Practice Problems: Linear First-Order ODEs

Lesson 14 - Prof. Ditkowski's ODE 1

Part A: Converting to Standard Form (5 problems)

1. Convert to standard form: $3ty' - 6y = t^2 e^t$
2. Convert to standard form: $\sin(t)y' + \cos(t)y = \tan(t)$
3. Convert to standard form: $(t^2 + 1)y' + ty = \ln t$
4. Convert to standard form: $e^t y' + 2e^t y = t$
5. Identify which equations are linear:
   (a) $y' + ty^2 = \sin t$
   (b) $yy' + t = e^t$
   (c) $y' + (\sin t)y = \cos t$
   (d) $y' + y = e^y$

Part B: Basic Integrating Factor Problems (6 problems)

6. Solve: $y' + 3y = e^{2t}$
7. Solve: $y' - 2y = t$
8. Solve: $y' + y = \sin t$
9. Solve: $y' + \frac{1}{t}y = t$ for $t > 0$
10. Solve: $y' - \frac{2}{t}y = t^2$ for $t > 0$
11. Solve: $ty' + y = t$ for $t > 0$

Part C: Complex Integration Required (5 problems)

12. Solve: $y' + (\tan t)y = \sec t$ on $(-\pi/2, \pi/2)$
13. Solve: $y' + \frac{2t}{1+t^2}y = \frac{1}{1+t^2}$
14. Solve: $y' + e^t y = e^{2t}$
15. Solve: $y' + \frac{1}{t \ln t}y = \frac{1}{t}$ for $t > e$
16. Solve: $y' + (\cot t)y = \csc t$ on $(0, \pi)$

Part D: Initial Value Problems (5 problems)

17. Solve: $y' + 2y = 4$, $y(0) = 3$
18. Solve: $y' - y = e^{2t}$, $y(0) = 2$
19. Solve: $ty' + 2y = t^3$, $y(1) = 2$ for $t > 0$
20. Solve: $y' + (\cos t)y = \cos t$, $y(0) = 0$
21. Find the solution of $y' + \frac{3}{t}y = \frac{1}{t^2}$ that remains bounded as $t \to \infty$

Part E: Theoretical and Proof Problems (4 problems)

22. Prove that if y_1 and y_2 are solutions of $y' + p(t)y = g(t)$, then $y_1 - y_2$ is a solution of the homogeneous equation.
23. Show that the integrating factor $\mu(t) = e^{\int p(t)dt}$ is never zero.
24. Prove that if $p(t)$ is continuous on $[a, b]$ and $p(t) > 0$ for all $t \in [a, b]$, then any solution of $y' + p(t)y = 0$ with $y(a) > 0$ satisfies $y(t) > 0$ for all $t \in [a, b]$.
25. Let $y' + p(t)y = g(t)$ where $p(t)$ and $g(t)$ are continuous and periodic with period T. Prove that there exists a unique periodic solution with period T.

Part F: Exam-Style Mixed Problems (5 problems)

26. Consider the equation $y' + \frac{2}{t}y = \frac{\sin t}{t^2}$ for $t > 0$.
    (a) Find the general solution
    (b) Find the solution satisfying $\lim_{t\to\infty} y(t) = 0$
    (c) Verify your solution by substitution
27. A tank contains 100 liters of water with 10 kg of salt dissolved. Fresh water enters at 2 L/min and the mixture leaves at 2 L/min.

(a) Set up the differential equation for salt amount $S(t)$

(b) Solve for $S(t)$

(c) When will the salt concentration reach 0.01 kg/L?

28. For the equation $y' + p(t)y = e^{-\int p(t)dt}$:

    (a) Show that $y_p = t \cdot e^{-\int p(t)dt}$ is a particular solution

    (b) Write the general solution

    (c) Explain why this forcing function is special

29. Consider $y' + ay = b$ where a, b are constants with $a \neq 0$.

    (a) Find the general solution

    (b) Find $\lim_{t\to\infty} y(t)$ if $a > 0$

    (c) Sketch solution curves for $a > 0$ and $a < 0$

30. The equation $ty' + (1 - t)y = e^{-t}$ for $t > 0$:

    (a) Convert to standard form

    (b) Identify any singular points

    (c) Find the general solution

    (d) Discuss behavior as $t \to 0^+$ and $t \to \infty$

Solutions

Part A Solutions

For Problem 1: Divide by $3t$ to get $y' - \frac{2}{t}y = \frac{te^t}{3}$

Solution: Complete solutions with detailed steps...

[Full solutions would be provided for all problems, showing complete work, alternative methods where applicable, and connections to exam techniques. Each solution would include verification by substitution where appropriate.]