Lesson 24: Special Integrating Factors - $\mu(xy)$ and Beyond

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1 Beyond Simple Integrating Factors

When neither $\mu(x)$ nor $\mu(y)$ exists, we look for integrating factors of special forms:

- $\mu(xy)$ - depends on the product xy
- $\mu(x^2 + y^2)$ - depends on radial distance
- $\mu = x^a y^b$ - power form
- $\mu = 1/(xM + yN)$ - for homogeneous equations

Each special form has its own existence test. Master the tests to quickly identify which form to use!

2 Case 3: Integrating Factor $\mu(xy)$

Theorem 1 (Existence of $\mu(xy)$)**.** *An integrating factor depending only on $z = xy$ exists if and only if*

$$\frac{1}{xN - yM}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = F(xy)$$

where F is a function of xy alone. The integrating factor is:

$$\mu(xy) = e^{\int F(z)\,dz} \text{ where } z = xy$$

Proof Sketch. If $\mu = \mu(z)$ where $z = xy$, then:

$$\frac{\partial \mu}{\partial x} = \frac{d\mu}{dz} \cdot y, \quad \frac{\partial \mu}{\partial y} = \frac{d\mu}{dz} \cdot x$$

Substituting into the exactness condition:

$$M \cdot \frac{d\mu}{dz} \cdot x - N \cdot \frac{d\mu}{dz} \cdot y = \mu(N_x - M_y)$$

$$\frac{1}{\mu}\frac{d\mu}{dz} = \frac{M_y - N_x}{xN - yM}$$

This requires the right side to be a function of $z = xy$ only. □

Testing for $\mu(xy)$:

1. Compute $M_y - N_x$ (numerator)
2. Compute $xN - yM$ (denominator - note the pattern!)
3. Form the ratio $\frac{M_y - N_x}{xN - yM}$
4. Check if this can be expressed as $F(xy)$
5. If yes, solve $\frac{d\mu}{dz} = \mu \cdot F(z)$ where $z = xy$

Example 1 ($\mu(xy)$ Application)**.** *Solve* $(2y^2 + 3xy)dx + (2xy + x^2)dy = 0$

Step 1: *Check exactness:* $M_y = 4y + 3x$, $N_x = 2y + 2x$. *Not exact.*

Step 2: *Test* $\mu(x)$*:* $\frac{M_y - N_x}{N} = \frac{2y+x}{2xy+x^2} = \frac{2y+x}{x(2y+x)} = \frac{1}{x}$

Actually, $\mu(x) = x$ *works here! But let's continue for illustration...*

Step 3: *Test* $\mu(xy)$*:*

$$\frac{M_y - N_x}{xN - yM} = \frac{2y + x}{x(2xy + x^2) - y(2y^2 + 3xy)}$$

$$= \frac{2y + x}{2x^2y + x^3 - 2y^3 - 3xy^2}$$

This expression is complex and not a clear function of xy*.*

3 Case 4: Integrating Factor $\mu(x^2 + y^2)$

Theorem 2 (Existence of $\mu(x^2 + y^2)$)**.** *An integrating factor depending only on* $r^2 = x^2 + y^2$ *exists if and only if*

$$\frac{1}{xM + yN}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = G(x^2 + y^2)$$

where G *is a function of* $x^2 + y^2$ *alone.*

The denominator $xM + yN$ appears naturally in polar coordinate transformations. This test often succeeds for equations with circular symmetry.

Example 2 (Radial Integrating Factor)**.** *Consider* $(x^2 + y^2 + x)dx + (x^2 + y^2 + y)dy = 0$

Let's test for $\mu(x^2 + y^2)$*:*

$$M_y - N_x = 2y - 2x$$

$$xM + yN = x(x^2 + y^2 + x) + y(x^2 + y^2 + y) = (x^2 + y^2)(x + y) + x^2 + y^2$$

$$= (x^2 + y^2)(x + y + 1)$$

The ratio doesn't simplify to a function of $x^2 + y^2$ *easily, but if we had* $(x^2 + y^2)$ *as a common factor throughout, it would work.*

4 Case 5: Power Form $\mu = x^a y^b$

Method 1 (Finding $\mu = x^a y^b$). *To find an integrating factor of the form $\mu = x^a y^b$:*

1. *Multiply the equation by $x^a y^b$*
2. *Apply the exactness condition*
3. *Compare powers of x and y on both sides*
4. *Solve the resulting system for a and b*

Example 3 (Power Form). *Find $\mu = x^a y^b$ for $ydx + 2xdy = 0$*

After multiplication: $x^a y^{b+1} dx + 2x^{a+1} y^b dy = 0$

Exactness requires:

$$\frac{\partial}{\partial y}(x^a y^{b+1}) = \frac{\partial}{\partial x}(2x^{a+1} y^b)$$

$$(b+1)x^a y^b = 2(a+1)x^a y^b$$

$$b+1 = 2(a+1)$$

$$b = 2a+1$$

Choosing $a = -1$ gives $b = -1$, so $\mu = \frac{1}{xy}$ works.

5 Case 6: Homogeneous Equations

Theorem 3 (Integrating Factor for Homogeneous Equations). *If $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree n, then*

$$\mu = \frac{1}{xM + yN}$$

is an integrating factor (provided $xM + yN \neq 0$).

Proof Outline. Using Euler's theorem for homogeneous functions of degree n:

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = nM$$

$$x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN$$

The exactness condition with $\mu = 1/(xM + yN)$ can be verified using these relations. □

Quick Recognition of Homogeneous Equations:

- All terms have the same total degree in x and y
- $M(tx, ty) = t^n M(x, y)$ for some n

- Common forms: rational functions where numerator and denominator have same degree

6 Strategy Flowchart

Complete Integrating Factor Strategy:

1. Test for exactness - if exact, solve directly
2. Test for $\mu(x)$: Is $(M_y - N_x)/N$ a function of x only?
3. Test for $\mu(y)$: Is $(N_x - M_y)/M$ a function of y only?
4. Check for homogeneity - if yes, use $\mu = 1/(xM + yN)$
5. Test for $\mu(xy)$: Is $(M_y - N_x)/(xN - yM)$ a function of xy?
6. Test for $\mu(x^2 + y^2)$: Is $(M_y - N_x)/(xM + yN)$ a function of $x^2 + y^2$?
7. Try $\mu = x^a y^b$ by comparing powers
8. Look for patterns or use inspection

7 Memory Aids and Patterns

Denominator Patterns:

Form	**Test Denominator**	**Mnemonic**
$\mu(x)$	N	"N for x"
$\mu(y)$	M	"M for y"
$\mu(xy)$	$xN - yM$	"Cross product"
$\mu(x^2 + y^2)$	$xM + yN$	"Dot product"
Homogeneous	$xM + yN$	"Euler's friend"

8 Common Exam Patterns

Prof. Ditkowski often gives hints about the form:

- "Find an integrating factor of the form $x^a y^b$"
- "Show that the equation has an integrating factor depending on xy"
- "Find μ assuming it depends only on $x^2 + y^2$"

When you see these hints, skip the testing phase and work directly with the given form!