Practice Problems: Implicit and Singular Solutions

Lesson 13 Exercises

August 23, 2025

Part A: Implicit Solutions - Recognition and Verification (6 problems)

1. Verify that $x^2 - xy + y^2 = C$ is an implicit solution of some ODE. Find the ODE.

2. Show that $e^y + e^{-x} = C$ solves $\frac{dy}{dx} = e^{y-x}$

3. For the circles $x^2 + y^2 = r^2$, find the differential equation they satisfy.

4. Verify that $\sin(x + y) = x$ defines an implicit solution. Find the corresponding ODE.

5. Can $xy + \ln(xy) = C$ be solved explicitly for y? Find the ODE it satisfies.

6. Show that the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (with $a/b = k$ constant) satisfies a certain ODE.

Part B: Finding Singular Solutions - Basic (6 problems)

7. Find all solutions of $(y')^2 = 4y$, including any singular solutions.

8. For the general solution $y = Cx + C^2$, find the singular solution using the C-discriminant.

9. Show that $y = 0$ is a singular solution of $y' = 2\sqrt{|y|}$

10. Find the singular solution of $(y')^2 - xy' + y = 0$ (Clairaut type).

11. Given general solution $(y - Cx)^2 = 1 + C^2$, find the envelope.

12. Determine if $y(y')^2 - 2xy' + 4x = 0$ has singular solutions.

Part C: Clairaut's Equation and Variants (6 problems)

13. Solve completely: $y = xy' + \sqrt{1 + (y')^2}$

14. Find general and singular solutions: $y = xy' - \frac{1}{y'}$

15. Solve: $y = xy' + a(y')^2$ where a is a constant.

16. Transform and solve: $2y = xy' + \ln(y')$

17. Find all solutions: $y - xy' = e^{y'}$

18. Solve the generalized Clairaut: $y = xf(y') + g(y')$

Part D: p-Discriminant Method (5 problems)

19. Use p-discriminant to find singular solutions of: $y = x(y')^2 - (y')^3$

20. Apply p-discriminant to: $(y')^3 - xy' + y = 0$

21. Find singular solutions: $x(y')^2 - 2yy' + 4x = 0$

22. Determine all solutions of: $8(y')^3 = 27y^2$ using p-discriminant.

23. Show the p-discriminant process for: $y^2(1 + (y')^2) = 1$

Part E: Parametric Solutions (5 problems)

24. Express solutions parametrically: $(y')^2 + 2y' = x$

25. Find parametric solution: $x = (y')^3 + y'$

26. Solve using parameter $p = y'$: $y = 2xy' + (y')^3$

27. Convert to parametric form: $(1 + (y')^2)^3 = (y')^2$

28. Find parametric solution: $x^2 + (y')^2 = 1$

Part F: Advanced and Mixed Problems (6 problems)

29. Show that the family $y^2 = 2Cx + C^2$ has envelope $y^2 + 2x = 0$. Verify both satisfy the same ODE.

30. For $(y')^2(1 - x^2) = (1 - y^2)$:

    (a) Find implicit general solution
    (b) Identify any singular solutions

(c) Sketch solution curves

31. The brachistochrone problem gives $(1 + (y')^2) = \frac{C}{y}$. Find parametric solution using θ where $y' = \cot(\theta/2)$.

32. Consider $y^2 = (x + c)^3$:

    (a) Find the differential equation
    (b) Show $y = 0$ is singular
    (c) Find the C-discriminant

33. For the Lagrange equation $y = xf(y') + g(y')$ where $f(p) \neq p$:

    (a) Derive the general solution method
    (b) Apply to $y = 2xy' + (y')^2$

34. Prove that if $F(x, y, C) = 0$ is the general solution and its envelope exists, the envelope satisfies the same ODE.

Part G: Exam-Style Problems (7 problems)

35. [**Prof. Ditkowski Special**] Consider $(y')^2 = (1 - y^2)$

    (a) Solve by separation of variables
    (b) Find all singular solutions
    (c) Show that $|y| \leq 1$ for all solutions
    (d) Sketch the complete solution family
    (e) Identify the envelope

36. [**Clairaut Comprehensive**] For $y = xy' - \frac{(y')^3}{3}$:

    (a) Find the general solution
    (b) Derive the singular solution
    (c) Show the singular solution is an envelope
    (d) Find points where general and singular solutions meet
    (e) Sketch several solution curves

37. [**Implicit Challenge**] The equation $x^3 + y^3 = 3axy$ (folium of Descartes):

    (a) Find the ODE it satisfies
    (b) Show solution exists implicitly near any non-singular point
    (c) Find where vertical tangents occur
    (d) Determine singular points

38. **[C-Discriminant Application]** Given $(y - cx)^2 = c^3$:

    (a) Apply C-discriminant method

    (b) Verify the discriminant satisfies the ODE

    (c) Interpret geometrically

    (d) Find where uniqueness fails

39. **[Verification Focus]** A student claims $y^2 = x$ is a singular solution of $2(y')^3 = 3y'x - 3y$:

    (a) Verify it satisfies the ODE

    (b) Find the general solution

    (c) Prove it's singular

    (d) Find the relationship between solutions

40. **[Parametric Excellence]** For $(y')^2 + (y')^3 = x$:

    (a) Find parametric solution with $p = y'$

    (b) Determine where solution has vertical tangent

    (c) Find range of x values

    (d) Express arc length parametrically

41. **[Prof. Ditkowski Comprehensive]** Consider Riccati-type: $y' = y^2 - x^2$

    (a) Show no elementary explicit solution exists

    (b) Find solution satisfying $y(0) = 0$ as power series (first 4 terms)

    (c) Determine if singular solutions exist

    (d) Find curves where $y' = 0$

    (e) Analyze behavior as $|x| \to \infty$

Answer Key with Essential Hints

1. ODE: $\frac{dy}{dx} = \frac{2x-y}{x-2y}$

    7. General: $y = (x + C)^2$; Singular: $y = 0$

    8. Singular solution: $y = -x^2/4$ (parabola)

    10. Singular: $x^2 + 4y = 0$ (use $p = x/2$)

    13. General: $y = Cx + \sqrt{1 + C^2}$; Singular: $x^2 + y^2 = 1$

    16. $y - xy' = e^{y'}$ not exactly Clairaut; use substitution

    19. Set $y' = p$: $y = xp^2 - p^3$; eliminate using $\partial/\partial p = 0$

    24. Parametric: $x = p^2 + 2p$, $y = \frac{2p^3}{3} + p^2 + C$

    31. C-discriminant gives $x = -C/2$, substitute back

    37. General: $y = \sin(x + C)$ or $y = -\sin(x + C)$; Singular: $y = \pm 1$

38. General: $y = Cx - C^3/3$; Singular: $y = \frac{2x^{3/2}}{3\sqrt{3}}$ for $x \geq 0$

41. Apply C-discriminant: $27y^2 = 4x^3$ (semicubical parabola)

43. Power series: $y = -\frac{x^3}{3} + \frac{2x^7}{63} + ...$