ODE Lesson 30: Constant Coefficient Systems - Distinct Eigenvalues

ODE 1 - Prof. Adi Ditkowski

1 Theory of Linear Systems

Definition 1 (Linear System with Constant Coefficients). *A system of linear ODEs with constant coefficients has the form:*

$$\mathbf{x}'(t) = A\mathbf{x}(t) \tag{1}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ *and* A *is an* $n \times n$ *constant matrix.*

Theorem 1 (Fundamental Solution Structure). *If* λ *is an eigenvalue of* A *with eigenvector* $\mathbf{v}$*, then* $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ *is a solution to* $\mathbf{x}' = A\mathbf{x}$*.*

Proof. Direct verification:

$$\frac{d}{dt}(e^{\lambda t}\mathbf{v}) = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}(\lambda \mathbf{v}) = e^{\lambda t}(A\mathbf{v}) = A(e^{\lambda t}\mathbf{v})$$

□

Eigenvalue Computation Algorithm:

1. Form the characteristic polynomial: $p(\lambda) = \det(A - \lambda I)$
2. Expand the determinant (cofactor expansion for 3×3 or larger)
3. Solve $p(\lambda) = 0$ for eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$
4. For each λ_i, solve $(A - \lambda_i I)\mathbf{v} = \mathbf{0}$ for eigenvector $\mathbf{v}_i$

Theorem 2 (General Solution for Distinct Eigenvalues). *If* A *has* n *distinct eigenvalues* $\lambda_1, \ldots, \lambda_n$ *with corresponding eigenvectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$*, then the general solution is:*

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n \tag{2}$$

Matrix Diagonalization Perspective:
If $P = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n]$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then:

- $A = PDP^{-1}$
- Solution: $\mathbf{x}(t) = Pe^{Dt}P^{-1}\mathbf{x}_0$

- Where $e^{Dt} = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$

2 Solution Method Step-by-Step

Method 1 (Complete Solution Algorithm). *Given* $\mathbf{x}' = A\mathbf{x}$ *with* $\mathbf{x}(0) = \mathbf{x}_0$*:*

Step 1: Find Eigenvalues

- *Compute* $\det(A - \lambda I) = 0$
- *For* 2×2*:* $\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0$
- *Expand:* $\lambda^2 - tr(A)\lambda + \det(A) = 0$

Step 2: Find Eigenvectors

- *For each* λ_i*, solve* $(A - \lambda_i I)\mathbf{v} = \mathbf{0}$
- *Row reduce to find null space*
- *Choose convenient basis vector*

Step 3: Form General Solution

- *Write* $\mathbf{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\mathbf{v}_i$

Step 4: Apply Initial Conditions

- *Set* $t = 0$*:* $\mathbf{x}_0 = \sum_{i=1}^{n} c_i \mathbf{v}_i$
- *Solve linear system for* $c_1, \ldots, c_n$

Example 1 (Complete 2×2 System). *Solve* $\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \mathbf{x}$ *with* $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Solution:

1. ***Eigenvalues:***

$$\det(A - \lambda I) = \det \begin{pmatrix} 4 - \lambda & 2 \\ 3 & -1 - \lambda \end{pmatrix} = (4 - \lambda)(-1 - \lambda) - 6$$

$$= -4 - 4\lambda + \lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda - 5)(\lambda + 2) = 0 \implies \lambda_1 = 5, \lambda_2 = -2$$

2. ***Eigenvectors:***

   For $\lambda_1 = 5$*:*

$$(A - 5I)\mathbf{v} = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Row 1: $-v_1 + 2v_2 = 0 \implies v_1 = 2v_2$. *Choose* $v_2 = 1$*:* $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda_2 = -2$*:*

$$(A + 2I)\mathbf{v} = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Row 2: $3v_1 + v_2 = 0 \implies v_2 = -3v_1$. *Choose* $v_1 = 1$*:* $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

3. ***General Solution:***

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

4. ***Initial Conditions:***

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

System: $2c_1 + c_2 = 1$ *and* $c_1 - 3c_2 = 2$

From equation 2: $c_1 = 2 + 3c_2$. *Substitute into equation 1:*

$$2(2 + 3c_2) + c_2 = 1 \implies 4 + 7c_2 = 1 \implies c_2 = -\frac{3}{7}$$

$$c_1 = 2 + 3(-\frac{3}{7}) = 2 - \frac{9}{7} = \frac{5}{7}$$

5. ***Final Solution:***

$$\mathbf{x}(t) = \frac{5}{7} e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{3}{7} e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Common Errors:

- Forgetting that eigenvectors are determined up to scalar multiplication
- Sign errors when computing $\det(A - \lambda I)$
- Not verifying eigenvectors: always check $A\mathbf{v} = \lambda\mathbf{v}$
- Arithmetic mistakes in the linear system for constants

Solution Behavior from Eigenvalues:

- All $\lambda_i < 0$: Stable node (all solutions $\to \mathbf{0}$)
- All $\lambda_i > 0$: Unstable node (all solutions grow)
- Mixed signs: Saddle point

- $\lambda_i = 0$: Non-isolated equilibrium (degenerate)

Prof. Ditkowski's exams typically include:

- One 2×2 system (always)
- Possibly one 3×3 system
- Initial value problems (60% of system problems)
- Questions about stability based on eigenvalues
- Verification of solutions

3 Phase Portrait Connection

The eigenvalues and eigenvectors completely determine the phase portrait:

- Eigenvectors give the principal directions
- Eigenvalues give the behavior along each direction
- Trajectories are tangent to eigenvector directions at the origin

Saddle Point: $\lambda_1 > 0, \lambda_2 < 0$