

Practice Problems - Lesson 39: 3D Systems and Higher Dimensions

ODE 1 Course

Part A: 3D Classification (Problems 1-6)

1. Classify the 3D equilibrium at the origin for:

(a) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

2. For each set of eigenvalues, classify the 3D equilibrium:

(a) $\lambda = -1, -2, -3$

(b) $\lambda = 2, -1 \pm 3i$

(c) $\lambda = 0, \pm 2i$

(d) $\lambda = 1, 1, 1$ (repeated)

3. Find the dimensions of stable and unstable manifolds for:

(a) $\lambda = -2, 1, 3$

(b) $\lambda = -1, -2 \pm i$

(c) $\lambda = 0, -1, 2$

(d) $\lambda = \pm 2i, 3$

4. For the system with $A = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$:

- (a) Find all eigenvalues and eigenvectors
 - (b) Classify the equilibrium
 - (c) Identify stable and unstable manifolds
 - (d) Describe the behavior near origin
5. Given a 3D system has one eigenvalue $\lambda_1 = -2$ and a complex pair with $\text{Re}(\lambda_{2,3}) = 1$:
- (a) What type of equilibrium is this?
 - (b) What are the manifold dimensions?
 - (c) Is the origin stable?
 - (d) Describe trajectory behavior
6. For the block diagonal system:

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

- (a) Find eigenvalues without full calculation
- (b) Classify the equilibrium
- (c) Which coordinate decouples?
- (d) Describe motion in each subspace

Part B: Manifold Analysis (Problems 7-11)

7. For $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$:

- (a) Find all eigenvalues
- (b) Find eigenvectors
- (c) Determine E^s and E^u explicitly. Write parametric equations for each manifold
- (8) Given eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$, $\lambda_3 = 1$ with eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

- (a) Find the stable manifold equation
- (b) Find the unstable manifold equation
- (c) Describe trajectories starting in E^u

9. For a system with 2D stable manifold in the xy-plane:
 - (a) What can you say about eigenvalues?
 - (b) What happens to trajectories starting at $(0, 0, 1)$?
 - (c) Sketch behavior in xz-projection
10. Consider the saddle with $\lambda = -2, -1, 3$:
 - (a) What type of saddle is this?
 - (b) If $v_3 = (1, 1, 1)^T$, describe escaped direction How fast do trajectories approach E^s ?
11. For a spiral-saddle system:
 - (a) What are possible eigenvalue configurations?
 - (b) Describe motion in the spiral plane
 - (c) What determines overall stability?

Part C: Projections (Problems 12-16)

12. For the system $\dot{x} = -x$, $\dot{y} = -2y + z$, $\dot{z} = -y - 2z$:
 - (a) Find eigenvalues
 - (b) Sketch xy-projection
 - (c) Sketch xz-projection
 - (d) Sketch yz-projection
 - (e) Which projection best shows the spiral?
13. Given a 3D center-line with eigenvalues $0, \pm 3i$:
 - (a) Which variable corresponds to the zero eigenvalue?
 - (b) What do trajectories look like?
 - (c) Draw the three projections
14. For a stable spiral-node with spiral in xy-plane:
 - (a) What does xz-projection show?
 - (b) What does yz-projection show?
 - (c) Which projection shows spiral clearly?
15. Project the trajectory of:

$$\dot{x} = -x, \quad \dot{y} = y, \quad \dot{z} = -2z$$

starting at $(1, 1, 1)$ onto all three coordinate planes.

16. For the system with solution:

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} \cos(2t) \\ \sin(2t) \\ 1 \end{pmatrix}$$

17. Find the system matrix A

18. Draw xy-projection

19. Draw xz-projection

20. What type of equilibrium?

Part D: Poincaré Sections (Problems 17-20)

21. For periodic motion in 3D with period $T = 2\pi$:

- (a) What does Poincaré section show?
- (b) How many points for one trajectory?
- (c) What if period doubles?

22. Consider the plane $z = 0$ as Poincaré section for:

$$\dot{x} = y, \quad \dot{y} = -x, \quad \dot{z} = -z + x^2$$

- (a) When do trajectories cross $z = 0$?
- (b) What pattern emerges?
- (c) Is motion periodic?

23. For quasi-periodic motion on a torus:

- (a) What does Poincaré section look like?
- (b) How to distinguish from periodic?
- (c) What if frequencies are rational?

24. Design a Poincaré section to best reveal:

- (a) Periodic orbits
- (b) Chaotic trajectories
- (c) Quasi-periodic motion

Part E: Higher Dimensions (Problems 21-24)

25. For a 4D system with eigenvalues $-1, -2, 3, 4$:
- (a) Find dimensions of invariant manifolds
 - (b) Is the origin stable?
 - (c) What's the dominant behavior?
26. In 5D with eigenvalues $-1 \pm i, -2, 3 \pm 2i$:
- (a) Classify stability
 - (b) Find $\dim(E^s)$ and $\dim(E^u)$. How many rotation planes?
27. For an n -dimensional system:
- (a) Maximum number of complex eigenvalues?
 - (b) When is origin globally stable?
 - (c) When do all trajectories escape?
28. Given a 10D system with 8 negative and 2 positive eigenvalues:
- (a) What are manifold dimensions?
 - (b) Describe long-term behavior
 - (c) What's the "most likely" outcome?

Part F: Special 3D Phenomena (Problems 25-27)

29. For the Rössler system:

$$\dot{x} = -y - z, \quad \dot{y} = x + ay, \quad \dot{z} = b + z(x - c)$$

- (a) Find equilibria when $a = b = 0.2, c = 5.7$
 - (b) Linearize at equilibria
 - (c) What type of attractor exists?
30. Consider a 3D system with a stable limit cycle:
- (a) Can this happen in a linear system?
 - (b) What do nearby trajectories do?
 - (c) How would you detect this numerically?
31. For motion on a 2-torus in 3D:
- (a) What are the two frequencies?
 - (b) When is motion periodic?
 - (c) When is trajectory dense on torus?

Part G: Exam-Style Problems (Problems 28-30)

32. [Prof. Ditkowski Style - 3D Complete Analysis] For the system:

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

- (a) Find all eigenvalues (show characteristic polynomial)
- (b) Verify using trace and determinant
- (c) Classify the equilibrium type
- (d) Find eigenvectors for all eigenvalues
- (e) Identify and state dimensions of E^s and E^u . Describe behavior in each invariant subspace
- (f) Sketch the xy-projection
- (g) Sketch the xz-projection
- (h) Which trajectories approach origin fastest?
- (i) Write the general solution

33. [Higher-D Application] A 6D linear system models coupled oscillators with eigenvalues:

$$\lambda_{1,2} = -0.1 \pm 2i, \quad \lambda_{3,4} = -0.2 \pm 3i, \quad \lambda_{5,6} = 0.1 \pm i$$

- (a) Is the system stable? Why?
 - (b) How many oscillation frequencies?
 - (c) Find dimensions of stable/unstable manifolds
 - (d) Which oscillator damps fastest?
 - (e) What happens as $t \rightarrow \infty$?
 - (f) Describe the attractor type
34. [Conceptual Understanding]
- (a) Prove: A 3D linear system cannot have all complex eigenvalues
 - (b) Show: If two eigenvalues are complex conjugates, the third must be real
 - (c) Explain: Why can 4D systems have two pairs of complex eigenvalues?
 - (d) True/False: A 3D saddle is always unstable
 - (e) Can a 3D linear system have a limit cycle?
 - (f) What's the minimum dimension for quasi-periodic motion?
 - (g) In 3D, if $\det(A) < 0$, what can you conclude?
 - (h) How many types of 3D saddles exist?

3D Visualization Strategy:

- Don't try to draw full 3D portraits
- Use projections to show key features
- Describe behavior verbally
- State manifold dimensions explicitly
- Use computer tools when possible
- Focus on eigenvalue analysis