Lesson 40: Critical Points and Linearization

ODE 1 - Prof. Adi Ditkowski

Nonlinear Systems Analysis

1 Introduction to Nonlinear Systems

Definition 1 (Nonlinear Autonomous System)**.** *A two-dimensional nonlinear autonomous system has the form:*

$$\dot{x} = f(x, y) \tag{1}$$

$$\dot{y} = g(x, y) \tag{2}$$

where f and g are continuously differentiable nonlinear functions.

Unlike linear systems where solutions can be found explicitly using eigenvalues and eigenvectors, nonlinear systems generally cannot be solved in closed form. Linearization provides a powerful tool to understand local behavior.

2 Critical Points

Definition 2 (Critical Point/Equilibrium)**.** *A point (x_0, y_0) is a critical point (or equilibrium point) of the system if:*

$$f(x_0, y_0) = 0 \quad \text{and} \quad g(x_0, y_0) = 0$$

Method 1 (Finding Critical Points)**.**

1. *Set $f(x, y) = 0$ and $g(x, y) = 0$*
2. *Solve the resulting algebraic system*
3. *Check each solution for validity*
4. *List all critical points systematically*

Nonlinear systems can have:

- No critical points
- Finitely many critical points (most common)
- Infinitely many critical points (degenerate cases)

Always verify you've found ALL critical points!

3 The Jacobian Matrix

Definition 3 (Jacobian Matrix). *For the system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, the Jacobian matrix is:*

$$J(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

Critical Computation Rule:

1. Compute partial derivatives *symbolically*
2. *Then* evaluate at the critical point (x_0, y_0)
3. The result $J(x_0, y_0)$ is a constant matrix

4 Linearization Process

Theorem 1 (Linearization near Equilibrium). *Near a critical point (x_0, y_0), the nonlinear system can be approximated by the linear system:*

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = J(x_0, y_0) \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where $\xi = x - x_0$ and $\eta = y - y_0$ represent small deviations from equilibrium.

The linearization is derived from the Taylor expansion:

$$\begin{aligned} f(x, y) &= f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0) + \text{h.o.t.} \qquad (3) \\ &= 0 + \text{linear terms} + \text{higher order terms} \qquad (4) \end{aligned}$$

Near the critical point, higher order terms are negligible!

5 Classification of Critical Points

Method 2 (Classification Procedure). *Given a critical point (x_0, y_0):*

1. *Compute $J(x_0, y_0)$*
2. *Find eigenvalues λ_1, λ_2 of $J(x_0, y_0)$*
3. *Classify using:*

Eigenvalues	**Type**	**Stability**
$\lambda_1, \lambda_2 < 0$ (real)	Node	Asymptotically stable
$\lambda_1, \lambda_2 > 0$ (real)	Node	Unstable
$\lambda_1 < 0 < \lambda_2$ (real)	Saddle	Unstable
$\lambda = \alpha \pm i\beta$, $\alpha < 0$	Spiral	Asymptotically stable
$\lambda = \alpha \pm i\beta$, $\alpha > 0$	Spiral	Unstable
$\lambda = \pm i\beta$ (pure imaginary)	Center	Linearization inconclusive

Prof. Ditkowski expects you to:

- State the type (node, saddle, spiral, center)
- State stability (stable, unstable, or inconclusive)
- Show eigenvalue computation explicitly

6 Complete Examples

Example 1 (Van der Pol Oscillator)**.** *Consider the system:*

$$\dot{x} = y \tag{5}$$
$$\dot{y} = -x + \mu(1 - x^2)y \tag{6}$$

where $\mu > 0$ is a parameter.

Step 1: *Find critical points*

Setting $y = 0$ and $-x + \mu(1 - x^2) \cdot 0 = -x = 0$ gives $(0, 0)$ as the only critical point.

Step 2: *Compute Jacobian*

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu xy & \mu(1 - x^2) \end{pmatrix}$$

Step 3: *Evaluate at $(0, 0)$*

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

Step 4: *Find eigenvalues*

Characteristic equation: $\lambda^2 - \mu\lambda + 1 = 0$

Eigenvalues: $\lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$

Step 5: *Classification*

- *If $0 < \mu < 2$: Complex with positive real part $\Rightarrow$ Unstable spiral*
- *If $\mu = 2$: Repeated positive eigenvalue $\Rightarrow$ Unstable node*
- *If $\mu > 2$: Two positive real eigenvalues $\Rightarrow$ Unstable node*

7 Validity of Linearization

Linearization provides accurate local behavior EXCEPT when:

- Eigenvalues have zero real part (centers, degenerate nodes)
- The system has special symmetries
- Nonlinear terms dominate near the critical point

The region where linearization is valid shrinks as:

- Eigenvalues approach the imaginary axis
- Nonlinearity becomes stronger
- Other critical points are nearby

8 Summary Flowchart

Nonlinear System

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Find Critical Points

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Compute Jacobian

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Evaluate at (x_0, y_0)

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Find Eigenvalues

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Classify & State Stability