Practice Problems: Integrating Factor Deep Dive

Lesson 15 - Prof. Ditkowski's ODE 1

Part A: Theoretical Understanding (5 problems)

1. Prove that if $\mu_1(t)$ and $\mu_2(t)$ are both integrating factors for $y' + p(t)y = g(t)$, then $\mu_2(t) = C\mu_1(t)$ for some constant C.
2. Show that the integrating factor $\mu(t) = e^{\int p(t)dt}$ is the unique positive integrating factor with $\mu(t_0) = 1$ for any fixed t_0.
3. For the equation $y' + p(t)y = g(t)$, prove that:

$$\frac{d}{dt}[\mu(t)W(t)] = 0 \tag{1}$$

   where $W(t)$ is the Wronskian of any two solutions and $\mu(t)$ is the integrating factor.
4. Explain why the integrating factor method always produces an exact equation, while not every first-order ODE can be made exact.
5. If $p(t)$ is periodic with period T, prove that $\mu(t + T) = \mu(t) \cdot \mu(T)$.

Part B: Computing Complex Integrating Factors (6 problems)

6. Find the integrating factor for: $y' + \frac{2t}{1+t^2}y = e^t$
7. Find the integrating factor for: $y' + (\tan t + \sec t)y = \cos t$
8. Find the integrating factor for: $y' + \frac{1}{t \ln t}y = \frac{1}{t^2}$ for $t > e$
9. Find the integrating factor for: $y' - \frac{3}{2t}y + \frac{1}{2t^2}y = 0$ (Hint: This isn't quite standard form!)
10. For the equation $(t^2 + 1)y' + 2ty = t$, find the integrating factor: a) After converting to standard form b) Directly without converting to standard form Compare your answers.
11. Find the integrating factor for the piecewise coefficient:

$$p(t) = \begin{cases} 2 & 0 \le t < 1 \\ \frac{1}{t} & t \ge 1 \end{cases} \tag{2}$$

Part C: Alternative Approaches (5 problems)

12. Given that $y_h = t^2 e^{-t}$ solves the homogeneous equation $y' + p(t)y = 0$, find: a) The function $p(t)$ b) The integrating factor $\mu(t)$ c) Verify that $\mu(t) = 1/y_h(t)$ up to a constant

13. For the equation $ty' + (2-t)y = t^2e^t$: a) Find the integrating factor without converting to standard form b) Show that multiplying by your integrating factor creates the exact form $d/dt[f(t)y] = h(t)$

14. Consider the equation $y' + p(t)y = p(t)$. a) Show that $y = 1$ is a particular solution b) Use this to find the general solution without computing $\mu(t)$ explicitly c) What does this tell you about the relationship between $p(t)$ and the solution?

15. The equation $(1+t^2)y' - 2ty = (1+t^2)^2$ can be solved by substitution. Let $v = y/(1+t^2)$. a) Show that this transforms the equation to $v' = 1 + t^2$ b) Solve for v and hence y c) Find the integrating factor and verify it gives the same solution

16. For the Bernoulli-like equation $y' + p(t)y = g(t)y^0$ (which is actually linear): a) Explain why the standard integrating factor works b) Show that if we mistakenly treat it as Bernoulli and use substitution $v = y^{1-0} = y$, we get the same equation

Part D: Connections to Exact Equations (5 problems)

17. Show that after multiplying $y' + \frac{2}{t}y = t^2$ by its integrating factor, the resulting equation can be written as an exact equation $M(t, y)dt + N(t, y)dy = 0$. Find M and N and verify exactness.

18. For the general equation $y' + p(t)y = g(t)$: a) After multiplying by $\mu(t)$, express as $M(t, y)dt + N(t, y)dy = 0$ b) Find the potential function $F(t, y)$ such that $\partial F/\partial t = M$ and $\partial F/\partial y = N$ c) Show that the solution curves are level sets of F

19. Consider the exact equation $(2ty + t^2)dt + t^2dy = 0$. a) Verify it's exact b) Find an integrating factor that would convert $y' + (2/t)y = -1$ to this exact form c) Solve both ways and verify you get the same answer

20. The equation $(\sin t)y' + (\cos t)y = 1$ becomes exact after multiplication by $\mu(t)$. a) Find $\mu(t)$ b) Write the exact form and find the potential function c) Solve using both the integrating factor method and the exact equation method

21. Prove that if an equation is already exact, then $\mu(t) = 1$ is an integrating factor, and any other integrating factor must be a constant.

Part E: Advanced Theory and Applications (5 problems)

22. (Adjoint Equation) For $L[y] = y' + p(t)y$: a) Show that the adjoint operator is $L^*[v] = -v' + p(t)v$ b) Prove that if μ satisfies $L^*[\mu] = 0$, then μ is an integrating factor c) Find the relationship between solutions of $L[y] = 0$ and $L^*[v] = 0$

23. (Green's Function Preview) For the equation $y' + p(t)y = g(t)$ with $y(0) = 0$: a) Show that the solution can be written as $y(t) = \int_0^t G(t,s)g(s)ds$ b) Find $G(t,s)$ in terms of the integrating factor $\mu(t)$ c) Verify that $G(t,s) = \mu(s)/\mu(t)$ for $s \leq t$

24. (Numerical Analysis) Consider $y' + 100y = e^{-t}$ on $[0, 10]$. a) Find the integrating factor $\mu(t)$ b) Estimate $\mu(10)$ and discuss potential numerical overflow c) Suggest a reformulation to avoid numerical issues

25. (Discontinuous Coefficients) For:

$$y' + p(t)y = 1, \quad p(t) = \begin{cases} 1 & t < 1 \\ 2 & t \geq 1 \end{cases} \tag{3}$$

a) Find the integrating factor on each interval b) Solve on each interval with arbitrary constants c) Find conditions for C^0 continuity at $t = 1$ d) Can the solution be C^1 at $t = 1$?

26. (Generalized Integrating Factors) Sometimes we seek $\mu(t, y)$ for non-linear equations. a) For $(y^2 + 2ty)dt + t^2dy = 0$, show that $\mu = 1/y^2$ makes it exact b) Explain why this doesn't contradict the uniqueness theorem for linear equations c) Find the general solution using this integrating factor

Part F: Exam-Style Comprehensive Problems (4 problems)

27. Consider the family of equations $y' + p_n(t)y = t^n$ where $p_n(t) = n/t$. a) Find the integrating factor $\mu_n(t)$ b) Solve the equation for general n c) For which values of n does the solution remain bounded as $t \to \infty$? d) For which values of n is the solution continuous at $t = 0$?

28. The equation $y' + (\cot t)y = \csc t$ on $(0, \pi)$: a) Find two different forms of the integrating factor using different antiderivatives b) Show both lead to the same general solution c) Find the solution satisfying $y(\pi/2) = 1$ d) Determine the behavior as $t \to 0^+$ and $t \to \pi^-$

29. For the equation $ty' + y = te^t$: a) Explain why you cannot use the standard form on $(-\infty, \infty)$ b) Solve separately for $t > 0$ and $t < 0$ c) Show that no solution exists that is continuous at $t = 0$ unless it satisfies a specific condition d) Find the integrating factor directly without converting to standard form

30. (Comprehensive Theory) Let $y' + p(t)y = g(t)$ where $p(t) = 2\alpha\cos(2t)$ and $g(t) = \sin(t)$. a) Find the integrating factor (Hint: Use the identity $\int \cos(2t)dt = \sin(2t)/2$) b) Show that if $\alpha = 0$, the solution is periodic c) For $\alpha \neq 0$, find the general solution d) Prove that no periodic solution exists when $\alpha \neq 0$ e) Discuss the physical interpretation if this represents a driven oscillator

Solutions and Hints

For Problem 1: Use the fact that both μ_1 and μ_2 satisfy $\mu' = p(t)\mu$.

For Problem 6: Remember that $\int \sec t\, dt = \ln|\sec t + \tan t|$.

For Problem 11: After converting to standard form, $p(t) = 2t/(t^2+1)$. For the direct approach, look for μ such that $d/dt[\mu(t^2+1)y] = \mu \cdot t$.

For Problem 22: The Green's function represents the influence of the forcing at time s on the solution at time $t > s$.

[Complete solutions would be provided showing multiple approaches where applicable, emphasizing the deep connections between different concepts.]