Practice Problems: Lesson 3 - IVP vs BVP

Critical distinction for exams!

Part A: Classification

Identify each as IVP or BVP, then predict existence/uniqueness:

1. $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$
2. $y'' + 4y = 0$, $y(0) = 1$, $y(\pi/4) = 0$
3. $y'' = 2y'$, $y(0) = 0$, $y(1) = 1$, $y'(1) = e$
4. $y'' + y' + y = e^x$, $y(0) + y'(0) = 1$, $y(1) = 0$
5. $y^{(4)} = 0$, $y(0) = y'(0) = 0$, $y(1) = y'(1) = 1$

Part B: Solving IVPs

Solve these IVPs completely:

6. $y'' - y = 0$, $y(0) = 2$, $y'(0) = 0$
7. $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 0$
8. $y'' = 6x$, $y(0) = 0$, $y'(0) = 1$

Part C: BVP Analysis

For each BVP, determine if solutions exist (none/unique/infinite):

9. $y'' = 0$, $y(0) = 0$, $y(1) = 1$
10. $y'' = 0$, $y'(0) = 1$, $y'(1) = 1$
11. $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$
12. $y'' + 4\pi^2 y = 0$, $y(0) = 0$, $y(1/2) = 0$

Part D: Shooting Method

13. For the BVP: $y'' + y = 0$, $y(0) = 0$, $y(\pi/2) = 2$
    (a) Set up the shooting method with parameter $s = y'(0)$
    (b) Find the value of s that satisfies the boundary conditions
    (c) Write the complete solution

14. Consider: $y'' = -4y$, $y(0) = 1$, $y(\pi/4) = 0$
    (a) Use shooting method to find $y'(0)$
    (b) Verify your solution satisfies both conditions

Part E: Eigenvalue Problems

15. Find all values of λ for which the BVP has non-trivial solutions:
$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

16. For what values of μ does this BVP have solutions?
$$y'' + \mu^2 y = 0, \quad y(0) = 0, \quad y(1) = \sin(\mu)$$

17. Consider the Sturm-Liouville problem:
$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$$
Find the eigenvalues and eigenfunctions.

Part F: Mixed Problems

18. Given $y'' = f(x)$ where f is continuous:
    (a) How many conditions do you need for a unique solution?
    (b) If given $y(0) = A$, $y(1) = B$, $y(2) = C$, when does a solution exist?
    (c) Write the compatibility condition for the three-point BVP

19. A beam equation: $y^{(4)} = 0$ (fourth-order)
    (a) How many initial conditions for an IVP?
    (b) How many boundary conditions for a well-posed BVP?
    (c) Give an example of each type

20. Consider the nonlinear BVP: $y'' = y^2$, $y(0) = 1$, $y(1) =?$
    (a) Can you always find a value at $x = 1$ to make this solvable?
    (b) What if $y(0) = 0$ instead?
    (c) Discuss existence based on the boundary values

Part G: Theoretical Questions

21. Explain why the IVP $y'' = y^{1/3}$, $y(0) = 0$, $y'(0) = 0$ might not have a unique solution.
22. For the BVP $y'' + y = 0$, $y(0) = 0$, $y(\alpha) = 0$:
    (a) For which values of α is the solution unique?
    (b) For which values are there infinitely many solutions?
    (c) Can there be no solution for some α?
23. Prove that if y_1 and y_2 both solve the linear BVP:
    $$y'' + p(x)y' + q(x)y = 0, \quad y(a) = A, \quad y(b) = B$$
    then $y_1 \equiv y_2$ (uniqueness for linear BVP).

Part H: Exam-Style Questions

24. Professor Ditkowski asks: "Give an example of a second-order linear BVP with:
    (a) No solution
    (b) Exactly one solution
    (c) Infinitely many solutions"
25. You're given: $y'' = -\lambda^2 y$, $y(0) = 0$, $\int_0^1 y(x)dx = 1$
    (a) Is this an IVP or BVP?
    (b) For which λ does a solution exist?
    (c) Find the solution when it exists
26. Compare and contrast:
    (a) IVP: $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$
    (b) BVP: $y'' + y = 0$, $y(0) = 1$, $y(\pi) = 1$
    (c) Which has a unique solution? Find both solutions.

Answer Guide

Key Points to Remember:

- IVP: All conditions at one point $\Rightarrow$ Usually unique solution
- BVP: Conditions at different points $\Rightarrow$ Check existence carefully
- Linear BVP can have 0, 1, or ∞ solutions
- Eigenvalue problems have solutions only for special λ values
- Shooting method converts BVP to sequence of IVPs