Implicit Solutions and Singular Solutions

ODE 1 - Prof. Adi Ditkowski

Lesson 13

1 Implicit Solutions

Definition 1 (Implicit Solution). *An* ***implicit solution*** *of the ODE* $\frac{dy}{dx} = f(x, y)$ *is a relation* $F(x, y) = C$ *that:*

1. *Defines y as a function of x (possibly multi-valued)*
2. *Satisfies the ODE when differentiated implicitly*

Method 1 (Verifying Implicit Solutions). *Given* $F(x, y) = C$ *and ODE* $\frac{dy}{dx} = f(x, y)$*:*

1. *Differentiate* $F(x, y) = C$ *implicitly:* $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$
2. *Solve for* $\frac{dy}{dx}$*:* $\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$
3. *Verify this equals* $f(x, y)$

Example 1 (Implicit Verification). *Verify that* $x^2 + xy + y^2 = C$ *solves some ODE and find it.*

Solution: *Differentiating implicitly:*

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$*:*

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Therefore, $x^2 + xy + y^2 = C$ *solves* $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$

Implicit solutions are preferable when:

- Explicit solution involves complex expressions
- Solution curves have vertical tangents
- Multiple branches exist

- The implicit form has geometric meaning

2 Singular Solutions

Definition 2 (Singular Solution). *A* ***singular solution*** *is a solution that:*

1. *Satisfies the ODE*
2. *Cannot be obtained from the general solution for any value of the arbitrary constant*
3. *Is typically the envelope of the general solution family*

Theorem 1 (Existence of Singular Solutions). *Singular solutions may exist when:*

- *The ODE is not linear in y'*
- *Uniqueness conditions fail*
- *The general solution involves parameters nonlinearly*

3 Methods for Finding Singular Solutions

3.1 C-Discriminant Method

Method 2 (C-Discriminant). *Given general solution $F(x, y, C) = 0$:*

1. *Form the system:*

$$F(x, y, C) = 0 \tag{1}$$
$$\frac{\partial F}{\partial C} = 0 \tag{2}$$

2. *Eliminate C between these equations*
3. *The result is the C-discriminant*
4. *Test if it satisfies the original ODE*

Example 2 (C-Discriminant Application). *For $(y - Cx)^2 = C^2 + 1$ (general solution):*
Step 1: *Compute $\frac{\partial F}{\partial C} = 0$:*

$$2(y - Cx)(-x) = 2C$$
$$-x(y - Cx) = C$$

Step 2: *Substitute into original: From $C = -x(y-Cx) = -xy+Cx^2$, we get $C(1-x^2) = -xy$*
If $x^2 \neq 1$: $C = \frac{-xy}{1-x^2}$
Step 3: *Substituting back gives the singular solution.*

3.2 p-Discriminant Method

Method 3 (p-Discriminant). *From ODE $F(x, y, y') = 0$:*

1. *Let $p = y'$ and write $F(x, y, p) = 0$*
2. *Form the system:*

$$F(x, y, p) = 0 \tag{3}$$
$$\frac{\partial F}{\partial p} = 0 \tag{4}$$

3. *Eliminate p to get the p-discriminant*
4. *Verify it satisfies the ODE*

4 Clairaut's Equation

Definition 3 (Clairaut's Equation). *An ODE of the form:*

$$y = xy' + f(y')$$

where f is a function of y' alone.

Theorem 2 (Solutions of Clairaut's Equation). *Clairaut's equation $y = xy' + f(y')$ has:*

1. ***General solution:*** *$y = Cx + f(C)$ (family of straight lines)*
2. ***Singular solution:*** *Obtained by eliminating p from:*

$$y = xp + f(p) \tag{5}$$
$$0 = x + f'(p) \tag{6}$$

Example 3 (Complete Clairaut Solution). *Solve: $y = xy' - (y')^2$*
General solution: *$y = Cx - C^2$ (family of lines)*
Singular solution: *From $x + f'(p) = 0$: $x - 2p = 0$, so $p = x/2$*
Substituting: $y = x(x/2) - (x/2)^2 = x^2/4$
The parabola $y = x^2/4$ is the envelope of all lines.

5 Envelope Theory

Definition 4 (Envelope). *The* ***envelope*** *of a family of curves $F(x, y, C) = 0$ is a curve that:*

1. *Is tangent to each member of the family*
2. *At each point, is tangent to exactly one family member*

Singular solutions are often envelopes of the general solution family. They represent the boundary of the region covered by all general solutions.

Method 4 (Finding Envelopes)**.** *To find the envelope of $F(x, y, C) = 0$:*

1. *Solve simultaneously:*
$$F(x, y, C) = 0, \quad \frac{\partial F}{\partial C} = 0$$

2. *Eliminate C to get envelope equation*

3. *Verify tangency conditions*

6 Parametric Solutions

Definition 5 (Parametric Solution)**.** *A solution expressed as:*

$$x = x(t), \quad y = y(t)$$

where t is a parameter.

Example 4 (When Parametric is Better)**.** *For $(y')^2 + y' = x$:*
Let $p = y'$, then $x = p^2 + p$
Differentiating: $dx = (2p + 1)dp$
Since $dy = p\,dx = p(2p + 1)dp$:

$$y = \int p(2p + 1)dp = \frac{2p^3}{3} + \frac{p^2}{2} + C$$

Parametric solution:
$$x = p^2 + p, \quad y = \frac{2p^3}{3} + \frac{p^2}{2} + C$$

7 Verification Techniques

Method 5 (Complete Verification Protocol)**.**

1. ***For implicit solutions:*** *Use implicit differentiation*

2. ***For singular solutions:***
   - *Direct substitution into ODE*
   - *Show it's not in general family*

3. ***For parametric solutions:*** *Use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$*

Prof. Ditkowski always awards points for proper verification, even if the solution is incorrect. Show all steps!

8 Common Errors

Critical mistakes:

1. Assuming all solutions can be made explicit
2. Missing singular solutions in nonlinear ODEs
3. Incorrect implicit differentiation
4. Confusing singular solutions with particular solutions
5. Not checking if p-discriminant satisfies ODE