Lesson 47: Practice Problems
Euler-Cauchy Equations

ODE 1 with Prof. Adi Ditkowski

Tel Aviv University

Part A: Basic Euler Equations (5 problems)

1. Solve: t^{2y}" - 2ty' + 2y = 0fort ¿ 0
2. Solve: t^{2y}" + 3ty' + y = 0fort ¿ 0
3. Solve: t^{2y}" - ty' - 3y = 0fort ¿ 0
4. Solve: t^{2y}" + 4ty' + 2y = 0fort ¿ 0
5. Solve: t^{2y}" - 3ty' + 4y = 0fort ¿ 0

Part B: Repeated Roots Cases (5 problems)

1. Solve: t^{2y}" - 3ty' + 4y = 0fort ¿ 0$(verify repeated root)$
2. Solve: t^{2y}" + 5ty' + 4y = 0fort ¿ 0
3. Solve: t^{2y}" - ty' + y = 0fort ¿ 0
4. Solve: 4t^{2y}" + 8ty' + y = 0fort ¿ 0
5. Solve: t^{2y}" + 3ty' + y = 0fort ¿ 0

Part C: Complex Roots Cases (5 problems)

1. Solve: t^{2y}" - ty' + 5y = 0fort ¿ 0
2. Solve: t^{2y}" + ty' + y = 0fort ¿ 0
3. Solve: t^{2y}" - 3ty' + 13y = 0fort ¿ 0
4. Solve: t^{2y}" + ty' + 4y = 0fort ¿ 0
5. Solve: t^{2y}" + 3ty' + 5y = 0fort ¿ 0

Part D: Using the Transform Method (5 problems)

1. Using $x = \ln t$, transform and solve: $t^{2y"}$ - 4ty' + 6y = 0
2. Transform and solve: $t^{2y"}$ + 2ty' - 2y = 0
3. Show that the transform method gives the same solution as the direct method for: $t^{2y"}$ - ty' - 8y = 0
4. Use the transform to solve: $t^{2y"}$ + 5ty' + 3y = 0
5. Transform the third-order equation: $t^{3y"'}$ + 3t$^{2y"}$ - 2ty' + 2y = 0

Part E: Initial Value Problems (5 problems)

1. Solve: $t^{2y"}$ - 2ty' + 2y = 0,y(1) = 3,y'(1) = 5
2. Solve: $t^{2y"}$ + 3ty' + y = 0,y(1) = 2,y'(1) = -1
3. Solve: $t^{2y"}$ - ty' + y = 0,y(1) = 0,y'(1) = 1
4. Solve: $t^{2y"}$ - ty' + 5y = 0,y(1) = 1,y'(1) = 2
5. Solve: $t^{2y"}$ - 3ty' + 4y = 0,y(2) = 8,y'(2) = 12

Part F: Exam-Style Problems (5 problems)

1. (Prof. Ditkowski style) Consider the equation $t^{2y"}$ + aty' + by = 0.

   2. For what values of a and b are all solutions bounded as $t \to \infty$?
   3. For what values do solutions oscillate on a logarithmic scale?
   4. Find conditions for polynomial solutions.

The equation $t^{2y"}$ - $2\alpha ty' + \alpha(\alpha+1)y = 0$ has $y_1 = t^\alpha$ as a solution.

(a) Verify this directly.
(b) Find the second solution.
(c) What is special about this equation?

Transform the equation $(t+1)^{2y"}$ + 3(t+1)y' + y = $0 to an Euler equation and solve.$

For the equation $t^{2y"}$ + ty' + (t$^2 - \nu^2)y = 0$ (Bessel's equation of order ν):

(a) Show this is NOT a pure Euler equation.
(b) Find the indicial equation at $t = 0$.
(c) What are the indices?

Consider the system of Euler equations:

$$\begin{cases} t^{2y''} \text{ - 2ty' + 2y} = \text{z} \\ t^{2z''} \text{ - 4tz' + 6z} = \text{y} \end{cases}$$

(a) Find the general solution for y and z.

(b) Determine if solutions can remain bounded as $t \to 0^+$.

Solutions

Part A: Basic Euler Equations

1. Characteristic equation: $r(r-1) - 2r + 2 = 0$
   $r^2 - 3r + 2 = 0$
   $(r-1)(r-2) = 0$
   Roots: $r = 1, 2$
   General solution: $y = c_{1t} + c_{2t}^2$

2. Characteristic equation: $r(r-1) + 3r + 1 = 0$
   $r^2 + 2r + 1 = 0$
   $(r+1)^2 = 0$
   Repeated root: $r = -1$
   General solution: $y = \frac{c_1 + c_2 \ln t}{t}$

3. Characteristic equation: $r(r-1) - r - 3 = 0$
   $r^2 - 2r - 3 = 0$
   $(r-3)(r+1) = 0$
   Roots: $r = 3, -1$
   General solution: $y = c_{1t}^3 + \frac{c_2}{t}$

4. Characteristic equation: $r(r-1) + 4r + 2 = 0$
   $r^2 + 3r + 2 = 0$
   $(r+1)(r+2) = 0$
   Roots: $r = -1, -2$
   General solution: $y = \frac{c_1}{t} + \frac{c_2}{t^2}$

5. Characteristic equation: $r(r-1) - 3r + 4 = 0$
   $r^2 - 4r + 4 = 0$
   $(r-2)^2 = 0$
   Repeated root: $r = 2$
   General solution: $y = (c_1 + c_2 \ln t)t^2$

Part B: Repeated Roots Cases

6. Already solved in Part A, 5:
   $r = 2$ (double root)
   General solution: $y = (c_1 + c_2 \ln t)t^2$

7. Characteristic equation: $r^2 + 4r + 4 = 0$
   $(r+2)^2 = 0$
   Repeated root: $r = -2$
   General solution: $y = \frac{c_1 + c_2 \ln t}{t^2}$

8. Characteristic equation: $r^2 - 2r + 1 = 0$
   $(r-1)^2 = 0$

Repeated root: $r = 1$
General solution: $y = (c_1 + c_2 \ln t)t$

9. Divide by 4: $t^{2y"}$ + 2ty' + $1\frac{}{4y=0}$
   Characteristic equation: $r^2 + r + \frac{1}{4} = 0$
   $(r + \frac{1}{2})^2 = 0$
   Repeated root: $r = -\frac{1}{2}$
   General solution: $y = \frac{c_1 + c_2 \ln t}{\sqrt{t}}$

10. Characteristic equation: $r^2 + 2r + 1 = 0$
    $(r + 1)^2 = 0$
    Repeated root: $r = -1$
    General solution: $y = \frac{c_1 + c_2 \ln t}{t}$

Part C: Complex Roots Cases

6. Characteristic equation: $r^2 - 2r + 5 = 0$
   $r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$
   General solution: $y = t[c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)]$

7. Characteristic equation: $r^2 + 1 = 0$
   $r = \pm i$
   General solution: $y = c_1 \cos(\ln t) + c_2 \sin(\ln t)$

8. Characteristic equation: $r^2 - 4r + 13 = 0$
   $r = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$
   General solution: $y = t^2[c_1 \cos(3 \ln t) + c_2 \sin(3 \ln t)]$

9. Characteristic equation: $r^2 + 4 = 0$
   $r = \pm 2i$
   General solution: $y = c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)$

10. Characteristic equation: $r^2 + 2r + 5 = 0$
    $r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$
    General solution: $y = \frac{1}{t}[c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)]$

Part D: Using the Transform Method

11. Let $x = \ln t$, $v(x) = y(t)$
    Transform: $v'' - 5v' + 6v = 0$
    Characteristic: $\lambda^2 - 5\lambda + 6 = 0$
    $(\lambda - 2)(\lambda - 3) = 0$
    $v = {c_{1e}}^{2x} + {c_{2e}}^{3x}$
    Back-transform: $y = {c_{1t}}^2 + {c_{2t}}^3$

12. Transform: $v'' + v' - 2v = 0$
    Characteristic: $\lambda^2 + \lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$
$v = c_{1e}^{-2x} + c_{2e}^{x}$
Back-transform: $y = \frac{c_1}{t^2} + c_{2t}$

13. Direct method: $r^2 - 2r - 8 = 0 \Rightarrow r = 4, -2$
Transform method: $v'' - 2v' - 8v = 0$
Same characteristic equation!
Solution: $y = c_{1t}^{4} + \frac{c_2}{t^2}$

14. Transform: $v'' + 4v' + 3v = 0$
$(\lambda + 1)(\lambda + 3) = 0$
$v = c_{1e}^{-x} + c_{2e}^{-3x}$
Back-transform: $y = \frac{c_1}{t} + \frac{c_2}{t^3}$

15. Let $D = d/dx$. Transform gives:
$(D^3 - 3D^2 + 2D) + 3(D^2 - D) - 2D + 2 = 0$
$D^3 - D - 2D + 2 = D^3 - 3D + 2 = 0$
Characteristic: $\lambda^3 - 3\lambda + 2 = 0$
$(\lambda - 1)^2(\lambda + 2) = 0$
$v = (c_1 + c_{2x})\mathrm{e}^x + \mathrm{c}_{3e}^{-2x}$
$y = (c_1 + c_2 \ln t)t + \frac{c_3}{t^2}$

Part E: Initial Value Problems

16. From Part A 1: $y = c_{1t} + \mathrm{c}_{2t}^{2}$
$y(1) = c_1 + c_2 = 3$
$y'(t) = c_1 + 2c_{2t}$
$\mathrm{y'}(1) = \mathrm{c}_1 + 2c_2 = 5$
Solving: $c_2 = 2$, $c_1 = 1$
Solution: $y = t + 2t^2$

17. From Part A 2: $y = \frac{c_1 + c_2 \ln t}{t}$
$y(1) = c_1 = 2$
$y'(t) = \frac{c_2 - c_1 - c_2 \ln t}{t^2}$
$y'(1) = c_2 - c_1 = -1$
$c_2 = 1$
Solution: $y = \frac{2 + \ln t}{t}$

18. From Part B 8: $y = (c_1 + c_2 \ln t)t$
$y(1) = c_1 = 0$
$y'(t) = c_1 + c_2(1 + \ln t)$
$y'(1) = c_2 = 1$
Solution: $y = t \ln t$

19. From Part C 11: $y = t[c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)]$
$y(1) = c_1 = 1$
$y'(t) = [c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t)] + t[-2c_1 \sin(2 \ln t)/t + 2c_2 \cos(2 \ln t)/t]$

$y'(1) = 1 + 2c_2 = 2$
$c_2 = \frac{1}{2}$
Solution: $y = t[\cos(2\ln t) + \frac{1}{2}\sin(2\ln t)]$

20. From Part A 5: $y = (c_1 + c_2 \ln t)t^2$
$y(2) = 4c_1 + 4c_2 \ln 2 = 8$
$y'(t) = 2(c_1 + c_2 \ln t)t + c_{2t}$
y'(2) $= 8c_1 + 8c_2 \ln 2 + 2c_2 = 12$
From first: $c_1 + c_2 \ln 2 = 2$
From second: $4c_1 + 4c_2 \ln 2 + c_2 = 6$
$c_2 = -2$, $c_1 = 2 + 2\ln 2$
Solution: $y = [2 + 2\ln 2 - 2\ln t]t^2 = 2t^2[1 + \ln(4/t)]$

Part F: Exam-Style Problems

21. (a) Bounded as $t \to \infty$: Need both roots to have negative real parts.
From $r^2 + (a-1)r + b = 0$: Need $a > 1$ and $b > 0$ (and $(a-1)^2 > 4b$ for real roots).
(b) Oscillation: Need complex roots, so $(a-1)^2 < 4b$.
(c) Polynomial solutions: Need positive integer roots. For example, $r = n$ requires $n^2 + (a-1)n + b = 0$.

22. (a) $y_1' = \alpha t^{\alpha-1}$, $y_1'' = \alpha(\alpha-1)t^{\alpha-2}$
Substitute: $\alpha(\alpha-1) - 2\alpha \cdot \alpha + \alpha(\alpha+1) = 0$

$$\checkmark$$

(b) Characteristic equation: $r^2 - (2\alpha+1)r + \alpha(\alpha+1) = 0$
$(r-\alpha)(r-(\alpha+1)) = 0$
Second solution: $y_2 = t^{\alpha+1}$
(c) This is the Euler equation whose solutions are consecutive powers of t.

23. Let $s = t + 1$, then the equation becomes:
$s^{2y"}$ + 3sy' + y = $0(standardEulerins)$
$Characteristic$:r$^2 + 2r + 1 = (r+1)^2 = 0$
$y = \frac{c_1 + c_2 \ln(t+1)}{t+1}$

24. (a) The t^2 in the last term makes this NOT a pure Euler equation.
(b) For the indicial equation, consider $y = t^r(1 +$ a$_{1t}$ + ...)
$Leadingtermsgive$:r(r-1) + r - $\nu^2 = 0$
$r^2 - \nu^2 = 0$
(c) Indices: $r = \pm\nu$

25. (a) Decouple: Fourth-order equations result.
For y: $(t^{2D2} - 2tD + 2)(t^{2D2} - 4tD + 6) - 1 = 0$
This gives $t^{4y(4)} - 6t^{3y"'}$ + 15t$^{2y"}$ - 15ty' + (12-1)y = 0
$Complexcalculationyieldsgeneralsolutions.$

(b) As $t \to 0^+$: Solutions behave like $t^r whererarethecharacteristicroots.Boundedonlyifall$Re(r) ≥ 0.