Lesson 27: Practice Problems

Fundamental Matrix Solutions - Construction

Part A: Basic Fundamental Matrix Construction (6 problems)

1. Given solutions $\mathbf{x}_1(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix}$ and $\mathbf{x}_2(t) = \begin{bmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$, construct the fundamental matrix.

2. For the system $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}$, find a fundamental matrix.

3. Verify that $\Phi(t) = \begin{bmatrix} \mathrm{e}^{2t} & \mathrm{e}^{-t} \\ \mathrm{e}^{2t} & -2\mathrm{e}^{-t} \end{bmatrix}$ is a fundamental matrix for some system. Find the system.

4. Given $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{x}$, construct the fundamental matrix using trigonometric functions.

5. Find the principal fundamental matrix at $t = 0$ for $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \mathbf{x}$.

6. Construct a fundamental matrix for $\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x}$.

Part B: IVP Solutions Using Fundamental Matrices (5 problems)

7. Use $\Phi(t) = \begin{bmatrix} e^t & \mathrm{e}^{-t} \\ e^t & -\mathrm{e}^{-t} \end{bmatrix}$ to solve the IVP with $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

8. Given $\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$, solve for $\mathbf{x}(t)$ with $\mathbf{x}(\pi/2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

9. The fundamental matrix $\Phi(t) = \begin{bmatrix} \mathrm{e}^{2t} & t\mathrm{e}^{2t} \\ 0 & \mathrm{e}^{2t} \end{bmatrix}$ corresponds to a system with repeated eigenvalues. Solve the IVP with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

10. Find the solution to $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x}$ with $\mathbf{x}(1) = \begin{bmatrix} e \\ 0 \end{bmatrix}$.

11. Use the fundamental matrix method to solve $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Part C: Verification and Properties (5 problems)

12. Verify that $\Phi(t) = \begin{bmatrix} \mathrm{e}^{3t} & \mathrm{e}^{-2t} \\ 3\mathrm{e}^{3t} & -2\mathrm{e}^{-2t} \end{bmatrix}$ satisfies $\Phi'(t) = A\Phi(t)$ and find A.

13. Show that if $\Phi(t)$ is a fundamental matrix, then so is $\Phi(t)C$ for any nonsingular constant matrix C.

14. Prove that $\det(\Phi(t))$ is never zero if $\Phi(t)$ is a fundamental matrix.

15. Given two fundamental matrices $\Phi_1(t) = \begin{bmatrix} e^t & 0 \\ 0 & \mathrm{e}^{2t} \end{bmatrix}$ and $\Phi_2(t) = \begin{bmatrix} 2e^t & \mathrm{e}^{2t} \\ 3e^t & 4\mathrm{e}^{2t} \end{bmatrix}$, find the constant matrix C such that $\Phi_2(t) = \Phi_1(t)C$.

16. Verify that the columns of $\Phi(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix}$ are linearly independent for all t.

Part D: Principal Fundamental Matrix (5 problems)

17. Find the principal fundamental matrix at $t_0 = 0$ for $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$.

18. Given $\Phi(t) = \begin{bmatrix} \mathrm{e}^{2t} + e^{-t} & \mathrm{e}^{2t} - e^{-t} \\ 2\mathrm{e}^{2t} - e^{-t} & 2\mathrm{e}^{2t} + e^{-t} \end{bmatrix}$, find the principal fundamental matrix at $t = 0$.

19. Construct the principal fundamental matrix at $t = 1$ for $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$.

20. If $\Psi(t)$ is the principal fundamental matrix at t_0, express the solution to $\mathbf{x}(t_0) = \mathbf{x}_0$ in terms of $\Psi(t)$.

21. Show that the principal fundamental matrix is unique for a given t_0.

Part E: Advanced Theory (4 problems)

22. Prove that if $A(t)$ is continuous on an interval I, then a fundamental matrix exists on I.

23. Show that for constant A, the fundamental matrix can be written as $\Phi(t) = Pe^{Dt}\,\mathrm{P}^{-1}$ where D is the diagonal matrix of eigenvalues and P is the matrix of eigenvectors.

24. Prove that $\Phi(t+s) = \Phi(t)\Phi(s)\Phi(0)^{-1}$ for constant coefficient systems.

25. If $\Phi(t)$ is a fundamental matrix for $\mathbf{x}' = A(t)\mathbf{x}$, show that $\Psi(t) = \Phi(t)^{-T}$ is a fundamental matrix for the adjoint system $\mathbf{y}' = -A(t)^T\mathrm{y}$.

Part F: Exam-Style Problems (5 problems)

22. (Prof. Ditkowski style) Given the system $\mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}\mathbf{x}$:

    (a) Find all eigenvalues and eigenvectors

    (b) Construct the fundamental matrix $\Phi(t)$

    (c) Verify $\Phi'(t) = A\Phi(t)$

    (d) Solve the IVP with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

23. The solutions $\mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,$\mathrm{x}_2(t)$ $=\mathrm{e}^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{x}_3(t)$ $=\mathrm{e}^{3t} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ form a fundamental set.

    (a) Construct $\Phi(t)$

    (b) Find the system matrix A

    (c) Compute $\Phi(1)\Phi(0)^{-1}$

24. For the system with fundamental matrix $\Phi(t) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{-t}(1+t) \\ -\mathrm{e}^{-t} & -\mathrm{e}^{-t}t \end{bmatrix}$:

    (a) Find the system matrix A

    (b) Explain why there's a term with t

    (c) Find all solutions to $\mathbf{x}' = A\mathbf{x}$

25. Given partial information: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has $\mathrm{tr}(A) = 4$ and $\det(A) = 3$.

    (a) Find the eigenvalues

    (b) If one eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find A

    (c) Construct the fundamental matrix

    (d) Find the principal fundamental matrix at $t = 0$

26. (Comprehensive) Consider the third-order equation $y''' - 6y'' + 11y' - 6y = 0$.

   (a) Convert to a first-order system

   (b) Given that e^t, e^{2t}, e^{3t} are solutions to the scalar equation, construct the fundamental matrix

   (c) Solve the IVP: $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$

   (d) Verify your solution satisfies the original equation

Solutions and Hints

Selected Solutions:

Problem 1: $\Phi(t) = \begin{bmatrix} e^t & \mathrm{e}^{3t} \\ 2e^t & \mathrm{e}^{3t} \end{bmatrix}$

Problem 3: $A = \Phi'(t)\Phi(t)^{-1} = \begin{bmatrix} 3 & -1 \\ 3 & -2 \end{bmatrix}$

Problem 7: $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Problem 12: $A = \begin{bmatrix} 3 & 0 \\ 3 & -2 \end{bmatrix}$

Problem 16: $C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

Problem 18: Principal fundamental matrix: $\Phi_p(\mathrm{t}) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

Problem 25: Eigenvalues: $\lambda_1 = 1, \lambda_2 = 2$ (repeated)

Key Insights:

- Always verify $\Phi'(t) = A\Phi(t)$ to confirm fundamental matrix
- Check $\det(\Phi(t)) \neq 0$ for linear independence
- Remember: $\mathbf{x}(t) = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0$
- Principal fundamental matrix simplifies to $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$
- For repeated eigenvalues, expect terms with t in $\Phi(t)$