ODE Lesson 3: Initial Value vs Boundary Value Problems

ODE 1 - Prof. Adi Ditkowski

1 Fundamental Distinction

Definition 1 (Initial Value Problem (IVP))**.** *An **IVP** specifies all conditions at a single point:*

$$\begin{cases} y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \\ y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

Definition 2 (Boundary Value Problem (BVP))**.** *A **BVP** specifies conditions at multiple points:*

$$\begin{cases} y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \\ \textit{Conditions at } x = a, b, \textit{ and possibly other points} \end{cases}$$

Physical Intuition:

- IVP: "Given where you are and your velocity now, where will you be?"
- BVP: "Given where you start and must end, what path connects them?"

2 Existence and Uniqueness Comparison

Property	**IVP**	**BVP**
Existence	Guaranteed (locally) under mild conditions	Not guaranteed
Uniqueness	Guaranteed under Lipschitz	May have 0, 1, or ∞ solutions
Solution method	Forward integration	Global methods
Numerical approach	Runge-Kutta, Euler	Shooting, finite differences

3 IVP: Detailed Analysis

3.1 Standard Form

Second-order IVP:

$$y'' = f(x, y, y'), \quad y(x_0) = \alpha, \quad y'(x_0) = \beta$$

3.2 Solution Structure

Theorem 1 (IVP Uniqueness). *Under Picard-Lindelöf conditions, an n-th order IVP has a unique solution with exactly n constants determined by the initial conditions.*

Example 1 (Simple Harmonic Oscillator - IVP).

$$y'' + \omega^2 y = 0, \quad y(0) = A, \quad y'(0) = B$$

General solution: $y = c_1 \cos(\omega x) + c_2 \sin(\omega x)$
Applying ICs:

$$y(0) = A \Rightarrow c_1 = A \tag{1}$$
$$y'(0) = B \Rightarrow \omega c_2 = B \Rightarrow c_2 = B/\omega \tag{2}$$

Unique solution: $y = A\cos(\omega x) + \frac{B}{\omega}\sin(\omega x)$

4 BVP: Detailed Analysis

4.1 Standard Forms

Dirichlet conditions: Specify function values

$$y(a) = \alpha, \quad y(b) = \beta$$

Neumann conditions: Specify derivative values

$$y'(a) = \alpha, \quad y'(b) = \beta$$

Mixed (Robin) conditions: Linear combination

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$

4.2 Solution Possibilities

Example 2 (BVP with No Solution).

$$y'' = 0, \quad y(0) = 0, \quad y(1) = 1, \quad y(2) = 5$$

General solution: $y = Ax + B$
From conditions:

$$y(0) = 0 \Rightarrow B = 0 \tag{3}$$
$$y(1) = 1 \Rightarrow A = 1 \tag{4}$$
$$y(2) = 5 \Rightarrow 2A = 5 \Rightarrow A = 2.5 \tag{5}$$

Contradiction! No solution exists.

Example 3 (BVP with Infinitely Many Solutions).

$$y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

General solution: $y = c_1 \cos x + c_2 \sin x$
Boundary conditions:

$$y(0) = 0 \Rightarrow c_1 = 0 \tag{6}$$
$$y(\pi) = 0 \Rightarrow -c_1 = 0 \quad \checkmark \tag{7}$$

Solution: $y = c_2 \sin x$ *for ANY* $c_2 \in \mathbb{R}$

5 The Shooting Method

Algorithm: Converting BVP to IVP

1. Start with BVP: $y'' = f(x, y, y')$, $y(a) = \alpha$, $y(b) = \beta$
2. Guess missing IC: $y'(a) = s$
3. Solve IVP: $y'' = f(x, y, y')$, $y(a) = \alpha$, $y'(a) = s$
4. Check: Does $y(b) = \beta$?
5. Adjust s and repeat until boundary condition is met

Example 4 (Shooting Method in Action). *BVP:* $y'' = -y$, $y(0) = 0$, $y(\pi/2) = 1$
Guess $y'(0) = s$ *and solve IVP:*

$$y = s \sin x$$

Require $y(\pi/2) = 1$*:*

$$s \sin(\pi/2) = s = 1$$

Solution: $y = \sin x$ *with* $y'(0) = 1$

6 Green's Functions for Linear BVPs

For linear BVP: $Ly = f$ with homogeneous BCs, solution is:

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi$$

where $G(x, \xi)$ is the Green's function.

7 Comparison Table: IVP vs BVP Methods

IVP Methods	BVP Methods
Euler's Method	Shooting Method
Runge-Kutta	Finite Differences
Taylor Series	Collocation
Picard Iteration	Green's Functions

8 Eigenvalue Problems (Special BVPs)

Definition 3 (Sturm-Liouville Problem)**.**

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0$$

with boundary conditions at $x = a, b$.

These have solutions only for special values of λ (eigenvalues).

Example 5 (Classic Eigenvalue Problem)**.**

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

Non-trivial solutions only when $\lambda = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$
Eigenfunctions: $y_n = \sin(n\pi x/L)$

9 Warning Signs

IVP Red Flags:

- Discontinuous f at initial point
- Non-Lipschitz at initial value
- Singular point at x_0

BVP Red Flags:

- Periodic boundary conditions
- Over-determined (too many conditions)

- Homogeneous equation with homogeneous BCs

10 Exam Strategy

Quick Identification:

1. Count the points where conditions are given
2. One point $\Rightarrow$ IVP $\Rightarrow$ Use forward integration
3. Multiple points $\Rightarrow$ BVP $\Rightarrow$ Check for existence first!
4. For BVP: Try to find general solution first, then apply BCs
5. If infinitely many solutions exist, write the family clearly