Lesson 24: Practice Problems - Special Integrating Factors

ODE 1 - Prof. Adi Ditkowski

Part A: Testing for $\mu(xy)$ (Problems 1-5)

1. Test for $\mu(xy)$: $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$
2. Check if $\mu(xy)$ exists: $(y - x^2y)dx + (x - xy^2)dy = 0$
3. Test the $\mu(xy)$ condition: $(xy^2 + y)dx + (x^2y + x)dy = 0$
4. Determine if $\mu(xy)$ works: $(3xy + y^3)dx + (x^3 + 3xy)dy = 0$
5. Check for $\mu(xy)$: $(y\cos(xy) + 1)dx + (x\cos(xy) + 1)dy = 0$

Part B: Power Form $\mu = x^a y^b$ (Problems 6-10)

6. Find $\mu = x^a y^b$ for: $(y^3)dx + (xy^2)dy = 0$
7. Determine powers: $(x^2y)dx + (xy^2)dy = 0$
8. Find a, b: $(y + x^2y^3)dx + (x + x^3y^2)dy = 0$
9. Power integrating factor: $(2xy^2)dx + (3x^2y)dy = 0$
10. Find $\mu = x^a y^b$: $(y^2 + 2xy^3)dx + (xy + x^2y^2)dy = 0$

Part C: Linear Combination $\mu = (ax + by)^n$ (Problems 11-15)

11. Try $\mu = (x + y)^n$: $(x + 2y)dx + (2x + y)dy = 0$
12. Find n for $\mu = (x - y)^n$: $(x - y + 1)dx + (x - y - 1)dy = 0$
13. Linear combination form: $(2x + y + 1)dx + (x + 2y + 1)dy = 0$
14. Test $\mu = (ax + by)^n$: $(3x + 2y)dx + (2x + 3y)dy = 0$
15. Find appropriate linear μ: $(x + y + xy)dx + (x + y - xy)dy = 0$

Part D: Mixed Special Forms (Problems 16-20)

16. Given $\mu = \frac{1}{xy}$, solve: $(y^2 - x^2)dx + (2xy)dy = 0$
17. Verify $\mu = e^{x+y}$ works: $(e^{-x-y} + y)dx + (x + e^{-x-y})dy = 0$
18. Use $\mu = \frac{1}{x+y}$: $(x + y)^2 dx + (x + y)^2 dy = 0$
19. Try $\mu = xy$: $\left(\frac{y}{x^2} + \frac{1}{y}\right) dx + \left(\frac{x}{y^2} - \frac{1}{x}\right) dy = 0$
20. Given hint $\mu = (x^2 + y^2)^{-1}$: $(x + y^3)dx + (y - x^3)dy = 0$

Part E: Complete Problem Solving (Problems 21-25)

21. Full analysis and solution: $(2y)dx + (3x + xy)dy = 0$
22. Systematic approach: $(xy + y^2)dx + (x^2 + xy)dy = 0$
23. Find any integrating factor and solve: $(x^2 + y^2)dx - (2xy)dy = 0$
24. Complete solution: $(y^2 \cos x + y)dx + (2y \sin x + x)dy = 0$
25. Challenge problem: $(y + x^2y^3)dx + (x - x^3y^2)dy = 0$

Solutions and Hints

Problem 1: $(M_y - N_x)/(xN - yM) = \frac{(2x+2y)-(2x+2y)}{x(x^2+2xy)-y(2xy+y^2)} = \frac{0}{x^3+2x^2y-2xy^2-y^3}$ Since numerator is 0, this suggests the equation may already be exact.

Problem 6: For $\mu = x^a y^b$ with $(y^3)dx + (xy^2)dy = 0$: $(x^a y^{b+3})dx + (x^{a+1}y^{b+2})dy = 0$ Exactness: $x^a(b + 3)y^{b+2} = x^a(a + 1)y^{b+2}$ So $b + 3 = a + 1$, giving $a = b + 2$ Try $b = -1$: $a = 1$, so $\mu = \frac{x}{y}$

Problem 11: For $\mu = (x + y)^n$ with $(x + 2y)dx + (2x + y)dy = 0$: After multiplication: $[(x + y)^n(x + 2y)]dx + [(x + y)^n(2x + y)]dy = 0$ The algebra becomes complex, but often $n = 1$ or $n = -1$ work.

Problem 16: With $\mu = \frac{1}{xy}$: $\left(\frac{y^2-x^2}{xy}\right) dx + \left(\frac{2xy}{xy}\right) dy = 0$ $\left(\frac{y}{x} - \frac{x}{y}\right) dx + 2dy = 0$

Key Strategy Tips:

- Always check exactness first
- Try simple cases: $\mu(x)$, $\mu(y)$ before special forms
- For power forms, work systematically with small integer values
- Verification is essential - substitute back into exactness condition
- If hint is given, use it! Prof. Ditkowski usually provides guidance for special forms