Practice Problems: Lesson 5 - Picard-Lindelöf Theorem

Master uniqueness conditions!

Part A: Lipschitz Verification

Determine if $f(x,y)$ is Lipschitz in y on the given domain:

1. $f(x,y) = 3y + x^2$ on $\mathbb{R}^2$
2. $f(x,y) = y^3$ on $|y| \leq 2$
3. $f(x,y) = \sqrt{y}$ on $y \geq 1$
4. $f(x,y) = e^y$ on $y \leq 5$
5. $f(x,y) = \frac{y}{1+y^2}$ on $\mathbb{R}$

Part B: Finding Lipschitz Constants

Find the smallest Lipschitz constant L for the given function and domain:

6. $f(x,y) = 2xy$ on $|x| \leq 3$, $|y| \leq 1$
7. $f(x,y) = \sin(y) + xy$ on $|x| \leq 2$, $y \in \mathbb{R}$
8. $f(x,y) = y^2 - 2y$ on $0 \leq y \leq 3$

Part C: Uniqueness Analysis

For each IVP, determine if Picard-Lindelöf guarantees a unique solution:

9. $y' = y\cos(x)$, $y(0) = 1$
10. $y' = |y|^{1/2}$, $y(0) = 0$
11. $y' = y^2 + x^2$, $y(0) = 0$ (on bounded domain)
12. $y' = 3y^{2/3}$, $y(0) = 0$

Part D: Picard Iteration

Apply Picard iteration (first 3 iterations) to approximate the solution:

13. $y' = x + y$, $y(0) = 0$
    (a) Find y_0, y_1, y_2, y_3
    (b) Guess the pattern for y_n
    (c) What's the exact solution?
14. $y' = 2xy$, $y(0) = 1$
    (a) Compute first three iterations
    (b) Compare with the exact solution $y = e^{x^2}$

Part E: Comparing Solutions

15. If $y' = 2y$ with $y(0) = 1$ and $z' = 2z$ with $z(0) = 1.01$:
    (a) Find the Lipschitz constant
    (b) Bound $|y(x) - z(x)|$ for $x \in [0, 1]$
    (c) Find the exact difference at $x = 1$
16. For $y' = -y + \sin(x)$ with different initial conditions:
    (a) If $|y_1(0) - y_2(0)| = 0.1$, bound the difference at $x = 5$
    (b) Is this bound sharp?

Part F: Local vs Global Lipschitz

17. Classify each as globally Lipschitz, locally Lipschitz, or not Lipschitz:
    (a) $f(y) = y^3$
    (b) $f(y) = \arctan(y)$
    (c) $f(y) = y + \sin(y)$
    (d) $f(y) = \sqrt{|y|}$
    (e) $f(y) = e^{-y^2}$
18. For $f(y) = y^n$:
    (a) When is it globally Lipschitz?
    (b) When is it locally Lipschitz?
    (c) Find the Lipschitz constant on $|y| \leq M$

Part G: Theoretical Questions

19. Prove that if f and g are Lipschitz with constants L_f and L_g, then:
    (a) $f + g$ is Lipschitz with constant $L_f + L_g$
    (b) cf is Lipschitz with constant $|c|L_f$
20. Show that $f(y) = |y|$ is Lipschitz with $L = 1$ even though it's not differentiable at $y = 0$.
21. Explain why linear ODEs $y' + p(x)y = q(x)$ always have unique solutions when p, q are continuous.

Part H: Exam-Style Problems

22. Professor Ditkowski gives: $y' = |y - 1|^\alpha$, $y(0) = 1$
    (a) For which $\alpha > 0$ is the solution unique?
    (b) Find all solutions when $\alpha = 1/2$
    (c) What happens as $\alpha \to 0^+$?
23. Consider the piecewise function:

$$f(x, y) = \begin{cases} \frac{y^2}{|y|} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

    (a) Is f continuous at $y = 0$?
    (b) Is f Lipschitz near $y = 0$?
    (c) Does the IVP with $y(0) = 0$ have a unique solution?
24. The Picard iteration for $y' = f(x, y)$, $y(0) = 1$ gives:

$$y_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

    (a) What is $f(x, y)$?
    (b) Verify it's Lipschitz
    (c) What's the exact solution?

Part I: Advanced Applications

25. For the delay equation $y'(x) = y(x - 1)$ with $y(x) = 1$ for $x \in [-1, 0]$:
    (a) Is this covered by Picard-Lindelöf?

(b) Find the solution on $[0, 1]$

(c) Is it unique?

26. Systems: For $\mathbf{y}' = A\mathbf{y}$ where A is a constant matrix:

    (a) Show the system is Lipschitz

    (b) What's the Lipschitz constant in terms of A?

    (c) Why is the solution always unique?

Answer Key

Part A: 1. Yes, $L = 3$ (linear in y) 2. Yes, $L = 12$ (bounded domain) 3. Yes, $L = 1/2$ (derivative bounded on $y \geq 1$) 4. Yes, $L = e^5$ (bounded domain) 5. Yes, $L = 1$ (derivative analysis)

Part C: 9. Yes (linear in y) 10. No (not Lipschitz at $y = 0$) 11. Yes (locally on bounded domain) 12. No ($\partial f/\partial y = 2y^{-1/3} \to \infty$ as $y \to 0$)

Key Insight: Lipschitz $\Rightarrow$ Unique solution. Check $\partial f/\partial y$ bounded!