ODE Lesson 8: Parameter-Dependent Existence Problems

ODE 1 - Prof. Adi Ditkowski

1 Introduction to Parametric ODEs

Definition 1 (Parametric ODE)**.** *A **parametric ODE** has the form:* $y' = f(x, y; \mu)$ *where* μ *is a parameter (or vector of parameters) that affects the equation's behavior.*

Key Questions for Parametric Problems:

1. For which μ does a solution exist?
2. For which μ is it unique?
3. How does the solution behavior change with μ?
4. Where are the bifurcation points?

2 Parameter Location Analysis

2.1 Where Parameters Appear

Location	**Example**	**Main Concern**
Coefficients	$y' = \mu y$	Usually safe
Denominators	$y' = \frac{y}{x-\mu}$	Singularities
Exponents	$y' = \|y\|^\mu$	Lipschitz condition
Initial conditions	$y(0) = \mu$	Starting point issues
Domain bounds	$y' = \sqrt{\mu - y^2}$	Real-valued constraints

3 Critical Parameter Values

Definition 2 (Critical Parameter)**.** *A **critical parameter value** μ_c is where:*

- *The number of equilibria changes*

- *Stability of equilibria changes*
- *Existence/uniqueness properties change*
- *Solution behavior undergoes qualitative change*

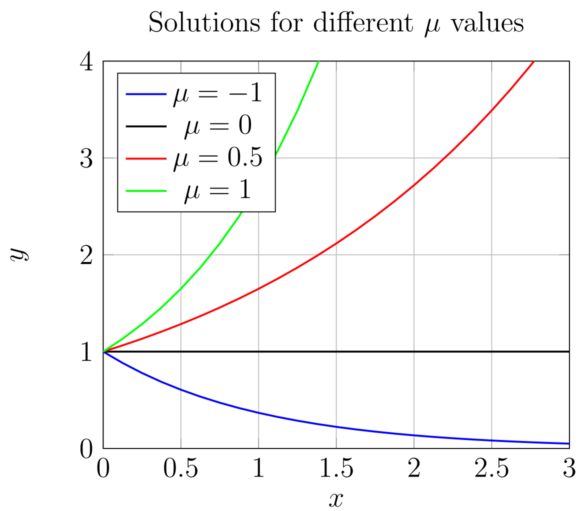
4 Standard Examples

4.1 Example 1: Linear with Parameter

Example 1 (Exponential Growth/Decay). $y' = \mu y, \quad y(0) = 1$

Analysis:

- *Solution exists for all* μ*:* $y = e^{\mu x}$
- *Behavior changes at* $\mu = 0$*:*
  - $\mu < 0$*: Exponential decay to 0*
  - $\mu = 0$*: Constant solution*
  - $\mu > 0$*: Exponential growth to* ∞



4.2 Example 2: Parameter in Denominator

Example 2 (Moving Singularity). $y' = \frac{y}{x-\mu}, \quad y(0) = 1$

Analysis by parameter:

- $\mu < 0$*: Solution exists for all* $x \geq 0$
- $\mu = 0$*: No solution through* $(0, 1)$ *(singularity at initial point)*
- $\mu > 0$*: Solution exists for* $x \in [0, \mu)$*, blows up at* $x = \mu$

General solution (when it exists): $y = \frac{C}{x-\mu}$

4.3 Example 3: Parameter Affects Lipschitz

Example 3 (Uniqueness Switch). $y' = \mu|y|^{1/2}, \quad y(0) = 0$

Analysis:

- $\mu = 0$: *Unique solution* $y \equiv 0$
- $\mu \neq 0$: *Not Lipschitz at* $y = 0$, *infinitely many solutions*

Solution family for $\mu \neq 0$: $y(x) = \begin{cases} 0 & \text{if } |x| \leq c \\ \frac{\mu^2}{4}(x-c)^2 & \text{if } x > c \geq 0 \end{cases}$

5 Bifurcation Analysis

5.1 Types of Bifurcations

Definition 3 (Bifurcation). *A* ***bifurcation*** *occurs at parameter value* μ_c *where the qualitative structure of solutions changes.*

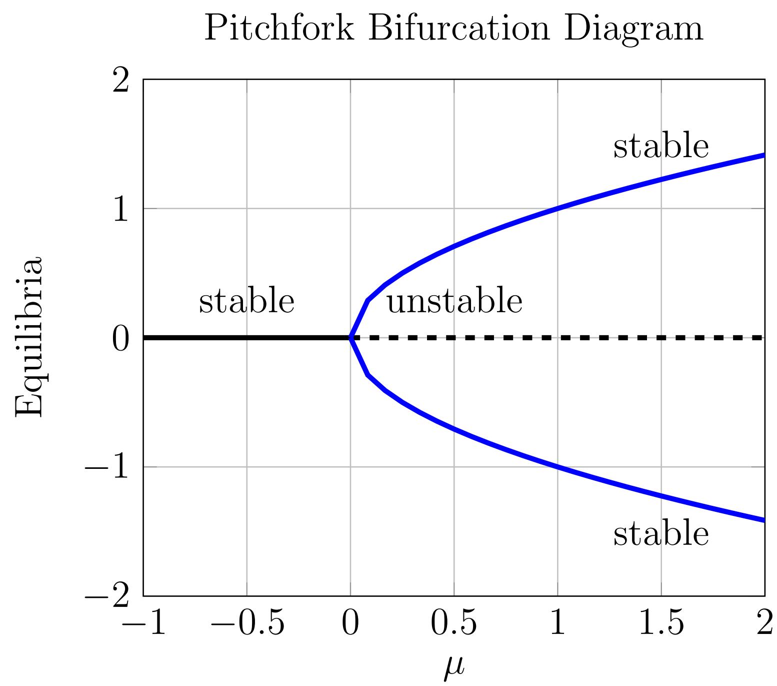
Type	**Equation Form**	**Behavior**
Saddle-node	$y' = \mu - y^2$	Equilibria appear/disappear
Transcritical	$y' = \mu y - y^2$	Equilibria exchange stability
Pitchfork	$y' = \mu y - y^3$	Symmetry breaking
Hopf	System of ODEs	Periodic orbits appear

5.2 Pitchfork Bifurcation Example

Example 4 (Pitchfork). $y' = \mu y - y^3$

Equilibria: $y = 0$ *and* $y = \pm\sqrt{\mu}$ *(if* $\mu > 0$*)*

- $\mu < 0$: *Only* $y = 0$ *(stable)*
- $\mu = 0$: *Only* $y = 0$ *(critically stable)*
- $\mu > 0$: *Three equilibria:*
  - $y = 0$ *(unstable)*
  - $y = \pm\sqrt{\mu}$ *(stable)*



6 Riccati Equation with Parameter

Example 5 (Parametric Riccati)**.** $y' = y^2 - \mu$

Analysis:

- $\mu < 0$: *No real equilibria, all solutions blow up*
- $\mu = 0$: *One equilibrium at* $y = 0$ *(semi-stable)*
- $\mu > 0$: *Two equilibria at* $y = \pm\sqrt{\mu}$

Blow-up analysis:

- *If* $|y_0| > \sqrt{\mu}$ *(when* $\mu > 0$*): Solution blows up*
- *If* $|y_0| < \sqrt{\mu}$*: Solution remains bounded*
- *If* $|y_0| = \sqrt{\mu}$*: Solution approaches equilibrium*

7 Global Existence Criteria

For global existence, check:

1. No finite-time blow-up (solutions remain bounded)
2. No singularities in the domain

3. Lipschitz condition holds globally

Example 6 (Global Existence Analysis). $y' = y^2 + \mu y + 1$
Discriminant: $\Delta = \mu^2 - 4$

- $|\mu| < 2$: *No real equilibria, all solutions blow up*
- $|\mu| = 2$: *One equilibrium (double root)*
- $|\mu| > 2$: *Two equilibria, bounded solutions possible*

For global existence: Need $|\mu| \geq 2$ *and appropriate initial conditions.*

8 Continuous Dependence on Parameters

Theorem 1 (Continuous Dependence). *If* f *and* $\frac{\partial f}{\partial y}$ *are continuous in* (x, y, μ)*, then the solution* $y(x, \mu)$ *is continuous in* μ*.*

Continuous $\neq$ Smooth! Small parameter changes can cause:

- Bifurcations (structure changes)
- Blow-up time shifts
- Stability switches
- Period changes (for oscillatory solutions)

9 Singular Perturbations

Definition 4 (Singular Perturbation). *When a small parameter* ϵ *multiplies the highest derivative:* $\epsilon y'' + f(y', y, x) = 0$ *As* $\epsilon \to 0$*, the order of the equation changes!*

Example 7 (Fast-Slow System). $\epsilon y' = -y + \mu$

- *For* $\epsilon > 0$: *First-order ODE with solution* $y = \mu + (y_0 - \mu)e^{-x/\epsilon}$
- *As* $\epsilon \to 0^+$: *Rapid transition to* $y = \mu$
- *At* $\epsilon = 0$: *Algebraic equation* $y = \mu$

Time scale: $\tau = x/\epsilon$ *shows fast dynamics.*

10 Parameter Identification

Exam Question Type: "Find all μ such that..."

1. The solution exists globally
2. All solutions are periodic
3. The equilibrium at origin is stable
4. The solution through $(0, 1)$ remains bounded

11 Systematic Analysis Strategy

Parameter Analysis Algorithm:

1. Identify where μ appears (coefficient, denominator, etc.)
2. Find critical values (singularities, Lipschitz failure, etc.)
3. Analyze each parameter regime separately
4. Determine equilibria and their stability
5. Check for bifurcations
6. Sketch bifurcation diagram
7. Consider limiting cases ($\mu \to 0, \pm\infty$)

12 Memory Device

PARAMETER Analysis:
Position (where is μ?)
Anomalies (singularities)
Regimes (different cases)
Analysis (each case)
Monotonicity (how solution changes)
Equilibria
Transitions (bifurcations)
Existence (global vs local)
Rate of change