Econ 521: Econometric Methods I
Assignment 1

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1. Consider the following regression model

$$\begin{aligned} &y = g(\mathbf{x}, \mathbf{z}_1) + \varepsilon; && \mathbf{z} = [\mathbf{z}_1', \mathbf{z}_2']', \\ &\mathbf{x} = \mathbf{h}(\mathbf{z}) + \mathbf{u}; && \mathbb{E}[\mathbf{u}|\mathbf{z}] = 0, \, \mathbb{E}[\varepsilon|\mathbf{z}, \mathbf{u}] = \mathbb{E}[\varepsilon|\mathbf{u}], \end{aligned}$$

where y is an observable scalar random variable, g denotes a known scalar function, $\mathbf{x}$ is a $d_{\mathbf{x}} \times 1$ vector of explanatory variables, $\mathbf{z}_1$ and $\mathbf{z}_2$ are $d_1 \times 1$ and $d_2 \times 1$ vectors of instrumental variables, $\mathbf{h} := [h_1, \ldots, h_{d_{\mathbf{x}}}]'$ is a $d_{\mathbf{x}} \times 1$ vector of functions of instruments $\mathbf{z}$, and $\mathbf{u}$ and ε are disturbances. What does $\mathbb{E}[y|\mathbf{x}, \mathbf{z}, \mathbf{u}]$ equal to?

Answer:

$\mathbb{E}[y|\mathbf{x}, \mathbf{z}, \mathbf{u}] = \mathbb{E}\left[\left(g\left(\mathbf{x}, \mathbf{z}_1\right) + \varepsilon\right)|\mathbf{x}, \mathbf{z}, \mathbf{u}\right]$

Substituting definitions above.

$\mathbb{E}[y|\mathbf{x}, \mathbf{z}, \mathbf{u}] = \mathbb{E}\left[g\left(\mathbf{x}, \mathbf{z}_1\right))|\mathbf{x}, \mathbf{z}, \mathbf{u}\right] + \mathbb{E}[\varepsilon|\mathbf{x}, \mathbf{z}, \mathbf{u}]$

Addition Rule of Expectation.

$\mathbb{E}[y|\mathbf{x}, \mathbf{z}, \mathbf{u}] = g\left(\mathbf{x}, \mathbf{z}_1\right) + \mathbb{E}[\varepsilon|\mathbf{x}, \mathbf{z}, \mathbf{u}]$

Because $g\left(\mathbf{x}, \mathbf{z}_1\right)$ is a function, its expectation is equal to the function.

By the law of iterated expectations:
$\mathbb{E}[y|\mathbf{x}, \mathbf{z}, \mathbf{u}] = g\left(\mathbf{x}, \mathbf{z}_1\right) + \mathbb{E}[\varepsilon|\mathbf{x}, \mathbf{u}]$
$\mathbb{E}[y|\mathbf{x}, \mathbf{z}, \mathbf{u}] = g\left(\mathbf{x}, \mathbf{z}_1\right) + \mathbb{E}[\varepsilon|(\mathbf{h}(\mathbf{z}) + \mathbf{u}), \mathbf{u}]$
$\mathbb{E}[y|\mathbf{x}, \mathbf{z}, \mathbf{u}] = g\left(\mathbf{x}, \mathbf{z}_1\right) + \mathbb{E}[\varepsilon|\mathbf{z}, \mathbf{u}]$
$\mathbb{E}[y|\mathbf{x}, \mathbf{z}, \mathbf{u}] = g\left(\mathbf{x}, \mathbf{z}_1\right) + \mathbb{E}[\varepsilon|\mathbf{u}]$

2. Let x be an absolutely continuous random variable with strictly increasing cdf F_x. Let $\widehat{q}$ be the value that minimizes $\mathbb{E}[\rho_\tau(x-q)]$ with respect to q, where $\rho_\tau(u) = u[\tau - \mathbb{I}(u<0)]$ and $\mathbb{I}(A)$ is called the indicator function that equals one if A is true and 0 otherwise. Show that $\widehat{q} \equiv F_x^{-1}(\tau)$. Hint: Use the Leibniz integral rule.

Answer:

Let $\widehat{q}$ be the value that minimizes $\mathbb{E}\left[\rho_\tau(x-q)\right]$ with respect to q, where $\rho_\tau(u) = u[\tau - \mathbb{I}(u<0)]$

By definition, $\rho_\tau(x-q) = (x-q)[\tau - \mathbb{I}((x-q)<0)]$
So then $\mathbb{E}\left[\rho_\tau(x-q)\right] = \mathbb{E}[(x-q)[\tau - \mathbb{I}((x-q)<0)]]$

Because $\mathbb{I}(u<0)$ is an indicator function,
$\mathbb{I}(x<q) = 1$ for $x<q$ and $\mathbb{I}(x<q) = 0$ for $x>q$.

$\mathbb{E}\left[\rho_\tau(x-q)\right] = \mathbb{E}[(x-q)[\tau - \mathbb{I}((x-q)<0)]]$

$\mathbb{E}\left[\rho_\tau(x-q)\right] = \int_{-\infty}^{\infty} \rho_\tau(x-q) f(x) dx$

$\mathbb{E}\left[\rho_\tau(x-q)\right] = \int_{-\infty}^{\infty} \rho_\tau(x-q) dF(x)$

$\mathbb{E}\left[\rho_\tau(x-q)\right] = \int_{-\infty}^{\infty} (x-q)[\tau - \mathbb{I}((x-q)<0)] dF(x)$

$\mathbb{E}\left[\rho_\tau(x-q)\right] = \int_{-\infty}^{\infty} (x-q)\tau dF(x) - \int_{-\infty}^{\infty} (x-q)\mathbb{I}((x-q)<0) dF(x)$

$\mathbb{E}\left[\rho_\tau(x-q)\right] = \int_{-\infty}^{\infty} (x-q)\tau dF(x)[\mathbb{I}(x<q) + \mathbb{I}(x>q)]$

$\mathbb{E}\left[\rho_\tau(x-q)\right] = \tau \int_{-\infty}^{\infty} (x-q)\mathbb{I}(x>q) dF(x) - (1-\tau) \int_{-\infty}^{\infty} (x-q)\mathbb{I}(x<q) dF(x)$

$\mathbb{E}\left[\rho_\tau(x-q)\right] = \tau \int_{q}^{\infty} (x-q) dF(x) - (1-\tau) \int_{-\infty}^{q} (x-q) dF(x)$

To find the value $\widehat{q}$ which minimizes $\mathbb{E}\left[\rho_\tau(x-q)\right]$, employ the first order condition:

$\frac{\partial}{\partial q}\mathbb{E}\left[\rho_\tau(x-q)\right] = \frac{\partial}{\partial q}\tau \int_{q}^{\infty} (x-q) dF(x) - (1-\tau) \int_{-\infty}^{q} (x-q) dF(x) = 0$

By the Leibniz integral rule,

$\frac{\partial}{\partial q}\mathbb{E}\left[\rho_\tau(x-q)\right] = \tau \int_{q}^{\infty} \frac{\partial}{\partial q}(x-q) dF(x) - (1-\tau) \int_{-\infty}^{q} \frac{\partial}{\partial q}(x-q) dF(x) = 0$

$\frac{\partial}{\partial q}\mathbb{E}\left[\rho_\tau(x-q)\right] = -\tau \int_{q}^{\infty} dF(x) + (1-\tau) \int_{-\infty}^{q} dF(x) = 0$

$\frac{\partial}{\partial q}\mathbb{E}\left[\rho_\tau(x-q)\right] = -\tau \int_{q}^{\infty} dF(x) - \tau \int_{-\infty}^{q} dF(x) + \int_{-\infty}^{q} dF(x) = 0$

$\frac{\partial}{\partial q}\mathbb{E}\left[\rho_\tau(x-q)\right] = -\tau \left(\int_{q}^{\infty} dF(x) + \int_{-\infty}^{q} dF(x) \right) + \int_{-\infty}^{q} dF(x) = 0$

$\frac{\partial}{\partial q}\mathbb{E}\left[\rho_\tau(x-q)\right] = -\tau + \int_{-\infty}^{q} dF(x) = 0$

Thus, $\int_{-\infty}^{q} dF(x) = \tau$

For $\min(\text{q}) = \widehat{q}$

$\int_{-\infty}^{q} f(x) = F_x(\widehat{q}) = \tau$

$\widehat{q} \equiv F_x^{-1}(\tau)$

3. Let y be the response variable variable, $\mathbf{x}$ a set of $d_{\mathbf{x}} \times 1$ conditioning variables, and s a scalar binary group indicator (such as gender, college graduate versus non-college graduate, and so on). Define $\mu_0(\mathbf{x}) = \mathbb{E}[y|\mathbf{x}, s = 0]$ and $\mu_1(\mathbf{x}) = \mathbb{E}[y|\mathbf{x}, s = 1]$ to be the regression functions for the two groups.

   (a) Show that

$$\begin{aligned}\mathbb{E}[y|s = 1] - \mathbb{E}[y|s = 0] =& \{\mathbb{E}[\mu_1(\mathbf{x})|s = 1] - \mathbb{E}[\mu_0(\mathbf{x})|s = 1]\} \\ &+ \{\mathbb{E}[\mu_0(\mathbf{x})|s = 1] - \mathbb{E}[\mu_0(\mathbf{x})|s = 0]\},\end{aligned}$$

   Hint: Use a suitable representation of $\mathbb{E}[y|\mathbf{x}, s]$ as a function of $\mu_0(\mathbf{x})$ <u>and</u> $\mu_1(\mathbf{x})$, and then apply the *Law of Iterated Expectations*.

   Answer:

   $\mu_0(\mathbf{x}) = \mathbb{E}[y|\mathbf{x}, s = 0]$
   $\mu_1(\mathbf{x}) = \mathbb{E}[y|\mathbf{x}, s = 1]$
   By the Law of Iterated Expectations:
   $\mathbb{E}[y|\mathbf{x}, s] = \mathbb{E}[\mathbb{E}[y|x, s|s]$
   $\mathbb{E}[y|s = 0] = \mathbb{E}\left[\mu_0|s = 0\right]$
   $\mathbb{E}[y|s = 1] = \mathbb{E}\left[\mu_1|s = 1\right]$
   So one can simplify:
   $\mathbb{E}[y|s = 1] - \mathbb{E}[y|s = 0] = \mathbb{E}\left[\mu_1|s = 1\right] - \mathbb{E}\left[\mu_0|s = 0\right]$
   $\mathbb{E}[y|s = 1] - \mathbb{E}[y|s = 0] = \mathbb{E}\left[\mu_1|s = 1\right] - \mathbb{E}\left[\mu_0|s = 1\right] + \mathbb{E}\left[\mu_0|s = 1\right] - \mathbb{E}\left[\mu_0|s = 0\right]$

   (b) Suppose both expectations are linear: $\mu_s(\mathbf{x}) = \mathbf{x}'\beta_s$, $s \in \{0, 1\}$. Show that

   $\mathbb{E}[y|s = 1] - \mathbb{E}[y|s = 0] = \mathbb{E}[\mathbf{x}'|s = 1] \times \{\beta_1 - \beta_0\} + \{\mathbb{E}[\mathbf{x}'|s = 1] - \mathbb{E}[\mathbf{x}'|s = 0]\} \times \beta_0.$

   Can you interpret this decomposition?

   Answer:

   $\mu_s(\mathbf{x}) = \mathbf{x}'\beta_s$

So:
$\mu_0(\mathbf{x}) = \mathbf{x}'\beta_0$
$\mu_1(\mathbf{x}) = \mathbf{x}'\beta_1$

And:
$\mathbb{E}[y|s=1] - \mathbb{E}[y|s=0] = \mathbb{E}\left[\mu_1|s=1\right] - \mathbb{E}\left[\mu_0|s=1\right] + \mathbb{E}\left[\mu_0|s=1\right] - \mathbb{E}\left[\mu_0|s=0\right]$

Substituting μ_0 and μ_1 :

$$\begin{aligned}\mathbb{E}[y|s=1] - \mathbb{E}[y|s=0] =& \mathbb{E}\left[\mathbf{x}'\beta_1|s=1\right] - \mathbb{E}\left[\mathbf{x}'\beta_0|s=1\right] \\ &+ \mathbb{E}\mathbf{x}'\beta_0|s=1] - \mathbb{E}\left[\mathbf{x}'\beta_0|s=0\right]\end{aligned}$$

Factoring yields:
$\mathbb{E}[y|s=1] - \mathbb{E}[y|s=0]$
$= \mathbb{E}\left[\mathbf{x}'|s=1\right] \times \{\beta_1 - \beta_0\} + \{\mathbb{E}\left[\mathbf{x}'|s=1\right] - \mathbb{E}\left[\mathbf{x}'|s=0\right]\} \times \beta_0$

One can interpret this decomposition as the change in the portion of outcome variable y, respectively due to underlying characteristics (β_0) and due to the group indicator $(\beta_1 - \beta_0))$.

It is clear that β_0 represents the change in outcome variable common between the two groups while $(\beta_1 - \beta_0)$ represents the difference between the two groups treatment characteristic (gender, college graduate vs. non-college graduate, etc.).

If $\beta_1 = 0$ then, the two groups' outcomes are similar.