Lesson 41: Practice Problems

Hartman-Grobman Theorem Applications

ODE 1 - Prof. Adi Ditkowski

Part A: Identifying Hyperbolic Equilibria (5 problems)

1. For the system $\dot{x} = x - y$, $\dot{y} = x^2 - 1$, determine which critical points are hyperbolic. **Solution Hint:** Find critical points at $(1, 1)$ and $(-1, -1)$. Check eigenvalues at each.

2. Consider $\dot{x} = y$, $\dot{y} = -\sin(x)$. Which equilibria at $x = n\pi$ are hyperbolic? **Solution Hint:** Check $\cos(n\pi) = (-1)^n$ in the Jacobian. Different behavior for even/odd n.

3. For what values of a is the origin a hyperbolic equilibrium?

$$\dot{x} = ax - y, \quad \dot{y} = x + ay$$

Solution Hint: Find when eigenvalues $\lambda = a \pm i$ have non-zero real parts.

4. Determine all hyperbolic critical points of:

$$\dot{x} = x(1 - x - y), \quad \dot{y} = y(2 - 3x - y)$$

Solution Hint: Four critical points to check. Compute eigenvalues at each.

5. For the gradient system with $V(x, y) = x^4 - 2x^2 + y^2$, identify hyperbolic equilibria. **Solution Hint:** Critical points where $\nabla V = 0$. Use Hessian for eigenvalues.

Part B: Applying Hartman-Grobman (6 problems)

6. Given the system:

$$\dot{x} = -x + y^2, \quad \dot{y} = -2y + x^2$$

Can linearization at the origin determine the local behavior? Justify using Hartman-Grobman. **Solution Hint:** Check eigenvalues of $J(0, 0)$. Both negative implies hyperbolic stable node.

7. For the system $\dot{x} = y + x^3$, $\dot{y} = -x + y^3$:

   (a) Show the origin is non-hyperbolic

   (b) Explain why Hartman-Grobman doesn't apply

   (c) What additional analysis would be needed?

Solution Hint: Eigenvalues are $\pm i$. Need Lyapunov function or normal form analysis.

8. Consider the damped Duffing oscillator:

$$\dot{x} = y, \quad \dot{y} = x - x^3 - \delta y$$

For $\delta > 0$, determine which equilibria have behavior determined by linearization. **Solution Hint:** Three equilibria. Check hyperbolicity at each.

9. A system has Jacobian at equilibrium:

$$J = \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$$

What does Hartman-Grobman tell us about the nonlinear behavior near this point? **Solution Hint:** Compute eigenvalues. Check if both have non-zero real parts.

10. For the predator-prey system:

$$\dot{x} = x(1-x) - \frac{xy}{1+x}, \quad \dot{y} = ry\left(1 - \frac{y}{x}\right)$$

Under what conditions on $r > 0$ is the coexistence equilibrium hyperbolic? **Solution Hint:** Find the equilibrium, compute Jacobian, determine eigenvalue conditions.

11. Given that a system's linearization at $(1, 2)$ has eigenvalues $\lambda = 2 \pm 3i$, what can you conclude about:

    (a) The validity of linearization?

    (b) The type of equilibrium?

    (c) The stability?

    Solution Hint: Positive real parts mean hyperbolic unstable spiral.

Part C: Non-Hyperbolic Cases (5 problems)

12. Show that the origin is non-hyperbolic for:

$$\dot{x} = y + ax^3, \quad \dot{y} = -x + by^3$$

What does this mean for stability analysis? **Solution Hint:** $J(0, 0)$ has purely imaginary eigenvalues. Linearization fails.

13. Consider the Hamiltonian system:

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

where $H(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{4}(x^4 + y^4)$. Why doesn't Hartman-Grobman apply at the origin? **Solution Hint:** Hamiltonian systems have centers; eigenvalues are purely imaginary.

14. For the system:

$$\dot{x} = -y + \mu x(x^2 + y^2), \quad \dot{y} = x + \mu y(x^2 + y^2)$$

    (a) Show the origin is non-hyperbolic for all μ

    (b) Determine stability using polar coordinates

    (c) Compare with linearization prediction

    Solution Hint: In polar: $\dot{r} = \mu r^3$. Sign of μ determines stability.

15. A system has a line of equilibria $y = 0$. Can Hartman-Grobman apply at any point on this line? **Solution Hint:** Zero eigenvalue guaranteed, so non-hyperbolic everywhere.

16. For what parameter values does Hartman-Grobman fail?

$$\dot{x} = \alpha x - y, \quad \dot{y} = x + \alpha y$$

    Solution Hint: Eigenvalues $\lambda = \alpha \pm i$. Fails when $\alpha = 0$.

Part D: Structural Stability (4 problems)

17. If a system has a hyperbolic saddle at the origin, what happens under small perturbations $\epsilon f_1(x, y)$ and $\epsilon g_1(x, y)$? **Solution Hint:** Saddle persists but may shift location slightly.

18. Consider the perturbed system:

$$\dot{x} = -x + \epsilon xy, \quad \dot{y} = -2y + \epsilon x^2$$

    For small ϵ, does the qualitative behavior near the origin change? **Solution Hint:** Origin remains hyperbolic stable node for small ϵ.

19. A system undergoes a Hopf bifurcation at $\mu = 0$:

$$\dot{x} = \mu x - y + x(x^2 + y^2), \quad \dot{y} = x + \mu y + y(x^2 + y^2)$$

    Explain using Hartman-Grobman why the behavior changes at $\mu = 0$. **Solution Hint:** Eigenvalues cross imaginary axis; hyperbolicity lost.

20. Two systems have the same linearization at their respective equilibria, with eigenvalues $-1 \pm 2i$. What can you conclude about their local phase portraits? **Solution Hint:** Both have topologically equivalent stable spirals near equilibrium.

Part E: Exam-Style Problems (5 problems)

21. [**Comprehensive**] Consider the system:

$$\dot{x} = y^2 - x, \quad \dot{y} = x^2 - y$$

   (a) Find all critical points
   (b) Determine which are hyperbolic
   (c) For hyperbolic equilibria, state the type and stability
   (d) For non-hyperbolic equilibria, explain why Hartman-Grobman fails

   Solution Hint: Critical points at $(0, 0)$, $(1, 1)$, and two others. Check each.

22. [**Prof. Ditkowski Style**] The van der Pol oscillator:

$$\dot{x} = y, \quad \dot{y} = -x + \epsilon(1 - x^2)y$$

   (a) Show the origin is the only equilibrium
   (b) For what values of $\epsilon > 0$ can we use linearization?
   (c) Classify the origin for small $\epsilon > 0$
   (d) What happens as $\epsilon \to 0$?

   Solution Hint: Always hyperbolic for $\epsilon > 0$, non-hyperbolic at $\epsilon = 0$.

23. [**Application**] A chemical reaction system:

$$\dot{x} = 1 - x - xy^2, \quad \dot{y} = \alpha(xy^2 - y)$$

   (a) Find the equilibrium with $x, y > 0$
   (b) Determine conditions on $\alpha > 0$ for hyperbolicity
   (c) When hyperbolic, what does linearization tell us?
   (d) Interpret chemically

   Solution Hint: Steady state exists. Check when Jacobian eigenvalues have non-zero real parts.

24. [**Multiple Equilibria**] For the system:

$$\dot{x} = x(1 - x^2 - y^2), \quad \dot{y} = y(1 - x^2 - y^2) + x$$

   (a) Show there are exactly two equilibria
   (b) Determine which (if any) are hyperbolic
   (c) Apply Hartman-Grobman where valid
   (d) Sketch the expected phase portrait

Solution Hint: One at origin, another on unit circle. Check hyperbolicity of each.

25. [**Theory and Application**] Consider:

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

where $f(1,2) = g(1,2) = 0$ and the Jacobian at $(1,2)$ is:

$$J(1,2) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

(a) Is $(1,2)$ hyperbolic? Explain.

(b) What does linearization predict?

(c) Why might this prediction be wrong?

(d) Give an example of f and g where linearization fails

Solution Hint: Purely imaginary eigenvalues. Could be center, spiral, or more complex.

Part F: Advanced Problems (5 problems)

26. Prove that all equilibria of a gradient system $\dot{x} = -\nabla V(x)$ with non-degenerate critical points are hyperbolic. **Solution Hint:** Hessian eigenvalues are real; non-degenerate means non-zero.

27. For the Lorenz system restricted to the x-y plane (setting $z = 0$):

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y$$

Determine for which values of $\sigma, r > 0$ the origin is hyperbolic.

28. Consider a system where the Jacobian at an equilibrium has characteristic polynomial $\lambda^2 + p\lambda + q = 0$. Give conditions on p and q for:

(a) Hyperbolic node

(b) Hyperbolic saddle

(c) Hyperbolic spiral

(d) Non-hyperbolic equilibrium

29. [**Challenge**] Show that if a 2D system has a homoclinic orbit (trajectory connecting a saddle to itself), then the saddle must be hyperbolic.

30. [**Research Preview**] Consider the system:

$$\dot{x} = y + \delta x - x^3, \quad \dot{y} = -x$$

(a) Find critical points as functions of δ

(b) Determine hyperbolicity conditions

(c) Show that bifurcations occur when hyperbolicity is lost

(d) Sketch the bifurcation diagram

Key Exam Strategies:

- Always check $\mathrm{Re}(\lambda) \neq 0$ for ALL eigenvalues
- State explicitly whether Hartman-Grobman applies
- If it applies, conclude definitively about behavior
- If it doesn't apply, state that additional analysis is needed
- Remember: "hyperbolic" = "linearization works"
- Common error: Forgetting that BOTH eigenvalues need non-zero real parts