Practice Problems: Separable Equations

Lesson 12 Exercises

August 23, 2025

Part A: Recognition and Classification (6 problems)

1. Determine which equations are separable:
   (a) $\frac{dy}{dx} = x + y$
   (b) $\frac{dy}{dx} = xy + x$
   (c) $\frac{dy}{dx} = \frac{x^2}{y^2}$
   (d) $\frac{dy}{dx} = \sin(x + y)$
   (e) $\frac{dy}{dx} = e^x \cdot e^y$
   (f) $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$

2. Rewrite in separable form if possible: $(x^2 + 1)y\,dx + (y^2 - 1)x\,dy = 0$

3. For the autonomous equation $\frac{dy}{dx} = y^3 - y$, identify all equilibrium points.

4. True or False: Every autonomous equation is separable. Explain.

5. Can $\frac{dy}{dx} = \sqrt{xy}$ be made separable? If so, how?

6. Explain why $\frac{dy}{dx} = x + y$ is NOT separable.

Part B: Basic Separable Equations (6 problems)

7. Solve: $\frac{dy}{dx} = 2xy$

8. Solve: $\frac{dy}{dx} = \frac{y}{x}$ for $x > 0$

9. Find the general solution: $\frac{dy}{dx} = y^2$

10. Solve: $\frac{dy}{dx} = (1 + x)(1 + y)$

11. Solve with initial condition: $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = 2$

12. Find all solutions: $\frac{dy}{dx} = y\cos(x)$

Part C: Singular Solutions and Lost Solutions (6 problems)

13. Find ALL solutions to: $\frac{dy}{dx} = 2\sqrt{y}$
14. Solve completely: $\frac{dy}{dx} = y^2 - 1$
15. Find general and singular solutions: $\frac{dy}{dx} = x(y^2 - 4y + 4)$
16. Solve: $y' = y^2(1-y)^2$ and identify all constant solutions
17. For $\frac{dy}{dx} = \frac{2xy}{1-x^2}$, find all solutions including any lost in separation
18. Determine all solutions: $(y-1)y' = x(y-1)^2$

Part D: Implicit Solutions and Special Cases (5 problems)

19. Solve (leave implicit if necessary): $\frac{dy}{dx} = \frac{1-y}{1+x}$
20. Find the solution curves: $x\,dy + y\,dx = xy^2\,dx$
21. Solve: $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ using substitution $v = y/x$
22. Show that $x^2 + xy + y^2 = C$ solves a certain separable equation. Find the equation.
23. Solve using appropriate substitution: $\frac{dy}{dx} = (x+y)^2$

Part E: Advanced Techniques (5 problems)

24. Solve the logistic equation: $\frac{dy}{dt} = y(1 - y/100)$ with $y(0) = 10$
25. Find the solution: $\frac{dy}{dx} = \frac{y^2-1}{x^2-1}$ passing through $(2, 2)$
26. Solve: $\frac{dy}{dx} = e^{x+y} + e^{x-y}$
27. For what values of n is $\frac{dy}{dx} = x^n y^{2-n}$ separable? Solve for those values.
28. Find the orthogonal trajectories of the family $y = Cx^2$

Part F: Exam-Style Problems (7 problems)

29. [**Prof. Ditkowski Special**] Consider $\frac{dy}{dx} = \frac{2y^2-y}{x}$
    (a) Find the general solution
    (b) Find all singular solutions
    (c) Solve the IVP with $y(1) = 1/2$
    (d) Sketch several solution curves
    (e) For what initial conditions does the solution blow up in finite time?

30. [**Comprehensive**] The population $P(t)$ of bacteria satisfies $\frac{dP}{dt} = P(2 - P/1000)$
    (a) Find all equilibrium populations
    (b) Solve for $P(t)$ if $P(0) = 500$
    (c) Find $\lim_{t\to\infty} P(t)$
    (d) When does the population reach 1500?

31. [**Implicit Challenge**] Solve $\frac{dy}{dx} = \frac{3x^2+2xy}{x^2+2y^2}$ and verify your solution

32. [**Lost Solutions Focus**] For $\frac{dy}{dx} = \frac{y^2(y-3)}{x}$:
    (a) Separate and find the general solution
    (b) Identify ALL constant solutions
    (c) Which constant solutions are singular?
    (d) Verify each constant solution

33. [**Domain Analysis**] Consider $\frac{dy}{dx} = \frac{1}{y\ln(y)}$ for $y > 1$
    (a) Solve the equation
    (b) If $y(0) = e$, find the particular solution
    (c) Determine the domain of existence
    (d) What happens as $x \to \pm\infty$?

34. [**Uniqueness Failure**] For $\frac{dy}{dx} = 3y^{2/3}$:
    (a) Show that $y = 0$ is a solution
    (b) Find the general solution for $y \neq 0$
    (c) Show that the IVP with $y(0) = 0$ has infinitely many solutions
    (d) Sketch several solutions through the origin

35. [**Prof. Ditkowski Comprehensive**] Chemical reaction kinetics gives $\frac{dy}{dt} = k(a - y)(b - y)$ where $a = 2$, $b = 1$, $k = 1$

(a) Use partial fractions to separate variables

(b) Find the general solution

(c) If $y(0) = 0$, find $y(t)$

(d) What is $\lim_{t\to\infty} y(t)$?

(e) Interpret physically: what does y represent?

Answer Key with Essential Hints

1. Separable: (b) factor as $x(y + 1)$, (c), (e)

3. Equilibria: $y = 0, \pm 1$

7. $y = Ce^{x^2}$

9. $y = -\frac{1}{x+C}$; blows up at $x = -C$

11. $y = Cx$; also $y = 0$ (singular)

13. $y = 2x^2 + C$; also $y = 0$ (lost solution)

14. General: $\arctan(y) = x^2/2 + C$; Singular: $y = \pm 1$

15. $y = (x - 2)^2$; also $y = 2$ (singular)

19. $(1 + x)(1 - y) = C$

24. Logistic solution: $y = \frac{100}{1+9e^{-t}}$

27. Separable for all n; Solution form depends on n

30. General: $y = \frac{Cx^2}{1+Cx}$; Singular: $y = 0, y = 1$

31. Equilibria at $P = 0, 2000$; Solution approaches 2000

35. Solution: $\ln(\ln(y)) = x + C$

37. Use $(a - y)(b - y) = (y - 1)(y - 2)$; $y \to 1$ as $t \to \infty$