Lesson 45: Practice Problems
Method of Undetermined Coefficients

ODE 1 with Prof. Adi Ditkowski

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Part A: Basic Non-Resonant Cases (5 problems)

1. Solve: $y'' - 5y' + 6y = 12$
2. Solve: $y'' + y' - 2y = 4t - 6$
3. Solve: $y'' - 4y = 3e^{-t}$
4. Solve: $y'' + 9y = 5\cos(2t)$
5. Solve: $y'' - 2y' + y = t^2 + 1$

Part B: Resonance Cases (5 problems)

6. Solve: $y'' - 4y = e^{2t}$
7. Solve: $y'' + 4y = \sin(2t)$
8. Solve: $y'' - 6y' + 9y = e^{3t}$
9. Solve: $y'' + y = \cos(t) + \sin(t)$
10. Solve: $y''' - y' = e^t$

Part C: Products and Combinations (5 problems)

11. Solve: $y'' - 3y' + 2y = te^t$
12. Solve: $y'' + 4y = e^{-t}\sin(t)$
13. Solve: $y'' - y = t^2e^t$
14. Solve: $y'' + 2y' + 5y = e^{-t}\cos(2t)$
15. Solve: $y''' - y'' = t^2 + e^t$

Part D: Superposition Problems (5 problems)

16. Solve: $y'' - 4y' + 3y = 2e^t + 3e^{2t}$
17. Solve: $y'' + y = t + \sin(t)$
18. Solve: $y'' - y' - 6y = 8e^{3t} - 5\sin(t)$
19. Solve: $y'' + 4y = 3\cos(2t) + 4\sin(3t)$
20. Solve: $y''' - y' = 2t + 3e^{-t} + \cos(t)$

Part E: Initial Value Problems (5 problems)

21. Solve: $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 2$
22. Solve: $y'' + 4y = 8t$, $y(0) = 0$, $y'(0) = 1$
23. Solve: $y'' + y = \sin(2t)$, $y(0) = 0$, $y'(\pi/4) = 1$
24. Solve: $y'' - 4y' + 4y = t^2e^{2t}$, $y(0) = 0$, $y'(0) = 0$
25. Solve: $y''' - y' = 4$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 2$

Part F: Exam-Style Problems (5 problems)

26. (Prof. Ditkowski style) Consider $y'' + 2y' + y = f(t)$.
    (a) If $f(t) = e^{-t}$, explain why standard undetermined coefficients fails.
    (b) Find the particular solution using the modified approach.
    (c) What physical phenomenon does this represent?
27. A mass-spring system satisfies $y'' + \omega_0^2 y = F_0\cos(\omega t)$.
    (a) Find the general solution when $\omega \neq \omega_0$.
    (b) Find the general solution when $\omega = \omega_0$ (resonance).
    (c) Describe the behavior as $\omega \to \omega_0$.
28. Find all values of a for which $y'' - 4y' + 4y = e^{at}$ has a particular solution of the form $y_p = Ae^{at}$.
29. For the equation $y'' + py' + qy = e^{rt}$, derive conditions on p, q, and r that determine whether resonance occurs.
30. Solve $y^{(4)} + 2y'' + y = \cos(t)$ and explain why this represents a doubly resonant system.

Solutions

Part A: Basic Non-Resonant Cases

1. Homogeneous: $r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$
   $y_h = c_1 e^{2t} + c_2 e^{3t}$
   Trial: $y_p = A$ (constant)
   Substitute: $6A = 12 \Rightarrow A = 2$
   General solution: $y = c_1 e^{2t} + c_2 e^{3t} + 2$

2. Homogeneous: $r^2 + r - 2 = 0 \Rightarrow r = 1, -2$
   $y_h = c_1 e^{t} + c_2 e^{-2t}$
   Trial: $y_p = At + B$
   $y_p' = A$, $y_p'' = 0$
   Substitute: $0 + A - 2(At + B) = 4t - 6$
   $-2At + (A - 2B) = 4t - 6$
   $A = -2$, $B = 2$
   General solution: $y = c_1 e^{t} + c_2 e^{-2t} - 2t + 2$

3. Homogeneous: $r^2 - 4 = 0 \Rightarrow r = \pm 2$
   $y_h = c_1 e^{2t} + c_2 e^{-2t}$
   Trial: $y_p = Ae^{-t}$
   $y_p' = -Ae^{-t}$, $y_p'' = Ae^{-t}$
   Substitute: $Ae^{-t} - 4Ae^{-t} = 3e^{-t}$
   $-3A = 3 \Rightarrow A = -1$
   General solution: $y = c_1 e^{2t} + c_2 e^{-2t} - e^{-t}$

4. Homogeneous: $r^2 + 9 = 0 \Rightarrow r = \pm 3i$
   $y_h = c_1 \cos(3t) + c_2 \sin(3t)$
   Trial: $y_p = A\cos(2t) + B\sin(2t)$
   $y_p'' = -4A\cos(2t) - 4B\sin(2t)$
   Substitute: $5A\cos(2t) + 5B\sin(2t) = 5\cos(2t)$
   $A = 1$, $B = 0$
   General solution: $y = c_1 \cos(3t) + c_2 \sin(3t) + \cos(2t)$

5. Homogeneous: $(r - 1)^2 = 0 \Rightarrow r = 1$ (double)
   $y_h = (c_1 + c_2 t)e^{t}$
   Trial: $y_p = At^2 + Bt + C$
   $y_p' = 2At + B$, $y_p'' = 2A$
   Substitute: $2A - 2(2At + B) + (At^2 + Bt + C) = t^2 + 1$
   $At^2 + (B - 4A)t + (2A - 2B + C) = t^2 + 1$
   $A = 1$, $B = 4$, $C = 7$
   General solution: $y = (c_1 + c_2 t)e^{t} + t^2 + 4t + 7$

Part B: Resonance Cases

6. Homogeneous: $r = \pm 2$, so $y_h = c_1 e^{2t} + c_2 e^{-2t}$
   Resonance! Trial: $y_p = Ate^{2t}$
   $y_p' = Ae^{2t} + 2Ate^{2t}$
   $y_p'' = 4Ae^{2t} + 4Ate^{2t}$
   Substitute: $4Ae^{2t} = e^{2t} \Rightarrow A = 1/4$
   General solution: $y = c_1 e^{2t} + c_2 e^{-2t} + \frac{t}{4} e^{2t}$

7. Homogeneous: $r = \pm 2i$, so $y_h = c_1 \cos(2t) + c_2 \sin(2t)$
   Resonance! Trial: $y_p = t[A\cos(2t) + B\sin(2t)]$
   After substitution: $-4A\sin(2t) + 4B\cos(2t) = \sin(2t)$
   $A = -1/4$, $B = 0$
   General solution: $y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{t}{4}\cos(2t)$

8. Homogeneous: $(r-3)^2 = 0$, so $y_h = (c_1 + c_2 t)e^{3t}$
   Double resonance! Trial: $y_p = At^2 e^{3t}$
   After substitution: $2Ae^{3t} = e^{3t} \Rightarrow A = 1/2$
   General solution: $y = (c_1 + c_2 t)e^{3t} + \frac{t^2}{2} e^{3t}$

9. Homogeneous: $r = \pm i$, so $y_h = c_1 \cos(t) + c_2 \sin(t)$
   Resonance! Trial: $y_p = t[A\cos(t) + B\sin(t)]$
   After substitution: $-2A\sin(t) + 2B\cos(t) = \cos(t) + \sin(t)$
   $A = -1/2$, $B = 1/2$
   General solution: $y = c_1 \cos(t) + c_2 \sin(t) + \frac{t}{2}[-\cos(t) + \sin(t)]$

10. Homogeneous: $r(r^2 - 1) = 0 \Rightarrow r = 0, \pm 1$
    $y_h = c_1 + c_2 e^t + c_3 e^{-t}$
    Resonance at $r = 1$! Trial: $y_p = Ate^t$
    After substitution: $2Ae^t = e^t \Rightarrow A = 1/2$
    General solution: $y = c_1 + c_2 e^t + c_3 e^{-t} + \frac{t}{2} e^t$

Part C: Products and Combinations

11. Homogeneous: $r = 1, 2$, so $y_h = c_1 e^t + c_2 e^{2t}$
    Resonance! Trial: $y_p = t(At + B)e^t = (At^2 + Bt)e^t$
    After substitution: $A = -1$, $B = -2$
    General solution: $y = c_1 e^t + c_2 e^{2t} - t(t+2)e^t$

12. Homogeneous: $r^2 + 4 = 0 \Rightarrow r = \pm 2i$
    Trial: $y_p = e^{-t}[A\cos(t) + B\sin(t)]$
    After substitution: $A = -1/6$, $B = 1/6$
    General solution: $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{e^{-t}}{6}[-\cos(t) + \sin(t)]$

13. Homogeneous: $r = \pm 1$
    Resonance! Trial: $y_p = t(At^2 + Bt + C)e^t$
    After substitution: $A = 1/6$, $B = 0$, $C = 0$
    General solution: $y = c_1 e^t + c_2 e^{-t} + \frac{t^3}{6} e^t$

14. Homogeneous: $r = -1 \pm 2i$
    Resonance! Trial: $y_p = te^{-t}[A\cos(2t) + B\sin(2t)]$
    After substitution: $A = 0$, $B = 1/4$
    General solution: $y = e^{-t}[c_1 \cos(2t) + c_2 \sin(2t)] + \frac{t}{4}e^{-t}\sin(2t)$

15. Homogeneous: $r^2(r-1) = 0 \Rightarrow r = 0$ (double), $r = 1$
    For t^2: Trial $t^2(At^2 + Bt + C)$ due to double root at 0
    For e^t: Trial Dte^t due to simple root at 1
    Combined: $y_p = At^4 + Bt^3 + Ct^2 + Dte^t$
    After substitution: $A = -1/12$, $B = 0$, $C = -1$, $D = -1$
    General solution: $y = c_1 + c_2 t + c_3 e^t - \frac{t^4}{12} - t^2 - te^t$

Part D: Superposition Problems

16. Homogeneous: $r = 1, 3$
    For $2e^t$: resonance, use $y_{p1} = Ate^t \Rightarrow A = -1$
    For $3e^{2t}$: no resonance, use $y_{p2} = Be^{2t} \Rightarrow B = -3$
    General solution: $y = c_1 e^t + c_2 e^{3t} - te^t - 3e^{2t}$

17. Homogeneous: $r = \pm i$
    For t: $y_{p1} = At + B \Rightarrow A = 1, B = 0$
    For $\sin(t)$: resonance, $y_{p2} = t[C\cos(t) + D\sin(t)] \Rightarrow C = 0, D = -1/2$
    General solution: $y = c_1 \cos(t) + c_2 \sin(t) + t - \frac{t}{2}\sin(t)$

18. Homogeneous: $r = 3, -2$
    For $8e^{3t}$: resonance, $y_{p1} = Ate^{3t} \Rightarrow A = 8/5$
    For $-5\sin(t)$: $y_{p2} = B\cos(t) + C\sin(t) \Rightarrow B = -1/2, C = 1/2$
    General solution: $y = c_1 e^{3t} + c_2 e^{-2t} + \frac{8t}{5}e^{3t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$

19. Homogeneous: $r = \pm 2i$
    For $3\cos(2t)$: resonance, $y_{p1} = t[A\cos(2t) + B\sin(2t)] \Rightarrow A = 0, B = 3/4$
    For $4\sin(3t)$: $y_{p2} = C\cos(3t) + D\sin(3t) \Rightarrow C = 0, D = -4/5$
    General solution: $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{3t}{4}\sin(2t) - \frac{4}{5}\sin(3t)$

20. Homogeneous: $r = 0, \pm 1$
    For $2t$: double resonance at 0, $y_{p1} = At^3 \Rightarrow A = -1/3$
    For $3e^{-t}$: resonance, $y_{p2} = Bte^{-t} \Rightarrow B = -3/2$
    For $\cos(t)$: $y_{p3} = C\cos(t) + D\sin(t) \Rightarrow C = -1/2, D = 0$
    General solution: $y = c_1 + c_2 e^t + c_3 e^{-t} - \frac{t^3}{3} - \frac{3t}{2}e^{-t} - \frac{1}{2}\cos(t)$

Part E: Initial Value Problems

21. General: $y = c_1 e^t + c_2 e^{2t} + e^{3t}$
    $y(0) = c_1 + c_2 + 1 = 1 \Rightarrow c_1 + c_2 = 0$
    $y'(0) = c_1 + 2c_2 + 3 = 2 \Rightarrow c_1 + 2c_2 = -1$
    Solving: $c_2 = -1$, $c_1 = 1$
    Solution: $y = e^t - e^{2t} + e^{3t}$

22. General: $y = c_1\cos(2t) + c_2\sin(2t) + 2t$
$y(0) = c_1 = 0$
$y'(0) = 2c_2 + 2 = 1 \Rightarrow c_2 = -1/2$
Solution: $y = -\frac{1}{2}\sin(2t) + 2t$

23. General: $y = c_1\cos(t) + c_2\sin(t) - \frac{1}{3}\sin(2t)$
$y(0) = c_1 = 0$
$y'(t) = -c_1\sin(t) + c_2\cos(t) - \frac{2}{3}\cos(2t)$
$y'(\pi/4) = c_2/\sqrt{2} = 1 \Rightarrow c_2 = \sqrt{2}$
Solution: $y = \sqrt{2}\sin(t) - \frac{1}{3}\sin(2t)$

24. General: $y = (c_1 + c_2 t)e^{2t} + \frac{t^4}{12}e^{2t}$
$y(0) = c_1 = 0$
$y'(0) = c_2 + 2c_1 = 0 \Rightarrow c_2 = 0$
Solution: $y = \frac{t^4}{12}e^{2t}$

25. General: $y = c_1 + c_2e^t + c_3e^{-t} - 2t^2$
$y(0) = c_1 + c_2 + c_3 = 0$
$y'(0) = c_2 - c_3 = 0$
$y''(0) = c_2 + c_3 - 4 = 2 \Rightarrow c_2 + c_3 = 6$
Solving: $c_2 = c_3 = 3$, $c_1 = -6$
Solution: $y = -6 + 3e^t + 3e^{-t} - 2t^2$

Part F: Exam-Style Problems

26. (a) $(r+1)^2 = 0$, so $r = -1$ is a double root. Standard trial Ae^{-t} is already in y_h.
    (b) Use $y_p = At^2e^{-t}$. After substitution: $A = 1/2$.
    (c) This represents critical damping with resonant forcing.

27. (a) Non-resonant: $y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{\omega_0^2-\omega^2}\cos(\omega t)$
    (b) Resonant: $y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0 t}{2\omega_0}\sin(\omega_0 t)$
    (c) As $\omega \to \omega_0$, amplitude $\to \infty$ (beats phenomenon)

28. $(r-2)^2 = 0$ gives double root at $r = 2$.
$y_p = Ae^{at}$ works only if $a \neq 2$.
All values except $a = 2$.

29. Characteristic: $r^2 + pr + q = 0$
Resonance occurs when r satisfies this, i.e., $r^2 + pr + q = 0$.

30. $(r^2+1)^2 = 0$ gives $r = \pm i$ with multiplicity 2.
For $\cos(t)$: double resonance!
$y_p = t^2[A\cos(t) + B\sin(t)]$
After substitution: $A = 0$, $B = 1/8$
$y = (c_1 + c_2t)\cos(t) + (c_3 + c_4t)\sin(t) + \frac{t^2}{8}\sin(t)$