Practice Problems: Lesson 7 - Non-Unique Solutions

Master these non-uniqueness patterns!

Part A: Identifying Non-Uniqueness

For each IVP, determine if solutions are unique. If not, explain why:

1. $y' = 3y^{2/3}$, $y(0) = 0$
2. $y' = |y|^{1/2}\text{sign}(y)$, $y(0) = 0$
3. $y' = 2\sqrt{y}$, $y(1) = 1$
4. $y' = y^{1/3}$, $y(0) = 1$
5. $y' = \begin{cases} 2\sqrt{y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$, $y(0) = 0$

Part B: Finding Multiple Solutions

Find at least two different solutions for each IVP:

6. $y' = 2|y|^{1/2}$, $y(0) = 0$
7. $(y')^2 = 4y$, $y(0) = 0$, with $y \geq 0$
8. $y' = 3y^{2/3}$, $y(1) = 0$
9. $|y'| = |y|^{1/2}$, $y(0) = 0$

Part C: Solution Families

10. For $y' = 2\sqrt{y}$, $y(0) = 0$:
    (a) Show that $y \equiv 0$ is a solution
    (b) Show that $y = \begin{cases} 0 & x \leq c \\ (x-c)^2 & x > c \end{cases}$ is a solution for any $c \geq 0$
    (c) Sketch at least 4 different solutions
    (d) Where does the Lipschitz condition fail?

11. Consider the general equation $y' = k|y|^\alpha$, $y(0) = 0$:
    (a) For which values of α is the solution unique?
    (b) Find the general solution family for $0 < \alpha < 1$
    (c) What happens as $\alpha \to 0^+$?
    (d) What happens as $\alpha \to 1^-$?

Part D: Lipschitz Analysis

12. Check the Lipschitz condition at $y = 0$ for:
    (a) $f(y) = y^2$
    (b) $f(y) = \sqrt{|y|}$
    (c) $f(y) = y \ln |y|$ (with $f(0) = 0$)
    (d) $f(y) = |y|^{3/2}$
    (e) $f(y) = y^{2/3} \sin(1/y)$ (with $f(0) = 0$)
13. For each function in Problem 12, state whether the IVP with $y(0) = 0$ has unique solutions.

Part E: Clairaut and Singular Solutions

14. Consider Clairaut's equation: $y = xy' + (y')^2$
    (a) Find the general solution (family of lines)
    (b) Find the singular solution (envelope)
    (c) Verify both satisfy the original equation
    (d) Sketch the solution family and envelope
15. For the equation $y = xy' - (y')^3$:
    (a) Find all solutions through the point $(0, 0)$
    (b) Are there infinitely many such solutions?

Part F: Lost Solutions

16. Solve $y' = 2y^{1/2}$ by separation of variables:
    (a) What solution do you get?
    (b) What solution is missed?
    (c) Why was it missed?

17. For $y' = y^2(1-y)^2$:
    (a) Find all equilibrium solutions
    (b) Solve by separation of variables
    (c) Which solutions might be missed in the process?

Part G: Solution Crossing

18. Can two solutions of $y' = y^2$ cross? Why or why not?
19. Can two solutions of $y' = |y|^{1/2}$ cross? If yes, where?
20. Consider $y' = f(y)$ where f is Lipschitz everywhere:
    (a) Prove that two solutions cannot cross
    (b) What does this imply about solution uniqueness?

Part H: Exam-Style Problems

21. Professor Ditkowski asks: "Give an example of an IVP with exactly 3 solutions."
    (a) Construct such an example
    (b) Verify all three solutions
    (c) Explain why there are exactly three
22. Consider the "raindrop equation": $y' = -k\sqrt{y}$, $y(T) = 0$
    (a) Interpret y physically (hint: radius)
    (b) Find all solutions backward in time from $t = T$
    (c) What does non-uniqueness mean physically?
23. For the IVP $y' = |y-1|^{1/2} + |y+1|^{1/2}$, $y(0) = 1$:
    (a) Where might uniqueness fail?
    (b) Is the solution unique through $(0, 1)$?
    (c) What about through $(0, -1)$?
    (d) What about through $(0, 0)$?
24. The equation $y' = g(y)$ has $g(0) = g(1) = g(2) = 0$ and g is not Lipschitz at these points:
    (a) Describe possible solution behaviors
    (b) Can a solution starting at $y(0) = 0.5$ reach $y = 2$?
    (c) How many solutions might connect $y(0) = 0$ to $y(10) = 2$?

Part I: Advanced Theory

25. Prove that if $f(y) = |y|^\alpha$ with $0 < \alpha < 1$, then f is not Lipschitz at $y = 0$.

26. Show that the IVP $y' = y \ln |y|$, $y(0) = 0$ (with $f(0) = 0$) has a unique solution despite f not being Lipschitz at 0. (Hint: Use Osgood's criterion)

27. Consider "patching" solutions:

    (a) If $y_1(x)$ solves the ODE for $x < a$ with $y_1(a) = 0$

    (b) And $y_2(x)$ solves it for $x > b$ with $y_2(b) = 0$

    (c) When can you "patch" them with $y \equiv 0$ on $[a, b]$?

Solution Hints

Part A: 1. Not unique ($\alpha = 2/3 < 1$) 2. Not unique (like $\sqrt{|y|}$) 3. Unique (initial value away from 0) 4. Unique (initial value away from 0) 5. Not unique (piecewise with flat spot)

Part B Key Ideas: - Always check $y \equiv 0$ first - Look for delayed start solutions - Consider both positive and negative branches

Remember: Non-uniqueness occurs when $\partial f / \partial y$ is unbounded or discontinuous!