Lesson 22: Practice Problems - Finding Potential Functions

ODE 1 - Prof. Adi Ditkowski

Part A: Basic Potential Functions (Problems 1-6)

1. Find the potential function: $(2x + 3y)dx + (3x + 4y)dy = 0$
2. Find the potential function: $(y^2 + 2xy)dx + (2xy + x^2)dy = 0$
3. Find the potential function: $(e^x \sin y + 2x)dx + (e^x \cos y)dy = 0$
4. Find the potential function: $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$
5. Find the potential function: $\left(\frac{1}{x} + y\right) dx + \left(x - \frac{1}{y}\right) dy = 0$
6. Find the potential function: $(2xy^3 + 1)dx + (3x^2y^2 + 2)dy = 0$

Part B: Method Comparison (Problems 7-11)

7. Solve using both Method 1 and Method 2, verify they agree: $(y \cos x + 2x)dx + (\sin x + 2y)dy = 0$
8. Solve using Method 1, then verify using Method 2: $(2xe^y + y)dx + (x^2e^y + x + 3y^2)dy = 0$
9. Solve using Method 2, then verify using Method 1: $(y^2e^{xy} + 2x)dx + (2xye^{xy} + 3y^2)dy = 0$
10. Use Method 3 (line integral from origin): $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$
11. Compare all three methods for: $(3x^2y + y^3)dx + (x^3 + 3xy^2)dy = 0$

Part C: Initial Value Problems (Problems 12-16)

12. Solve $(2xy + 3)dx + (x^2 - 1)dy = 0$ with $y(1) = 2$
13. Solve $(ye^{xy} + 2x)dx + (xe^{xy} + 2y)dy = 0$ with $y(0) = 1$
14. Solve $(\cos y - y \sin x)\, dx + (-x \sin y - \cos x)\, dy = 0$ with $y(\pi/2) = \pi$
15. Solve $(2x + y^2)dx + (2xy - 3y^2)dy = 0$ with $y(2) = 1$
16. Find the solution curve passing through $(1, 1)$ for: $(3x^2y^2 + 2x)dx + (2x^3y + 3y^2)dy = 0$

Part D: Complex Expressions (Problems 17-21)

17. Find H for: $\left(e^x + \frac{y}{x^2}\right) dx + \left(\frac{1}{x} + e^y\right) dy = 0$
18. Find H for: $(y \sec^2 x + \sec x \tan x)dx + (\tan x + 2y)dy = 0$
19. Find H for: $\left(\frac{2x}{x^2+y^2} + 3x^2\right) dx + \left(\frac{2y}{x^2+y^2} + 2y\right) dy = 0$
20. Find H for: $(\ln y + 2xy)dx + \left(\frac{x}{y} + x^2 + y^2\right) dy = 0$
21. Find H for: $(y^2 \cos x - 3x^2 y)dx + (2y \sin x - x^3 + \ln y)dy = 0$

Part E: Verification Problems (Problems 22-25)

22. Given $H(x, y) = x^3 y + xy^3 + x^2$, find M and N such that $Mdx + Ndy = 0$ is exact with this potential function.
23. A student claims $H = x^2 y + y^3$ is the potential function for $(2xy)dx + (x^2 + 3y^2)dy = 0$. Verify or correct this.
24. Find the error: For $(2xy + 1)dx + (x^2 + 2y)dy = 0$, a student got $H = x^2 y + x + y^2 + 5$. Is this correct?
25. Show that if H_1 and H_2 are both potential functions for the same exact equation, then $H_1 - H_2 =$ constant.

Part F: Exam-Style Problems (Problems 26-30)

26. (Prof. Ditkowski 2023) Find the potential function using two methods: $(2xy + y^2 \cos x)dx + (x^2 + 2y \sin x)dy = 0$
27. Given that $(ax^2 y + 2xy^2)dx + (x^3 + bx^2 y)dy = 0$ is exact with $a = 3$, find b and solve the equation.
28. The equation $(P(x) + y^2)dx + (Q(y) + 2xy)dy = 0$ is exact.
    (a) Find the relationship between $P(x)$ and $Q(y)$
    (b) If $P(x) = x^2$, find $Q(y)$ and the potential function
29. Find all functions $f(x)$ such that $(f(x) + y)dx + (x + e^y)dy = 0$ is exact, then solve for $f(x) = x$.
30. A potential function satisfies $H(0, y) = y^2$ and $H(x, 0) = x^3$. If the equation is $(3x^2 + g(y))dx + (h(x) + 2y)dy = 0$, find $g(y)$, $h(x)$, and $H(x, y)$.

Solutions and Key Insights

Problem 1: Using Method 1: $H = \int(2x+3y)dx = x^2+3xy+g(y)$ $\frac{\partial H}{\partial y} = 3x+g'(y) = 3x+4y$ So $g'(y) = 4y$, giving $g(y) = 2y^2$ Answer: $H = x^2 + 3xy + 2y^2$

Problem 7: Method 1: $H = \int(y\cos x + 2x)dx = y\sin x + x^2 + g(y)$ $\frac{\partial H}{\partial y} = \sin x + g'(y) = \sin x + 2y$ So $g(y) = y^2$. Thus $H = y\sin x + x^2 + y^2$

Method 2: $H = \int(\sin x + 2y)dy = y\sin x + y^2 + f(x)$ $\frac{\partial H}{\partial x} = y\cos x + f'(x) = y\cos x + 2x$ So $f(x) = x^2$. Same answer!

Problem 12: First find H: $H = \int(2xy+3)dx = x^2y+3x+g(y)$ $\frac{\partial H}{\partial y} = x^2+g'(y) = x^2-1$ So $g(y) = -y$, giving $H = x^2y + 3x - y$

Using $y(1) = 2$: $C = (1)^2(2) + 3(1) - 2 = 3$ Solution: $x^2y + 3x - y = 3$

Problem 22: Given $H = x^3y + xy^3 + x^2$ $M = \frac{\partial H}{\partial x} = 3x^2y + y^3 + 2x$ $N = \frac{\partial H}{\partial y} = x^3 + 3xy^2$

Key Strategy Note: For Problem 19, use the fact that $d(\ln(x^2+y^2)) = \frac{2x\,dx + 2y\,dy}{x^2+y^2}$

Warning for Problem 30: Use the boundary conditions to build H systematically.