Separable Equations: Complete Method and Analysis

ODE 1 - Prof. Adi Ditkowski

Lesson 12

1 Definition and Recognition

Definition 1 (Separable Equation)**.** *A first-order ODE is* ***separable*** *if it can be written in the form:*

$$\frac{dy}{dx} = g(x)h(y)$$

where $g(x)$ is a function of x only and $h(y)$ is a function of y only.

Alternative forms that are separable:

- $M(x)N(y)dx + P(x)Q(y)dy = 0$
- $\frac{dy}{dx} = \frac{f(x)}{g(y)}$
- Autonomous: $\frac{dy}{dx} = f(y)$

2 Solution Method

Method 1 (Separation of Variables)**.**

1. *Write in standard form:* $\frac{dy}{dx} = g(x)h(y)$
2. *Separate variables:* $\frac{dy}{h(y)} = g(x)dx \quad$ *(assuming* $h(y) \neq 0$*)*
3. *Integrate both sides:* $\int \frac{dy}{h(y)} = \int g(x)dx$
4. *Include integration constant:* $H(y) = G(x) + C$
5. *Solve for* y *if possible (explicit solution)*
6. *Check for singular solutions where* $h(y) = 0$

Division by $h(y)$ loses solutions where $h(y) = 0$. These must be checked separately!

3 Complete Solution Structure

Theorem 1 (General and Singular Solutions)**.** *For* $\frac{dy}{dx} = g(x)h(y)$*:*

- ***General solution****: Obtained by separation (family of curves)*
- ***Singular solutions****: Constants* $y = c$ *where* $h(c) = 0$
- ***Complete solution****: Union of general and singular solutions*

Example 1 (Finding All Solutions)**.** *Solve:* $\frac{dy}{dx} = 2x(y^2 - 1)$

Step 1: *Separate (assuming* $y^2 - 1 \neq 0$*):*

$$\frac{dy}{y^2 - 1} = 2x\,dx$$

Step 2: *Use partial fractions:*

$$\frac{1}{y^2 - 1} = \frac{1}{2}\left(\frac{1}{y-1} - \frac{1}{y+1}\right)$$

Step 3: *Integrate:*

$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = x^2 + C$$

Step 4: *General solution:*

$$\frac{y-1}{y+1} = Ae^{2x^2}$$

where $A = \pm e^{2C}$

Step 5: *Check* $y^2 - 1 = 0$*:*

- *If* $y = 1$*:* $\frac{dy}{dx} = 0 = 2x(1-1)$ ✓
- *If* $y = -1$*:* $\frac{dy}{dx} = 0 = 2x(1-1)$ ✓

Complete solution: $\frac{y-1}{y+1} = Ae^{2x^2}$ *and* $y = \pm 1$

4 Autonomous Equations

Definition 2 (Autonomous Equation)**.** *An ODE* $\frac{dy}{dx} = f(y)$ *depending only on* y *is* ***autonomous****.*

Method 2 (Solving Autonomous Equations)**.** *For* $\frac{dy}{dx} = f(y)$*:*

1. *Find equilibria: solve* $f(y) = 0$
2. *For non-equilibrium solutions:* $\int \frac{dy}{f(y)} = \int dx = x + C$
3. *This gives* x *as a function of* y *(inverse solution)*
4. *Invert if possible to get* $y(x)$

Autonomous equations have time-translation symmetry: if $y(x)$ is a solution, so is $y(x - x_0)$ for any constant x_0.

5 Common Integration Patterns

5.1 Essential Integrals for Separable Equations

Integral	Result	Notes
$\int \frac{dy}{y}$	$\ln\lvert y\rvert + C$	$y \neq 0$
$\int \frac{dy}{y^2}$	$-\frac{1}{y} + C$	$y \neq 0$
$\int \frac{dy}{1-y^2}$	$\frac{1}{2}\ln\left\lvert\frac{1+y}{1-y}\right\rvert + C$	$\lvert y\rvert \neq 1$
$\int \frac{dy}{1+y^2}$	$\arctan(y) + C$	All y
$\int \frac{dy}{\sqrt{1-y^2}}$	$\arcsin(y) + C$	$\lvert y\rvert < 1$
$\int \frac{dy}{y(1-y)}$	$\ln\left\lvert\frac{y}{1-y}\right\rvert + C$	$y \neq 0, 1$

6 Implicit Solutions

Definition 3 (Implicit Solution). *A relation $F(x, y) = C$ that cannot be solved explicitly for y but satisfies the ODE.*

Example 2 (Necessarily Implicit). *For $\frac{dy}{dx} = \frac{-2xy}{x^2+2y^2}$:*
Separating gives: $x^2 + 2y^2 = C$ (circles and ellipses)
Cannot solve explicitly for y as a single-valued function.

Prof. Ditkowski accepts implicit solutions. Don't waste time trying to solve for y if the algebra becomes messy!

7 Existence and Uniqueness Issues

Theorem 2 (Uniqueness Violation at Equilibria). *For $\frac{dy}{dx} = f(y)$ with $f(y_0) = 0$:*

- *The constant solution $y = y_0$ exists*
- *Non-constant solutions may touch $y = y_0$*
- *Uniqueness fails if $f'(y_0) = 0$*

Example 3 (Non-unique Solution). $\frac{dy}{dx} = 2\sqrt{|y|}$ *with* $y(0) = 0$
Solutions include:

- $y = 0$ *(singular)*
- $y = x^2$ *for* $x \geq 0$

- $y = -x^2$ *for* $x \leq 0$
- *Combinations thereof*

8 Solution Verification

Method 3 (Verification Protocol)**.** *1. Differentiate the solution (implicit differentiation if needed)*

2. Substitute into original ODE

3. Verify algebraic identity

4. Check initial conditions

5. Verify singular solutions separately

6. State domain of validity

9 Common Errors

Critical mistakes that cost points:

1. Forgetting to check for singular solutions
2. Losing absolute value signs in logarithms
3. Incorrect partial fraction decomposition
4. Not including integration constant
5. Domain errors (e.g., $\ln(y)$ requires $y > 0$)
6. Sign errors when separating