Lesson 49: Practice Problems - Frobenius Method

ODE 1 - Prof. Adi Ditkowski

Part A: Classifying Singular Points (6 problems)

1. Classify all singular points (regular or irregular) for:

   $$x$$

   ^{2}(x-1)y” + 2xy’ + y = 0

2. For Bessel’s equation $x^{2y”}$ + xy’ + (x^2-4)y = $0, verifythat$x = $0isaregularsingularpoint.$
3. Determine the nature of $x = 0$ for:

   $$x$$

   $^{3y”}$ + xy’ + y = 0

4. Show that both $x = 0$ and $x = 1$ are regular singular points of:

   $$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

   (This is the hypergeometric equation)

5. Classify the singular point at $x = 0$ for:

   $$x$$

   $^{2y”}$ $+ \sin(x)y' + xy = 0$

6. Find and classify all singular points:

   $$(x$$

   ^{2}-1)$^{2y”}$ + (x-1)y’ + y = 0

Part B: Finding Indicial Equations (6 problems)

1. Find the indicial equation at $x = 0$ for:

$$xy'' + y' - y = 0$$

2. Determine the indicial equation for:

$$x$$

$^{2y”}$ + x(1+x)y’ - y = 0

3. Find the indicial equation and its roots for:

$$2x$$

$^{2y”}$ + xy’ - (1+x)y = 0

4. For the equation $x^{2y”}$ + 3xy’ + (1-x)y = 0, $findtheindicialequationandidentifywhichcaseapplies.$

5. Find the indicial equation at $x = 1$ for:

$$(x-1)$$

$^{2y”}$ + (x-1)y’ + y = 0

6. Determine the indicial equation for Laguerre’s equation:

$$xy'' + (1-x)y' + ny = 0$$

Part C: Determining Solution Forms (5 problems)

7. For $x^{2y”}$ + xy’ + (x^2-1/4)y = 0, $findtherootsoftheindicialequationandstatetheformofthegeneralsol$

8. Given the indicial roots r_1 = $3and$r$_2$ = -2, $writethegeneralformofthesolution.$

9. If the indicial equation has repeated root $r = 1/2$, write the form of both linearly independent solutions.

10. For Bessel’s equation of order 2, explain why the second solution must contain $\ln(x)$.

11. Given roots $r_1 = 1and$r$_2 = 0, determinewhetheralogarithmictermisneededbycheckingtherecurrenceat$n = 1.

Part D: Computing Frobenius Series (5 problems)

7. Find the first three non-zero terms of the Frobenius series solution for:

$$xy'' + 2y' + xy = 0$$

using the larger root.

8. For $x^{2y"}$ + xy' + (x^2-1)y = 0, $findtherecurrencerelationfortheroot$r = 1.

9. Solve using Frobenius method:

$$2xy'' + (1 + 2x)y' + y = 0$$

Find coefficients a_0, a$_1$, a$_2$$forthelargerroot.$

10. For the equation $x^{2y"}$ + x$^{2y'}$ - 2y = 0:

    11. Find the indicial equation

    12. Find the recurrence relation for $r = 2$

    13. Compute the first four coefficients

Apply Frobenius method to find one solution of:

$$x$$

$^{2y"}$ + x(x+1)y' - y = 0

Part E: Special Cases and Logarithmic Solutions (3 problems)

14. Show that for $x^{2y"}$ + 3xy' + (1+x)y = $0withrepeatedroot$r = -1, $thesecondsolutionmustcontain$ln($x$).

15. For the equation with roots differing by an integer:

$$x$$

$^{2y"}$ + xy' - y = 0

$Determineifbothsolutionscanbepure Frobeniusseriesoriflogarithmsareneeded.$

16. Verify that Euler's equation $x^{2y"}$ - xy' + y = $0hassolutions$y$_1$ = xandy$_2$ = xln(x).

Part F: Exam-Style Problems (5 problems)

17. [**10 points**] Consider the modified Bessel equation:

x

$^{2y"}$ + xy' - (x^2 + n^2)y = 0

where

n = 2.

2 pts Show that $x = 0$ is a regular singular point

3 pts Find the indicial equation and its roots

2 pts Which case applies for the general solution?

3 pts Write the form of both linearly independent solutions

[**9 points**] For the equation:

$$x(x-1)y'' + 3y' + y = 0$$

3 pts Find and classify all singular points

3 pts Find the indicial equation at $x = 0$

3 pts Find the first three terms of the Frobenius series for the larger root

[**8 points**] Given:

$2x$

$^{2y"}$ + x(1+x)y' - 2y = 0

Find the indicial equation and roots

Set up the recurrence relation

Determine if the second solution requires logarithms

[**10 points**] *Comprehensive Problem*

x

$^{2y"}$ + x(1-x)y' - (1+3x)y = 0

Verify $x = 0$ is a regular singular point

Find the indicial equation and solve for r

Find the recurrence relation for the larger root

State the form of the general solution

[**12 points**] *Prof. Ditkowski Special - Hypergeometric Type* Consider: $x(1-x)y'' + [2-(3+x)]y' - y = 0$

3 pts Show both $x = 0$ and $x = 1$ are regular singular points

3 pts Find the indicial equation at $x = 0$

3 pts Find the indicial equation at $x = 1$

3 pts Around which point would you prefer to expand and why?

Solutions and Hints

Selected Solutions:

Problem 1: - $x = 0$: Check $xp(x) = 2x^2/$(x(x-1))andx^{2q}(x) = x^2/(x(x-1))atx $=$ 0$\rightarrow$ $Regular-$x = 1: $Check$(x-1)p(x)and(x-1)^{2q}(x)atx = 1$\rightarrow$ $Regular$

Problem 7: Standard form: $y'' + (1/x)y' - (1/x)y = 0$ - $p_0 = 1$,q$_0 = 0 - Indicialequation$:r(r-1) + r = 0$\rightarrow$r^2 = 0$\rightarrow$r = 0$(repeated)$

Problem 13: - Indicial equation: r^2 - 1/4 = 0$-Roots$:r$_1$ = 1/2,r$_2$ = -1/2$(differby1) - Form : CheckifpureFrobeniusworksfor$r$_2$$orneedslogterm$

Problem 18: For larger root $r = 0$: - $y = a_0$(1 - x^2/2 + x^4/24 - $\cdots$) - This gives the Bessel function J_0(x)$series$

Problem 26: - Indicial roots: $r = 2, -1$ (differ by 3) - At $x = 0$: Regular singular point - Second solution likely needs no logarithm (check recurrence)

Key Insights: - Always check r_1 - r$_2$$first - Integerdifferencesrequirecarefulanalysis - Bessel-typeequationsareexamfavorites - When$r = 0$appears, onesolutionisaregularpowerseries$