Lesson 43: Practice Problems
Characteristic Equation Method

ODE 1 with Prof. Adi Ditkowski

Tel Aviv University

Part A: Characteristic Equation Setup (5 problems)

1. Write the characteristic equation for: $y''' - 6y'' + 11y' - 6y = 0$
2. Write the characteristic equation for: $2y^{(4)} + 3y'' - y = 0$
3. Write the characteristic equation for: $y'' + 4y' + 4y = 0$
4. Write the characteristic equation for: $y''' + y' = 0$
5. Write the characteristic equation for: $y^{(4)} + 2y'' + y = 0$

Part B: Distinct Real Roots (5 problems)

6. Solve: $y'' - 7y' + 12y = 0$
7. Solve: $y''' - 6y'' + 11y' - 6y = 0$
8. Solve the IVP: $y'' - y' - 2y = 0$, $y(0) = 3$, $y'(0) = 1$
9. Solve: $y^{(4)} - 5y'' + 4y = 0$
10. Find the solution of $y'' - 9y = 0$ that satisfies $y(0) = 2$ and remains bounded as $t \to -\infty$.

Part C: Repeated Roots (5 problems)

11. Solve: $y'' - 4y' + 4y = 0$
12. Solve: $y''' - 3y'' + 3y' - y = 0$
13. Solve the IVP: $y'' + 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = -2$
14. Solve: $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$
15. Find all solutions of $y''' - 6y'' + 12y' - 8y = 0$ that satisfy $\lim_{t\to\infty} \mathrm{e}^{-2t} y(t) = \mathrm{L} forsomefinite \mathrm{L} \neq 0$.

Part D: Complex Roots (5 problems)

16. Solve: $y'' + y = 0$
17. Solve: $y'' - 2y' + 2y = 0$
18. Solve the IVP: $y'' + 4y' + 13y = 0$, $y(0) = 1$, $y'(0) = -2$
19. Solve: $y^{(4)} + 4y'' = 0$
20. Find the solution of $y'' + 2y' + 5y = 0$ with $y(0) = 0$ that has maximum amplitude.

Part E: Mixed Cases (5 problems)

21. Solve: $y''' - y'' + y' - y = 0$
22. Solve: $y^{(4)} + y''' - y' - y = 0$
23. Solve: $y^{(4)} + 8y'' + 16y = 0$
24. Find all solutions of $y''' + y'' - y' - y = 0$ that are periodic.
25. Solve the IVP: $y^{(4)} - y = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$

Part F: Exam-Style Problems (5 problems)

26. (Prof. Ditkowski style) Consider the equation $y'' + py' + qy = 0$ where $p, q \in \mathbb{R}$.
    (a) For what values of p and q are all solutions bounded as $t \to \infty$?
    (b) For what values do all solutions oscillate?
    (c) When do all non-zero solutions tend to infinity as $t \to \infty$?
27. Let $y'' + ay' + by = 0$ have solutions $y_1(t) =\mathrm{e}^{2t} andy_2(t) =\mathrm{e}^{-3t}.Findaand\mathrm{b}$.
28. The characteristic equation of a third-order ODE has roots $r_1 = 2$, $r_2 = 3$, and $r_3 = -1$. If $y(0) = 1$, $y'(0) = 5$, and $y''(0) = 7$, find the solution.
29. A fourth-order equation has characteristic polynomial $(r^2+4)(r-1)^2 = 0$. Write the general real-valued solution.
30. Show that if $y(t)$ is a solution of $y'' + py' + qy = 0$ with constant coefficients, then $z(t) = y(at + b)$ is a solution of a similar equation. Find the new equation.

Solutions

Part A: Characteristic Equation Setup

1. $r^3 - 6r^2 + 11r - 6 = 0$
2. $2r^4 + 3r^2 - 1 = 0$
3. $r^2 + 4r + 4 = 0$
4. $r^3 + r = 0$
5. $r^4 + 2r^2 + 1 = 0$

Part B: Distinct Real Roots

6. Characteristic equation: $r^2 - 7r + 12 = 0$
   Factor: $(r-3)(r-4) = 0$
   Roots: $r_1 = 3, r_2 = 4$
   General solution: $y(t) = c_1 \mathrm{e}^{3t} + \mathrm{c}_2\mathrm{e}^{4t}$

7. Characteristic equation: $r^3 - 6r^2 + 11r - 6 = 0$
   Factor: $(r-1)(r-2)(r-3) = 0$
   Roots: $r_1 = 1, r_2 = 2, r_3 = 3$
   General solution: $y(t) = c_1 \mathrm{e}^{t} + \mathrm{c}_2\mathrm{e}^{2t} + \mathrm{c}_3\mathrm{e}^{3t}$

8. Characteristic equation: $r^2 - r - 2 = 0$
   Factor: $(r-2)(r+1) = 0$
   Roots: $r_1 = 2, r_2 = -1$
   General solution: $y(t) = c_1 \mathrm{e}^{2t} + \mathrm{c}_2\mathrm{e}^{-t}$
   $Apply ICs$:$\mathrm{c}_1 + c_2 = 3$, $2c_1 - c_2 = 1$
   Solving: $c_1 = 4/3$, $c_2 = 5/3$
   Solution: $y(t) = \frac{4}{3}\mathrm{e}^{2t} + 5_{\overline{3}}\mathrm{e}^{-t}$

9. Characteristic equation: $r^4 - 5r^2 + 4 = 0$
   Let $s = r^2$: $s^2 - 5s + 4 = 0$
   Factor: $(s-1)(s-4) = 0$, so $s = 1$ or $s = 4$
   Thus $r^2 = 1$ or $r^2 = 4$
   Roots: $r = \pm 1, \pm 2$
   General solution: $y(t) = c_1 \mathrm{e}^{-2t} + \mathrm{c}_2\mathrm{e}^{-t} + \mathrm{c}_3\mathrm{e}^{t} + \mathrm{c}_4\mathrm{e}^{2t}$

10. Characteristic equation: $r^2 - 9 = 0$
    Roots: $r = \pm 3$
    General solution: $y(t) = c_1 \mathrm{e}^{3t} + \mathrm{c}_2\mathrm{e}^{-3t}$
    $For boundedness as$t $\to -\infty$, need $c_1 = 0$
    Apply $y(0) = 2$: $c_2 = 2$
    Solution: $y(t) = 2\mathrm{e}^{-3t}$

Part C: Repeated Roots

6. Characteristic equation: $r^2 - 4r + 4 = 0$
   Factor: $(r-2)^2 = 0$
   Root: $r = 2$ (multiplicity 2)
   General solution: $y(t) = (c_1 + c_2 t)\mathrm{e}^{2t}$

7. Characteristic equation: $r^3 - 3r^2 + 3r - 1 = 0$
   Factor: $(r-1)^3 = 0$
   Root: $r = 1$ (multiplicity 3)
   General solution: $y(t) = (c_1 + c_2 t + c_3 t^2)e^t$

8. Characteristic equation: $r^2 + 6r + 9 = 0$
   Factor: $(r+3)^2 = 0$
   Root: $r = -3$ (multiplicity 2)
   General solution: $y(t) = (c_1 + c_2 t)\mathrm{e}^{-3t}$
   $Apply ICs:$c$_1 = 1$, $c_2 - 3c_1 = -2$
   Thus $c_2 = 1$
   Solution: $y(t) = (1 + t)\mathrm{e}^{-3t}$

9. Characteristic equation: $r^4 - 4r^3 + 6r^2 - 4r + 1 = 0$
   This is $(r-1)^4 = 0$
   Root: $r = 1$ (multiplicity 4)
   General solution: $y(t) = (c_1 + c_2 t + c_3 t^2 + c_4 t^3)e^t$

10. Characteristic equation: $r^3 - 6r^2 + 12r - 8 = 0$
    Factor: $(r-2)^3 = 0$
    Root: $r = 2$ (multiplicity 3)
    General solution: $y(t) = (c_1 + c_2 t + c_3 t^2)\mathrm{e}^{2t}$
    $For finite limit, need$c$_3 = 0$ and $c_2 = 0$
    Solution: $y(t) = c_1\mathrm{e}^{2t} for any$c$_1 \neq 0$

Part D: Complex Roots

11. Characteristic equation: $r^2 + 1 = 0$
    Roots: $r = \pm i$
    General solution: $y(t) = c_1 \cos(t) + c_2 \sin(t)$

12. Characteristic equation: $r^2 - 2r + 2 = 0$
    Roots: $r = 1 \pm i$
    General solution: $y(t) =$e$^t(c_1 \cos(t)$ + c$_2 \sin(t))$

13. Characteristic equation: $r^2 + 4r + 13 = 0$
    Roots: $r = -2 \pm 3i$
    General solution: $y(t) =$e$^{-2t}(c_1 \cos(3t)$ + c$_2 \sin(3t))$
    Apply ICs: $c_1 = 1$, $-2c_1 + 3c_2 = -2$
    Thus $c_2 = 0$
    Solution: $y(t) =$e$^{-2t} \cos(3t)$

14. Characteristic equation: $r^4 + 4r^2 = 0$
Factor: $r^2(r^2 + 4) = 0$
Roots: $r = 0$ (multiplicity 2), $r = \pm 2i$
General solution: $y(t) = c_1 + c_2 t + c_3 \cos(2t) + c_4 \sin(2t)$

15. Characteristic equation: $r^2 + 2r + 5 = 0$
Roots: $r = -1 \pm 2i$
General solution: $y(t) = \mathrm{e}^{-t}(c_1 \cos(2t) + \mathrm{c}_2 \sin(2t))$
Apply $y(0) = 0$: $c_1 = 0$
For maximum amplitude, choose c_2 as large as needed
Solution: $y(t) = A\mathrm{e}^{-t} \sin(2t) for any \mathrm{A} \neq 0$

Part E: Mixed Cases

16. Characteristic equation: $r^3 - r^2 + r - 1 = 0$
Factor: $r^2(r - 1) + (r - 1) = (r^2 + 1)(r - 1) = 0$
Roots: $r = 1, r = \pm i$
General solution: $y(t) = c_1\mathrm{e}^t + \mathrm{c}_2 \cos(t) + c_3 \sin(t)$

17. Characteristic equation: $r^4 + r^3 - r - 1 = 0$
Factor: $r^3(r + 1) - (r + 1) = (r^3 - 1)(r + 1) = (r - 1)(r^2 + r + 1)(r + 1) = 0$
Roots: $r = 1, -1, \frac{-1 \pm i\sqrt{3}}{2}$
General solution: $y(t) = c_1\mathrm{e}^t + \mathrm{c}_2\mathrm{e}^{-t} + e^{-t/2}(c_3 \cos(\frac{\sqrt{3}}{2}t) + \mathrm{c}_4 \sin(\frac{\sqrt{3}}{2}t))$

18. Characteristic equation: $r^4 + 8r^2 + 16 = 0$
This is $(r^2 + 4)^2 = 0$
Roots: $r = \pm 2i$ (each with multiplicity 2)
General solution: $y(t) = (c_1 + c_2 t) \cos(2t) + (c_3 + c_4 t) \sin(2t)$

19. Characteristic equation: $r^3 + r^2 - r - 1 = 0$
Factor: $(r + 1)^2(r - 1) = 0$
Roots: $r = -1$ (multiplicity 2), $r = 1$
No periodic solutions (all involve exponentials)

20. Characteristic equation: $r^4 - 1 = 0$
Roots: $r = \pm 1, \pm i$
General solution: $y(t) = c_1\mathrm{e}^t + \mathrm{c}_2\mathrm{e}^{-t} + \mathrm{c}_3 \cos(t) + c_4 \sin(t)$
Apply ICs: System gives $c_1 = c_2 = c_3 = 1/2$, $c_4 = 0$
Solution: $y(t) = \frac{1}{2}(\mathrm{e}^t + e^{-t} + \cos(t))$

Part F: Exam-Style Problems

21. (a) All solutions bounded: Need both roots to have negative real parts. This occurs when $p > 0$ and $q > 0$ (by Routh-Hurwitz).
    (b) Solutions oscillate: Need complex roots, so discriminant $p^2 - 4q < 0$, i.e., $q > p^2/4$.

(c) Solutions tend to infinity: At least one root has positive real part. This occurs when either $p < 0$ or $q < 0$.

22. The characteristic equation must be $(r-2)(r+3) = r^2 + r - 6 = 0$
    Thus $a = 1$ and $b = -6$.

23. Characteristic equation: $(r-2)(r-3)(r+1) = r^3 - 4r^2 + r + 6 = 0$
    General solution: $y(t) = c_1\mathrm{e}^{2t}$ + $\mathrm{c}_2\mathrm{e}^{3t}$ + $\mathrm{c}_3\mathrm{e}^{-t}$
    $ApplyICstogetsystem:$
    $\mathrm{c}_1 + c_2 + c_3 = 1$
    $2c_1 + 3c_2 - c_3 = 5$
    $4c_1 + 9c_2 + c_3 = 7$
    Solving: $c_1 = 3/2$, $c_2 = 0$, $c_3 = -1/2$
    Solution: $y(t) = \frac{3}{2}\mathrm{e}^{2t}$ - $1_{\overline{2}}\mathrm{e}^{-t}$

24. Roots: $r = \pm 2i, r = 1$ (multiplicity 2)
    General solution: $y(t) = c_1\cos(2t) + c_2\sin(2t) + (c_3 + c_4 t)e^t$

25. Let $s = at + b$, so $t = (s-b)/a$.
    Then $\frac{dy}{dt} = a\frac{dy}{ds}$ and $\frac{d^2y}{dt^2} = a^2\frac{d^2y}{ds^2}$
    Substituting: $a^2y'' + apy' + qy = 0$
    Dividing by a^2: $y'' + \frac{p}{a}y' + \frac{q}{a^2}y = 0$
    New equation has coefficients $p' = p/a$ and $q' = q/a^2$.