

# Lesson 42: Lyapunov Functions and Global Stability

ODE 1 - Prof. Adi Ditkowski

## Nonlinear Systems Analysis

### 1 Lyapunov Stability Concepts

**Definition 1** (Lyapunov Stability). *The equilibrium point  $\mathbf{x} = \mathbf{0}$  of  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  is:*

- **Lyapunov stable** if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that:

$$\|\mathbf{x}(0)\| < \delta \implies \|\mathbf{x}(t)\| < \epsilon \text{ for all } t \geq 0$$

- **Asymptotically stable** if it is Lyapunov stable AND there exists  $\delta_0 > 0$  such that:

$$\|\mathbf{x}(0)\| < \delta_0 \implies \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$$

- **Globally asymptotically stable** if it is asymptotically stable for ALL initial conditions:

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0} \text{ for all } \mathbf{x}(0) \in \mathbb{R}^2$$

#### Intuitive Understanding:

- Lyapunov stable: "Stay nearby forever"
- Asymptotically stable: "Stay nearby and converge"
- Globally asymptotically stable: "Converge from anywhere"

### 2 Lyapunov Functions

**Definition 2** (Lyapunov Function). *A continuously differentiable function  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a **Lyapunov function** for the system  $\dot{x} = f(x, y)$ ,  $\dot{y} = g(x, y)$  at the origin if:*

1.  $V(0, 0) = 0$
2.  $V(x, y) > 0$  for all  $(x, y) \neq (0, 0)$  in some neighborhood
3.  $\dot{V}(x, y) \leq 0$  along trajectories in that neighborhood

where the derivative along trajectories is:

$$\dot{V} = \frac{\partial V}{\partial x} f(x, y) + \frac{\partial V}{\partial y} g(x, y) = \nabla V \cdot \mathbf{f}$$

**Critical Distinction:**

$$\dot{V} \neq \frac{\partial V}{\partial t}$$

Since  $V$  doesn't explicitly depend on time,  $\frac{\partial V}{\partial t} = 0$ . Instead:

$$\dot{V} = \left. \frac{dV}{dt} \right|_{\text{along trajectories}} = \nabla V \cdot \mathbf{f}$$

### 3 Main Stability Theorems

**Theorem 1** (Lyapunov's Direct Method). *For the equilibrium at the origin:*

1. *If there exists a Lyapunov function with  $\dot{V} \leq 0$ , then the origin is **Lyapunov stable***
2. *If additionally  $\dot{V} < 0$  for  $(x, y) \neq (0, 0)$ , then the origin is **asymptotically stable***
3. *If additionally  $V(x, y) \rightarrow \infty$  as  $\|(x, y)\| \rightarrow \infty$  (radial unboundedness), then the origin is **globally asymptotically stable***

**Theorem 2** (Instability Theorem). *If there exists a function  $V$  with  $V(0, 0) = 0$  and:*

- *$V$  can take positive values arbitrarily close to the origin*
- *$\dot{V} > 0$  where  $V > 0$*

*then the origin is **unstable**.*

### 4 LaSalle's Invariance Principle

**Theorem 3** (LaSalle's Principle). *Let  $V$  be a Lyapunov function with  $\dot{V} \leq 0$ . Define:*

$$E = \{(x, y) : \dot{V}(x, y) = 0\}$$

*Let  $M$  be the largest invariant set contained in  $E$ . Then all trajectories starting in a neighborhood of the origin approach  $M$  as  $t \rightarrow \infty$ .*

LaSalle's principle often upgrades Lyapunov stability to asymptotic stability! If the only invariant set in  $E$  is the origin itself, then we have asymptotic stability even when  $\dot{V} \leq 0$  (not strictly negative).

### 5 Construction Methods

### Standard Approaches for Finding Lyapunov Functions:

1. **Quadratic Forms:** Try  $V = ax^2 + cy^2$  with  $a, c > 0$

2. **General Quadratic:** Try  $V = ax^2 + bxy + cy^2$  with:

$$a > 0, \quad c > 0, \quad ac - b^2/4 > 0$$

3. **Energy Methods:** For mechanical systems, use:

$$V = \text{kinetic energy} + \text{potential energy}$$

4. **Gradient Systems:** If  $\dot{\mathbf{x}} = -\nabla U(\mathbf{x})$ , use  $V = U$

5. **First Integrals:** If  $H(x, y)$  is conserved, modifications of  $H$  often work

6. **Trial and Error:** Combine terms based on system structure

## 6 Complete Examples

**Example 1** (Quadratic Lyapunov Function). *Consider the system:*

$$\dot{x} = -x^3 + xy^2, \quad \dot{y} = -y^3 + x^2y$$

*Try  $V(x, y) = x^2 + y^2$ :*

**Check conditions:**

1.  $V(0, 0) = 0$  ✓
2.  $V(x, y) = x^2 + y^2 > 0$  for  $(x, y) \neq (0, 0)$  ✓
3. Compute  $\dot{V}$ :

$$\dot{V} = 2x\dot{x} + 2y\dot{y} \tag{1}$$

$$= 2x(-x^3 + xy^2) + 2y(-y^3 + x^2y) \tag{2}$$

$$= -2x^4 + 2x^2y^2 - 2y^4 + 2x^2y^2 \tag{3}$$

$$= -2(x^4 + y^4) + 4x^2y^2 \tag{4}$$

$$= -2(x^4 + y^4 - 2x^2y^2) \tag{5}$$

$$= -2(x^2 - y^2)^2 \leq 0 \tag{6}$$

*So the origin is Lyapunov stable. For asymptotic stability, apply LaSalle:  $\dot{V} = 0$  only when  $x^2 = y^2$ , i.e.,  $x = \pm y$ .*

*On the line  $x = y$ :  $\dot{x} = 0$ ,  $\dot{y} = 0$  only at origin. On the line  $x = -y$ :  $\dot{x} = -2x^3$ ,  $\dot{y} = 2y^3$ , so trajectories leave unless at origin.*

*Therefore,  $M = \{(0, 0)\}$  and the origin is asymptotically stable.*

**Example 2** (Energy Function for Pendulum). *Damped pendulum:  $\ddot{\theta} + c\dot{\theta} + \sin \theta = 0$*

*As a system:  $\dot{x} = y, \dot{y} = -\sin x - cy$  (where  $x = \theta, y = \dot{\theta}$ )*

*Energy function:  $V(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$*

**Analysis:**

1.  $V(0, 0) = 0$  ✓
2. For small  $x$ :  $1 - \cos x \approx \frac{x^2}{2} > 0$ , so  $V > 0$  near origin ✓
3.  $\dot{V} = y\dot{y} + \sin x \cdot \dot{x} = y(-\sin x - cy) + \sin x \cdot y = -cy^2 \leq 0$

*By LaSalle:  $\dot{V} = 0$  when  $y = 0$ . The invariant set on  $\{y = 0\}$  is just the equilibria. Near the origin, this is only  $(0, 0)$ , giving asymptotic stability.*

## 7 Basin of Attraction

**Definition 3** (Basin of Attraction). *The basin of attraction of an asymptotically stable equilibrium  $\mathbf{x}^*$  is:*

$$\mathcal{B}(\mathbf{x}^*) = \{\mathbf{x}_0 : \lim_{t \rightarrow \infty} \phi(t, \mathbf{x}_0) = \mathbf{x}^*\}$$

*where  $\phi(t, \mathbf{x}_0)$  is the solution starting at  $\mathbf{x}_0$ .*

**Method 1** (Estimating Basins with Lyapunov Functions). *If  $V$  is a Lyapunov function with  $\dot{V} < 0$  for  $\mathbf{x} \neq \mathbf{0}$ , and  $\Omega_c = \{\mathbf{x} : V(\mathbf{x}) \leq c\}$  is bounded and contains only the origin as an equilibrium, then  $\Omega_c \subseteq \mathcal{B}(\mathbf{0})$ .*

## 8 Comparison: Linearization vs Lyapunov

Aspect	Linearization	Lyapunov
Scope	Local only	Local or global
Applicability	Hyperbolic equilibria	Any equilibrium
Information	Qualitative behavior	Stability + basin
Construction	Automatic (Jacobian)	Requires creativity
Limitations	Fails at centers	May not exist/find

### Prof. Ditkowski's Exam Strategy:

1. If given  $V$ , just verify conditions and compute  $\dot{V}$
2. If asked to find  $V$ :
  - Start with  $V = x^2 + y^2$
  - Try  $V = ax^2 + cy^2$  with suitable  $a, c$
  - Look for physical energy if mechanical system
  - Check if gradient system

3. Always apply LaSalle when  $\dot{V} \leq 0$  (not strictly negative)
4. State stability conclusion explicitly

## 9 Summary Algorithm

### Lyapunov Analysis Procedure:

1. Identify/construct candidate  $V(x, y)$
2. Verify  $V(0, 0) = 0$
3. Check  $V(x, y) > 0$  for  $(x, y) \neq (0, 0)$  nearby
4. Compute  $\dot{V} = \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial y}g$
5. Analyze sign of  $\dot{V}$ :
  - $\dot{V} \leq 0$ : Lyapunov stable
  - $\dot{V} < 0$  (except origin): Asymptotically stable
  - $\dot{V} > 0$  somewhere: Check instability theorem
6. If  $\dot{V} \leq 0$ , apply LaSalle's principle
7. Check radial unboundedness for global stability