Lesson 23: Integrating Factors - $\mu(x)$ and $\mu(y)$ Cases

ODE 1 - Prof. Adi Ditkowski

1 Introduction to Integrating Factors

Definition 1 (Integrating Factor)**.** *For a non-exact equation $M(x,y)dx + N(x,y)dy = 0$, an **integrating factor** $\mu(x,y)$ is a function such that*

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact.

An integrating factor converts a non-exact equation into an exact one without changing the solution curves.

2 Integrating Factor Depending Only on x

Theorem 1 (Test for $\mu(x)$)**.** *If $\frac{\partial M/\partial y - \partial N/\partial x}{N}$ depends only on x, then there exists an integrating factor $\mu(x)$ satisfying:*

$$\frac{d\mu}{dx} = \mu \cdot \frac{\partial M/\partial y - \partial N/\partial x}{N}$$

Finding $\mu(x)$:

1. Calculate $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$
2. Compute $\frac{\partial M/\partial y - \partial N/\partial x}{N}$
3. Check if this expression contains only x (no y terms)
4. If yes, solve $\frac{d\mu}{dx} = \mu \cdot \frac{\partial M/\partial y - \partial N/\partial x}{N}$
5. Multiply original equation by $\mu(x)$
6. Verify the result is exact and solve

3 Integrating Factor Depending Only on y

Theorem 2 (Test for $\mu(y)$)**.** *If* $\frac{\partial N/\partial x - \partial M/\partial y}{M}$ *depends only on* y*, then there exists an integrating factor* $\mu(y)$ *satisfying:*

$$\frac{d\mu}{dy} = \mu \cdot \frac{\partial N/\partial x - \partial M/\partial y}{M}$$

Memory Aid:

- For $\mu(x)$: "$(M_y - N_x)/N$" $\to$ function of x only
- For $\mu(y)$: "$(N_x - M_y)/M$" $\to$ function of y only

Note the sign flip and denominator change!

4 Complete Examples

Example 1 (Finding $\mu(x)$)**.** *Solve:* $(3xy + y^2)dx + (x^2 + xy)dy = 0$

Step 1: Check exactness $\frac{\partial M}{\partial y} = 3x + 2y$, $\frac{\partial N}{\partial x} = 2x + y$ *Not equal* $\Rightarrow$ *Not exact*

Step 2: Test for $\mu(x)$ $\frac{M_y - N_x}{N} = \frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}$

This depends only on x*!*

Step 3: Find $\mu(x)$ $\frac{d\mu}{dx} = \frac{\mu}{x} \Rightarrow \mu = x$

Step 4: Multiply and solve $x(3xy + y^2)dx + x(x^2 + xy)dy = 0$ $(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$

Verify exactness: $\frac{\partial}{\partial y}(3x^2y + xy^2) = 3x^2 + 2xy = \frac{\partial}{\partial x}(x^3 + x^2y)$ ✓

Example 2 (Finding $\mu(y)$)**.** *Solve:* $(2y)dx + (3x)dy = 0$

Test for $\mu(y)$***:*** $\frac{N_x - M_y}{M} = \frac{3-2}{2y} = \frac{1}{2y}$

This depends only on y*!*

$\frac{d\mu}{dy} = \frac{\mu}{2y} \Rightarrow \mu = y^{1/2}$

Multiplied equation: $(2y^{3/2})dx + (3xy^{1/2})dy = 0$

Solution: $H = 2xy^{3/2} = C$

5 Decision Flow

Complete Strategy:

1. Test for exactness first
2. If not exact, test for $\mu(x)$: check if $(M_y - N_x)/N$ depends only on x
3. If not, test for $\mu(y)$: check if $(N_x - M_y)/M$ depends only on y
4. If neither works, try special forms (Lesson 24)

5. After finding μ, multiply equation and solve as exact

Common Mistakes:

- Forgetting to multiply both M and N by the integrating factor
- Sign errors in $(M_y - N_x)$ vs $(N_x - M_y)$
- Confusing which denominator to use (N vs M)
- Not verifying exactness after multiplication

Prof. Ditkowski's Patterns:

- Often gives equations where both $\mu(x)$ and $\mu(y)$ exist
- May ask you to find the integrating factor, not the full solution
- Watch for simple integrating factors like x, y, x^2, y^{-1}
- Sometimes disguises as "Show that $\mu = \ldots$ is an integrating factor"