Lesson 21: Exact Equations - Theory and Recognition

ODE 1 - Prof. Adi Ditkowski

1 Introduction to Exact Equations

Definition 1 (Exact Differential Equation)**.** *A first-order differential equation of the form*

$$M(x,y)dx + N(x,y)dy = 0$$

is called ***exact*** *if there exists a function $H(x,y)$ such that*

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy = M(x,y)dx + N(x,y)dy$$

When an equation is exact, its solution curves are the level curves $H(x,y) = C$ of the potential function H.

2 The Exactness Criterion

Theorem 1 (Test for Exactness)**.** *Let $M(x,y)$ and $N(x,y)$ have continuous partial derivatives in a simply connected domain D. The equation $M(x,y)dx + N(x,y)dy = 0$ is exact if and only if*

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Proof. ($\Rightarrow$) If the equation is exact, then $M = \frac{\partial H}{\partial x}$ and $N = \frac{\partial H}{\partial y}$ for some H. By Schwarz's theorem:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 H}{\partial y \partial x} = \frac{\partial^2 H}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

($\Leftarrow$) The converse requires constructing H from the condition, shown in Lesson 22. $\square$

3 Geometric Interpretation

The vector field $\mathbf{F} = (M, N)$ is conservative (has a potential function) if and only if its curl vanishes:

$$\text{curl}(\mathbf{F}) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

4 Connection to Physics

In thermodynamics, exact differentials correspond to state functions:

- Internal Energy: $dU = TdS - PdV$ (exact)
- Enthalpy: $dH = TdS + VdP$ (exact)
- Work: $\delta W = PdV$ (not exact - path dependent)
- Heat: $\delta Q = TdS$ (not exact - path dependent)

5 Algorithm for Testing Exactness

Step-by-Step Exactness Test:

1. Write equation in standard form: $M(x,y)dx + N(x,y)dy = 0$
2. Identify $M(x,y)$ and $N(x,y)$ explicitly
3. Compute $\frac{\partial M}{\partial y}$ (show all steps)
4. Compute $\frac{\partial N}{\partial x}$ (show all steps)
5. Compare the results:
   - If equal $\Rightarrow$ equation is exact
   - If not equal $\Rightarrow$ equation is not exact
6. State conclusion explicitly

6 Common Forms and Patterns

Recognize these patterns that often appear in Prof. Ditkowski's exams:

1. **Polynomial Forms:** $(ax^n y^m + bx^p y^q)dx + (cx^r y^s + dx^t y^u)dy = 0$
2. **Exponential Forms:** $(ae^{x+y} + bx)dx + (ce^{x+y} + dy)dy = 0$
3. **Trigonometric:** $(\cos(xy) \cdot y + f(x))dx + (\cos(xy) \cdot x + g(y))dy = 0$
4. **Mixed:** $(x^2 y + \sin x)dx + (x^3/3 + e^y)dy = 0$

7 Domain Considerations

The exactness condition guarantees existence of a potential function only in simply connected domains. Watch for:

- Punctured plane: $\mathbb{R}^2 \setminus \{(0, 0)\}$
- Domains with holes or discontinuities
- Multi-valued potential functions

8 Quick Reference

Memory Aid: "My Nexus"

M_y	Derivative of M with respect to y
N_x	Derivative of N with respect to x

If $M_y = N_x$, the equation is exact!