Lesson 19: Riccati Equations with Known Particular Solution

ODE 1 - Prof. Adi Ditkowski

1 Definition and Recognition

Definition 1 (Riccati Equation). *A Riccati differential equation has the form:*

$$\frac{dy}{dx} = q_0(x) + q_1(x)y + q_2(x)y^2$$

where $q_0(x)$, $q_1(x)$, and $q_2(x)$ are continuous functions and $q_2(x) \not\equiv 0$.

Special Cases:

- If $q_2(x) \equiv 0$: Linear equation
- If $q_0(x) \equiv 0$: Bernoulli equation with $n = 2$
- If $q_1(x) \equiv 0$: Separable after substitution
- General case: Requires particular solution or transformation to second-order

2 The Fundamental Transformation

Theorem 1 (Riccati to Bernoulli Reduction). *If $y_p(x)$ is a particular solution of the Riccati equation, then the substitution*

$$y = y_p + v$$

transforms the Riccati equation into the Bernoulli equation:

$$\frac{dv}{dx} = (q_1 + 2q_2y_p)v + q_2v^2$$

Proof. Given: $y' = q_0 + q_1y + q_2y^2$ and $y_p' = q_0 + q_1y_p + q_2y_p^2$

Substitute $y = y_p + v$:

$$y_p' + v' = q_0 + q_1(y_p + v) + q_2(y_p + v)^2 \quad (1)$$

$$y_p' + v' = q_0 + q_1y_p + q_1v + q_2y_p^2 + 2q_2y_pv + q_2v^2 \quad (2)$$

$$v' = q_1v + 2q_2y_pv + q_2v^2 \quad (\text{using } y_p' = q_0 + q_1y_p + q_2y_p^2) \quad (3)$$

$$v' = (q_1 + 2q_2y_p)v + q_2v^2 \quad \square \quad (4)$$

□

Complete Solution Algorithm:

1. Find or verify particular solution y_p
2. Substitute $y = y_p + v$
3. Obtain Bernoulli equation: $v' = (q_1 + 2q_2 y_p)v + q_2 v^2$
4. Use substitution $w = v^{-1}$ (since $n = 2$)
5. Solve linear equation: $w' = -(q_1 + 2q_2 y_p)w - q_2$
6. Back-substitute: $v = 1/w$, then $y = y_p + v$

3 Finding Particular Solutions

Method 1 (Inspection Techniques)**.** ***Common forms to try:***

1. ***Constants:*** *Try* $y_p = c$ *when coefficients allow*
2. ***Linear:*** *Try* $y_p = ax + b$ *for polynomial coefficients*
3. ***Rational:*** *Try* $y_p = a/x$ *or* $y_p = a/(x + b)$
4. ***Exponential:*** *Try* $y_p = ae^{bx}$ *for constant coefficients*
5. ***Trigonometric:*** *Try* $y_p = a\tan(bx)$ *or* $a\cot(bx)$
6. ***Special:*** $y_p = -q_1/(2q_2)$ *when this ratio is constant*

Example 1 (Polynomial Particular Solution)**.** *Solve:* $y' = \frac{2}{x^2} - \frac{2y}{x} + y^2$

Finding y_p***:*** *Try* $y_p = \frac{a}{x}$

$$-\frac{a}{x^2} = \frac{2}{x^2} - \frac{2a}{x^2} + \frac{a^2}{x^2}$$

$$-a = 2 - 2a + a^2 \implies a^2 - a + 2 = 0$$

This gives $a = 2$ *or* $a = -1$*. Use* $y_p = \frac{2}{x}$*.*

Transformation: *Let* $y = \frac{2}{x} + v$

$$v' = -\frac{2v}{x} + \frac{4v}{x} + v^2 = \frac{2v}{x} + v^2$$

Bernoulli to Linear: *Let* $w = v^{-1}$

$$w' = -\frac{2w}{x} - 1$$

Solution: $w = \frac{C}{x^2} - \frac{x}{3}$

Final Answer: $y = \frac{2}{x} + \frac{1}{C/x^2 - x/3}$

4 Special Riccati Forms

Constant Coefficient Riccati: $y' = a + by + cy^2$

- If $b^2 - 4ac > 0$: Two constant particular solutions
- If $b^2 - 4ac = 0$: One constant particular solution
- If $b^2 - 4ac < 0$: No real constant solutions

For $b^2 - 4ac < 0$, try $y_p = \alpha \tan(\beta x)$ where $\beta = \sqrt{4ac - b^2}/(2c)$

Example 2 (Trigonometric Particular Solution). *Solve:* $y' = 1 + y^2$

Observation: *This matches* $\frac{d}{dx}[\tan x] = \sec^2 x = 1 + \tan^2 x$

Particular solution: $y_p = \tan x$

General solution: *Let* $y = \tan x + v$

$$v' = 2\tan x \cdot v + v^2$$

After solving the Bernoulli equation:

$$y = \tan x + \frac{\sin x}{C - \cos x}$$

5 Connection to Linear Second-Order

Theorem 2 (Riccati-Linear Duality). *The Riccati equation* $y' = q_0 + q_1 y + q_2 y^2$ *is equivalent to the second-order linear equation:*

$$u'' - (q_1 + \frac{q_2'}{q_2})u' + q_0 q_2 \cdot u = 0$$

via the transformation $y = -\frac{1}{q_2} \cdot \frac{u'}{u}$

Prof. Ditkowski's Patterns:

- Often provides y_p or strong hints ("verify that...")
- Tests connection to Bernoulli reduction
- Likes rational particular solutions $y_p = a/x$
- May ask for multiple particular solutions
- Tests the second-order connection
- Partial credit for correct transformation setup

6 Geometric Interpretation

Lemma 1 (Cross-Ratio Property). *If y_1, y_2, y_3, y_4 are four solutions of a Riccati equation, their cross-ratio:*

$$\frac{(y_1 - y_3)(y_2 - y_4)}{(y_1 - y_4)(y_2 - y_3)}$$

is constant (independent of x).

Common Pitfalls:

- Not verifying that y_p satisfies the equation
- Sign errors in the transformation to Bernoulli
- Forgetting that Bernoulli with $n = 2$ needs $w = v^{-1}$
- Missing singular solutions when $v = 0$

7 Solution Structure

General Solution Form:

$$y = y_p + \frac{1}{w(x)}$$

where $w(x)$ satisfies the linear equation:

$$w' + (q_1 + 2q_2y_p)w = -q_2$$

The general solution has the structure:

$$y = y_p + \frac{1}{\phi(x) + C\psi(x)}$$

where ϕ and ψ depend on the particular solution chosen.