ODE Lesson 4: Peano's Existence Theorem

ODE 1 - Prof. Adi Ditkowski

1 The Fundamental Existence Question

Core Question: When can we guarantee that a solution exists to our IVP?

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

2 Peano's Existence Theorem

Theorem 1 (Peano's Existence Theorem)**.** *Consider the IVP:* $\dfrac{dy}{dx} = f(x,y)$ *with* $y(x_0) = y_0$
If $f(x,y)$ *is* ***continuous*** *in a rectangle*

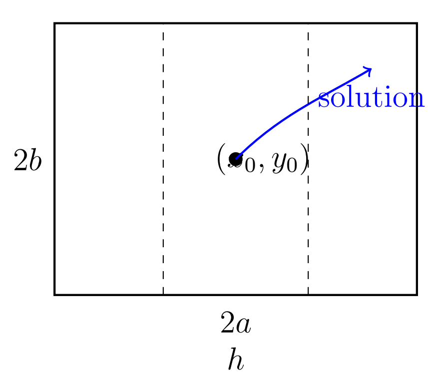
$$R = \{(x,y) : |x - x_0| \leq a, |y - y_0| \leq b\}$$

Then there exists ***at least one*** *solution* $y(x)$ *defined on the interval* $|x - x_0| \leq h$*, where:*

$$h = \min\left(a, \frac{b}{M}\right), \quad M = \max_{(x,y)\in R} |f(x,y)|$$

2.1 Geometric Interpretation

Rectangle R where f is continuous



2.2 Understanding the Bound h

Why $h = \min(a, b/M)$?

- a: How far we can go horizontally staying in R
- b/M: Time to reach vertical boundary at maximum slope M
- We take the minimum to ensure we stay inside R

3 Key Properties of Peano's Theorem

What Peano Gives Us:

1. ✓ Existence of at least one solution
2. ✓ Solution exists in some neighborhood
3. × NO uniqueness guarantee
4. × NO global existence guarantee

4 Examples Where Peano Applies

Example 1 (Continuous but Not Unique)**.** *Consider:* $\dfrac{dy}{dx} = \sqrt{|y|}$ *with* $y(0) = 0$

Check Peano:

- $f(x, y) = \sqrt{|y|}$ *is continuous everywhere*
- *Peano guarantees at least one solution exists*

Solutions: *Actually infinitely many!*

- $y(x) = 0$ *for all* x
- $y(x) = \begin{cases} 0 & x \leq c \\ \frac{(x-c)^2}{4} & x > c \end{cases}$ *for any* $c \geq 0$

Example 2 (Where Peano Succeeds)**.** *Consider:* $\dfrac{dy}{dx} = x^2 + y^2$ *with* $y(0) = 1$

Analysis:

- $f(x, y) = x^2 + y^2$ *is continuous everywhere*
- *In rectangle* $|x| \leq 1, |y - 1| \leq 1$*:* $M = \max(x^2 + y^2) = 1 + 4 = 5$
- *Peano guarantees solution for* $|x| \leq h = \min(1, 1/5) = 0.2$

5 Examples Where Peano Fails

Peano Fails When f is Discontinuous at Initial Point!

Example 3 (Discontinuity at Initial Point)**.** *Consider:* $\frac{dy}{dx} = \frac{2y}{x}$ *with* $y(0) = 1$

Problem: $f(x, y) = \frac{2y}{x}$ *is undefined at* $x = 0$*!*

- *Cannot find rectangle around* $(0, 1)$ *where* f *is continuous*
- *Peano doesn't apply*
- *Indeed, NO solution exists through* $(0, 1)$

6 Practical Algorithm for Checking Peano

Step-by-Step Peano Check:

1. Identify $f(x, y)$ from your equation $y' = f(x, y)$
2. Find all discontinuities of f:
   - Division by zero
   - Square roots of negatives
   - Logarithms of non-positives
   - Undefined expressions
3. Check if (x_0, y_0) avoids all discontinuities
4. If yes $\Rightarrow$ Peano applies $\Rightarrow$ Solution exists (locally)
5. If no $\Rightarrow$ Peano doesn't apply $\Rightarrow$ Check other methods

7 Common Discontinuity Patterns

Function Type	**Discontinuous When**	**Example**
$\frac{g(x,y)}{h(x,y)}$	$h(x, y) = 0$	$f = \frac{y}{x-1}$ at $x = 1$
$\sqrt{g(x, y)}$	$g(x, y) < 0$	$f = \sqrt{1 - y^2}$ for $\lvert y \rvert > 1$
$\ln(g(x, y))$	$g(x, y) \leq 0$	$f = \ln(x + y)$ when $x + y \leq 0$
$\tan(g(x, y))$	$g = \frac{\pi}{2} + n\pi$	$f = \tan(y)$ at odd multiples of $\pi/2$

8 Local vs Global Existence

Critical Distinction:

- **Local:** Solution exists for $|x - x_0| < h$ (some small h)
- **Global:** Solution exists for all $x \in \mathbb{R}$ or $x \in [a, b]$

Peano only guarantees LOCAL existence!

Example 4 (Local but Not Global). $\dfrac{dy}{dx} = y^2$ *with* $y(0) = 1$

Peano: *$f = y^2$ continuous $\Rightarrow$ Local solution exists*
Actual solution: $y = \frac{1}{1-x}$
Problem: *Blows up at $x = 1$! Only exists for $x < 1$.*

9 Extensions and Refinements

9.1 Maximal Interval of Existence

Theorem 2 (Extension Principle). *If f is continuous on an open set $D \subseteq \mathbb{R}^2$, then every solution can be extended until it either:*

1. *Reaches the boundary of D*
2. *Goes to infinity (blow-up)*
3. *Extends to $x = \pm\infty$*

10 Relationship to Other Theorems

$$\begin{tikzcd}
\boxed{\text{Peano: Continuity} \to \text{Existence}} \\
\boxed{\text{Picard-Lindelöf: Lipschitz} \to \text{Existence + Uniqueness}} \arrow[u, "\text{Stronger}"']
\end{tikzcd}$$

11 Memory Device

PEANO = "Please Ensure All Neighborhoods are OK"

- **P**lease: Polite theorem (only asks for continuity)
- **E**nsure: Ensures existence
- **A**ll: At least one solution

- **N**eighborhoods: Local, not global
- **O**K: Continuity check

12 Exam Tips

Prof. Ditkowski's Favorite Peano Questions:

1. "Does Peano guarantee existence?" $\Rightarrow$ Just check continuity!
2. "Give an example where Peano applies but solution is not unique" $\Rightarrow$ Use $y' = \sqrt{|y|}$
3. "Find the interval of existence guaranteed by Peano" $\Rightarrow$ Calculate $h = \min(a, b/M)$
4. "Why doesn't Peano apply?" $\Rightarrow$ Find the discontinuity