	–	0.057
	Ga	–	0.065
	In	–	0.160
	Tl	–	0.260
Ge	P	0.012	–
	As	0.013	–
	Sb	0.096	–
	B	–	0.010
	Al	–	0.010
	Ga	–	0.010
	In	–	0.011
	Tl	–	0.01
GaAs	Se	0.005	–
	Te	0.003	–
	Zn	–	0.024
	Cd	–	0.021

Runyan, W.R., and S.B. Watelski, *Handbook of Materials and Processes for Electronics*, C.A. Harper, ed., New York: McGraw-Hill, 1970.

Stress is defined as force per unit area; for pure tension the following apply:

Engineering stress

$$\sigma = \frac{F}{A_0}$$

where

σ = engineering stress

F = applied force

A_0 = initial cross-sectional area

True stress

$$\sigma_T = \frac{F}{A}$$

where

σ_T = true stress

F = applied force

A = actual cross-sectional area

The elastic modulus (also called modulus, modulus of elasticity, Young's modulus) describes the relationship between engineering stress and engineering strain during elastic loading. Hooke's Law applies in such a case.

$$\sigma = E\varepsilon$$

where E = elastic modulus

Key mechanical properties obtained from a tensile test curve:

- Elastic modulus
- Ductility (also called percent elongation): Permanent engineering strain after failure
- Ultimate tensile strength (also called tensile strength): Maximum engineering stress
- Yield strength: Engineering stress at which permanent deformation is first observed, calculated by 0.2% offset method.

Other mechanical properties:

- Creep: Time-dependent deformation under load. Usually measured by strain rate. For steady-state creep this is:

$$\frac{d\varepsilon}{dt} = A\sigma^n e^{-\frac{Q}{RT}}$$

where

A = pre-exponential constant

n = stress sensitivity

Q = activation energy for creep

R = ideal gas law constant

T = absolute temperature

- Fatigue: Time-dependent failure under cyclic load. Fatigue life is the number of cycles to failure. The endurance limit is the stress below which fatigue failure is unlikely.

$$R = \frac{\sigma_{\min}}{\sigma_{\max}}$$

where R = stress ratio

For $R = -1$ and high cycle fatigue, based on the Basquin equation:

$$N = \left(\frac{\sigma_r}{A}\right)^{\frac{1}{B}}$$

where

N = cycles to failure

σ_r = completely (fully) reversed stress

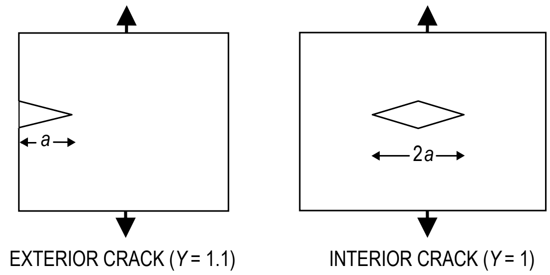
A and B = material constants

• Fracture toughness: The combination of applied stress and the crack length in a brittle material. It is the stress intensity when the material will fail.

$$K_{IC} = Y\sigma\sqrt{\pi a}$$

where

K_{IC} = fracture toughness
σ = applied engineering stress
a = crack length
Y = geometrical factor



The critical value of stress intensity at which catastrophic crack propagation occurs, K_{Ic}, is a material property.

Representative Values of Fracture Toughness

Material	K_{Ic} (MPa•m$^{1/2}$)	K_{Ic} (ksi-in$^{1/2}$)
A1 2014-T651	24.2	22
A1 2024-T3	44	40
52100 Steel	14.3	13
4340 Steel	46	42
Alumina	4.5	4.1
Silicon Carbide	3.5	3.2

Relationship Between Hardness and Tensile Strength

For plain carbon steels, there is a general relationship between Brinell hardness and tensile strength as follows:

$$\text{TS}(\text{psi}) \simeq 500\ \text{BHN}$$

$$\text{TS}(\text{MPa}) \simeq 3.5\ \text{BHN}$$

ASTM Grain Size

$$S_V = 2P_L$$

$$N_{(0.0645\ \text{mm}^2)} = 2^{(n-1)}$$

$$\frac{N_{\text{actual}}}{\text{Actual Area}} = \frac{N}{(0.0645\ \text{mm}^2)}$$

where

S_V = grain-boundary surface per unit volume
P_L = number of points of intersection per unit length between the line and the boundaries
N = number of grains observed in an area of 0.0645 mm^2
n = grain size (nearest integer > 1)

Composite Materials

$$\rho_c = \Sigma f_i \rho_i$$

$$C_c = \Sigma f_i c_i$$

$$\left[\Sigma \frac{f_i}{E_i}\right]^{-1} \le E_c \le \Sigma f_i E_i$$

$$\sigma_c = \Sigma f_i \sigma_i$$

where

ρ_c = density of composite

C_c = heat capacity of composite per unit volume

E_c = Young's modulus of composite

f_i = volume fraction of individual material

c_i = heat capacity of individual material per unit volume

E_i = Young's modulus of individual material

σ_c = strength parallel to fiber direction

Also, for axially oriented, long, fiber-reinforced composites, the strains of the two components are equal.

$$(\Delta L/L)_1 = (\Delta L/L)_2$$

where

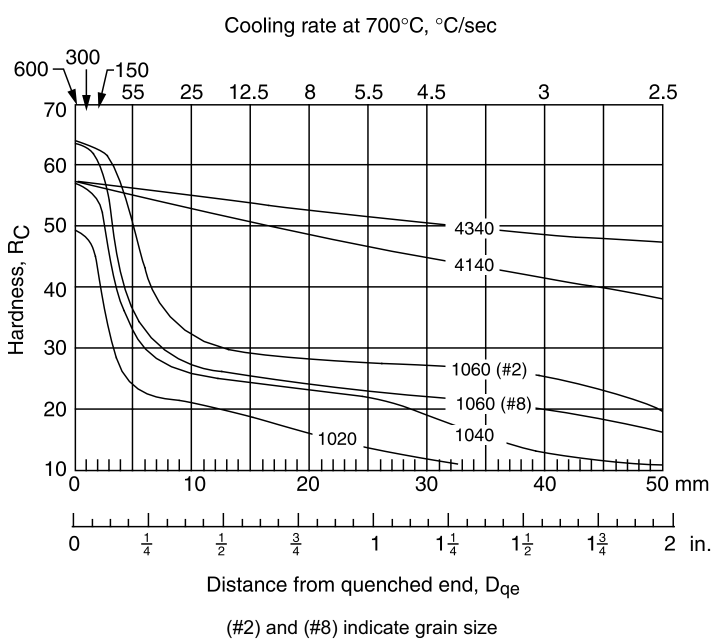
ΔL = change in length of the composite

L = original length of the composite

Hardenability

Hardness: Resistance to penetration. Measured by denting a material under known load and measuring the size of the dent.

Hardenability: The "ease" with which hardness can be obtained.



JOMINY HARDENABILITY CURVES FOR SIX STEELS

The following two graphs show cooling curves for four different positions in the bar.

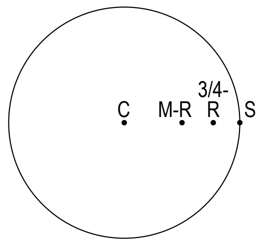
C = Center

M-R = Halfway between center and surface

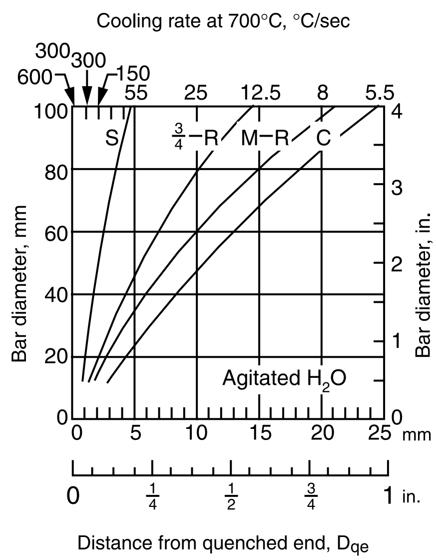
3/4-R = 75% of the distance between the center and the surface

S = Surface

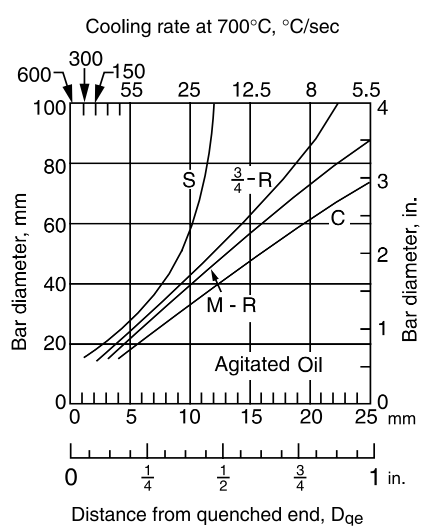
These positions are shown in the following figure.



COOLING RATES FOR BARS QUENCHED IN AGITATED WATER

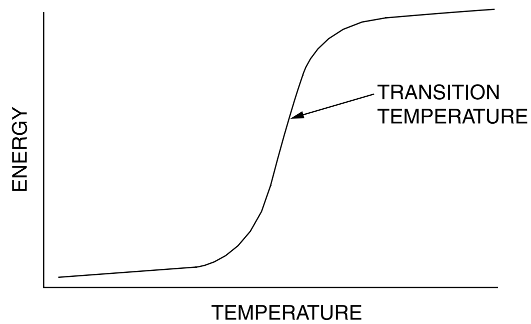


COOLING RATES FOR BARS QUENCHED IN AGITATED OIL



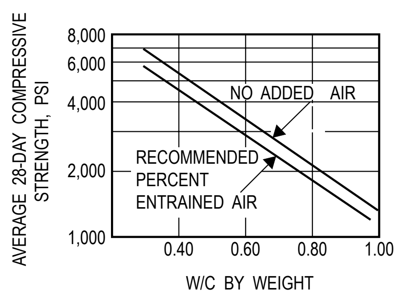
Impact Test

The *Charpy Impact Test* is used to find energy required to fracture and to identify ductile to brittle transition.



Impact tests determine the amount of energy required to cause failure in standardized test samples. The tests are repeated over a range of temperatures to determine the *ductile to brittle transition temperature.*

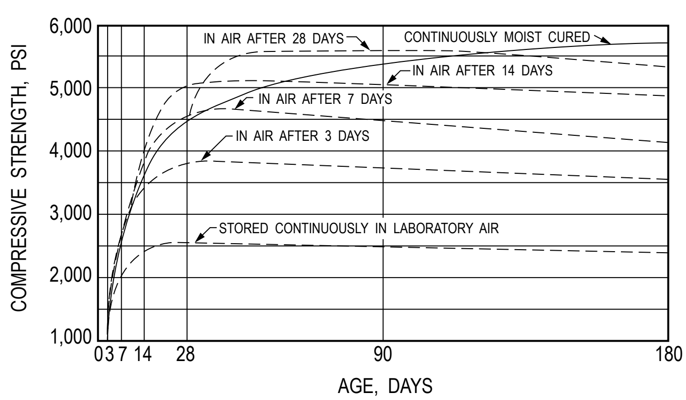
Concrete



Concrete strength decreases with increases in water-cement ratio for concrete with and without entrained air.

Concrete Manual, 8th ed., U.S. Bureau of Reclamation, 1975.

Water-cement (W/C) ratio is the primary factor affecting the strength of concrete. The figure above shows how W/C expressed as a ratio of weight of water and cement by weight of concrete mix affects the compressive strength of both air-entrained and non-air-entrained concrete.



Concrete compressive strength varies with moist-curing conditions. Mixes tested had a water-cement ratio of 0.50, a slump of 3.5 in., cement content of 556 lb/yd^3, sand content of 36%, and air content of 4%.

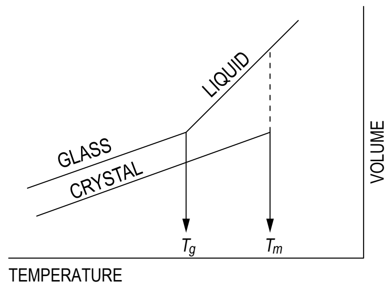
Merritt, Frederick S., *Standard Handbook for Civil Engineers*, 3rd ed., McGraw-Hill, 1983.

Water content affects workability. However, an increase in water without a corresponding increase in cement reduces the concrete strength. Superplasticizers are the most typical way to increase workability. Air entrainment is used to improve durability.

Amorphous Materials

Amorphous materials such as glass are non-crystalline solids.
Thermoplastic polymers are either semicrystalline or amorphous.

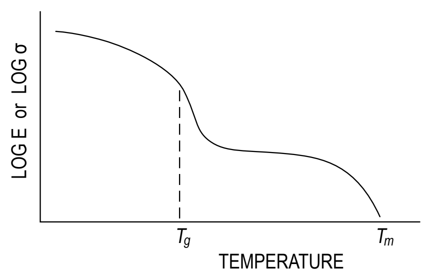
Below the glass transition temperature (T_g) the amorphous material will be a brittle solid.



The volume temperature curve as shown above is often used to show the difference between amorphous and crystalline solids.

Polymers

Polymers are classified as thermoplastics that can be melted and reformed. Thermosets cannot be melted and reformed.



The above curve shows the temperature dependent strength (σ) or modulus (E) for a thermoplastic polymer.

Polymer Additives

Chemicals and compounds are added to polymers to improve properties for commercial use. These substances, such as plasticizers, improve formability during processing, while others increase strength or durability.

Examples of common additives are:

Plasticizers: vegetable oils, low molecular weight polymers or monomers

Fillers: talc, chopped glass fibers

Flame retardants: halogenated paraffins, zinc borate, chlorinated phosphates

Ultraviolet or visible light resistance: carbon black

Oxidation resistance: phenols, aldehydes

Thermal Properties

The thermal expansion coefficient is the ratio of engineering strain to the change in temperature.

$$\alpha = \frac{\varepsilon}{\Delta T}$$

where

α = thermal expansion coefficient

ε = engineering strain

ΔT = change in temperature

Specific heat (also called heat capacity) is the amount of heat required to raise the temperature of something or an amount of something by 1 degree.

At constant pressure the amount of heat (Q) required to increase the temperature of something by ΔT is $C_p\Delta T$, where C_p is the constant pressure heat capacity.

At constant volume the amount of heat (Q) required to increase the temperature of something by ΔT is $C_v\Delta T$, where C_v is the constant volume heat capacity.

An object can have a heat capacity that would be expressed as energy/degree.

The heat capacity of a material can be reported as energy/degree per unit mass or per unit volume.

Binary Phase Diagrams

Allows determination of (1) what phases are present at equilibrium at any temperature and average composition, (2) the compositions of those phases, and (3) the fractions of those phases.

Eutectic reaction (liquid → two solid phases)

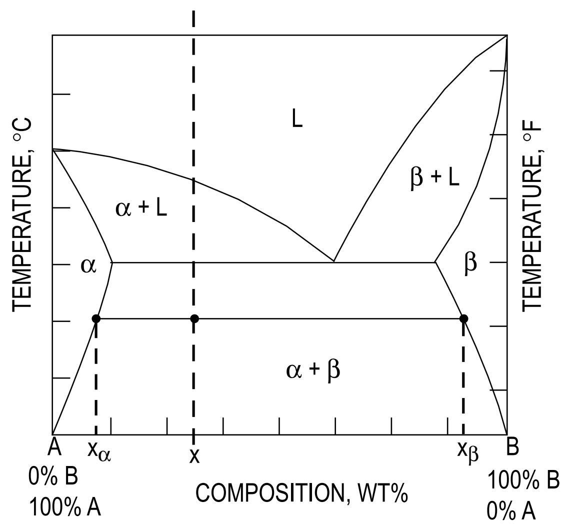
Eutectoid reaction (solid → two solid phases)

Peritectic reaction (liquid + solid → solid)

Peritectoid reaction (two solid phases → solid)

Lever Rule

The following phase diagram and equations illustrate how the weight of each phase in a two-phase system can be determined:

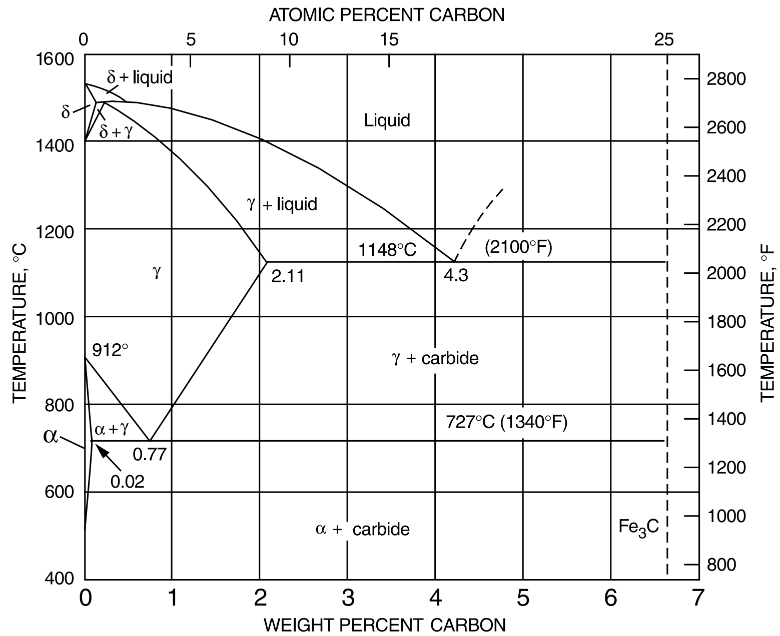


(In diagram, L = liquid.) If x = the average composition at temperature T, then

$$\text{wt\%}\ \alpha = \frac{x_\beta - x}{x_\beta - x_\alpha} \times 100$$

$$\text{wt\%}\ \beta = \frac{x - x_\alpha}{x_\beta - x_\alpha} \times 100$$

Iron-Iron Carbide Phase Diagram



Van Vlack, L.H., *Elements of Materials Science and Engineering*, 6th ed., ©1989. Reprinted by permission of Pearson Education, Inc., New, New York.

Statics

Force (Two Dimensions)

A *force* is a *vector* quantity. It is defined when its **(1)** magnitude, **(2)** point of application, and **(3)** direction are known.

The vector form of a force is

$$\boldsymbol{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j}$$

Resultant (Two Dimensions)

The *resultant*, F, of n forces with components $F_{x,i}$ and $F_{y,i}$ has the magnitude of

$$F = \left[\left(\sum_{i=1}^{n} F_{x,i}\right)^2 + \left(\sum_{i=1}^{n} F_{y,i}\right)^2\right]^{1/2}$$

The resultant direction with respect to the x-axis is

$$\theta = \arctan\left(\sum_{i=1}^{n} F_{y,i} \Big/ \sum_{i=1}^{n} F_{x,i}\right)$$

Resolution of a Force

$$F_x = F\cos\theta_x \qquad F_y = F\cos\theta_y \qquad F_z = F\cos\theta_z$$

$$\cos\theta_x = F_x/F \qquad \cos\theta_y = F_y/F \qquad \cos\theta_z = F_z/F$$

Separating a force into components when the geometry of force is known and $R = \sqrt{x^2 + y^2 + z^2}$

$$F_x = (x/R)F \qquad F_y = (y/R)F \qquad F_z = (z/R)F$$

Moments (Couples)

A system of two forces that are equal in magnitude, opposite in direction, and parallel to each other is called a *couple*. A *moment* $\boldsymbol{M}$ is defined as the cross product of the *radius vector* $\boldsymbol{r}$ and the *force* $\boldsymbol{F}$ from a point to the line of action of the force.

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \qquad \begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned}$$

Systems of Forces

$$\boldsymbol{F} = \Sigma\,\boldsymbol{F}_n$$

$$\boldsymbol{M} = \Sigma\,(\boldsymbol{r}_n \times \boldsymbol{F}_n)$$

Equilibrium Requirements

$$\Sigma\,\boldsymbol{F}_n = 0$$

$$\Sigma\,\boldsymbol{M}_n = 0$$

Centroids of Masses, Areas, Lengths, and Volumes

The following formulas are for discrete masses, areas, lengths, and volumes:

$\boldsymbol{r}_c = \Sigma\, m_n \boldsymbol{r}_n / \Sigma\, m_n$

where

m_n = mass of each particle making up the system

$\boldsymbol{r}_n$ = radius vector to each particle from a selected reference point

$\boldsymbol{r}_c$ = radius vector to the centroid of the total mass from the selected reference point

The *moment of area* (M_a) is defined as

$M_{ay} = \Sigma\, x_n a_n$

$M_{ax} = \Sigma\, y_n a_n$

The *centroid of area* is defined as

$x_{ac} = M_{ay}/A = \Sigma\, x_n\, a_n/A$

$y_{ac} = M_{ax}/A = \Sigma\, y_n\, a_n/A$

where $A = \Sigma\, a_n$

The following equations are for an area, bounded by the axes and the function $y = f(x)$. The centroid of area is defined as

$$x_c = \frac{\int x dA}{A}$$

$$y_c = \frac{\int y dA}{A}$$

$$A = \int f(x)\, dx$$

$$dA = f(x)\, dx = g(y)\, dy$$

The *first moment of area* with respect to the *y*-axis and the *x*-axis, respectively, are:

$M_y = \int x\, dA = x_c A$

$M_x = \int y\, dA = y_c A$

Moment of Inertia

The *moment of inertia*, or the second moment of area, is defined as

$I_y = \int x^2\, dA$

$I_x = \int y^2\, dA$

The *polar moment of inertia* J of an area about a point is equal to the sum of the moments of inertia of the area about any two perpendicular axes in the area and passing through the same point.

$I_z = J = I_y + I_x = \int (x^2 + y^2)\, dA$

$= r_p^{\,2} A$

where r_p = the radius of gyration (as defined below)

Moment of Inertia Parallel Axis Theorem

The moment of inertia of an area about any axis is defined as the moment of inertia of the area about a parallel centroidal axis plus a term equal to the area multiplied by the square of the perpendicular distance *d* from the centroidal axis to the axis in question.

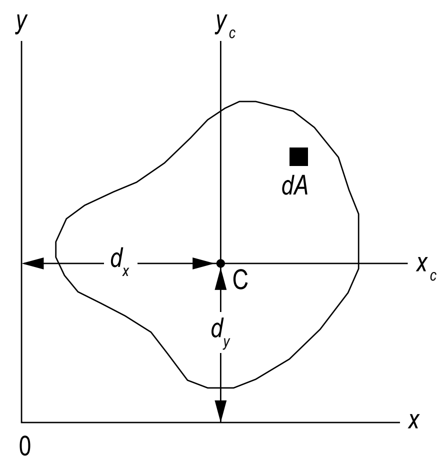
$$I_x = I_{x_c} + d_y^2 A$$
$$I_y = I_{y_c} + d_x^2 A$$

where

d_x, d_y = distance between the two axes in question

I_{x_c}, I_{y_c} = moment of inertia about the centroidal axis

I_x, I_y = moment of inertia about the new axis



Hibbeler, R.C., *Engineering Mechanics: Statics and Dynamics*, 10 ed., Pearson Prentice Hall, 2004.

Radius of Gyration

The *radius of gyration* r_p, r_x, r_y is the distance from a reference axis at which all of the area can be considered to be concentrated to produce the moment of inertia.

$$r_x = \sqrt{I_x/A} \quad r_y = \sqrt{I_y/A} \quad r_p = \sqrt{J/A}$$

Product of Inertia

The *product of inertia* (I_{xy}, etc.) is defined as:

$I_{xy} = \int xydA$, with respect to the xy-coordinate system

The *parallel-axis theorem* also applies:

$I'_{xy} = I_{x_c y_c} + d_x d_y A$ for the xy-coordinate system, etc.

where

d_x = x-axis distance between the two axes in question

d_y = y-axis distance between the two axes in question

Friction

The largest frictional force is called the *limiting friction*.
Any further increase in applied forces will cause motion.

$$F \le \mu_s N$$

where

F = friction force

μ_s = coefficient of static friction

N = normal force between surfaces in contact

Screw Thread

For a *screw-jack*, *square thread*,

$$M = Pr \tan(\alpha \pm \phi)$$

where

+ is for screw tightening

– is for screw loosening

M = external moment applied to axis of screw

P = load on jack applied along and on the line of the axis

r = mean thread radius

α = pitch angle of the thread

μ = $\tan \phi$ = appropriate coefficient of friction

Belt Friction

$$F_1 = F_2 e^{\mu\theta}$$

where

F_1 = force being applied in the direction of impending motion

F_2 = force applied to resist impending motion

μ = coefficient of static friction

θ = total angle of contact between the surfaces expressed in radians

Statically Determinate Truss

Plane Truss: Method of Joints

The method consists of solving for the forces in the members by writing the two equilibrium equations for each joint of the truss.

$$\Sigma F_H = 0 \text{ and } \Sigma F_V = 0$$

where

F_H = horizontal forces and member components

F_V = vertical forces and member components

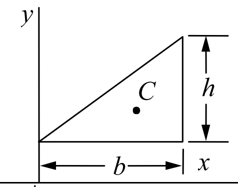
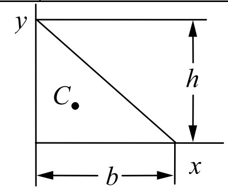
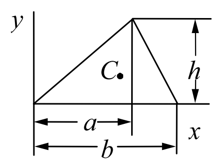
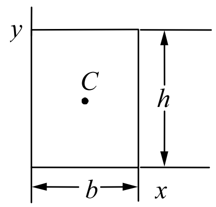
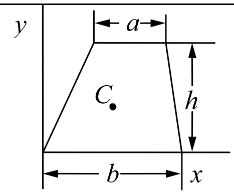
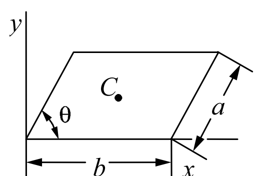
Plane Truss: Method of Sections

The method consists of drawing a free-body diagram of a portion of the truss in such a way that the unknown truss member force is exposed as an external force.

Concurrent Forces

A concurrent-force system is one in which the lines of action of the applied forces all meet at one point.

A *two-force* body in static equilibrium has two applied forces that are equal in magnitude, opposite in direction, and collinear.

Figure	Area & Centroid	Area Moment of Inertia	(Radius of Gyration)^2	Product of Inertia
	$A = bh/2$ $x_c = 2b/3$ $y_c = h/3$	$I_{x_c} = bh^3/36$ $I_{y_c} = b^3h/36$ $I_x = bh^3/12$ $I_y = b^3h/4$	$r_{x_c}^2 = h^2/18$ $r_{y_c}^2 = b^2/18$ $r_x^2 = h^2/6$ $r_y^2 = b^2/2$	$I_{x_cy_c} = Abh/36 = b^2h^2/72$ $I_{xy} = Abh/4 = b^2h^2/8$
	$A = bh/2$ $x_c = b/3$ $y_c = h/3$	$I_{x_c} = bh^3/36$ $I_{y_c} = b^3h/36$ $I_x = bh^3/12$ $I_y = b^3h/12$	$r_{x_c}^2 = h^2/18$ $r_{y_c}^2 = b^2/18$ $r_x^2 = h^2/6$ $r_y^2 = b^2/6$	$I_{x_cy_c} = -Abh/36 = -b^2h^2/72$ $I_{xy} = Abh/12 = b^2h^2/24$
	$A = bh/2$ $x_c = (a+b)/3$ $y_c = h/3$	$I_{x_c} = bh^3/36$ $I_{y_c} = \left[bh\left(b^2 - ab + a^2\right)\right]/36$ $I_x = bh^3/12$ $I_y = \left[bh\left(b^2 + ab + a^2\right)\right]/12$	$r_{x_c}^2 = h^2/18$ $r_{y_c}^2 = \left(b^2 - ab + a^2\right)/18$ $r_x^2 = h^2/6$ $r_y^2 = \left(b^2 + ab + a^2\right)/6$	$I_{x_cy_c} = \left[Ah(2a-b)\right]/36$ $= \left[bh^2(2a-b)\right]/72$ $I_{xy} = \left[Ah(2a+b)\right]/12$ $= \left[bh^2(2a+b)\right]/24$
	$A = bh$ $x_c = b/2$ $y_c = h/2$	$I_{x_c} = bh^3/12$ $I_{y_c} = b^3h/12$ $I_x = bh^3/3$ $I_y = b^3h/3$ $J = \left[bh\left(b^2 + h^2\right)\right]/12$	$r_{x_c}^2 = h^2/12$ $r_{y_c}^2 = b^2/12$ $r_x^2 = h^2/3$ $r_y^2 = b^2/3$ $r_p^2 = \left(b^2 + h^2\right)/12$	$I_{x_cy_c} = 0$ $I_{xy} = Abh/4 = b^2h^2/4$
	$A = h(a+b)/2$ $y_c = \dfrac{h(2a+b)}{3(a+b)}$	$I_{x_c} = \dfrac{h^3\left(a^2 + 4ab + b^2\right)}{36(a+b)}$ $I_x = \dfrac{h^3(3a+b)}{12}$	$r_{x_c}^2 = \dfrac{h^2\left(a^2 + 4ab + b^2\right)}{18(a+b)}$ $r_x^2 = \dfrac{h^2(3a+b)}{6(a+b)}$	
	$A = ab\sin\theta$ $x_c = (b + a\cos\theta)/2$ $y_c = (a\sin\theta)/2$	$I_{x_c} = \left(a^3b\sin^3\theta\right)/12$ $I_{y_c} = \left[ab\sin\theta\left(b^2 + a^2\cos^2\theta\right)\right]/12$ $I_x = \left(a^3b\sin^3\theta\right)/3$ $I_y = \left[ab\sin\theta(b + a\cos\theta)^2\right]/3$ $-\left(a^2b^2\sin\theta\cos\theta\right)/6$	$r_{x_c}^2 = (a\sin\theta)^2/12$ $r_{y_c}^2 = \left(b^2 + a^2\cos^2\theta\right)/12$ $r_x^2 = (a\sin\theta)^2/3$ $r_y^2 = (b + a\cos\theta)^2/3$ $-(ab\cos\theta)/6$	$I_{x_cy_c} = \left(a^3b\sin^2\theta\cos\theta\right)/12$

Housner, George W., and Donald E. Hudson, *Applied Mechanics Dynamics*, D. Van Nostrand Company, Inc., Princeton, NJ, 1959. Table reprinted by permission of G.W. Housner & D.E. Hudson.

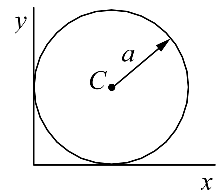
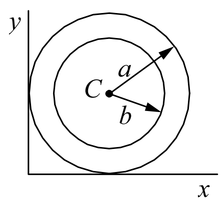
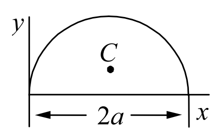
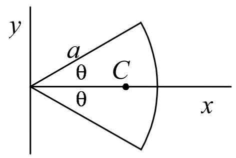
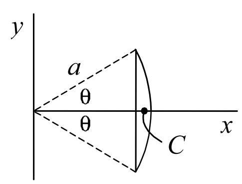
Figure	Area & Centroid	Area Moment of Inertia	(Radius of Gyration)^2	Product of Inertia
	$A = \pi a^2$ $x_c = a$ $y_c = a$	$I_{x_c} = I_{y_c} = \pi a^4/4$ $I_x = I_y = 5\pi a^4/4$ $J = \pi a^4/2$	$r_{x_c}^2 = r_{y_c}^2 = a^2/4$ $r_x^2 = r_y^2 = 5a^2/4$ $r_p^2 = a^2/2$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2$
	$A = \pi(a^2 - b^2)$ $x_c = a$ $y_c = a$	$I_{x_c} = I_{y_c} = \pi\left(a^4 - b^4\right)/4$ $I_x = I_y = \frac{5\pi a^4}{4} - \pi a^2 b^2 - \frac{\pi b^4}{4}$ $J = \pi\left(a^4 - b^4\right)/2$	$r_{x_c}^2 = r_{y_c}^2 = \left(a^2 + b^2\right)/4$ $r_x^2 = r_y^2 = \left(5a^2 + b^2\right)/4$ $r_p^2 = \left(a^2 + b^2\right)/2$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2$ $= \pi a^2\left(a^2 - b^2\right)$
	$A = \pi a^2/2$ $x_c = a$ $y_c = 4a/(3\pi)$	$I_{x_c} = \frac{a^4\left(9\pi^2 - 64\right)}{72\pi}$ $I_{y_c} = \pi a^4/8$ $I_x = \pi a^4/8$ $I_y = 5\pi a^4/8$	$r_{x_c}^2 = \frac{a^2\left(9\pi^2 - 64\right)}{36\pi^2}$ $r_{y_c}^2 = a^2/4$ $r_x^2 = a^2/4$ $r_y^2 = 5a^2/4$	$I_{x_c y_c} = 0$ $I_{xy} = 2a^4/3$
 CIRCULAR SECTOR	$A = a^2\theta$ $x_c = \frac{2a}{3}\frac{\sin\theta}{\theta}$ $y_c = 0$	$I_x = a^4(\theta - \sin\theta\cos\theta)/4$ $I_y = a^4(\theta + \sin\theta\cos\theta)/4$	$r_x^2 = \frac{a^2}{4}\frac{(\theta - \sin\theta\cos\theta)}{\theta}$ $r_y^2 = \frac{a^2}{4}\frac{(\theta + \sin\theta\cos\theta)}{\theta}$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
 CIRCULAR SEGMENT	$A = a^2\left[\theta - \frac{\sin 2\theta}{2}\right]$ $x_c = \frac{2a}{3}\frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}$ $y_c = 0$	$I_x = \frac{Aa^2}{4}\left[1 - \frac{2\sin^3\theta\cos\theta}{3\theta - 3\sin\theta\cos\theta}\right]$ $I_y = \frac{Aa^2}{4}\left[1 + \frac{2\sin^3\theta\cos\theta}{\theta - \sin\theta\cos\theta}\right]$	$r_x^2 = \frac{a^2}{4}\left[1 - \frac{2\sin^3\theta\cos\theta}{3\theta - 3\sin\theta\cos\theta}\right]$ $r_y^2 = \frac{a^2}{4}\left[1 + \frac{2\sin^3\theta\cos\theta}{\theta - \sin\theta\cos\theta}\right]$	$I_{x_c y_c} = 0$ $I_{xy} = 0$

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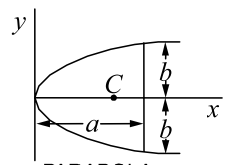
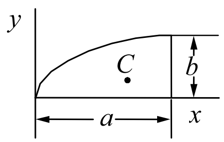
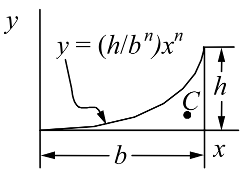
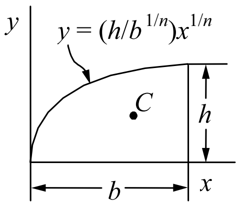
Figure	Area & Centroid	Area Moment of Inertia	(Radius of Gyration)^2	Product of Inertia
 PARABOLA	$A = 4ab/3$ $x_c = 3a/5$ $y_c = 0$	$I_{x_c} = I_x = 4ab^3/15$ $I_{y_c} = 16a^3b/175$ $I_y = 4a^3b/7$	$r_{x_c}^2 = r_x^2 = b^2/5$ $r_{y_c}^2 = 12a^2/175$ $r_y^2 = 3a^2/7$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
 HALF A PARABOLA	$A = 2ab/3$ $x_c = 3a/5$ $y_c = 3b/8$	$I_x = 2ab^3/15$ $I_y = 2ba^3/7$	$r_x^2 = b^2/5$ $r_y^2 = 3a^2/7$	$I_{xy} = Aab/4 = a^2b^2$
 nth DEGREE PARABOLA	$A = bh/(n+1)$ $x_c = \frac{n+1}{n+2}b$ $y_c = \frac{h}{2}\frac{n+1}{2n+1}$	$I_x = \frac{bh^3}{3(3n+1)}$ $I_y = \frac{hb^3}{n+3}$	$r_x^2 = \frac{h^2(n+1)}{3(3n+1)}$ $r_y^2 = \frac{n+1}{n+3}b^2$	
 nth DEGREE PARABOLA	$A = \frac{n}{n+1}bh$ $x_c = \frac{n+1}{2n+1}b$ $y_c = \frac{n+1}{2(n+2)}h$	$I_x = \frac{n}{3(n+3)}bh^3$ $I_y = \frac{n}{3n+1}b^3h$	$r_x^2 = \frac{n+1}{3(n+1)}h^2$ $r_y^2 = \frac{n+1}{3n+1}b^2$	

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Dynamics

Common Nomenclature

t = time
s = position coordinate, measured along a curve from an origin
v = velocity
a = acceleration
a_n = normal acceleration
a_t = tangential acceleration
θ = angular position coordinate
ω = angular velocity
α = angular acceleration
Ω = angular velocity of x,y,z reference axis measured from the X,Y,Z reference
$\dot{\Omega}$ = angular acceleration of x,y,z reference axis measured from the X,Y,Z reference
$\mathbf{r}_{A/B}$ = relative position of "A" with respect to "B"
$\mathbf{v}_{A/B}$ = relative velocity of "A" with respect to "B"
$\mathbf{a}_{A/B}$ = relative acceleration of "A" with respect to "B"

Particle Kinematics

Kinematics is the study of motion without consideration of the mass of, or the forces acting on, a system. For particle motion, let $\mathbf{r}(t)$ be the position vector of the particle in an inertial reference frame. The velocity and acceleration of the particle are defined, respectively, as

$\mathbf{v} = d\mathbf{r}/dt$
$\mathbf{a} = d\mathbf{v}/dt$

where

$\mathbf{v}$ = instantaneous velocity
$\mathbf{a}$ = instantaneous acceleration
t = time

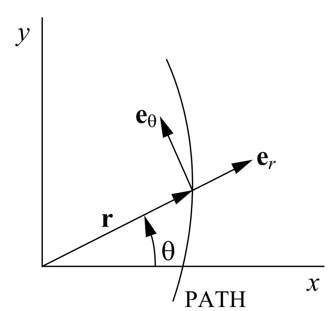
Cartesian Coordinates

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$
$$\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$$

where

$\dot{x} = dx/dt = v_x$, etc.
$\ddot{x} = d^2x/dt^2 = a_x$, etc.

Radial and Transverse Components for Planar Motion



Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are, respectively, collinear with and normal to the position vector **r**. Thus:

$$\mathbf{r} = r\mathbf{e}_r$$
$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$
$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

where

r = radial position coordinate

θ = angle from the x axis to **r**

$\dot{r} = dr/dt$, etc.

$\ddot{r} = d^2r/dt^2$, etc.

Particle Rectilinear Motion

Variable a	*Constant a = a_0*
$a = \dfrac{dv}{dt}$	$v = v_0 + a_0 t$
$v = \dfrac{ds}{dt}$	$s = s_0 + v_0 t + \dfrac{1}{2} a_0 t^2$
$a\,ds = v\,dv$	$s = s_0 + \dfrac{1}{2}\left(v_0 + v\right)t$
	$v^2 = v_0^2 + 2a_0\left(s - s_0\right)$

Particle Curvilinear Motion

x, y, z Coordinates		*r, θ, z Coordinates*	
$v_x = \dot{x}$	$a_x = \ddot{x}$	$v_r = \dot{r}$	$a_r = \ddot{r} - r\dot{\theta}^2$
$v_y = \dot{y}$	$a_y = \ddot{y}$	$v_\theta = r\dot{\theta}$	$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$v_z = \dot{z}$	$a_z = \ddot{z}$	$v_z = \dot{z}$	$a_z = \ddot{z}$

n, t, b Coordinates

$$v = \dot{s} \qquad a_t = \dot{v} = \frac{dv}{dt} = v\frac{dv}{ds}$$

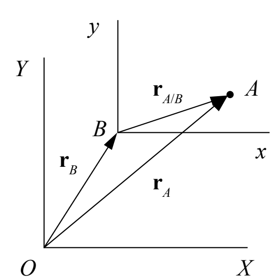
$$a_n = \frac{v^2}{\rho} \qquad \rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|\dfrac{d^2y}{dx^2}\right|}$$

Relative Motion

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B} \qquad \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \qquad \mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

Translating Axes x-y

The equations that relate the absolute and relative position, velocity, and acceleration vectors of two particles A and B, in plane motion, and separated at a constant distance, may be written as



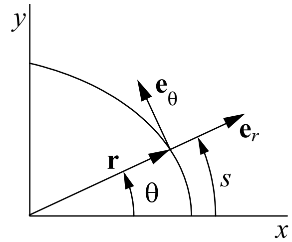
$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$
$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B} = \mathbf{v}_B + \mathbf{v}_{A/B}$$
$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}) = \mathbf{a}_B + \mathbf{a}_{A/B}$$

where ω and α are the absolute angular velocity and absolute angular acceleration of the relative position vector $\mathbf{r}_{A/B}$ of constant length, respectively.

Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

Plane Circular Motion

A special case of radial and transverse components is for constant radius rotation about the origin, or plane circular motion.



Here the vector quantities are defined as

$\mathbf{r} = r\mathbf{e}_r$

$\mathbf{v} = r\omega\mathbf{e}_\theta$

$\mathbf{a} = \left(-r\omega^2\right)\mathbf{e}_r + r\alpha\mathbf{e}_\theta$

where

r = radius of the circle

θ = angle from the x axis to $\mathbf{r}$

The values of the angular velocity and acceleration, respectively, are defined as

$\omega = \dot{\theta}$

$\alpha = \dot{\omega} = \ddot{\theta}$

Arc length, transverse velocity, and transverse acceleration, respectively, are

$s = r\theta$

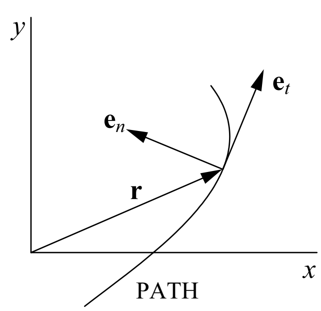
$v_\theta = r\omega$

$a_\theta = r\alpha$

The radial acceleration is given by

$a_r = -r\omega^2$ (towards the center of the circle)

Normal and Tangential Components



Unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$ are, respectively, tangent and normal to the path with $\mathbf{e}_n$ pointing to the center of curvature. Thus

$\mathbf{v} = v(t)\mathbf{e}_t$

$\mathbf{a} = a(t)\mathbf{e}_t + \left(v_t^2/\rho\right)\mathbf{e}_n$

where

ρ = instantaneous radius of curvature

Constant Acceleration

The equations for the velocity and displacement when acceleration is a constant are given as

$$a(t) = a_0$$
$$v(t) = a_0 (t - t_0) + v_0$$
$$s(t) = a_0 (t - t_0)^2/2 + v_0 (t - t_0) + s_0$$

where

s = displacement at time t, along the line of travel
s_0 = displacement at time t_0
v = velocity along the direction of travel
v_0 = velocity at time t_0
a_0 = constant acceleration
t = time
t_0 = some initial time

For a free-falling body, $a_0 = g$ (downward towards earth).

An additional equation for velocity as a function of position may be written as

$$v^2 = v_0^2 + 2a_0(s - s_0)$$

For constant angular acceleration, the equations for angular velocity and displacement are

$$\alpha(t) = \alpha_0$$
$$\omega(t) = \alpha_0(t - t_0) + \omega_0$$
$$\theta(t) = \alpha_0(t - t_0)^2/2 + \omega_0(t - t_0) + \theta_0$$

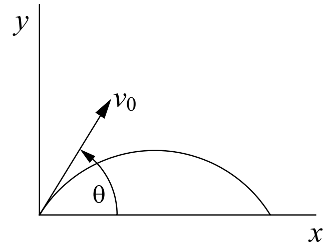
where

θ = angular displacement
θ_0 = angular displacement at time t_0
ω = angular velocity
ω_0 = angular velocity at time t_0
α_0 = constant angular acceleration
t = time
t_0 = some initial time

An additional equation for angular velocity as a function of angular position may be written as

$$\omega^2 = \omega_0^2 + 2\alpha_0(\theta - \theta_0)$$

Projectile Motion



The equations for common projectile motion may be obtained from the constant acceleration equations as

$a_x = 0$

$v_x = v_0 \cos(\theta)$

$x = v_0 \cos(\theta) t + x_0$

$a_y = -g$

$v_y = -gt + v_0 \sin(\theta)$

$y = -gt^2/2 + v_0 \sin(\theta) t + y_0$

Non-constant Acceleration

When non-constant acceleration, $a(t)$, is considered, the equations for the velocity and displacement may be obtained from

$$v(t) = \int_{t_0}^{t} a(\tau) d\tau + v_{t_0}$$

$$s(t) = \int_{t_0}^{t} v(\tau) d\tau + s_{t_0}$$

For variable angular acceleration

$$\omega(t) = \int_{t_0}^{t} \alpha(\tau) d\tau + \omega_{t_0}$$

$$\theta(t) = \int_{t_0}^{t} \omega(\tau) d\tau + \theta_{t_0}$$

where τ is the variable of integration

Concept of Weight

$W = mg$

where

W = weight (N or lbf)

m = mass (kg or lbf-sec^2/ft)

g = local acceleration of gravity (m/s^2 or ft/sec^2)

Particle Kinetics

Newton's second law for a particle is

$$\Sigma \mathbf{F} = d(m\boldsymbol{v})/dt$$

where

$\Sigma \mathbf{F}$ = sum of the applied forces acting on the particle

m = mass of the particle

$\boldsymbol{v}$ = velocity of the particle

For constant mass,

$$\Sigma \mathbf{F} = m\, d\boldsymbol{v}/dt = m\mathbf{a}$$

One-Dimensional Motion of a Particle (Constant Mass)

When motion exists only in a single dimension then, without loss of generality, it may be assumed to be in the x direction, and

$$a_x = F_x/m$$

where F_x = the resultant of the applied forces, which in general can depend on t, x, and v_x.

If F_x only depends on t, then

$$a_x(t) = F_x(t)/m$$

$$v_x(t) = \int_{t_0}^{t} a_x(\tau)d\tau + v_{xt_0}$$

$$x(t) = \int_{t_0}^{t} v_x(\tau)d\tau + x_{t_0}$$

where τ is the variable of integration.

If the force is constant (i.e., independent of time, displacement, and velocity) then

$$a_x = F_x/m$$

$$v_x = a_x(t - t_0) + v_{xt_0}$$

$$x = a_x(t - t_0)^2/2 + v_{xt_0}(t - t_0) + x_{t_0}$$

Normal and Tangential Kinetics for Planar Problems

When working with normal and tangential directions, the scalar equations may be written as

$$\Sigma F_t = ma_t = mdv_t/dt$$

$$\Sigma F_n = ma_n = m\left(v_t^2/\rho\right)$$

Principle of Work and Energy

If T_i and V_i are, respectively, the kinetic and potential energy of a particle at state i, then for conservative systems (no energy dissipation or gain), the law of conservation of energy is

$$T_2 + V_2 = T_1 + V_1$$

If nonconservative forces are present, then the work done by these forces must be accounted for. Hence

$$T_2 + V_2 = T_1 + V_1 + U_{1\to2}\text{, where}$$

$U_{1\to2}$ = the work done by the nonconservative forces in moving between state 1 and state 2. Care must be exercised during computations to correctly compute the algebraic sign of the work term. If the forces serve to increase the energy of the system, $U_{1\to2}$ is positive. If the forces, such as friction, serve to dissipate energy, $U_{1\to2}$ is negative.

Kinetic Energy

Particle	$T = \frac{1}{2}mv^2$
Rigid Body (Plane Motion)	$T = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\,\omega^2$

subscript *c* represents the center of mass

Potential Energy

$V = V_g + V_e$, where $V_g = Wy$, $V_e = 1/2\ ks^2$

The work done by an external agent in the presence of a conservative field is termed the change in potential energy.

Potential Energy in Gravity Field

$V_g = mgh$

where h = the elevation above some specified datum.

Elastic Potential Energy

For a linear elastic spring with modulus, stiffness, or spring constant, k, the force in the spring is

$F_s = k\,s$

where s = the change in length of the spring from the undeformed length of the spring.

In changing the deformation in the spring from position s_1 to s_2, the change in the potential energy stored in the spring is

$V_2 - V_1 = k\left(s_2^2 - s_1^2\right)/2$

Work

Work U is defined as

$\underline{U} = \int \mathbf{F} \cdot d\mathbf{r}$

Variable force	$U_F = \int F \cos\theta\, ds$
Constant force	$U_F = \left(F_c \cos\theta\right)\Delta s$
Weight	$U_W = -W\Delta y$
Spring	$U_s = -\left(\frac{1}{2}k s_2^2 - \frac{1}{2}k s_1^2\right)$
Couple moment	$U_M = M\Delta\theta$

Power and Efficiency

$$P = \frac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad \varepsilon = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{U_{\text{out}}}{U_{\text{in}}}$$

Adapted from Hibbeler, R.C., *Engineering Mechanics*, 10th ed., Prentice Hall, 2003.

Impulse and Momentum

Linear Momentum

Assuming constant mass, the equation of motion of a particle may be written as

$$m d\mathbf{v}/dt = \mathbf{F}$$

$$m d\mathbf{v} = \mathbf{F} dt$$

For a system of particles, by integrating and summing over the number of particles, this may be expanded to

$$\Sigma m_i (\boldsymbol{v}_i)_{t_2} = \Sigma m_i (\boldsymbol{v}_i)_{t_1} + \Sigma \int_{t_1}^{t_2} \mathbf{F}_i \, dt$$

The term on the left side of the equation is the linear momentum of a system of particles at time t_2. The first term on the right side of the equation is the linear momentum of a system of particles at time t_1. The second term on the right side of the equation is the impulse of the force $\boldsymbol{F}$ from time t_1 to t_2. It should be noted that the above equation is a vector equation. Component scalar equations may be obtained by considering the momentum and force in a set of orthogonal directions.

Angular Momentum or Moment of Momentum

The angular momentum or the moment of momentum about point 0 for a particle is defined as

$$\mathbf{H}_0 = \mathbf{r} \times m\mathbf{v}, \text{ or}$$

$$\mathbf{H}_0 = I_0 \omega$$

Taking the time derivative of the above, the equation of motion may be written as

$$\dot{\mathbf{H}}_0 = d(I_0 \omega)/dt = \mathbf{M}_0$$

where $\mathbf{M}_0$ is the moment applied to the particle. Now by integrating and summing over a system of any number of particles, this may be expanded to

$$\Sigma(\mathbf{H}_{0i})_{t_2} = \Sigma(\mathbf{H}_{0i})_{t_1} + \Sigma \int_{t_1}^{t_2} \mathbf{M}_{0i} dt$$

The term on the left side of the equation is the angular momentum of a system of particles at time t_2. The first term on the right side of the equation is the angular momentum of a system of particles at time t_1. The second term on the right side of the equation is the angular impulse of the moment $\mathbf{M}_0$ from time t_1 to t_2.

Impact

During an impact, momentum is conserved while energy may or may not be conserved. For direct central impact with no external forces

$$m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2 = m_1 \boldsymbol{v}'_1 + m_2 \boldsymbol{v}'_2$$

where

m_1, m_2 = masses of the two bodies

$\boldsymbol{v}_1, \boldsymbol{v}_2$ = velocities of the bodies just before impact

$\boldsymbol{v}'_1, \boldsymbol{v}'_2$ = velocities of the bodies just after impact

For impacts, the relative velocity expression is

$$e = \frac{(v'_2)_n - (v'_1)_n}{(v_1)_n - (v_2)_n}$$

where

e = coefficient of restitution

$(v_i)_n$ = velocity normal to the plane of impact just **before** impact

$(v'_i)_n$ = velocity normal to the plane of impact just **after** impact

The value of e is such that

$0 \le e \le 1$, with limiting values

$e = 1$, perfectly elastic (energy conserved)

$e = 0$, perfectly plastic (no rebound)

Knowing the value of e, the velocities after the impact are given as

$$(v'_1)_n = \frac{m_2(v_2)_n(1 + e) + (m_1 - em_2)(v_1)_n}{m_1 + m_2}$$

$$(v'_2)_n = \frac{m_1(v_1)_n(1 + e) - (em_1 - m_2)(v_2)_n}{m_1 + m_2}$$

Friction

The Laws of Friction are

1. The total friction force F that can be developed is independent of the magnitude of the area of contact.
2. The total friction force F that can be developed is proportional to the normal force N.
3. For low velocities of sliding, the total frictional force that can be developed is practically independent of the sliding velocity, although experiments show that the force F necessary to initiate slip is greater than that necessary to maintain the motion.

The formula expressing the Laws of Friction is

$F \le \mu N$

where μ = the coefficient of friction.

In general

$F < \mu_s N$, no slip occurring

$F = \mu_s N$, at the point of impending slip

$F = \mu_k N$, when slip is occurring

Here,

μ_s = coefficient of static friction

μ_k = coefficient of kinetic friction

Plane Motion of a Rigid Body

Kinematics of a Rigid Body

Rigid Body Rotation

For rigid body rotation θ

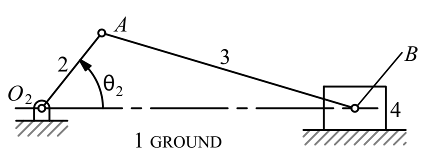
$\omega = d\theta/dt$

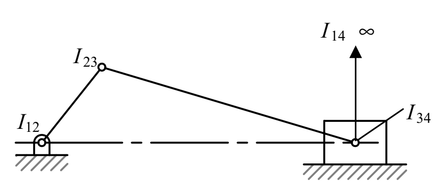
$\alpha = d\omega/dt$

$\alpha d\theta = \omega d\omega$

Instantaneous Center of Rotation (Instant Centers)

An instantaneous center of rotation (instant center) is a point, common to two bodies, at which each has the same velocity (magnitude and direction) at a given instant. It is also a point in space about which a body rotates, instantaneously.

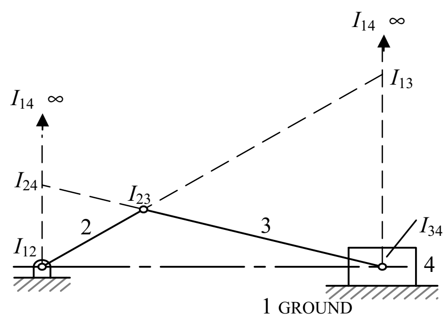




The figure shows a fourbar slider-crank. Link 2 (the crank) rotates about the fixed center, O_2. Link 3 couples the crank to the slider (link 4), which slides against ground (link 1). Using the definition of an instant center (*IC*), we see that the pins at O_2, A, and B are *IC*s that are designated I_{12}, I_{23}, and I_{34}. The easily observable *IC* is I_{14}, which is located at infinity with its direction perpendicular to the interface between links 1 and 4 (the direction of sliding). To locate the remaining two *IC*s (for a fourbar) we must make use of Kennedy's rule.

Kennedy's Rule: When three bodies move relative to one another they have three instantaneous centers, all of which lie on the same straight line.

To apply this rule to the slider-crank mechanism, consider links 1, 2, and 3 whose *IC*s are I_{12}, I_{23}, and I_{13}, all of which lie on a straight line. Consider also links 1, 3, and 4 whose *IC*s are I_{13}, I_{34}, and I_{14}, all of which lie on a straight line. Extending the line through I_{12} and I_{23} and the line through I_{34} and I_{14} to their intersection locates I_{13}, which is common to the two groups of links that were considered.



Similarly, if body groups 1, 2, 4 and 2, 3, 4 are considered, a line drawn through known *IC*s I_{12} and I_{14} to the intersection of a line drawn through known *IC*s I_{23} and I_{34} locates I_{24}.

The number of *IC*s, c, for a given mechanism is related to the number of links, n, by

$$c = \frac{n(n-1)}{2}$$

Kinetics of a Rigid Body

In general, Newton's second law for a rigid body, with constant mass and mass moment of inertia, in plane motion may be written in vector form as

$$\Sigma \mathbf{F} = m\mathbf{a}_c$$

$$\Sigma \mathbf{M}_c = I_c \boldsymbol{\alpha}$$

$$\Sigma \mathbf{M}_p = I_c \boldsymbol{\alpha} + \boldsymbol{\rho}_{pc} \times m\mathbf{a}_c$$

where $\mathbf{F}$ are forces and $\mathbf{a}_c$ is the acceleration of the body's mass center both in the plane of motion, $\mathbf{M}_c$ are moments and α is the angular acceleration both about an axis normal to the plane of motion, I_c is the mass moment of inertia about the normal axis through the mass center, and $\boldsymbol{\rho}_{pc}$ is a vector from point p to point c.

Mass Moment of Inertia

$$I = \int r^2 \, dm$$

Parallel-Axis Theorem $I = I_c + md^2$

Radius of Gyration $r_m = \sqrt{\frac{I}{m}}$

Equations of Motion

$$\begin{array}{l|l} \textit{Rigid Body} & \Sigma F_x = m(a_c)_x \\ (\textit{Plane Motion}) & \Sigma F_y = m(a_c)_y \\ & \Sigma M_c = I_c\alpha \text{ or } \Sigma M_P = \Sigma(M_k)_P \end{array}$$

Subscript c indicates center of mass.

Mass Moment of Inertia

The definitions for the mass moments of inertia are

$$I_x = \int \left(y^2 + z^2\right) dm$$
$$I_y = \int \left(x^2 + z^2\right) dm$$
$$I_z = \int \left(x^2 + y^2\right) dm$$

A table listing moment of inertia formulas for some standard shapes is at the end of this section.

Parallel-Axis Theorem
The mass moments of inertia may be calculated about any axis through the application of the above definitions. However, once the moments of inertia have been determined about an axis passing through a body's mass center, it may be transformed to another parallel axis. The transformation equation is

$$I_{\text{new}} = I_c + md^2$$

where

I_{new} = mass moment of inertia about any specified axis
I_c = mass moment of inertia about an axis that is parallel to the above specified axis but passes through the body's mass center
m = mass of the body
d = normal distance from the body's mass center to the above-specified axis

Mass Radius of Gyration
The mass radius of gyration is defined as

$$r_m = \sqrt{I/m}$$

Without loss of generality, the body may be assumed to be in the x-y plane. The scalar equations of motion may then be written as

$$\Sigma F_x = ma_{xc}$$
$$\Sigma F_y = ma_{yc}$$
$$\Sigma M_{zc} = I_{zc}\alpha$$

where zc indicates the z axis passing through the body's mass center, a_{xc} and a_{yc} are the acceleration of the body's mass center in the x and y directions, respectively, and α is the angular acceleration of the body about the z axis.

Rigid Body Motion About a Fixed Axis

Variable α	*Constant* $\alpha = \alpha_0$
$\alpha = \dfrac{d\omega}{dt}$	$\omega = \omega_0 + \alpha_0 t$
$\omega = \dfrac{d\theta}{dt}$	$\theta = \theta_0 + \omega_0 t + \dfrac{1}{2}\alpha_0 t^2$
$\omega\, d\omega = \alpha\, d\theta$	$\omega^2 = \omega_0^2 + 2\alpha_0(\theta - \theta_0)$

For rotation about some arbitrary fixed axis q

$$\Sigma M_q = I_q \alpha$$

If the applied moment acting about the fixed axis is constant then integrating with respect to time, from $t = 0$ yields

$$\alpha = M_q / I_q$$
$$\omega = \omega_0 + \alpha t$$
$$\theta = \theta_0 + \omega_0 t + \alpha t^2/2$$

where ω_0 and θ_0 are the values of angular velocity and angular displacement at time $t = 0$, respectively.

The change in kinetic energy is the work done in accelerating the rigid body from ω_0 to ω

$$I_q \omega^2/2 = I_q \omega_0^2/2 + \int_{\theta_0}^{\theta} M_q d\theta$$

Kinetic Energy

In general the kinetic energy for a rigid body may be written as

$$T = mv^2/2 + I_c \omega^2/2$$

For motion in the xy plane this reduces to

$$T = m\left(v_{cx}^2 + v_{cy}^2\right)/2 + I_c \omega_z^2/2$$

For motion about an instant center,

$$T = I_{IC} \omega^2/2$$

Principle of Angular Impulse and Momentum

Rigid Body (*Plane Motion*)	$(\mathbf{H}_c)_1 + \Sigma\int \mathbf{M}_c\, dt = (\mathbf{H}_c)_2$ where $\mathbf{H}_c = I_c \omega$ $(\mathbf{H}_0)_1 + \Sigma\int \mathbf{M}_0\, dt = (\mathbf{H}_0)_2$ where $\mathbf{H}_0 = I_0 \omega$

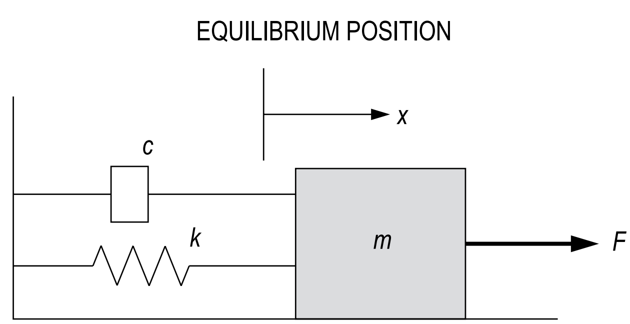
Subscript c indicates center of mass.

Conservation of Angular Momentum

$$\Sigma(\text{syst. } \mathbf{H})_1 = \Sigma(\text{syst. } \mathbf{H})_2$$

Free and Forced Vibration

A single degree-of-freedom vibration system, containing a mass m, a spring k, a viscous damper c, and an external applied force F can be diagrammed as shown:



The equation of motion for the displacement of x is:

$$m\ddot{x} = -kx - c\dot{x} + F$$

or in terms of x,

$$m\ddot{x} + c\dot{x} + kx = F$$

One can define

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2\sqrt{km}}$$

$$K = \frac{1}{k}$$

Then:

$$\frac{1}{\omega_n^2}\ddot{x} + \frac{2\zeta}{\omega_n}\dot{x} + x = KF$$

If the externally applied force is 0, this is a free vibration, and the motion of x is solved as the solution to a homogeneous ordinary differential equation.

In a forced vibration system, the externally applied force F is typically periodic (for example, $F = F_0 \sin \omega t$). The solution is the sum of the homogeneous solution and a particular solution.

For forced vibrations, one is typically interested in the steady state behavior (i.e. a long time after the system has started), which is the particular solution.

For $F = F_0 \sin \omega t$, the particular solution is:

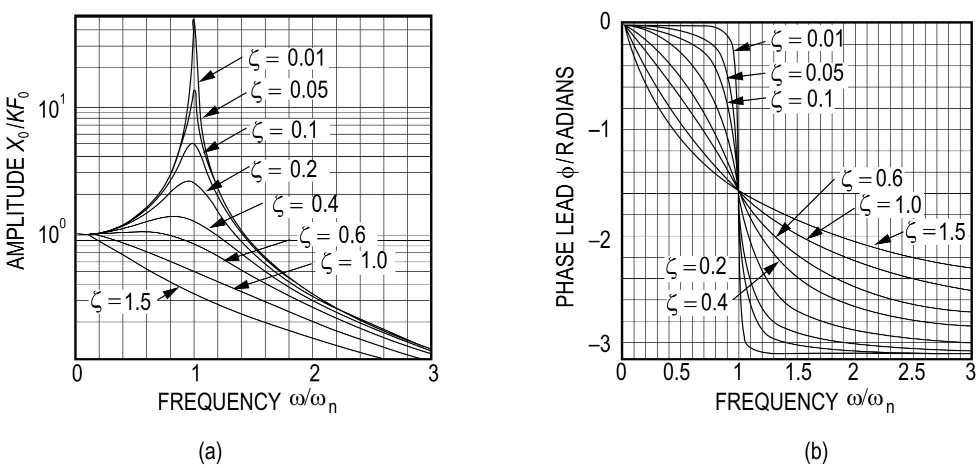
$$x(t) = X_0 \sin(\omega t + \phi)$$

where

$$X_0 = \frac{KF_0}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

$$\phi = \tan^{-1}\frac{-\frac{2\zeta\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

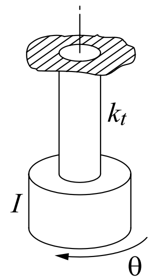
The following figures provide illustrative plots of relative amplitude and phase, depending on ω and ω_n.



Steady state vibration of a force spring-mass system (a) amplitude (b) phase.

From Brown University School of Engineering, Introduction to Dynamics and Vibrations, as posted on www.brown.edu/Departments/Engineering/Courses/En4/Notes/vibrations_forced/vibrations_forced.htm, April 2019.

Torsional Vibration



For torsional free vibrations it may be shown that the differential equation of motion is

$$\ddot{\theta} + \left(k_t / I\right)\theta = 0$$

where

θ = angular displacement of the system

k_t = torsional stiffness of the massless rod

I = mass moment of inertia of the end mass

The solution may now be written in terms of the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$ as

$$\theta(t) = \theta_0 \cos(\omega_n t) + \left(\dot{\theta}_0 / \omega_n\right) \sin(\omega_n t)$$

where the undamped natural circular frequency is given by

$$\omega_n = \sqrt{k_t / I}$$

The torsional stiffness of a solid round rod with associated polar moment-of-inertia *J*, length *L,* and shear modulus of elasticity *G* is given by

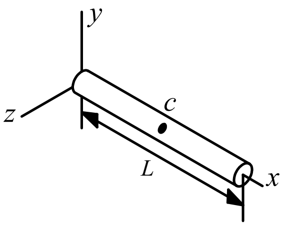
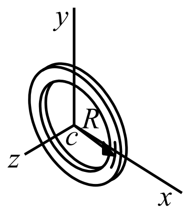
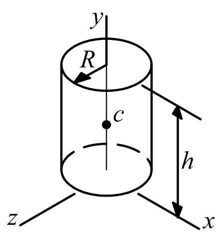
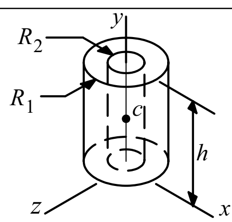
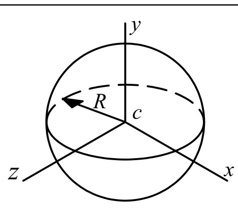
$$k_t = GJ/L$$

Thus the undamped circular natural frequency for a system with a solid round supporting rod may be written as

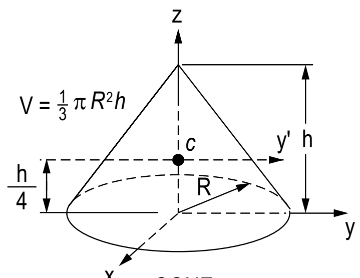
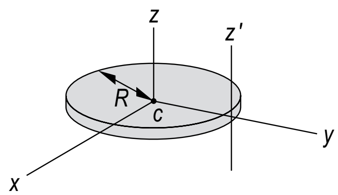
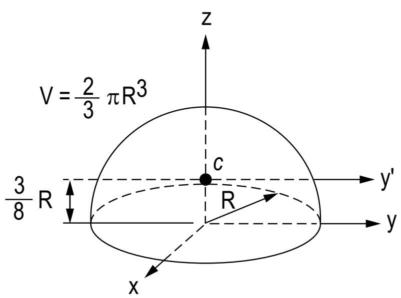
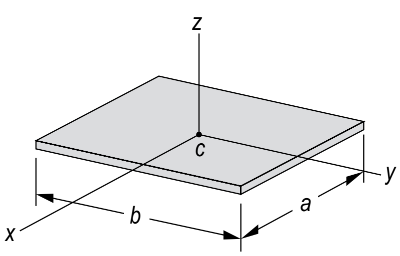
$$\omega_n = \sqrt{GJ/IL}$$

Similar to the linear vibration problem, the undamped natural period may be written as

$$\tau_n = 2\pi/\omega_n = \frac{2\pi}{\sqrt{\frac{k_t}{I}}} = \frac{2\pi}{\sqrt{\frac{GJ}{IL}}}$$

Figure	Mass & Centroid	Mass Moment of Inertia	(Radius of Gyration)^2
	$M = \rho LA$ $x_c = L/2$ $y_c = 0$ $z_c = 0$ A = cross-sectional area of rod ρ = mass/vol.	$I_x = I_{x_c} = 0$ $I_{y_c} = I_{z_c} = ML^2/12$ $I_y = I_z = ML^2/3$	$r_x^2 = r_{x_c}^2 = 0$ $r_{y_c}^2 = r_{z_c}^2 = L^2/12$ $r_y^2 = r_z^2 = L^2/3$
	$M = \rho_s A$ x_c = R = mean radius y_c = R = mean radius $z_c = 0$ A = cross-sectional area of ring ρ = mass/area	$I_{x_c} = I_{y_c} = MR^2/2$ $I_{z_c} = MR^2$ $I_x = I_y = 3MR^2/2$ $I_z = 3MR^2$	$r_{x_c}^2 = r_{y_c}^2 = R^2/2$ $r_{2_c}^2 = R^2$ $r_x^2 = r_y^2 = 3R^2/2$ $r_z^2 = 3R^2$
	$M = \pi R^2 \rho h$ $x_c = 0$ $y_c = h/2$ $z_c = 0$ ρ = mass/vol.	$I_{x_c} = I_{z_c} = M\left(3R^2 + h^2\right)/12$ $I_{y_c} = I_y = MR^2/2$ $I_x = I_z = M\left(3R^2 + 4h^2\right)/12$	$r_{x_c}^2 = r_{z_c}^2 = \left(3R^2 + h^2\right)/12$ $r_{y_c}^2 = r_y^2 = R^2/2$ $r_x^2 = r_z^2 = \left(3R^2 + 4h^2\right)/12$
	$M = \pi\left(R_1^2 - R_2^2\right)\rho h$ $x_c = 0$ $y_c = h/2$ $z_c = 0$ ρ = mass/vol.	$I_{x_c} = I_{z_c}$ $= M\left(3R_1^2 + 3R_2^2 + h^2\right)/12$ $I_{y_c} = I_y = M\left(R_1^2 + R_2^2\right)/2$ $I_x = I_z$ $= M\left(3R_1^2 + 3R_2^2 + 4h^2\right)/12$	$r_{x_c}^2 = r_{z_c}^2 = \left(3R_1^2 + 3R_2^2 + h^2\right)/12$ $r_{y_c}^2 = r_y^2 = \left(R_1^2 + R_2^2\right)/2$ $r_x^2 = r_z^2$ $= \left(3R_1^2 + 3R_2^2 + 4h^2\right)/12$
	$M = \frac{4}{3}\pi R^3 \rho$ $x_c = 0$ $y_c = 0$ $z_c = 0$ ρ = mass/vol.	$I_{x_c} = I_x = 2MR^2/5$ $I_{y_c} = I_y = 2MR^2/5$ $I_{z_c} = I_z = 2MR^2/5$	$r_{x_c}^2 = r_x^2 = 2R^2/5$ $r_{y_c}^2 = r_y^2 = 2R^2/5$ $r_{z_c}^2 = r_z^2 = 2R^2/5$

Housner, George W., and Donald E. Hudson, *Applied Mechanics Dynamics*, D. Van Nostrand Company, Inc., Princeton, NJ, 1959. Table reprinted by permission of G.W. Housner & D.E. Hudson.

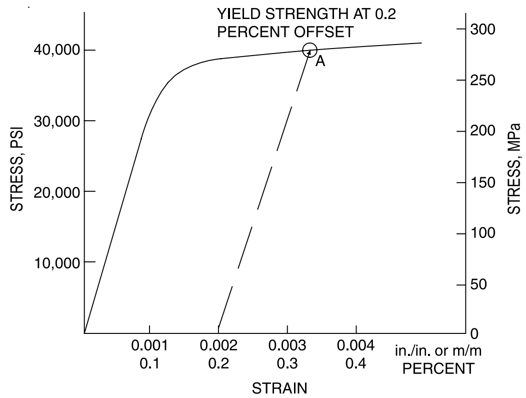
Figure	Mass & Centroid	Mass Moment of Inertia	(Radius of Gyration)^2
 CONE	$M=\frac{1}{3}\pi R^2 h\rho$ $x_c=y_c=0$ $z_c=\frac{h}{4}$ ρ = mass/vol.	$I_{x'x'}=I_{y'y'}=\frac{3}{80}M\left(4R^2+h^2\right)$ $I_{zz}=\frac{3}{10}MR^2$ $I_{yy}=I_{xx}=\frac{1}{20}M\left(3R^2+2h^2\right)$	$r_{xx}^2=r_{yy}^2=\frac{3}{80}\left(4R^2+h^2\right)$ $r_{zz}^2=\frac{3}{10}R^2$
 THIN CIRCULAR DISK	$M=\pi R^2\rho_s$ $x_c=y_c=z_c=0$ ρ_s = mass/area	$I_{xx}=I_{yy}=\frac{1}{4}MR^2$ $I_{zz}=\frac{1}{2}MR^2$ $I_{z'z'}=\frac{3}{2}MR^2$	$r_{xx}^2=r_{yy}^2=\frac{1}{4}R^2$ $r_{zz}^2=\frac{1}{2}R^2$ $r_{z'z'}^2=\frac{3}{2}R^2$
 HEMISPHERE	$M=\frac{2}{3}\pi R^3\rho$ $x_c=y_c=0$ $z_c=\frac{3}{8}R$ ρ = mass/vol.	$I_{x'x'}=I_{y'y'}=\frac{83}{320}MR^2$ $I_{zz}=\frac{2}{5}MR^2$	$r_{xx}^2=r_{yy}^2=0.259R^2$ $r_{zz}^2=\frac{2}{5}R^2$
 THIN PLATE	$M=ab\rho_s$ $x_c=y_c=z_c=0$ ρ_s = mass/area	$I_{xx}=\frac{1}{12}Mb^2$ $I_{yy}=\frac{1}{12}Ma^2$ $I_{zz}=\frac{1}{12}M\left(a^2+b^2\right)$	$r_{xx}^2=\frac{1}{12}b^2$ $r_{yy}^2=\frac{1}{12}a^2$ $r_{zz}^2=\frac{1}{12}\left(a^2+b^2\right)$

Housner, George W., and Donald E. Hudson, *Applied Mechanics Dynamics*, D. Van Nostrand Company, Inc., Princeton, NJ, 1959. Table reprinted by permission of G.W. Housner & D.E. Hudson.

Mechanics of Materials

Uniaxial Stress-Strain

Stress-Strain Curve for Mild Steel



Flinn, Richard A., and Paul K. Trojan, *Engineering Materials & Their Applications,* 4th ed., Houghton Mifflin Co., Boston, 1990.

The slope of the linear portion of the curve equals the modulus of elasticity.

Definitions

Engineering Strain

$\varepsilon = \Delta L / L_o$

where

ε = engineering strain (units per unit)

ΔL = change in length (units) of member

L_o = original length (units) of member

Percent Elongation

$$\%\text{ Elongation} = \left(\frac{\Delta L}{L_o}\right) \times 100$$

Percent Reduction in Area (RA)

The % reduction in area from initial area, A_i, to final area, A_f, is:

$$\%RA = \left(\frac{A_i - A_f}{A_i}\right) \times 100$$

Shear Stress-Strain

$\gamma = \tau / G$

where

γ = shear strain

τ = shear stress

G = shear modulus (constant in linear torsion-rotation relationship)

$$G = \frac{E}{2(1+v)}$$

where

E = modulus of elasticity (Young's modulus)

ν = Poisson's ratio

= – (lateral strain)/(longitudinal strain)

Bulk (Volume) Modulus of Elasticity

$$K = \frac{E}{3(1-2\nu)}$$

where

K = bulk modulus

E = modulus of elasticity

ν = Poisson's ratio

Uniaxial Loading and Deformation

$\sigma = P/A$

where

σ = stress on the cross section

P = loading

A = cross-sectional area

$\varepsilon = \delta/L$

where

δ = elastic longitudinal deformation

L = length of member

$$E = \sigma/\varepsilon = \frac{P/A}{\delta/L}$$

$$\delta = \frac{PL}{AE}$$

True stress is load divided by actual cross-sectional area whereas engineering stress is load divided by the initial area.

Thermal Deformations

$\delta_t = \alpha L(T - T_o)$

where

δ_t = deformation caused by a change in temperature

α = temperature coefficient of expansion

L = length of member

T = final temperature

T_o = initial temperature

Cylindrical Pressure Vessel

For internal pressure only, the stresses at the inside wall are:

$$\sigma_t = P_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad \sigma_r = -P_i$$

For external pressure only, the stresses at the outside wall are:

$$\sigma_t = -P_o \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad \sigma_r = -P_o$$

where

σ_t = tangential (hoop) stress
σ_r = radial stress
P_i = internal pressure
P_o = external pressure
r_i = inside radius
r_o = outside radius

For vessels with end caps, the axial stress is:

$$\sigma_a = P_i \frac{r_i^2}{r_o^2 - r_i^2}$$

where σ_t, σ_r, and σ_a are principal stresses.

When the thickness of the cylinder wall is about one-tenth or less of inside radius, the cylinder can be considered as thin-walled. In which case, the internal pressure is resisted by the hoop stress and the axial stress.

$$\sigma_t = \frac{P_i r}{t} \quad \text{and} \quad \sigma_a = \frac{P_i r}{2t}$$

where

t = wall thickness
$r = \frac{r_i + r_o}{2}$

Stress and Strain

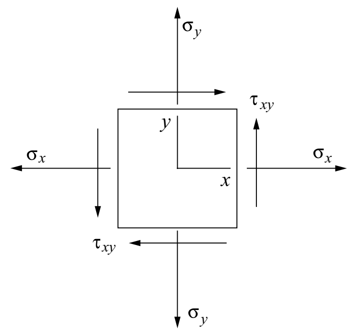
Principal Stresses

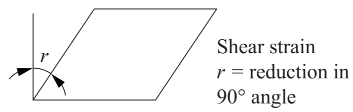
For the special case of a *two-dimensional* stress state, the equations for principal stress reduce to

$$\sigma_a, \sigma_b = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_c = 0$$

The two nonzero values calculated from this equation are temporarily labeled σ_a and σ_b and the third value σ_c is always zero in this case. Depending on their values, the three roots are then labeled according to the convention:
algebraically largest = σ_1, *algebraically smallest* = σ_3, *other* = σ_2. A typical 2D stress element is shown below with all indicated components shown in their positive sense.





Crandall, S.H., and N.C. Dahl, *An Introduction to Mechanics of Solids*, McGraw-Hill, New York, 1959.

Mohr's Circle—Stress, 2D

To construct a Mohr's circle, the following sign conventions are used.

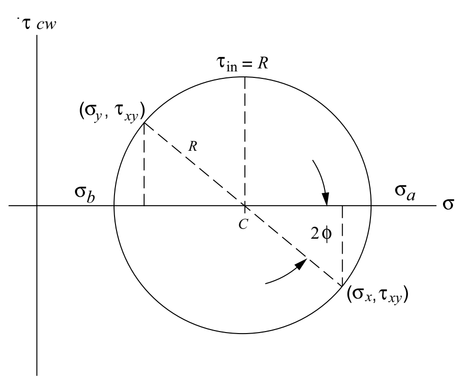
1. Tensile normal stress components are plotted on the horizontal axis and are considered positive. Compressive normal stress components are negative.
2. For constructing Mohr's circle only, shearing stresses are plotted above the normal stress axis when the pair of shearing stresses, acting on opposite and parallel faces of an element, forms a clockwise couple. Shearing stresses are plotted below the normal axis when the shear stresses form a counterclockwise couple.

The circle drawn with the center on the normal stress (horizontal) axis with center, *C*, and radius, *R,* where

$$C = \frac{\sigma_x + \sigma_y}{2}, \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The two nonzero principal stresses are then:

$$\sigma_a = C + R$$
$$\sigma_b = C - R$$



Crandall, S.H., and N.C. Dahl, *An Introduction to Mechanics of Solids*, McGraw-Hill, New York, 1959.

The maximum *inplane* shear stress is $\tau_{in} = R$. However, the maximum shear stress considering three dimensions is always

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}.$$

Hooke's Law

Three-dimensional case:

$\varepsilon_x = (1/E)[\sigma_x - v(\sigma_y + \sigma_z)]$ $\quad\gamma_{xy} = \tau_{xy}/G$

$\varepsilon_y = (1/E)[\sigma_y - v(\sigma_z + \sigma_x)]$ $\quad\gamma_{yz} = \tau_{yz}/G$

$\varepsilon_z = (1/E)[\sigma_z - v(\sigma_x + \sigma_y)]$ $\quad\gamma_{zx} = \tau_{zx}/G$

Plane stress case ($\sigma_z = 0$):

$\varepsilon_x = (1/E)(\sigma_x - v\sigma_y)$

$\varepsilon_y = (1/E)(\sigma_y - v\sigma_x)$

$\varepsilon_z = -(1/E)(v\sigma_x + v\sigma_y)$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-v^2}\begin{bmatrix}1 & v & 0\\ v & 1 & 0\\ 0 & 0 & \frac{1-v}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

Uniaxial case ($\sigma_y = \sigma_z = 0$):

$\sigma_x = E\varepsilon_x$ or $\sigma = E\varepsilon$

where

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ = normal strain
$\sigma_x, \sigma_y, \sigma_z$ = normal stress
$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ = shear strain
$\tau_{xy}, \tau_{yz}, \tau_{zx}$ = shear stress
E = modulus of elasticity
G = shear modulus
ν = Poisson's ratio

When there is a temperature change from an initial temperature T_i to a final temperature T_f there are also thermally-induced normal strains. In this case, ε_x, ε_y, and ε_z require modification. Thus,

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - v\left(\sigma_y + \sigma_z\right)\right] + \alpha\left(T_f - T_i\right)$$

and similarly for ε_y and ε_z, where α = coefficient of thermal expansion (CTE).

Torsion

Torsion stress in circular solid or thick-walled (t > 0.1 r) shafts:

$$\tau = \frac{Tr}{J}$$

where J = polar moment of inertia

Torsional Strain

$$\gamma_{\phi z} = \lim_{\Delta z \to 0} r\left(\Delta\phi/\Delta z\right) = r\left(d\phi/dz\right)$$

The shear strain varies in direct proportion to the radius, from zero strain at the center to the greatest strain at the outside of the shaft. $d\phi/dz$ is the twist per unit length or the rate of twist.

$$\tau_{\phi z} = G\gamma_{\phi z} = Gr\left(d\phi/dz\right)$$

$$T = G\left(d\phi/dz\right)\int_A r^2 dA = GJ\left(d\phi/dz\right)$$

$$\phi = \int_o^L \frac{T}{GJ}dz = \frac{TL}{GJ}$$

where

ϕ = total angle (radians) of twist
T = torque
L = length of shaft

T/ϕ gives the *twisting moment per radian of twist*. This is called the *torsional stiffness* and is often denoted by the symbol k or c.

For Hollow, Thin-Walled Shafts

$$\tau = \frac{T}{2A_m t}$$

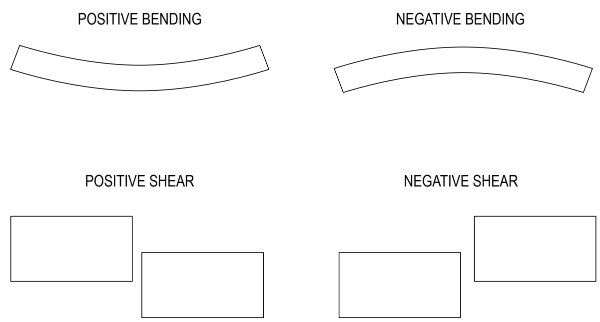
where

t = thickness of shaft wall
A_m = area of a solid shaft of radius equal to the mean radius of the hollow shaft

Beams

Shearing Force and Bending Moment Sign Conventions

1. The bending moment is *positive* if it produces bending of the beam *concave upward* (compression in top fibers and tension in bottom fibers).
2. The shearing force is *positive* if the *right portion of the beam tends to shear downward with respect to the left*.



Timoshenko, S., and Gleason H. MacCullough, *Elements of Strengths of Materials*, K. Van Nostrand Co./Wadsworth Publishing Co., 1949.

The relationship between the load (w), shear (V), and moment (M) equations are:

$$w(x) = -\frac{dV(x)}{dx}$$

$$V = \frac{dM(x)}{dx}$$

$$V_2 - V_1 = \int_{x_1}^{x_2} [-w(x)]\,dx$$

$$M_2 - M_1 = \int_{x_1}^{x_2} V(x)\,dx$$

Stresses in Beams

The normal stress in a beam due to bending:

$\sigma_x = -My/I$

where

M = moment at the section

I = moment of inertia of the cross section

y = distance from the neutral axis to the fiber location above or below the neutral axis

The maximum normal stresses in a beam due to bending:

$\sigma_x = \pm Mc/I$

where

c = distance from the neutral axis to the outermost fiber of a symmetrical beam section

$\sigma_x = -M/s$

where

s = I/c: the elastic section modulus of the beam

Transverse shear stress:

$\tau_{xy} = VQ/(Ib)$

where

V = shear force

$Q = A'\overline{y'}$ = first moment of area above or below the point where shear stress is to be determined

Hibbeler, Russel C., *Mechanics of Materials*, 10th ed., Pearson, 2015, pp. 386 –387.

where

A' = area above the layer (or plane) upon which the desired transverse shear stress acts

$\overline{y'}$ = distance from neutral axis to area centroid

b = width or thickness or the cross-section

Transverse shear flow:

$q = VQ/I$

Deflection of Beams

Using $1/\rho = M/(EI)$,

$$EI\frac{d^2y}{dx^2} = M, \text{ differential equation of deflection curve}$$

$$EI\frac{d^3y}{dx^3} = dM(x)/dx = V$$

$$EI\frac{d^4y}{dx^4} = dV(x)/dx = -w$$

Determine the deflection curve equation by double integration (apply boundary conditions applicable to the deflection and/or slope).

$EI\,(dy/dx) = \int M(x)\,dx$

$EIy = \int[\int M(x)\,dx]\,dx$

The constants of integration can be determined from the physical geometry of the beam.

Composite Sections

The bending stresses in a beam composed of dissimilar materials (Material 1 and Material 2) where $E_1 > E_2$ are:

$\sigma_1 = -nMy/I_T$

$\sigma_2 = -My/I_T$

where

I_T = moment of inertia of the transformed section

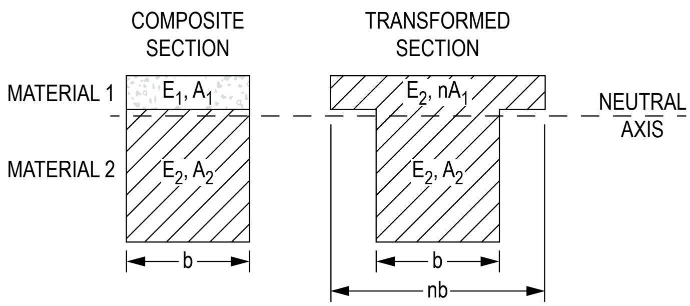
n = modular ratio E_1/E_2

E_1 = elastic modulus of Material 1

E_2 = elastic modulus of Material 2

y = distance from the neutral axis to the fiber location above or below the neutral axis

The composite section is transformed into a section composed of a single material. The centroid and then the moment of inertia are found on the transformed section for use in the bending stress equations.



Columns

Critical axial load for long column subject to buckling:
Euler's Formula

$$P_{cr} = \frac{\pi^2 EI}{(K\ell)^2}$$

where

ℓ = unbraced column length

K = effective-length factor to account for end supports

Theoretical effective-length factors for columns include:

Pinned-pinned, $K = 1.0$

Fixed-fixed, $K = 0.5$

Fixed-pinned, $K = 0.7$

Fixed-free, $K = 2.0$

Critical buckling stress for long columns:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(K\ell/r)^2}$$

where

r = radius of gyration = $\sqrt{I/A}$

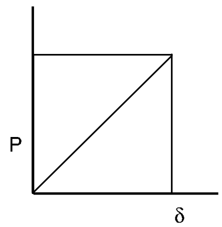
$K\ell/r$ = effective slenderness ratio for the column

Elastic Strain Energy

If the strain remains within the elastic limit, the work done during deflection (extension) of a member will be transformed into potential energy and can be recovered.

If the final load is P and the corresponding elongation of a tension member is δ, then the total energy U stored is equal to the work W done during loading.

$$U = W = P\delta/2$$



The strain energy per unit volume is

$$u = U/AL = \sigma^2/2E \qquad \text{(for tension)}$$

Material Properties

Table 1 - Typical Material Properties
(Use these values if the specific alloy and temper are not listed on Table 2 below)

Material	Modulus of Elasticity, E [Mpsi (GPa)]	Modulus of Rigidity, G [Mpsi (GPa)]	Poisson's Ratio, v	Coefficient of Thermal Expansion, α [10^{-6}/ºF (10^{-6}/ºC)]	Density, ρ [lb/in^3 (Mg/m^3)]
Steel	29.0 (200.0)	11.5 (80.0)	0.30	6.5 (11.7)	0.282 (7.8)
Aluminum	10.0 (69.0)	3.8 (26.0)	0.33	13.1 (23.6)	0.098 (2.7)
Cast Iron	14.5 (100.0)	6.0 (41.4)	0.21	6.7 (12.1)	0.246–0.282 (6.8–7.8)
Wood (Fir)	1.6 (11.0)	0.6 (4.1)	0.33	1.7 (3.0)	–
Brass	14.8–18.1 (102–125)	5.8 (40)	0.33	10.4 (18.7)	0.303–0.313 (8.4–8.7)
Copper	17 (117)	6.5 (45)	0.36	9.3 (16.6)	0.322 (8.9)
Bronze	13.9–17.4 (96–120)	6.5 (45)	0.34	10.0 (18.0)	0.278–0.314 (7.7–8.7)
Magnesium	6.5 (45)	2.4 (16.5)	0.35	14 (25)	0.061 (1.7)
Glass	10.2 (70)	–	0.22	5.0 (9.0)	0.090 (2.5)
Polystyrene	0.3 (2)	–	0.34	38.9 (70.0)	0.038 (1.05)
Polyvinyl Chloride (PVC)	<0.6 (<4)	–	–	28.0 (50.4)	0.047 (1.3)
Alumina Fiber	58 (400)	–	–	–	0.141 (3.9)
Aramide Fiber	18.1 (125)	–	–	–	0.047 (1.3)
Boron Fiber	58 (400)	–	–	–	0.083 (2.3)
Beryllium Fiber	43.5 (300)	–	–	–	0.069 (1.9)
BeO Fiber	58 (400)	–	–	–	0.108 (3.0)
Carbon Fiber	101.5 (700)	–	–	–	0.083 (2.3)
Silicon Carbide Fiber	58 (400)	–	–	–	0.116 (3.2)

Hibbeler, R.C., *Mechanics of Materials*, 4 ed., 2000. Reprinted by permission of Pearson Education, Inc., New York, New York.

Table 2 - Average Mechanical Properties of Typical Engineering Materials
(U.S. Customary Units)
(Use these values for the specific alloys and temper listed. For all other materials refer to Table 1 above.)

Materials		Specific Weight γ (lb/in^3)	Modulus of Elasticity E (10^3 ksi)	Modulus of Rigidity G (10^3 ksi)	Yield Strength (ksi) σ_y			Ultimate Strength (ksi) σ_u			% Elongation in 2 in. specimen	Poisson's Ratio v	Coef. of Therm. Expansion α $(10^{-6})/°\Phi$
					Tens.	Comp.	Shear	Tens.	Comp.	Shear			
Metallic													
Aluminum Wrought Alloys	2014-T6	0.101	10.6	3.9	60	60	25	68	68	42	10	0.35	12.8
	6061-T6	0.098	10.0	3.7	37	37	19	42	42	27	12	0.35	13.1
Cast Iron Alloys	Gray ASTM 20	0.260	10.0	3.9	—	—	—	26	97	—	0.6	0.28	6.70
	Malleable ASTM A-197	0.263	25.0	9.8	—	—	—	40	83	—	5	0.28	6.60
Copper Alloys	Red Brass C83400	0.316	14.6	5.4	11.4	11.4	—	35	35	—	35	0.35	9.80
	Bronze C86100	0.319	15.0	5.6	50	50	—	95	95	—	20	0.34	9.60
Magnesium Alloy	[Am 1004-T611]	0.066	6.48	2.5	22	22	—	40	40	22	1	0.30	14.3
Steel Alloys	Structural A36	0.284	29.0	11.0	36	36	—	58	58	—	30	0.32	6.60
	Stainless 304	0.284	28.0	11.0	30	30	—	75	75	—	40	0.27	9.60
	Tool L2	0.295	29.0	11.0	102	102	—	116	116	—	22	0.32	6.50
Titanium Alloy	[Ti-6Al-4V]	0.160	17.4	6.4	134	134	—	145	145	—	16	0.36	5.20
Nonmetallic													
Concrete	Low Strength	0.086	3.20	—	—	—	1.8	—	—	—	—	0.15	6.0
	High Strength	0.086	4.20	—	—	—	5.5	—	—	—	—	0.15	6.0
Plastic Reinforced	Kevlar 49	0.0524	19.0	—	—	—	—	104	70	10.2	2.8	0.34	—
	30% Glass	0.0524	10.5	—	—	—	—	13	19	—	—	0.34	—
Wood Select Structural Grade	Douglas Fir	0.017	1.90	—	—	—	—	0.30[c]	3.78[d]	0.90[d]	—	0.29[c]	—
	White Spruce	0.130	1.40	—	—	—	—	0.36[c]	5.18[d]	0.97[d]	—	0.31[c]	—

[a] SPECIFIC VALUES MAY VARY FOR A PARTICULAR MATERIAL DUE TO ALLOY OR MINERAL COMPOSITION, MECHANICAL WORKING OF THE SPECIMEN, OR HEAT TREATMENT. FOR A MORE EXACT VALUE REFERENCE BOOKS FOR THE MATERIAL SHOULD BE CONSULTED.

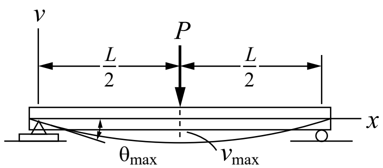
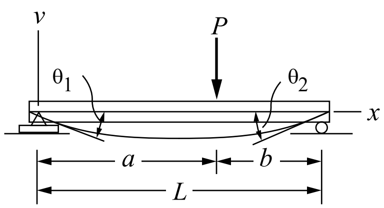
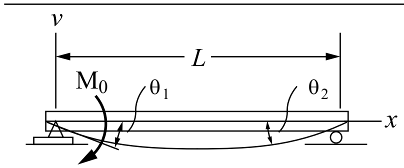
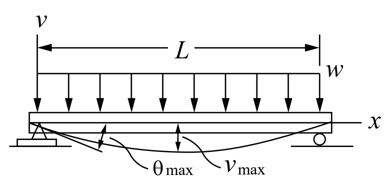
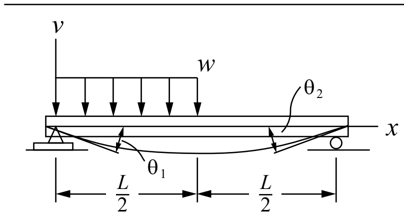
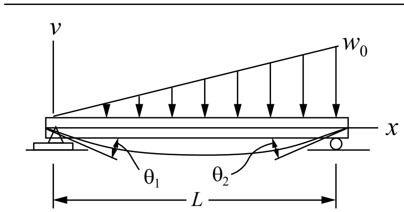
[b] THE YIELD AND ULTIMATE STRENGTHS FOR DUCTILE MATERIALS CAN BE ASSUMED EQUAL FOR BOTH TENSION AND COMPRESSION.

[c] MEASURED PERPENDICULAR TO THE GRAIN.

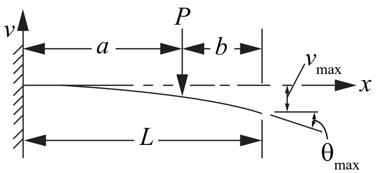
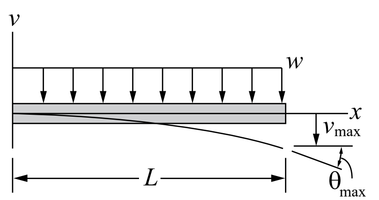
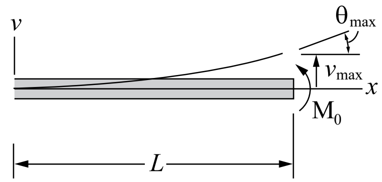
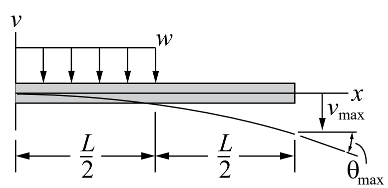
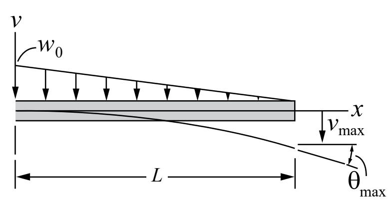
[d] MEASURED PARALLEL TO THE GRAIN.

[e] DEFORMATION MEASURED PERPENDICULAR TO THE GRAIN WHEN THE LOAD IS APPLIED ALONG THE GRAIN.

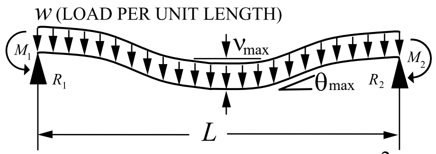
Simply Supported Beam Slopes and Deflections

BEAM	SLOPE	DEFLECTION	ELASTIC CURVE	MAXIMUM MOMENT
	$\theta_{max} = \dfrac{-PL^2}{16EI}$	$v_{max} = \dfrac{-PL^3}{48EI}$	$v = \dfrac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$	M_{max} (at center) $= \dfrac{PL}{4}$
	$\theta_1 = \dfrac{-Pab(L+b)}{6EIL}$ $\theta_2 = \dfrac{Pab(L+a)}{6EIL}$	$v\Big\vert_{x=a} = \dfrac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \dfrac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$	M_{max} (at point of load) $= \dfrac{Pab}{L}$
	$\theta_1 = \dfrac{-M_0L}{3EI}$ $\theta_2 = \dfrac{M_0L}{6EI}$	$v_{max} = \dfrac{-M_0L^2}{\sqrt{243}EI}$	$v = \dfrac{-M_0x}{6EIL}(x^2 - 3Lx + 2L^2)$	M_{max} (at $x = 0$) $= M_0$
	$\theta_{max} = \dfrac{-wL^3}{24EI}$	$v_{max} = \dfrac{-5wL^4}{384EI}$	$v = \dfrac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$	M_{max} (at center) $= \dfrac{wL^2}{8}$
	$\theta_1 = \dfrac{-3wL^3}{128EI}$ $\theta_2 = \dfrac{7wL^3}{384EI}$	$v\Big\vert_{x=L/2} = \dfrac{-5wL^4}{768EI}$ $v_{max} = -0.006563\dfrac{wL^4}{EI}$ at $x = 0.4598L$	$v = \dfrac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \dfrac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$	$M_{max}\left(\text{at } x = \dfrac{3}{8}l\right) = \dfrac{9}{128}wL^2$
	$\theta_1 = \dfrac{-7w_0L^3}{360EI}$ $\theta_2 = \dfrac{w_0L^3}{45EI}$	$v_{max} = -0.00652\dfrac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \dfrac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$	$M_{max}\left(\text{at } x = \dfrac{L}{\sqrt{3}}\right) = \dfrac{w_0L^2}{9\sqrt{3}}$

Cantilevered Beam Slopes and Deflections

BEAM	SLOPE	DEFLECTION	ELASTIC CURVE	MAXIMUM MOMENT
	$\theta_{max} = \dfrac{-Pa^2}{2EI}$	$v_{max} = \dfrac{-Pa^2}{6EI}(3L-a)$	$v = \dfrac{-Pa^2}{6EI}(3x-a)$, for $x > a$ $v = \dfrac{-Px^2}{6EI}(-x+3a)$, for $x \le a$	$M_{max}(\text{at } x=0) = Pa$
	$\theta_{max} = \dfrac{-wL^3}{6EI}$	$v_{max} = \dfrac{-wL^4}{8EI}$	$v = \dfrac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$	$M_{max}(\text{at } x=0) = \dfrac{wL^2}{2}$
	$\theta_{max} = \dfrac{M_0 L}{EI}$	$v_{max} = \dfrac{M_0 L^2}{2EI}$	$v = \dfrac{M_0 x^2}{2EI}$	$M_{max}(\text{at all } x) = M_0$
	$\theta_{max} = \dfrac{-wL^3}{48EI}$	$v_{max} = \dfrac{-7wL^4}{384EI}$	$v = \dfrac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $0 \le x \le L/2$ $v = \dfrac{-wL^3}{192EI}(4x - L/2)$ $L/2 \le x \le L$	$M_{max}(\text{at } x=0) = \dfrac{wL^2}{8}$
	$\theta_{max} = \dfrac{-w_0 L^3}{24EI}$	$v_{max} = \dfrac{-w_0 L^4}{30EI}$	$v = \dfrac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$	$M_{max}(\text{at } x=0) = \dfrac{w_0 L^2}{6}$

Piping Segment Slopes and Deflections

PIPE	SLOPE	DEFLECTION	ELASTIC CURVE	MAXIMUM MOMENT
 $R_1 = R_2 = \frac{wL}{2}$ and $M_1 = M_2 = \frac{wL^2}{12}$	$\lvert\theta_{max}\rvert = 0.008\frac{wL^3}{24EI}$ at $x = \frac{1}{2} \pm \frac{L}{\sqrt{12}}$	$\lvert\nu_{max}\rvert = \frac{wL^4}{384EI}$ at $x = \frac{L}{2}$	$\nu(x) = \frac{wx^2}{24EI}(L^2 - 2Lx + x^2)$	M_{max} (at $x = 0$) $= \frac{wL^2}{12}$

Adapted from Crandall, S.H. and N.C. Dahl, *An Introduction to Mechanics of Solids*, McGraw-Hill, New York, 1959.

Thermodynamics

Properties of Single-Component Systems

Nomenclature

1. Intensive properties are independent of mass.
2. Extensive properties are proportional to mass.
3. Specific properties are lowercase (extensive/mass).

State Functions (properties)

Functions and Their Symbols and Units

Function	Symbol(s)	Unit (I-P *or* SI)
Absolute pressure	P	$\frac{\text{lbf}}{\text{in}^2}$ *or* Pa
Absolute temperature	T	°R *or* K
Volume	V	ft^3 *or* m^3
Specific volume	$v = \frac{V}{m}$	$\frac{\text{ft}^3}{\text{lbm}}$ *or* $\frac{\text{m}^3}{\text{kg}}$
Internal energy	U	Btu *or* kJ
Specific internal energy	$u = \frac{U}{m}$	$\frac{\text{Btu}}{\text{lbm}}$ *or* $\frac{\text{kJ}}{\text{kg}}$
Enthalpy	H	Btu *or* kJ
Specific enthalpy	$h = u + Pv = \frac{H}{m}$	$\frac{\text{Btu}}{\text{lbm}}$ *or* $\frac{\text{kJ}}{\text{kg}}$
Entropy	S	$\frac{\text{Btu}}{°\text{R}}$ *or* $\frac{\text{kJ}}{\text{K}}$
Specific entropy	$s = \frac{S}{m}$	$\frac{\text{Btu}}{\text{lbm-°R}}$ *or* $\frac{\text{kJ}}{\text{kg} \cdot \text{K}}$
Gibbs free energy	$G = h - Ts$	$\frac{\text{Btu}}{\text{lbm}}$ *or* $\frac{\text{kJ}}{\text{kg}}$
Helmholtz free energy	$A = u - Ts$	$\frac{\text{Btu}}{\text{lbm}}$ *or* $\frac{\text{kJ}}{\text{kg}}$

For a single-phase pure component, specification of any two intensive, independent properties is sufficient to fix all the rest.

Specific Heat (Heat Capacity) at Constant Pressure,

$$c_p = \left(\frac{\partial h}{\partial T}\right)_P \qquad \text{[Btu/(lbm-°R) or kJ/(kg•K)]}$$

Specific Heat (Heat Capacity) at Constant Volume,

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v \qquad \text{[Btu/(lbm-°R) or kJ/(kg•K)]}$$

The steam tables in this section provide T, P, v, u, h, and s data for saturated and superheated water.

P-h diagrams and tables for Refrigerant 134A and 410A, providing T, P, v, h, and s data, are included in this section.

Thermal and physical property tables for selected gases, liquids, and solids are included in this section.

Properties for Two-Phase (vapor-liquid) Systems

Quality x (for liquid-vapor systems at saturation) is defined as the mass fraction of the vapor phase:

$x = m_g/(m_g + m_f)$

where

m_g = mass of vapor
m_f = mass of liquid

Specific volume of a two-phase system can be written:

$v = xv_g + (1 - x)v_f$ or $v = v_f + xv_{fg}$

where

v_f = specific volume of saturated liquid
v_g = specific volume of saturated vapor
v_{fg} = specific volume change upon vaporization
$= v_g - v_f$

Similar expressions exist for u, h, and s:

$u = xu_g + (1 - x)\, u_f$ or $u = \mathrm{u_f} + \mathrm{xu_{fg}}$
$h = xh_g + (1 - x)\, h_f$ or $h = h_f + xh_{fg}$
$s = xs_g + (1 - x)\, s_f$ or $s = s_f + xs_{fg}$

PVT Behavior

Ideal Gas

For an ideal gas

$Pv = RT$ or $PV = mRT$, and

$P_1v_1/T_1 = P_2v_2/T_2$

where

P = pressure
v = specific volume
m = mass of gas
R = gas constant
T = absolute temperature
V = volume

R is *specific to each gas* but can be found from

$$R_i = \frac{\overline{R}}{(mol.\, wt)_i}$$

where

$\overline{R}$ = universal gas constant
= 1,545 ft-lbf/(lbmol-°R) = 8,314 J/(kmol•K)
= 8.314 kPa•m^3/(kmol•K) = 0.08206 L•atm/(mole•K)

For *ideal gases*, $c_p - c_v = R$

Ideal gas behavior is characterized by:

- no intermolecular interactions
- molecules occupy zero volume

The properties of an ideal gas reflect those of a single molecule and are attributable entirely to the structure of the molecule and the system T.

For *ideal gases*:

$$\left(\frac{\partial h}{\partial P}\right)_T = 0 \qquad \left(\frac{\partial u}{\partial v}\right)_T = 0$$

For cold air standard, *heat capacities are assumed to be constant* at their room temperature values. In that case, the following are true:

$$\Delta u = c_v \Delta T; \quad \Delta h = c_p \Delta T$$
$$\Delta s = c_p \ln (T_2/T_1) - R \ln (P_2/P_1)$$
$$\Delta s = c_v \ln (T_2/T_1) + R \ln (v_2/v_1)$$

Also, for *constant entropy* processes:

$$\frac{P_2}{P_1} = \left(\frac{v_1}{v_2}\right)^k; \qquad \frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}}$$
$$\frac{T_2}{T_1} = \left(\frac{v_1}{v_2}\right)^{k-1}, \text{ where } k = c_p/c_v$$

Ideal Gas Mixtures

$i = 1, 2, \ldots, n$ constituents. Each constituent is an ideal gas.
Mole Fraction:

$$x_i = N_i/N; \; N = \Sigma N_i; \; \Sigma x_i = 1$$

where N_i = number of moles of component i

N = total moles in the mixture

Mass Fraction: $y_i = m_i/m; \; m = \Sigma m_i; \; \Sigma y_i = 1$

Molecular Weight: $M = m/N = \Sigma x_i M_i$

To convert *mole fractions* x_i to *mass fractions* y_i:

$$y_i = \frac{x_i M_i}{\Sigma (x_i M_i)}$$

To convert *mass fractions* to *mole fractions*:

$$x_i = \frac{y_i / M_i}{\Sigma (y_i / M_i)}$$

Partial Pressures: $P_i = \dfrac{m_i R_i T}{V}$ and $P = \Sigma P_i$

Partial Volumes: $V_i = \dfrac{m_i R_i T}{P}$ and $V = \Sigma V_i$

where P, V, T = pressure, volume, and temperature of the mixture and $R_i = \overline{R}/M_i$

Combining the above generates the following additional expressions for mole fraction.

$$x_i = P_i/P = V_i/V$$

Other Properties:

$$c_p = \Sigma (y_i c_{p_i})$$
$$c_v = \Sigma (y_i c_{v_i})$$
$$u = \Sigma (y_i u_i); \; h = \Sigma (y_i h_i); \; s = \Sigma (y_i s_i)$$

u_i and h_i are evaluated at T
s_i is evaluated at T and P_i

Real Gas

Most gases exhibit ideal gas behavior when the system pressure is less than 3 atm since the distance between molecules is large enough to produce negligible molecular interactions. The behavior of a real gas deviates from that of an ideal gas at higher pressures due to molecular interactions.

For a real gas, $Pv = ZRT$

where

Z = compressibility factor

$Z = 1$ for an ideal gas

$Z \neq 1$ for a real gas

Equations of State (EOS)

EOS are used to quantify PvT behavior

Ideal Gas EOS (applicable only to ideal gases)

$$P = \left(\frac{RT}{v}\right)$$

Generalized Compressibility EOS (applicable to all systems as gases, liquids, and/or solids)

$$P = \left(\frac{RT}{v}\right)Z$$

Virial EOS (applicable only to gases)

$$P = \left(\frac{RT}{v}\right)\left(1 + \frac{B}{v} + \frac{C}{v^2} + \dots\right)$$

where B, C, ...are virial coefficients obtained from PvT measurements or statistical mechanics.

Cubic EOS (theoretically motivated with intent to predict gas and liquid thermodynamic properties)

$$P = \frac{RT}{v - b} - \frac{a(T)}{(v + c_1 b)(v + c_2 b)}$$

where $a(T)$, b, and c_1 and c_2 are species specific.

An example of a cubic EOS is the Van der Waals equation with constants based on the critical point:

$$\left(P + \frac{a}{\bar{v}^2}\right)(\bar{v} - b) = \bar{R}T$$

where $a = \left(\frac{27}{64}\right)\left(\frac{\bar{R}^2 T_c^2}{P_c}\right)$, $b = \frac{\bar{R}T_c}{8P_c}$

where P_c and T_c are the pressure and temperature at the critical point, respectively, and $\bar{v}$ is the molar specific volume.

EOS are used to predict:

- P, v, or T when two of the three are specified
- other thermodynamic properties based on analytic manipulation of the EOS
- mixture properties using appropriate mixing rules to create a pseudo-component that mimics the mixture properties

The Theorem of Corresponding States asserts that all normal fluids have the same value of Z at the same reduced temperature T_r and pressure P_r.

$$T_r = \frac{T}{T_c} \quad P_r = \frac{P}{P_c}$$

where T_c and P_c are the critical temperature and pressure, respectively, expressed in absolute units.

First Law of Thermodynamics

The *First Law of Thermodynamics* is a statement of conservation of energy in a thermodynamic system. The net energy crossing the system boundary is equal to the change in energy inside the system.

Heat Q ($q = Q/m$) is *energy transferred* due to temperature difference and is considered positive if it is inward or added to the system.

Work W ($w = W/m$) is considered *positive if it is outward* or *work done* by the system.

Closed Thermodynamic System

No mass crosses system boundary

$$Q - W = \Delta U + \Delta KE + \Delta PE$$

where

ΔU = change in internal energy

ΔKE = change in kinetic energy

ΔPE = change in potential energy

Energy can cross the boundary only in the form of heat or work. Work can be boundary work, w_b, or other work forms (electrical work, etc.)

Reversible boundary work is given by $w_b = \int P\, dv$.

Special Cases of Closed Systems (with no change in kinetic or potential energy)

Constant System Pressure process (***Charles' Law***):

$w_b = P\Delta v$

(ideal gas) T/v = constant

Constant Volume process:

$w_b = 0$

(ideal gas) T/P = constant

Isentropic process (ideal gas):

Pv^k = constant

$w = (P_2v_2 - P_1v_1)/(1 - k)$

$\quad = R(T_2 - T_1)/(1 - k)$

Constant Temperature process (***Boyle's Law***): (ideal gas) Pv = constant

$w_b = RT\ln(v_2/v_1) = RT\ln(P_1/P_2)$

Polytropic process (ideal gas):

Pv^n = constant

$w = (P_2v_2 - P_1v_1)/(1 - n),\ n \neq 1$

Open Thermodynamic System

Mass crosses the system boundary.
There is flow work (Pv) done by mass entering the system.
The reversible flow work is given by:

$w_{\text{rev}} = -\int v\, dP + \Delta KE + \Delta PE$

First Law applies whether or not processes are reversible.

Open System First Law (energy balance)

$$\Sigma\dot{m}_i\left[h_i + V_i^2/2 + gZ_i\right] - \Sigma\dot{m}_e\left[h_e + V_e^2/2 + gZ_e\right] + \dot{Q}_{in} - \dot{W}_{net} = d\left(m_s u_s\right)/dt$$

where

$\dot{W}_{net}$ = rate of net or shaft work

$\dot{m}$ = mass flowrate (subscripts i and e refer to inlet and exit states of system)

g = acceleration of gravity

Z = elevation

V = velocity

m_s = mass of fluid within the system

u_s = specific internal energy of system

$\dot{Q}_{in}$ = rate of heat transfer (neglecting kinetic and potential energy of the system)

<u>Special Cases of Open Systems (with no change in kinetic or potential energy)</u>

Constant Volume process:

$w_{\text{rev}} = -v\,(P_2 - P_1)$

Constant System Pressure process:

$w_{\text{rev}} = 0$

Constant Temperature process: (ideal gas) Pv = constant

$w_{\text{rev}} = RT \ln (v_2/v_1) = RT \ln (P_1/P_2)$

Isentropic process (ideal gas):

Pv^k = constant

$w_{\text{rev}} = k\,(P_2v_2 - P_1v_1)/(1-k)$

$= kR\,(T_2 - T_1)/(1-k)$

$$w_{rev} = \frac{k}{k-1} RT_1\left[1 - \left(\frac{P_2}{P_1}\right)^{(k-1)/k}\right]$$

Polytropic process (ideal gas):

Pv^n = constant

Closed system

$w_{\text{rev}} = (P_2v_2 - P_1v_1)/(1-n)$

One-inlet, one-exit control volume

$w_{\text{rev}} = n\,(P_2v_2 - P_1v_1)/(1-n)$

Steady-Flow Systems

The system does not change state with time. This assumption is valid for steady operation of turbines, pumps, compressors, throttling valves, nozzles, and heat exchangers, including boilers and condensers.

$$\Sigma\dot{m}_i\left(h_i + V_i^2/2 + gZ_i\right) - \Sigma\dot{m}_e\left(h_e + V_e^2/2 + gZ_e\right) + \dot{Q}_{in} - \dot{W}_{out} = 0$$

and

$$\Sigma\dot{m}_i = \Sigma\dot{m}_e$$

where

$\dot{m}$ = mass flowrate (subscripts i and e refer to inlet and exit states of system)

g = acceleration of gravity

Z = elevation

V = velocity

$\dot{Q}_{in}$ = net rate of heat transfer into the system

$\dot{W}_{out}$ = net rate of work out of the system

Special Cases of Steady-Flow Energy Equation

Nozzles, Diffusers: Velocity terms are significant. No elevation change, no heat transfer, and no work. Single-mass stream.

$$h_i + V_i^2/2 = h_e + V_e^2/2$$

Isentropic Efficiency (nozzle) = $\dfrac{V_e^2 - V_i^2}{2\left(h_i - h_{es}\right)}$

where h_{es} = enthalpy at isentropic exit state.

Turbines, Pumps, Compressors: Often considered adiabatic (no heat transfer). Velocity terms usually can be ignored. There are significant work terms and a single-mass stream.

$$h_i = h_e + w$$

Isentropic Efficiency (turbine) = $\dfrac{h_i - h_e}{h_i - h_{es}}$

Isentropic Efficiency (compressor, pump) = $\dfrac{h_{es} - h_i}{h_e - h_i}$

For pump only, $h_{es} - h_i = v_i(P_e - P_i)$

Throttling Valves and Throttling Processes: No work, no heat transfer, and single-mass stream. Velocity terms are often insignificant.

$$h_i = h_e$$

Boilers, Condensers, Evaporators, One Side in a Heat
Exchanger: Heat transfer terms are significant. For a single-mass stream, the following applies:

$$h_i + q = h_e$$

Heat Exchangers: No heat loss to the surroundings or work. Two separate flowrates $\dot{m}_1$ and $\dot{m}_2$:

$$\dot{m}_1\left(h_{1i} - h_{1e}\right) = \dot{m}_2\left(h_{2e} - h_{2i}\right)$$

Mixers, Separators, Open or Closed Feedwater Heaters:

$$\Sigma \dot{m}_i h_i = \Sigma \dot{m}_e h_e \quad \text{and}$$

$$\Sigma \dot{m}_i = \Sigma \dot{m}_e$$

Basic Cycles

Heat engines take in heat Q_H at a high temperature T_H, produce a net amount of work W, and reject heat Q_L at a low temperature T_L. The efficiency η of a heat engine is given by:

$$\eta = W/Q_H = (Q_H - Q_L)/Q_H$$

The most efficient engine possible is the *Carnot Cycle*. Its efficiency is given by:

$$\eta_c = (T_H - T_L)/T_H$$

where T_H and T_L = absolute temperatures (Kelvin or Rankine).

The following heat-engine cycles are plotted on *P-v* and *T-s* diagrams in this section:

Carnot, Otto, Rankine

Refrigeration cycles are the reverse of heat-engine cycles. Heat is moved from low to high temperature requiring work, W. Cycles can be used either for refrigeration or as heat pumps.

Coefficient of Performance (COP) is defined as:

COP = Q_H/W for heat pumps, and as

COP = Q_L/W for refrigerators and air conditioners.

Upper limit of COP is based on reversed Carnot Cycle:

$COP_c = T_H/(T_H - T_L)$ for heat pumps and

$COP_c = T_L/(T_H - T_L)$ for refrigeration.

1 ton refrigeration = 12,000 Btu/hr = 3,516 W

The following refrigeration cycles are plotted on *T-s* diagrams in this section: reversed rankine, two-stage refrigeration, air refrigeration

Psychrometrics

Properties of an air-water vapor mixture at a fixed pressure are given in graphical form on a psychrometric chart as provided in this section. When the system pressure is 1 atm, an ideal-gas mixture is assumed.

The definitions that follow use subscript *a* for dry air and *v* for water vapor.

P = pressure of the air-water mixture, normally 1 atm

T = dry-bulb temp (air/water mixture temperature)

P_a = partial pressure of dry air

P_v = partial pressure of water vapor

$P = P_a + P_v$

Specific Humidity (absolute humidity, humidity ratio) ω:

$\omega = m_v/m_a$

where

m_v = mass of water vapor

m_a = mass of dry air

$\omega = 0.622P_v/P_a = 0.622P_v/(P - P_v)$

Relative Humidity (rh) ϕ:

$\phi = P_v/P_g$

where P_g = saturation pressure of water at T.

Enthalpy h:

$h = h_a + \omega h_v$

Dew-Point Temperature T_{dp}:

$T_{dp} = T_{sat}$ at $P_g = P_v$

Wet-bulb temperature T_{wb} is the temperature indicated by a thermometer covered by a wick saturated with liquid water and in contact with moving air.

Humid Volume: Volume of moist air/mass of dry air.

Second Law of Thermodynamics

Thermal Energy Reservoirs

$\Delta S_{reservoir} = Q/T_{reservoir}$

where Q is measured with respect to the reservoir.

Kelvin-Planck Statement of Second Law

No heat engine can operate in a cycle while transferring heat with a single heat reservoir.

COROLLARY to Kelvin-Planck: No heat engine can have a higher efficiency than a Carnot Cycle operating between the same reservoirs.

Clausius' Statement of Second Law

No refrigeration or heat pump cycle can operate without a net work input.

COROLLARY: No refrigerator or heat pump can have a higher COP than a Carnot Cycle refrigerator or heat pump.

Entropy

$$ds = (1/T)\delta q_{\text{rev}}$$
$$s_2 - s_1 = \int_1^2 (1/T)\delta q_{\text{rev}}$$

Inequality of Clausius

$$\oint (1/T)\delta q_{\text{rev}} \leq 0$$
$$\int_1^2 (1/T)\delta q \leq s_2 - s_1$$

Isothermal, Reversible Process

$$\Delta s = s_2 - s_1 = q/T$$

Isentropic Process

$$\Delta s = 0;\ ds = 0$$

A reversible adiabatic process is isentropic.

Adiabatic Process

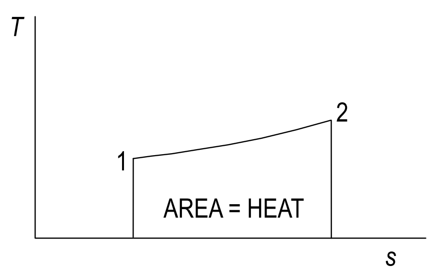
$$\delta q = 0;\ \Delta s \geq 0$$

Increase of Entropy Principle

$$\Delta s_{\text{total}} = \Delta s_{\text{system}} + \Delta s_{\text{surroundings}} \geq 0$$
$$\Delta \dot{s}_{\text{total}} = \Sigma \dot{m}_{\text{out}} s_{\text{out}} - \Sigma \dot{m}_{\text{in}} s_{\text{in}} - \Sigma\left(\dot{q}_{\text{external}}/T_{\text{external}}\right) \geq 0$$

Temperature-Entropy (T-s) Diagram

$$q_{rev} = \int_1^2 T\,ds$$



Entropy Change for Solids and Liquids

$$ds = c\,(dT/T)$$
$$s_2 - s_1 = \int c\,(dT/T) = c_{\text{mean}} \ln\,(T_2/T_1)$$

where c equals the heat capacity of the solid or liquid.

Exergy (Availability)

Exergy (also known as availability) is the maximum possible work that can be obtained from a cycle of a heat engine. The maximum possible work is obtained in a reversible process.

Closed-System Exergy (Availability)

(no chemical reactions)

$$\phi = (u - u_L) - T_L(s - s_L) + p_L(v - v_L)$$

where the subscript L designates environmental conditions and ϕ is availability function.

$$w_{\max} = w_{\text{rev}} = \phi_{\text{i}} - \phi_2$$

Open-System Exergy (Availability)

$$\Psi = (h - h_L) - T_L(s - s_L) + V^2/2 + gZ$$

where V is velocity, g is acceleration of gravity, Z is elevation and Ψ is availability function.

$$w_{\max} = w_{\text{rev}} = \Psi_{\text{i}} - \Psi_2$$

Gibbs Free Energy, Δ*G*

Energy released or absorbed in a reaction occurring reversibly at constant pressure and temperature.

Helmholtz Free Energy, Δ*A*

Energy released or absorbed in a reaction occurring reversibly at constant volume and temperature.

Irreversibility, *I*

$$I = w_{\text{rev}} - w_{\text{actual}} = T_L \Delta s_{\text{total}}$$

Heats of Reaction

For a chemical reaction the associated energy can be defined in terms of heats of formation of the individual species ΔH_f° at the standard state

$$\left(\Delta H_r^\circ\right) = \sum_{\text{products}} \nu_i \left(\Delta H_f^\circ\right)_i - \sum_{\text{reactants}} \nu_i \left(\Delta H_f^\circ\right)_i$$

ν_i = stoichiometric coefficient for species "i"

The standard state is 25°C and 1 bar.

The heat of formation is defined as the enthalpy change associated with the formation of a compound from its atomic species as they normally occur in nature [i.e., O_2(g), H_2(g), C(solid), etc.]

The heat of reaction varies with the temperature as follows:

$$\Delta H_r^\circ(T) = \Delta H_r^\circ(T_{\text{ref}}) + \int_{T_{\text{ref}}}^{T} \Delta c_p dT$$

where T_{ref} is some reference temperature (typically 25°C or 298 K), and:

$$\Delta c_p = \sum_{\text{products}} \nu_i c_{p,i} - \sum_{\text{reactants}} \nu_i c_{p,i}$$

and $c_{p,i}$ is the molar heat capacity of component i.

The heat of reaction for a combustion process using oxygen is also known as the heat of combustion. The principal products are CO_2(g) and $H_2O(l)$.

Combustion Processes

First, the combustion equation should be written and balanced. For example, for the stoichiometric combustion of methane in oxygen:

$$CH_4 + 2\ O_2 \rightarrow CO_2 + 2\ H_2O$$

Combustion in Air

For each mole of oxygen, there will be 3.76 moles of nitrogen. For stoichiometric combustion of methane in air:

$$CH_4 + 2\ O_2 + 2(3.76)\ N_2 \rightarrow CO_2 + 2\ H_2O + 7.52\ N_2$$

Combustion in Excess Air

The excess oxygen appears as oxygen on the right side of the combustion equation.

Incomplete Combustion

Some carbon is burned to create carbon monoxide (CO).

$$\text{Molar Air-Fuel Ratio, } \overline{A/F} = \frac{\text{No. of moles of air}}{\text{No. of moles of fuel}}$$

$$\text{Air-Fuel Ratio, } A/F = \frac{\text{Mass of air}}{\text{Mass of fuel}} = \left(\overline{A/F}\right)\left(\frac{M_{\text{air}}}{M_{\text{fuel}}}\right)$$

Stoichiometric (theoretical) air-fuel ratio is the air-fuel ratio calculated from the stoichiometric combustion equation.

$$\text{Percent Theoretical Air} = \frac{(A/F)_{\text{actual}}}{(A/F)_{\text{stoichiometric}}} \times 100$$

$$\text{Percent Excess Air} = \frac{(A/F)_{\text{actual}} - (A/F)_{\text{stoichiometric}}}{(A/F)_{\text{stoichiometric}}} \times 100$$

Vapor-Liquid Equilibrium (VLE)

Henry's Law at Constant Temperature

At equilibrium, the partial pressure of a gas is proportional to its concentration in a liquid. Henry's Law is valid for low concentrations; i.e., $x \approx 0$.

$$P_i = Py_i = hx_i$$

where

- h = Henry's Law constant
- P_i = partial pressure of a gas in contact with a liquid
- x_i = mol fraction of the gas in the liquid
- y_i = mol fraction of the gas in the vapor
- P = total pressure

Raoult's Law for Vapor-Liquid Equilibrium

Valid for concentrations near 1; i.e., $x_i \approx 1$ at low pressure (ideal gas behavior)

$$P_i = x_i P_i^*$$

where

- P_i = partial pressure of component i
- x_i = mol fraction of component i in the liquid
- P_i^* = vapor pressure of pure component i at the temperature of the mixture

Rigorous Vapor-Liquid Equilibrium

For a multicomponent mixture at equilibrium

$$\hat{f}_i^V = \hat{f}_i^L$$

where

$\hat{f}_i^V$ = fugacity of component i in the vapor phase

$\hat{f}_i^L$ = fugacity of component i in the liquid phase

Fugacities of component *i* in a mixture are commonly calculated in the following ways:

For a liquid $\hat{f}_i^L = x_i \gamma_i f_i^L$

where

x_i = mole fraction of component *i*

γ_i = activity coefficient of component *i*

f_i^L = fugacity of pure liquid component *i*

For a vapor $\hat{f}_i^V = y_i \hat{\Phi}_i P$

where

y_i = mole fraction of component *i* in the vapor

$\hat{\Phi}_i$ = fugacity coefficient of component *i* in the vapor

P = system pressure

The activity coefficient γ_i is a correction for liquid phase nonideality. Many models have been proposed for γ_i such as the Van Laar model:

$$\ln \gamma_1 = A_{12}\left(1 + \frac{A_{12}x_1}{A_{21}x_2}\right)^{-2}$$

$$\ln \gamma_2 = A_{21}\left(1 + \frac{A_{21}x_2}{A_{12}x_1}\right)^{-2}$$

where

γ_1 = activity coefficient of component 1 in a two-component system

γ_2 = activity coefficient of component 2 in a two-component system

A_{12}, A_{21} = constants, typically fitted from experimental data

The pure component fugacity is calculated as:

$$f_i^L = \Phi_i^{\text{sat}} P_i^{\text{sat}} \exp\left\{v_i^L\left(P - P_i^{\text{sat}}\right)/\left(RT\right)\right\}$$

where

Φ_i^{sat} = fugacity coefficient of pure saturated *i*

P_i^{sat} = saturation pressure of pure *i*

v_i^L = specific volume of pure liquid *i*

R = Ideal Gas Law Constant

T = absolute temperature

Often at system pressures close to atmospheric:

$$f_i^L \cong P_i^{\text{sat}}$$

The fugacity coefficient $\hat{\Phi}_i$ for component *i* in the vapor is calculated from an equation of state (e.g., Virial). Sometimes it is approximated by a pure component value from a correlation. Often at pressures close to atmospheric, $\hat{\Phi}_i = 1$. The fugacity coefficient is a correction for vapor phase nonideality.

For sparingly soluble gases the liquid phase is sometimes represented as:

$$\hat{f}_i^L = x_i k_i$$

where k_i is a constant set by experiment (Henry's constant). Sometimes other concentration units are used besides mole fraction with a corresponding change in k_i.

Phase Relations

Clapeyron Equation for phase transitions:

$$\left(\frac{dP}{dT}\right)_{\text{sat}} = \frac{h_{fg}}{Tv_{fg}} = \frac{s_{fg}}{v_{fg}}$$

where

h_{fg} = enthalpy change for phase transitions
v_{fg} = volume change
s_{fg} = entropy change
T = absolute temperature
$(dP/dT)_{\text{sat}}$ = slope of phase transition (e.g.,vapor-liquid) saturation line

Clausius-Clapeyron Equation
This equation results if it is assumed that (1) the volume change (v_{fg}) can be replaced with the vapor volume (v_g), (2) the latter can be replaced with $P/\bar{R}\,T$ from the ideal gas law, and (3) h_{fg} is independent of the temperature (T).

$$\ln_e\left(\frac{P_2}{P_1}\right) = \frac{h_{fg}}{\bar{R}} \bullet \frac{T_2 - T_1}{T_1 T_2}$$

Gibbs Phase Rule (*non-reacting systems*)

$$P + F = C + 2$$

where

P = number of phases making up a system
F = degrees of freedom
C = number of components in a system

Chemical Reaction Equilibria

Definitions

Conversion – moles reacted/moles fed
Extent – For each species in a reaction, the mole balance may be written:

$$\text{moles}_{i,\text{out}} = \text{moles}_{i,\text{in}} + v_i \xi$$

where ξ is the extent in moles and v_i is the stoichiometric coefficient of the ith species, the sign of which is negative for reactants and positive for products.

Limiting reactant – Reactant that would be consumed first if the reaction proceeded to completion. Other reactants are excess reactants.

Selectivity – Moles of desired product formed/moles of undesired product formed.

Yield – Moles of desired product formed/moles that would have been formed if there were no side reactions and the limiting reactant had reacted completely.

Chemical Reaction Equilibrium

For the reaction

$$aA + bB \rightleftharpoons cC + dD$$

$$\Delta G^{\circ} = -RT \ln K_a$$

$$K_a = \frac{\left(\hat{a}_C^c\right)\left(\hat{a}_D^d\right)}{\left(\hat{a}_A^a\right)\left(\hat{a}_B^b\right)} = \prod_i \left(\hat{a}_i\right)^{v_i}$$

where

$\hat{a}_i$ = activity of component $i = \frac{\hat{f}_i}{f_i^{\circ}}$

f_i° = fugacity of pure i in its standard state at the equilibrium reaction temperature T

v_i = stoichiometric coefficient of component i

ΔG° = standard Gibbs energy change of reaction

K_a = chemical equilibrium constant

For mixtures of ideal gases:

f_i° = unit pressure, often 1 bar

$\hat{f}_i = y_i P = p_i$

where p_i = partial pressure of component i

Then $K_a = K_p = \frac{\left(p_C^c\right)\left(p_D^d\right)}{\left(p_A^a\right)\left(p_B^b\right)} = P^{c+d-a-b}\frac{\left(y_C^c\right)\left(y_D^d\right)}{\left(y_A^a\right)\left(y_B^b\right)}$

For solids $\hat{a}_i = 1$

For liquids $\hat{a}_i = x_i \gamma_i$

The effect of temperature on the equilibrium constant is

$$\frac{d \ln K}{dT} = \frac{\Delta H^{\circ}}{RT^2}$$

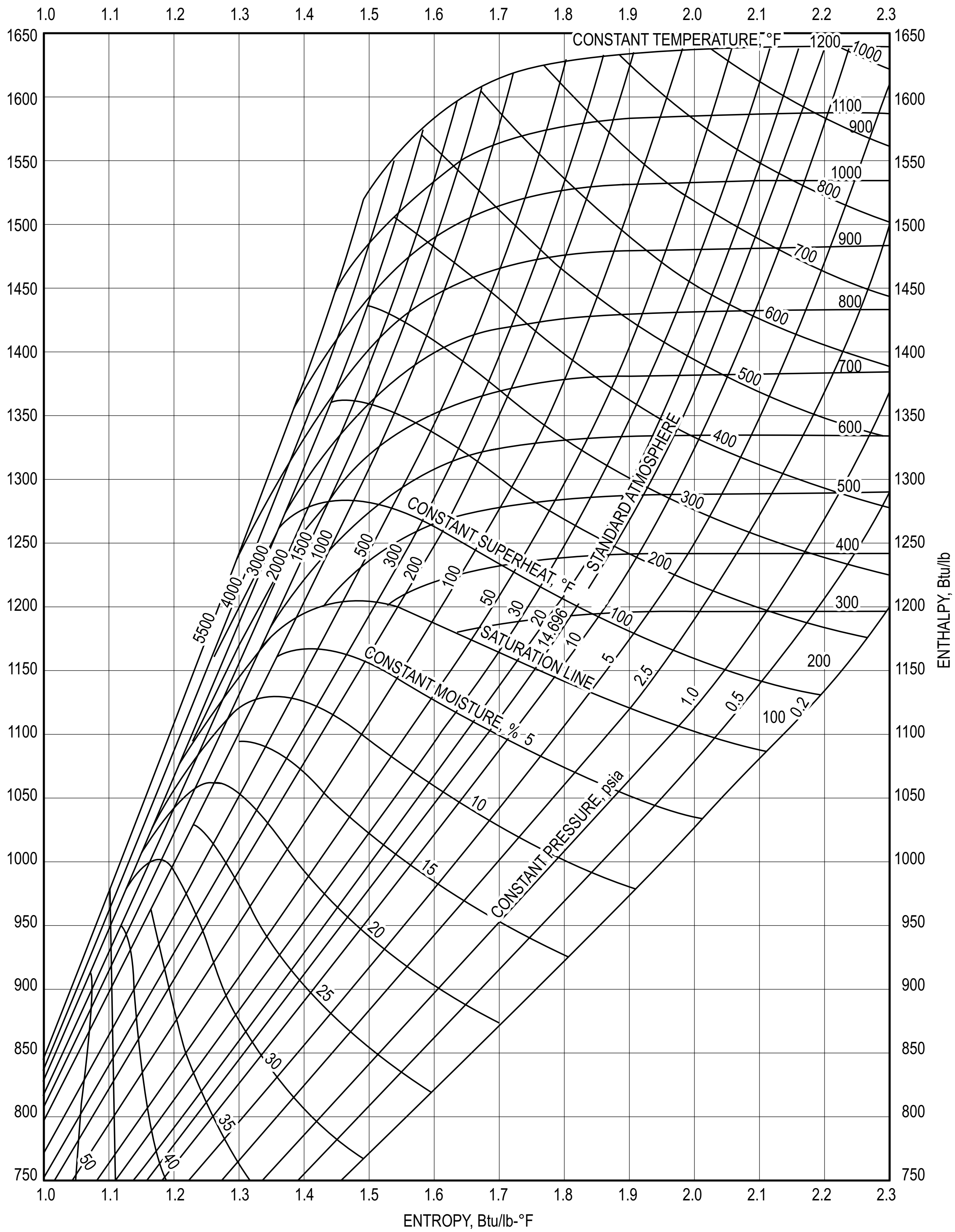
where ΔH° = standard enthalpy change of reaction

STEAM TABLES Saturated Water - Temperature Table												
Temp. °C T	Sat. Press. kPa	Specific Volume m^3/kg		Internal Energy kJ/kg			Enthalpy kJ/kg			Entropy kJ/(kg·K)		
		Sat. liquid	Sat. vapor	Sat. liquid	Evap.	Sat. vapor	Sat. liquid	Evap.	Sat. vapor	Sat. liquid	Evap.	Sat. vapor
	p_{sat}	v_f	v_g	u_f	u_{fg}	u_g	h_f	h_{fg}	h_g	s_f	s_{fg}	s_g
0.01	0.6113	0.001 000	206.14	0.00	2375.3	2375.3	0.01	2501.3	2501.4	0.0000	9.1562	9.1562
5	0.8721	0.001 000	147.12	20.97	2361.3	2382.3	20.98	2489.6	2510.6	0.0761	8.9496	9.0257
10	1.2276	0.001 000	106.38	42.00	2347.2	2389.2	42.01	2477.7	2519.8	0.1510	8.7498	8.9008
15	1.7051	0.001 001	77.93	62.99	2333.1	2396.1	62.99	2465.9	2528.9	0.2245	8.5569	8.7814
20	**2.339**	**0.001 002**	**57.79**	**83.95**	**2319.0**	**2402.9**	**83.96**	**2454.1**	**2538.1**	**0.2966**	**8.3706**	**8.6672**
25	3.169	0.001 003	43.36	104.88	2304.9	2409.8	104.89	2442.3	2547.2	0.3674	8.1905	8.5580
30	4.246	0.001 004	32.89	125.78	2290.8	2416.6	125.79	2430.5	2556.3	0.4369	8.0164	8.4533
35	5.628	0.001 006	25.22	146.67	2276.7	2423.4	146.68	2418.6	2565.3	0.5053	7.8478	8.3531
40	7.384	0.001 008	19.52	167.56	2262.6	2430.1	167.57	2406.7	2574.3	0.5725	7.6845	8.2570
45	**9.593**	**0.001 010**	**15.26**	**188.44**	**2248.4**	**2436.8**	**188.45**	**2394.8**	**2583.2**	**0.6387**	**7.5261**	**8.1648**
50	12.349	0.001 012	12.03	209.32	2234.2	2443.5	209.33	2382.7	2592.1	0.7038	7.3725	8.0763
55	15.758	0.001 015	9.568	230.21	2219.9	2450.1	230.23	2370.7	2600.9	0.7679	7.2234	7.9913
60	19.940	0.001 017	7.671	251.11	2205.5	2456.6	251.13	2358.5	2609.6	0.8312	7.0784	7.9096
65	25.03	0.001 020	6.197	272.02	2191.1	2463.1	272.06	2346.2	2618.3	0.8935	6.9375	7.8310
70	**31.19**	**0.001 023**	**5.042**	**292.95**	**2176.6**	**2569.6**	**292.98**	**2333.8**	**2626.8**	**0.9549**	**6.8004**	**7.7553**
75	38.58	0.001 026	4.131	313.90	2162.0	2475.9	313.93	2321.4	2635.3	1.0155	6.6669	7.6824
80	47.39	0.001 029	3.407	334.86	2147.4	2482.2	334.91	2308.8	2643.7	1.0753	6.5369	7.6122
85	57.83	0.001 033	2.828	355.84	2132.6	2488.4	355.90	2296.0	2651.9	1.1343	6.4102	7.5445
90	70.14	0.001 036	2.361	376.85	2117.7	2494.5	376.92	2283.2	2660.1	1.1925	6.2866	7.4791
95	**84.55**	**0.001 040**	**1.982**	**397.88**	**2102.7**	**2500.6**	**397.96**	**2270.2**	**2668.1**	**1.2500**	**6.1659**	**7.4159**
	MPa											
100	0.101 35	0.001 044	1.6729	418.94	2087.6	2506.5	419.04	2257.0	2676.1	1.3069	6.0480	7.3549
105	0.120 82	0.001 048	1.4194	440.02	2072.3	2512.4	440.15	2243.7	2683.8	1.3630	5.9328	7.2958
110	0.143 27	0.001 052	1.2102	461.14	2057.0	2518.1	461.30	2230.2	2691.5	1.4185	5.8202	7.2387
115	0.169 06	0.001 056	1.0366	482.30	2041.4	2523.7	482.48	2216.5	2699.0	1.4734	5.7100	7.1833
120	**0.198 53**	**0.001 060**	**0.8919**	**503.50**	**2025.8**	**2529.3**	**503.71**	**2202.6**	**2706.3**	**1.5276**	**5.6020**	**7.1296**
125	0.2321	0.001 065	0.7706	524.74	2009.9	2534.6	524.99	2188.5	2713.5	1.5813	5.4962	7.0775
130	0.2701	0.001 070	0.6685	546.02	1993.9	2539.9	546.31	2174.2	2720.5	1.6344	5.3925	7.0269
135	0.3130	0.001 075	0.5822	567.35	1977.7	2545.0	567.69	2159.6	2727.3	1.6870	5.2907	6.9777
140	0.3613	0.001 080	0.5089	588.74	1961.3	2550.0	589.13	2144.7	2733.9	1.7391	5.1908	6.9299
145	**0.4154**	**0.001 085**	**0.4463**	**610.18**	**1944.7**	**2554.9**	**610.63**	**2129.6**	**2740.3**	**1.7907**	**5.0926**	**6.8833**
150	0.4758	0.001 091	0.3928	631.68	1927.9	2559.5	632.20	2114.3	2746.5	1.8418	4.9960	6.8379
155	0.5431	0.001 096	0.3468	653.24	1910.8	2564.1	653.84	2098.6	2752.4	1.8925	4.9010	6.7935
160	0.6178	0.001 102	0.3071	674.87	1893.5	2568.4	675.55	2082.6	2758.1	1.9427	4.8075	6.7502
165	0.7005	0.001 108	0.2727	696.56	1876.0	2572.5	697.34	2066.2	2763.5	1.9925	4.7153	6.7078
170	**0.7917**	**0.001 114**	**0.2428**	**718.33**	**1858.1**	**2576.5**	**719.21**	**2049.5**	**2768.7**	**2.0419**	**4.6244**	**6.6663**
175	0.8920	0.001 121	0.2168	740.17	1840.0	2580.2	741.17	2032.4	2773.6	2.0909	4.5347	6.6256
180	1.0021	0.001 127	0.194 05	762.09	1821.6	2583.7	763.22	2015.0	2778.2	2.1396	4.4461	6.5857
185	1.1227	0.001 134	0.174 09	784.10	1802.9	2587.0	785.37	1997.1	2782.4	2.1879	4.3586	6.5465
190	1.2544	0.001 141	0.156 54	806.19	1783.8	2590.0	807.62	1978.8	2786.4	2.2359	4.2720	6.5079
195	**1.3978**	**0.001 149**	**0.141 05**	**828.37**	**1764.4**	**2592.8**	**829.98**	**1960.0**	**2790.0**	**2.2835**	**4.1863**	**6.4698**
200	1.5538	0.001 157	0.127 36	850.65	1744.7	2595.3	852.45	1940.7	2793.2	2.3309	4.1014	6.4323
205	1.7230	0.001 164	0.115 21	873.04	1724.5	2597.5	875.04	1921.0	2796.0	2.3780	4.0172	6.3952
210	1.9062	0.001 173	0.104 41	895.53	1703.9	2599.5	897.76	1900.7	2798.5	2.4248	3.9337	6.3585
215	2.104	0.001 181	0.094 79	918.14	1682.9	2601.1	920.62	1879.9	2800.5	2.4714	3.8507	6.3221
220	**2.318**	**0.001 190**	**0.086 19**	**940.87**	**1661.5**	**2602.4**	**943.62**	**1858.5**	**2802.1**	**2.5178**	**3.7683**	**6.2861**
225	2.548	0.001 199	0.078 49	963.73	1639.6	2603.3	966.78	1836.5	2803.3	2.5639	3.6863	6.2503
230	2.795	0.001 209	0.071 58	986.74	1617.2	2603.9	990.12	1813.8	2804.0	2.6099	3.6047	6.2146
235	3.060	0.001 219	0.065 37	1009.89	1594.2	2604.1	1013.62	1790.5	2804.2	2.6558	3.5233	6.1791
240	3.344	0.001 229	0.059 76	1033.21	1570.8	2604.0	1037.32	1766.5	2803.8	2.7015	3.4422	6.1437
245	**3.648**	**0.001 240**	**0.054 71**	**1056.71**	**1546.7**	**2603.4**	**1061.23**	**1741.7**	**2803.0**	**2.7472**	**3.3612**	**6.1083**
250	3.973	0.001 251	0.050 13	1080.39	1522.0	2602.4	1085.36	1716.2	2801.5	2.7927	3.2802	6.0730
255	4.319	0.001 263	0.045 98	1104.28	1596.7	2600.9	1109.73	1689.8	2799.5	2.8383	3.1992	6.0375
260	4.688	0.001 276	0.042 21	1128.39	1470.6	2599.0	1134.37	1662.5	2796.9	2.8838	3.1181	6.0019
265	5.081	0.001 289	0.038 77	1152.74	1443.9	2596.6	1159.28	1634.4	2793.6	2.9294	3.0368	5.9662
270	**5.499**	**0.001 302**	**0.035 64**	**1177.36**	**1416.3**	**2593.7**	**1184.51**	**1605.2**	**2789.7**	**2.9751**	**2.9551**	**5.9301**
275	5.942	0.001 317	0.032 79	1202.25	1387.9	2590.2	1210.07	1574.9	2785.0	3.0208	2.8730	5.8938
280	6.412	0.001 332	0.030 17	1227.46	1358.7	2586.1	1235.99	1543.6	2779.6	3.0668	2.7903	5.8571
285	6.909	0.001 348	0.027 77	1253.00	1328.4	2581.4	1262.31	1511.0	2773.3	3.1130	2.7070	5.8199
290	7.436	0.001 366	0.025 57	1278.92	1297.1	2576.0	1289.07	1477.1	2766.2	3.1594	2.6227	5.7821
295	**7.993**	**0.001 384**	**0.023 54**	**1305.2**	**1264.7**	**2569.9**	**1316.3**	**1441.8**	**2758.1**	**3.2062**	**2.5375**	**5.7437**
300	8.581	0.001 404	0.021 67	1332.0	1231.0	2563.0	1344.0	1404.9	2749.0	3.2534	2.4511	5.7045
305	9.202	0.001 425	0.019 948	1359.3	1195.9	2555.2	1372.4	1366.4	2738.7	3.3010	2.3633	5.6643
310	9.856	0.001 447	0.018 350	1387.1	1159.4	2546.4	1401.3	1326.0	2727.3	3.3493	2.2737	5.6230
315	10.547	0.001 472	0.016 867	1415.5	1121.1	2536.6	1431.0	1283.5	2714.5	3.3982	2.1821	5.5804
320	**11.274**	**0.001 499**	**0.015 488**	**1444.6**	**1080.9**	**2525.5**	**1461.5**	**1238.6**	**2700.1**	**3.4480**	**2.0882**	**5.5362**
330	12.845	0.001 561	0.012 996	1505.3	993.7	2498.9	1525.3	1140.6	2665.9	3.5507	1.8909	5.4417
340	14.586	0.001 638	0.010 797	1570.3	894.3	2464.6	1594.2	1027.9	2622.0	3.6594	1.6763	5.3357
350	16.513	0.001 740	0.008 813	1641.9	776.6	2418.4	1670.6	893.4	2563.9	3.7777	1.4335	5.2112
360	18.651	0.001 893	0.006 945	1725.2	626.3	2351.5	1760.5	720.3	2481.0	3.9147	1.1379	5.0526
370	**21.03**	**0.002 213**	**0.004 925**	**1844.0**	**384.5**	**2228.5**	**1890.5**	**441.6**	**2332.1**	**4.1106**	**0.6865**	**4.7971**
374.14	22.09	0.003 155	0.003 155	2029.6	0	2029.6	2099.3	0	2099.3	4.4298	0	4.4298

Superheated Water Tables

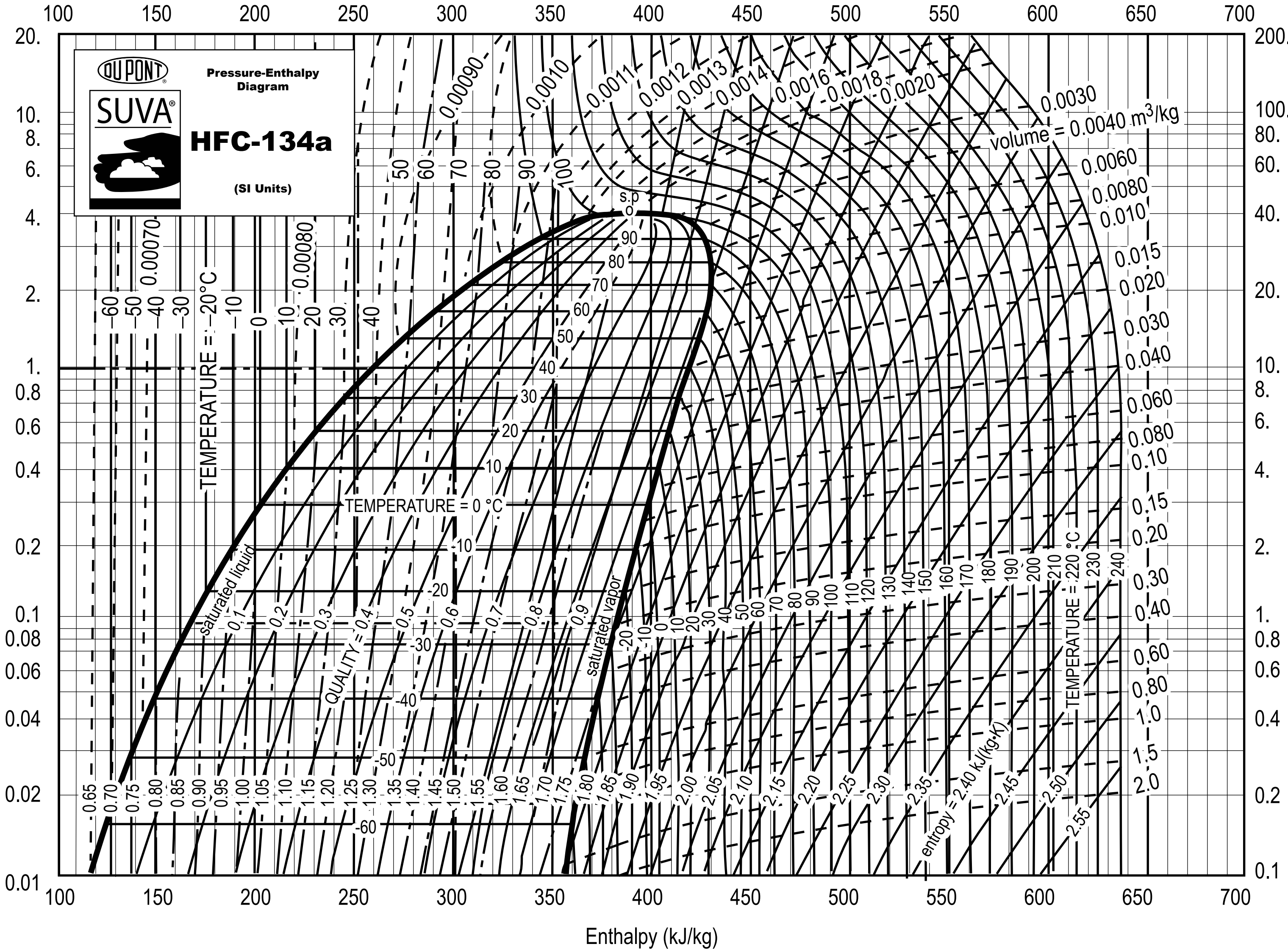
T Temp. ^oC	*v* m^3/kg	*u* kJ/kg	*h* kJ/kg	*s* kJ/(kg·K)	*v* m^3/kg	*u* kJ/kg	*h* kJ/kg	*s* kJ/(kg·K)
	***p* = 0.01 MPa (45.81^{o}C)**				***p* = 0.05 MPa (81.33^{o}C)**			
Sat.	14.674	2437.9	2584.7	8.1502	3.240	2483.9	2645.9	7.5939
50	14.869	2443.9	2592.6	8.1749				
100	17.196	2515.5	2687.5	8.4479	3.418	2511.6	2682.5	7.6947
150	19.512	2587.9	2783.0	8.6882	3.889	2585.6	2780.1	7.9401
200	**21.825**	**2661.3**	**2879.5**	**8.9038**	**4.356**	**2659.9**	**2877.7**	**8.1580**
250	24.136	2736.0	2977.3	9.1002	4.820	2735.0	2976.0	8.3556
300	26.445	2812.1	3076.5	9.2813	5.284	2811.3	3075.5	8.5373
400	31.063	2968.9	3279.6	9.6077	6.209	2968.5	3278.9	8.8642
500	35.679	3132.3	3489.1	9.8978	7.134	3132.0	3488.7	9.1546
600	**40.295**	**3302.5**	**3705.4**	**10.1608**	**8.057**	**3302.2**	**3705.1**	**9.4178**
700	44.911	3479.6	3928.7	10.4028	8.981	3479.4	3928.5	9.6599
800	49.526	3663.8	4159.0	10.6281	9.904	3663.6	4158.9	9.8852
900	54.141	3855.0	4396.4	10.8396	10.828	3854.9	4396.3	10.0967
1000	58.757	4053.0	4640.6	11.0393	11.751	4052.9	4640.5	10.2964
1100	**63.372**	**4257.5**	**4891.2**	**11.2287**	**12.674**	**4257.4**	**4891.1**	**10.4859**
1200	67.987	4467.9	5147.8	11.4091	13.597	4467.8	5147.7	10.6662
1300	72.602	4683.7	5409.7	11.5811	14.521	4683.6	5409.6	10.8382
	***p* = 0.10 MPa (99.63^{o}C)**				***p* = 0.20 MPa (120.23^{o}C)**			
Sat.	1.6940	2506.1	2675.5	7.3594	0.8857	2529.5	2706.7	7.1272
100	1.6958	2506.7	2676.2	7.3614				
150	1.9364	2582.8	2776.4	7.6134	0.9596	2576.9	2768.8	7.2795
200	2.172	2658.1	2875.3	7.8343	1.0803	2654.4	2870.5	7.5066
250	**2.406**	**2733.7**	**2974.3**	**8.0333**	**1.1988**	**2731.2**	**2971.0**	**7.7086**
300	2.639	2810.4	3074.3	8.2158	1.3162	2808.6	3071.8	7.8926
400	3.103	2967.9	3278.2	8.5435	1.5493	2966.7	3276.6	8.2218
500	3.565	3131.6	3488.1	8.8342	1.7814	3130.8	3487.1	8.5133
600	4.028	3301.9	3704.4	9.0976	2.013	3301.4	3704.0	8.7770
700	**4.490**	**3479.2**	**3928.2**	**9.3398**	**2.244**	**3478.8**	**3927.6**	**9.0194**
800	4.952	3663.5	4158.6	9.5652	2.475	3663.1	4158.2	9.2449
900	5.414	3854.8	4396.1	9.7767	2.705	3854.5	4395.8	9.4566
1000	5.875	4052.8	4640.3	9.9764	2.937	4052.5	4640.0	9.6563
1100	6.337	4257.3	4891.0	10.1659	3.168	4257.0	4890.7	9.8458
1200	**6.799**	**4467.7**	**5147.6**	**10.3463**	**3.399**	**4467.5**	**5147.5**	**10.0262**
1300	7.260	4683.5	5409.5	10.5183	3.630	4683.2	5409.3	10.1982
	***p* = 0.40 MPa (143.63^{o}C)**				***p* = 0.60 MPa (158.85^{o}C)**			
Sat.	0.4625	2553.6	2738.6	6.8959	0.3157	2567.4	2756.8	6.7600
150	0.4708	2564.5	2752.8	6.9299				
200	0.5342	2646.8	2860.5	7.1706	0.3520	2638.9	2850.1	6.9665
250	0.5951	2726.1	2964.2	7.3789	0.3938	2720.9	2957.2	7.1816
300	**0.6548**	**2804.8**	**3066.8**	**7.5662**	**0.4344**	**2801.0**	**3061.6**	**7.3724**
350	0.7137	2884.6	3170.1	7.7324	0.4742	2881.2	3165.7	7.5464
400	0.7726	2964.4	3273.4	7.8985	0.5137	2962.1	3270.3	7.7079
500	0.8893	3129.2	3484.9	8.1913	0.5920	3127.6	3482.8	8.0021
600	1.0055	3300.2	3702.4	8.4558	0.6697	3299.1	3700.9	8.2674
700	**1.1215**	**3477.9**	**3926.5**	**8.6987**	**0.7472**	**3477.0**	**3925.3**	**8.5107**
800	1.2372	3662.4	4157.3	8.9244	0.8245	3661.8	4156.5	8.7367
900	1.3529	3853.9	4395.1	9.1362	0.9017	3853.4	4394.4	8.9486
1000	1.4685	4052.0	4639.4	9.3360	0.9788	4051.5	4638.8	9.1485
1100	1.5840	4256.5	4890.2	9.5256	1.0559	4256.1	4889.6	9.3381
1200	**1.6996**	**4467.0**	**5146.8**	**9.7060**	**1.1330**	**4466.5**	**5146.3**	**9.5185**
1300	1.8151	4682.8	5408.8	9.8780	1.2101	4682.3	5408.3	9.6906
	***p* = 0.80 MPa (170.43^{o}C)**				***p* = 1.00 MPa (179.91^{o}C)**			
Sat.	0.2404	2576.8	2769.1	6.6628	0.194 44	2583.6	2778.1	6.5865
200	0.2608	2630.6	2839.3	6.8158	0.2060	2621.9	2827.9	6.6940
250	0.2931	2715.5	2950.0	7.0384	0.2327	2709.9	2942.6	6.9247
300	0.3241	2797.2	3056.5	7.2328	0.2579	2793.2	3051.2	7.1229
350	**0.3544**	**2878.2**	**3161.7**	**7.4089**	**0.2825**	**2875.2**	**3157.7**	**7.3011**
400	0.3843	2959.7	3267.1	7.5716	0.3066	2957.3	3263.9	7.4651
500	0.4433	3126.0	3480.6	7.8673	0.3541	3124.4	3478.5	7.7622
600	0.5018	3297.9	3699.4	8.1333	0.4011	3296.8	3697.9	8.0290
700	0.5601	3476.2	3924.2	8.3770	0.4478	3475.3	3923.1	8.2731
800	**0.6181**	**3661.1**	**4155.6**	**8.6033**	**0.4943**	**3660.4**	**4154.7**	**8.4996**
900	0.6761	3852.8	4393.7	8.8153	0.5407	3852.2	4392.9	8.7118
1000	0.7340	4051.0	4638.2	9.0153	0.5871	4050.5	4637.6	8.9119
1100	0.7919	4255.6	4889.1	9.2050	0.6335	4255.1	4888.6	9.1017
1200	0.8497	4466.1	5145.9	9.3855	0.6798	4465.6	5145.4	9.2822
1300	**0.9076**	**4681.8**	**5407.9**	**9.5575**	**0.7261**	**4681.3**	**5407.4**	**9.4543**

Mollier (h, s) Diagram for Steam



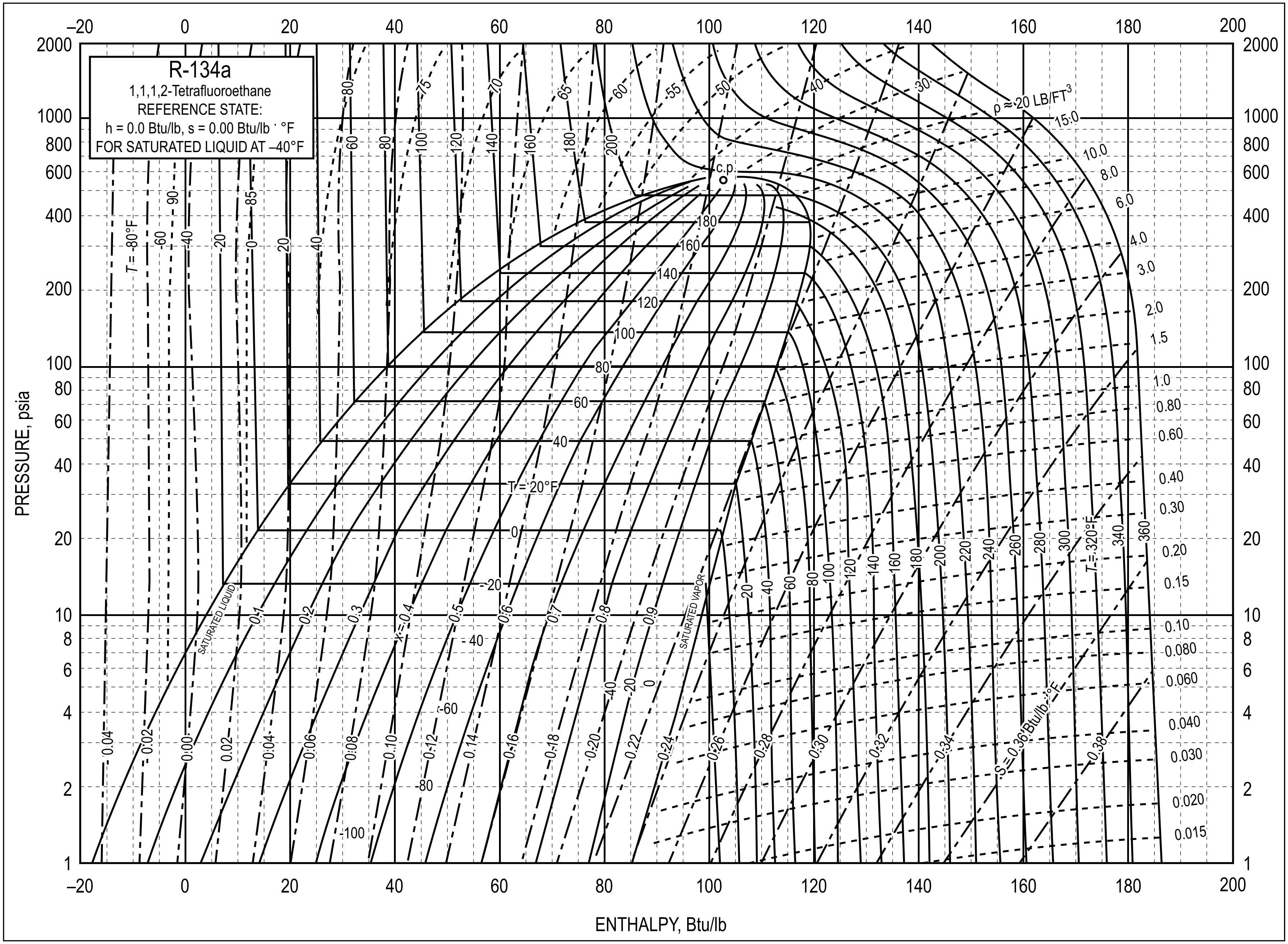
Howell, Ronald, H., William J. Coad, Harry J. Sauer, Jr., *Principles of Heating, Ventilating and Air Conditioning*, 6th ed., American Society of Heating, Refrigerating and Air-Conditioning Engineers, 2009, p. 21.

P-*h* Diagram for Refrigerant HFC-134a
(metric units)



Data provided by DuPont Refrigerants, a division of E.I. duPont de Nemours and Co., Inc.

Pressure Versus Enthalpy Curves for Refrigerant 134a
(USCS units)



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Refrigerant 134a (1,1,1,2-Tetrafluoroethane) Properties of Saturated Liquid and Saturated Vapor

Temp.,* °F	Pressure, psia	Density, lb/ft³	Volume, ft³/lb	Enthalpy, Btu/lb-°F		Entropy, Btu/lb-°F		Specific Heat c_p Btu/lb-°F		C_p/C_v	Thermal Conductivity Btu/hr-ft-°F		Temp.,* °F
		Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Vapor	Liquid	Vapor	
–153.94[a]	0.057	99.33	568.59	–32.992	80.362	–0.09154	0.27923	0.2829	0.1399	1.1637	0.0840	0.00178	–153.94[a]
–150.00	0.072	98.97	452.12	–31.878	80.907	–0.08791	0.27629	0.2830	0.1411	1.1623	0.0832	0.00188	–150.00
–140.00	0.129	98.05	260.63	–29.046	82.304	–0.07891	0.26941	0.2834	0.1443	1.1589	0.0813	0.00214	–140.00
–130.00	0.221	97.13	156.50	–26.208	83.725	–0.07017	0.26329	0.2842	0.1475	1.1559	0.0794	0.00240	–130.00
–120.00	0.365	96.20	97.48	–23.360	85.168	–0.06166	0.25784	0.2853	0.1508	1.1532	0.0775	0.00265	–120.00
–110.00	0.583	95.27	62.763	–20.500	86.629	–0.05337	0.25300	0.2866	0.1540	1.1509	0.0757	0.00291	–110.00
–100.00	0.903	94.33	41.637	–17.626	88.107	–0.04527	0.24871	0.2881	0.1573	1.1490	0.0739	0.00317	–100.00
–90.00	1.359	93.38	28.381	–14.736	89.599	–0.03734	0.24490	0.2898	0.1607	1.1475	0.0722	0.00343	–90.00
–80.00	1.993	92.42	19.825	–11.829	91.103	–0.02959	0.24152	0.2916	0.1641	1.1465	0.0705	0.00369	–80.00
–75.00	2.392	91.94	16.711	–10.368	91.858	–0.02577	0.23998	0.2925	0.1658	1.1462	0.0696	0.00382	–75.00
–70.00	2.854	91.46	14.161	–8.903	92.614	–0.02198	0.23854	0.2935	0.1676	1.1460	0.0688	0.00395	–70.00
–65.00	3.389	90.97	12.060	–7.432	93.372	–0.01824	0.23718	0.2945	0.1694	1.1459	0.0680	0.00408	–65.00
–60.00	4.002	90.49	10.321	–5.957	94.131	–0.01452	0.23590	0.2955	0.1713	1.1460	0.0671	0.00420	–60.00
–55.00	4.703	90.00	8.873	–4.476	94.890	–0.01085	0.23470	0.2965	0.1731	1.1462	0.0663	0.00433	–55.00
–50.00	5.501	89.50	7.662	–2.989	95.650	–0.00720	0.23358	0.2976	0.1751	1.1466	0.0655	0.00446	–50.00
–45.00	6.406	89.00	6.6438	–1.498	96.409	–0.00358	0.23252	0.2987	0.1770	1.1471	0.0647	0.00460	–45.00
–40.00	7.427	88.50	5.7839	0.000	97.167	0.00000	0.23153	0.2999	0.1790	1.1478	0.0639	0.00473	–40.00
–35.00	8.576	88.00	5.0544	1.503	97.924	0.00356	0.23060	0.3010	0.1811	1.1486	0.0632	0.00486	–35.00
–30.00	9.862	87.49	4.4330	3.013	98.679	0.00708	0.22973	0.3022	0.1832	1.1496	0.0624	0.00499	–30.00
–25.00	11.299	86.98	3.9014	4.529	99.433	0.01058	0.22892	0.3035	0.1853	1.1508	0.0616	0.00512	–25.00
–20.00	12.898	86.47	3.4449	6.051	100.184	0.01406	0.22816	0.3047	0.1875	1.1521	0.0608	0.00525	–20.00
–15.00	14.671	85.95	3.0514	7.580	100.932	0.01751	0.22744	0.3060	0.1898	1.1537	0.0601	0.00538	–15.00
–14.93[b]	14.696	85.94	3.0465	7.600	100.942	0.01755	0.22743	0.3061	0.1898	1.1537	0.0601	0.00538	–14.93[b]
–10.00	16.632	85.43	2.7109	9.115	101.677	0.02093	0.22678	0.3074	0.1921	1.1554	0.0593	0.00552	–10.00
–5.00	18.794	84.90	2.4154	10.657	102.419	0.02433	0.22615	0.3088	0.1945	1.1573	0.0586	0.00565	–5.00

Refrigerant 134a (1,1,1,2-Tetrafluoroethane) Properties of Saturated Liquid and Saturated Vapor (cont'd)

Temp.,* °F	Pressure, psia	Density, lb/ft^3	Volume, ft^3/lb	Enthalpy, Btu/lb-°F		Entropy, Btu/lb-°F		Specific Heat c_p Btu/lb-°F		C_p/C_v	Thermal Conductivity Btu/hr-ft-°F		Temp.,* °F
		Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Vapor	Liquid	Vapor	
0.00	21.171	84.37	2.1579	12.207	103.156	0.02771	0.22557	0.3102	0.1969	1.1595	0.0578	0.00578	0.00
5.00	23.777	83.83	1.9330	13.764	103.889	0.03107	0.22502	0.3117	0.1995	1.1619	0.0571	0.00592	5.00
10.00	26.628	83.29	1.7357	15.328	104.617	0.03440	0.22451	0.3132	0.2021	1.1645	0.0564	0.00605	10.00
15.00	29.739	82.74	1.5623	16.901	105.339	0.03772	0.22403	0.3147	0.2047	1.1674	0.0556	0.00619	15.00
20.00	33.124	82.19	1.4094	18.481	106.056	0.04101	0.22359	0.3164	0.2075	1.1705	0.0549	0.00632	20.00
25.00	36.800	81.63	1.2742	20.070	106.767	0.04429	0.22317	0.3181	0.2103	1.1740	0.0542	0.00646	25.00
30.00	40.784	81.06	1.1543	21.667	107.471	0.04755	0.22278	0.3198	0.2132	1.1777	0.0535	0.00660	30.00
35.00	45.092	80.49	1.0478	23.274	108.167	0.05079	0.22241	0.3216	0.2163	1.1818	0.0528	0.00674	35.00
40.00	49.741	79.90	0.9528	24.890	108.856	0.05402	0.22207	0.3235	0.2194	1.1862	0.0521	0.00688	40.00
45.00	54.749	79.32	0.8680	26.515	109.537	0.05724	0.22174	0.3255	0.2226	1.1910	0.0514	0.00703	45.00
50.00	60.134	78.72	0.7920	28.150	110.209	0.06044	0.22144	0.3275	0.2260	1.1961	0.0507	0.00717	50.00
55.00	65.913	78.11	0.7238	29.796	110.871	0.06362	0.22115	0.3297	0.2294	1.2018	0.0500	0.00732	55.00
60.00	72.105	77.50	0.6625	31.452	111.524	0.06680	0.22088	0.3319	0.2331	1.2079	0.0493	0.00747	60.00
65.00	78.729	76.87	0.6072	33.120	112.165	0.06996	0.22062	0.3343	0.2368	1.2145	0.0486	0.00762	65.00
70.00	85.805	76.24	0.5572	34.799	112.796	0.07311	0.22037	0.3368	0.2408	1.2217	0.0479	0.00777	70.00
75.00	93.351	75.59	0.5120	36.491	113.414	0.07626	0.22013	0.3394	0.2449	1.2296	0.0472	0.00793	75.00
80.00	101.390	74.94	0.4710	38.195	114.019	0.07939	0.21989	0.3422	0.2492	1.2382	0.0465	0.00809	80.00
85.00	109.930	74.27	0.4338	39.913	114.610	0.08252	0.21966	0.3451	0.2537	1.2475	0.0458	0.00825	85.00
90.00	119.010	73.58	0.3999	41.645	115.186	0.08565	0.21944	0.3482	0.2585	1.2578	0.0451	0.00842	90.00
95.00	128.650	72.88	0.3690	43.392	115.746	0.08877	0.21921	0.3515	0.2636	1.2690	0.0444	0.00860	95.00
100.00	138.850	72.17	0.3407	45.155	116.289	0.09188	0.21898	0.3551	0.2690	1.2813	0.0437	0.00878	100.00
105.00	149.650	71.44	0.3148	46.934	116.813	0.09500	0.21875	0.3589	0.2747	1.2950	0.0431	0.00897	105.00
110.00	161.070	70.69	0.2911	48.731	117.317	0.09811	0.21851	0.3630	0.2809	1.3101	0.0424	0.00916	110.00
115.00	173.140	69.93	0.2693	50.546	117.799	0.10123	0.21826	0.3675	0.2875	1.3268	0.0417	0.00936	115.00
120.00	185.860	69.14	0.2493	52.382	118.258	0.10435	0.21800	0.3723	0.2948	1.3456	0.0410	0.00958	120.00

Refrigerant 134a (1,1,1,2-Tetrafluoroethane) Properties of Saturated Liquid and Saturated Vapor (cont'd)

Temp.,* °F	Pressure, psia	Density, lb/ft^3	Volume, ft^3/lb	Enthalpy, Btu/lb-°F		Entropy, Btu/lb-°F		Specific Heat c_p Btu/lb-°F		C_p/C_v	Thermal Conductivity Btu/hr-ft-°F		Temp.,* °F
		Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Vapor	Liquid	Vapor	
125.00	199.280	68.32	0.2308	54.239	118.690	0.10748	0.21772	0.3775	0.3026	1.3666	0.0403	0.00981	125.00
130.00	213.410	67.49	0.2137	56.119	119.095	0.11062	0.21742	0.3833	0.3112	1.3903	0.0396	0.01005	130.00
135.00	228.280	66.62	0.1980	58.023	119.468	0.11376	0.21709	0.3897	0.3208	1.4173	0.0389	0.01031	135.00
140.00	243.920	65.73	0.1833	59.954	119.807	0.11692	0.21673	0.3968	0.3315	1.4481	0.0382	0.01058	140.00
145.00	260.360	64.80	0.1697	61.915	120.108	0.12010	0.21634	0.4048	0.3435	1.4837	0.0375	0.01089	145.00
150.00	277.610	63.83	0.1571	63.908	120.366	0.12330	0.21591	0.4138	0.3571	1.5250	0.0368	0.01122	150.00
155.00	295.730	62.82	0.1453	65.936	120.576	0.12653	0.21542	0.4242	0.3729	1.5738	0.0361	0.01158	155.00
160.00	314.730	61.76	0.1343	68.005	120.731	0.12979	0.21488	0.4362	0.3914	1.6318	0.0354	0.01199	160.00
165.00	334.650	60.65	0.1239	70.118	120.823	0.13309	0.21426	0.4504	0.4133	1.7022	0.0346	0.01245	165.00
170.00	355.530	59.47	0.1142	72.283	120.842	0.13644	0.21356	0.4675	0.4400	1.7889	0.0339	0.01297	170.00
175.00	377.410	58.21	0.1051	74.509	120.773	0.13985	0.21274	0.4887	0.4733	1.8984	0.0332	0.01358	175.00
180.00	400.340	56.86	0.0964	76.807	120.598	0.14334	0.21180	0.5156	0.5159	2.0405	0.0325	0.01430	180.00
185.00	424.360	55.38	0.0881	79.193	120.294	0.14693	0.21069	0.5512	0.5729	2.2321	0.0318	0.01516	185.00
190.00	449.520	53.76	0.0801	81.692	119.822	0.15066	0.20935	0.6012	0.6532	2.5041	0.0311	0.01623	190.00
195.00	475.910	51.91	0.0724	84.343	119.123	0.15459	0.20771	0.6768	0.7751	2.9192	0.0304	0.01760	195.00
200.00	503.590	49.76	0.0647	87.214	118.097	0.15880	0.20562	0.8062	0.9835	3.6309	0.0300	0.01949	200.00
205.00	532.680	47.08	0.0567	90.454	116.526	0.16353	0.20275	1.0830	1.4250	5.1360	0.0300	0.02240	205.00
210.00	563.350	43.20	0.0477	94.530	113.746	0.16945	0.19814	2.1130	3.0080	10.5120	0.0316	0.02848	210.00
213.91[c]	588.750	31.96	0.0313	103.894	103.894	0.18320	0.18320	∞	∞	∞	∞	∞	213.91c

* Temperature on ITS-90 scale [a] Triple point [b] Normal boiling point [c] Critical point

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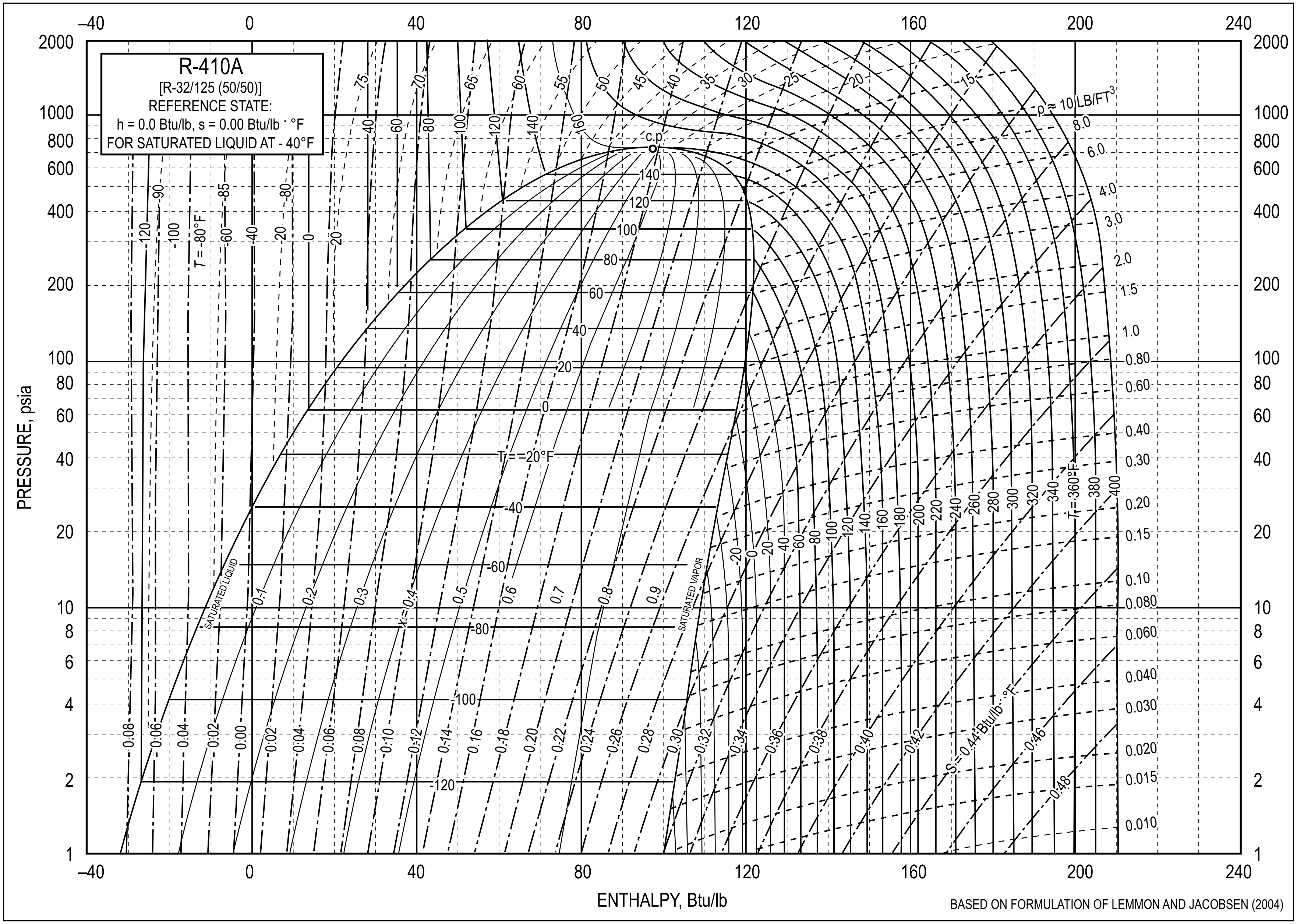
Pressure Versus Enthalpy Curves for Refrigerant 410A
(USCS units)



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Refrigerant 410A [R-32/125 (50/50)] Properties of Liquid on Bubble Line and Vapor on Dew Line

Pressure, psia	Temp.,* °F		Density, lb/ft³	Volume, ft³/lb	Enthalpy, Btu/lb-°F		Entropy, Btu/lb-°F		Specific Heat c_p Btu/lb-°F		C_p/C_v	Thermal Conductivity Btu/hr-ft-°F		Pressure, psia
	Bubble	Dew	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Vapor	Liquid	Vapor	
1	–135.16	–134.98	92.02	47.6458	–30.90	100.62	–0.08330	0.32188	0.3215	0.1568	1.228	0.1043	0.00421	1
1.5	–126.03	–125.87	91.10	32.5774	–27.97	101.90	–0.07439	0.31477	0.3212	0.1600	1.227	0.1023	0.00431	1.5
2	–119.18	–119.02	90.41	24.8810	–25.76	102.86	–0.06786	0.30981	0.3213	0.1626	1.227	0.1008	0.00439	2
2.5	–113.63	–113.48	89.84	20.1891	–23.98	103.63	–0.06267	0.30602	0.3214	0.1648	1.228	0.0996	0.00446	2.5
3	–108.94	–108.78	89.36	17.0211	–22.47	104.27	–0.05834	0.30296	0.3216	0.1668	1.228	0.0985	0.00451	3
4	–101.22	–101.07	88.57	13.0027	–19.98	105.33	–0.05133	0.29820	0.3221	0.1703	1.229	0.0968	0.00461	4
5	–94.94	–94.80	87.92	10.5514	–17.96	106.18	–0.04574	0.29455	0.3226	0.1733	1.230	0.0954	0.00469	5
6	–89.63	–89.48	87.36	8.8953	–16.24	106.89	–0.04107	0.29162	0.3231	0.1760	1.232	0.0942	0.00476	6
7	–84.98	–84.84	86.87	7.6992	–14.74	107.50	–0.03704	0.28916	0.3236	0.1785	1.233	0.0931	0.00482	7
8	–80.85	–80.71	86.44	6.7935	–13.40	108.05	–0.03349	0.28705	0.3241	0.1807	1.234	0.0922	0.00488	8
10	–73.70	–73.56	85.67	5.5105	–11.08	108.97	–0.02743	0.28356	0.3251	0.1848	1.237	0.0905	0.00498	10
12	–67.62	–67.48	85.02	4.6434	–9.10	109.75	–0.02235	0.28075	0.3261	0.1884	1.240	0.0891	0.00507	12
14	–62.31	–62.16	84.44	4.0168	–7.36	110.42	–0.01795	0.27840	0.3270	0.1917	1.243	0.0879	0.00515	14
14.70[b]	–60.60	–60.46	84.26	3.8375	–6.80	110.63	–0.01655	0.27766	0.3274	0.1928	1.244	0.0875	0.00517	14.70[b]
16	–57.56	–57.42	83.93	3.5423	–5.80	111.01	–0.01407	0.27638	0.3279	0.1947	1.245	0.0868	0.00522	16
18	–53.27	–53.13	83.45	3.1699	–4.39	111.54	–0.01059	0.27461	0.3288	0.1975	1.248	0.0858	0.00528	18
20	–49.34	–49.19	83.02	2.8698	–3.09	112.01	–0.00743	0.27305	0.3297	0.2002	1.251	0.0849	0.00535	20
22	–45.70	–45.56	82.61	2.6225	–1.89	112.45	–0.00452	0.27164	0.3305	0.2027	1.254	0.0841	0.00540	22
24	–42.32	–42.18	82.23	2.4151	–0.77	112.85	–0.00184	0.27036	0.3313	0.2050	1.256	0.0833	0.00546	24
26	–39.15	–39.01	81.87	2.2386	0.28	113.22	0.0007	0.26919	0.3321	0.2073	1.259	0.0826	0.00551	26
28	–36.17	–36.02	81.54	2.0865	1.27	113.56	0.0030	0.26811	0.3329	0.2094	1.261	0.0819	0.00556	28
30	–33.35	–33.20	81.21	1.9540	2.22	113.88	0.0052	0.26711	0.3337	0.2115	1.264	0.0813	0.00561	30
32	–30.68	–30.53	80.90	1.8375	3.11	114.19	0.0073	0.26617	0.3345	0.2135	1.267	0.0806	0.00565	32
34	–28.13	–27.98	80.61	1.7343	3.97	114.47	0.0093	0.26530	0.3352	0.2154	1.269	0.0801	0.00570	34
36	–25.69	–25.54	80.33	1.6422	4.79	114.74	0.0112	0.26448	0.3360	0.2173	1.272	0.0795	0.00574	36

Refrigerant 410A [R-32/125 (50/50)] Properties of Liquid on Bubble Line and Vapor on Dew Line (con't)

Pressure, psia	Temp.,* °F		Density, lb/ft^3	Volume, ft^3/lb	Enthalpy, Btu/lb-°F		Entropy, Btu/lb-°F		Specific Heat c_p Btu/lb-°F		C_p/C_v	Thermal Conductivity Btu/hr-ft-°F		Pressure, psia
	Bubble	Dew	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Vapor	Liquid	Vapor	
38	–23.36	–23.20	80.05	1.5594	5.57	115.00	0.0130	0.26371	0.3367	0.2191	1.274	0.0790	0.00578	38
40	–21.12	–20.96	79.79	1.4847	6.33	115.24	0.0147	0.26297	0.3374	0.2208	1.277	0.0785	0.00582	40
42	–18.96	–18.81	79.54	1.4168	7.06	115.47	0.0163	0.26228	0.3382	0.2226	1.279	0.0780	0.00586	42
44	–16.89	–16.73	79.29	1.3549	7.76	115.69	0.0179	0.26162	0.3389	0.2242	1.282	0.0775	0.00589	44
46	–14.88	–14.73	79.05	1.2982	8.45	115.90	0.0194	0.26098	0.3396	0.2259	1.284	0.0771	0.00593	46
48	–12.94	–12.79	78.82	1.2460	9.11	116.10	0.0209	0.26038	0.3403	0.2275	1.287	0.0766	0.00597	48
50	–11.07	–10.91	78.59	1.1979	9.75	116.30	0.0223	0.25980	0.3410	0.2290	1.289	0.0762	0.00600	50
55	–6.62	–6.45	78.05	1.0925	11.27	116.75	0.0257	0.25845	0.3427	0.2328	1.295	0.0752	0.00610	55
60	–2.46	–2.30	77.54	1.0040	12.70	117.16	0.0288	0.25722	0.3445	0.2365	1.301	0.0743	0.00619	60
65	1.43	1.60	77.06	0.9287	14.05	117.53	0.0317	0.25610	0.3462	0.2400	1.308	0.0734	0.00628	65
70	5.10	5.27	76.60	0.8638	15.33	117.88	0.0344	0.25505	0.3478	0.2434	1.314	0.0726	0.00636	70
75	8.58	8.75	76.15	0.8073	16.54	118.20	0.0370	0.25408	0.3495	0.2467	1.320	0.0719	0.00645	75
80	11.88	12.06	75.73	0.7576	17.70	118.49	0.0395	0.25316	0.3512	0.2499	1.326	0.0711	0.00653	80
85	15.03	15.21	75.32	0.7135	18.81	118.77	0.0418	0.25231	0.3528	0.2531	1.333	0.0704	0.00661	85
90	18.05	18.22	74.93	0.6742	19.88	119.02	0.0440	0.25149	0.3545	0.2562	1.339	0.0698	0.00669	90
95	20.93	21.11	74.54	0.6389	20.91	119.26	0.0461	0.25072	0.3561	0.2592	1.345	0.0692	0.00677	95
100	23.71	23.89	74.17	0.6070	21.90	119.48	0.0482	0.24999	0.3578	0.2622	1.352	0.0685	0.00684	100
110	28.96	29.14	73.46	0.5515	23.79	119.89	0.0520	0.24862	0.3611	0.2681	1.365	0.0674	0.00700	110
120	33.86	34.05	72.78	0.5051	25.57	120.24	0.0556	0.24736	0.3644	0.2738	1.378	0.0664	0.00715	120
130	38.46	38.65	72.13	0.4655	27.25	120.56	0.0589	0.24618	0.3678	0.2795	1.392	0.0654	0.00730	130
140	42.80	42.99	71.51	0.4314	28.85	120.83	0.0621	0.24508	0.3712	0.2852	1.406	0.0645	0.00745	140
150	46.91	47.11	70.90	0.4016	30.38	121.08	0.0650	0.24403	0.3746	0.2908	1.420	0.0636	0.00760	150
160	50.82	51.02	70.32	0.3755	31.85	121.29	0.0679	0.24304	0.3781	0.2965	1.435	0.0628	0.00775	160
170	54.56	54.76	69.75	0.3523	33.27	121.48	0.0706	0.24210	0.3816	0.3022	1.451	0.0620	0.00791	170
180	58.13	58.33	69.20	0.3316	34.63	121.65	0.0732	0.24119	0.3851	0.3080	1.467	0.0612	0.00807	180

Refrigerant 410A [R-32/125 (50/50)] Properties of Liquid on Bubble Line and Vapor on Dew Line (con't)

Pressure, psia	Temp.,* °F		Density, lb/ft^3	Volume, ft^3/lb	Enthalpy, Btu/lb-°F		Entropy, Btu/lb-°F		Specific Heat c_p Btu/lb-°F		C_p/C_v	Thermal Conductivity Btu/hr-ft-°F		Pressure, psia
	Bubble	Dew	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Liquid	Vapor	Vapor	Liquid	Vapor	
190	61.55	61.76	68.66	0.3130	35.95	121.79	0.0757	0.24031	0.3888	0.3139	1.483	0.0605	0.00823	190
200	64.84	65.05	68.13	0.2962	37.22	121.91	0.0780	0.23946	0.3925	0.3200	1.500	0.0598	0.00839	200
220	71.07	71.28	67.10	0.2669	39.67	122.09	0.0826	0.23783	0.4001	0.3325	1.537	0.0585	0.00873	220
240	76.89	77.10	66.11	0.2424	41.99	122.20	0.0868	0.23628	0.4081	0.3457	1.576	0.0573	0.00908	240
260	82.35	82.57	65.14	0.2215	44.21	122.25	0.0908	0.23478	0.4165	0.3599	1.619	0.0562	0.00945	260
280	87.51	87.73	64.19	0.2034	46.34	122.24	0.0946	0.23333	0.4255	0.3751	1.665	0.0552	0.00983	280
300	92.40	92.61	63.26	0.1876	48.40	122.18	0.0983	0.23190	0.4350	0.3915	1.716	0.0542	0.01024	300
320	97.04	97.26	62.34	0.1736	50.38	122.07	0.1018	0.23049	0.4452	0.4094	1.772	0.0533	0.01067	320
340	101.48	101.69	61.42	0.1613	52.31	121.91	0.1051	0.22909	0.4564	0.4290	1.833	0.0524	0.01113	340
360	105.71	105.93	60.52	0.1501	54.19	121.70	0.1083	0.22769	0.4685	0.4507	1.901	0.0515	0.01162	360
380	109.78	109.99	59.61	0.1401	56.03	121.44	0.1115	0.22629	0.4820	0.4747	1.977	0.0507	0.01214	380
400	113.68	113.89	58.70	0.1310	57.83	121.13	0.1145	0.22488	0.4971	0.5016	2.063	0.0499	0.01271	400
450	122.82	123.01	56.39	0.1114	62.23	120.14	0.1218	0.22124	0.5443	0.5857	2.333	0.0481	0.01433	450
500	131.19	131.38	53.97	0.0952	66.54	118.80	0.1289	0.21732	0.6143	0.7083	2.728	0.0465	0.01636	500
550	138.93	139.09	51.32	0.0814	70.89	117.02	0.1359	0.21295	0.7303	0.9059	3.367	0.0451	0.01902	550
600	146.12	146.25	48.24	0.0690	75.47	114.59	0.1432	0.20777	0.9603	1.2829	4.579	0.0440	0.02275	600
692.78[c]	158.40	158.40	34.18	0.0293	90.97	90.97	0.1678	0.16781	—	—	—	—	—	692.78[c]

* Temperature on ITS-90 scale   [b] Bubble and dew point at one standard atmosphere   [c] Critical point

Thermal and Physical Property Tables
(at room temperature)

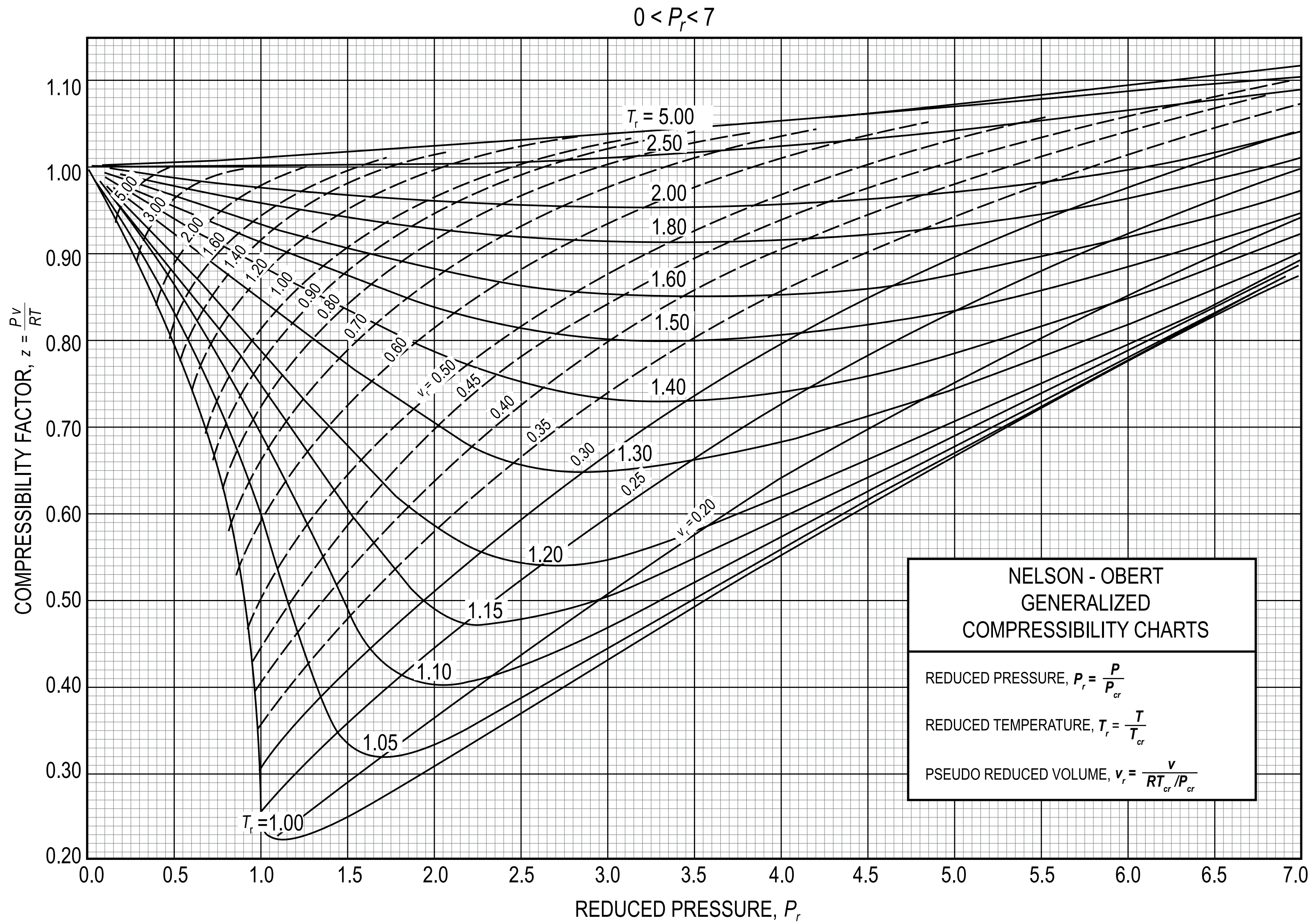
GASES								
Substance	Mol wt	c_p		c_v		k	R	
		kJ/(kg·K)	Btu/(lbm-°R)	kJ/(kg·K)	Btu/(lbm-°R)		kJ/(kg·K)	ft-lbf/(lbm-°R)
Gases								
Air	29	1.00	0.240	0.718	0.171	1.40	0.2870	53.34
Argon	40	0.520	0.125	0.312	0.0756	1.67	0.2081	38.68
Butane	58	1.72	0.415	1.57	0.381	1.09	0.1430	26.58
Carbon dioxide	44	0.846	0.203	0.657	0.158	1.29	0.1889	35.10
Carbon monoxide	28	1.04	0.249	0.744	0.178	1.40	0.2968	55.16
Ethane	30	1.77	0.427	1.49	0.361	1.18	0.2765	51.38
Helium	4	5.19	1.25	3.12	0.753	1.67	2.0769	386.0
Hydrogen	2	14.3	3.43	10.2	2.44	1.40	4.1240	766.4
Methane	16	2.25	0.532	1.74	0.403	1.30	0.5182	96.35
Neon	20	1.03	0.246	0.618	0.148	1.67	0.4119	76.55
Nitrogen	28	1.04	0.248	0.743	0.177	1.40	0.2968	55.15
Octane vapor	114	1.71	0.409	1.64	0.392	1.04	0.0729	13.53
Oxygen	32	0.918	0.219	0.658	0.157	1.40	0.2598	48.28
Propane	44	1.68	0.407	1.49	0.362	1.12	0.1885	35.04
Steam	18	1.87	0.445	1.41	0.335	1.33	0.4615	85.76

GASES						
Substance	Critical Temperature, T_{cr}		Critical Pressure, P_{cr}		Critical Volume, V_{cr}	
	K	°R	MPa	atm	m³/kmol	ft³/lbmol
Air	132.5	238.5	3.77	37.2	—	—
Argon	150.8	271.4	4.87	48.1	0.0749	1.20
Butane	425.0	765.4	3.80	37.5	0.255	4.08
Carbon dioxide	304.1	547.4	7.38	72.8	0.0939	1.50
Carbon monoxide	132.9	239.2	3.50	34.5	0.09325	1.49
Ethane	305.4	549.7	4.88	48.2	0.1483	2.376
Helium	5.19	9.34	0.227	2.24	0.0574	0.9195
Hydrogen	33.2	59.8	1.30	12.8	0.0651	1.043
Methane	190.4	342.7	4.60	45.4	0.0992	1.59
Neon	44.4	79.9	2.76	27.2	0.0416	0.666
Nitrogen	126.2	227.2	3.39	33.5	0.0898	1.44
Octane vapor	568.8	1024.0	2.49	24.6	0.492	7.88
Oxygen	154.6	278.3	5.04	49.7	0.0734	1.18
Propane	369.8	665.6	4.25	41.9	0.203	3.25
Steam	647.1	1165.0	22.06	217.7	0.0560	0.8971

Howell, John R., and Richard O. Buckius, *Fundamentals of Engineering Thermodynamics,* 2nd ed., 1992, McGraw Hill, adapted from Table C.4 Critical Constants, pp. 870-872.

SELECTED LIQUIDS AND SOLIDS				
Substance	c_p		**Density**	
	kJ/(kg·K)	**Btu/(lbm-°R)**	**kg/m^3**	**lbm/ft^3**
Liquids				
Ammonia	4.80	1.146	602	38
Mercury	0.139	0.033	13,560	847
Water	4.18	1.000	997	62.4
Solids				
Aluminum	0.900	0.215	2,700	170
Copper	0.386	0.092	8,900	555
Ice (0°C; 32°F)	2.11	0.502	917	57.2
Iron	0.450	0.107	7,840	490
Lead	0.128	0.030	11,310	705

Howell, John, R. and Richard O. Bukins, *Fundamentals of Engineering Thermodynamics*, 2nd ed., McGraw-Hill, 1992, p. 896.



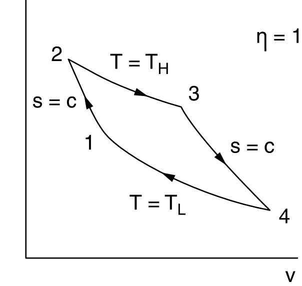
Definition of Compressibility Factor

The compressibility factor z is the ratio of the volume actually occupied by a gas at given temperature and pressure to the volume the gas would occupy if it behaved like an ideal gas at the same temperature and pressure. The compressibility factor is not a constant but varies with changes in gas composition, temperature, and pressure. It must be determined experimentally.

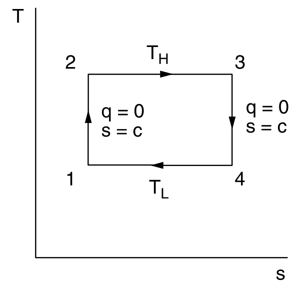
$$z = \frac{V_{\text{actual}}}{V_{\text{ideal}}}$$

COMMON THERMODYNAMIC CYCLES

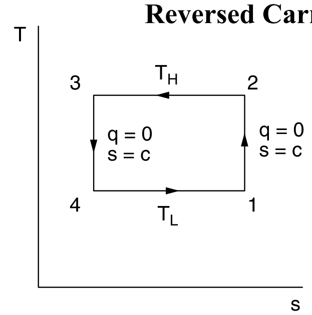
Carnot Cycle



$$\eta = 1 - \frac{T_L}{T_H}$$

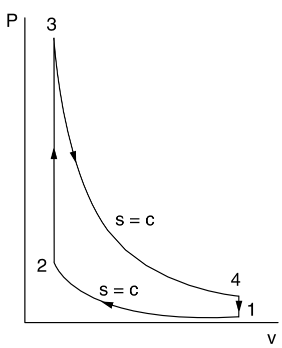


Reversed Carnot



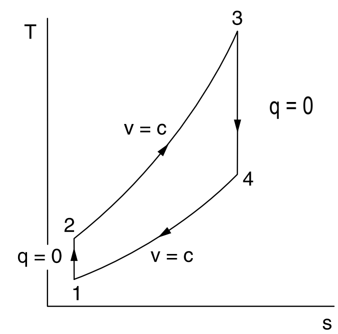
Otto Cycle

(Gasoline Engine)

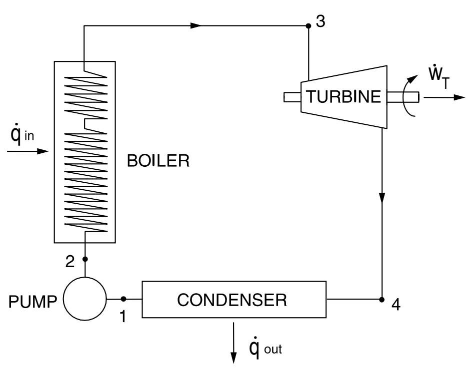


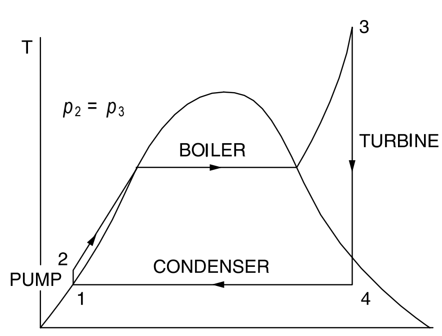
$$\eta = 1 - r^{1-k}$$

$$r = v_1/v_2$$



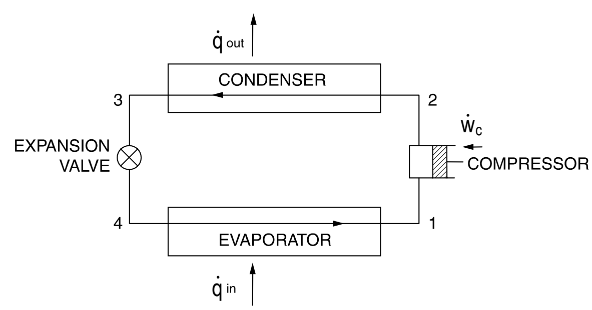
Rankine Cycle

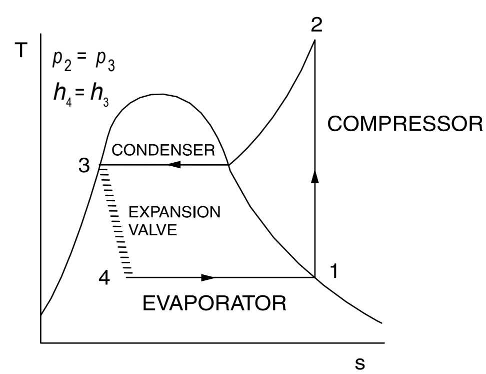




$$\eta = \frac{(h_3 - h_4) - (h_2 - h_1)}{h_3 - h_2}$$

Refrigeration



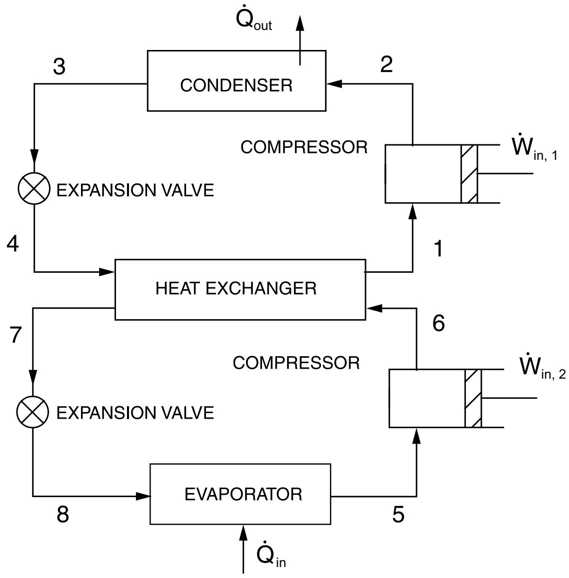


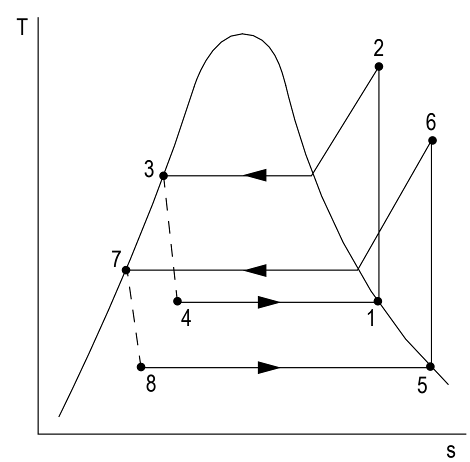
$$COP_{ref} = \frac{h_1 - h_4}{h_2 - h_1} \qquad COP_{HP} = \frac{h_2 - h_3}{h_2 - h_1}$$

Refrigeration and HVAC

Cycles

Refrigeration and HVAC
Two-Stage Cycle



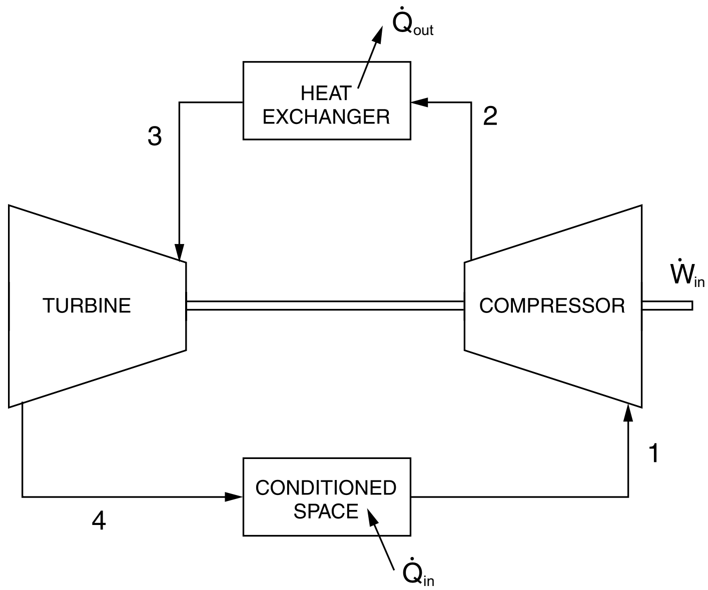


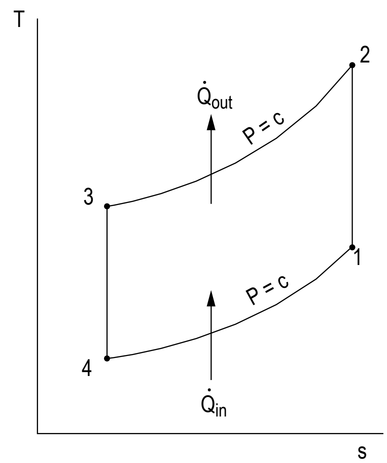
The following equations are valid if the mass flows are the same in each stage.

$$COP_{\text{ref}} = \frac{\dot{Q}_{\text{in}}}{\dot{W}_{\text{in},1} + \dot{W}_{\text{in},2}} = \frac{h_5 - h_8}{h_2 - h_1 + h_6 - h_5}$$

$$COP_{\text{HP}} = \frac{\dot{Q}_{\text{out}}}{\dot{W}_{\text{in},1} + \dot{W}_{\text{in},2}} = \frac{h_2 - h_3}{h_2 - h_1 + h_6 - h_5}$$

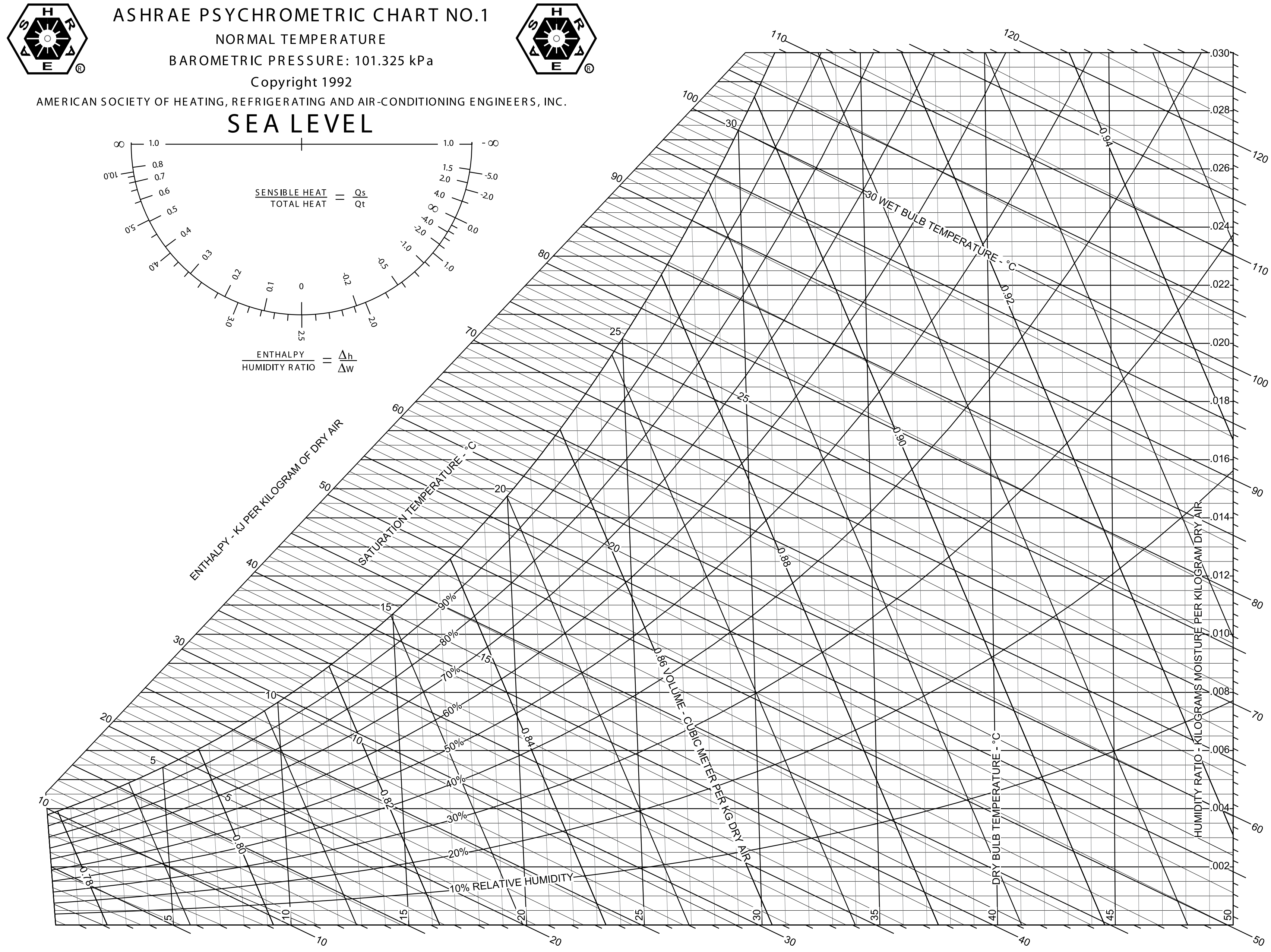
Air Refrigeration Cycle



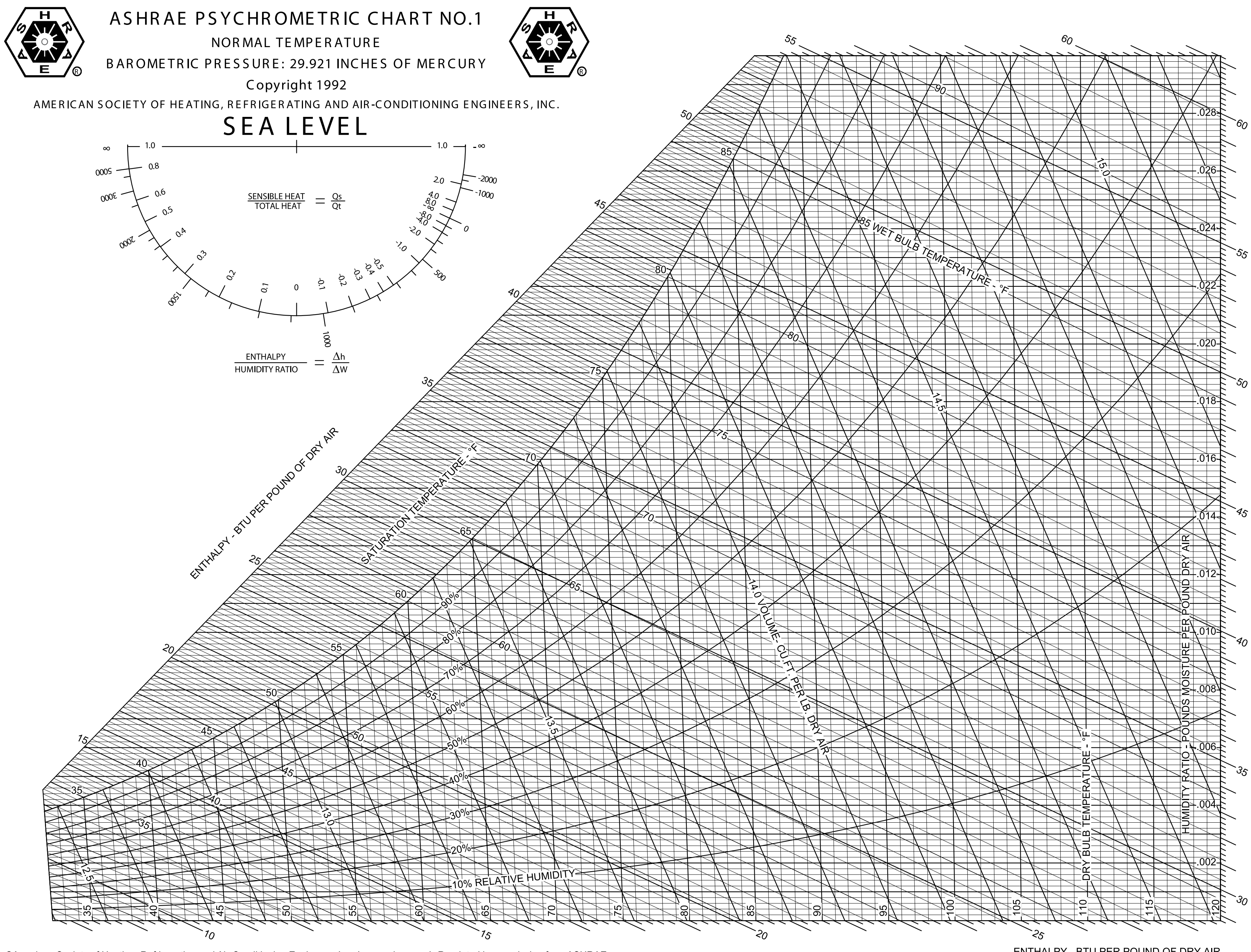


$$COP_{ref} = \frac{h_1 - h_4}{(h_2 - h_1) - (h_3 - h_4)}$$

$$COP_{HP} = \frac{h_2 - h_3}{(h_2 - h_1) - (h_3 - h_4)}$$


ASHRAE PSYCHROMETRIC CHART NO.1
NORMAL TEMPERATURE
BAROMETRIC PRESSURE: 101.325 kPa
Copyright 1992
AMERICAN SOCIETY OF HEATING, REFRIGERATING AND AIR-CONDITIONING ENGINEERS, INC.
SEA LEVEL
SENSIBLE HEAT / TOTAL HEAT = Qs / Qt
ENTHALPY / HUMIDITY RATIO = Δh / Δw
ENTHALPY - KJ PER KILOGRAM OF DRY AIR
SATURATION TEMPERATURE - °C
30 WET BULB TEMPERATURE - °C
0.86 VOLUME - CUBIC METER PER KG DRY AIR
10% RELATIVE HUMIDITY
DRY BULB TEMPERATURE - °C
HUMIDITY RATIO - KILOGRAMS MOISTURE PER KILOGRAM DRY AIR
ENTHALPY - KJ PER KILOGRAM OF DRY AIR

ASHRAE Psychrometric Chart No. 1
(English units)



Fluid Mechanics

Definitions

Density, Specific Volume, Specific Weight, and Specific Gravity

The definitions of density, specific weight, and specific gravity follow:

$$\rho = \lim_{\Delta V \to 0} \Delta m/\Delta V$$

$$\gamma = \lim_{\Delta V \to 0} \Delta W/\Delta V$$

$$\gamma = \lim_{\Delta V \to 0} g \cdot \Delta m/\Delta V = \rho g$$

also

$$SG = \gamma/\gamma_w = \rho/\rho_w$$

where

ρ = density (also called mass density)

Δm = mass of infinitesimal volume

ΔV = volume of infinitesimal object considered

γ = specific weight

= ρg

ΔW = weight of an infinitesimal volume

SG = specific gravity

ρ_w = density of water at standard conditions
= 1,000 kg/m^3 (62.4 lbm/ft^3)

γ_ω = specific weight of water at standard conditions
= 9,810 N/m^3 (62.4 lbf/ft^3)
= 9,810 kg/(m^2•s^2)

Stress, Pressure, and Viscosity

Stress is defined as

$$\tau(1) = \lim_{\Delta A \to 0} \Delta F/\Delta A$$

where

$\tau(1)$ = surface stress vector at Point 1

ΔF = force acting on infinitesimal area ΔA

ΔA = infinitesimal area at Point 1

τ_n = – P

τ_t = $\mu(dv/dy)$ (one-dimensional; i.e., y)

where

τ_n and τ_t = normal and tangential stress components at Point 1, respectively

P = pressure at Point 1

μ = absolute dynamic viscosity of the fluid
N•s/m^2 [lbm/(ft-sec)]

dv = differential velocity

dy = differential distance, normal to boundary

v = velocity at boundary condition

y = normal distance, measured from boundary

ν = kinematic viscosity (m^2/s *or* ft^2/sec)

where $\nu = \frac{\mu}{\rho}$

For a thin Newtonian fluid film and a linear velocity profile,

$$v(y) = vy/\delta;\ dv/dy = v/\delta$$

where

v = velocity of plate on film

δ = thickness of fluid film

For a power law (non-Newtonian) fluid

$$\tau_t = K\,(dv/dy)^n$$

where

K = consistency index

n = power law index

$n < 1 \equiv$ pseudo plastic

$n > 1 \equiv$ dilatant

Surface Tension and Capillarity

Surface tension σ is the force per unit contact length

$$\sigma = F/L$$

where

σ = surface tension, force/length

F = surface force at the interface

L = length of interface

The *capillary rise* h is approximated by

$$h = (4\sigma \cos \beta)/(\gamma d)$$

where

h = height of the liquid in the vertical tube

σ = surface tension

β = angle made by the liquid with the wetted tube wall

γ = specific weight of the liquid

d = diameter of the capillary tube

Characteristics of a Static Liquid

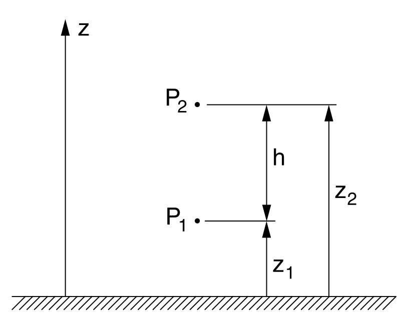
The Pressure Field in a Static Liquid

The difference in pressure between two different points is

$$P_2 - P_1 = -\gamma\,(z_2 - z_1) = -\gamma h = -\rho g h$$

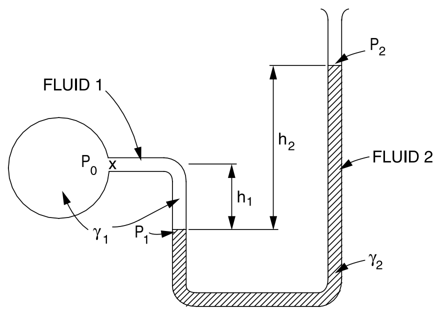
Absolute pressure = atmospheric pressure + gauge pressure reading

Absolute pressure = atmospheric pressure – vacuum gauge pressure reading



Bober, W., and R.A. Kenyon, *Fluid Mechanics*, Wiley, 1980. Diagrams reprinted by permission of William Bober and Richard A. Kenyon.

Manometers



Bober, W., and R.A. Kenyon, *Fluid Mechanics*, Wiley, 1980. Diagrams reprinted by permission of William Bober and Richard A. Kenyon.

For a simple manometer,

$$P_0 = P_2 + \gamma_2 h_2 - \gamma_1 h_1 = P_2 + g\,(\rho_2\, h_2 - \rho_1\, h_1)$$

If $h_1 = h_2 = h$

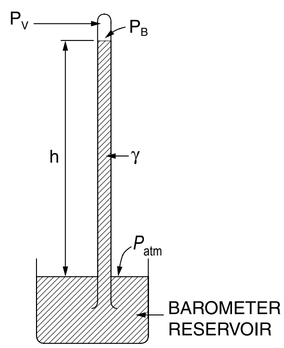
$$P_0 = P_2 + (\gamma_2 - \gamma_1)h = P_2 + (\rho_2 - \rho_1)gh$$

Note that the difference between the two densities is used.

- P = pressure
- γ = specific weight of fluid
- h = height
- g = acceleration of gravity
- ρ = fluid density

Another device that works on the same principle as the manometer is the simple barometer.

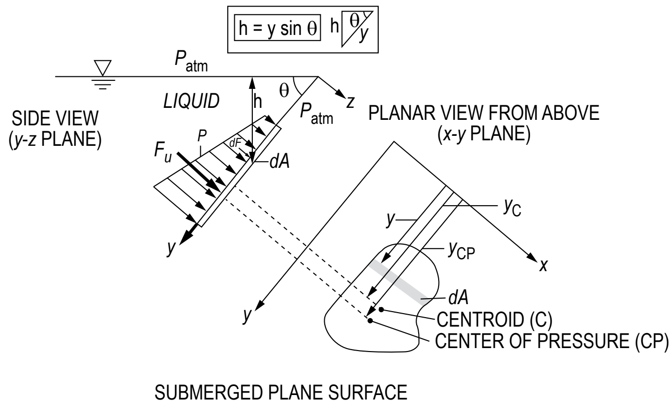
$$P_{\text{atm}} = P_A = P_v + \gamma h = P_B + \gamma h = P_B + \rho g h$$



P_v = vapor pressure of the barometer fluid

Bober, W., and R.A. Kenyon, *Fluid Mechanics*, Wiley, 1980. Diagrams reprinted by permission of William Bober and Richard A. Kenyon.

Forces on Submerged Surfaces and the Center of Pressure



Elger, Donald F., et al, *Engineering Fluid Mechanics*, 10th ed., 2012. Reproduced with permission of John Wiley & Sons, Inc.

The pressure on a point at a vertical distance h below the surface is:

$P = P_{atm} + \rho g h$, for $h \geq 0$

where

P = pressure

P_{atm} = atmospheric pressure

P_C = pressure at the centroid of area

P_{CP} = pressure at center of pressure

y_C = slant distance from liquid surface to the centroid of area

y_C = $h_C/\sin\theta$

h_C = vertical distance from liquid surface to centroid of area

y_{CP} = slant distance from liquid surface to center of pressure

h_{CP} = vertical distance from liquid surface to center of pressure

θ = angle between liquid surface and edge of submerged surface

I_{xC} = moment of inertia about the centroidal x-axis

If atmospheric pressure acts above the liquid surface and on the non-wetted side of the submerged surface:

$y_{CP} = y_C + I_{xC}/y_C A$

$y_{CP} = y_C + \rho g \sin\theta \, I_{xC}/P_C A$

Wetted side: $F_R = (P_{atm} + \rho g y_C \sin\theta)A$

P_{atm} acting both sides: $F_{R_{net}} = (\rho g y_C \sin\theta)A$

Archimedes Principle and Buoyancy

1. The buoyant force exerted on a submerged or floating body is equal to the weight of the fluid displaced by the body.
2. A floating body displaces a weight of fluid equal to its own weight; i.e., a floating body is in equilibrium.

The *center of buoyancy* is located at the centroid of the displaced fluid volume.

In the case of a body lying at the *interface of two immiscible fluids*, the buoyant force equals the sum of the weights of the fluids displaced by the body.

Principles of One-Dimensional Fluid Flow

The Continuity Equation

So long as the flow Q is continuous, the *continuity equation*, as applied to one-dimensional flows, states that the flow passing two points (1 and 2) in a stream is equal at each point, $A_1v_1 = A_2v_2$.

$$Q = Av$$

$$\dot{m} = \rho Q = \rho Av$$

where

Q = volumetric flowrate

$\dot{m}$ = mass flowrate

A = cross-sectional area of flow

v = average flow velocity

ρ = fluid density

For steady, one-dimensional flow, $\dot{m}$ is a constant. If, in addition, the density is constant, then Q is constant.

Energy Equation

The energy equation for steady incompressible flow with no energy input (e.g., no pump) is:

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g} + h_f \quad \text{or}$$

$$\frac{P_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\rho g} + z_2 + \frac{v_2^2}{2g} + h_f$$

where h_f = the head loss, considered a friction effect, and all remaining terms are defined above.

If the cross-sectional area and the elevation of the pipe are the same at both sections (1 and 2), then $z_1 = z_2$ and $v_1 = v_2$.

The pressure drop $P_1 - P_2$ is given by the following:

$$P_1 - P_2 = \gamma\, h_f = \rho g h_f$$

Bernoulli Equation

The field equation is derived when the energy equation is applied to one-dimensional flows. Assuming no friction losses and that no pump or turbine exists between sections 1 and 2 in the system,

$$\frac{P_2}{\gamma} + \frac{v_2^2}{2g} + z_2 = \frac{P_1}{\gamma} + \frac{v_1^2}{2g} + z_1 \quad \text{or}$$

$$\frac{P_2}{\rho} + \frac{v_2^2}{2} + z_2 g = \frac{P_1}{\rho} + \frac{v_1^2}{2} + z_1 g$$

where

P_1, P_2 = pressure at sections 1 and 2

v_1, v_2 = average velocity of the fluid at the sections

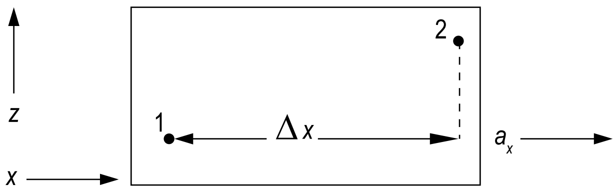
z_1, z_2 = vertical distance from a datum to the sections (the potential energy)

γ = specific weight of the fluid (ρg)

g = acceleration of gravity

ρ = fluid density

Euler's Equation



For unsteady flow due to local acceleration (i.e., temporal acceleration) in the x-direction, the change in pressure between two points in a fluid can be determined by Euler's equation:

$$\left(P_2 + \gamma \bullet z_2\right) - \left(P_1 + \gamma \bullet z_1\right) = -\Delta x \bullet \rho \bullet a_x$$

where

P_1, P_2 = pressure at Locations 1 and 2
γ = specific weight of the fluid (ρg)
z_1, z_2 = elevation at Locations 1 and 2
ρ = fluid density
a_x = local (temporal) acceleration of fluid in the x-direction
Δx = distance between Locations 1 and 2 in the x-direction

Crowe, Clayton T., *Engineering Fluid Mechanics*, 2nd ed., New York: John Wiley and Sons, 1980, p. 144.

Hydraulic Gradient (Grade Line)

Hydraulic grade line is the line connecting the sum of pressure and elevation heads at different points in conveyance systems. If a row of piezometers were placed at intervals along the pipe, the grade line would join the water levels in the piezometer water columns.

Energy Line (Bernoulli Equation)

The Bernoulli equation states that the sum of the pressure, velocity, and elevation heads is constant. The energy line is this sum or the "total head line" above a horizontal datum. The difference between the hydraulic grade line and the energy line is the $v^2/2g$ term.

Fluid flow characterization

Reynolds Number

$$\mathrm{Re} = \frac{vD\rho}{\mu} = \frac{vD}{\nu}$$

$$\mathrm{Re}' = \frac{v^{(2-n)}D^n\rho}{K\left(\frac{3n+1}{4n}\right)^n 8^{(n-1)}}$$

where

v = fluid velocity
ρ = mass density
D = diameter of the pipe, dimension of the fluid streamline, or characteristic length
μ = dynamic viscosity
ν = kinematic viscosity
Re = Reynolds number (Newtonian fluid)
Re′ = Reynolds number (Power law fluid)

K and n are defined in the Stress, Pressure, and Viscosity section.

The critical Reynolds number $(\mathrm{Re})_c$ is defined to be the minimum Reynolds number at which a flow will turn turbulent.

Flow through a pipe is generally characterized as laminar for Re < 2,100 and fully turbulent for Re > 10,000, and transitional flow for 2,100 < Re < 10,000.

The velocity distribution for *laminar flow* in circular tubes or between planes is

$$v(r) = v_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

where

- r = distance (m) from the centerline
- R = radius (m) of the tube or half the distance between the parallel planes
- v = local velocity (m/s) at r
- $v_{\max}$ = velocity (m/s) at the centerline of the duct
- $v_{\max}$ = $1.18\overline{v}$, for fully turbulent flow
- $v_{\max}$ = $2\overline{v}$, for circular tubes in laminar flow and
- $v_{\max}$ = $1.5\overline{v}$, for parallel planes in laminar flow, where
- $\overline{v}$ = average velocity (m/s) in the duct

The shear stress distribution is

$$\frac{\tau}{\tau_w} = \frac{r}{R}$$

where τ and τ_w are the shear stresses at radii r and R, respectively.

Consequences of Fluid Flow

Head Loss Due to Flow

The *Darcy-Weisbach equation* is

$$h_f = f\frac{L}{D}\frac{v^2}{2g}$$

where

- f = $f(\mathrm{Re}, \varepsilon/D)$, the Moody, Darcy, or Stanton friction factor
- D = diameter of the pipe
- L = length over which the pressure drop occurs
- ε = roughness factor for the pipe, and other symbols are defined as before

An alternative formulation employed by chemical engineers is

$$h_f = \left(4f_{\text{Fanning}}\right)\frac{Lv^2}{D2g} = \frac{2f_{\text{Fanning}}\,Lv^2}{Dg}$$

Fanning friction factor, $f_{\text{Fanning}} = \dfrac{f}{4}$

A chart that gives f versus Re for various values of ε/D, known as a *Moody, Darcy,* or *Stanton diagram*, is available in this section.

Minor Losses in Pipe Fittings, Contractions, and Expansions

Head losses also occur as the fluid flows through pipe fittings (i.e., elbows, valves, couplings, etc.) and sudden pipe contractions and expansions.

$$\frac{P_1}{\gamma} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\gamma} + z_2 + \frac{v_2^2}{2g} + h_f + h_{f,\,\text{fitting}}$$

$$\frac{P_1}{\rho g} + z_1 + \frac{v_1^2}{2g} = \frac{P_2}{\rho g} + z_2 + \frac{v_2^2}{2g} + h_f + h_{f,\,\text{fitting}}$$

where

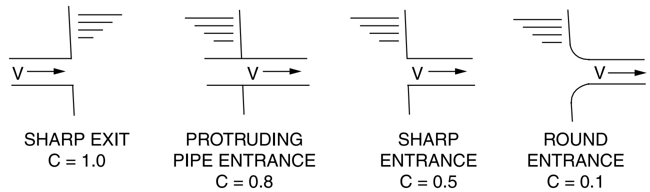
$$h_{f,\,\text{fitting}} = C\frac{v^2}{2g}$$

$$\frac{v^2}{2g} = 1 \text{ velocity head}$$

Specific fittings have characteristic values of C, which will be provided in the problem statement. A generally accepted *nominal value* for head loss in *well-streamlined gradual contractions* is

$$h_{f,\ \text{fitting}} = 0.04\ v^2/\ 2g$$

The *head loss* at either an *entrance* or *exit* of a pipe from or to a reservoir is also given by the $h_{f,\ \text{fitting}}$ equation. Values for C for various cases are shown as follows.



Bober, W., and R.A. Kenyon, *Fluid Mechanics*, Wiley, 1980. Diagrams reprinted by permission of William Bober and Richard A. Kenyon.

Pressure Drop for Laminar Flow

The equation for Q in terms of the pressure drop ΔP_f is the Hagen-Poiseuille equation. This relation is valid only for flow in the laminar region.

$$Q = \frac{\pi R^4 \Delta P_f}{8\mu L} = \frac{\pi D^4 \Delta P_f}{128\mu L}$$

Flow in Noncircular Conduits

Analysis of flow in conduits having a noncircular cross section uses the *hydraulic radius* R_H, or the *hydraulic diameter* D_H, as follows:

$$R_H = \frac{\text{cross-sectional area}}{\text{wetted perimeter}} = \frac{D_H}{4}$$

Drag Force

The *drag force* F_D on objects immersed in a large body of flowing fluid or objects moving through a stagnant fluid is

$$F_D = \frac{C_D \rho v^2 A}{2}$$

where

C_D = drag coefficient

v = velocity (m/s) of the flowing fluid or moving object

A = projected area (m^2) of blunt objects such as spheres, ellipsoids, disks, and plates, cylinders, ellipses, and air foils with axes perpendicular to the flow

ρ = fluid density

For flat plates placed parallel with the flow:

$C_D = 1.33/\text{Re}^{0.5}\ (10^4 < \text{Re} < 5 \times 10^5)$

$C_D = 0.031/\text{Re}^{1/7}\ (10^6 < \text{Re} < 10^9)$

The characteristic length in the Reynolds Number (Re) is the length of the plate parallel with the flow. For blunt objects, the characteristic length is the largest linear dimension (diameter of cylinder, sphere, disk, etc.) that is perpendicular to the flow.

Characteristics of Selected Flow Configurations

Open-Channel Flow and/or Pipe Flow of Water

Manning's Equation

$$Q = \frac{K}{n} A R_H^{2/3} S^{1/2}$$

$$v = \frac{K}{n} R_H^{2/3} S^{1/2}$$

where

Q = discharge (ft^3/sec or m^3/s)

v = velocity (ft/sec or m/s)

K = 1.486 for USCS units, 1.0 for SI units

n = roughness coefficient

A = cross-sectional area of flow (ft^2 or m^2)

R_H = hydraulic radius (ft or m) = $\frac{A}{P}$

P = wetted perimeter (ft or m)

S = slope (ft/ft or m/m)

Hazen-Williams Equation

$$v = k_1 C R_H^{0.63} S^{0.54}$$

$$Q = k_1 C A R_H^{0.63} S^{0.54}$$

where

k_1 = 0.849 for SI units, 1.318 for USCS units

C = roughness coefficient, as tabulated in the Civil Engineering section. Other symbols are defined as before.

Flow Through a Packed Bed

A porous, fixed bed of solid particles can be characterized by

L = length of particle bed (m)

D_p = average particle diameter (m)

Φ_s = sphericity of particles, dimensionless (0–1)

ε = porosity or void fraction of the particle bed, dimensionless (0–1)

The Ergun equation can be used to estimate pressure loss through a packed bed under laminar and turbulent flow conditions.

$$\frac{\Delta P}{L} = \frac{150 v_o \mu (1-\varepsilon)^2}{\Phi_s^2 D_p^2 \varepsilon^3} + \frac{1.75 \rho v_o^2 (1-\varepsilon)}{\Phi_s D_p \varepsilon^3}$$

where

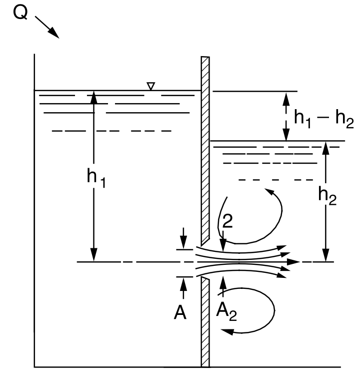
ΔP = pressure loss across packed bed (Pa)

v_o = superficial (flow through empty vessel) fluid velocity (m/s)

ρ = fluid density (kg/m^3)

μ = fluid viscosity [kg/(m•s)]

Submerged Orifice Operating under Steady-Flow Conditions:



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

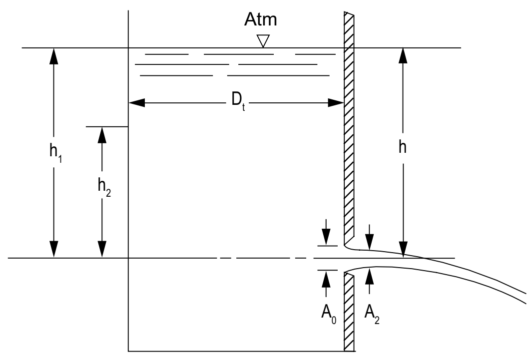
$$Q = A_2 v_2 = C_c C_v A \sqrt{2g(h_1 - h_2)} = CA\sqrt{2g(h_1 - h_2)}$$

in which the product of C_c and C_v is defined as the *coefficient of discharge* of the orifice.

where

v_2 = velocity of fluid exiting orifice

Orifice Discharging Freely into Atmosphere



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

$$Q = CA_0\sqrt{2gh}$$

in which h is measured from the liquid surface to the centroid of the orifice opening.

Q = volumetric flow

A_0 = cross-sectional area of flow

g = acceleration of gravity

h = height of fluid above orifice

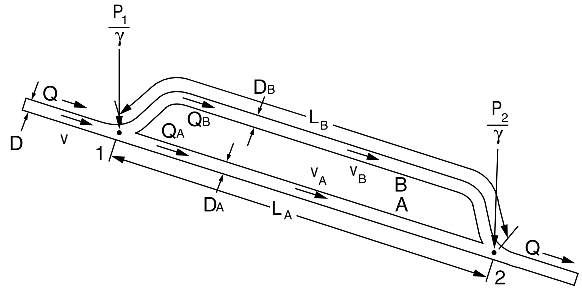
Time required to drain a tank

$$\Delta t = \frac{2(A_t/A_0)}{\sqrt{2g}}\left(h_1^{1/2} - h_2^{1/2}\right)$$

where

A_t = cross-sectional area of tank = $\dfrac{\pi D_t^2}{4}$

Multipath Pipeline Problems



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

For pipes in parallel, the head loss is the same in each pipe.

$$h_L = f_A \frac{L_A}{D_A}\frac{v_A^2}{2g} = f_B \frac{L_B}{D_B}\frac{v_B^2}{2g}$$

$$\left(\pi D^2/4\right)v = \left(\pi D_A^2/4\right)v_A + \left(\pi D_B^2/4\right)v_B$$

The total flowrate Q is the sum of the flowrates in the parallel pipes.

The Impulse-Momentum Principle

The resultant force in a given direction acting on the fluid equals the rate of change of momentum of the fluid.

$$\Sigma \boldsymbol{F} = \Sigma Q_2 \rho_2 v_2 - \Sigma Q_1 \rho_1 v_1$$

where

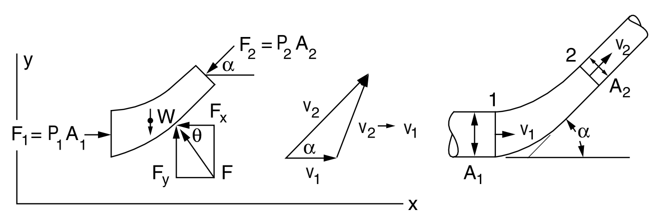
$\Sigma \boldsymbol{F}$ = resultant of all external forces acting on the control volume

$\Sigma Q_1 \rho_1 v_1$ = rate of momentum of the fluid flow entering the control volume in the same direction of the force

$\Sigma Q_2 \rho_2 v_2$ = rate of momentum of the fluid flow leaving the control volume in the same direction of the force

Pipe Bends, Enlargements, and Contractions

The force exerted by a flowing fluid on a bend, enlargement, or contraction in a pipeline may be computed using the impulse-momentum principle.



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

$$P_1 A_1 - P_2 A_2 \cos\alpha - \boldsymbol{F}_x = Q\rho\,(v_2 \cos\alpha - v_1)$$

$$\boldsymbol{F}_y - W - P_2 A_2 \sin\alpha = Q\rho\,(v_2 \sin\alpha - 0)$$

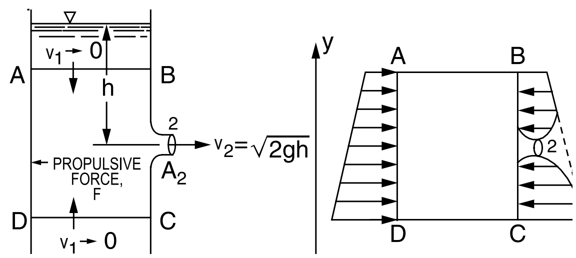
where

$\boldsymbol{F}$ = force exerted by the bend on the fluid (the force exerted by the fluid on the bend is equal in magnitude and opposite in sign), $\boldsymbol{F}_x$ and $\boldsymbol{F}_y$ are the x-component and y-component of the force $\boldsymbol{F} = \sqrt{\boldsymbol{F}_x^2 + \boldsymbol{F}_y^2}$ and $\theta = tan^{-1}\left(\frac{\boldsymbol{F}_y}{\boldsymbol{F}_x}\right)$

where

P = internal pressure in the pipe line
A = cross-sectional area of the pipe line
W = weight of the fluid
v = velocity of the fluid flow
α = angle the pipe bend makes with the horizontal
ρ = density of the fluid
Q = fluid volumetric flowrate

Jet Propulsion



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

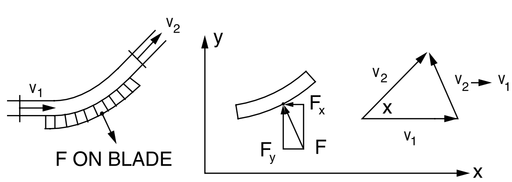
$\boldsymbol{F} = Q\rho(v_2 - 0)$
$\boldsymbol{F} = 2\gamma h A_2$

where

$\boldsymbol{F}$ = propulsive force
γ = specific weight of the fluid
h = height of the fluid above the outlet
A_2 = area of the nozzle tip
$Q = A_2\sqrt{2gh}$
$v_2 = \sqrt{2gh}$

Deflectors and Blades

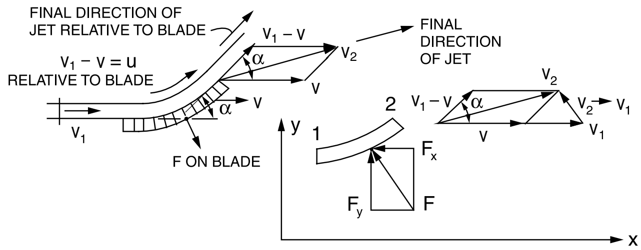
Fixed Blade



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

$-\boldsymbol{F}_x = Q\rho(v_2\cos\alpha - v_1)$
$\boldsymbol{F}_y = Q\rho(v_2\sin\alpha - 0)$

Moving Blade

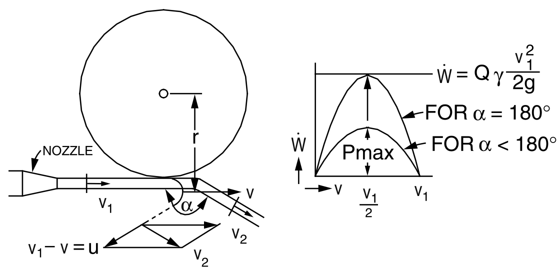


$$-\boldsymbol{F}_x = Q\rho(v_{2x} - v_{1x})$$
$$= -Q\rho(v_1 - v)(1 - \cos\alpha)$$
$$\boldsymbol{F}_y = Q\rho(v_{2y} - v_{1y})$$
$$= +Q\rho(v_1 - v)\sin\alpha$$

where v = velocity of the blade

Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

Impulse Turbine



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

$$\dot{W} = Q\rho(v_1 - v)(1 - \cos\alpha)v$$
$\dot{W}$ = power of the turbine.

where $\dot{W}_{max} = Q\rho\left(v_1^2/4\right)(1 - \cos\alpha)$

When α = 180°,

$$\dot{W}_{max} = \left(Q\rho v_1^2\right)/2 = \left(Q\gamma v_1^2\right)/2g$$

Compressible Flow

Mach Number

The local *speed of sound* in an ideal gas is given by:

$$c = \sqrt{kRT}$$

where

$c \equiv$ local speed of sound

$k \equiv$ ratio of specific heats $= \dfrac{c_p}{c_v}$

$R \equiv$ specific gas constant $= \overline{R}$/(molecular weight)

$T \equiv$ absolute temperature

Example: speed of sound in dry air at 1 atm 20°C is 343.2 m/s.

This shows that the acoustic velocity in an ideal gas depends only on its temperature. The *Mach number* (Ma) is the ratio of the fluid velocity to the speed of sound.

$$\text{Ma} \equiv \frac{V}{c}$$

V ≡ mean fluid velocity

Isentropic Flow Relationships

In an ideal gas for an isentropic process, the following relationships exist between static properties at any two points in the flow.

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{(k-1)}} = \left(\frac{\rho_2}{\rho_1}\right)^k$$

The stagnation temperature, T_0, at a point in the flow is related to the static temperature as follows:

$$T_0 = T + \frac{V^2}{2 \cdot c_p}$$

Energy relation between two points:

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

The relationship between the static and stagnation properties (T_0, P_0, and ρ_0) at any point in the flow can be expressed as a function of the Mach number as follows:

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} \cdot \text{Ma}^2$$

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\frac{k}{(k-1)}} = \left(1 + \frac{k-1}{2} \cdot \text{Ma}^2\right)^{\frac{k}{(k-1)}}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{(k-1)}} = \left(1 + \frac{k-1}{2} \cdot \text{Ma}^2\right)^{\frac{1}{(k-1)}}$$

Compressible flows are often accelerated or decelerated through a nozzle or diffuser. For subsonic flows, the velocity decreases as the flow cross-sectional area increases and vice versa. For supersonic flows, the velocity increases as the flow cross-sectional area increases and decreases as the flow cross-sectional area decreases. The point at which the Mach number is sonic is called the throat and its area is represented by the variable, A^*. The following area ratio holds for any Mach number.

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left[\frac{1 + \frac{1}{2}(k-1)\text{Ma}^2}{\frac{1}{2}(k+1)}\right]^{\frac{(k+1)}{2(k-1)}}$$

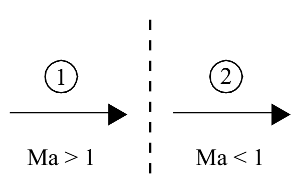
where

A ≡ area [length^2]

A^* ≡ area at the sonic point (Ma = 1.0)

Normal Shock Relationships

A normal shock wave is a physical mechanism that slows a flow from supersonic to subsonic. It occurs over an infinitesimal distance. The flow upstream of a normal shock wave is always supersonic and the flow downstream is always subsonic as depicted in the figure.



NORMAL SHOCK

The following equations relate downstream flow conditions to upstream flow conditions for a normal shock wave.

$$\mathrm{Ma}_2 = \sqrt{\frac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\,\mathrm{Ma}_1^2 - (k-1)}}$$

$$\frac{T_2}{T_1} = \left[2 + (k-1)\mathrm{Ma}_1^2\right]\frac{2k\,\mathrm{Ma}_1^2 - (k-1)}{(k+1)^2\mathrm{Ma}_1^2}$$

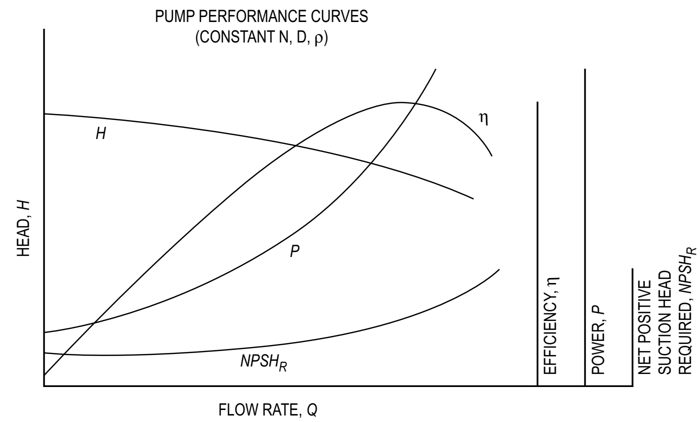
$$\frac{P_2}{P_1} = \frac{1}{k+1}\left[2k\,\mathrm{Ma}_1^2 - (k-1)\right]$$

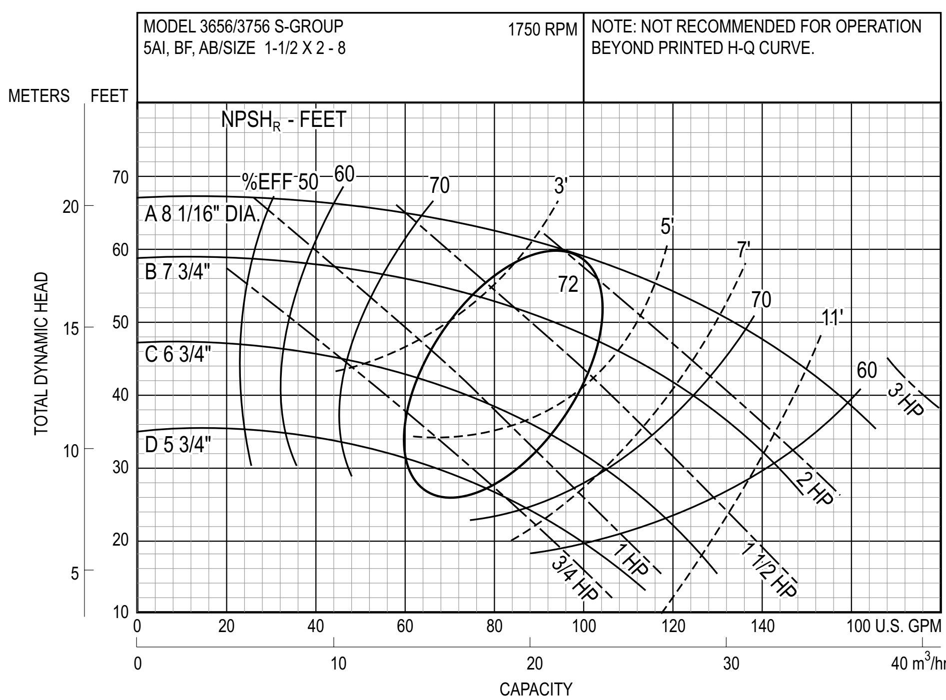
$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(k+1)\mathrm{Ma}_1^2}{(k-1)\mathrm{Ma}_1^2 + 2}$$

$$T_{01} = T_{02}$$

Fluid Flow Machinery

Centrifugal Pump Characteristics





CENTRIFUGAL PUMP CURVE FOR A GOULD MODEL 3656/3756 PUMP

Net Positive Suction Head Available ($NPSH_A$)

$$NPSH_A = H_{pa} + H_s - \sum h_L - H_{vp} = \frac{P_{\text{inlet}}}{\rho g} + \frac{v_{\text{inlet}}^2}{2g} - \frac{P_{\text{vapor}}}{\rho g}$$

where

H_{pa} = atmospheric pressure head on the surface of the liquid in the sump (ft or m)

H_s = static suction head of liquid. This is the height of the surface of the liquid above the centerline of the pump impeller (ft or m).

Σh_L = total friction losses in the suction line (ft or m)

H_{vp} = vapor pressure head of the liquid at the operating temperature (ft or m)

v = fluid velocity at pump inlet

P_{vapor} = fluid vapor pressure at pump inlet

ρ = fluid density

g = acceleration due to gravity

Fluid power $\dot{W}_{\text{fluid}} = \rho g H Q$

Pump (brake) power $\dot{W} = \dfrac{\rho g H Q}{\eta_{\text{pump}}}$

Purchased power $\dot{W}_{\text{purchased}} = \dfrac{\dot{W}}{\eta_{\text{motor}}}$

where

η_{pump} = pump efficiency (0 to 1)

η_{motor} = motor efficiency (0 to 1)

H = head increase provided by pump

Pump Power Equation

$$\dot{W} = Q\gamma h/\eta_t = Q\rho g h/\eta_t$$

where

Q = volumetric flow (m^3/s or cfs)

h = head (m or ft) the fluid has to be lifted

η_t = total efficiency $(\eta_{\text{pump}} \times \eta_{\text{motor}})$

$\dot{W}$ = power ($kg{\bullet}m^2/sec^3$ or ft-lbf/sec)

Fan Characteristics

Typical Backward Curved Fans

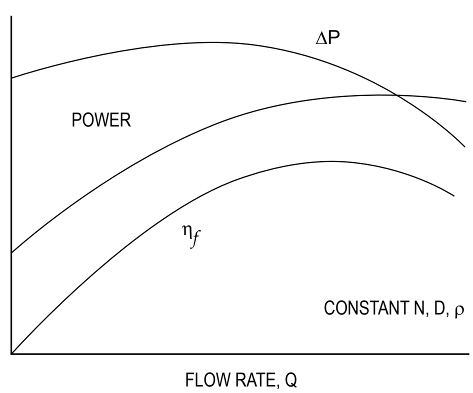
$$\dot{W} = \frac{\Delta P Q}{\eta_f}$$

where

$\dot{W}$ = fan power

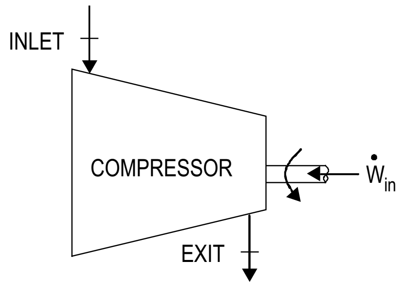
ΔP = pressure rise

η_f = fan efficiency



Compressors

Compressors consume power to add energy to the working fluid. This energy addition results in an increase in fluid pressure (head).



For an adiabatic compressor with $\Delta PE = 0$ and negligible ΔKE:

$$\dot{W}_{\text{comp}} = -\dot{m}\left(h_e - h_i\right)$$

For an ideal gas with constant specific heats:

$$\dot{W}_{\text{comp}} = -\dot{m}c_p\left(T_e - T_i\right)$$

Per unit mass:

$$w_{\text{comp}} = -c_p\left(T_e - T_i\right)$$

Compressor Isentropic Efficiency

$$\eta_C = \frac{w_s}{w_a} = \frac{T_{es} - T_i}{T_e - T_i}$$

where

$w_a \equiv$ actual compressor work per unit mass
$w_s \equiv$ isentropic compressor work per unit mass
$T_{es} \equiv$ isentropic exit temperature

For a compressor where ΔKE is included:

$$\dot{W}_{\text{comp}} = -\dot{m}\left(h_e - h_i + \frac{V_e^2 - V_i^2}{2}\right)$$
$$= -\dot{m}\left(c_p\left(T_e - T_i\right) + \frac{V_e^2 - V_i^2}{2}\right)$$

Adiabatic Compression

$$\dot{W}_{\text{comp}} = \frac{\dot{m}\,P_i\,k}{(k-1)\,\rho_i\,\eta_c}\left[\left(\frac{P_e}{P_i}\right)^{1-1/k} - 1\right]$$

where

$\dot{W}_{\text{comp}}$ = fluid or gas power (W)
P_i = inlet or suction pressure (N/m^2)
P_e = exit or discharge pressure (N/m^2)
k = ratio of specific heats = c_p/c_v
ρ_i = inlet gas density (kg/m^3)
η_c = isentropic compressor efficiency

Isothermal Compression

$$\dot{W}_{\text{comp}} = \frac{\overline{R}T_i}{M\eta_c}\ln\frac{P_e}{P_i}(\dot{m})$$

where

$\dot{W}_{\text{comp}}$, P_i, P_e, and η_c as defined for adiabatic compression

$\overline{R}$ = universal gas constant

T_i = inlet temperature of gas (K)

M = molecular weight of gas (kg/kmol)

Blowers

$$P_w = \frac{WRT_1}{Cne}\left[\left(\frac{P_2}{P_1}\right)^{0.283} - 1\right]$$

where

C = 29.7 (constant for SI unit conversion)

= 550 ft-lbf/(sec-hp) (U.S. Customary Units)

P_W = power requirement (hp)

W = weight of flow of air (lb/sec)

R = engineering gas constant for air = 53.3 ft-lbf/(lb air-°R)

T_1 = absolute inlet temperature (°R)

P_1 = absolute inlet pressure (lbf/in^2)

P_2 = absolute outlet pressure (lbf/in^2)

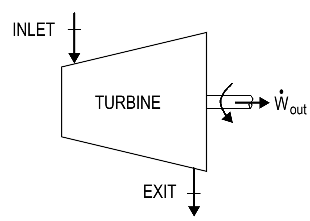
n = (k − 1)/k = 0.283 for air

e = efficiency (usually 0.70 < e < 0.90)

Metcalf and Eddy, *Wastewater Engineering: Treatment, Disposal, and Reuse*, 3rd ed., McGraw-Hill, 1991.

Turbines

Turbines produce power by extracting energy from a working fluid. The energy loss shows up as a decrease in fluid pressure (head).



For an adiabatic turbine with ΔPE = 0 and negligible ΔKE:

$$\dot{W}_{\text{turb}} = \dot{m}(h_i - h_e)$$

For an ideal gas with constant specific heats:

$$\dot{W}_{\text{turb}} = \dot{m}c_p(T_i - T_e)$$

Per unit mass:

$$w_{\text{turb}} = c_p(T_i - T_e)$$

Turbine Isentropic Efficiency

$$\eta_T = \frac{w_a}{w_s} = \frac{T_i - T_e}{T_i - T_{es}}$$

For a turbine where ΔKE is included:

$$\dot{W}_{\text{turb}} = \dot{m}\left(h_i - h_e + \frac{V_i^2 - V_e^2}{2}\right) = \dot{m}\left(c_p(T_i - T_e) + \frac{V_i^2 - V_e^2}{2}\right)$$

Performance of Components

<u>Fans, Pumps, and Compressors</u>

Scaling Laws; *Affinity Laws*

$$\left(\frac{Q}{ND^3}\right)_2 = \left(\frac{Q}{ND^3}\right)_1$$

$$\left(\frac{\dot{m}}{\rho ND^3}\right)_2 = \left(\frac{\dot{m}}{\rho ND^3}\right)_1$$

$$\left(\frac{H}{N^2D^2}\right)_2 = \left(\frac{H}{N^2D^2}\right)_1$$

$$\left(\frac{P}{\rho N^2D^2}\right)_2 = \left(\frac{P}{\rho N^2D^2}\right)_1$$

$$\left(\frac{\dot{W}}{\rho N^3D^5}\right)_2 = \left(\frac{\dot{W}}{\rho N^3D^5}\right)_1$$

where

Q = volumetric flowrate
$\dot{m}$ = mass flowrate
H = head
P = pressure rise
$\dot{W}$ = power
ρ = fluid density
N = rotational speed
D = impeller diameter

Subscripts 1 and 2 refer to different but similar machines or to different operating conditions of the same machine.

Fluid Flow Measurement

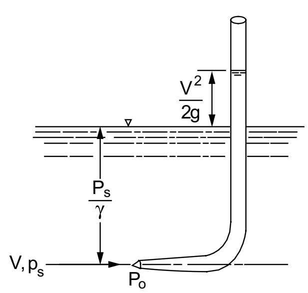
Pitot Tubes

From the stagnation pressure equation for an incompressible fluid,

$$v = \sqrt{(2/\rho)(P_0 - P_s)} = \sqrt{2g(P_0 - P_s)/\gamma}$$

where

v = velocity of the fluid
P_0 = stagnation pressure
P_s = static pressure of the fluid at the elevation where the measurement is taken



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

For a *compressible fluid*, use the above incompressible fluid equation if the Mach number ≤ 0.3.

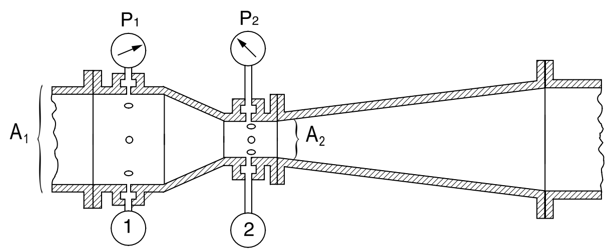
Venturi Meters

$$Q = \frac{C_v A_2}{\sqrt{1 - \left(A_2/A_1\right)^2}} \quad \sqrt{2g\left(\frac{P_1}{\gamma} + z_1 - \frac{P_2}{\gamma} - z_2\right)}$$

where

Q = volumetric flowrate
C_v = coefficient of velocity
A = cross-sectional area of flow
P = pressure
γ = ρg
z_1 = elevation of venturi entrance
z_2 = elevation of venturi throat

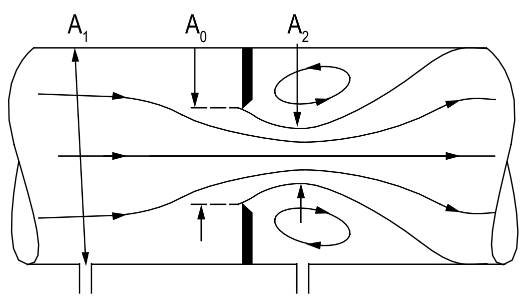
The above equation is for *incompressible fluids*.



Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

Orifices

The cross-sectional area at the vena contracta A_2 is characterized by a *coefficient of contraction* C_c and given by $C_c A_0$.

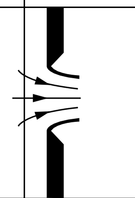
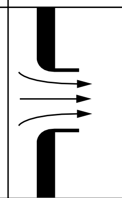
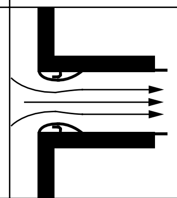
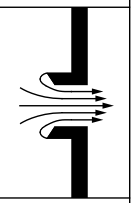


Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

$$Q = CA_0\sqrt{2g\left(\frac{P_1}{\gamma} + z_1 - \frac{P_2}{\gamma} - z_2\right)}$$

where C, the *coefficient of the meter* (*orifice coefficient*), is given by

$$C = \frac{C_v C_c}{\sqrt{1 - C_c^2\left(A_0/A_1\right)^2}}$$

ORIFICES AND THEIR NOMINAL COEFFICIENTS				
	SHARP EDGED	ROUNDED	SHORT TUBE	BORDA
				
C	0.61	0.98	0.80	0.51
C_c	0.62	1.00	1.00	0.52
C_v	0.98	0.98	0.80	0.98

Vennard, J.K., *Elementary Fluid Mechanics*, 6th ed., John Wiley and Sons, 1982.

For incompressible flow through a horizontal orifice meter installation

$$Q = CA_0\sqrt{\frac{2}{\rho}\left(P_1 - P_2\right)}$$

Dimensional Homogeneity

Dimensional Analysis

A dimensionally homogeneous equation has the same dimensions on the left and right sides of the equation. Dimensional analysis involves the development of equations that relate dimensionless groups of variables to describe physical phemona.

Buckingham Pi Theorem: The *number of independent dimensionless groups* that may be employed to describe a phenomenon known to involve n variables is equal to the number $(n - \bar{r})$, where $\bar{r}$ is the number of basic dimensions (e.g., M, L, T) needed to express the variables dimensionally.

Similitude

In order to use a model to simulate the conditions of the prototype, the model must be *geometrically*, *kinematically*, and *dynamically similar* to the prototype system.

To obtain dynamic similarity between two flow pictures, all independent force ratios that can be written must be the same in both the model and the prototype. Thus, dynamic similarity between two flow pictures (when all possible forces are acting) is expressed in the five simultaneous equations below.

$$\left[\frac{F_I}{F_P}\right]_p = \left[\frac{F_I}{F_P}\right]_m = \left[\frac{\rho v^2}{P}\right]_p = \left[\frac{\rho v^2}{P}\right]_m$$

$$\left[\frac{F_I}{F_V}\right]_p = \left[\frac{F_I}{F_V}\right]_m = \left[\frac{vl\rho}{\mu}\right]_p = \left[\frac{vl\rho}{\mu}\right]_m = \left[\mathrm{Re}\right]_p = \left[\mathrm{Re}\right]_m$$

$$\left[\frac{F_I}{F_G}\right]_p = \left[\frac{F_I}{F_G}\right]_m = \left[\frac{v^2}{lg}\right]_p = \left[\frac{v^2}{lg}\right]_m = \left[\mathrm{Fr}\right]_p = \left[\mathrm{Fr}\right]_m$$

$$\left[\frac{F_I}{F_E}\right]_p = \left[\frac{F_I}{F_E}\right]_m = \left[\frac{\rho v^2}{E_v}\right]_p = \left[\frac{\rho v^2}{E_v}\right]_m = \left[\mathrm{Ca}\right]_p = \left[\mathrm{Ca}\right]_m$$

$$\left[\frac{F_I}{F_T}\right]_p = \left[\frac{F_I}{F_T}\right]_m = \left[\frac{\rho l v^2}{\sigma}\right]_p = \left[\frac{\rho l v^2}{\sigma}\right]_m = \left[\mathrm{We}\right]_p = \left[\mathrm{We}\right]_m$$

where the subscripts p and m stand for *prototype* and *model* respectively, and

F_I = inertia force
F_P = pressure force
F_V = viscous force
F_G = gravity force
F_E = elastic force
F_T = surface tension force
Re = Reynolds number
We = Weber number
Ca = Cauchy number
Fr = Froude number
l = characteristic length
v = velocity
ρ = density
σ = surface tension
E_v = bulk modulus
μ = dynamic viscosity
P = pressure
g = acceleration of gravity

Aerodynamics

Airfoil Theory

The lift force on an airfoil F_L is given by

$$F_L = \frac{C_L \rho v^2 A_P}{2}$$

where

C_L = lift coefficient
ρ = fluid density
v = velocity (m/s) of the undisturbed fluid and
A_P = projected area of the airfoil as seen from above (plan area). This same area is used in defining the drag coefficient for an airfoil.

The lift coefficient C_L can be approximated by the equation

$C_L = 2\pi k_1 \sin(\alpha + \beta)$, which is valid for small values of α and β

where

k_1 = constant of proportionality
α = angle of attack (angle between chord of airfoil and direction of flow)
β = negative of angle of attack for zero lift

The drag coefficient C_D may be approximated by

$$C_D = C_{D\infty} + \frac{C_L^2}{\pi AR}$$

where $C_{D\infty}$ = infinite span drag coefficient

The aspect ratio AR is defined

$$AR = \frac{b^2}{A_p} = \frac{A_p}{c^2}$$

where

b = span length

A_p = plan area

c = chord length

The aerodynamic moment M is given by

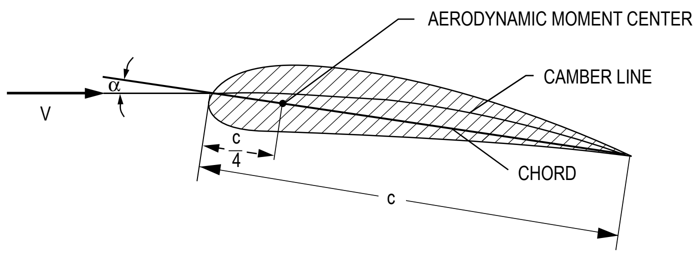
$$M = \frac{C_M \rho v^2 A_p c}{2}$$

where the moment is taken about the front quarter point of the airfoil.

C_M = moment coefficient

ρ = fluid density

v = velocity



Properties of Water (SI Metric Units)

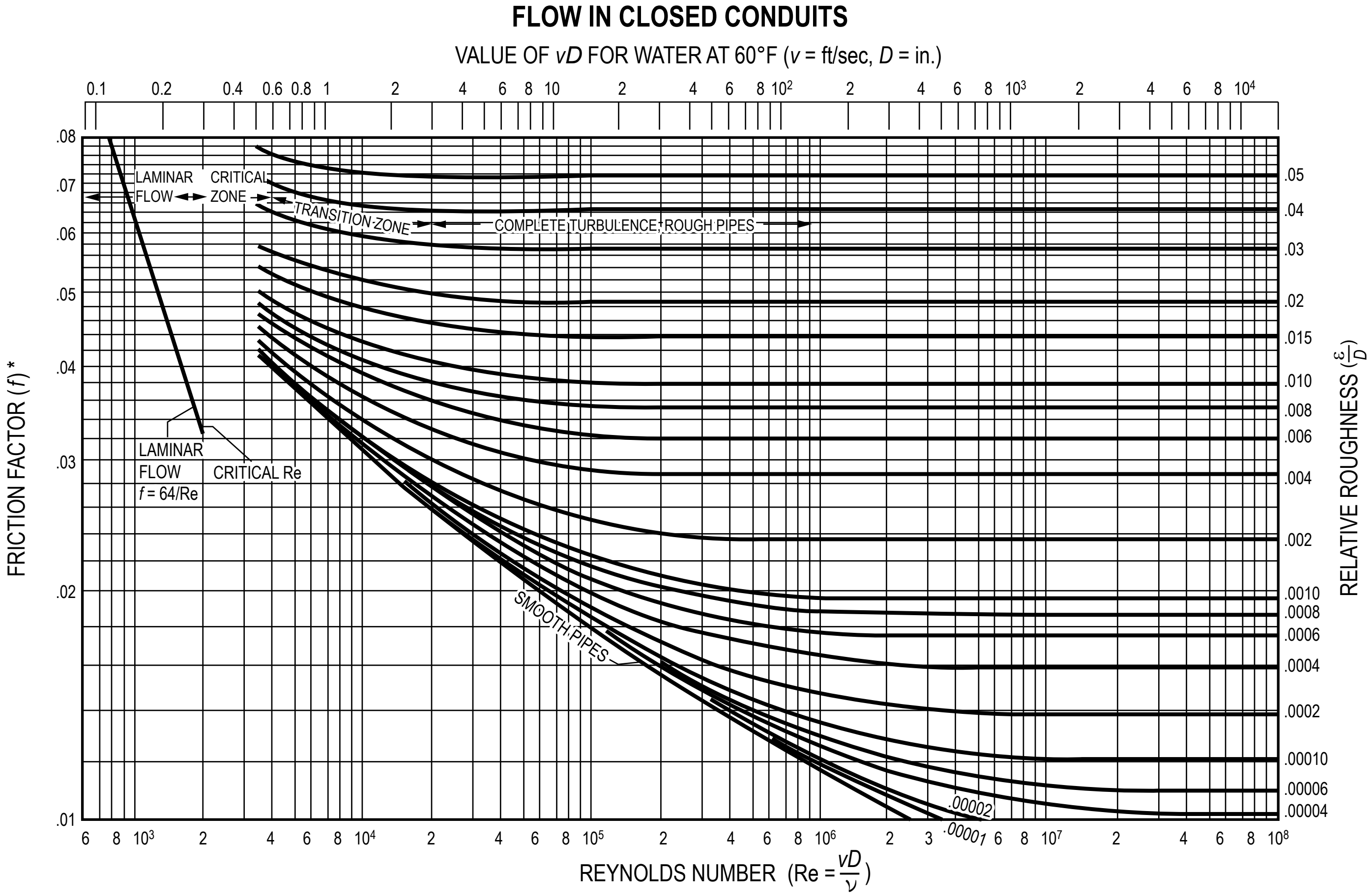
Temperature (°C)	Specific Weight γ (kN/m^3)	Density ρ (kg/m^3)	Absolute Dynamic Viscosity μ (Pa•s)	Kinematic Viscosity ν (m^2/s)	Vapor Pressure P_v (kPa)
0	9.805	999.8	0.001781	0.000001785	0.61
5	9.807	1000.0	0.001518	0.000001518	0.87
10	9.804	999.7	0.001307	0.000001306	1.23
15	9.798	999.1	0.001139	0.000001139	1.70
20	9.789	998.2	0.001002	0.000001003	2.34
25	9.777	997.0	0.000890	0.000000893	3.17
30	9.764	995.7	0.000798	0.000000800	4.24
40	9.730	992.2	0.000653	0.000000658	7.38
50	9.689	988.0	0.000547	0.000000553	12.33
60	9.642	983.2	0.000466	0.000000474	19.92
70	9.589	977.8	0.000404	0.000000413	31.16
80	9.530	971.8	0.000354	0.000000364	47.34
90	9.466	965.3	0.000315	0.000000326	70.10
100	9.399	958.4	0.000282	0.000000294	101.33

Properties of Water (English Units)

Temperature (°F)	Specific Weight γ (lbf/ft^3)	Mass Density ρ (lbf-sec^2/ft^4)	Absolute Dynamic Viscosity μ ($\times 10^{-5}$ lbf-sec/ft^2)	Kinematic Viscosity ν ($\times 10^{-5}$ ft^2/sec)	Vapor Pressure P_v (psi)
32	62.42	1.940	3.746	1.931	0.09
40	62.43	1.940	3.229	1.664	0.12
50	62.41	1.940	2.735	1.410	0.18
60	62.37	1.938	2.359	1.217	0.26
70	62.30	1.936	2.050	1.059	0.36
80	62.22	1.934	1.799	0.930	0.51
90	62.11	1.931	1.595	0.826	0.70
100	62.00	1.927	1.424	0.739	0.95
110	61.86	1.923	1.284	0.667	1.24
120	61.71	1.918	1.168	0.609	1.69
130	61.55	1.913	1.069	0.558	2.22
140	61.38	1.908	0.981	0.514	2.89
150	61.20	1.902	0.905	0.476	3.72
160	61.00	1.896	0.838	0.442	4.74
170	60.80	1.890	0.780	0.413	5.99
180	60.58	1.883	0.726	0.385	7.51
190	60.36	1.876	0.678	0.362	9.34
200	60.12	1.868	0.637	0.341	11.52
212	59.83	1.860	0.593	0.319	14.70

Vennard, John K., and Robert L. Street, *Elementary Fluid Mechanics*, 6th ed., New York: Wiley, 1982, p. 663. Reproduced with permission of John Wiley & Sons, Inc.

Moody, Darcy, or Stanton Friction Factor Diagram

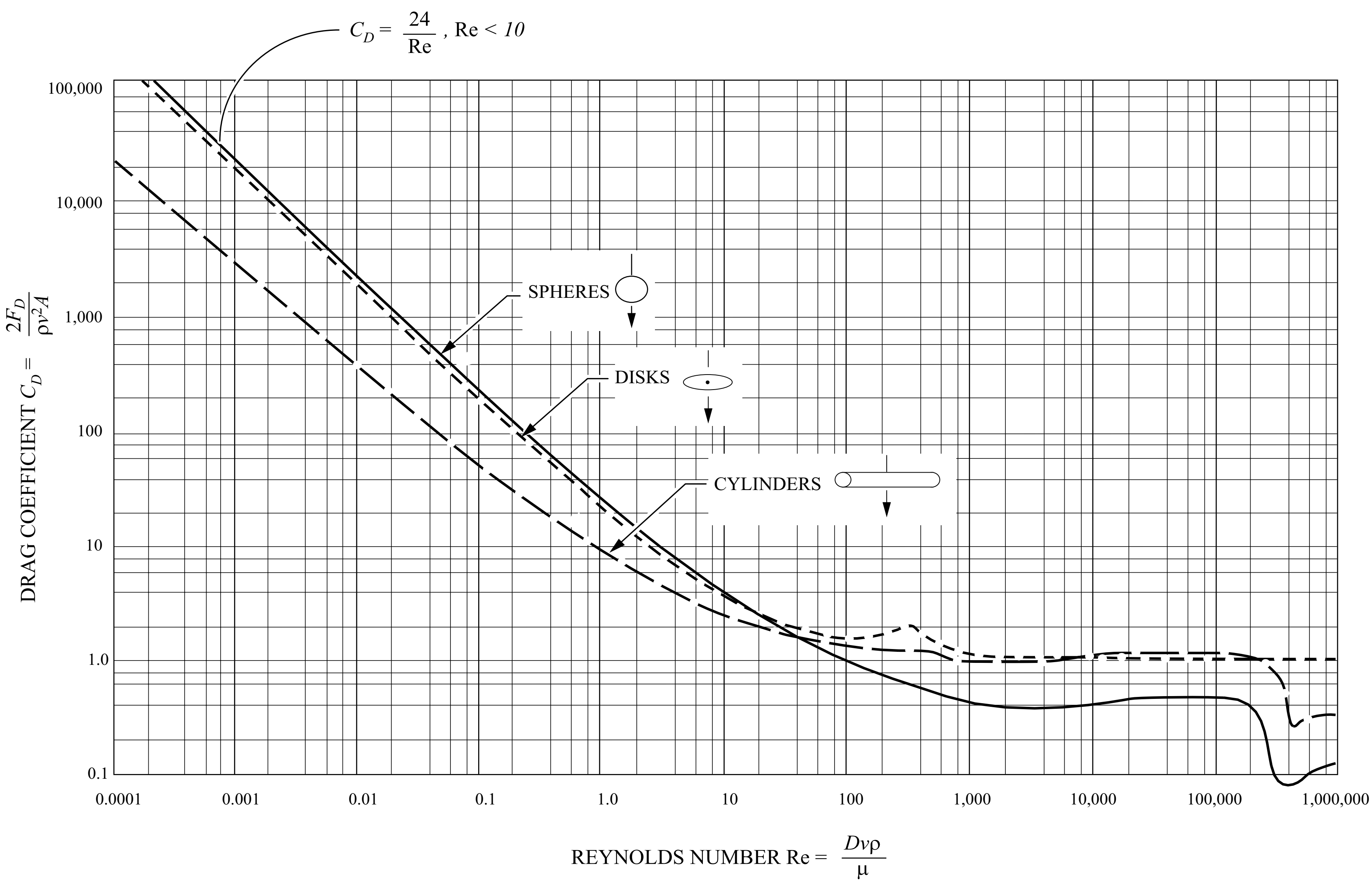


* The Fanning Friction is this factor divided by 4.

	ε (ft)	ε (mm)
GLASS, DRAWN BRASS, COPPER, LEAD	SMOOTH	SMOOTH
COMMERCIAL STEEL, WROUGHT IRON	0.0001–0.0003	0.03–0.09
ASPHALTED CAST IRON	0.0002–0.0006	0.06–0.18
GALVANIZED IRON	0.0002–0.0008	0.06–0.24
CAST IRON	0.0006–0.003	0.18–0.91
CONCRETE	0.001–0.01	0.30–3.0
RIVETED STEEL	0.003–0.03	0.91–9.1
CORRUGATED METAL PIPE	0.1–0.2	30–61
LARGE TUNNEL, CONCRETE OR STEEL LINED	0.002–0.004	0.61–1.2
BLASTED ROCK TUNNEL	1.0–2.0	300–610

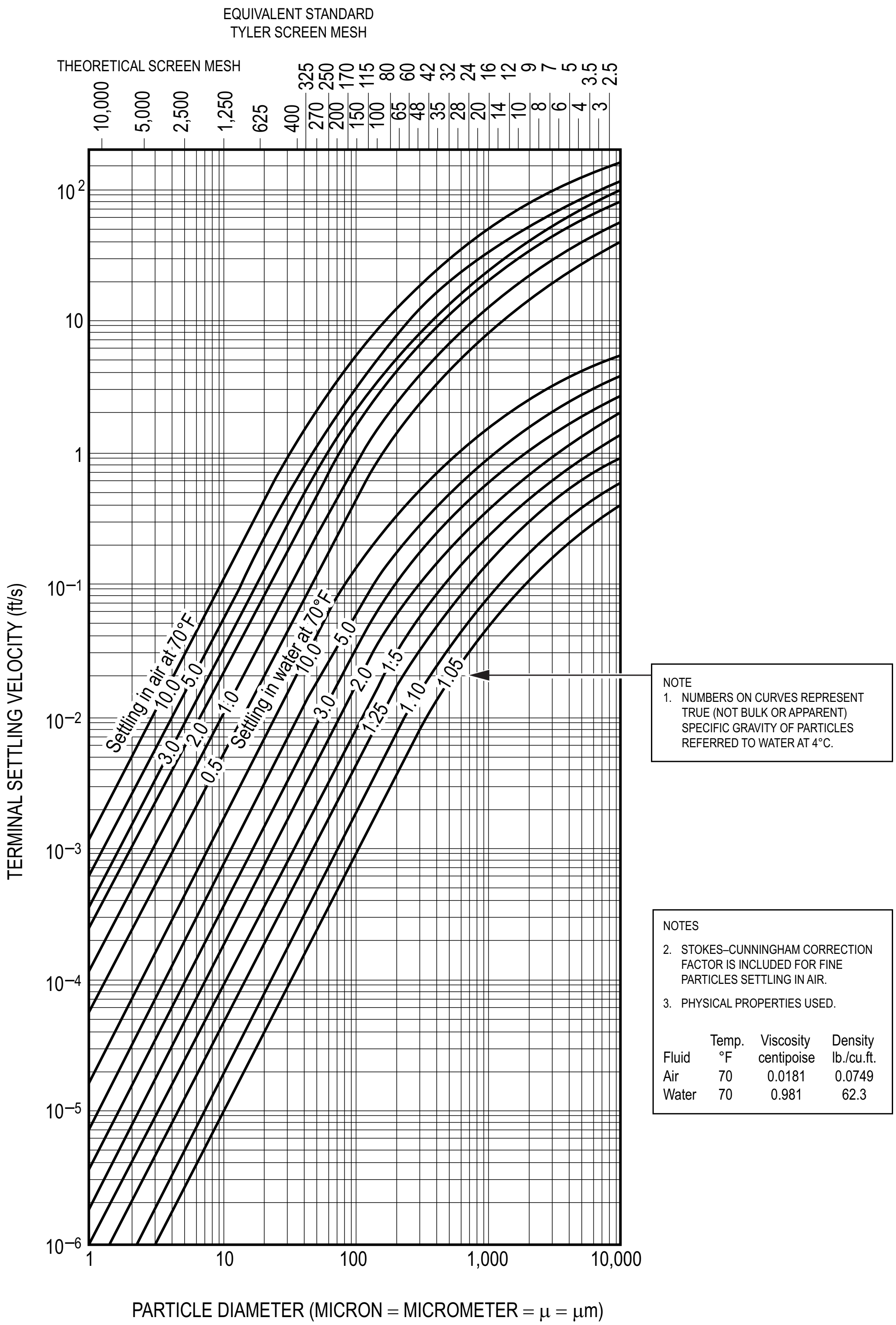
Chow, Ven Te, *Handbook of Applied Hydrology,* McGraw-Hill, 1964.

Drag Coefficient for Spheres, Disks, and Cylinders



Note: Intermediate divisions are 2, 4, 6, and 8

Terminal Velocities of Spherical Particles of Different Densities



De Nevers, Noel, *Fluid Mechanics for Chemical Engineers*, 3rd ed., New York: McGraw-Hill, 2004, p. 225.

Heat Transfer

There are three modes of heat transfer: conduction, convection, and radiation.

Basic Heat-Transfer Rate Equations

Conduction

Fourier's Law of Conduction

$$\dot{Q} = -kA\frac{dT}{dx}$$

where

$\dot{Q}$ = rate of heat transfer (W)

k = thermal conductivity [W/(m•K)]

A = surface area perpendicular to direction of heat transfer (m^2)

Convection

Newton's Law of Cooling

$$\dot{Q} = hA\left(T_w - T_\infty\right)$$

where

h = convection heat-transfer coefficient of the fluid [W/(m^2•K)]

A = convection surface area (m^2)

T_w = wall surface temperature (K)

T_∞ = bulk fluid temperature (K)

Radiation

The radiation emitted by a body is given by

$$\dot{Q} = \varepsilon\sigma A T^4$$

where

ε = emissivity of the body

σ = Stefan-Boltzmann constant

= 5.67×10^{-8} W/(m^2•K^4)

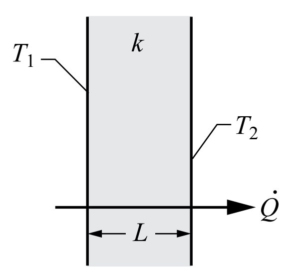
A = body surface area (m^2)

T = absolute temperature (K)

Conduction

Conduction Through a Plane Wall

$$\dot{Q} = \frac{-kA\left(T_2 - T_1\right)}{L}$$



where

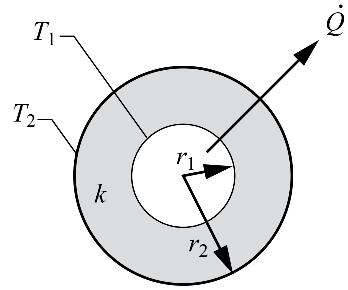
A = wall surface area normal to heat flow (m^2)

L = wall thickness (m)

T_1 = temperature of one surface of the wall (K)

T_2 = temperature of the other surface of the wall (K)

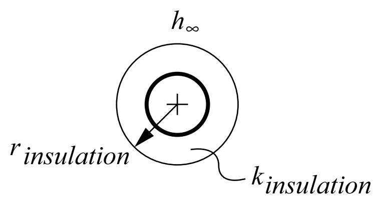
Conduction Through a Cylindrical Wall



Cylinder (Length = L)

$$\dot{Q} = \frac{2\pi kL(T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$

<u>Critical Insulation Radius</u>



$$r_{cr} = \frac{k_{insulation}}{h_\infty}$$

Thermal Resistance (R)

$$\dot{Q} = \frac{\Delta T}{R_{total}}$$

Resistances in series are added:

$$R_{total} = \Sigma R$$

where

Plane Wall Conduction Resistance (K/W):

$$R = \frac{L}{kA}$$

where L = wall thickness

Cylindrical Wall Conduction Resistance (K/W):

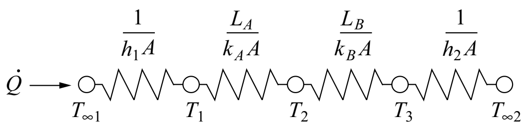
$$R = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi kL}$$

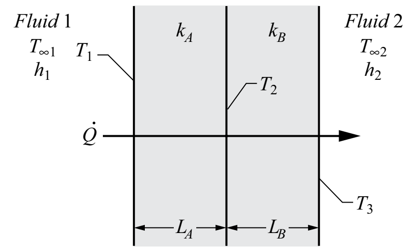
where L = cylinder length

Convection Resistance (K/W) :

$$R = \frac{1}{hA}$$

Composite Plane Wall

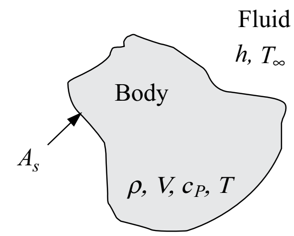




To evaluate surface or intermediate temperatures:

$$\dot{Q} = \frac{T_1 - T_2}{R_A} = \frac{T_2 - T_3}{R_B}$$

Transient Conduction Using the Lumped Capacitance Model



The lumped capacitance model is valid if

$$\text{Biot number, Bi} = \frac{hV}{kA_s} < 0.1$$

where

h = convection heat-transfer coefficient of the fluid [W/(m^2•K)]
V = volume of the body (m^3)
k = thermal conductivity of the body [W/(m•K)]
A_s = surface area of the body (m^2)

Constant Fluid Temperature

If the temperature may be considered uniform within the body at any time, the heat-transfer rate at the body surface is given by

$$\dot{Q} = hA_s\left(T - T_\infty\right) = -\rho V\left(c_P\right)\left(\frac{dT}{dt}\right)$$

where

T = body temperature (K)
T_∞ = fluid temperature (K)
ρ = density of the body (kg/m^3)
c_P = heat capacity of the body [J/(kg•K)]
t = time (s)

The temperature variation of the body with time is

$$T - T_\infty = \left(T_i - T_\infty\right)e^{-\beta t}$$

$$\beta = \frac{hA_s}{\rho V c_P}$$

where

$$\beta = \frac{1}{\tau}$$

τ = time constant (s)

The total heat transferred (Q_{total}) up to time t is

$$Q_{\text{total}} = \rho V c_P\left(T_i - T\right)$$

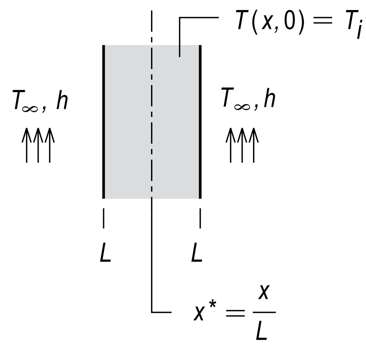
where T_i = initial body temperature (K)

Approximate Solution for Solid with Sudden Convection

The time dependence of the temperature at any location within the solid is the same as that of the midplane/centerline/centerpoint temperature T_o.

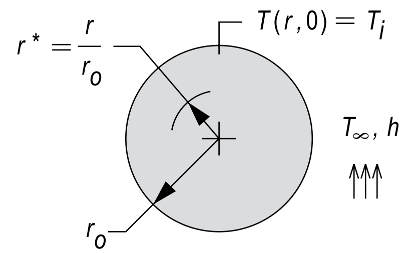
PLANE WALL

For $Fo = \dfrac{\alpha t}{L^2} > 0.2$



INFINITE CYLINDER AND SPHERE

For $Fo = \dfrac{\alpha t}{r_o^2} > 0.2$



where

T_∞ = bulk fluid temperature
T_i = initial uniform temperature of solid
T_o = temperature at midplane of wall, centerline of cylinder, centerpoint of sphere at time t
L = half-thickness of plane wall
x = distance from midplane of wall
r_o = radius of cylinder/sphere
r = radial distance from centerline of cylinder/centerpoint of sphere
h = convective heat transfer coefficient
t = time
α = thermal diffusivity = $\dfrac{k}{\rho c}$
k = thermal conductivity of solid
ρ = density of solid
c = specific heat of solid

$$\left(T_o - T_\infty\right)/\left(T_i - T_\infty\right) = C_1 \exp\left(-\zeta_1^2 Fo\right)$$

where C_1 and ζ are obtained from the following table

Coefficients used in the one-term approximation to the series solutions for transient one-dimensional conduction

	Plane Wall		Infinite Cylinder		Sphere	
Bi*	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1	ζ_1 (rad)	C_1
0.01	0.0998	1.0017	0.1412	1.0025	0.1730	1.0030
0.02	0.1410	1.0033	0.1995	1.0050	0.2445	1.0060
0.03	0.1732	1.0049	0.2439	1.0075	0.2989	1.0090
0.04	0.1987	1.0066	0.2814	1.0099	0.3450	1.0120
0.05	0.2217	1.0082	0.3142	1.0124	0.3852	1.0149
0.06	0.2425	1.0098	0.3438	1.0148	0.4217	1.0179
0.07	0.2615	1.0114	0.3708	1.0173	0.4550	1.0209
0.08	0.2791	1.0130	0.3960	1.0197	0.4860	1.0239
0.09	0.2956	1.0145	0.4195	1.0222	0.5150	1.0268
0.10	0.3111	1.0160	0.4417	1.0246	0.5423	1.0298
0.15	0.3779	1.0237	0.5376	1.0365	0.6608	1.0445
0.20	0.4328	1.0311	0.6170	1.0483	0.7593	1.0592
0.25	0.4801	1.0382	0.6856	1.0598	0.8448	1.0737
0.30	0.5218	1.0450	0.7465	1.0712	0.9208	1.0880
0.40	0.5932	1.0580	0.8516	1.0932	1.0528	1.1164
0.50	0.6533	1.0701	0.9408	1.1143	1.1656	1.1441
0.60	0.7051	1.0814	1.0185	1.1346	1.2644	1.1713
0.70	0.7506	1.0919	1.0873	1.1539	1.3525	1.1978
0.80	0.7910	1.1016	1.1490	1.1725	1.4320	1.2236
0.90	0.8274	1.1107	1.2048	1.1902	1.5044	1.2488
1.0	0.8603	1.1191	1.2558	1.2071	1.5708	1.2732
2.0	1.0769	1.1795	1.5995	1.3384	2.0288	1.4793
3.0	1.1925	1.2102	1.7887	1.4191	2.2889	1.6227
4.0	1.2646	1.2287	1.9081	1.4698	2.4556	1.7201
5.0	1.3138	1.2402	1.9898	1.5029	2.5704	1.7870
6.0	1.3496	1.2479	2.0490	1.5253	2.6537	1.8338
7.0	1.3766	1.2532	2.0937	1.5411	2.7165	1.8674
8.0	1.3978	1.2570	2.1286	1.5526	1.7654	1.8921
9.0	1.4149	1.2598	2.1566	1.5611	2.8044	1.9106
10.0	1.4289	1.2620	2.1795	1.5677	2.8363	1.9249
20.0	1.4961	1.2699	2.2881	1.5919	2.9857	1.9781
30.0	1.5202	1.2717	2.3261	1.5973	3.0372	1.9898
40.0	1.5325	1.2723	2.3455	1.5993	3.0632	1.9942
50.0	1.5400	1.2727	2.3572	1.6002	3.0788	1.9962
100.0	1.5552	1.2731	2.3809	1.6015	3.1102	1.9990
∞	1.5707	1.2733	2.4050	1.6018	3.1415	2.0000

*$Bi = hL/k$ for the plane wall and hr_o/k for the infinite cylinder and sphere.

Incropera, Frank P. and David P. DeWitt, *Introduction to Heat Transfer*, 4th ed., John Wiley and Sons, 2002, pp. 256–261.

Fins

For a straight fin with uniform cross section (assuming negligible heat transfer from tip),

$$\dot{Q} = \sqrt{hPkA_c}\left(T_b - T_\infty\right)\tanh\left(mL_c\right)$$

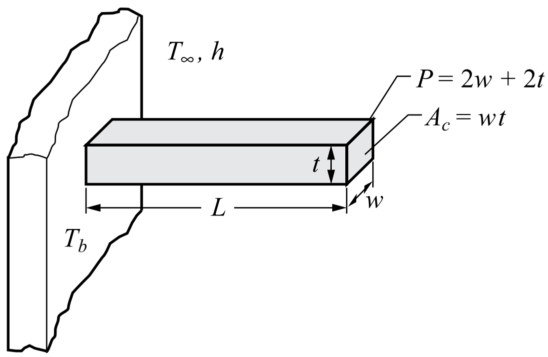
where

h = convection heat-transfer coefficient of the fluid [W/(m^2•K)]
P = perimeter of exposed fin cross section (m)
k = fin thermal conductivity [W/(m•K)]
A_c = fin cross-sectional area (m^2)
T_b = temperature at base of fin (K)
T_∞ = fluid temperature (K)

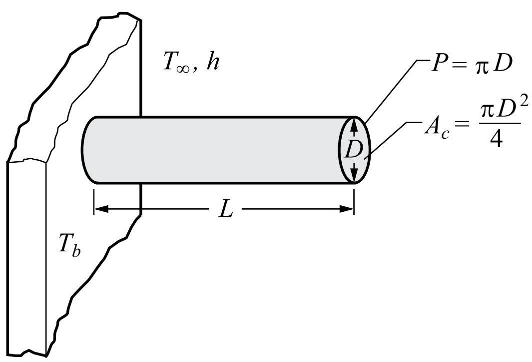
$$m = \sqrt{\frac{hP}{kA_c}}$$

$L_c = L + \frac{A_c}{P}$, corrected length of fin (m)

Rectangular Fin



Pin Fin



Convection

Terms

D = diameter (m)
$\bar{h}$ = average convection heat-transfer coefficient of the fluid [W/(m^2•K)]
L = length (m)
$\overline{Nu}$ = average Nusselt number
Pr = Prandtl number = $\frac{c_P \mu}{k}$
u_m = mean velocity of fluid (m/s)

u_∞ = free stream velocity of fluid (m/s)
μ = dynamic viscosity of fluid [kg/(m•s)]
ρ = density of fluid (kg/m^3)

External Flow

In all cases, evaluate fluid properties at average temperature between that of the body and that of the flowing fluid.

Flat Plate of Length L in Parallel Flow

$$\mathrm{Re}_L = \frac{\rho u_\infty L}{\mu}$$

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.6640\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3} \qquad \left(\mathrm{Re}_L < 10^5\right)$$

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.0366\,\mathrm{Re}_L^{0.8}\,\mathrm{Pr}^{1/3} \qquad \left(\mathrm{Re}_L > 10^5\right)$$

Cylinder of Diameter D in Cross Flow

$$\mathrm{Re}_D = \frac{\rho u_\infty D}{\mu}$$

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = C\,\mathrm{Re}_D^n\,\mathrm{Pr}^{1/3}$$

where

Re_D	C	n
1 – 4	0.989	0.330
4 – 40	0.911	0.385
40 – 4,000	0.683	0.466
4,000 – 40,000	0.193	0.618
40,000 – 250,000	0.0266	0.805

Flow Over a Sphere of Diameter, D

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = 2.0 + 0.60\,\mathrm{Re}_D^{1/2}\,\mathrm{Pr}^{1/3}$$

$$\left(1 < \mathrm{Re}_D < 70{,}000; 0.6 < \mathrm{Pr} < 400\right)$$

Internal Flow

$$\mathrm{Re}_D = \frac{\rho u_m D}{\mu}$$

Laminar Flow in Circular Tubes

For laminar flow ($\mathrm{Re}_D < 2300$), fully developed conditions

$\mathrm{Nu}_D = 4.36$ (uniform heat flux)

$\mathrm{Nu}_D = 3.66$ (constant surface temperature)

For laminar flow ($\mathrm{Re}_D < 2300$), combined entry length with constant surface temperature

$$Nu_D = 1.86\left(\frac{\mathrm{Re}_D\,\mathrm{Pr}}{\frac{L}{D}}\right)^{1/3}\left(\frac{\mu_b}{\mu_s}\right)^{0.14}$$

where

L = length of tube (m)
D = tube diameter (m)
μ_b = dynamic viscosity of fluid [kg/(m•s)] at bulk temperature of fluid T_b
μ_s = dynamic viscosity of fluid [kg/(m•s)] at inside surface temperature of the tube T_s

Turbulent Flow in Circular Tubes

Dittus-Boelter Equation

$$Nu_D = 0.023\,\mathrm{Re}_D^{4/5}\,\mathrm{Pr}^n \qquad \text{where}\begin{bmatrix} 0.7 \le \mathrm{Pr} \le 160 \\ \mathrm{Re}_D \gtrsim 10{,}000 \\ \dfrac{L}{D} \gtrsim 10 \end{bmatrix}$$

where

$n = 0.4$ for heating

$n = 0.3$ for cooling

should be used for small to moderate temperature differences

Sieder-Tate Equation

$$Nu_D = 0.027\,\mathrm{Re}_D^{4/5}\,\mathrm{Pr}^{1/3}\left(\frac{\mu}{\mu_s}\right)^{0.14} \qquad \text{where}\begin{bmatrix} 0.7 \le \mathrm{Pr} \le 16{,}700 \\ \mathrm{Re}_D \gtrsim 10{,}000 \\ \dfrac{L}{D} \gtrsim 10 \end{bmatrix}$$

should be used for flows characterized by large property variations.

Incropera, Frank P. and David P. DeWitt, *Fundamentals of Heat and Mass Transfer*, 3rd ed., Wiley, 1990, p. 496.

Noncircular Ducts

In place of the diameter, D, use the equivalent (hydraulic) diameter (D_H) defined as

$$D_H = \frac{4 \times \text{cross-sectional area}}{\text{wetted perimeter}}$$

Circular Annulus ($D_o > D_i$)

In place of the diameter, D, use the equivalent (hydraulic) diameter (D_H) defined as

$$D_H = D_o - D_i$$

Liquid Metals ($0.003 < \mathrm{Pr} < 0.05$)

$$Nu_D = 6.3 + 0.0167\,\mathrm{Re}_D^{0.85}\,\mathrm{Pr}^{0.93} \text{ (uniform heat flux)}$$

$$Nu_D = 7.0 + 0.025\,\mathrm{Re}_D^{0.8}\,\mathrm{Pr}^{0.8} \text{ (constant wall temperature)}$$

Boiling

Evaporation occurring at a solid-liquid interface when

$T_{solid} > T_{sat,\ liquid}$

$q'' = h(T_s - T_{sat}) = h\Delta T_e$

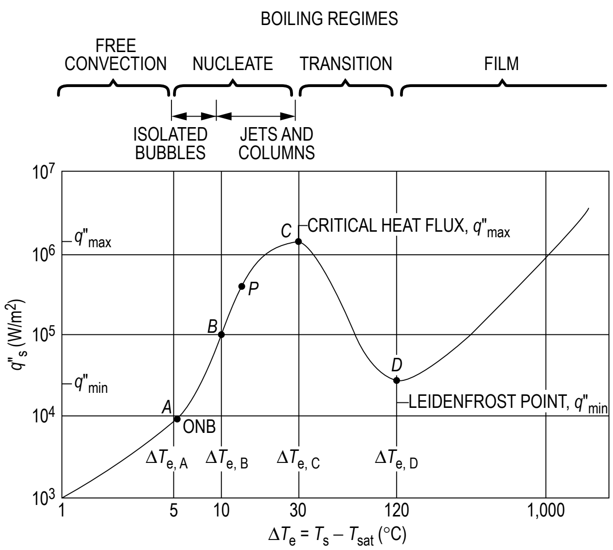
where ΔT_e = excess temperature

Pool Boiling – Liquid is quiescent; motion near solid surface is due to free convection and mixing induced by bubble growth and detachment.

Forced Convection Boiling – Fluid motion is induced by external means in addition to free convection and bubble-induced mixing.

Sub-Cooled Boiling – Temperature of liquid is below saturation temperature; bubbles forming at surface may condense in the liquid.

Saturated Boiling – Liquid temperature slightly exceeds the saturation temperature; bubbles forming at the surface are propelled through liquid by buoyancy forces.



Incropera, Frank P. and David P. DeWitt, *Fundamentals of Heat and Mass Transfer*, 3rd ed., Wiley, 1990. Reproduced with permission of John Wiley & Sons, Inc.

Typical boiling curve for water at one atmosphere: surface heat flux q''_s as a function of excess temperature, $\Delta T_e = T_s - T_{sat}$

Free Convection Boiling – Insufficient vapor is in contact with the liquid phase to cause boiling at the saturation temperature.

Nucleate Boiling – Isolated bubbles form at nucleation sites and separate from surface; vapor escapes as jets or columns.

For nucleate boiling a widely used correlation was proposed in 1952 by Rohsenow:

$$\dot{q}_{\text{nucleate}} = \mu_l \, h_{fg}\left[\frac{g(\rho_l - \rho_v)}{\sigma}\right]^{1/2}\left[\frac{c_{pl}(T_s - T_{\text{sat}})}{C_{sf}\, h_{fg}\, \text{Pr}_l^n}\right]^3$$

where

$\dot{q}_{\text{nucleate}}$ = nucleate boiling heat flux (W/m^2)
μ_l = viscosity of the liquid [kg/(m•s)]
h_{fg} = enthalpy of vaporization (J/kg)
g = gravitational acceleration (m/s^2)
ρ_l = density of the liquid (kg/m^3)
ρ_v = density of the vapor (kg/m^3)
σ = surface tension of liquid–vapor interface (N/m)
c_{pl} = specific heat of the liquid [J/(kg•°C)]
T_s = surface temperature of the heater (°C)
T_{sat} = saturation temperature of the fluid (°C)
C_{sf} = experimental constant that depends on surface–fluid combination
Pr_l = Prandtl number of the liquid
n = experimental constant that depends on the fluid

Çengel, Yunus A., *Heat and Mass Transfer: A Practical Approach*, 3rd ed., New York: McGraw-Hill, 2007.

Peak Heat Flux

The maximum (or critical) heat flux (CHF) in nucleate pool boiling:

$$\dot{q}_{\max} = C_{cr}\, h_{fg}\left[\sigma g \rho^2_v(\rho_l - \rho_v)\right]^{1/4}$$

C_{cr} is a constant whose value depends on the heater geometry, but generally is about 0.15.

The CHF is independent of the fluid–heating surface combination, as well as the viscosity, thermal conductivity, and specific heat of the liquid.

The CHF increases with pressure up to about one-third of the critical pressure, and then starts to decrease and becomes zero at the critical pressure.

The CHF is proportional to h_{fg}, and large maximum heat fluxes can be obtained using fluids with a large enthalpy of vaporization, such as water.

Values of the coefficient C_{cr} for maximum heat flux (dimensionless parameter $L^* = L[g(\rho_l - \rho_v)/\sigma]^{1/2}$

Heater Geometry	C_{cr}	Charac. Dimension of Heater, L	Range of L^*
Large horizontal flat heater	0.149	Width or diameter	$L^* > 27$
Small horizontal flat heater[1]	$18.9\ K_1$	Width or diameter	$9 < L^* < 20$
Large horizontal cyclinder	0.12	Radius	$L^* > 1.2$
Small horizontal cyclinder	$0.12\ L^{*-0.25}$	Radius	$0.15 < L^* < 1.2$
Large sphere	0.11	Radius	$L^* > 4.26$
Small sphere	$0.227\ L^{*-0.5}$	Radius	$0.15 < L^* < 4.26$

[1] $K_1 = \sigma/[g(\rho_l - \rho_v)A_{\text{heater}}]$

Çengel, Yunus A., *Heat and Mass Transfer: A Practical Approach*, 3rd ed., New York: McGraw-Hill, 2007.

Minimum Heat Flux

Minimum heat flux, which occurs at the Leidenfrost point, it represents the lower limit for the heat flux in the film boiling regime.

Zuber derived the following expression for the minimum heat flux for a large horizontal plate

$$\dot{q}_{\min} = 0.09\ \rho_v\ h_{fg}\left[\frac{\sigma g(\rho_l - \rho_v)}{(\rho_l + \rho_v)^2}\right]^{1/4}$$

The relation above can be in error by 50% or more.

Transition Boiling – Rapid bubble formation results in vapor film on surface and oscillation between film and nucleate boiling.

Film Boiling – Surface completely covered by vapor blanket; includes significant radiation through vapor film.

Çengel, Yunus A., *Heat and Mass Transfer: A Practical Approach*, 3rd ed., New York: McGraw-Hill, 2007.

Film Boiling

The heat flux for film boiling on a horizontal cylinder or sphere of diameter D is given by

$$\dot{q}_{\text{film}} = C_{\text{film}}\left[\frac{g k_v^3\ \rho_v(\rho_l - \rho_v)\left[h_{fg} + 0.4 c_{pv}(T_s - T_{\text{sat}})\right]}{\mu_v D(T_s - T_{\text{sat}})}\right]^{1/4}(T_s - T_{\text{sat}})$$

$$C_{\text{film}} = \begin{Bmatrix} 0.62 \text{ for horizontal cylinders} \\ 0.67 \text{ for spheres} \end{Bmatrix}$$

Çengel, Yunus A., *Heat and Mass Transfer: A Practical Approach*, 3rd ed., New York: McGraw-Hill, 2007.

Film Condensation of a Pure Vapor

On a Vertical Surface

$$\overline{Nu}_L = \frac{\bar{h}L}{k_l} = 0.943\left[\frac{\rho_l^2 g h_{fg} L^3}{\mu_l k_l \left(T_{\text{sat}} - T_s\right)}\right]^{0.25}$$

where

ρ_l = density of liquid phase of fluid (kg/m^3)
g = gravitational acceleration (9.81 m/s^2)
h_{fg} = latent heat of vaporization (J/kg)
L = length of surface (m)
μ_l = dynamic viscosity of liquid phase of fluid [kg/(s•m)]
k_l = thermal conductivity of liquid phase of fluid [W/(m•K)]
T_{sat} = saturation temperature of fluid (K)
T_s = temperature of vertical surface (K)

Note: Evaluate all liquid properties at the average temperature between the saturated temperature T_{sat} and the surface temperature T_s.

Outside Horizontal Tubes

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = 0.729\left[\frac{\rho_l^2 g h_{fg} D^3}{\mu_l k_l \left(T_{\text{sat}} - T_s\right)}\right]^{0.25}$$

where D = tube outside diameter (m)

Note: Evaluate all liquid properties at the average temperature between the saturated temperature T_{sat} and the surface temperature T_s.

Natural (Free) Convection

Vertical Flat Plate in Large Body of Stationary Fluid

Equation also can apply to vertical cylinder of sufficiently large diameter in large body of stationary fluid.

$$\bar{h} = C\left(\frac{k}{L}\right)\text{Ra}_L^n$$

where

L = length of the plate (cylinder) in the vertical direction

Ra_L = Rayleigh Number = $\dfrac{g\beta\left(T_s - T_\infty\right)L^3}{\nu^2}Pr$

T_s = surface temperature (K)
T_∞ = fluid temperature (K)
β = coefficient of thermal expansion (1/K)

(For an ideal gas: $\beta = \dfrac{2}{T_s + T_\infty}$ with T in absolute temperature)

ν = kinematic viscosity (m^2/s)

Range of Ra_L	C	n
$10^4 - 10^9$	0.59	1/4
$10^9 - 10^{13}$	0.10	1/3

Long Horizontal Cylinder in Large Body of Stationary Fluid

$$\overline{h} = C\left(\frac{k}{D}\right)\mathrm{Ra}_D^n$$

$$\mathrm{Ra}_D = \frac{g\beta\left(T_s - T_\infty\right)D^3}{\nu^2}Pr$$

Ra_D	C	n
$10^{-3} - 10^{2}$	1.02	0.148
$10^{2} - 10^{4}$	0.850	0.188
$10^{4} - 10^{7}$	0.480	0.250
$10^{7} - 10^{12}$	0.125	0.333

Heat Exchangers

The rate of heat transfer associated with either stream in a heat exchanger in which incompressible fluid or ideal gas with constant specific heats flows is

$$\dot{Q} = \dot{m}c_p\left(T_{\text{exit}} - T_{\text{inlet}}\right)$$

where

c_p = specific heat (at constant pressure)

$\dot{m}$ = mass flow rate

The rate of heat transfer in a heat exchanger is

$$\dot{Q} = UAF\Delta T_{lm}$$

where

A = any convenient reference area (m^2)

F = correction factor for log mean temperature difference for more complex heat exchangers (shell and tube arrangements with several tube or shell passes or cross-flow exchangers with mixed and unmixed flow); otherwise $F = 1$.

U = overall heat-transfer coefficient based on area A and the log mean temperature difference [$W/(m^2 \bullet K)$]

ΔT_{lm} = log mean temperature difference (K)

Log Mean Temperature Difference (LMTD)

For *counterflow* in tubular heat exchangers

$$\Delta T_{lm} = \frac{\left(T_{Ho} - T_{Ci}\right) - \left(T_{Hi} - T_{Co}\right)}{\ln\left(\dfrac{T_{Ho} - T_{Ci}}{T_{Hi} - T_{Co}}\right)}$$

For *parallel flow* in tubular heat exchangers

$$\Delta T_{lm} = \frac{\left(T_{Ho} - T_{Co}\right) - \left(T_{Hi} - T_{Ci}\right)}{\ln\left(\dfrac{T_{Ho} - T_{Co}}{T_{Hi} - T_{Ci}}\right)}$$

where

ΔT_{lm} = log mean temperature difference (K)

T_{Hi} = inlet temperature of the hot fluid (K)

T_{Ho} = outlet temperature of the hot fluid (K)

T_{Ci} = inlet temperature of the cold fluid (K)

T_{Co} = outlet temperature of the cold fluid (K)

Heat Exchanger Effectiveness, ε

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} = \frac{\text{actual heat transfer rate}}{\text{maximum possible heat transfer rate}}$$

$$\varepsilon = \frac{C_H\left(T_{Hi} - T_{Ho}\right)}{C_{\min}\left(T_{Hi} - T_{Ci}\right)} \quad \text{or} \quad \varepsilon = \frac{C_C\left(T_{Co} - T_{Ci}\right)}{C_{\min}\left(T_{Hi} - T_{Ci}\right)}$$

where

$C = \dot{m}c_P =$ heat capacity rate (W/K)

$C_{\min}$ = smaller of C_C or C_H

Number of Transfer Units (*NTU*)

$$NTU = \frac{UA}{C_{\min}}$$

Effectiveness-NTU Relations

$$C_r = \frac{C_{\min}}{C_{\max}} = \text{heat capacity ratio}$$

For *parallel flow concentric tube* heat exchanger

$$\varepsilon = \frac{1 - \exp\left[-NTU\left(1 + C_r\right)\right]}{1 + C_r}$$

$$NTU = -\frac{\ln\left[1 - \varepsilon\left(1 + C_r\right)\right]}{1 + C_r}$$

For *counterflow concentric tube* heat exchanger

$$\varepsilon = \frac{1 - \exp\left[-NTU\left(1 - C_r\right)\right]}{1 - C_r\exp\left[-NTU\left(1 - C_r\right)\right]} \qquad \left(C_r < 1\right)$$

$$\varepsilon = \frac{NTU}{1 + NTU} \qquad \left(C_r = 1\right)$$

$$NTU = \frac{1}{C_r - 1}\ln\left(\frac{\varepsilon - 1}{\varepsilon C_r - 1}\right) \qquad \left(C_r < 1\right)$$

$$NTU = \frac{\varepsilon}{1 - \varepsilon} \qquad \left(C_r = 1\right)$$

Overall Heat-Transfer Coefficient for Concentric Tube and Shell-and-Tube Heat Exchangers

$$\frac{1}{UA} = \frac{1}{h_iA_i} + \frac{R_{fi}}{A_i} + \frac{\ln\left(\frac{D_o}{D_i}\right)}{2\pi kL} + \frac{R_{fo}}{A_o} + \frac{1}{h_oA_o}$$

where

A_i = inside area of tubes (m^2)

A_o = outside area of tubes (m^2)

D_i = inside diameter of tubes (m)

D_o = outside diameter of tubes (m)

h_i = convection heat-transfer coefficient for inside of tubes [W/(m^2•K)]

h_o = convection heat-transfer coefficient for outside of tubes [W/(m^2•K)]

k = thermal conductivity of tube material [W/(m•K)]

R_{fi} = fouling factor for inside of tube [(m^2•K)/W]

R_{fo} = fouling factor for outside of tube [(m^2•K)/W]

Radiation

Types of Bodies

Any Body

For any body

$\alpha + \rho + \tau = 1$

where

α = absorptivity (ratio of energy absorbed to incident energy)

ρ = reflectivity (ratio of energy reflected to incident energy)

τ = transmissivity (ratio of energy transmitted to incident energy)

Opaque Body

For an opaque body

$\alpha + \rho = 1$

Gray Body

A gray body is one for which

$\alpha = \varepsilon$, $(0 < \alpha < 1; 0 < \varepsilon < 1)$

where

ε = the emissivity of the body

For a gray body

$\varepsilon + \rho = 1$

Real bodies are frequently approximated as gray bodies.

Black body

A black body is defined as one that absorbs all energy incident upon it. It also emits radiation at the maximum rate for a body of a particular size at a particular temperature. For such a body

$\alpha = \varepsilon = 1$

Shape Factor (View Factor, Configuration Factor) Relations

Reciprocity Relations

$$A_i F_{ij} = A_j F_{ji}$$

where

A_i = surface area (m^2) of surface i

F_{ij} = shape factor (view factor, configuration factor); fraction of the radiation leaving surface *i* that is intercepted by surface *j*; $0 \le F_{ij} \le 1$

Summation Rule for *N* Surfaces

$$\sum_{j=1}^{N} F_{ij} = 1$$

Net Energy Exchange by Radiation between Two Bodies

Body Small Compared to its Surroundings

$$\dot{Q}_{12} = \varepsilon\sigma A\left(T_1^4 - T_2^4\right)$$

where

$\dot{Q}_{12}$ = net heat-transfer rate from the body (W)

ε = emissivity of the body

σ = Stefan-Boltzmann constant [$\sigma = 5.67 \times 10^{-8}$ W/(m^2•K^4)]

A = body surface area (m^2)

T_1 = absolute temperature (K) of the body surface

T_2 = absolute temperature (K) of the surroundings

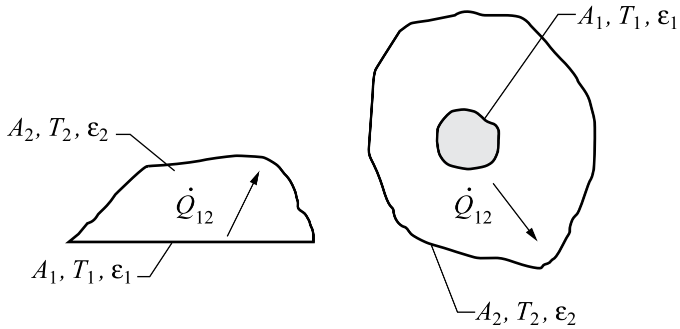
Net Energy Exchange by Radiation between Two Black Bodies

The net energy exchange by radiation between two black bodies that see each other is given by

$$\dot{Q}_{12} = A_1 F_{12}\,\sigma\left(T_1^4 - T_2^4\right)$$

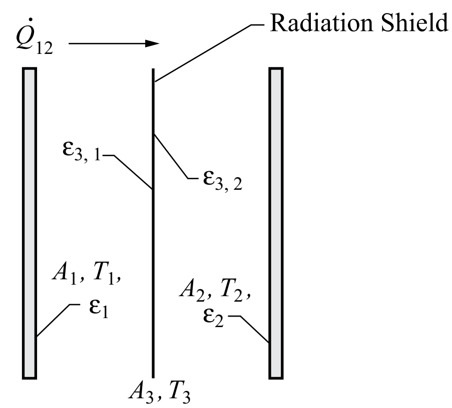
Net Energy Exchange by Radiation between Two Diffuse-Gray Surfaces that Form an Enclosure

Generalized Cases



$$\dot{Q}_{12} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\dfrac{1-\varepsilon_1}{\varepsilon_1 A_1} + \dfrac{1}{A_1 F_{12}} + \dfrac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

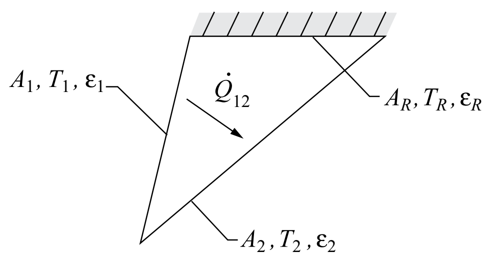
One-Dimensional Geometry with Thin Low-Emissivity Shield Inserted between Two Parallel Plates



$$\dot{Q}_{12} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{13}} + \frac{1-\varepsilon_{3,1}}{\varepsilon_{3,1} A_3} + \frac{1-\varepsilon_{3,2}}{\varepsilon_{3,2} A_3} + \frac{1}{A_3 F_{32}} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

Reradiating Surface

Reradiating Surfaces are considered to be insulated or adiabatic $\left(\dot{Q}_R = 0\right)$.



$$\dot{Q}_{12} = \frac{\sigma\left(T_1^4 - T_2^4\right)}{\frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1 F_{12} + \left[\left(\frac{1}{A_1 F_{1R}}\right) + \left(\frac{1}{A_2 F_{2R}}\right)\right]^{-1}} + \frac{1-\varepsilon_2}{\varepsilon_2 A_2}}$$

Instrumentation, Measurement, and Control

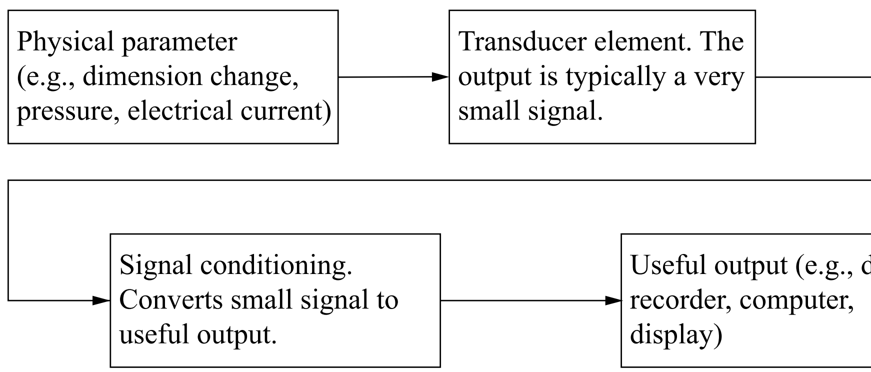
Measurement

Definitions

Calibration – the comparison of an instrument's output to accepted input reference values (for example, using a different instrument with known accuracy), including an evaluation of all the associated uncertainties. The formal definition of calibration is published in ISO/JCGM 200:2012.

Transducer – a device used to convert a physical parameter such as temperature, pressure, flow, light intensity, etc. into an electrical signal (also called a *sensor*).

Transducer Sensitivity – the ratio of change in electrical signal magnitude to the change in magnitude of the physical parameter being measured.



Temperature Sensors

Resistance Temperature Detector (RTD) – a device used to relate change in resistance to change in temperature. Typically made from platinum, the controlling equation for an RTD is given by:

$$R_T = R_0\left[1 + \alpha\left(T - T_0\right)\right]$$

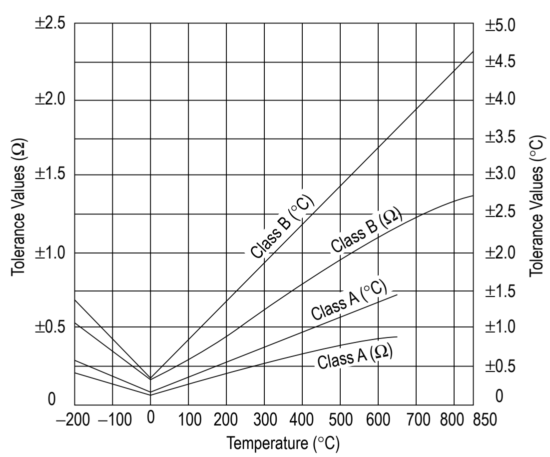
where

R_T = resistance of the RTD at temperature T (°C)

R_0 = resistance of the RTD at the reference temperature T_0 (usually 0°C)

α = resistance temperature coefficient of the RTD (typically 0.00385 Ω/Ω per °C for platinum)

The following graph shows tolerance values as a function of temperature for 100-Ω RTDs.



From Tempco Manufactured Products, as posted on www.tempco.com, July 2013.

Thermistors – Typically manufactured from a semiconductor, with a negative temperature coefficient.

The thermistor resistance is:

$$R_T = R_0 e^{\beta\left(\frac{1}{T} - \frac{1}{T_0}\right)}$$

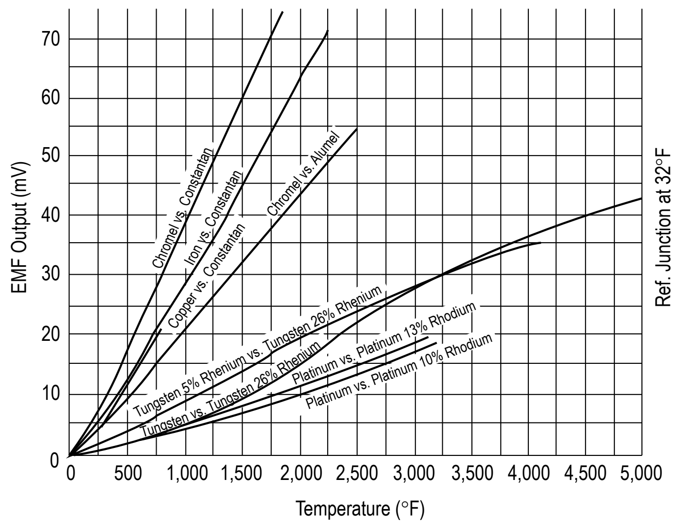
where β is a material dependent value and T is in Kelvin.

The Steinhart-Hart equation is often provided as a more precise model for thermistors:

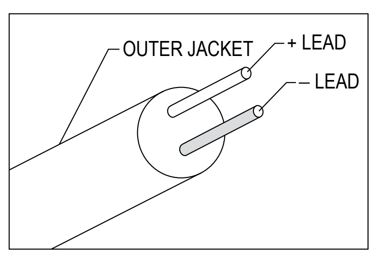
$$\frac{1}{T} = A + B\ln(R) + C(\ln(R))^3$$

Where the thermistor manufacturer will provide the coefficients *A*, *B*, and *C*. When *R* is in Ω and T is in Kelvin, a typical thermistor might have A = 1.403×10^{-3}; B = 2.373×10^{-4}; C = 9.827×10^{-8}.

Thermocouple (*TC*) – a device using the Seebeck effect to sense temperature differences. A thermocouple consists of two dissimilar conductors in electrical contact a measured point and also at a reference junction; the voltage output is proportional to the difference in temperature between the measured point and the reference junction.



From Convectronics Inc., as posted on www.convectronics.com, July 2013.



Typical Thermocouple (TC) Cable

From Convectronics Inc., as posted on www.convectronics.com, July 2013.

ANSI Code	Alloy Combination and Color		Outer Jacket Color		Maximum Thermocouple Temperature Range	Environment
	+ Lead	– Lead	Thermocouple Leads	Extension Cable		
J	IRON Fe (magnetic) White	CONSTANTAN COPPER-NICKEL Cu-Ni Red	Brown	Black	–346 to 2,193°F –210 to 1,200°C	Reducing, Vacuum, Inert. Limited Use in Oxidizing at High Temperatures. Not Recommended for Low Temperatures
K	NICKELCHROMIUM Ni-Cr Yellow	NICKEL-ALUMINUM Ni-Al (magnetic) Red	Brown	Yellow	–454 to 2,501°F –270 to 1,372°C	Clean Oxidizing and Inert. Limited Use in Vacuum or Reducing.
T	COPPER Cu Blue	CONSTANTAN COPPER-NICKEL Cu-Ni Red	Brown	Blue	–454 to 752°F –270 to 400°C	Mild Oxidizing, Reducing Vacuum or Inert. Good where moisture is present.
E	NICKELCHROMIUM Ni-Cr Purple	CONSTANTAN COPPER-NICKEL Cu-Ni Red	Brown	Purple	–454 to 1,832°F –270 to 1,000°C	Oxidizing or Inert. Limited Use in Vacuum or Reducing.

Strain Transducers

Strain Gauge – a device whose electrical resistance varies in proportion to the amount of strain in the device.

Gauge Factor (*GF*) – the ratio of fractional change in electrical resistance to the fractional change in length (strain):

$$GF = \frac{\Delta R/R}{\Delta L/L} = \frac{\Delta R/R}{\varepsilon}$$

where

R = nominal resistance of the strain gauge at nominal length L

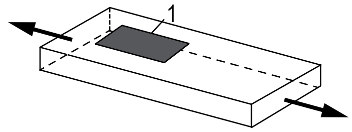
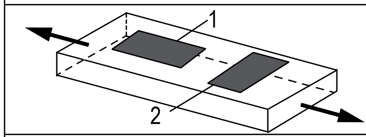
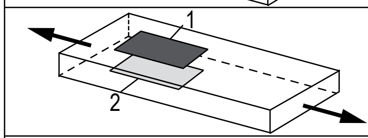
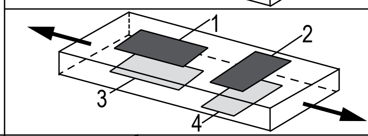
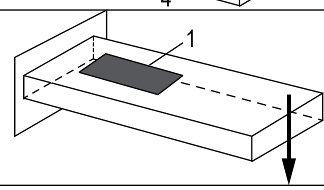
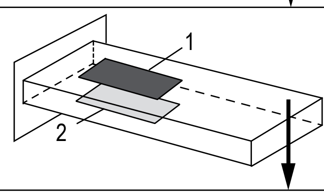
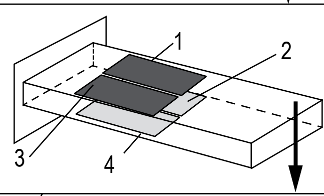
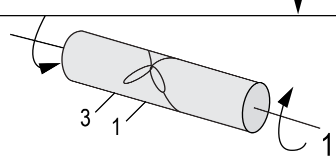
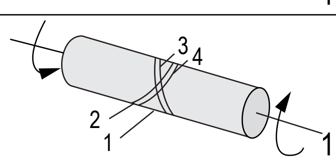
ΔR = change in resistance due the change in length ΔL

ε = normal strain sensed by the gauge

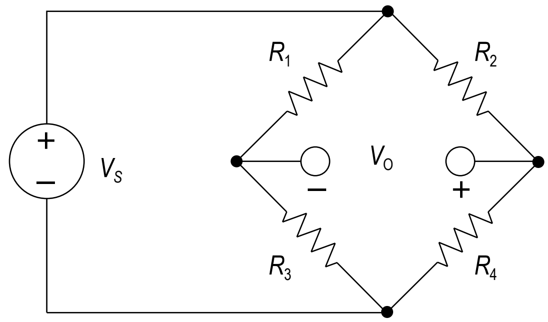
For metals, the change in resistance is due primarily to geometry. The gauge factor for metallic strain gauges is typically around 2.

Piezoresistive effect – a change in the intrinsic electrical conductivity of a material due to a mechanical strain. For many semiconductors, this leads to a gauge factor between 30 and 200 in strain transducers.

Piezoelectric effect – many crystalline or special ceramic materials convert mechanical energy to electrical energy. When a mechanical force is applied, the material changes dimension and an electric field is produced. Piezoelectric transducers can have many different geometries, including using multiple layers to increase gain. A simple peizoelectric transducer generates electrical charge that is proportional to the change in its ceramic's volume or will change volume proportional to an applied electric field. Dimensional changes are usually very small and can be predominantly in one dimension.

Strain	Gauge Setup	Bridge Type	Sensitivity mV/V @ 1,000 $\mu\varepsilon$	Details
Axial		1/4	0.5	Good: Simplest to implement, but must use a dummy gauge if compensating for temperature. Also responds to bending strain.
		1/2	0.65	Better: Temperature compensated, but it is sensitive to bending strain.
		1/2	1.0	Better: Rejects bending strain, but not temperature. Must use dummy gauges if compensating for temperature.
		Full	1.3	Best: More sensitive and compensates for both temperature and bending strain.
Bending		1/4	0.5	Good: Simplest to implement, but must use a dummy gauge if compensating for temperature. Responds equally to axial strain.
		1/2	1.0	Better: Rejects axial strain and is temperature compensated.
		Full	2.0	Best: Rejects axial strain and is temperature compensated. Most sensitive to bending strain.
Torsional and Shear		1/2	1.0	Good: Gauges must be mounted at 45 degrees from centerline.
		Full	2.0	Best: Most sensitive full-bridge version of previous setup. Rejects both axial and bending strains.

Wheatstone Bridge – an electrical circuit used to measure changes in resistance.



WHEATSTONE BRIDGE

If $\frac{R_1}{R_3} = \frac{R_2}{R_4}$ then $V_0 = 0\ V$ and the bridge is said to be balanced.

If $R_1 = R_2 = R_3 = R$ and $R_4 = R + \Delta R$, where $\Delta R \ll R$, then

$V_0 \approx \frac{\Delta R}{4R} \cdot V_S$

Pressure Sensors

Pressure Sensors – can alternatively be called pressure transducers, pressure transmitters, pressure senders, pressure indicators, piezometers, and manometers. They are typically based on measuring the strain on a thin membrane due to an applied pressure.

Pressure Relative Measurement Types	Comparison
Absolute	Relative to 0 Pa, the pressure in a vacuum
Gauge	Relative to local atmospheric pressure
Differential	Relative to another pressurized source

From National Instruments Corporation, as posted on www.ni.com, July 2013.

pH Sensors

pH Sensor – a typical pH meter consists of a special measuring probe connected to an electronic meter that measures and displays the pH reading.

$E_{el} = E^0 - S(\mathrm{pH}_a - \mathrm{pH}_i)$

where

E_{el} = electrode potential
E^0 = zero potential
S = slope (mV per pH unit)
pH_a = pH value of the measured solution
pH_i = pH value of the internal buffer

From Alliance Technical Sales, Inc., as posted on www.alliancets.com, July 2013.

Examples of Common Chemical Sensors

Sensor Type	Principle	Materials	Analyte
Semiconducting oxide sensor	Conductivity impedance	SnO_2, TiO_2, ZnO_2, WO_3, polymers	O_2, H_2, CO, SO_x, NO_x, combustible hydrocarbons, alcohol, H_2S, NH_3
Electrochemical sensor (liquid electrolyte)	Amperiometric	composite Pt, Au catalyst	H_2, O_2, O_3, CO, H_2S, SO_2, NO_x, NH_3, glucose, hydrazine
Ion-selective electrode (ISE)	Potentiometric	glass, LaF_3, CaF_2	pH, K^+, Na^+, Cl^-, Ca^2, Mg^{2+}, F^-, Ag^+
Solid electrode sensor	Amperiometric Potentiometric	YSZ, H^+-conductor YSZ, β-alumina, Nasicon, Nafion	O_2, H_2, CO, combustible hydrocarbons, O_2, H_2, CO_2, CO, NO_x, SO_x, H_2S, Cl_2 H_2O, combustible hydrocarbons
Piezoelectric sensor	Mechanical w/ polymer film	quartz	combustible hydrocarbons, VOCs
Catalytic combustion sensor	Calorimetric	Pt/Al_2O_3, Pt-wire	H_2, CO, combustible hydrocarbons
Pyroelectric sensor	Calorimetric	Pyroelectric + film	Vapors
Optical sensors	Colorimetric fluorescence	optical fiber/indicator dye	Acids, bases, combustible hydrocarbons, biologicals

Sampling

When a continuous-time or analog signal is sampled using a discrete-time method, certain basic concepts should be considered. The sampling rate or frequency is given by

$$f_s = \frac{1}{\Delta t}$$

Nyquist's (Shannon's) sampling theorem states that in order to accurately reconstruct the analog signal from the discrete sample points, the sample rate must be larger than twice the highest frequency contained in the measured signal. Denoting this frequency, which is called the Nyquist frequency, as f_N, the sampling theorem requires that

$$f_s > 2f_N$$

When the above condition is not met, the higher frequencies in the measured signal will not be accurately represented and will appear as lower frequencies in the sampled data. These are known as alias frequencies.

Analog-to-Digital Conversion

When converting an analog signal to digital form, the resolution of the conversion is an important factor. For a measured analog signal over the nominal range $[V_L, V_H]$, where V_L is the low end of the voltage range and V_H is the nominal high end of the voltage range, the voltage resolution is given by

$$\varepsilon_V = \frac{V_H - V_L}{2^n}$$

where n is the number of conversion bits of the A/D converter with typical values of 4, 8, 10, 12, or 16. This number is a key design parameter. After converting an analog signal, the A/D converter produces an integer number of n bits. Call this number N. Note that the range of N is $[0\ ,\ 2^n - 1]$. When calculating the discrete voltage, V, using the reading, N, from the A/D converter the following equation is used.

$$V = \varepsilon_V N + V_L$$

Note that with this strategy, the highest measurable voltage is one voltage resolution less than V_H, or $V_H - \varepsilon_V$.

Signal Conditioning

Signal conditioning of the measured analog signal is often required to prevent alias frequencies from being measured, and to reduce measurement errors.

Measurement Uncertainty

Measurement Accuracy is defined as "closeness of agreement between a measured quantity value and a true quantity value of a measurand." [cite ISO JCGM 200:2012, definition 2.13]

Measurement Precision is defined as "closeness of agreement between indications or measured quantity values obtained by replicate measurements on the same or similar objects under specified conditions." [cite ISO JCGM 200:2012, definition 2.15]

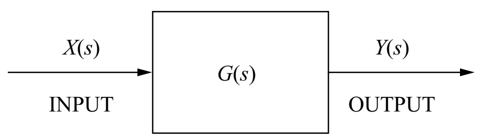
It is critical to always consider the measurement uncertainty of your instrumentation and processes when performing measurements. When reporting measurement results, it is necessary to provide an associated uncertainty so that those who use it may assess its reliability. The Engineering Probability and Statistics section provides a high-level overview of measurement uncertainty.

Suppose that a calculated result R depends on measurements whose values are $x_1 \pm w_1$, $x_2 \pm w_2$, $x_3 \pm w_3$, etc., where $R = f(x_1, x_2, x_3, \ldots x_n)$, x_i is the measured value, and w_i is the uncertainty in that value. The uncertainty in R, w_R, can be estimated using the Kline-McClintock equation:

$$w_R = \sqrt{\left(w_1 \frac{\partial f}{\partial x_1}\right)^2 + \left(w_2 \frac{\partial f}{\partial x_2}\right)^2 + \cdots + \left(w_n \frac{\partial f}{\partial x_n}\right)^2}$$

Control Systems

The linear time-invariant transfer function model represented by the block diagram

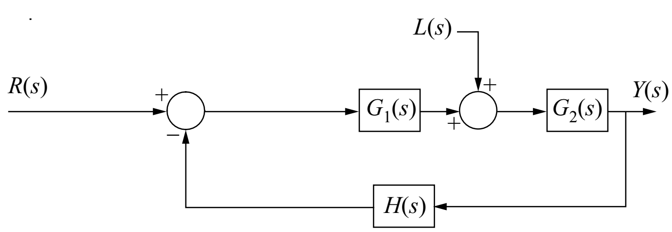


can be expressed as the ratio of two polynomials in the form

$$\frac{Y(s)}{X(s)} = G(s) = \frac{N(s)}{D(s)} = K\frac{\prod_{m=1}^{M}(s - z_m)}{\prod_{n=1}^{N}(s - p_n)}$$

where the M zeros, z_m, and the N poles, p_n, are the roots of the numerator polynomial, $N(s)$, and the denominator polynomial, $D(s)$, respectively.

One classical negative feedback control system model block diagram is



where $G_1(s)$ is a controller or compensator, $G_2(s)$ represents a plant model, and $H(s)$ represents the measurement dynamics. $Y(s)$ represents the controlled variable, $R(s)$ represents the reference input, and $L(s)$ represents a disturbance. $Y(s)$ is related to $R(s)$ and $L(s)$ by

$$Y(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}R(s) + \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}L(s)$$

$G_1(s)\, G_2(s)\, H(s)$ is the open-loop transfer function. The closed-loop characteristic equation is

$$1 + G_1(s)\, G_2(s)\, H(s) = 0$$

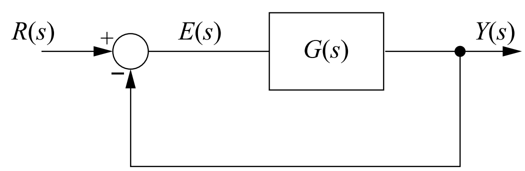
System performance studies normally include

1. Steady-state analysis using constant inputs based on the Final Value Theorem. If all poles of a $G(s)$ function have negative real parts, then

$$\text{dc gain} = \lim_{s \to 0} G(s)$$

Note that $G(s)$ could refer to either an open-loop or a closed-loop transfer function.

For the unity feedback control system model



with the open-loop transfer function defined by

$$G(s) = \frac{K_B}{s^T} \times \frac{\prod_{m=1}^{M}\left(1 + s/\omega_m\right)}{\prod_{n=1}^{N}\left(1 + s/\omega_n\right)}$$

The following steady-state error analysis table can be constructed where T denotes the type of system, i.e., type 0, type 1, etc.

Steady-State Error e_{ss}			
Input \ Type	$T = 0$	$T = 1$	$T = 2$
Unit Step	$1/(K_B + 1)$	0	0
Ramp	∞	$1/K_B$	0
Acceleration	∞	∞	$1/K_B$

2. Frequency response evaluations to determine dynamic performance and stability. For example, relative stability can be quantified in terms of

    a. Gain margin (GM), which is the additional gain required to produce instability in the unity gain feedback control system. If at $\omega = \omega_{180}$,

    $\angle G(j\omega_{180}) = -180°$; then

    $$\text{GM} = -20\log_{10}\left(\left|G(j\omega_{180})\right|\right)$$

    b. Phase margin (PM), which is the additional phase required to produce instability. Thus,

    $$\text{PM} = 180° + \angle G(j\omega_{0\text{dB}})$$

    where $\omega_{0\text{dB}}$ is the ω that satisfies $\left|G(j\omega)\right| = 1$.

3. Transient responses are obtained by using Laplace transforms or computer solutions with numerical integration.

Common Compensator/Controller forms are

PID Controller $G_C(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right)$

Lag or Lead Compensator $G_C(s) = K\left(\frac{1 + sT_1}{1 + sT_2}\right)$ depending on the ratio of T_1/T_2.

First-Order Control System Models

The transfer function model for a first-order system is

$$\frac{Y(s)}{R(s)} = \frac{K}{\tau s + 1}$$

where

K = steady-state gain

τ = time constant

The step response of a first-order system to a step input of magnitude M is

$$y(t) = y_0 e^{-t/\tau} + KM\left(1 - e^{-t/\tau}\right)$$

In the chemical process industry, y_0 is typically taken to be zero, and $y(t)$ is referred to as a deviation variable.

For systems with time delay (dead time or transport lag) θ, the transfer function is

$$\frac{Y(s)}{R(s)} = \frac{Ke^{-\theta s}}{\tau s + 1}$$

The step response for $t \geq \theta$ to a step of magnitude M is

where

$$y(t) = \left[y_0 e^{-(t-\theta)/\tau} + KM\left(1 - e^{-(t-\theta)/\tau}\right)\right]u(t-\theta)$$

$u(t)$ is the unit step function.

Second-Order Control System Models

One standard second-order control system model is

$$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where

K = steady-state gain

ζ = damping ratio

ω_n = undamped natural ($\zeta = 0$) frequency

$\omega_d = \omega_n\sqrt{1 - \zeta^2}$, the damped natural frequency

$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$, the damped resonant frequency

If the damping ratio ζ is less than unity, the system is said to be underdamped; if ζ is equal to unity, it is said to be critically damped; and if ζ is greater than unity, the system is said to be overdamped.

For a unit step input to a normalized underdamped second-order control system, the time required to reach a peak value t_p and the value of that peak M_p are given by

$$t_p = \pi\Big/\left(\omega_n\sqrt{1 - \zeta^2}\right)$$

$$M_p = 1 + e^{-\pi\zeta/\sqrt{1 - \zeta^2}}$$

The percent overshoot (% OS) of the response is given by

$$\%\,\mathrm{OS} = 100e^{-\pi\zeta/\sqrt{1 - \zeta^2}}$$

For an underdamped second-order system, the logarithmic decrement is

$$\delta = \frac{1}{m}\ln\left(\frac{x_k}{x_{k+m}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

where x_k and x_{k+m} are the amplitudes of oscillation at cycles k and $k + m$, respectively. The period of oscillation τ is related to ω_d by

$$\omega_d \tau = 2\pi$$

The time required for the output of a second-order system to settle to within 2% of its final value (2% settling time) is defined to be

$$T_s = \frac{4}{\zeta\omega_n}$$

An alternative form commonly employed in the chemical process industry is

$$\frac{Y(s)}{R(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

where

K = steady-state gain

ζ = the damping ratio

τ = the inverse natural frequency

Engineering Economics

Factor Name	Converts	Symbol	Formula
Single Payment Compound Amount	to F given P	$(F/P, i\%, n)$	$(1+i)^n$
Single Payment Present Worth	to P given F	$(P/F, i\%, n)$	$(1+i)^{-n}$
Uniform Series Sinking Fund	to A given F	$(A/F, i\%, n)$	$\frac{i}{(1+i)^n-1}$
Capital Recovery	to A given P	$(A/P, i\%, n)$	$\frac{i(1+i)^n}{(1+i)^n-1}$
Uniform Series Compound Amount	to F given A	$(F/A, i\%, n)$	$\frac{(1+i)^n-1}{i}$
Uniform Series Present Worth	to P given A	$(P/A, i\%, n)$	$\frac{(1+i)^n-1}{i(1+i)^n}$
Uniform Gradient Present Worth	to P given G	$(P/G, i\%, n)$	$\frac{(1+i)^n-1}{i^2(1+i)^n}-\frac{n}{i(1+i)^n}$
Uniform Gradient † Future Worth	to F given G	$(F/G, i\%, n)$	$\frac{(1+i)^n-1}{i^2}-\frac{n}{i}$
Uniform Gradient Uniform Series	to A given G	$(A/G, i\%, n)$	$\frac{1}{i}-\frac{n}{(1+i)^n-1}$

Nomenclature and Definitions

A ....Uniform amount per interest period
B ....Benefit
BV...........Book value
C............Cost
d.............Inflation adjusted interest rate per interest period
D_j...........Depreciation in year j
EV...........Expected value
F ....Future worth, value, or amount
f ....General inflation rate per interest period
G............Uniform gradient amount per interest period
i ....Interest rate per interest period
i_e ....Annual effective interest rate
MARR....Minimum acceptable/attractive rate of return
m............Number of compounding periods per year
n.............Number of compounding periods; or the expected life of an asset
P ....Present worth, value, or amount
r.............Nominal annual interest rate
S_n ....Expected salvage value in year n

Subscripts

j .... at time j
n........... at time n
†........... $F/G = (F/A - n)/i = (F/A) \times (A/G)$

Non-Annual Compounding

$$i_e = \left(1 + \frac{r}{m}\right)^m - 1$$

Breakeven Analysis

By altering the value of any one of the variables in a situation, holding all of the other values constant, it is possible to find a value for that variable that makes the two alternatives equally economical. This value is the breakeven point.

Breakeven analysis is used to describe the percentage of capacity of operation for a manufacturing plant at which income will just cover expenses.

The payback period is the period of time required for the profit or other benefits of an investment to equal the cost of the investment.

Inflation

To account for inflation, the dollars are deflated by the general inflation rate per interest period f, and then they are shifted over the time scale using the interest rate per interest period i. Use an inflation adjusted interest rate per interest period d for computing present worth values P. The formula for d is $d = i + f + (i \times f)$

Depreciation

Straight Line

$$D_j = \frac{C - S_n}{n}$$

Modified Accelerated Cost Recovery System (MACRS)

$$D_j = (\text{factor})\, C$$

A table of MACRS factors is provided below.

Book Value

$$BV = \text{initial cost} - \Sigma\, D_j$$

Taxation

Income taxes are paid at a specific rate on taxable income. Taxable income is total income less depreciation and ordinary expenses. Expenses do not include capital items, which should be depreciated.

Capitalized Costs

Capitalized costs are present worth values using an assumed perpetual period of time.

$$\text{Capitalized Costs} = P = \frac{A}{i}$$

Bonds

Bond value equals the present worth of the payments the purchaser (or holder of the bond) receives during the life of the bond at some interest rate i.

Bond yield equals the computed interest rate of the bond value when compared with the bond cost.

Rate-of-Return

The minimum acceptable rate-of-return (MARR) is that interest rate that one is willing to accept, or the rate one desires to earn on investments. The rate-of-return on an investment is the interest rate that makes the benefits and costs equal.

Benefit-Cost Analysis

In a benefit-cost analysis, the benefits B of a project should exceed the estimated costs C.

$B - C \geq 0$, or $B/C \geq 1$

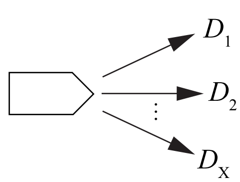
Modified Accelerated Cost Recovery System (MACRS)

MACRS FACTORS				
Year	Recovery Period (Years)			
	3	5	7	10
	Recovery Rate (Percent)			
1	33.33	20.00	14.29	10.00
2	44.45	32.00	24.49	18.00
3	14.81	19.20	17.49	14.40
4	7.41	11.52	12.49	11.52
5		11.52	8.93	9.22
6		5.76	8.92	7.37
7			8.93	6.55
8			4.46	6.55
9				6.56
10				6.55
11				3.28

Economic Decision Trees

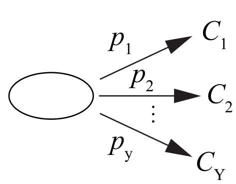
The following symbols are used to model decisions with decision trees:

Decision node



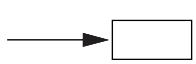
Decision maker chooses 1 of the available paths.

Chance node



Represents a probabilistic (chance) event. Each possible outcome (C_1, C_2,..., C_y) has a probability (p_1, p_2,..., p_y) associated with it.

Outcome node



Shows result for a particular path through the decision tree.

Expected Value: $EV = (C_1)(p_1) + (C_2)(p_2) + \ldots$

Interest Rate Tables
Factor Table - *i* = 0.50%

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9950	0.9950	0.0000	1.0050	1.0000	1.0050	1.0000	0.0000
2	0.9901	1.9851	0.9901	1.0100	2.0050	0.5038	0.4988	0.4988
3	0.9851	2.9702	2.9604	1.0151	3.0150	0.3367	0.3317	0.9967
4	0.9802	3.9505	5.9011	1.0202	4.0301	0.2531	0.2481	1.4938
5	**0.9754**	**4.9259**	**9.8026**	**1.0253**	**5.0503**	**0.2030**	**0.1980**	**1.9900**
6	0.9705	5.8964	14.6552	1.0304	6.0755	0.1696	0.1646	2.4855
7	0.9657	6.8621	20.4493	1.0355	7.1059	0.1457	0.1407	2.9801
8	0.9609	7.8230	27.1755	1.0407	8.1414	0.1278	0.1228	3.4738
9	0.9561	8.7791	34.8244	1.0459	9.1821	0.1139	0.1089	3.9668
10	**0.9513**	**9.7304**	**43.3865**	**1.0511**	**10.2280**	**0.1028**	**0.0978**	**4.4589**
11	0.9466	10.6770	52.8526	1.0564	11.2792	0.0937	0.0887	4.9501
12	0.9419	11.6189	63.2136	1.0617	12.3356	0.0861	0.0811	5.4406
13	0.9372	12.5562	74.4602	1.0670	13.3972	0.0796	0.0746	5.9302
14	0.9326	13.4887	86.5835	1.0723	14.4642	0.0741	0.0691	6.4190
15	**0.9279**	**14.4166**	**99.5743**	**1.0777**	**15.5365**	**0.0694**	**0.0644**	**6.9069**
16	0.9233	15.3399	113.4238	1.0831	16.6142	0.0652	0.0602	7.3940
17	0.9187	16.2586	128.1231	1.0885	17.6973	0.0615	0.0565	7.8803
18	0.9141	17.1728	143.6634	1.0939	18.7858	0.0582	0.0532	8.3658
19	0.9096	18.0824	160.0360	1.0994	19.8797	0.0553	0.0503	8.8504
20	**0.9051**	**18.9874**	**177.2322**	**1.1049**	**20.9791**	**0.0527**	**0.0477**	**9.3342**
21	0.9006	19.8880	195.2434	1.1104	22.0840	0.0503	0.0453	9.8172
22	0.8961	20.7841	214.0611	1.1160	23.1944	0.0481	0.0431	10.2993
23	0.8916	21.6757	233.6768	1.1216	24.3104	0.0461	0.0411	10.7806
24	0.8872	22.5629	254.0820	1.1272	25.4320	0.0443	0.0393	11.2611
25	**0.8828**	**23.4456**	**275.2686**	**1.1328**	**26.5591**	**0.0427**	**0.0377**	**11.7407**
30	0.8610	27.7941	392.6324	1.1614	32.2800	0.0360	0.0310	14.1265
40	0.8191	36.1722	681.3347	1.2208	44.1588	0.0276	0.0226	18.8359
50	0.7793	44.1428	1,035.6966	1.2832	56.6452	0.0227	0.0177	23.4624
60	0.7414	51.7256	1,448.6458	1.3489	69.7700	0.0193	0.0143	28.0064
100	**0.6073**	**78.5426**	**3,562.7934**	**1.6467**	**129.3337**	**0.0127**	**0.0077**	**45.3613**

Factor Table - *i* = 1.00%

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9901	0.9901	0.0000	1.0100	1.0000	1.0100	1.0000	0.0000
2	0.9803	1.9704	0.9803	1.0201	2.0100	0.5075	0.4975	0.4975
3	0.9706	2.9410	2.9215	1.0303	3.0301	0.3400	0.3300	0.9934
4	0.9610	3.9020	5.8044	1.0406	4.0604	0.2563	0.2463	1.4876
5	**0.9515**	**4.8534**	**9.6103**	**1.0510**	**5.1010**	**0.2060**	**0.1960**	**1.9801**
6	0.9420	5.7955	14.3205	1.0615	6.1520	0.1725	0.1625	2.4710
7	0.9327	6.7282	19.9168	1.0721	7.2135	0.1486	0.1386	2.9602
8	0.9235	7.6517	26.3812	1.0829	8.2857	0.1307	0.1207	3.4478
9	0.9143	8.5650	33.6959	1.0937	9.3685	0.1167	0.1067	3.9337
10	**0.9053**	**9.4713**	**41.8435**	**1.1046**	**10.4622**	**0.1056**	**0.0956**	**4.4179**
11	0.8963	10.3676	50.8067	1.1157	11.5668	0.0965	0.0865	4.9005
12	0.8874	11.2551	60.5687	1.1268	12.6825	0.0888	0.0788	5.3815
13	0.8787	12.1337	71.1126	1.1381	13.8093	0.0824	0.0724	5.8607
14	0.8700	13.0037	82.4221	1.1495	14.9474	0.0769	0.0669	6.3384
15	**0.8613**	**13.8651**	**94.4810**	**1.1610**	**16.0969**	**0.0721**	**0.0621**	**6.8143**
16	0.8528	14.7179	107.2734	1.1726	17.2579	0.0679	0.0579	7.2886
17	0.8444	15.5623	120.7834	1.1843	18.4304	0.0643	0.0543	7.7613
18	0.8360	16.3983	134.9957	1.1961	19.6147	0.0610	0.0510	8.2323
19	0.8277	17.2260	149.8950	1.2081	20.8109	0.0581	0.0481	8.7017
20	**0.8195**	**18.0456**	**165.4664**	**1.2202**	**22.0190**	**0.0554**	**0.0454**	**9.1694**
21	0.8114	18.8570	181.6950	1.2324	23.2392	0.0530	0.0430	9.6354
22	0.8034	19.6604	198.5663	1.2447	24.4716	0.0509	0.0409	10.0998
23	0.7954	20.4558	216.0660	1.2572	25.7163	0.0489	0.0389	10.5626
24	0.7876	21.2434	234.1800	1.2697	26.9735	0.0471	0.0371	11.0237
25	**0.7798**	**22.0232**	**252.8945**	**1.2824**	**28.2432**	**0.0454**	**0.0354**	**11.4831**
30	0.7419	25.8077	355.0021	1.3478	34.7849	0.0387	0.0277	13.7557
40	0.6717	32.8347	596.8561	1.4889	48.8864	0.0305	0.0205	18.1776
50	0.6080	39.1961	879.4176	1.6446	64.4632	0.0255	0.0155	22.4363
60	0.5504	44.9550	1,192.8061	1.8167	81.6697	0.0222	0.0122	26.5333
100	**0.3697**	**63.0289**	**2,605.7758**	**2.7048**	**170.4814**	**0.0159**	**0.0059**	**41.3426**

Interest Rate Tables

Factor Table - *i* = 1.50%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9852	0.9852	0.0000	1.0150	1.0000	1.0150	1.0000	0.0000
2	0.9707	1.9559	0.9707	1.0302	2.0150	0.5113	0.4963	0.4963
3	0.9563	2.9122	2.8833	1.0457	3.0452	0.3434	0.3284	0.9901
4	0.9422	3.8544	5.7098	1.0614	4.0909	0.2594	0.2444	1.4814
5	**0.9283**	**4.7826**	**9.4229**	**1.0773**	**5.1523**	**0.2091**	**0.1941**	**1.9702**
6	0.9145	5.6972	13.9956	1.0934	6.2296	0.1755	0.1605	2.4566
7	0.9010	6.5982	19.4018	1.1098	7.3230	0.1516	0.1366	2.9405
8	0.8877	7.4859	26.6157	1.1265	8.4328	0.1336	0.1186	3.4219
9	0.8746	8.3605	32.6125	1.1434	9.5593	0.1196	0.1046	3.9008
10	**0.8617**	**9.2222**	**40.3675**	**1.1605**	**10.7027**	**0.1084**	**0.0934**	**4.3772**
11	0.8489	10.0711	48.8568	1.1779	11.8633	0.0993	0.0843	4.8512
12	0.8364	10.9075	58.0571	1.1956	13.0412	0.0917	0.0767	5.3227
13	0.8240	11.7315	67.9454	1.2136	14.2368	0.0852	0.0702	5.7917
14	0.8118	12.5434	78.4994	1.2318	15.4504	0.0797	0.0647	6.2582
15	**0.7999**	**13.3432**	**89.6974**	**1.2502**	**16.6821**	**0.0749**	**0.0599**	**6.7223**
16	0.7880	14.1313	101.5178	1.2690	17.9324	0.0708	0.0558	7.1839
17	0.7764	14.9076	113.9400	1.2880	19.2014	0.0671	0.0521	7.6431
18	0.7649	15.6726	126.9435	1.3073	20.4894	0.0638	0.0488	8.0997
19	0.7536	16.4262	140.5084	1.3270	21.7967	0.0609	0.0459	8.5539
20	**0.7425**	**17.1686**	**154.6154**	**1.3469**	**23.1237**	**0.0582**	**0.0432**	**9.0057**
21	0.7315	17.9001	169.2453	1.3671	24.4705	0.0559	0.0409	9.4550
22	0.7207	18.6208	184.3798	1.3876	25.8376	0.0537	0.0387	9.9018
23	0.7100	19.3309	200.0006	1.4084	27.2251	0.0517	0.0367	10.3462
24	0.6995	20.0304	216.0901	1.4295	28.6335	0.0499	0.0349	10.7881
25	**0.6892**	**20.7196**	**232.6310**	**1.4509**	**30.0630**	**0.0483**	**0.0333**	**11.2276**
30	0.6398	24.0158	321.5310	1.5631	37.5387	0.0416	0.0266	13.3883
40	0.5513	29.9158	524.3568	1.8140	54.2679	0.0334	0.0184	17.5277
50	0.4750	34.9997	749.9636	2.1052	73.6828	0.0286	0.0136	21.4277
60	0.4093	39.3803	988.1674	2.4432	96.2147	0.0254	0.0104	25.0930
100	**0.2256**	**51.6247**	**1,937.4506**	**4.4320**	**228.8030**	**0.0194**	**0.0044**	**37.5295**

Factor Table - *i* = 2.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9804	0.9804	0.0000	1.0200	1.0000	1.0200	1.0000	0.0000
2	0.9612	1.9416	0.9612	1.0404	2.0200	0.5150	0.4950	0.4950
3	0.9423	2.8839	2.8458	1.0612	3.0604	0.3468	0.3268	0.9868
4	0.9238	3.8077	5.6173	1.0824	4.1216	0.2626	0.2426	1.4752
5	**0.9057**	**4.7135**	**9.2403**	**1.1041**	**5.2040**	**0.2122**	**0.1922**	**1.9604**
6	0.8880	5.6014	13.6801	1.1262	6.3081	0.1785	0.1585	2.4423
7	0.8706	6.4720	18.9035	1.1487	7.4343	0.1545	0.1345	2.9208
8	0.8535	7.3255	24.8779	1.1717	8.5830	0.1365	0.1165	3.3961
9	0.8368	8.1622	31.5720	1.1951	9.7546	0.1225	0.1025	3.8681
10	**0.8203**	**8.9826**	**38.9551**	**1.2190**	**10.9497**	**0.1113**	**0.0913**	**4.3367**
11	0.8043	9.7868	46.9977	1.2434	12.1687	0.1022	0.0822	4.8021
12	0.7885	10.5753	55.6712	1.2682	13.4121	0.0946	0.0746	5.2642
13	0.7730	11.3484	64.9475	1.2936	14.6803	0.0881	0.0681	5.7231
14	0.7579	12.1062	74.7999	1.3195	15.9739	0.0826	0.0626	6.1786
15	**0.7430**	**12.8493**	**85.2021**	**1.3459**	**17.2934**	**0.0778**	**0.0578**	**6.6309**
16	0.7284	13.5777	96.1288	1.3728	18.6393	0.0737	0.0537	7.0799
17	0.7142	14.2919	107.5554	1.4002	20.0121	0.0700	0.0500	7.5256
18	0.7002	14.9920	119.4581	1.4282	21.4123	0.0667	0.0467	7.9681
19	0.6864	15.6785	131.8139	1.4568	22.8406	0.0638	0.0438	8.4073
20	**0.6730**	**16.3514**	**144.6003**	**1.4859**	**24.2974**	**0.0612**	**0.0412**	**8.8433**
21	0.6598	17.0112	157.7959	1.5157	25.7833	0.0588	0.0388	9.2760
22	0.6468	17.6580	171.3795	1.5460	27.2990	0.0566	0.0366	9.7055
23	0.6342	18.2922	185.3309	1.5769	28.8450	0.0547	0.0347	10.1317
24	0.6217	18.9139	199.6305	1.6084	30.4219	0.0529	0.0329	10.5547
25	**0.6095**	**19.5235**	**214.2592**	**1.6406**	**32.0303**	**0.0512**	**0.0312**	**10.9745**
30	0.5521	22.3965	291.7164	1.8114	40.5681	0.0446	0.0246	13.0251
40	0.4529	27.3555	461.9931	2.2080	60.4020	0.0366	0.0166	16.8885
50	0.3715	31.4236	642.3606	2.6916	84.5794	0.0318	0.0118	20.4420
60	0.3048	34.7609	823.6975	3.2810	114.0515	0.0288	0.0088	23.6961
100	**0.1380**	**43.0984**	**1,464.7527**	**7.2446**	**312.2323**	**0.0232**	**0.0032**	**33.9863**

Interest Rate Tables

Factor Table - i = 4.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9615	0.9615	0.0000	1.0400	1.0000	1.0400	1.0000	0.0000
2	0.9246	1.8861	0.9246	1.0816	2.0400	0.5302	0.4902	0.4902
3	0.8890	2.7751	2.7025	1.1249	3.1216	0.3603	0.3203	0.9739
4	0.8548	3.6299	5.2670	1.1699	4.2465	0.2755	0.2355	1.4510
5	**0.8219**	**4.4518**	**8.5547**	**1.2167**	**5.4163**	**0.2246**	**0.1846**	**1.9216**
6	0.7903	5.2421	12.5062	1.2653	6.6330	0.1908	0.1508	2.3857
7	0.7599	6.0021	17.0657	1.3159	7.8983	0.1666	0.1266	2.8433
8	0.7307	6.7327	22.1806	1.3686	9.2142	0.1485	0.1085	3.2944
9	0.7026	7.4353	27.8013	1.4233	10.5828	0.1345	0.0945	3.7391
10	**0.6756**	**8.1109**	**33.8814**	**1.4802**	**12.0061**	**0.1233**	**0.0833**	**4.1773**
11	0.6496	8.7605	40.3772	1.5395	13.4864	0.1141	0.0741	4.6090
12	0.6246	9.3851	47.2477	1.6010	15.0258	0.1066	0.0666	5.0343
13	0.6006	9.9856	54.4546	1.6651	16.6268	0.1001	0.0601	5.4533
14	0.5775	10.5631	61.9618	1.7317	18.2919	0.0947	0.0547	5.8659
15	**0.5553**	**11.1184**	**69.7355**	**1.8009**	**20.0236**	**0.0899**	**0.0499**	**6.2721**
16	0.5339	11.6523	77.7441	1.8730	21.8245	0.0858	0.0458	6.6720
17	0.5134	12.1657	85.9581	1.9479	23.6975	0.0822	0.0422	7.0656
18	0.4936	12.6593	94.3498	2.0258	25.6454	0.0790	0.0390	7.4530
19	0.4746	13.1339	102.8933	2.1068	27.6712	0.0761	0.0361	7.8342
20	**0.4564**	**13.5903**	**111.5647**	**2.1911**	**29.7781**	**0.0736**	**0.0336**	**8.2091**
21	0.4388	14.0292	120.3414	2.2788	31.9692	0.0713	0.0313	8.5779
22	0.4220	14.4511	129.2024	2.3699	34.2480	0.0692	0.0292	8.9407
23	0.4057	14.8568	138.1284	2.4647	36.6179	0.0673	0.0273	9.2973
24	0.3901	15.2470	147.1012	2.5633	39.0826	0.0656	0.0256	9.6479
25	**0.3751**	**15.6221**	**156.1040**	**2.6658**	**41.6459**	**0.0640**	**0.0240**	**9.9925**
30	0.3083	17.2920	201.0618	3.2434	56.0849	0.0578	0.0178	11.6274
40	0.2083	19.7928	286.5303	4.8010	95.0255	0.0505	0.0105	14.4765
50	0.1407	21.4822	361.1638	7.1067	152.6671	0.0466	0.0066	16.8122
60	0.0951	22.6235	422.9966	10.5196	237.9907	0.0442	0.0042	18.6972
100	**0.0198**	**24.5050**	**563.1249**	**50.5049**	**1,237.6237**	**0.0408**	**0.0008**	**22.9800**

Factor Table - i = 6.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9434	0.9434	0.0000	1.0600	1.0000	1.0600	1.0000	0.0000
2	0.8900	1.8334	0.8900	1.1236	2.0600	0.5454	0.4854	0.4854
3	0.8396	2.6730	2.5692	1.1910	3.1836	0.3741	0.3141	0.9612
4	0.7921	3.4651	4.9455	1.2625	4.3746	0.2886	0.2286	1.4272
5	**0.7473**	**4.2124**	**7.9345**	**1.3382**	**5.6371**	**0.2374**	**0.1774**	**1.8836**
6	0.7050	4.9173	11.4594	1.4185	6.9753	0.2034	0.1434	2.3304
7	0.6651	5.5824	15.4497	1.5036	8.3938	0.1791	0.1191	2.7676
8	0.6274	6.2098	19.8416	1.5938	9.8975	0.1610	0.1010	3.1952
9	0.5919	6.8017	24.5768	1.6895	11.4913	0.1470	0.0870	3.6133
10	**0.5584**	**7.3601**	**29.6023**	**1.7908**	**13.1808**	**0.1359**	**0.0759**	**4.0220**
11	0.5268	7.8869	34.8702	1.8983	14.9716	0.1268	0.0668	4.4213
12	0.4970	8.3838	40.3369	2.0122	16.8699	0.1193	0.0593	4.8113
13	0.4688	8.8527	45.9629	2.1329	18.8821	0.1130	0.0530	5.1920
14	0.4423	9.2950	51.7128	2.2609	21.0151	0.1076	0.0476	5.5635
15	**0.4173**	**9.7122**	**57.5546**	**2.3966**	**23.2760**	**0.1030**	**0.0430**	**5.9260**
16	0.3936	10.1059	63.4592	2.5404	25.6725	0.0990	0.0390	6.2794
17	0.3714	10.4773	69.4011	2.6928	28.2129	0.0954	0.0354	6.6240
18	0.3505	10.8276	75.3569	2.8543	30.9057	0.0924	0.0324	6.9597
19	0.3305	11.1581	81.3062	3.0256	33.7600	0.0896	0.0296	7.2867
20	**0.3118**	**11.4699**	**87.2304**	**3.2071**	**36.7856**	**0.0872**	**0.0272**	**7.6051**
21	0.2942	11.7641	93.1136	3.3996	39.9927	0.0850	0.0250	7.9151
22	0.2775	12.0416	98.9412	3.6035	43.3923	0.0830	0.0230	8.2166
23	0.2618	12.3034	104.7007	3.8197	46.9958	0.0813	0.0213	8.5099
24	0.2470	12.5504	110.3812	4.0489	50.8156	0.0797	0.0197	8.7951
25	**0.2330**	**12.7834**	**115.9732**	**4.2919**	**54.8645**	**0.0782**	**0.0182**	**9.0722**
30	0.1741	13.7648	142.3588	5.7435	79.0582	0.0726	0.0126	10.3422
40	0.0972	15.0463	185.9568	10.2857	154.7620	0.0665	0.0065	12.3590
50	0.0543	15.7619	217.4574	18.4202	290.3359	0.0634	0.0034	13.7964
60	0.0303	16.1614	239.0428	32.9877	533.1282	0.0619	0.0019	14.7909
100	**0.0029**	**16.6175**	**272.0471**	**339.3021**	**5,638.3681**	**0.0602**	**0.0002**	**16.3711**

Interest Rate Tables

Factor Table - *i* = 8.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9259	0.9259	0.0000	1.0800	1.0000	1.0800	1.0000	0.0000
2	0.8573	1.7833	0.8573	1.1664	2.0800	0.5608	0.4808	0.4808
3	0.7938	2.5771	2.4450	1.2597	3.2464	0.3880	0.3080	0.9487
4	0.7350	3.3121	4.6501	1.3605	4.5061	0.3019	0.2219	1.4040
5	**0.6806**	**3.9927**	**7.3724**	**1.4693**	**5.8666**	**0.2505**	**0.1705**	**1.8465**
6	0.6302	4.6229	10.5233	1.5869	7.3359	0.2163	0.1363	2.2763
7	0.5835	5.2064	14.0242	1.7138	8.9228	0.1921	0.1121	2.6937
8	0.5403	5.7466	17.8061	1.8509	10.6366	0.1740	0.0940	3.0985
9	0.5002	6.2469	21.8081	1.9990	12.4876	0.1601	0.0801	3.4910
10	**0.4632**	**6.7101**	**25.9768**	**2.1589**	**14.4866**	**0.1490**	**0.0690**	**3.8713**
11	0.4289	7.1390	30.2657	2.3316	16.6455	0.1401	0.0601	4.2395
12	0.3971	7.5361	34.6339	2.5182	18.9771	0.1327	0.0527	4.5957
13	0.3677	7.9038	39.0463	2.7196	21.4953	0.1265	0.0465	4.9402
14	0.3405	8.2442	43.4723	2.9372	24.2149	0.1213	0.0413	5.2731
15	**0.3152**	**8.5595**	**47.8857**	**3.1722**	**27.1521**	**0.1168**	**0.0368**	**5.5945**
16	0.2919	8.8514	52.2640	3.4259	30.3243	0.1130	0.0330	5.9046
17	0.2703	9.1216	56.5883	3.7000	33.7502	0.1096	0.0296	6.2037
18	0.2502	9.3719	60.8426	3.9960	37.4502	0.1067	0.0267	6.4920
19	0.2317	9.6036	65.0134	4.3157	41.4463	0.1041	0.0241	6.7697
20	**0.2145**	**9.8181**	**69.0898**	**4.6610**	**45.7620**	**0.1019**	**0.0219**	**7.0369**
21	0.1987	10.0168	73.0629	5.0338	50.4229	0.0998	0.0198	7.2940
22	0.1839	10.2007	76.9257	5.4365	55.4568	0.0980	0.0180	7.5412
23	0.1703	10.3711	80.6726	5.8715	60.8933	0.0964	0.0164	7.7786
24	0.1577	10.5288	84.2997	6.3412	66.7648	0.0950	0.0150	8.0066
25	**0.1460**	**10.6748**	**87.8041**	**6.8485**	**73.1059**	**0.0937**	**0.0137**	**8.2254**
30	0.0994	11.2578	103.4558	10.0627	113.2832	0.0888	0.0088	9.1897
40	0.0460	11.9246	126.0422	21.7245	259.0565	0.0839	0.0039	10.5699
50	0.0213	12.2335	139.5928	46.9016	573.7702	0.0817	0.0017	11.4107
60	0.0099	12.3766	147.3000	101.2571	1,253.2133	0.0808	0.0008	11.9015
100	**0.0005**	**12.4943**	**155.6107**	**2,199.7613**	**27,484.5157**	**0.0800**		**12.4545**

Factor Table - *i* = 10.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.9091	0.9091	0.0000	1.1000	1.0000	1.1000	1.0000	0.0000
2	0.8264	1.7355	0.8264	1.2100	2.1000	0.5762	0.4762	0.4762
3	0.7513	2.4869	2.3291	1.3310	3.3100	0.4021	0.3021	0.9366
4	0.6830	3.1699	4.3781	1.4641	4.6410	0.3155	0.2155	1.3812
5	**0.6209**	**3.7908**	**6.8618**	**1.6105**	**6.1051**	**0.2638**	**0.1638**	**1.8101**
6	0.5645	4.3553	9.6842	1.7716	7.7156	0.2296	0.1296	2.2236
7	0.5132	4.8684	12.7631	1.9487	9.4872	0.2054	0.1054	2.6216
8	0.4665	5.3349	16.0287	2.1436	11.4359	0.1874	0.0874	3.0045
9	0.4241	5.7590	19.4215	2.3579	13.5735	0.1736	0.0736	3.3724
10	**0.3855**	**6.1446**	**22.8913**	**2.5937**	**15.9374**	**0.1627**	**0.0627**	**3.7255**
11	0.3505	6.4951	26.3962	2.8531	18.5312	0.1540	0.0540	4.0641
12	0.3186	6.8137	29.9012	3.1384	21.3843	0.1468	0.0468	4.3884
13	0.2897	7.1034	33.3772	3.4523	24.5227	0.1408	0.0408	4.6988
14	0.2633	7.3667	36.8005	3.7975	27.9750	0.1357	0.0357	4.9955
15	**0.2394**	**7.6061**	**40.1520**	**4.1772**	**31.7725**	**0.1315**	**0.0315**	**5.2789**
16	0.2176	7.8237	43.4164	4.5950	35.9497	0.1278	0.0278	5.5493
17	0.1978	8.0216	46.5819	5.0545	40.5447	0.1247	0.0247	5.8071
18	0.1799	8.2014	49.6395	5.5599	45.5992	0.1219	0.0219	6.0526
19	0.1635	8.3649	52.5827	6.1159	51.1591	0.1195	0.0195	6.2861
20	**0.1486**	**8.5136**	**55.4069**	**6.7275**	**57.2750**	**0.1175**	**0.0175**	**6.5081**
21	0.1351	8.6487	58.1095	7.4002	64.0025	0.1156	0.0156	6.7189
22	0.1228	8.7715	60.6893	8.1403	71.4027	0.1140	0.0140	6.9189
23	0.1117	8.8832	63.1462	8.9543	79.5430	0.1126	0.0126	7.1085
24	0.1015	8.9847	65.4813	9.8497	88.4973	0.1113	0.0113	7.2881
25	**0.0923**	**9.0770**	**67.6964**	**10.8347**	**98.3471**	**0.1102**	**0.0102**	**7.4580**
30	0.0573	9.4269	77.0766	17.4494	164.4940	0.1061	0.0061	8.1762
40	0.0221	9.7791	88.9525	45.2593	442.5926	0.1023	0.0023	9.0962
50	0.0085	9.9148	94.8889	117.3909	1,163.9085	0.1009	0.0009	9.5704
60	0.0033	9.9672	97.7010	304.4816	3,034.8164	0.1003	0.0003	9.8023
100	**0.0001**	**9.9993**	**99.9202**	**13,780.6123**	**137,796.1234**	**0.1000**		**9.9927**

Interest Rate Tables

Factor Table - *i* = 12.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.8929	0.8929	0.0000	1.1200	1.0000	1.1200	1.0000	0.0000
2	0.7972	1.6901	0.7972	1.2544	2.1200	0.5917	0.4717	0.4717
3	0.7118	2.4018	2.2208	1.4049	3.3744	0.4163	0.2963	0.9246
4	0.6355	3.0373	4.1273	1.5735	4.7793	0.3292	0.2092	1.3589
5	**0.5674**	**3.6048**	**6.3970**	**1.7623**	**6.3528**	**0.2774**	**0.1574**	**1.7746**
6	0.5066	4.1114	8.9302	1.9738	8.1152	0.2432	0.1232	2.1720
7	0.4523	4.5638	11.6443	2.2107	10.0890	0.2191	0.0991	2.5515
8	0.4039	4.9676	14.4714	2.4760	12.2997	0.2013	0.0813	2.9131
9	0.3606	5.3282	17.3563	2.7731	14.7757	0.1877	0.0677	3.2574
10	**0.3220**	**5.6502**	**20.2541**	**3.1058**	**17.5487**	**0.1770**	**0.0570**	**3.5847**
11	0.2875	5.9377	23.1288	3.4785	20.6546	0.1684	0.0484	3.8953
12	0.2567	6.1944	25.9523	3.8960	24.1331	0.1614	0.0414	4.1897
13	0.2292	6.4235	28.7024	4.3635	28.0291	0.1557	0.0357	4.4683
14	0.2046	6.6282	31.3624	4.8871	32.3926	0.1509	0.0309	4.7317
15	**0.1827**	**6.8109**	**33.9202**	**5.4736**	**37.2797**	**0.1468**	**0.0268**	**4.9803**
16	0.1631	6.9740	36.3670	6.1304	42.7533	0.1434	0.0234	5.2147
17	0.1456	7.1196	38.6973	6.8660	48.8837	0.1405	0.0205	5.4353
18	0.1300	7.2497	40.9080	7.6900	55.7497	0.1379	0.0179	5.6427
19	0.1161	7.3658	42.9979	8.6128	63.4397	0.1358	0.0158	5.8375
20	**0.1037**	**7.4694**	**44.9676**	**9.6463**	**72.0524**	**0.1339**	**0.0139**	**6.0202**
21	0.0926	7.5620	46.8188	10.8038	81.6987	0.1322	0.0122	6.1913
22	0.0826	7.6446	48.5543	12.1003	92.5026	0.1308	0.0108	6.3514
23	0.0738	7.7184	50.1776	13.5523	104.6029	0.1296	0.0096	6.5010
24	0.0659	7.7843	51.6929	15.1786	118.1552	0.1285	0.0085	6.6406
25	**0.0588**	**7.8431**	**53.1046**	**17.0001**	**133.3339**	**0.1275**	**0.0075**	**6.7708**
30	0.0334	8.0552	58.7821	29.9599	241.3327	0.1241	0.0041	7.2974
40	0.0107	8.2438	65.1159	93.0510	767.0914	0.1213	0.0013	7.8988
50	0.0035	8.3045	67.7624	289.0022	2,400.0182	0.1204	0.0004	8.1597
60	0.0011	8.3240	68.8100	897.5969	7,471.6411	0.1201	0.0001	8.2664
100		**8.3332**	**69.4336**	**83,522.2657**	**696,010.5477**	**0.1200**		**8.3321**

Factor Table - *i* = 18.00%

n	*P/F*	*P/A*	*P/G*	*F/P*	*F/A*	*A/P*	*A/F*	*A/G*
1	0.8475	0.8475	0.0000	1.1800	1.0000	1.1800	1.0000	0.0000
2	0.7182	1.5656	0.7182	1.3924	2.1800	0.6387	0.4587	0.4587
3	0.6086	2.1743	1.9354	1.6430	3.5724	0.4599	0.2799	0.8902
4	0.5158	2.6901	3.4828	1.9388	5.2154	0.3717	0.1917	1.2947
5	**0.4371**	**3.1272**	**5.2312**	**2.2878**	**7.1542**	**0.3198**	**0.1398**	**1.6728**
6	0.3704	3.4976	7.0834	2.6996	9.4423	0.2859	0.1059	2.0252
7	0.3139	3.8115	8.9670	3.1855	12.1415	0.2624	0.0824	2.3526
8	0.2660	4.0776	10.8292	3.7589	15.3270	0.2452	0.0652	2.6558
9	0.2255	4.3030	12.6329	4.4355	19.0859	0.2324	0.0524	2.9358
10	**0.1911**	**4.4941**	**14.3525**	**5.2338**	**23.5213**	**0.2225**	**0.0425**	**3.1936**
11	0.1619	4.6560	15.9716	6.1759	28.7551	0.2148	0.0348	3.4303
12	0.1372	4.7932	17.4811	7.2876	34.9311	0.2086	0.0286	3.6470
13	0.1163	4.9095	18.8765	8.5994	42.2187	0.2037	0.0237	3.8449
14	0.0985	5.0081	20.1576	10.1472	50.8180	0.1997	0.0197	4.0250
15	**0.0835**	**5.0916**	**21.3269**	**11.9737**	**60.9653**	**0.1964**	**0.0164**	**4.1887**
16	0.0708	5.1624	22.3885	14.1290	72.9390	0.1937	0.0137	4.3369
17	0.0600	5.2223	23.3482	16.6722	87.0680	0.1915	0.0115	4.4708
18	0.0508	5.2732	24.2123	19.6731	103.7403	0.1896	0.0096	4.5916
19	0.0431	5.3162	24.9877	23.2144	123.4135	0.1881	0.0081	4.7003
20	**0.0365**	**5.3527**	**25.6813**	**27.3930**	**146.6280**	**0.1868**	**0.0068**	**4.7978**
21	0.0309	5.3837	26.3000	32.3238	174.0210	0.1857	0.0057	4.8851
22	0.0262	5.4099	26.8506	38.1421	206.3448	0.1848	0.0048	4.9632
23	0.0222	5.4321	27.3394	45.0076	244.4868	0.1841	0.0041	5.0329
24	0.0188	5.4509	27.7725	53.1090	289.4944	0.1835	0.0035	5.0950
25	**0.0159**	**5.4669**	**28.1555**	**62.6686**	**342.6035**	**0.1829**	**0.0029**	**5.1502**
30	0.0070	5.5168	29.4864	143.3706	790.9480	0.1813	0.0013	5.3448
40	0.0013	5.5482	30.5269	750.3783	4,163.2130	0.1802	0.0002	5.5022
50	0.0003	5.5541	30.7856	3,927.3569	21,813.0937	0.1800		5.5428
60	0.0001	5.5553	30.8465	20,555.1400	114,189.6665	0.1800		5.5526
100		**5.5556**	**30.8642**	**15,424,131.91**	**85,689,616.17**	**0.1800**		**5.5555**

Electrical and Computer Engineering

Units

The basic electrical units are coulombs for charge, volts for voltage, amperes for current, ohms for resistance and impedance, and siemens for conductance and admittance.

Electrostatics

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon r^2}\mathbf{a}_{r12}$$

where

$\mathbf{F}_2$ = force on charge 2 due to charge 1
Q_i = the ith point charge
r = distance between charges 1 and 2
$\mathbf{a}_{r12}$ = a unit vector directed from 1 to 2
ε = permittivity of the medium

For free space or air:

$\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}$ farads/meter

Electrostatic Fields

Electric field intensity **E** (volts/meter) at point 2 due to a point charge Q_1 at point 1 is

$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon r^2}\mathbf{a}_{r12}$$

For a line charge of density ρ_L coulombs/meter on the z-axis, the radial electric field is

$$\mathbf{E}_L = \frac{\rho_L}{2\pi\varepsilon r}\mathbf{a}_r$$

For a sheet charge of density ρ_s coulombs/meter^2 in the x-y plane:

$$\mathbf{E}_s = \frac{\rho_s}{2\varepsilon}\mathbf{a}_z, z > 0$$

Gauss' law states that the integral of the electric flux density $\mathbf{D} = \varepsilon\mathbf{E}$ over a closed surface is equal to the charge enclosed or

$$Q_{encl} = \oiint_s \varepsilon\mathbf{E} \cdot d\mathbf{S}$$

The force on a point charge Q in an electric field with intensity **E** is $\mathbf{F} = Q\mathbf{E}$.

The work done by an external agent in moving a charge Q in an electric field from point p_1 to point p_2 is

$$W = -Q\int_{p_1}^{p_2}\mathbf{E} \cdot d\mathbf{l}$$

The energy W_E stored in an electric field **E** is

$$W_E = (1/2)\iiint_V \varepsilon|\mathbf{E}|^2 dV$$

Voltage

The potential difference V between two points is the work per unit charge required to move the charge between the points.

For two parallel plates with potential difference V, separated by distance d, the strength of the E field between the plates is

$$E = \frac{V}{d}$$

directed from the + plate to the – plate.

Current

Electric current $i(t)$ through a surface is defined as the rate of charge transport through that surface or

$i(t) = dq(t)/dt$, which is a function of time t

since $q(t)$ denotes instantaneous charge.

A constant current $i(t)$ is written as I, and the vector current density in amperes/m^2 is defined as $\mathbf{J}$.

Magnetic Fields

For a current-carrying wire on the z-axis

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{I\mathbf{a}_\phi}{2\pi r}$$

where

$\mathbf{H}$ = magnetic field strength (amperes/meter)
$\mathbf{B}$ = magnetic flux density (tesla)
$\mathbf{a}_\phi$ = unit vector in positive ϕ direction in cylindrical coordinates
I = current
μ = permeability of the medium

For air: $\mu = \mu_o = 4\pi \times 10^{-7}$ H/m

Force on a current-carrying conductor in a uniform magnetic field is

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}$$

where $\mathbf{L}$= length vector of a conductor

The energy stored W_H in a magnetic field $\mathbf{H}$ is

$$W_H = (1/2) \iiint_V \mu |\mathbf{H}|^2 dv$$

Induced Voltage

Faraday's Law states for a coil of N turns enclosing flux ϕ:

$$v = -N\, d\phi/dt$$

where

v = induced voltage
ϕ = average flux (webers) enclosed by each turn
$\phi = \int_S \mathbf{B} \cdot d\mathbf{S}$

Resistivity

For a conductor of length L, electrical resistivity ρ, and cross-sectional area A, the resistance is

$$R = \frac{\rho L}{A}$$

For metallic conductors, the resistivity and resistance vary linearly with changes in temperature according to the following relationships:

$$\rho = \rho_0\left[1 + \alpha\left(T - T_0\right)\right]$$

and

$$R = R_0\left[1 + \alpha\left(T - T_0\right)\right]$$

where

ρ_0 = resistivity at T_0
R_0 = resistance at T_0
α = temperature coefficient
Ohm's Law: $V = IR$; $v(t) = i(t)\,R$

Resistors in Series and Parallel

For series connections, the current in all resistors is the same and the equivalent resistance for n resistors in series is

$$R_S = R_1 + R_2 + \ldots + R_n$$

For parallel connections of resistors, the voltage drop across each resistor is the same and the equivalent resistance for n resistors in parallel is

$$R_P = 1/(1/R_1 + 1/R_2 + \ldots + 1/R_n)$$

For two resistors R_1 and R_2 in parallel

$$R_P = \frac{R_1 R_2}{R_1 + R_2}$$

Power Absorbed by a Resistive Element

$$P = VI = \frac{V^2}{R} = I^2 R$$

Kirchhoff's Laws

Kirchhoff's voltage law for a closed path is expressed by

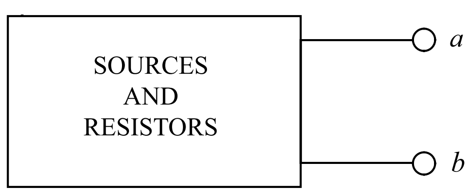
$$\Sigma\, V_{\text{rises}} = \Sigma\, V_{\text{drops}}$$

Kirchhoff's current law for a closed surface is

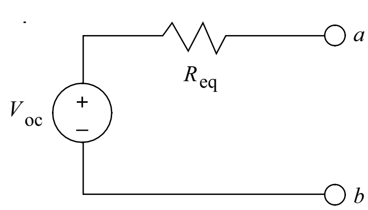
$$\Sigma\, I_{\text{in}} = \Sigma\, I_{\text{out}}$$

Source Equivalents

For an arbitrary circuit



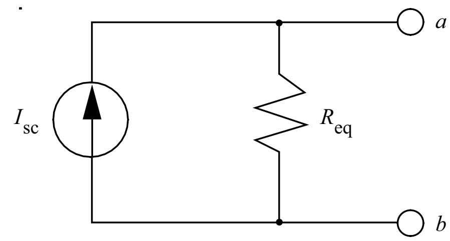
The Thévenin equivalent is



$$R_{\text{eq}} = \frac{V_{\text{oc}}}{I_{\text{sc}}}$$

The open circuit voltage V_{oc} is $V_a - V_b$, and the short circuit current is I_{sc} from a to b.

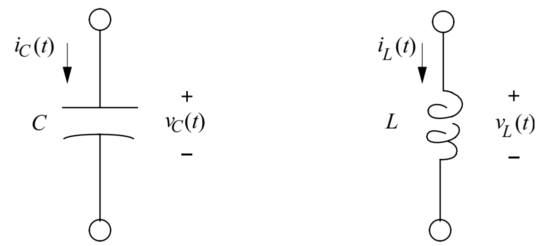
The Norton equivalent circuit is



where I_{sc} and R_{eq} are defined above.

A load resistor R_L connected across terminals a and b will draw maximum power when $R_L = R_{eq}$.

Capacitors and Inductors



The charge $q_C(t)$ and voltage $v_C(t)$ relationship for a capacitor C in farads is

$$C = q_C(t)/v_C(t) \quad \text{or} \quad q_C(t) = Cv_C(t)$$

A parallel plate capacitor of area A with plates separated a distance d by an insulator with a permittivity ε has a capacitance

$$C = \frac{\varepsilon A}{d}$$

ε is often given as $\varepsilon = \varepsilon_r(\varepsilon_o)$ where ε_r is the relative permittivity or dielectric constant and $\varepsilon_o = 8.85 \times 10^{-12}$ F/m.

The current-voltage relationships for a capacitor are

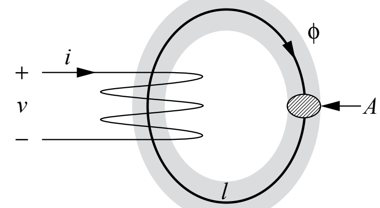
$$v_C(t) = v_C(0) + \frac{1}{C}\int_0^t i_C(\tau)\,d\tau$$

and $i_C(t) = C\,(dv_C/dt)$

The energy stored in a capacitor is expressed in joules and given by

$$\text{Energy} = Cv_C^2/2 = q_C^2/2C = q_Cv_C/2$$

Inductor



The inductance L (henrys) of a coil of N turns wound on a core with cross-sectional area A (m^2), permeability μ and flux ϕ with a mean path of l (m) is given as:

$$L = N^2 \mu A/l = N^2/\Re$$

$$N\phi = Li$$

where $\Re$ = reluctance = $l/\mu A$ (H^{-1}).

μ is sometimes given as $\mu = \mu_r \cdot \mu_o$ where μ_r is the relative permeability and $\mu_o = 4\pi \times 10^{-7}$ H/m.

Using Faraday's law, the voltage-current relations for an inductor are

$$v_L(t) = L\,(di_L/dt)$$

$$i_L(t) = i_L(0) + \frac{1}{L}\int_0^t v_L(\tau)\,d\tau$$

where

v_L = inductor voltage
L = inductance (henrys)
i_L = inductor current (amperes)

The energy stored in an inductor is expressed in joules and given by

$$\text{Energy} = Li_L^2/2$$

Capacitors and Inductors in Parallel and Series

Capacitors in Parallel

$$C_P = C_1 + C_2 + \ldots + C_n$$

Capacitors in Series

$$C_S = \frac{1}{1/C_1 + 1/C_2 + \ldots + 1/C_n}$$

Inductors in Parallel

$$L_P = \frac{1}{1/L_1 + 1/L_2 + \ldots + 1/L_n}$$

Inductors in Series

$$L_S = L_1 + L_2 + \ldots + L_n$$

AC Circuits

For a sinusoidal voltage or current of frequency f (Hz) and period T (seconds),

$$f = 1/T = \omega/(2\pi)$$

where ω = the angular frequency in radians/*s*

Average Value

For a periodic waveform (either voltage or current) with period T,

$$X_{\text{ave}} = (1/T)\int_0^T x(t)\,dt$$

The average value of a full-wave rectified sinusoid is

$$X_{\text{ave}} = (2X_{\max})/\pi$$

and half this for half-wave rectification, where

$X_{\max}$ = the peak amplitude of the sinusoid.

Effective or RMS Values

For a periodic waveform with period T, the rms or effective value is

$$X_{\text{eff}} = X_{\text{rms}} = \left[(1/T)\int_0^T x^2(t)\,dt\right]^{1/2}$$

For a sinusoidal waveform and full-wave rectified sine wave,

$$X_{\text{eff}} = X_{\text{rms}} = X_{\max}/\sqrt{2}$$

For a half-wave rectified sine wave,

$$X_{\text{eff}} = X_{\text{rms}} = X_{\max}/2$$

For a periodic signal,

$$X_{\text{rms}} = \sqrt{X_{\text{dc}}^2 + \sum_{n=1}^{\infty} X_n^2}$$

where

X_{dc} = dc component of $x(t)$

X_n = rms value of the nth harmonic

Sine-Cosine Relations and Trigonometric Identities

$$\cos(\omega t) = \sin(\omega t + \pi/2) = -\sin(\omega t - \pi/2)$$

$$\sin(\omega t) = \cos(\omega t - \pi/2) = -\cos(\omega t + \pi/2)$$

Other trigonometric identities for sinusoids are given in the section on Trigonometry.

Phasor Transforms of Sinusoids

$$P\left[V_{\max}\cos(\omega t + \phi)\right] = V_{\text{rms}}\angle\phi = \mathbf{V}$$

$$P\left[I_{\max}\cos(\omega t + \theta)\right] = I_{\text{rms}}\angle\theta = \mathbf{I}$$

For a circuit element, the impedance is defined as the ratio of phasor voltage to phasor current.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R + jX$$

where

R = resistance

X = reactance

The admittance is defined as the ratio of phasor current to phasor voltage or the inverse of impedance.

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}} = G + jB$$

where

G = conductance

B = susceptance

Circuit Element	Impedance	Resistance	Reactance	Admittance	Conductance	Susceptance
Resistor	R	R	0	$\frac{1}{R}$	$\frac{1}{R}$	0
Capacitor	$\frac{1}{j\omega C}$	0	$-\frac{1}{\omega C}$	$j\omega C$	0	ωC
Inductor	$j\omega L$	0	ωL	$\frac{1}{j\omega L}$	0	$-\frac{1}{\omega L}$

Impedances in series combine additively while those in parallel combine as the reciprocal of the sum of reciprocals, just as in the case of resistors.

Admittances in series combine as the reciprocal of the sum of reciprocals while those in parallel combine additively.

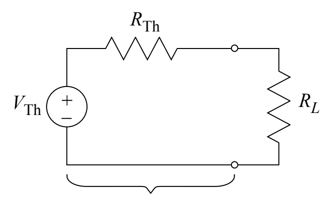
Maximum Power-Transfer Theorem

DC Circuits

Maximum power transfer to the load R_L occurs when $R_L = R_{\text{Th}}$.

$$P_{\max} = \frac{V_{\text{Th}}^2}{4\,R_{\text{Th}}}$$

$$\text{Efficiency: } \eta = \frac{P_L}{P_S} = \frac{R_L}{R_L + R_{\text{Th}}}$$

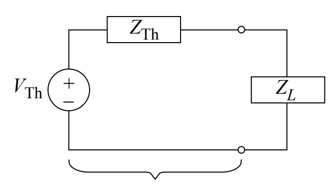


Thevenin Equivalent Circuit

AC Circuits

In an ac circuit maximum power transfer to the load impedance Z_L occurs when the load impedance equals the complex conjugate of the Thevenin equivalent impedance:

$$Z_L = Z_{\text{Th}}^*$$

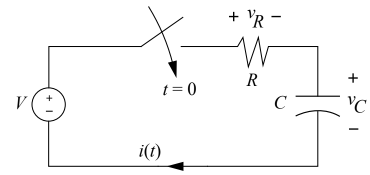


Thevenin Equivalent Circuit

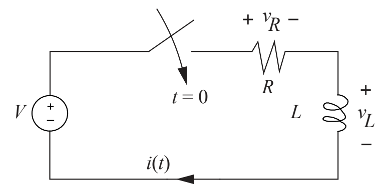
*If the load is purely resistive (R_L) then for maximum power transfer $R_L = |Z_{\text{Th}}|$

RC and RL Transients

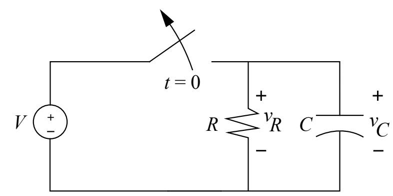
$$t \geq 0; v_C(t) = v_C(0)e^{-t/RC} + V\left(1 - e^{-t/RC}\right)$$
$$i(t) = \left\{\left[V - v_C(0)\right]/R\right\}e^{-t/RC}$$
$$v_R(t) = i(t)R = \left[V - v_C(0)\right]e^{-t/RC}$$



$$t \geq 0; i(t) = i(0)e^{-Rt/L} + \frac{V}{R}\left(1 - e^{-Rt/L}\right)$$
$$v_R(t) = i(t)R = i(0)Re^{-Rt/L} + V\left(1 - e^{-Rt/L}\right)$$
$$v_L(t) = L(di/dt) = -i(0)Re^{-Rt/L} + Ve^{-Rt/L}$$



$$t \geq 0; v_C(t) = v_R(t) = Ve^{-t/RC}$$
$$i_R(t) = -i_C(t) = \frac{V}{R}e^{-t/RC}$$



where $v(0)$ and $i(0)$ denote the initial conditions and the parameters RC and L/R are termed the respective circuit time constants.

Resonance

The radian resonant frequency for both parallel and series resonance situations is

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2\pi f_0 \text{ rad/s}$$

Series Resonance

$$\omega_0 L = \frac{1}{\omega_0 C}$$
$$Z = R \text{ at resonance}$$
$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$
$$BW = \frac{\omega_0}{Q} \text{ rad/s}$$

Parallel Resonance

$$\omega_0 L = \frac{1}{\omega_0 C}$$
$$Z = R \text{ at resonance}$$
$$Q = \omega_0 RC = \frac{R}{\omega_0 L}$$
$$BW = \frac{\omega_0}{Q} \text{ rad/s}$$

AC Power

Complex Power

Real power P (watts) is defined by

$$\begin{aligned} P &= (½)V_{\text{max}} I_{\text{max}} \cos\theta \\ &= V_{\text{rms}} I_{\text{rms}} \cos\theta \end{aligned}$$

where θ is the angle measured from **V** to **I**. If **I** leads **V**, then the power factor (*pf*),

$$pf = \cos\theta$$

is said to be a leading *pf*.

If **I** lags **V**, then the power factor (*pf*) is said to be a lagging *pf*.

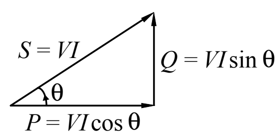
Reactive power Q (vars) is defined by

$$\begin{aligned} Q &= (½)V_{\text{max}} I_{\text{max}} \sin\theta \\ &= V_{\text{rms}} I_{\text{rms}} \sin\theta \end{aligned}$$

Complex power **S** (volt-amperes) is defined by

$$\mathbf{S} = \mathbf{VI}^* = P + jQ,$$

where $\mathbf{I}^*$ is the complex conjugate of the phasor current.



Complex Power Triangle (Inductive Load)

For resistors, θ = 0, so the real power is

$$P = V_{\text{rms}}\, I_{\text{rms}} = V^2_{\text{rms}}/R = I^2_{\text{rms}}R$$

Balanced Three-Phase (3-ϕ) Systems

The 3-phase line-phase relations are

for a delta	for a wye
$V_L = V_P$	$V_L = \sqrt{3}\,V_P = \sqrt{3}\,V_{LN}$
$I_L = \sqrt{3}\,I_P$	$I_L = I_P$

where subscripts L and P denote line and phase respectively.

A balanced 3-ϕ, delta-connected load impedance can be converted to an equivalent wye-connected load impedance using the following relationship

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_\text{Y}$$

The following formulas can be used to determine 3-ϕ power for balanced systems.

$$\begin{aligned} &\text{S} = P + jQ \\ &|\text{S}| = 3V_P I_P = \sqrt{3}\,V_L I_L \\ &\text{S} = 3V_P I_P^* = \sqrt{3}\,V_L I_L\left(\cos\theta_P + j\sin\theta_P\right) \end{aligned}$$

For balanced 3-ϕ, wye- and delta-connected loads

$$\mathbf{S} = \frac{V_L^2}{Z_Y^*} \qquad \mathbf{S} = 3\frac{V_L^2}{Z_\Delta^*}$$

where

$\mathbf{S}$ = total 3-ϕ complex power (VA)
$|\mathbf{S}|$ = total 3-ϕ apparent power (VA)
P = total 3-ϕ real power (W)
Q = total 3-ϕ reactive power (var)
θ_P = power factor angle of each phase
V_L = rms value of the line-to-line voltage
V_{LN} = rms value of the line-to-neutral voltage
I_L = rms value of the line current
I_P = rms value of the phase current

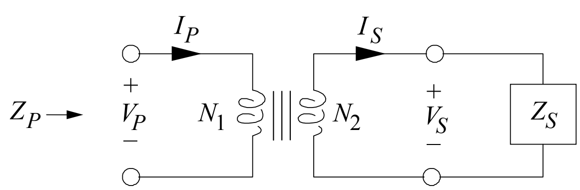
For a 3-ϕ, wye-connected source or load with line-to-neutral voltages and a positive phase sequence

$$\mathbf{V}_{an} = V_P\angle 0°$$
$$\mathbf{V}_{bn} = V_P\angle -120°$$
$$\mathbf{V}_{cn} = V_P\angle 120°$$

The corresponding line-to-line voltages are

$$\mathbf{V}_{ab} = \sqrt{3}\, V_P\angle 30°$$
$$\mathbf{V}_{bc} = \sqrt{3}\, V_P\angle -90°$$
$$\mathbf{V}_{ca} = \sqrt{3}\, V_P\angle 150°$$

Transformers (Ideal)



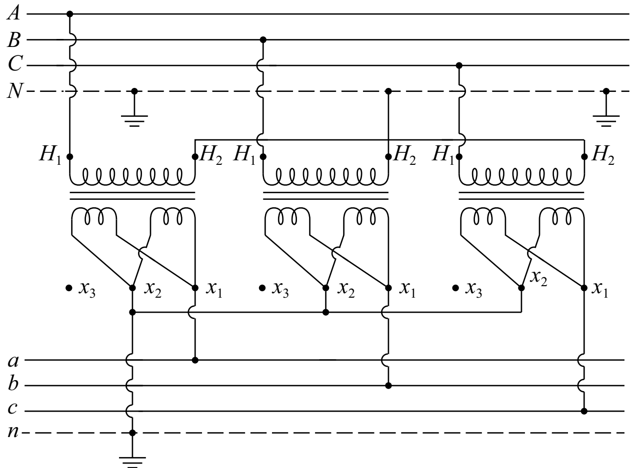
Turns Ratio

$$a = N_1/N_2$$
$$a = \left|\frac{\mathbf{V}_P}{\mathbf{V}_S}\right| = \left|\frac{\mathbf{I}_S}{\mathbf{I}_P}\right|$$

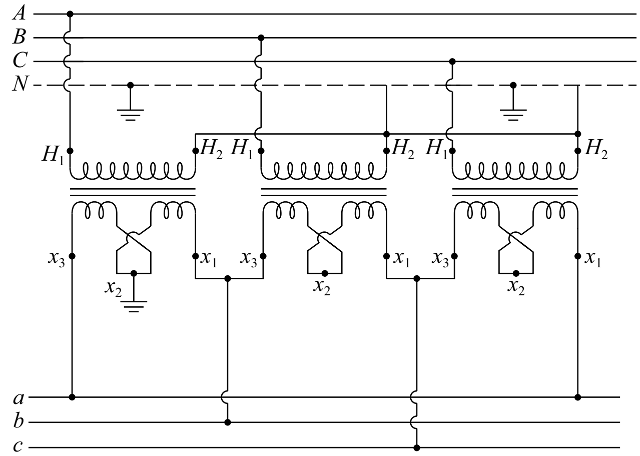
The impedance seen at the input is

$$\mathbf{Z}_P = a^2\mathbf{Z}_S$$

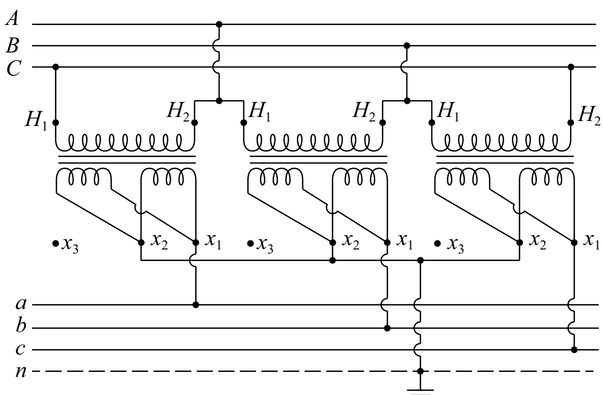
Three-Phase Transformer Connection Diagrams



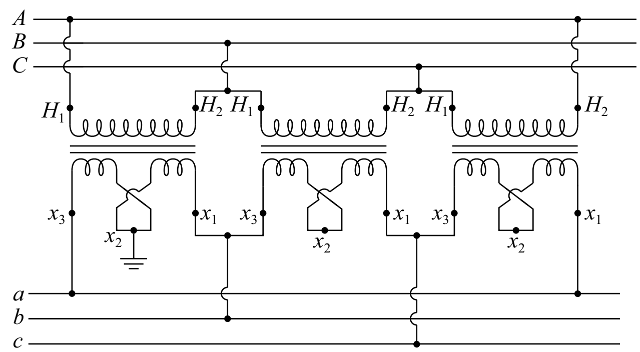
(A)
WYE-WYE CONNECTION



(B)
WYE-DELTA CONNECTION



(C)
DELTA-WYE CONNECTION



(D)
DELTA-DELTA CONNECTION

Gonen, Turan, *Electric Power Distribution Engineering*, 3rd ed., Boca Raton, Florida: CRC Press, 2014.

Rotating Machines (General)

Efficiency of a machine is defined as:

$$\eta = P_{\text{out}}/P_{\text{in}}$$

where

P_{out} = power output of the machine (W)

P_{in} = power input to the machine (W)

For a motor, P_{in} is the active component of the electrical input power and P_{out} is the mechanical output power. For a generator, vice versa.

The losses in a machine can be attributed to core, copper, friction and windage, and stray losses, and:

$$P_{\text{out}} = P_{\text{in}} - P_{\text{loss}}$$

Mechanical power in a rotating machine is given by:

$$P = T\omega_m$$

where

P = mechanical power (W)
T = mechanical torque (N•m)
ω_m = angular velocity (rad/s)

The angular velocity in rad/s is related to the speed in rpm by:

$\omega_m = (2\pi/60)n$

where n is the rotor's speed in rpm.

AC Machines

The synchronous speed n_s for ac motors is given by

$n_s = 120f/p$

where

f = the line voltage frequency (Hz)
p = the number of poles

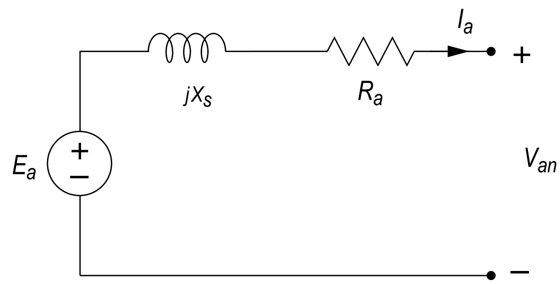
The slip for an induction motor is

slip = $(n_s - n)/n_s$

where n = the rotational speed (rpm)

Synchronous Machines

The single-phase equivalent circuit of a Y-connected synchronous machine is shown below. The induced voltage is $\boldsymbol{E_a} = E_a\angle\delta$ where the magnitude is proportional to the excitation (e.g., field current) and the angle δ is the torque or power angle. The direction for the current $\boldsymbol{I_a}$ is shown for a generator in this circuit. The resistance R_a is the armature circuit resistance and the reactance X_s is the synchronous reactance.



The power developed by the synchronous machine is:

$P_d = 3E_aI_a \cos(\delta + \theta)$

where θ is the power factor angle when the terminal voltage V_{an} is used as the reference.

If the armature resistance is negligible, the power developed by the synchronous machine is:

$P_d = 3(E_aV_a/\mathrm{X_s}) \sin\delta$

and maximum power capability of the synchronous machine is:

$P_d = 3(E_aV_a/\mathrm{X_s})$

Induction Machines

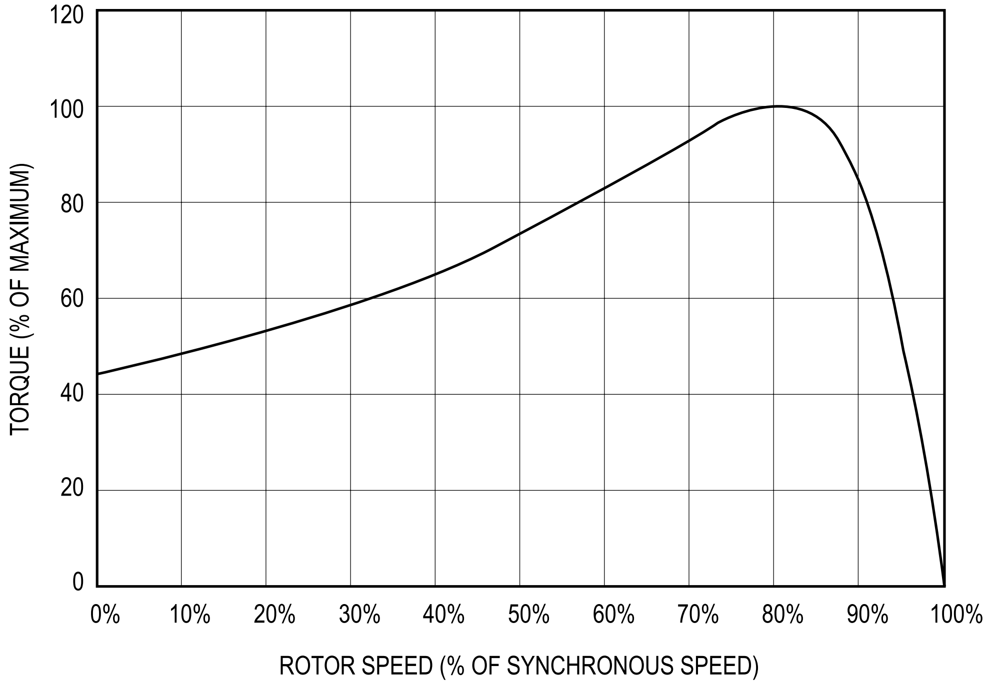
The slip s of an induction machine is defined as:

$s = (n_s - n)/n_s$

where

n_s = synchronous speed (rpm)
n = speed of the rotor (rpm)

A sample torque-speed characteristic of an induction motor is shown below, normalized to maximum (break down) torques.



DC Machines

The electrical input power (motor) or output power (generator) of the armature circuit is given by:

$$P = V_T I_a$$

where

V_T = armature circuit terminal voltage

I_a = armature current

The armature circuit of a dc machine is approximated by a series connection of the armature resistance R_a, the armature inductance L_a, and a dependent voltage source of value

$$V_a = K_a n \phi \text{ volts}$$

where

K_a = constant depending on the design

n = armature speed (rpm)

ϕ = magnetic flux generated by the field

The field circuit is approximated by the field resistance R_f in series with the field inductance L_f. Neglecting saturation, the magnetic flux generated by the field current I_f is

$$\phi = K_f I_f \text{ webers}$$

The mechanical power generated by the armature is

$$P_m = V_a I_a \text{ watts}$$

where I_a is the armature current.

The mechanical torque produced is

$$T_m = (60/2\pi) K_a \phi I_a \text{ newton-meters}$$

Servomotors and Generators

Servomotors are electrical motors tied to a feedback system to obtain precise control. Smaller servomotors typically are dc motors.

A permanent magnet dc generator can be used to convert mechanical energy to electrical energy, as in a tachometer.

DC motor suppliers may provide data sheets with speed torque curves, motor torque constants (K_T), and motor voltage constants (K_E). An idealized dc motor at steady state exhibits the following relationships:

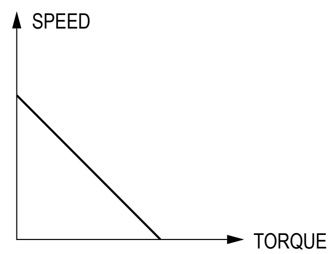
$$V = I\,R + K_E\,\omega$$
$$T = K_T I$$

where

V = voltage at the motor terminals
I = current through the motor
T = torque applied by the motor
R = resistance of the windings
ω = rotational speed

When using consistent SI units [N•m/A and V/(rad/s)], $K_T = K_E$.

An ideal speed-torque curve for a servomotor, with constant V, would look like this:



Voltage Regulation

The percent voltage regulation of a power supply is defined as

$$\%\ \text{Regulation} = \frac{|V_{NL}| - |V_{FL}|}{|V_{FL}|} \times 100\%$$

where

V_{NL} = voltage under no load conditions
V_{FL} = voltage under full load conditions (assumes that the source voltage remains constant)

Electromagnetic Dynamic Fields

The integral and point form of Maxwell's equations are

$$\oint \mathbf{E} \cdot d\ell = -\iint_S (\partial \mathbf{B}/\partial t) \cdot d\mathbf{S}$$
$$\oint \mathbf{H} \cdot d\ell = I_{\text{enc}} + \iint_S (\partial \mathbf{D}/\partial t) \cdot d\mathbf{S}$$
$$\oiint_{S_V} \mathbf{D} \cdot d\mathbf{S} = \iiint_V \rho\, dv$$
$$\oiint_{S_V} \mathbf{B} \cdot d\mathbf{S} = 0$$
$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$$
$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$

Lossless Transmission Lines

The wavelength, λ, of a sinusoidal signal is defined as the distance the signal will travel in one period.

$$\lambda = \frac{U}{f}$$

where

U = velocity of propagation

f = frequency of the sinusoid

The characteristic impedance, $\mathbf{Z}_0$, of a transmission line is the input impedance of an infinite length of the line and is given by

$$\mathbf{Z}_0 = \sqrt{L/C}$$

where L and C are the per unit length inductance and capacitance of the line.

The reflection coefficient at the load is defined as

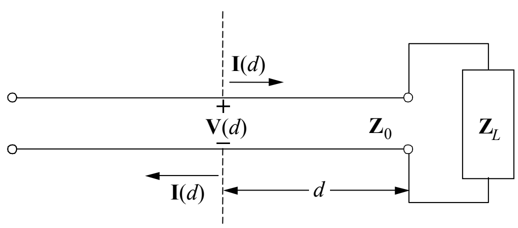
$$\Gamma = \frac{\boldsymbol{V}^-}{\boldsymbol{V}^+} = \frac{\mathbf{Z}_L - \mathbf{Z}_0}{\mathbf{Z}_L + \mathbf{Z}_0}$$

and the standing wave ratio SWR is

$$\text{SWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

β = Propagation constant = $\dfrac{2\pi}{\lambda}$

For sinusoidal voltages and currents:



Voltage across the transmission line:

$$\mathbf{V}(d) = \mathbf{V}^+ e^{j\beta d} + \mathbf{V}^- e^{-j\beta d}$$

Current along the transmission line:

$$\mathbf{I}(d) = \mathbf{I}^+ e^{j\beta d} + \mathbf{I}^- e^{-j\beta d}$$

where $\mathbf{I}^+ = \mathbf{V}^+/\mathbf{Z}_0$ and $\mathbf{I}^- = -\mathbf{V}^-/\mathbf{Z}_0$

Input impedance at d

$$\mathbf{Z}_{\text{in}}(d) = \mathbf{Z}_0 \frac{\mathbf{Z}_L + j\mathbf{Z}_0 \tan(\beta d)}{\mathbf{Z}_0 + j\mathbf{Z}_L \tan(\beta d)}$$

Difference Equations

Difference equations are used to model discrete systems. Systems which can be described by difference equations include computer program variables iteratively evaluated in a loop, sequential circuits, cash flows, recursive processes, systems with time-delay components, etc. Any system whose input $x(t)$ and output $y(t)$ are defined only at the equally spaced intervals $t = kT$ can be described by a difference equation.

First-Order Linear Difference Equation

A first-order difference equation is

$$y[k] + a_1 y[k-1] = x[k]$$

Second-Order Linear Difference Equation

A second-order difference equation is

$$y[k] + a_1 y[k-1] + a_2 y[k-2] = x[k]$$

z-Transforms

The transform definition is

$$F(z) = \sum_{k=0}^{\infty} f[k] z^{-k}$$

The inverse transform is given by the contour integral

$$f[k] = \frac{1}{2\pi j} \oint_{\Gamma} F(z) z^{k-1} dz$$

and it represents a powerful tool for solving linear shift-invariant difference equations. A limited unilateral list of z-transform pairs assuming zero initial conditions follows:

$f[k]$	$F(z)$
$\delta[k]$, Impulse at $k = 0$	1
$u[k]$, Step at $k = 0$	$1/(1 - z^{-1})$
β^k	$1/(1 - \beta z^{-1})$
$y[k-1]$	$z^{-1}Y(z)$
$y[k-2]$	$z^{-2}Y(z)$
$y[k+1]$	$zY(z) - zy[0]$
$y[k+2]$	$z^2Y(z) - z^2y[0] - zy[1]$
$\sum_{m=0}^{\infty} x[k-m]h[m]$	$H(z)X(z)$
$\lim_{k\to 0} f[k]$	$\lim_{z\to\infty} F(z)$
$\lim_{k\to\infty} f[k]$	$\lim_{z\to 1}(1 - z^{-1})F(z)$

[Note: The last two transform pairs represent the Initial Value Theorem (I.V.T.) and the Final Value Theorem (F.V.T.) respectively.]

Convolution

Continuous-time convolution:

$$v(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau$$

Discrete-time convolution:

$$v[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k] y[n - k]$$

Digital Signal Processing

A discrete-time, linear, time-invariant (DTLTI) system with a single input $x[n]$ and a single output $y[n]$ can be described by a linear difference equation with constant coefficients of the form

$$y[n] + \sum_{i=1}^{k} b_i y[n-i] = \sum_{i=0}^{l} a_i x[n-i]$$

If all initial conditions are zero, taking a z-transform yields a transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{l} a_i z^{-i}}{1 + \sum_{i=1}^{k} b_i z^{-i}}$$

Two common discrete inputs are the unit-step function $u[n]$ and the unit impulse function $\delta[n]$, where

$$u[n] = \begin{Bmatrix} 0 & n < 0 \\ 1 & n \geq 0 \end{Bmatrix} \text{ and } \delta[n] = \begin{Bmatrix} 1 & n = 0 \\ 0 & n \neq 0 \end{Bmatrix}$$

The impulse response $h[n]$ is the response of a discrete-time system to $x[n] = \delta[n]$.

A finite impulse response (FIR) filter is one in which the impulse response $h[n]$ is limited to a finite number of points:

$$h[n] = \sum_{i=0}^{k} a_i \delta[n-i]$$

The corresponding transfer function is given by

$$H(z) = \sum_{i=0}^{k} a_i z^{-i}$$

where k is the order of the filter.

An infinite impulse response (IIR) filter is one in which the impulse response $h[n]$ has an infinite number of points:

$$h[n] = \sum_{i=0}^{\infty} a_i \delta[n-i]$$

Communication Theory and Concepts

The following concepts and definitions are useful for communications systems analysis.

Functions

Unit step, $u(t)$	$u(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases}$
Rectangular pulse, $\Pi(t/\tau)$	$\Pi(t/\tau)=\begin{cases}1 & \lvert t/\tau\rvert<\frac{1}{2}\\ 0 & \lvert t/\tau\rvert>\frac{1}{2}\end{cases}$
Triangular pulse, $\Lambda(t/\tau)$	$\Lambda(t/\tau)=\begin{cases}1-\lvert t/\tau\rvert & \lvert t/\tau\rvert<1\\ 0 & \lvert t/\tau\rvert>1\end{cases}$
Sinc, $\text{sinc}(at)$	$\text{sinc}(at)=\dfrac{\sin(a\pi t)}{a\pi t}$
Unit impulse, $\delta(t)$	$\int_{-\infty}^{+\infty} x(t+t_0)\delta(t)dt = x(t_0)$ for every $x(t)$ defined and continuous at $t = t_0$. This is equivalent to $\int_{-\infty}^{+\infty} x(t)\delta(t-t_0)dt = x(t_0)$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$$
$$= h(t) * x(t) = \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)d\lambda$$

In particular,

$$x(t)*\delta(t-t_0) = x(t-t_0)$$

The Fourier Transform and its Inverse

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$$
$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df$$

$x(t)$ and $X(f)$ form a *Fourier transform pair*:

$$x(t) \leftrightarrow X(f)$$

Frequency Response and Impulse Response

The *frequency response* $H(f)$ of a system with input $x(t)$ and output $y(t)$ is given by

$$H(f) = \frac{Y(f)}{X(f)}$$

This gives

$$Y(f) = H(f)X(f)$$

The response $h(t)$ of a linear time-invariant system to a unit-impulse input $\delta(t)$ is called the *impulse response* of the system. The response $y(t)$ of the system to any input $x(t)$ is the convolution of the input $x(t)$ with the impulse response $h(t)$:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)\,d\lambda$$
$$= h(t) * x(t) = \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)\,d\lambda$$

Therefore, the impulse response $h(t)$ and frequency response $H(f)$ form a Fourier transform pair:

$$h(t) \leftrightarrow H(f)$$

Parseval's Theorem

The total energy in an energy signal (finite energy) $x(t)$ is given by

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

Parseval's Theorem for Fourier Series

A periodic signal $x(t)$ with period T_0 and fundamental frequency $f_0 = 1/T_0 = \omega_0/2\pi$ can be represented by a complex-exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{n=+\infty} X_n e^{jn2\pi f_0 t}$$

The average power in the dc component and the first N harmonics is

$$P = \sum_{n=-N}^{n=+N} |X_n|^2 = X_0^2 + 2\sum_{n=0}^{n=N} |X_n|^2$$

The total average power in the periodic signal $x(t)$ is given by *Parseval's theorem*:

$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{n=+\infty} |X_n|^2$$

Decibels and Bode Plots

Decibels is a technique to measure the ratio of two powers:

$$\text{dB} = 10\log_{10}(P_2/P_1)$$

The definition can be modified to measure the ratio of two voltages:

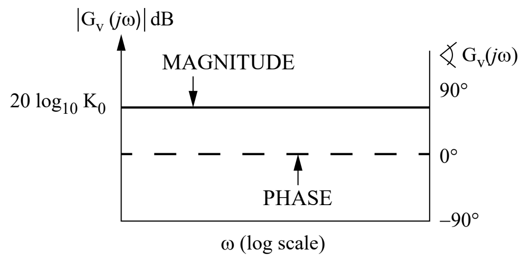
$$\text{dB} = 20\log_{10}(V_2/V_1)$$

Bode plots use a logarithmic scale for the frequency when plotting magnitude and phase response, where the magnitude is plotted in dB using a straight-line (asymptotic) approximation.

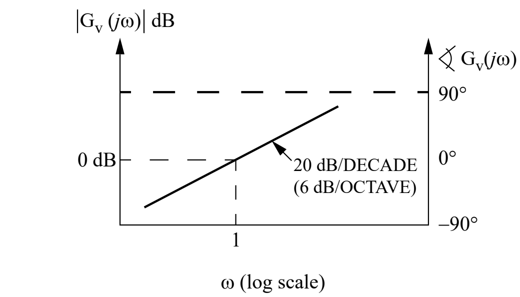
The information below summarizes Bode plots for several terms commonly encountered when determining voltage gain, $G_v(j\omega)$. Since logarithms are used to convert gain to decibels, the decibel response when these various terms are multiplied together can be added to determine the overall response.

Term	**Magnitude Response** $\lvert G_v(j\omega)\rvert_{dB}$	**Phase Response** $< G_v(j\omega)$	**Plot**
K_0	$20\log_{10}(K_0)$	0°	a
$(j\omega)^{\pm 1}$	$\pm 20\log_{10}(\omega)$	±90°	b & c
$(1 + j\omega/\omega_c)^{\pm 1}$	0 for $\omega << \omega_c$ ±3 dB for $\omega = \omega_c$ $\pm 20\log_{10}(\omega)$ for $\omega >> \omega_c$	0° for $\omega << \omega_c$ ±45° for $\omega = \omega_c$ ±90° for $\omega >> \omega_c$	d & e

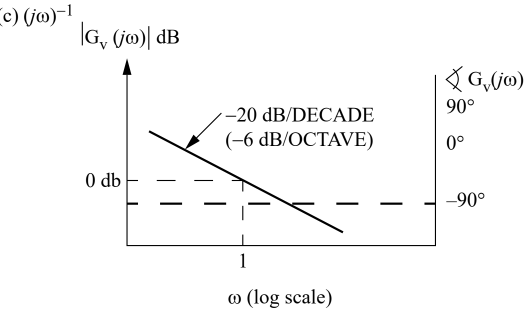
(a) K_o



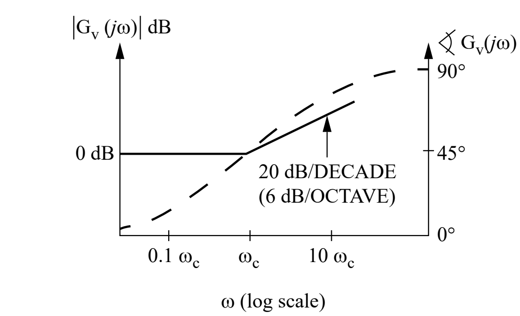
(b) $(j\omega)$



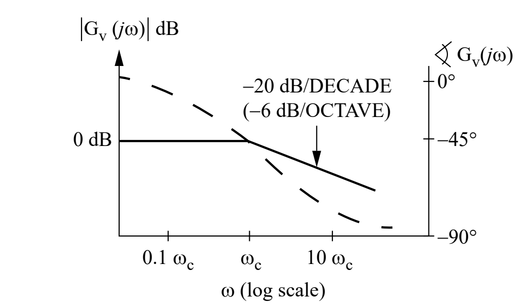
(c) $(j\omega)^{-1}$



(d) $(1 + j\omega/\omega_c)$



(e) $(1 + j\omega/\omega_c)^{-1}$



Amplitude Modulation (AM)

$$x_{AM}(t) = A_c\left[A + m(t)\right]\cos(2\pi f_c t)$$
$$= A'_c\left[1 + a m_n(t)\right]\cos(2\pi f_c t)$$

The *modulation index* is a, and the normalized message is

$$m_n(t) = \frac{m(t)}{\max|m(t)|}$$

The *efficiency* η is the percent of the total transmitted power that contains the message.

$$\eta = \frac{a^2 < m_n^2(t) >}{1 + a^2 < m_n^2(t) >} 100 \text{ percent}$$

where the mean-squared value or normalized average power in $m_n(t)$ is

$$< m_n^2(t) > = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} |m_n(t)|^2 dt$$

If $M(f) = 0$ for $|f| > W$, then the *bandwidth* of $x_{AM}(t)$ is $2W$.
AM signals can be demodulated with an envelope detector or a synchronous demodulator.

Double-Sideband Modulation (DSB)

$$x_{DSB}(t) = A_c m(t)\cos(2\pi f_c t)$$

If $M(f) = 0$ for $|f| > W$, then the bandwidth of $m(t)$ is W and the bandwidth of $x_{DSB}(t)$ is $2W$. DSB signals must be demodulated with a synchronous demodulator. A Costas loop is often used.

Single-Sideband Modulation (SSB)

Lower Sideband:

$$x_{LSB}(t) \longleftrightarrow X_{LSB}(f) = X_{DSB}(f)\Pi\left(\frac{f}{2f_c}\right)$$

Upper Sideband:

$$x_{USB}(t) \longleftrightarrow X_{USB}(f) = X_{DSB}(f)\left[1 - \Pi\left(\frac{f}{2f_c}\right)\right]$$

In either case, if $M(f) = 0$ for $|f| > W$, then the bandwidth of $x_{LSB}(t)$ or of $x_{USB}(t)$ is W. SSB signals can be demodulated with a synchronous demodulator or by carrier reinsertion and envelope detection.

Angle Modulation

$$x_{Ang}(t) = A_c\cos\left[2\pi f_c t + \phi(t)\right]$$

The *phase deviation* $\phi(t)$ is a function of the message $m(t)$.

The *instantaneous phase* is

$$\phi_i(t) = 2\pi f_c t + \phi(t) \quad \text{rad}$$

The *instantaneous frequency* is

$$\omega_i(t) = \frac{d}{dt}\phi_i(t) = 2\pi f_c + \frac{d}{dt}\phi(t) \quad \text{rad/s}$$

The *frequency deviation* is

$$\Delta\omega(t) = \frac{d}{dt}\phi(t) \quad \text{rad/s}$$

The *phase deviation* is

$$\phi(t) = k_P m(t) \quad \text{rad}$$

The *complete* bandwidth of an angle-modulated signal is infinite.

A discriminator or a phase-lock loop can demodulate angle-modulated signals.

Frequency Modulation (FM)

The *phase deviation* is

$$\phi(t) = k_F \int_{-\infty}^{t} m(\lambda) d\lambda \quad \text{rad}$$

The *frequency-deviation ratio* is

$$D = \frac{k_F \max|m(t)|}{2\pi W}$$

where W is the message bandwidth. If $D << 1$ (narrowband FM), the 98% power bandwidth B is

$$B \cong 2W$$

If $D > 1$, (wideband FM) the 98% power bandwidth B is given by *Carson's rule*:

$$B \cong 2(D + 1)W$$

Sampled Messages

A low-pass message $m(t)$ can be exactly reconstructed from uniformly spaced samples taken at a sampling frequency of $f_s = 1/T_s$

$$f_s > 2W \text{ where } M(f) = 0 \text{ for } f > W$$

The frequency $2W$ is called the *Nyquist frequency*. Sampled messages are typically transmitted by some form of pulse modulation. The minimum bandwidth B required for transmission of the pulse modulated message is inversely proportional to the pulse length τ.

$$B \propto \frac{1}{\tau}$$

Frequently, for approximate analysis

$$B \cong \frac{1}{2\tau}$$

is used as the *minimum* bandwidth of a pulse of length τ.

Ideal-Impulse Sampling

$$x_\delta(t) = m(t) \sum_{n=-\infty}^{n=+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{n=+\infty} m(nT_s)\delta(t - nT_s)$$

$$X_\delta(f) = M(f) * \left[f_s \sum_{k=-\infty}^{k=+\infty} \delta(f - kf_s) \right]$$

$$= f_s \sum_{k=-\infty}^{k=+\infty} M(f - kf_s)$$

The message $m(t)$ can be recovered from $x_\delta(t)$ with an ideal low-pass filter of bandwidth W if $f_s > 2W$.

(PAM) Pulse-Amplitude Modulation—Natural Sampling

A PAM signal can be generated by multiplying a message by a pulse train with pulses having duration τ and period

$T_s = 1/f_s$

$$x_N(t) = m(t)\sum_{n=-\infty}^{n=+\infty}\Pi\left[\frac{t-nT_s}{\tau}\right] = \sum_{n=-\infty}^{n=+\infty}m(t)\Pi\left[\frac{t-nT_s}{\tau}\right]$$

$$X_N(f) = \tau f_s\sum_{k=-\infty}^{k=+\infty}\operatorname{sinc}(k\tau f_s)M(f-kf_s)$$

The message $m(t)$ can be recovered from $x_N(t)$ with an ideal low-pass filter of bandwidth W.

Pulse-Code Modulation (PCM)

PCM is formed by sampling a message $m(t)$ and digitizing the sample values with an A/D converter. For an n-bit binary word length, transmission of a pulse-code-modulated low-pass message $m(t)$, with $M(f) = 0$ for $f \geq W$, requires the transmission of at least $2nW$ binary pulses per second. A binary word of length n bits can represent q quantization levels:

$$q = 2^n$$

The minimum bandwidth required to transmit the PCM message will be

$$B \propto 2nW = 2W\log_2 q$$

Error Coding

Error coding is a method of detecting and correcting errors that may have been introduced into a frame during data transmission. A system that is capable of detecting errors may be able to detect single or multiple errors at the receiver based on the error coding method. Below are a few examples of error detecting error coding methods.

Parity – For parity bit coding, a parity bit value is added to the transmitted frame to make the total number of ones odd (odd parity) or even (even parity). Parity bit coding can detect single bit errors.

Cyclical Redundancy Code (*CRC*) – CRC can detect multiple errors. To generate the transmitted frame from the receiver, the following equation is used:

$$T(x)/G(x) = E(x)$$

where

$T(x)$ = frame

$G(x)$ = generator

$E(x)$ = remainder

The transmitted code is $T(x) + E(x)$

On the receiver side, if

$$[T(x) + E(x)]/G(x) = 0$$

then no errors were detected.

To detect and correct errors, redundant bits need to be added to the transmitted data. Some error detecting and correcting algorithms include block code, Hamming code, and Reed Solomon.

Delays in Computer Networks

Transmission Delay – The time it takes to transmit the bits in the packet on the transmission link:

$d_{\text{trans}} = L/R$

where

L = packet size (bits/packet)

R = rate of transmission (bits/sec)

Propagation Delay – The time taken for a bit to travel from one end of the link to the other:

$d_{\text{prop}} = d/s$

where

d = distance or length of the link

s = propagation speed

The propagation speed is usually somewhere between the speed of light c and 2/3 c.

Nodal Processing Delay – It takes time to examine the packet's header and determine where to direct the packet to its destination.

Queueing Delay – The packet may experience delay as it waits to be transmitted onto the link. Ignoring nodal and queueing delays, the round-trip delay of delivering a packet from one node to another in the stop-and-wait system is

$D = 2\, d_{\text{prop}} + d_{\text{transAck}} + d_{\text{transData}}$

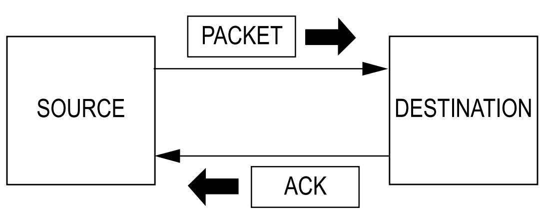
Because the sending host must wait until the ACK packet is received before sending another packet, this leads to a very poor utilization, U, of resources for stop-and-wait links with relatively large propagation delays:

$U = d_{\text{trans}}/D$

For this reason, for paths with large propagation delays, most computer networking systems use a pipelining system called go-back-N, in which N packets are transmitted in sequence before the transmitter receives an ACK for the first packet.

Automatic Request for Retransmission (ARQ)

Links in the network are most often twisted pair, optical fiber, coaxial cable, or wireless channels. These are all subject to errors and are often unreliable. The ARQ system is designed to provide reliable communications over these unreliable links. In ARQ, each packet contains an error detection process (at the link layer). If no errors are detected in the packet, the host (or intermediate switch) transmits a positive acknowledgement (ACK) packet back to the transmitting element indicating that the packet was received correctly. If any error is detected, the receiving host (or switch) automatically discards the packet and sends a negative acknowledgement (NAK) packet back to the originating element (or stays silent, allowing the transmitter to timeout). Upon receiving the NAK packet or by the trigger of a timeout, the transmitting host (or switch) retransmits the message packet that was in error. A diagram of a simple stop-and-wait ARQ system with a positive acknowledgement is shown below.



Transmission Algorithms

Sliding window protocol is used where delivery of data is required while maximizing channel capacity. In the sliding window protocol, each outbound frame contains a sequence number. When the transmitted frame is received, the receiver is required to transmit an ACK for each received frame before an additional frame can be transmitted. If the frame is not received, the receiver will transmit a NAK message indicating the frame was not received after an appropriate time has expired. Sliding window protocols automatically adjust the transmission speed to both the speed of the network and the rate at which the receiver sends new acknowledgements.

Shannon Channel Capacity Formula

$C = BW \log_2 (1+S/N)$

where

C = channel capacity in Hz (bits/sec)

BW = bandwidth in Hz (bits/sec)

S = power of the signal at the receiving device (watts)

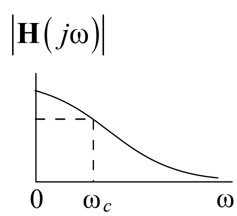
N = noise power at the receiving device (watts)

$\frac{S}{N}$ = Signal-to-Noise Ratio

Analog Filter Circuits

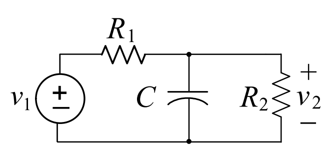
Analog filters are used to separate signals with different frequency content. The following circuits represent simple analog filters used in communications and signal processing.

First-Order Low-Pass Filters



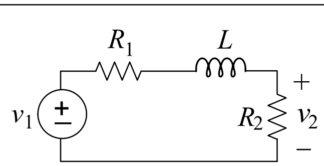
$$|\mathbf{H}(j\omega_c)| = \frac{1}{\sqrt{2}}|\mathbf{H}(0)|$$

Frequency Response



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_P}{R_1} \bullet \frac{1}{1+sR_PC}$$

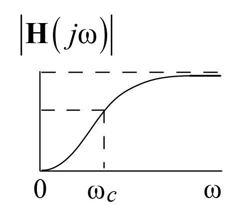
$$R_P = \frac{R_1R_2}{R_1+R_2} \qquad \omega_c = \frac{1}{R_PC}$$



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2}{R_S} \bullet \frac{1}{1+s\,L/R_S}$$

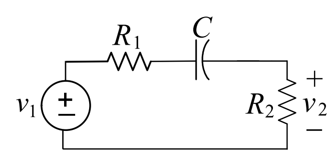
$$R_S = R_1 + R_2 \qquad \omega_c = \frac{R_S}{L}$$

First-Order High-Pass Filters



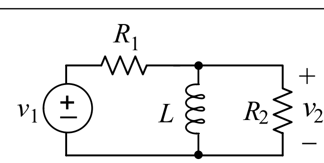
$$|\mathbf{H}(j\omega_c)| = \frac{1}{\sqrt{2}}|\mathbf{H}(j\infty)|$$

Frequency Response



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2}{R_S} \bullet \frac{sR_SC}{1+sR_SC}$$

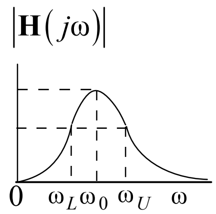
$$R_S = R_1 + R_2 \qquad \omega_c = \frac{1}{R_SC}$$



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_P}{R_1} \bullet \frac{s\,L/R_P}{1+s\,L/R_P}$$

$$R_P = \frac{R_1R_2}{R_1+R_2} \qquad \omega_c = \frac{R_P}{L}$$

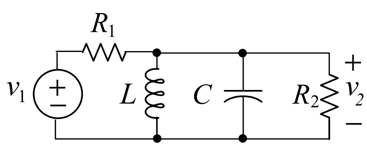
Band-Pass Filters



$$|\mathbf{H}(j\omega_L)| = |\mathbf{H}(j\omega_U)| = \frac{1}{\sqrt{2}}|\mathbf{H}(j\omega_0)|$$

3-dB Bandwidth = $BW = \omega_U - \omega_L$

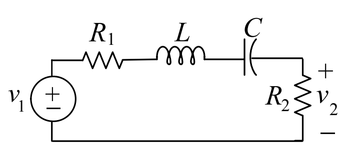
Frequency Response



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{1}{R_1 C} \bullet \frac{s}{s^2 + s/R_P C + 1/LC}$$

$$R_P = \frac{R_1 R_2}{R_1 + R_2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$|\mathbf{H}(j\omega_0)| = \frac{R_2}{R_1 + R_2} = \frac{R_P}{R_1} \qquad BW = \frac{1}{R_P C}$$

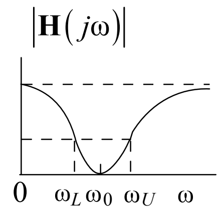


$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2}{L} \bullet \frac{s}{s^2 + sR_S/L + 1/LC}$$

$$R_S = R_1 + R_2 \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$|\mathbf{H}(j\omega_0)| = \frac{R_2}{R_1 + R_2} = \frac{R_2}{R_S} \qquad BW = \frac{R_S}{L}$$

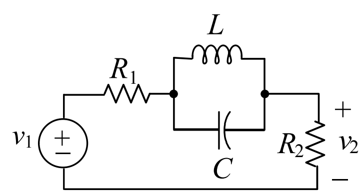
Band-Reject Filters



$$|\mathbf{H}(j\omega_L)| = |\mathbf{H}(j\omega_U)| = \left[1 - \frac{1}{\sqrt{2}}\right]|\mathbf{H}(0)|$$

3-dB Bandwidth = $BW = \omega_U - \omega_L$

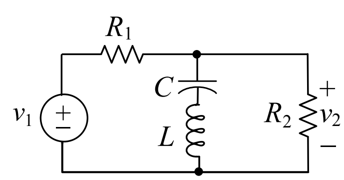
Frequency Response



$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2}{R_S} \bullet \frac{s^2 + 1/LC}{s^2 + s/R_S C + 1/LC}$$

$$R_S = R_1 + R_2 \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$|\mathbf{H}(0)| = \frac{R_2}{R_1 + R_2} = \frac{R_2}{R_S} \qquad BW = \frac{1}{R_S C}$$



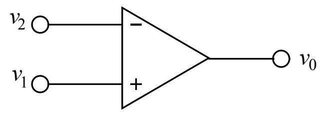
$$\mathbf{H}(s) = \frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_P}{R_1} \bullet \frac{s^2 + 1/LC}{s^2 + sR_P/L + 1/LC}$$

$$R_P = \frac{R_1 R_2}{R_1 + R_2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$|\mathbf{H}(0)| = \frac{R_2}{R_1 + R_2} = \frac{R_P}{R_1} \qquad BW = \frac{R_P}{L}$$

Operational Amplifiers

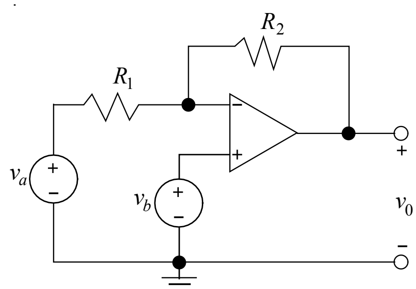
Ideal



$v_0 = A(v_1 - v_2)$

where A is large ($> 10^4$), and $v_1 - v_2$ is small enough so as not to saturate the amplifier.
For the ideal operational amplifier, assume that the input currents are zero and that the gain A is infinite so when operating linearly $v_2 - v_1 = 0$.

For the two-source configuration with an ideal operational amplifier,



$$v_0 = -\frac{R_2}{R_1} v_a + \left(1 + \frac{R_2}{R_1}\right) v_b$$

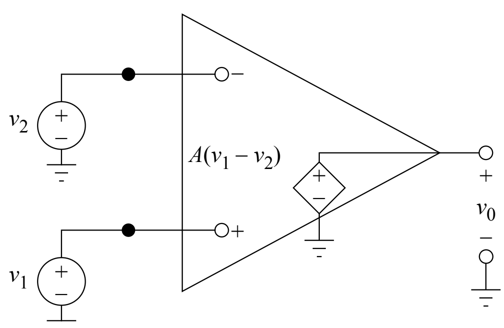
If $v_a = 0$, we have a non-inverting amplifier with

$$v_0 = \left(1 + \frac{R_2}{R_1}\right) v_b$$

If $v_b = 0$, we have an inverting amplifier with

$$v_0 = -\frac{R_2}{R_1} v_a$$

Common Mode Rejection Ratio (CMRR)



Equivalent Circuit of an Ideal Op Amp

In the op-amp circuit shown, the differential input is defined as:

$$v_{id} = v_1 - v_2$$

The common-mode input voltage is defined as:

$$v_{icm} = (v_1 + v_2)/2$$

The output voltage is given by:

$$v_O = Av_{id} + A_{cm}v_{icm}$$

In an ideal op amp, $A_{cm} = 0$. In a nonideal op amp, the *CMRR* is used to measure the relative degree of rejection between the differential gain and common-mode gain.

$$CMRR = \frac{|A|}{|A_{cm}|}$$

CMRR is usually expressed in decibels as:

$$CMRR = 20\log_{10}\left[\frac{|A|}{|A_{cm}|}\right]$$

Solid-State Electronics and Devices

Conductivity of a semiconductor material:

$$\sigma = q\,(n\mu_n + p\mu_p)$$

where

μ_n ≡ electron mobility
μ_p ≡ hole mobility
n ≡ electron concentration
p ≡ hole concentration
q ≡ charge on an electron (1.6×10^{-19} C)

Doped material:

p-type material; $p_p \approx N_a$
n-type material; $n_n \approx N_d$

Carrier concentrations at equilibrium

$$(p)(n) = n_i^2$$

where $n_i \equiv$ intrinsic concentration.

Built-in potential (contact potential) of a p-n junction:

$$V_0 = \frac{kT}{q}\ln\frac{N_aN_d}{n_i^2}$$

Thermal voltage

$$V_T = \frac{kT}{q} \approx 0.026 \text{ V at } 300 \text{ K}$$

N_a = acceptor concentration
N_d = donor concentration
T = temperature (K)
k = Boltzmann's constant = 1.38×10^{-23} J/K

Capacitance of abrupt p–n junction diode

$$C(V) = C_0\Big/\sqrt{1 - V/V_{bi}}$$

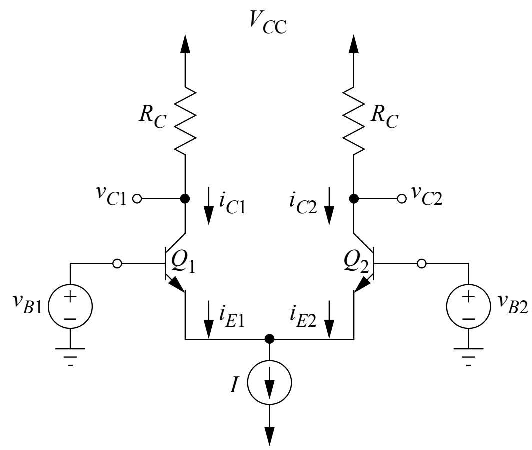
C_0 = junction capacitance at $V = 0$
V = potential of anode with respect to cathode
V_{bi} = junction contact potential

Resistance of a diffused layer is $R = R_s\,(L/W)$

where

R_s = sheet resistance = ρ/d in ohms per square
ρ = resistivity
d = thickness
L = length of diffusion
W = width of diffusion

Differential Amplifier



A Basic BJT Differential Amplifier

A basic BJT differential amplifier consists of two matched transistors whose emitters are connected and that are biased by a constant-current source. The following equations govern the operation of the circuit given that neither transistor is operating in the saturation region:

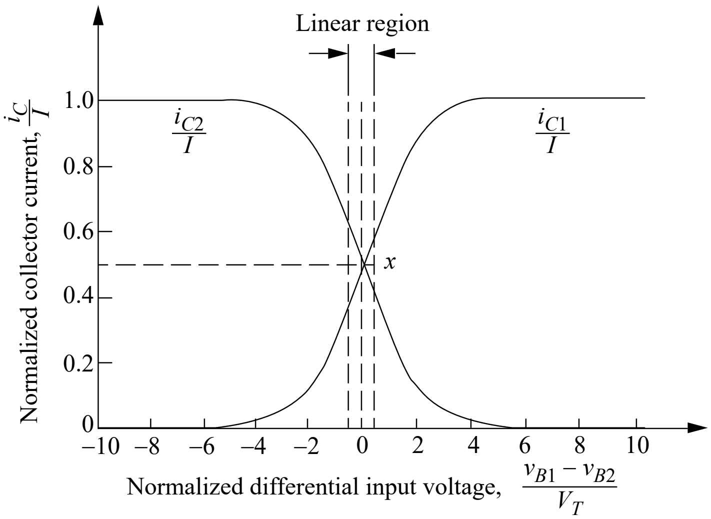
$$\frac{i_{E1}}{i_{E2}} = e^{(v_{B1} - v_{B2})/V_T}$$

$$i_{E1} + i_{E2} = I$$

$$i_{E1} = \frac{I}{1 + e^{(v_{B2} - v_{B1})/V_T}} \qquad i_{E2} = \frac{I}{1 + e^{(v_{B1} - v_{B2})/V_T}}$$

$$i_{C1} = \alpha I_{E1} \qquad i_{C2} = \alpha I_{E2}$$

The following figure shows a plot of two normalized collector currents versus normalized differential input voltage for a circuit using transistors with $\alpha \cong 1$.



Transfer characteristics of the BJT differential amplifier with $\alpha \cong 1$

Power Conversion

In the following figure, D represents the duty ratio, f represents the switching frequency, and T represents the switching period. The voltage gain of an ideal switching dc-dc converter with this gate command is:

Buck Converter: D

Boost Converter: $\frac{1}{1-D}$

Buck-Boost Converter: $-\frac{D}{1-D}$

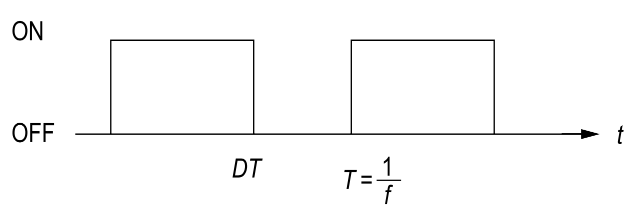
For an n-pulse rectifier with a line-to-line RMS input voltage of V_{rms} and no output filter, the average output voltage is

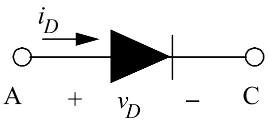
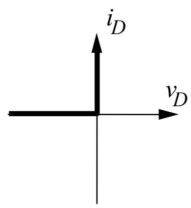
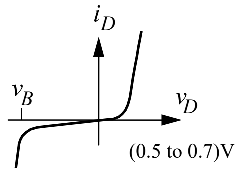
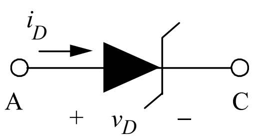
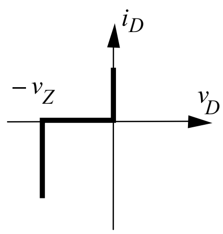
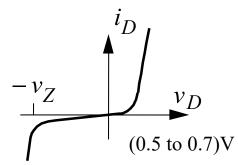
$$V_{dc} = V_{rms} \times \frac{n\sqrt{2}}{\pi} \sin\frac{\pi}{n}$$

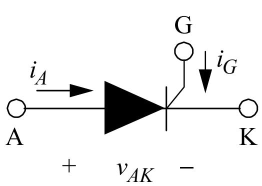
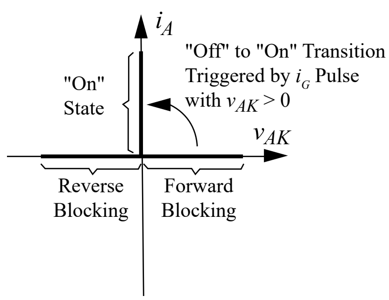
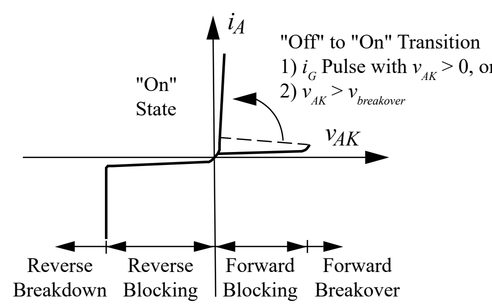
For a three-phase voltage-source inverter with an input voltage of V_{dc} and sine-triangle pulsewidth modulation with a peak modulation index of m, the line-to-line RMS fundamental output voltage is

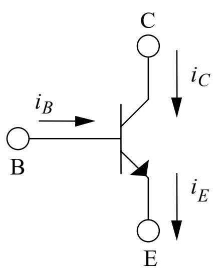
$$V_{rms} = mV_{dc} \times \frac{1}{2}\sqrt{\frac{3}{2}}$$

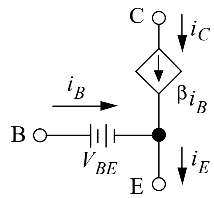
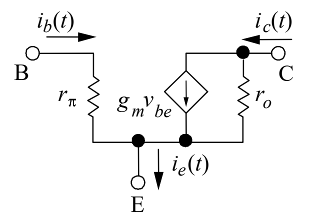
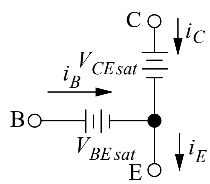
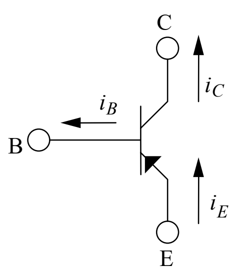
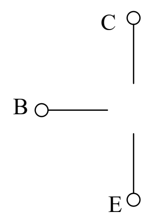
This is valid for $0 \le m \le 1$, or with third-harmonic injection $0 \le m \le 1.15$.

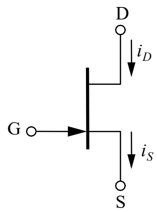
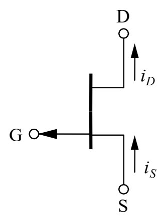
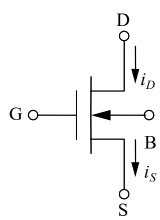
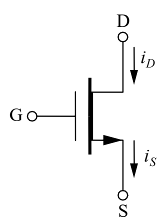
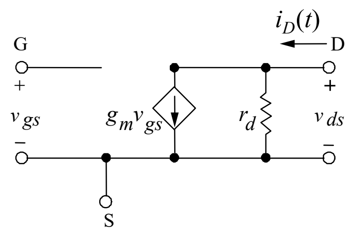
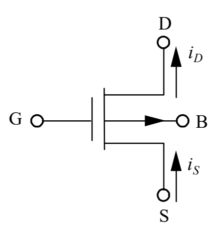
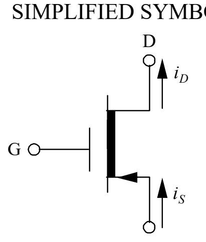


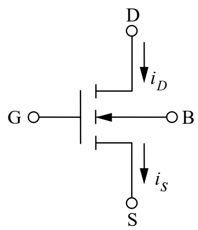
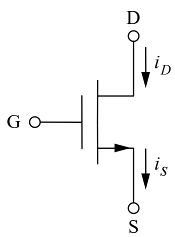
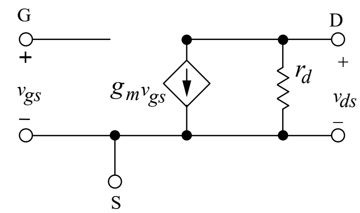
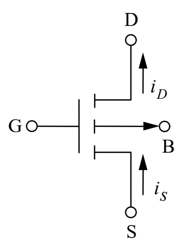
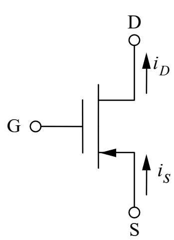
DIODES			
Device and Schematic Symbol	**Ideal $I-V$ Relationship**	**Realistic $I-V$ Relationship**	**Mathematical $I-V$ Relationship**
(Junction Diode) 		 v_B = breakdown voltage	Shockley Equation $i_D \approx I_s\left[e^{(v_D/\eta V_T)}-1\right]$ where I_s = saturation current η = emission coefficient, typically 1 for Si V_T = thermal voltage $= \dfrac{kT}{q}$
(Zener Diode) 		 v_Z = Zener voltage	Same as above.

Thyristor or Silicon Controlled Rectifier (SCR)		
Schematic Symbol	**Ideal *I-V* Relationship**	**Realistic *I-V* Relationship**
		

<table>
<tr><th colspan="4">Bipolar Junction Transistor (BJT)</th></tr>
<tr><th>Schematic Symbol</th><th>Mathematical Relationships</th><th>Large-Signal (DC) Equivalent Circuit</th><th>Low-Frequency Small-Signal (AC) Equivalent Circuit</th></tr>
<tr>
<td rowspan="2">
NPN – Transistor</td>
<td rowspan="2">$i_E \;= i_B + i_C$
$i_C \;= \beta i_B$
$i_C \;= \alpha\, i_E$
$\alpha \;= \beta/(\beta + 1)$
$i_C \;\approx I_S e^{(V_{BE}/V_T)}$
I_S =emitter saturation current
V_T =thermal voltage

Note: These relationships are valid in the active mode of operation.</td>
<td>Active Region:
base emitter junction forward biased; base collector juction reverse biased
</td>
<td rowspan="2">Low Frequency:
$g_m \;\approx I_{CQ}/V_T$
$r_\pi \;\approx \beta/g_m,$
$$r_o = \left[\frac{\partial v_{CE}}{\partial i_c}\right]_{Q_{point}} \approx \frac{V_A}{I_{CQ}}$$
where
I_{CQ} = dc collector current at the Q_{point}
V_A = Early voltage
</td>
</tr>
<tr>
<td>Saturation Region:
both junctions forward biased
</td>
</tr>
<tr>
<td rowspan="2">
PNP – Transistor</td>
<td rowspan="2">Same as for NPN with current directions and voltage polarities reversed.</td>
<td>Cutoff Region:
both junctions reverse biased
</td>
<td>Same as for NPN.</td>
</tr>
<tr>
<td>Same as NPN with current directions and voltage polarities reversed</td>
<td></td>
</tr>
</table>

Junction Field Effect Transistors (JFETs) and Depletion MOSFETs (Low and Medium Frequency)		
Schematic Symbol	**Mathematical Relationships**	**Small-Signal (AC) Equivalent Circuit**
N-CHANNEL JFET  P-CHANNEL JFET  N-CHANNEL DEPLETION MOSFET (NMOS)  SIMPLIFIED SYMBOL 	Cutoff Region: $v_{GS} < V_p$ $i_D = 0$ Triode Region: $v_{GS} > V_p$ and $v_{GD} > V_p$ $i_D = (I_{DSS}/V_p^2)[2v_{DS}(v_{GS} - V_p) - v_{DS}^2]$ Saturation Region: $v_{GS} > V_p$ and $v_{GD} < V_p$ $i_D = I_{DSS}(1 - v_{GS}/V_p)^2$ where I_{DSS} = drain current with $v_{GS} = 0$ (in the saturation region) $= KV_p^2$, K = conductivity factor For JFETs, V_p = pinch-off voltage For MOSFETs, $V_p = V_T$ = threshold voltage	$g_m = \dfrac{2\sqrt{I_{DSS} I_D}}{\lvert V_p \rvert}$ in saturation region  where $r_d = \left\lvert \dfrac{\partial v_{ds}}{\partial i_d} \right\rvert_{Q_{point}}$
P-CHANNEL DEPLETION MOSFET (PMOS)  SIMPLIFIED SYMBOL 	Same as for N-Channel with current directions and voltage polarities reversed	Same as for N-Channel

Enhancement MOSFET (Low and Medium Frequency)		
Schematic Symbol	**Mathematical Relationships**	**Small-Signal (AC) Equivalent Circuit**
N-CHANNEL ENHANCEMENT MOSFET (NMOS)  SIMPLIFIED SYMBOL 	<u>Cutoff Region</u>: $v_{GS} < V_t$ $i_D = 0$ <u>Triode Region</u>: $v_{GS} > V_t$ and $v_{GD} > V_t$ $i_D = K\,[2v_{DS}\,(v_{GS} - V_t) - v_{DS}^{\,2}\,]$ <u>Saturation Region</u>: $v_{GS} > V_t$ and $v_{GD} < V_t$ $i_D = K\,(v_{GS} - V_t)^2$ where $K =$ conductivity factor $V_t =$ threshold voltage	$g_m = 2K(v_{GS} - V_t)$ in saturation region  where $r_d = \left\lvert \dfrac{\partial v_{ds}}{\partial i_d} \right\rvert_{Q_{\text{point}}}$
P-CHANNEL ENHANCEMENT MOSFET (PMOS)  SIMPLIFIED SYMBOL 	Same as for N-channel with current directions and voltage polarities reversed	Same as for N-channel

Number Systems and Codes

An unsigned number of base-r has a decimal equivalent D defined by

$$D = \sum_{k=0}^{n} a_k r^k + \sum_{i=1}^{m} a_i r^{-i}$$

where

a_k = the (k + 1) digit to the left of the radix point

a_i = the ith digit to the right of the radix point

Binary Number System

In digital computers, the base-2, or binary, number system is normally used. Thus the decimal equivalent, D, of a binary number is given by

$$D = a_k 2^k + a_{k-1} 2^{k-1} + \ldots + a_0 + a_{-1} 2^{-1} + \ldots$$

Since this number system is so widely used in the design of digital systems, we use a shorthand notation for some powers of two:

2^{10} = 1,024 is abbreviated "K" or "kilo"

2^{20} = 1,048,576 is abbreviated "M" or "mega"

Signed numbers of base-r are often represented by the radix complement operation. If M is an N-digit value of base-r, the radix complement $R(M)$ is defined by

$$R(M) = r^N - M$$

The 2's complement of an N-bit binary integer can be written

$$\text{2's Complement } (M) = 2^N - M$$

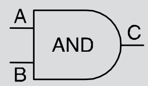
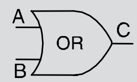
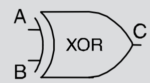
This operation is equivalent to taking the 1's complement (inverting each bit of M) and adding one.

The following table contains equivalent codes for a four-bit binary value.

Binary Base-2	Decimal Base-10	Hexa-decimal Base-16	Octal Base-8	Packed BCD Code	Gray Code
0000	0	0	0	0000	0000
0001	1	1	1	0001	0001
0010	2	2	2	0010	0011
0011	3	3	3	0011	0010
0100	4	4	4	0100	0110
0101	5	5	5	0101	0111
0110	6	6	6	0110	0101
0111	7	7	7	0111	0100
1000	8	8	10	1000	1100
1001	9	9	11	1001	1101
1010	10	A	12	---	1111
1011	11	B	13	---	1110
1100	12	C	14	---	1010
1101	13	D	15	---	1011
1110	14	E	16	---	1001
1111	15	F	17	---	1000

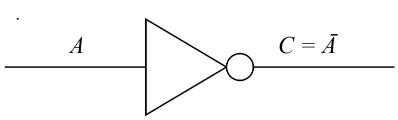
Logic Operations and Boolean Algebra

Three basic logic operations are the "AND (•)," "OR (+)," and "Exclusive-OR ⊕" functions. The definition of each function, its logic symbol, and its Boolean expression are given in the following table.

Function Inputs			
$A\ B$	$C = A \bullet B$	$C = A + B$	$C = A \oplus B$
0 0	0	0	0
0 1	0	1	1
1 0	0	1	1
1 1	1	1	0

As commonly used, A AND B is often written AB or $A \bullet B$.

The not operator inverts the sense of a binary value ($0 \rightarrow 1$, $1 \rightarrow 0$)



LOGIC SYMBOL

NOT OPERATOR

Input	Output
A	$C = \bar{A}$
0 1	1 0

De Morgan's Theorems

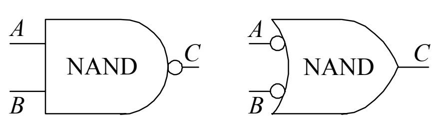
First theorem: $\overline{A + B} = \overline{A} \bullet \overline{B}$

Second theorem: $\overline{A \bullet B} = \overline{A} + \overline{B}$

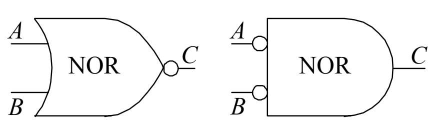
These theorems define the NAND gate and the NOR gate.

Logic symbols for these gates are shown below.

NAND Gates: $\overline{A \bullet B} = \overline{A} + \overline{B}$

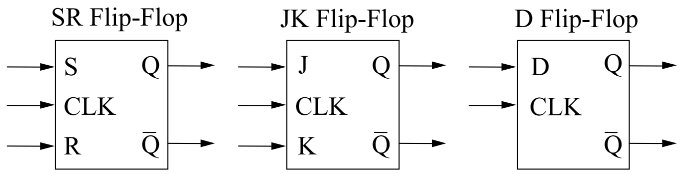


NOR Gates: $\overline{A + B} = \overline{A} \bullet \overline{B}$



Flip-Flops

A flip-flop is a device whose output can be placed in one of two states, 0 or 1. The flip-flop output is synchronized with a clock (CLK) signal. Q_n represents the value of the flip-flop output before CLK is applied, and Q_{n+1} represents the output after CLK has been applied. Three basic flip-flops are described below.



SR	Q_{n+1}
00	Q_n no change
01	0
10	1
11	x invalid

JK	Q_{n+1}
00	Q_n no change
01	0
10	1
11	$\overline{Q}_n$ toggle

D	Q_{n+1}
0	0
1	1

Composite Flip-Flop State Transition						
$\boldsymbol{Q_n}$	$\boldsymbol{Q_{n+1}}$	***S***	***R***	***J***	***K***	***D***
0	0	0	x	0	x	0
0	1	1	0	1	x	1
1	0	0	1	x	1	0
1	1	x	0	x	0	1

Switching Function Terminology

Minterm, m_i – A product term which contains an occurrence of every variable in the function.

Maxterm, M_i – A sum term which contains an occurrence of every variable in the function.

Implicant – A Boolean algebra term, either in sum or product form, which contains one or more minterms or maxterms of a function.

Prime Implicant – An implicant which is not entirely contained in any other implicant.

Essential Prime Implicant – A prime implicant which contains a minterm or maxterm which is not contained in any other prime implicant.

A function can be described as a sum of minterms using the notation

$$F(ABCD) = \Sigma m(h, i, j, \ldots)$$
$$= m_h + m_i + m_j + \ldots$$

A function can be described as a product of maxterms using the notation

$$G(ABCD) = \Pi M(h, i, j, \ldots)$$
$$= M_h \bullet M_i \bullet M_j \bullet \ldots$$

A function represented as a sum of minterms only is said to be in *canonical sum of products* (SOP) form. A function represented as a product of maxterms only is said to be in *canonical product of sums* (POS) form. A function in canonical SOP form is often represented as a *minterm list*, while a function in canonical POS form is often represented as a *maxterm list*.

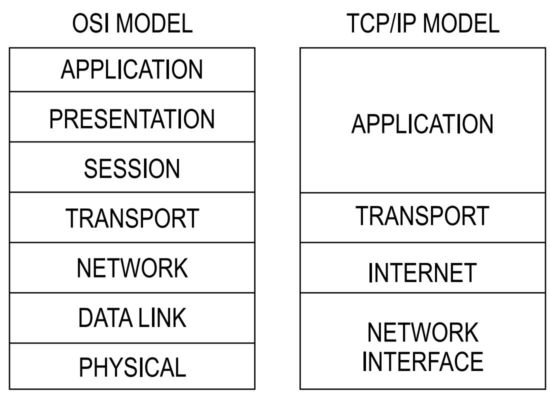
A *Karnaugh Map* (K-Map) is a graphical technique used to represent a truth table. Each square in the K-Map represents one minterm, and the squares of the K-Map are arranged so that the adjacent squares differ by a change in exactly one variable. A four-variable K-Map with its corresponding minterms is shown below. K-Maps are used to simplify switching functions by visually identifying all essential prime implicants.

AB \ CD	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

Computer Networking

Modern computer networks are primarily packet switching networks. This means that the messages in the system are broken down, or segmented into packets, and the packets are transmitted separately into the network. The primary purpose of the network is to exchange messages between endpoints of the network called hosts or nodes, typically computers, servers, or handheld devices. At the host, the packets are reassembled into the message and delivered to a software application, e.g., a browser, email, or video player.

Two widely used abstract models for modern computer networks are the open systems interconnect (OSI) model and the TCP/IP model shown in the figure below.



Tanenbaum, Andrew S., *Computer Networks*, 3rd ed., Prentice Hall, 1996, p. 36.

The application layer on the TCP/IP model corresponds to the three upper layers (application, presentation, and session) of the OSI model. The network interface layer of the TCP/IP model corresponds to the bottom two layers (data link and physical) of the OSI model.

The application layer is the network layer closest to the end user, which means both the application layer and the user interact directly with the software application. This layer interacts with software applications that implement a communicating component.

In the OSI model, the application layer interacts with the presentation layer. The presentation layer is responsible for the delivery and formatting of information to the application layer for further processing or display. It relieves the application layer of concern regarding syntactical differences in data representation within the end-user systems.

The OSI session layer provides the mechanism for opening, closing, and managing a session between end-user application processes. It provides for full-duplex, half-duplex, or simplex operation, and establishes checkpointing, adjournment, termination, and restart procedures.

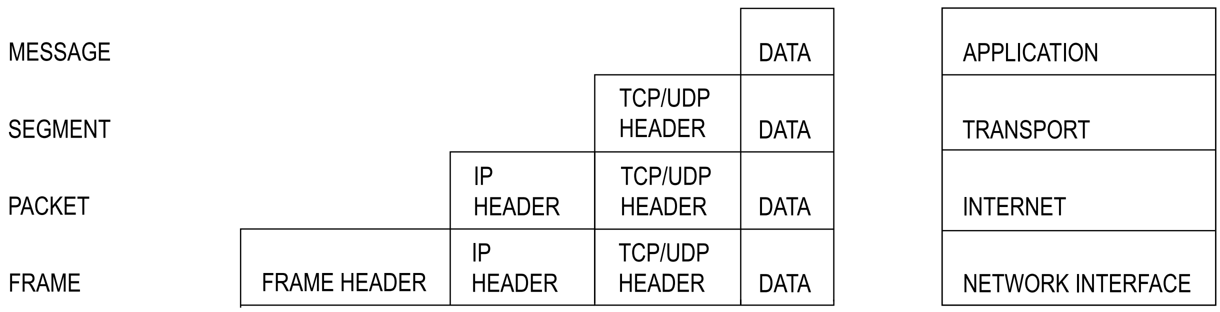
The transport layer adds a transport header normally containing TCP and UDP protocol information. The transport layer provides logical process-to-process communication primitives. Optionally, it may provide other services, such as reliability, in-order delivery, flow control, and congestion control.

The network layer or Internet layer adds another header normally containing the IP protocol; the main role of the networking layer is finding appropriate routes between end hosts, and forwarding the packets along these routes.

The link layer or data link layer contains protocols for transmissions between devices on the same link and usually handles error detection and correction and medium-access control.

The physical layer specifies physical transmission parameters (e.g., modulation, coding, channels, data rates) and governs the transmission of frames from one network element to another sharing a common link.

Hosts, routers, and link-layer switches showing the four-layer protocol stack with different sets of layers for hosts, a switch, and a router are shown in the figure below.



ENCAPSULATION OF APPLICATION DATA THROUGH EACH LAYER

In computer networking, encapsulation is a method of designing modular communication protocols in which logically separate functions in the network are abstracted from their underlying structures by inclusion or information hiding within higher-level objects. For example, a network layer packet is encapsulated in a data link layer frame.

Abbreviation

ACK	Acknowledge
ARQ	Automatic request
BW	Bandwidth
CRC	Cyclic redundancy code
DHCP	Dynamic host configuration protocol
IP	Internet protocol
LAN	Local area network
NAK	Negative acknowledgement
OSI	Open systems interconnect
TCP	Transmission control protocol

Protocol Definitions

- TCP/IP is the basic communication protocol suite for communication over the Internet.
- Internet Protocol (IP) provides end-to-end addressing and is used to encapsulate TCP or UDP datagrams. Both version 4 (IPv4) and version 6 (IPv6) are used and can coexist on the same network.
- Transmission Control Protocol (TCP) is a connection-oriented protocol that detects lost packets, duplicated packets, or packets that are received out of order and has mechanisms to correct these problems.
- User Datagram Protocol (UDP) is a connectionless-oriented protocol that has less network overhead than TCP but provides no guarantee of delivery, ordering, or duplicate protection.
- Internet Control Message Protocol (ICMP) is a supporting protocol used to send error messages and operational information.

Internet Protocol Addressing

This section from Hinden, R., and S. Deering, eds., *RFC 1884--IP Version 6 Addressing Architecture*, 1995, as found on https://tools.ietf.org/html/rfc1884 on October 16, 2019; and Information Science Institute, University of Southern California, RFC 791--*Internet Protocol*, 1981, as found on https://tools.ietf.org/html/rfc791 on Ocotober 16, 2019

IPv4 addresses are 32 bits in length and represented in dotted-decimal format using 4 decimal numbers separated by dots, e.g., 192.268.1.1. IPv6 addresses are 128 bits and are represented by eight groups of 4 hexadecimal digits separated by colons. Each group of digits is separated by a colon, e.g., 2001:0db8:85a3:0000:0000:8a2e:0370:7334. Optionally, leading zeros in a group may be dropped in order to shorten the representation, e.g., 2001:db8:85a3:0:0:8a2e:370:7334. One or more consecutive groups containing zeros only may be replaced with a single empty group, using two consecutive colons (::), e.g., 2001:db8:85a3::8a2e:370:7334. For both IPv4 and IPv6, the network address ranges can be specified in slash (/) - CIDR (Classless Inter-Domain Routing) notation after the address. The integer following the slash indicates the number of leftmost bits that are common to all addresses on the network. Alternately, for IPv4, the address range may also be specified by a network mask, a 32-bit dotted decimal number with ones for all bits common to the address space, e.g., 192.168.5.0/24 can be represented by 192.168.5.0/255.255.255.0.

IPv4 Special Address Blocks				
Address block	**Address range**	**Number of addresses**	**Scope**	**Description**
0.0.0.0/8	0.0.0.0–0.255.255.255	16777216	Software	Current network (only valid as source address).
10.0.0.0/8	10.0.0.0–10.255.255.255	16777216	Private network	Used for local communications within a private network.
100.64.0.0/10	100.64.0.0–100.127.255.255	4194304	Private network	Shared address space for communications between a service provider and its subscribers when using a carrier-grade NAT.
127.0.0.0/8	127.0.0.0–127.255.255.255	16777216	Host	Used for loopback addresses to the local host.
169.254.0.0/16	169.254.0.0–169.254.255.255	65536	Subnet	Used for link-local addresses between two hosts on a single link when no IP address is otherwise specified, such as would have normally been retrieved from a DHCP server.
172.16.0.0/12	172.16.0.0–172.31.255.255	1048576	Private network	Used for local communications within a private network.
192.0.0.0/24	192.0.0.0–192.0.0.255	256	Private network	IETF Protocol Assignments.
192.0.2.0/24	192.0.2.0–192.0.2.255	256	Documentation	Assigned as TEST-NET-1, documentation and examples.
192.88.99.0/24	192.88.99.0–192.88.99.255	256	Internet	Reserved. Formerly used for IPv6 to IPv4 relay (included IPv6 address block 2002::/16).
192.168.0.0/16	192.168.0.0–192.168.255.255	65536	Private network	Used for local communications within a private network.
198.18.0.0/15	198.18.0.0–198.19.255.255	131072	Private network	Used for benchmark testing of inter-network communications between two separate subnets.
198.51.100.0/24	198.51.100.0–198.51.100.255	256	Documentation	Assigned as TEST-NET-2, documentation and examples.
203.0.113.0/24	203.0.113.0–203.0.113.255	256	Documentation	Assigned as TEST-NET-3, documentation and examples.
224.0.0.0/4	224.0.0.0–239.255.255.255	268435456	Internet	In use for IP multicast. (Former Class D network.)
240.0.0.0/4	240.0.0.0–255.255.255.254	268435456	Internet	Reserved for future use. (Former Class E network.)
255.255.255.255/32	255.255.255.255	1	Subnet	Reserved for the "limited broadcast" destination address.

IPv6 Special Address Blocks					
Address block (CIDR)	**First address**	**Last address**	**Number of addresses**	**Usage**	**Purpose**
::/0	::	ffff:ffff:ffff:ffff:ffff:ffff:ffff:ffff	2^{128}	Routing	Default route.
::/128	::		1	Software	Unspecified address.
::1/128	::1		1	Host	Loopback address to the local host.
::ffff:0:0/96	::ffff:0.0.0.0	::ffff:255.255.255.255	2128−96 = 232= 4294967296	Software	IPv4 mapped addresses.
::ffff:0:0:0/96	::ffff:0:0.0.0.0	::ffff:0:255.255.255.255	2^{32}	Software	IPv4 translated addresses.
64:ff9b::/96	64:ff9b::0.0.0.0	64:ff9b::255.255.255.255	2^{32}	Global Internet	IPv4/IPv6 translation.
100::/64	100::	100::ffff:ffff:ffff:ffff	2^{64}	Routing	Discard prefix.
2001::/32	2001::	2001::ffff:ffff:ffff:ffff:ffff:ffff	2^{96}	Global Internet	Teredo tunneling.
2001:20::/28	2001:20::	2001:2f:ffff:ffff:ffff:ffff:ffff:ffff	2^{100}	Software	ORCHIDv2.
2001:db8::/32	2001:db8::	2001:db8:ffff:ffff:ffff:ffff:ffff:ffff	2^{96}	Documentation	Addresses used in documentation and example source code.
fc00::/7	fc00::	fdff:ffff:ffff:ffff:ffff:ffff:ffff:ffff	2^{121}	Private network	Unique local address.
fe80::/10	fe80::	febf:ffff:ffff:ffff:ffff:ffff:ffff:ffff	2^{118}	Link	Link-local address.
ff00::/8	ff00::	ffff:ffff:ffff:ffff:ffff:ffff:ffff:ffff	2^{120}	Global Internet	Multicast address.

Internet Protocol version 4 Header

The IPv4 packet header consists of 14 fields, of which 13 are required. The 14th field is optional and is named options. The fields in the header are packed with the most significant byte first (big endian), and for the diagram and discussion, the most significant bits are considered to come first (MSB 0 bit numbering). The most significant bit is numbered 0, so the version field is actually found in the four most significant bits of the first byte, for example.

<table>
<tr><th colspan="34">IPv4 Header Format</th></tr>
<tr><th>Offsets</th><th>Octet</th><th colspan="8">0</th><th colspan="8">1</th><th colspan="8">2</th><th colspan="8">3</th></tr>
<tr><th>Octet</th><th>Bit</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>12</th><th>13</th><th>14</th><th>15</th><th>16</th><th>17</th><th>18</th><th>19</th><th>20</th><th>21</th><th>22</th><th>23</th><th>24</th><th>25</th><th>26</th><th>27</th><th>28</th><th>29</th><th>30</th><th>31</th></tr>
<tr><td>0</td><td>0</td><td colspan="4">Version</td><td colspan="4">IHL</td><td colspan="6">DSCP</td><td colspan="2">ECN</td><td colspan="16">Total Length</td></tr>
<tr><td>4</td><td>32</td><td colspan="16">Identification</td><td colspan="3">Flags</td><td colspan="13">Fragment Offset</td></tr>
<tr><td>8</td><td>64</td><td colspan="8">Time To Live</td><td colspan="8">Protocol</td><td colspan="16">Header Checksum</td></tr>
<tr><td>12</td><td>96</td><td colspan="32">Source IP Address</td></tr>
<tr><td>16</td><td>128</td><td colspan="32">Destination IP Address</td></tr>
<tr><td>20</td><td>160</td><td colspan="32" rowspan="4">Options (if IHL > 5)</td></tr>
<tr><td>24</td><td>192</td></tr>
<tr><td>28</td><td>224</td></tr>
<tr><td>32</td><td>256</td></tr>
</table>

Version

The first header field in an IP packet is the four-bit version field. For IPv4, this is always equal to 4.

Internet Header Length (IHL)

The Internet Header Length (IHL) field has 4 bits, which is the number of 32-bit words. Since an IPv4 header may contain a variable number of options, this field specifies the size of the header (this also coincides with the offset to the data). The minimum value for this field is 5, which indicates a length of 5 × 32 bits = 160 bits = 20 bytes. As a 4-bit field, the maximum value is 15 words (15 × 32 bits, or 480 bits = 60 bytes).

Differentiated Services Code Point (DSCP)

Originally defined as the type of service (ToS), this field specifies differentiated services (DiffServ). New technologies are emerging that require real-time data streaming and therefore make use of the DSCP field. An example is Voice over IP (VoIP), which is used for interactive voice services.

Explicit Congestion Notification (ECN)

This field allows end-to-end notification of network congestion without dropping packets. ECN is an optional feature that is only used when both endpoints support it and are willing to use it. It is effective only when supported by the underlying network.

Total Length

This 16-bit field defines the entire packet size in bytes, including header and data. The minimum size is 20 bytes (header without data) and the maximum is 65,535 bytes. All hosts are required to be able to reassemble datagrams of size up to 576 bytes, but most modern hosts handle much larger packets. Sometimes links impose further restrictions on the packet size, in which case datagrams must be fragmented. Fragmentation in IPv4 is handled in either the host or in routers.

Identification

This field is an identification field and is primarily used for uniquely identifying the group of fragments of a single IP datagram.

Flags

A three-bit field follows and is used to control or identify fragments. They are (in order, from most significant to least significant):

- bit 0: Reserved; must be zero
- bit 1: Don't Fragment (DF)
- bit 2: More Fragments (MF)

If the DF flag is set, and fragmentation is required to route the packet, then the packet is dropped. This can be used when sending packets to a host that does not have resources to handle fragmentation. It can also be used for path MTU discovery, either automatically by the host IP software, or manually using diagnostic tools such as ping or traceroute. For unfragmented packets, the MF flag is cleared. For fragmented packets, all fragments except the last have the MF flag set. The last fragment has a non-zero Fragment Offset field, differentiating it from an unfragmented packet.

Fragment Offset

The fragment offset field is measured in units of eight-byte blocks. It is 13 bits long and specifies the offset of a particular fragment relative to the beginning of the original unfragmented IP datagram. The first fragment has an offset of zero. This allows a maximum offset of (213 – 1) × 8 = 65,528 bytes, which would exceed the maximum IP packet length of 65,535 bytes with the header length included (65,528 + 20 = 65,548 bytes).

Time To Live (TTL)

An eight-bit time to live field helps prevent datagrams from persisting (e.g., going in circles) on an internet. This field limits a datagram's lifetime. It is specified in seconds, but time intervals less than 1 second are rounded up to 1. In practice, the field has become a hop count—when the datagram arrives at a router, the router decrements the TTL field by one. When the TTL field hits zero, the router discards the packet and typically sends an ICMP Time Exceeded message to the sender. The program traceroute uses these ICMP Time Exceeded messages to print the routers used by packets to go from the source to the destination.

Protocol

This field defines the protocol used in the data portion of the IP datagram.

Header Checksum

The 16-bit IPv4 header checksum field is used for error-checking of the header. When a packet arrives at a router, the router calculates the checksum of the header and compares it to the checksum field. If the values do not match, the router discards the packet. Errors in the data field must be handled by the encapsulated protocol. Both UDP and TCP have checksum fields. When a packet arrives at a router, the router decreases the TTL field. Consequently, the router must calculate a new checksum.

Source Address

This field is the IPv4 address of the sender of the packet. Note that this address may be changed in transit by a network address translation device.

Destination Address

This field is the IPv4 address of the receiver of the packet. As with the source address, this may be changed in transit by a network address translation device.

Options

The options field is not often used. Note that the value in the IHL field must include enough extra 32-bit words to hold all the options (plus any padding needed to ensure that the header contains an integer number of 32-bit words). The list of options may be terminated with an EOL (End of Options List, 0x00) option; this is only necessary if the end of the options would not otherwise coincide with the end of the header. The possible options that can be put in the header are as follows:

Field	Size (bits)	Description
Copied	1	Set to 1 if the options need to be copied into all fragments of a fragmented packet.
Option Class	2	A general options category. 0 is for "control" options, and 2 is for "debugging and measurement." 1 and 3 are reserved.
Option Number	5	Specifies an option.
Option Length	8	Indicates the size of the entire option (including this field). This field may not exist for simple options.
Option Data	Variable	Option-specific data. This field may not exist for simple options.

Internet Protocol version 6 Header

The fixed header starts an IPv6 packet and has a size of 40 octets (320 bits). It has the following format:

Offsets	Octet	0								1								2								3							
Octet	Bit	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	0	Version				Traffic Class								Flow Label																			
4	32	Payload Length																Next Header								Hop Limit							
8	64	Source Address																															
12	96																																
16	128																																
20	160																																
24	192	Destination Address																															
28	224																																
32	256																																
36	288																																

Version (4 bits)

The constant 6 (bit sequence 0110).

Traffic Class (6+2 bits)

The bits of this field hold two values. The six most-significant bits hold the Differentiated Services (DS) field, which is used to classify packets. Currently, all standard DS fields end with a '0' bit. Any DS field that ends with two '1' bits is intended for local or experimental use.

The remaining two bits are used for Explicit Congestion Notification (ECN); priority values subdivide into ranges: traffic where the source provides congestion control and non-congestion control traffic.

Flow Label (20 bits)

Originally created for giving real-time applications special service. When set to a non-zero value, it serves as a hint to routers and switches with multiple outbound paths that these packets should stay on the same path, so that they will not be reordered. It has further been suggested that the flow label be used to help detect spoofed packets.

Payload Length (16 bits)

The size of the payload in octets, including any extension headers. The length is set to zero when a Hop-by-Hop extension header carries a Jumbo Payload option.

Next Header (8 bits)

Specifies the type of the next header. This field usually specifies the transport layer protocol used by a packet's payload. When extension headers are present in the packet, this field indicates which extension header follows. The values are shared with those used for the IPv4 protocol field, as both fields have the same function.

Hop Limit (8 bits)

Replaces the time to live field of IPv4. This value is decremented by one at each forwarding node and packet discarded if it becomes 0. However destination node should process the packet normally even if hop limit becomes 0.

Source Address (128 bits)

The IPv6 address of the sending node.

Destination Address (128 bits)

The IPv6 address of the destination node(s).

Transmission Control Protocol

<table>
<tr><td colspan="34">TCP Header</td></tr>
<tr><td>Offsets</td><td>Octet</td><td colspan="8">0</td><td colspan="8">1</td><td colspan="8">2</td><td colspan="8">3</td></tr>
<tr><td>Octet</td><td>Bit</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td><td>16</td><td>17</td><td>18</td><td>19</td><td>20</td><td>21</td><td>22</td><td>23</td><td>24</td><td>25</td><td>26</td><td>27</td><td>28</td><td>29</td><td>30</td><td>31</td></tr>
<tr><td>0</td><td>0</td><td colspan="16">Source port</td><td colspan="16">Destination port</td></tr>
<tr><td>4</td><td>32</td><td colspan="32">Sequence number</td></tr>
<tr><td>8</td><td>64</td><td colspan="32">Acknowledgment number (if ACK set)</td></tr>
<tr><td>12</td><td>96</td><td colspan="4">Data offset</td><td colspan="3">Reserved
0 0 0</td><td>NS</td><td>CWR</td><td>ECE</td><td>URG</td><td>ACK</td><td>PSH</td><td>RST</td><td>SYN</td><td>FIN</td><td colspan="16">Window Size</td></tr>
<tr><td>16</td><td>128</td><td colspan="16">Checksum</td><td colspan="16">Urgent pointer (if URG set)</td></tr>
<tr><td>20 ...</td><td>160 ...</td><td colspan="32">Options (if data offset > 5. Padded at the end with "0" bytes if necessary.) ...</td></tr>
</table>

Source Port (16 bits)

Identifies the sending port.

Destination Port (16 bits)

Identifies the receiving port.

Sequence Number (32 bits)

Has a dual role:

- If the SYN flag is set (1), then this is the initial sequence number. The sequence number of the actual first data byte and the acknowledged number in the corresponding ACK are then this sequence number plus 1.
- If the SYN flag is clear (0), then this is the accumulated sequence number of the first data byte of this segment for the current session.

Acknowledgment Number (32 bits)

If the ACK flag is set then the value of this field is the next sequence number that the sender of the ACK is expecting. This acknowledges receipt of all prior bytes (if any). The first ACK sent by each end acknowledges the other end's initial sequence number itself, but no data.

Data Offset (4 bits)

Specifies the size of the TCP header in 32-bit words. The minimum size header is 5 words and the maximum is 15 words thus giving the minimum size of 20 bytes and maximum of 60 bytes, allowing for up to 40 bytes of options in the header. This field gets its name from the fact that it is also the offset from the start of the TCP segment to the actual data.

Reserved (3 bits)

For future use and should be set to zero.

Flags (9 bits) (aka Control bits)

Contains 9 1-bit flags

- NS (1 bit): ECN-nonce - concealment protection (experimental).
- CWR (1 bit): Congestion Window Reduced (CWR) flag is set by the sending host to indicate that it received a TCP segment with the ECE flag set and had responded in congestion control mechanism.
- ECE (1 bit): ECN-Echo has a dual role, depending on the value of the SYN flag. It indicates:
  - If the SYN flag is set (1), that the TCP peer is ECN capable.
  - If the SYN flag is clear (0), that a packet with Congestion Experienced flag set (ECN=11) in the IP header was received during normal transmission. This serves as an indication of network congestion (or impending congestion) to the TCP sender.
- URG (1 bit): indicates that the Urgent pointer field is significant
- ACK (1 bit): indicates that the Acknowledgment field is significant. All packets after the initial SYN packet sent by the client should have this flag set.
- PSH (1 bit): Push function. Asks to push the buffered data to the receiving application.
- RST (1 bit): Reset the connection
- SYN (1 bit): Synchronize sequence numbers. Only the first packet sent from each end should have this flag set. Some other flags and fields change meaning based on this flag, and some are only valid when it is set, and others when it is clear.
- FIN (1 bit): Last packet from sender.

Window Size (16 bits)

The size of the receive window, which specifies the number of window size units (by default, bytes) (beyond the segment identified by the sequence number in the acknowledgment field) that the sender of this segment is currently willing to receive.

Checksum (16 bits)

The 16-bit checksum field is used for error-checking of the header, the Payload and a Pseudo-Header. The Pseudo-Header consists of the Source IP Address, the Destination IP Address, the protocol number for the TCP-Protocol (0x0006) and the length of the TCP-Headers including Payload (in Bytes).

Urgent Pointer (16 bits)

If the URG flag is set, then this 16-bit field is an offset from the sequence number indicating the last urgent data byte.

Options (Variable 0–320 bits, divisible by 32)

The length of this field is determined by the data offset field. Options have up to three fields: Option-Kind (1 byte), Option-Length (1 byte), Option-Data (variable). The Option-Kind field indicates the type of option, and is the only field that is not optional. Depending on what kind of option we are dealing with, the next two fields may be set: the Option-Length field indicates the total length of the option, and the Option-Data field contains the value of the option, if applicable. For example, an Option-Kind byte of 0x01 indicates that this is a No-Op option used only for padding, and does not have an Option-Length or Option-Data byte following it. An Option-Kind byte of 0 is the End Of Options option, and is also only one byte. An Option-Kind byte of 0x02 indicates that this is the Maximum Segment Size option, and will be followed by a byte specifying the length of the MSS field (should be 0x04). This length is the total length of the given options field, including Option-Kind and Option-Length bytes. So while the MSS value is typically expressed in two bytes, the length of the field will be 4 bytes (+2 bytes of kind and length). In short, an MSS option field with a value of 0x05B4 will show up as (0x02 0x04 0x05B4) in the TCP options section.

Some options may only be sent when SYN is set; they are indicated below as. Option-Kind and standard lengths given as (Option-Kind, Option-Length).

- 0 (8 bits): End of options list
- 1 (8 bits): No operation (NOP, Padding) This may be used to align option fields on 32-bit boundaries for better performance.
- 2,4,SS (32 bits): Maximum segment size
- 3,3,S (24 bits): Window scale
- 4,2 (16 bits): Selective Acknowledgement permitted.
- 5,N,BBBB,EEEE,... (variable bits, N is either 10, 18, 26, or 34)- Selective ACKnowledgement (SACK) These first two bytes are followed by a list of 1–4 blocks being selectively acknowledged, specified as 32-bit begin/end pointers.
- 8,10,TTTT,EEEE (80 bits)- Timestamp and echo of previous timestamp
- The remaining options are historical, obsolete, experimental, not yet standardized, or unassigned. Option number assignments are maintained by the IANA.

Padding

The TCP header padding is used to ensure that the TCP header ends, and data begins, on a 32 bit boundary. The padding is composed of zeros.

User Datagram Protocol

UDP Header																																	
Offsets	Octet	0								1								2								3							
Octet	Bit	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	0	Source port																Destination port															
4	32	Length																Checksum															

The UDP header consists of four fields, each of which is 2 bytes (16 bits). The use of the checksum and source port fields is optional in IPv4 (gray background in table). In IPv6 only the source port field is optional.

Source Port Number

This field identifies the sender's port, when used. If not used, it should be zero.

Destination Port Number

This field identifies the receiver's port and is required.

Length

This field that specifies the length in bytes of the UDP header and UDP data. The minimum length is 8 bytes, the length of the header. The field size sets a theoretical limit of 65,535 bytes (8 byte header + 65,527 bytes of data) for a UDP datagram. However, the actual limit for the data length, which is imposed by the underlying IPv4 protocol, is 65,507 bytes (65,535 − 8 byte UDP header − 20 byte IP header).

Using IPv6 jumbograms, it is possible to have UDP packets of size greater than 65,535 bytes. RFC 2675 specifies that the length field is set to zero if the length of the UDP header plus UDP data is greater than 65,535.

Checksum

The checksum field may be used for error-checking of the header and data. This field is optional in IPv4 and mandatory in IPv6. The field carries all-zeros if unused.

Internet Control Message Protocol

The Internet Control Message Protocol (ICMP) is a supporting protocol in the Internet protocol suite and is used for Internet Protocol version 4 (IPv4). It is used by network devices, including routers, to send error messages and operational information indicating, for example, that a requested service is not available or that a host or router could not be reached. Internet Control Message Protocol version 6 (ICMPv6) is the implementation of ICMP for Internet Protocol version 6 (IPv6).

ICMP and ICMPv6 Header Format																																	
Offsets	Octet	0								1								2								3							
Octet	Bit	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	0	Type								Code								Checksum															
4	32	Rest of Header																															

Partial List of ICMP Type and Code Values (IPv4)

ICMP Type	ICMP Code
0 = Echo Reply	0
3 = Destination Unreachable	0 = net unreachable
	1 = host unreachable
	2 = protocol unreachable
	4 = fragmentation needed and DF set
	5 = source route failed
5 = Redirect Message	0 = Redirect Datagram for the Network
	1 = Redirect Datagram for the Host
	2 = Redirect Datagram for the ToS and network
	3 = Redirect Datagram for the ToS and host
8 = Echo Request	0
9 = Router Advertisement	0
10 = Router Solicitation	0
11 = Time Exceeded	0 = TTL expired in transit
	1 = Fragment reassembly time exceeded

Partial List of ICMPv6 Type and Code Values

ICMPv6 Type	ICMPv6 Code
1 = Destination Unreachable	0 = no router to destination
	1 = communication with destination administratively prohibited
	2 = Beyond scope of source address
	3 = address unreachable
	4 = port unreachable
	5 = source address failed ingress/egress policy
	6 = reject route to destination
	7 = Error in Source Routing Header
2 = Packet Too Big	0
3 = Time exceeded	0 = hop limit exceeded in transit
	1 = fragment reassembly time exceeded
4 = Parameter problem	0 = erroneous header field encountered
	1 = unrecognized Next Header type encountered
	2 = unrecognized IPv6 option encountered
128 = Echo Request	0
129 = Echo Reply	0
130 = Multicast Listener Query	0
131 = Multicast listener Done	0
133 = Router Solicitation	0
134 = Router Advertisement	0
135 = Neighbor Solicitation	0
136 = Neighbor Advertisement	0
137 = Redirect Message	0
138 = Router Renumbering	0 = Router Renumbering Command
	1 = Router Renumbering Result
	255 = Sequence Number Reset
139 = ICMP Node Information Query	0 = The Data field contains an IPv6 address which is the Subject of this Query
	1 = The Data field contains a name which is the Subject of this Query, or is empty, as in the case of a NOOP.
	2 = The Data field contains an IPv4 address which is the Subject of this Query
140 = ICMP Node Information Response	0 = A successful reply. The Reply Data field may or may not be empty.
	1 = The Responder refuses to supply the answer. The Reply Data field will be empty.
	2 = The Qtype of the Query is unknown to the Responder. The Reply Data field will be empty.
141 = Inverse Neighbor Discovery Solicitation Message	0
142 = Inverse Neighbor Discovery Advertisement Message	0
143 = Multicast Listener Discovery (MLDv2) reports	0
144 = Home Agent Address Discovery Request Message	0
145 = Home Agent Address Discovery Reply Message	0
146 = Mobile Prefix Solicitation	0
147 = Mobile Prefix Advertisement	0
148 = Certification Path Solicitation	0
149 = Certification Path Advertisement	0
151 = Multicast Router Advertisement	0
152 = Multicast Router Solicitation	0
153 = Multicast Router Termination	0
155 = RPL Control Message	0

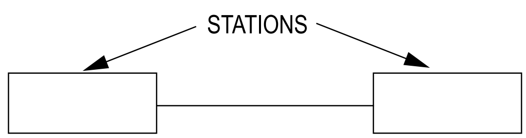
Local Area Network (LAN)

There are different methods for assigning IP addresses for devices entering a network.

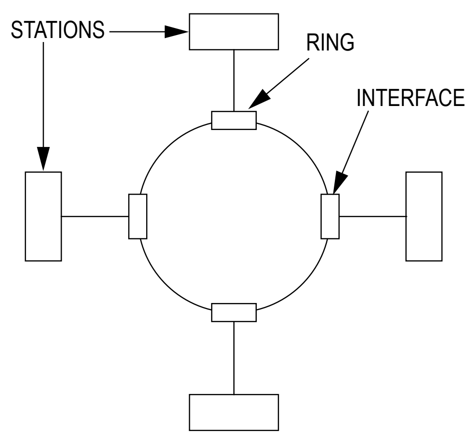
- Dynamic host configuration protocol (DHCP) is a networking protocol that allows a router to assign the IP address and other configuration information for all stations joining a network.
- Static IP addressing implies each station joining a network is manually configured with its own IP address.
- Stateless address autoconfiguration (SLAAC) allows for hosts to automatically configure themselves when connecting to an IPv6 network.

Network Topologies

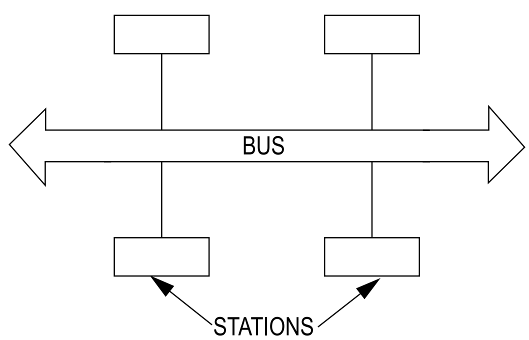
Point-to-Point



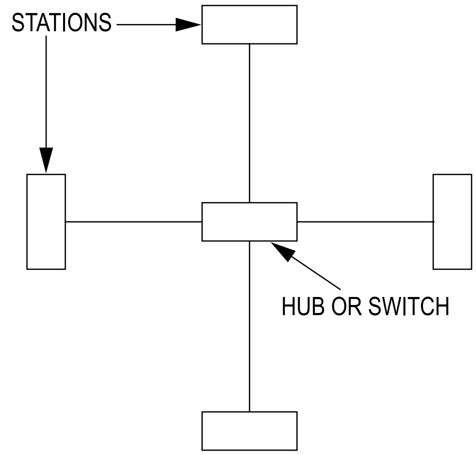
Token Ring



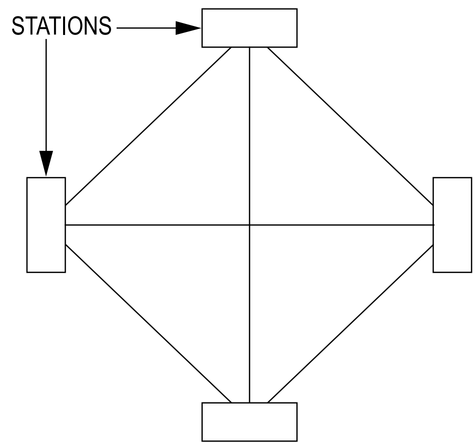
Bus



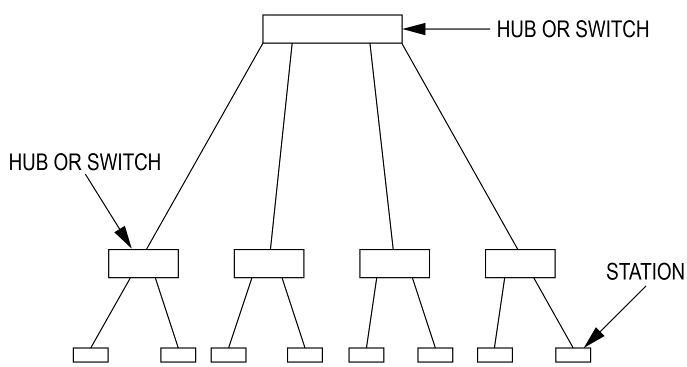
Star



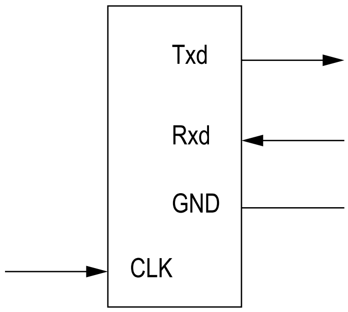
Mesh



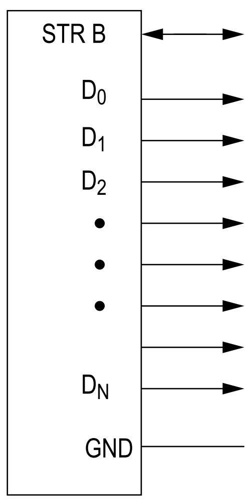
Tree



Communication Methodologies



SERIAL



PARALLEL

Serial

A communications channel where data is sent sequentially one bit at a time. RS-232 and RS-485 are common interfaces of this type.

Parallel

A communications channel where data is sent several bits as a whole. IEEE 1284 is a common interface.

Simplex

A single channel where communications is one direction only.

Half-Duplex

Provides communications in two directions but only one at a time

Full Duplex (Duplex)

Allows communications in both directions simultaneously

Computer Systems

Memory/Storage Types

RAM – Primary memory system in computing systems, volatile

Cache – faster, but smaller segment of memory used for buffering immediate data from slower memories

- L1: Level 1 cache, fastest memory available
- L2: Level 2 cache, next level away from CPU. May or may not be exclusive of L1 depending on architecture

ROM – nonvolatile. Contains system instructions or constant data for the system

Replacement Policy – For set associative and fully associative caches, if there is a miss and the set or cache (respectively) is full, then a block must be selected for replacement. The replacement policy determines which block is replaced. Common replacement policies are:

- Least recently used (LRU): Replace the least recently used block.
- Most recently used (MRU): Replace the most recently used block.
- First-in, first-out (FIFO): Also referred to as first come, first serve (FCFS) queue. Data is processed in the order it entered the buffer.
- Last-in, first-out (LIFO): Also referred to as a stack. Youngest (last) item is processed first.
- Random: Choose a block at random for replacement.
- Least frequently used (LFU): Replace the block that had the fewest references among the candidate blocks.

Write Policy – With caches, multiple copies of a memory block may exist in the system (e.g., a copy in the cache and a copy in main memory). There are two possible write policies.

- Write-through: Write to both the cache's copy and the main memory's copy.
- Write-back: Write only to the cache's copy. This requires adding a "dirty bit" for each block in the cache. When a block in the cache is written to, its dirty bit is set to indicate that the main memory's copy is stale. When a dirty block is evicted from the cache (due to a replacement), the entire block must be written back to main memory. Clean blocks need not be written back when they are evicted.

Cache Size – C (bytes) = S*A*B

where

S = Number of sets
A = Set associativity
B = Block size (bytes)

To search for the requested block in the cache, the CPU will generally divide the address into three fields: the tag, index, and block offset.

TAG	INDEX	BLOCK OFFSET

- *Tag* – These are the most significant bits of the address, which are checked against the current row (the row that has been retrieved by index) to see if it is the one needed or another, irrelevant memory location that happened to have the same index bits as the one wanted.

  # tag bits = # address bits – # index bits – # block offset bits

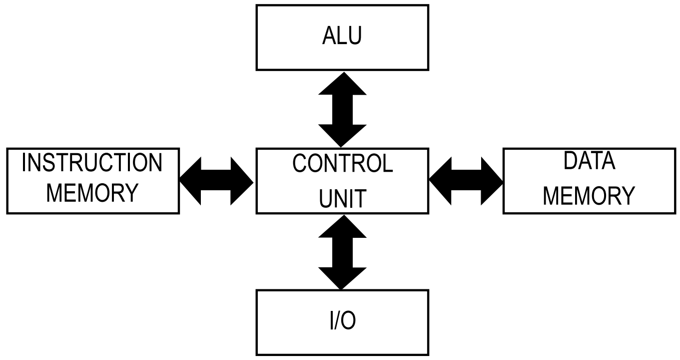
- *Index* – These bits specify which cache row (set) that the data has been put in.

  # index bits = $\log_2$(# sets) = $\log_2$(S)

- *Block Offset* – These are the lower bits of the address that select a byte within the block.

  # block offset bits = $\log_2$(block size) = $\log_2$(B)

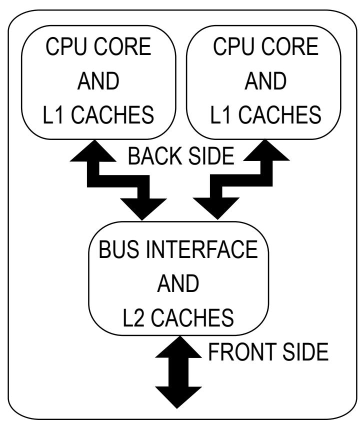
Microprocessor Architecture – Harvard



Multicore

A multicore processor is a single computing component with two or more independent actual processing units (called cores), which are the units that read and execute program instructions. The instructions are ordinary CPU instructions such as Add, Move Data, and Branch, but the multiple cores can run multiple instructions at the same time, increasing overall speed for programs amenable to parallel computing.

A multicore processor implements multiprocessing in a single physical package. Designers may couple cores in a multicore device tightly or loosely. For example, cores may or may not share caches, and they may implement message passing or shared memory intercore communication methods. Common network topologies to interconnect cores include bus, ring, two-dimensional mesh, and crossbar. Homogeneous multicore systems include only identical cores; heterogeneous multicore systems have cores that are not identical. Just as with single-processor systems, cores in multicore systems may implement architectures such as superscalar, VLIW, vector processing, SIMD, or multithreading.



Generic dual-core processor, with CPU-local level 1 caches, and a shared, on-die level 2 cache

Threading

In computer science, a thread of execution is the smallest sequence of programmed instructions that can be managed independently by a scheduler, which is typically a part of the operating system. The implementation of threads and processes differs between operating systems, but in most cases a thread is a component of a process. Multiple threads can exist within the same process and share resources such as memory, while different processes do not share these resources. In particular, the threads of a process share its instructions (executable code) and its context (the values of its variables at any given moment).

On a single processor, multithreading is generally implemented by time-division multiplexing (as in multitasking), and the CPU switches between different software threads. This context switching generally happens frequently enough that the user perceives the threads or tasks as running at the same time. On a multiprocessor or multicore system, threads can be executed in a true concurrent manner, with every processor or core executing a separate thread simultaneously. To implement multiprocessing, the operating system may use hardware threads that exist as a hardware-supported method for better utilization of a particular CPU. These are different from the software threads that are a pure software construct with no CPU-level representation.

Abbreviation

CISC Complex instruction set computing

CPU Central processing unit

FIFO First-in, first-out

LIFO Last-in, first-out

I/O Input/output

LFU Least frequently used

LRU Least recently used

MRU Most recently used

RISC Reduced instruction set computing

RAM Random access memory

ROM Read only memory

Software Engineering

Endianness

MSB – most significant bit first. Also known as Big-endian.

LSB – least significant bit first. Also known as Little-endian.

Pointers

A pointer is a reference to an object. The literal value of a pointer is the object's location in memory. Extracting the object referenced by a pointer is defined as dereferencing.

Algorithms

An algorithm is a specific sequence of steps that describe a process.

Sorting Algorithm – an algorithm that transforms a random collection of elements into a sorted collection of elements.

Examples include:

Bubble Sort: continuously steps through a list, swapping items until they appear in the correct order.

Insertion Sort: takes elements from a list one by one and inserts them in their correct position into a new sorted list.

Merge Sort: divides the list into the smallest unit (e.g., 1 element), then compares each element with the adjacent list to sort and merge the two adjacent lists. This process continues with larger lists until at last, two lists are merged into the final sorted list.

Heap Sort: divides a list into sorted and an unsorted lists and extracts the largest element from the unsorted list and moves it to the bottom of the sorted list.

Quick Sort: partitions list using a pivot value, placing elements smaller than the pivot before the pivot value and greater elements after it. The lesser and greater sublists are then recursively sorted.

Searching Algorithm – an algorithm that determines if an element exists in a collection of elements. If the element does exist, its location is also returned. Examples include:

Binary search: finds a search value within a sorted list by comparing the search value to the middle element of the array. If they are not equal, the half in which the target cannot lie is eliminated and the search continues on the remaining half, again taking the middle element to compare to the target value, and repeating this until the target value is found.

Hashing: uses a hashing function that maps data of arbitrary size (e.g., a string of characters) to data of a fixed size (e.g., an integer) and then to compute an index that suggests where the entry can be found in a hash table (an array of buckets or slots, from which the desired value can be found through the index).

Data Structures

Collection – a grouping of elements that are stored and accessed using algorithms. Examples include:

Array: collection of elements, typically of the same type, where each individual element can be accessed using an integer index.

Linked list: collection of nodes, where each node contains an element and a pointer to the next node in the linked list (and sometimes back to the previous node).

Stack: collection of elements that are kept in order and can only be accessed at one end of the set (e.g., last in, first out (LIFO))

Queue: collection of elements that are kept in order and can be accessed at both ends of the set where one is used to insert elements and the other end is used to remove elements.

Map: collection of key, value pairs, such that each possible key appears at most once in the collection. Also known as an associative array.

Set: collection of elements, without any particular order, that can be queried (static sets) and/or modified by inserting or deleting elements (dynamic set).

Graph: collection of nodes and a set of edges that connect a pair of nodes.

Tree: collection of nodes and a set of edges that connect the nodes hierarchically. One node is distinguished as a root and every other node is connected by a directed edge from exactly one other node in a parent to child relationship. A binary tree is a specialized case where each parent node can have no more than two children nodes.

Graph Traversal

There are primarily two algorithms used to parse through each node in a graph.

Breadth First Search – Beginning at a given node, the algorithm visits all connected nodes that have not been visited. The algorithm repeats for each visited node. The output of the algorithm is a list of nodes in the order that they have been visited. A queue data structure can be used to facilitate this algorithm.

Depth First Search – Beginning at a given node, the algorithm visits one connected node that has not been visited. This is repeated until a node does not have any connected nodes that have not been visited. At this point the algorithm backtracks to the last visited node and repeats the algorithm. The output of the algorithm is a list of nodes in the order that they have been visited. A stack can be used to facilitate this algorithm.

Tree Traversal

There are three primary algorithms that are used to traverse a binary tree data structure.

In-Order Traversal

1. Traverse the left sub-tree.
2. Visit the root node.
3. Traverse the right sub-tree.

Preorder Traversal

1. Visit the root node.
2. Traverse the left sub-tree.
3. Traverse the right sub-tree.

Postorder Traversal

1. Traverse the left sub-tree.
2. Traverse the right sub-tree.
3. Visit the root node.

Algorithm Efficiency (Big-O)

The concept of Big O Notation is used in software engineering to determine the efficiency of an algorithm. Big O equations are written as:

$$O(n) = f(n)$$

When comparing the efficiency of two algorithms, compare two O(n) values as n approaches infinity.

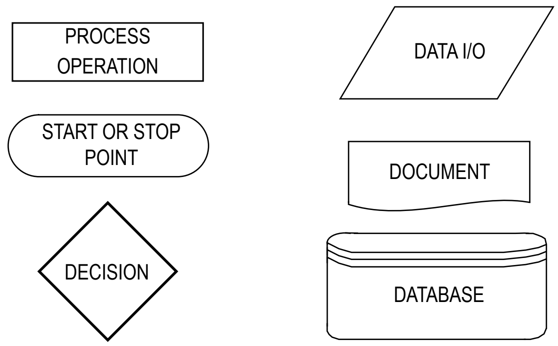
Notation	Name	Example (Worst Case)
$O(\log n)$	Logarithmic	Binary tree traversal, Hash table search
$O(n \log(n)) = O(\log n!)$	Loglinear	Merge sort, Heap sort, Fast Fourier Transform
$O(n^2)$	Quadratic	Insertion sort, Bubble sort, Quick sort

Software Syntax Guidelines

- Code is pseudocode, no specific language
- No end-of-line punctuation (e.g., semicolon) is used
- Comments are indicated with "--" double hyphen
- Loop structures end with "end" followed by structure name, e.g., "end while"
- "do-while" begins with "do" and ends with "while"—no "end" per se
- "if-then" statements have both "if" and "then"
- "else if" is a substitute for the "end" on the preceding "if"
- "=" is used to designate assignment. "==" refers to comparison in a conditional statement.
- Not equals is represented by <>
- Logical "and" and "or" are spelled out as "and" and "or"
- Variable and argument declarations are Pascal style—"name: type"
- Numeric data types are "integer" and "float"
- Text is a procedural variable, unless specified to be an object of type String
- Variables can be constant, and are declared with the "const" modifier
- Variables whose type is object and the exact specification of that object is not critical to the problem must have the data type obj
- Array indices are designated with square brackets [], not parentheses
- Unless otherwise specified, arrays begin at 1 (one)
- Compilation units are "procedure" and "function". "Module" is not a compilation unit
- Function parameters are designated with parentheses ()
- Unless specified, procedures and functions must have the return type "void"
- Arguments in a function/procedure call are separated by semicolons
- Class definitions start with "cls" (e.g., clsClassName)
- Classes, properties, and procedures are by default public and may be optionally modified by "private" or "protected"
- To instantiate an object, the follow syntax must be used: new clsName objName
- For input, read ("filename.ext", <variable list>)—if reading from console, do not use the first argument
- For output, write ("filename.ext", <expression list>)—if writing to console, do not use the first argument
- The Boolean data type is "boolean"; the return result of all comparison operators is a boolean type

- The operator "*" in front of a variable is used to return the data at the address location within that variable
- The operator "&" in front of a variable is used to return the address of a given variable. The declaration of "pointer_to" is used to define a variable of a pointer type

Flow Chart Definition



Software Testing

There are many approaches to software testing but they are typically split into static testing versus dynamic and black box versus white box testing.

Static Testing: techniques that do not execute the code but concentrate on checking the code, requirement documents and design documents. Examples: code reviews and walkthroughs and compiler syntax and structure checks.

Dynamic Testing: techniques that take place when the code is executed and is performed in the runtime environment. Examples: unit, integration, system, and acceptance testing.

Black Box Testing: examines functionality without knowledge of the internal code. Also known as functional testing, the approach oftentimes concentrates on checking performance against specifications and also avoids programmer bias.

White Box Testing: verifies the internal structures and workings of a code. The approach is a necessary part of software testing at the unit, integration and system levels, needed to uncover errors or problems, but does not detect unimplemented parts of the specification or missing requirements.

Computer Network Security

Source for material in Computer Network Security: Barrett, Diane, Martin M. Weiss, and Kirk Hausman, *CompTIA Security+™ SYO-401 Exam Cram*, 4th ed., Pearson IT Certification, Pearson Education, Inc., 2015.

Firewalls

A network security system that monitors and controls incoming and outgoing network traffic based on predetermined security rules. A firewall typically establishes a barrier between a trusted internal network and untrusted external network, such as the Internet.

Nmap

Usage: nmap [Scan Type(s)] [Options] {target specification}

Target Specification

Can pass hostnames, IP addresses, networks, etc.

Ex: scanme.nmap.org, microsoft.com/24, 192.168.0.1; 10.0.0-255.1-254

Host Discovery

sL: List Scan - simply list targets to scan

sn: Ping Scan - disable port scan

PS/PA/PU/PY[portlist]: TCP SYN/ACK, UDP or SCTP discovery to given ports

PE/PP/PM: ICMP echo, timestamp, and netmask request discovery probes

PO[protocol list]: IP Protocol Ping

dns-servers: Specify custom DNS servers

system-dns: Use OS's DNS resolver

traceroute: Trace hop path to each host

Scan Techniques

sS/sT/sA/sW/sM: TCP SYN/Connect()/ACK/Window/Maimon scans

sU: UDP Scan

sN/sF/sX: TCP Null, FIN, and Xmas scans

scanflags: Customize TCP scan flags

sO: IP protocol scan

b: FTP bounce scan

Port Specification and Scan Order

p: Only scan specified ports

Ex: -p22; -p1-65535; -p U:53,111,137,T:21-25,80,139,8080,S:9

Service/Version Detection

sV: Probe open ports to determine service/version info

OS Detection

O: Enable OS detection

Timing and Performance

Options which take <time> are in seconds, or append 'ms' (milliseconds),

's' (seconds), 'm' (minutes), or 'h' (hours) to the value (e.g., 30m).

max-retries: Caps number of port scan probe retransmissions.

host-timeout: Give up on target after this long

scan-delay/--max-scan-delay: Adjust delay between probes

min-rate: Send packets no slower than per second

max-rate: Send packets no faster than per second

Firewall/IDS Evasion and Spoofing

S: Spoof source address

e: Use specified interface

g/--source-port: Use given port number

data-length: Append random data to sent packets

Output

-oN/-oX/-oS/-oG: Output scan in normal, XML, s|: Output in the three major formats at once

open: Only show open (or possibly open) ports

packet-trace: Show all packets sent and received

Misc.

6: Enable IPv6 scanning

A: Enable OS detection, version detection, script scanning, and traceroute

V: Print version number

h: Print this help summary page.

Examples

nmap -v -A scanme.nmap.org

nmap -v -sn 192.168.0.0/16 10.0.0.0/8

nmap -v -iR 10000 -Pn -p 80

Port Scanning

Generally either TCP or UDP ports are scanned. Types of TCP scans include SYN, TCP Connect, NULL, FIN, XMAS

Common TCP Ports

Protocol	Port Number
FTP	20, 21
Telnet	23
HTTP	80
HTTPS	443
POP3	110
SMTP	25
TLS	587

Web Vulnerability Testing

OWASP – Open Web Application Security Project. Online community that provides many open source resources for web application security

Cross Site Scripting(XSS) – script injection attack, using a web application to send an attack to another user

Cross Site Request Forgery(CRSF) – an attack that forces user to perform unwanted actions with current authorizations. Usually coupled with a social engineering attack.

SQL Injection(SQLi) – injection attack, by inserting SQL query via input data from the client to the application for execution. The statements usually insert, select, delete or update stored data in the SQL database.

Endpoint Detection – collection and storage of endpoint data activity to help network administrators analyze, investigate and prevent cyber threats on a network.

WEP – Wired Equivalent Privacy – Uses 40 bit(10 hex digits) or 104(26 hex digits) bit key

WPA– Wifi Protected Access – Replacement for WPA, added TKIP and MIC

WPA2 – Replaced WPA and implements all mandatory elements of 802.11i, particularly mandatory support for CCMP(AES encryption mode)

WPA3 – Replaces WPA2. Replaces PSK with Simultaneous Exchange of Equals

Penetration Testing—Authorized Vulnerability Testing

Phases

1. Reconnaissance
2. Scanning
3. Gaining Access
4. Maintaining Access
5. Covering Tracks

Methods

External testing—Only systems and assets that are visible on the internet, such as the web application itself, are targeted. The goal of the testing is to gain access to the application and its data.

Internal testing—The pen tester has access to the application behind the firewall.

Blind testing—The pen tester is given the name of the company, but nothing else. This simulates an actual application attack in real-time.

Double-blind testing—This is similar to a blind test, but the security team is not made aware of the simulation.

Targeted testing—The penetration tester and security team work together, informing each other of steps taken to attack the application and to defend against the attack. (Red Team vs Blue Team)

Security Triad

AIC—Availability, Integrity, Confidentiality (also referred to as CIA Triad)

Availability—guarantee of reliable access to information by authorized entities

Integrity—assurance information is trustworthy and accurate

Confidentiality—set of rules that limits access to information

Authentication

Three factors for authentication

Something you know (password, PIN, etc)
Something you have (token, smart card, etc)
Something you are (biometrics, etc)

AAA protocols (Authentication, Authorization, Accounting)

TACACS, XTACACS, TACACS+—Terminal Access Controller Access Control System
RADIUS—Remote Authentication Dial In User Service
DIAMETER—Enhancement for RADIUS.

PPP protocols

PAP—Password Authentication Protocol
CHAP—Challenge Handshake Authentication Protocol
EAP—Extensible Authentication Protocol

Other protocols

Kerberos—authentication system using a Key Distribution Center

Key Equations

Assume that "*" implies multiplication.

<u>McCabe's Cyclomatic Complexity</u>

$$c = e - n + 2$$

where for a single program graph, n is the number of nodes, e is the number of edges, and c is the cyclomatic complexity.

<u>The RSA Public-Key Cryptosystem</u>

$$n = p * q$$

where p and q are both primes.

$$e * d = 1 \pmod{t}$$

where t = least common multiple ($p - 1$, $q - 1$)

- The encrypted cyphertext c of a message m is $c = m^e \pmod{n}$
- The decrypted message is $m = c^d \pmod{n}$
- The signature s of a message m is $s = m^d \pmod{n}$

<u>Diffie-Hellman Key-Exchange Protocol</u>

A sender and receiver separately select private keys x and y. Generator value g and prime number p is shared between the two. Their shared secret key k is:

$$k = (g^x)^y \pmod{p} = (g^y)^x \pmod{p}$$