Lesson 44: Repeated Roots - Why We Get t^k Terms

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1 Introduction: The Problem of Repeated Roots

When the characteristic equation of a linear ODE with constant coefficients has repeated roots, the naive approach of using only exponential solutions fails to generate enough linearly independent solutions.

Definition 1 (Multiplicity of a Root)**.** *If the characteristic polynomial $p(r)$ can be factored as $p(r) = (r - r_0)^m q(r)$ where $q(r_0) \neq 0$, then r_0 is a root of multiplicity m.*

Theorem 1 (Dimension of Solution Space)**.** *An n-th order linear homogeneous ODE has an n-dimensional solution space. Therefore, we need exactly n linearly independent solutions.*

2 Failure of Simple Exponentials

Example 1 (The Problem)**.** *Consider $y'' - 4y' + 4y = 0$ with characteristic equation $(r-2)^2 = 0$.*

If we only use $y = e^{2t}$, we have just one solution. But we need two linearly independent solutions for this second-order equation.

When a root r has multiplicity $m > 1$, the solution e^{rt} alone cannot span the m-dimensional subspace of solutions corresponding to that root.

3 Derivation via Reduction of Order

Method 1 (Reduction of Order for Repeated Roots)**.** *Given one solution $y_1 = e^{rt}$ where r is a repeated root, we seek a second solution of the form:*

$$y_2 = v(t)y_1 = v(t)e^{rt}$$

Theorem 2 (Second Solution for Double Root)**.** *If r is a double root of the characteristic equation for $y'' + py' + qy = 0$, and $y_1 = e^{rt}$ is one solution, then $y_2 = te^{rt}$ is a second, linearly independent solution.*

Proof. Let $y_2 = v(t)e^{rt}$. Then:

$$y_2' = v'e^{rt} + rve^{rt} \tag{1}$$
$$y_2'' = v''e^{rt} + 2rv'e^{rt} + r^2ve^{rt} \tag{2}$$

Substituting into the differential equation:

$$v''e^{rt} + 2rv'e^{rt} + r^2ve^{rt} + p(v'e^{rt} + rve^{rt}) + qve^{rt} = 0$$

Factoring out e^{rt}:

$$v'' + v'(2r + p) + v(r^2 + pr + q) = 0$$

Since r is a double root of $\lambda^2 + p\lambda + q = 0$:

1. $r^2 + pr + q = 0$ (r is a root)
2. $2r + p = 0$ (derivative of characteristic polynomial at r equals 0)

Therefore: $v'' = 0$, giving $v = c_1t + c_2$.
Taking $c_1 = 1, c_2 = 0$ yields $y_2 = te^{rt}$. □

4 The Differential Operator Approach

The differential equation $(D - r)^m[y] = 0$ naturally produces solutions t^ke^{rt} for $k = 0, 1, \ldots, m - 1$.

Theorem 3 (Operator Factorization)**.** *If the characteristic polynomial factors as $p(\lambda) = (\lambda - r)^m$, then the differential operator factors as:*

$$L = (D - r)^m$$

where $D = \frac{d}{dt}$.

Lemma 1 (Kernel of $(D - r)^m$)**.** *The kernel (null space) of the operator $(D - r)^m$ is:*

$$ker((D - r)^m) = span\{e^{rt}, te^{rt}, t^2e^{rt}, \ldots, t^{m-1}e^{rt}\}$$

Proof by Induction. Base case ($m = 1$): $(D - r)[y] = 0 \Rightarrow y = ce^{rt}$.

Inductive step: Assume true for $m = k$. For $m = k + 1$: If $(D - r)^{k+1}[y] = 0$, let $w = (D - r)[y]$. Then $(D - r)^k[w] = 0$, so $w = p_k(t)e^{rt}$ where p_k is a polynomial of degree $\leq k - 1$.

Solving $(D - r)[y] = p_k(t)e^{rt}$: Using integrating factor e^{-rt}: $\frac{d}{dt}[e^{-rt}y] = p_k(t)$
Integrating: $e^{-rt}y = P_{k+1}(t) + C$ where P_{k+1} has degree $\leq k$.
Therefore: $y = [P_{k+1}(t) + C]e^{rt} = p_{k+1}(t)e^{rt}$ where $\deg(p_{k+1}) \leq k$. □

5 The Limit Perspective

Theorem 4 (Repeated Roots as Limits). *The solution te^{rt} can be understood as the limit of solutions for nearby distinct roots.*

Heuristic Derivation. Consider two roots $r_1 = r$ and $r_2 = r + \epsilon$ with solutions:

$$y = c_1 e^{rt} + c_2 e^{(r+\epsilon)t}$$

Rewriting:

$$y = c_1 e^{rt} + c_2 e^{rt} e^{\epsilon t}$$

For small ϵ: $e^{\epsilon t} \approx 1 + \epsilon t$
Thus:

$$y \approx (c_1 + c_2)e^{rt} + c_2 \epsilon t e^{rt}$$

As $\epsilon \to 0$, setting $A = c_1 + c_2$ and $B = \lim_{\epsilon \to 0} c_2 \epsilon$:

$$y = Ae^{rt} + Bte^{rt}$$

□

6 Higher Multiplicities

Theorem 5 (General Multiplicity Case). *If r is a root of multiplicity m, then the m linearly independent solutions are:*

$$e^{rt}, te^{rt}, t^2 e^{rt}, \ldots, t^{m-1} e^{rt}$$

For a root r of multiplicity m:

$$\text{Solutions} = \{t^k e^{rt} : k = 0, 1, 2, \ldots, m-1\}$$

7 Complex Repeated Roots

Theorem 6 (Complex Repeated Roots). *If $\alpha \pm i\beta$ are complex conjugate roots each of multiplicity m, the $2m$ real-valued linearly independent solutions are:*

$$t^k e^{\alpha t} \cos(\beta t), \, t^k e^{\alpha t} \sin(\beta t) \quad \textit{for } k = 0, 1, \ldots, m-1$$

For complex repeated roots, both the sine AND cosine terms need all powers of t up to t^{m-1}.

8 The Wronskian for Repeated Roots

Theorem 7 (Linear Independence via Wronskian). *The functions* $\{t^k e^{rt} : k = 0, 1, \ldots, m-1\}$ *are linearly independent.*

Wronskian Calculation. For simplicity, consider $m = 2$ with solutions e^{rt} and te^{rt}:

$$W(t) = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & (1+rt)e^{rt} \end{vmatrix}$$

$$W(t) = e^{rt} \cdot (1+rt)e^{rt} - te^{rt} \cdot re^{rt} = e^{2rt}(1 + rt - rt) = e^{2rt} \neq 0$$

For general m, the Wronskian is:

$$W(t) = e^{mrt} \cdot \prod_{0 \leq i < j < m} (j - i) \neq 0$$

□

9 Connection to Jordan Normal Form

The appearance of t^k terms in solutions corresponds exactly to the Jordan block structure of the companion matrix.

Theorem 8 (Jordan Form and Solutions). *For a system* $\mathbf{x}' = A\mathbf{x}$ *where* A *has a Jordan block:*

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

The matrix exponential is:

$$e^{Jt} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

10 Algorithm for Repeated Roots

Method 2 (Complete Solution Construction). *1. Factor the characteristic polynomial completely.*

2. For each distinct root r_i *with multiplicity* m_i*:*

- *If r_i is real: Include solutions $t^k e^{r_i t}$ for $k = 0, 1, \ldots, m_i - 1$*
- *If $r_i = \alpha + i\beta$ is complex (with conjugate $\alpha - i\beta$): Include solutions $t^k e^{\alpha t}\cos(\beta t)$ and $t^k e^{\alpha t}\sin(\beta t)$ for $k = 0, 1, \ldots, m_i - 1$*

3. *Form the general solution as a linear combination of all these functions.*

11 Worked Examples

Example 2 (Triple Root)**.** *Solve: $y''' - 3y'' + 3y' - y = 0$*

Solution: *Characteristic equation: $r^3 - 3r^2 + 3r - 1 = 0$*
Recognize this as $(r-1)^3 = 0$ (expand to verify).
Root: $r = 1$ with multiplicity 3.
Solutions: e^t, te^t, t^2e^t
General solution: $y(t) = (c_1 + c_2t + c_3t^2)e^t$

Example 3 (Complex Double Roots)**.** *Solve: $y^{(4)} + 8y'' + 16y = 0$*

Solution: *Characteristic equation: $r^4 + 8r^2 + 16 = 0$*
This is $(r^2+4)^2 = 0$, giving $(r-2i)^2(r+2i)^2 = 0$
Roots: $\pm 2i$, each with multiplicity 2.
Real solutions:

- *From $2i$ (mult. 2): $\cos(2t), t\cos(2t), \sin(2t), t\sin(2t)$*

General solution: $y(t) = (c_1 + c_2t)\cos(2t) + (c_3 + c_4t)\sin(2t)$

Prof. Ditkowski often presents the characteristic polynomial already factored. Don't waste time trying to factor what's already factored - just identify multiplicities and write solutions systematically.