ODE Lesson 27: Fundamental Matrix Solutions - Construction

ODE 1 - Prof. Adi Ditkowski

1 Definition and Properties

Definition 1 (Fundamental Matrix). *A matrix $\Phi(t)$ is a **fundamental matrix** for the system $\mathbf{x}' = A(t)\mathbf{x}$ if:*

1. *Each column of $\Phi(t)$ is a solution to the system*
2. *The columns are linearly independent for all t*
3. $\Phi'(t) = A(t)\Phi(t)$

For an $n \times n$ system, the fundamental matrix $\Phi(t)$ is $n \times n$ with structure:

$$\Phi(t) = [\mathbf{x}_1(t) \mid \mathbf{x}_2(t) \mid \cdots \mid \mathbf{x}_n(t)]$$

where each $\mathbf{x}_i(t)$ is a linearly independent solution vector.

Theorem 1 (Fundamental Matrix Properties). *If $\Phi(t)$ is a fundamental matrix, then:*

1. $\det(\Phi(t)) \neq 0$ *for all t (never singular)*
2. $\Phi'(t) = A(t)\Phi(t)$ *(matrix differential equation)*
3. *General solution: $\mathbf{x}(t) = \Phi(t)\mathbf{c}$ for any constant vector $\mathbf{c}$*
4. $\Phi(t)^{-1}$ *exists for all t*

2 Principal Fundamental Matrix

Definition 2 (Principal Fundamental Matrix). *The **principal fundamental matrix** $\Phi(t)$ at t_0 satisfies:*

$$\Phi(t_0) = I$$

where I is the identity matrix.

The principal fundamental matrix provides the simplest IVP solution formula:

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$$

for the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

3 Solution of Initial Value Problems

Theorem 2 (IVP Solution Formula)**.** *For the IVP:*

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

The unique solution is:

$$\mathbf{x}(t) = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0$$

where $\Phi(t)$ is any fundamental matrix.

Solution procedure:

1. Find n linearly independent solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$
2. Construct $\Phi(t) = [\mathbf{x}_1(t) \mid \cdots \mid \mathbf{x}_n(t)]$
3. Compute $\Phi(t_0)$
4. Find $\Phi(t_0)^{-1}$
5. Calculate $\mathbf{x}(t) = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0$

4 Construction Methods

Method 1 (For Constant Coefficient Systems)**.** *For $\mathbf{x}' = A\mathbf{x}$ with constant A:*

1. *Find eigenvalues $\lambda_1, \ldots, \lambda_n$ of A*
2. *Find corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$*
3. *Solutions: $\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$*
4. *Fundamental matrix: $\Phi(t) = [e^{\lambda_1 t}\mathbf{v}_1 \mid \cdots \mid e^{\lambda_n t}\mathbf{v}_n]$*

5 Examples

Example 1 (2×2 System)**.** *Consider $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\mathbf{x}$.*

Step 1: *Find eigenvalues*

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 4 = 0$$

$$\lambda_1 = 4, \quad \lambda_2 = -1$$

Step 2: *Find eigenvectors For $\lambda_1 = 4$: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ For $\lambda_2 = -1$: $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$*

Step 3: *Construct fundamental matrix*

$$\Phi(t) = \begin{bmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{bmatrix}$$

Step 4: *Verify $\Phi'(t) = A\Phi(t)$*

$$\Phi'(t) = \begin{bmatrix} 8e^{4t} & -e^{-t} \\ 12e^{4t} & e^{-t} \end{bmatrix}$$

$$A\Phi(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{bmatrix} = \begin{bmatrix} 8e^{4t} & -e^{-t} \\ 12e^{4t} & e^{-t} \end{bmatrix} \checkmark$$

Example 2 (IVP Solution)**.** *Using the fundamental matrix from Example 1, solve:*

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Solution:

$$\Phi(0) = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\Phi(0)^{-1} = \frac{1}{-5} \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix}$$

$$\Phi(0)^{-1}\mathbf{x}_0 = \begin{bmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 12/5 \\ 1/5 \end{bmatrix}$$

$$\mathbf{x}(t) = \Phi(t) \begin{bmatrix} 12/5 \\ 1/5 \end{bmatrix} = \begin{bmatrix} \frac{24}{5}e^{4t} + \frac{1}{5}e^{-t} \\ \frac{36}{5}e^{4t} - \frac{1}{5}e^{-t} \end{bmatrix}$$

6 Relationship Between Fundamental Matrices

Theorem 3 (Fundamental Matrix Relationship)**.** *If $\Phi_1(t)$ and $\Phi_2(t)$ are both fundamental matrices for the same system, then:*

$$\Phi_2(t) = \Phi_1(t)C$$

where C is a constant nonsingular matrix.

Common errors:

- Forgetting to verify linear independence of solutions
- Wrong matrix multiplication order in IVP formula
- Not checking $\Phi'(t) = A(t)\Phi(t)$
- Confusing $\Phi(t)$ with individual solution vectors

7 Special Cases and Extensions

For repeated eigenvalues, the fundamental matrix includes terms with t:

$$\Phi(t) = e^{\lambda t}[I + tN + \frac{t^2}{2!}N^2 + \cdots]$$

where N is the nilpotent part of $A - \lambda I$.

Prof. Ditkowski's favorite exam questions:

- Construct $\Phi(t)$ from given solutions
- Verify fundamental matrix property
- Use $\Phi(t)$ to solve specific IVP
- Find relationship between two fundamental matrices
- Compute $\Phi(t)\Phi(s)^{-1}$ for various s, t

8 Matrix Exponential Connection

Proposition 1. *For constant matrix A, the principal fundamental matrix at $t_0 = 0$ is:*

$$\Phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$