Lesson 19: Practice Problems - Riccati Equations with Known Solution

ODE 1 - Prof. Adi Ditkowski

Part A: Recognition and Classification (5 problems)

1. Identify which equations are Riccati and find q_0, q_1, q_2:
   (a) $y' = x^2 + 2xy - y^2$
   (b) $y' = e^x + y^2$
   (c) $xy' = 1 + xy + y^2$
   (d) $y' + y = xy^2$
   (e) $y' = \sin x + 2y \cos x + y^2$
2. Show that the equation $y' = \frac{a+by}{c+dy}$ can be written in Riccati form if and only if $ad - bc \neq 0$.
3. Verify that if y_1 and y_2 are two solutions of a Riccati equation, then $y = y_1 + \frac{1}{z}$ satisfies a linear equation in z.
4. Prove that the sum of two particular solutions of a Riccati equation does not generally give another solution.
5. For the Riccati equation $y' = q_0 + q_1 y + q_2 y^2$, show that $y = -q_1/(2q_2)$ is a solution if and only if $q_0 = q_1^2/(4q_2)$.

Part B: Finding Particular Solutions (6 problems)

6. Find a particular solution by inspection:
   (a) $y' = 2 + y - y^2$
   (b) $y' = \frac{1}{x^2} - \frac{y}{x} + y^2$
   (c) $y' = 1 + 2y + y^2$
   (d) $y' = e^{2x} + e^x y - y^2$
7. Verify that $y_p = \tan x$ satisfies $y' = 1 + y^2$ and find the general solution.
8. Given that $y_p = 1/x$ is a solution of $y' = -1/x^2 + 2y/x - y^2$, find all solutions.

9. Show that $y_p = x$ satisfies $y' = 1 - x^2 + 2xy - y^2$ and solve completely.

10. Find two different particular solutions of $y' = 6 - y - y^2$ and use each to find the general solution.

11. For $y' = 2\cos^2 x + (\sin 2x)y - y^2$, verify that $y_p = \sin x$ is a solution.

Part C: Complete Solution Process (5 problems)

12. Solve the Riccati equation $y' = \frac{2}{x^2} - \frac{2y}{x} + y^2$ given that $y_p = 2/x$.

13. Find all solutions of $y' = -2 + y + y^2$ given one solution $y_p = 1$.

14. Solve $y' = e^{2x} + (1 - 2e^x)y + y^2$ with the particular solution $y_p = e^x$.

15. Given $y_p = \cot x$ solves $y' = -1 - y^2$, find the solution satisfying $y(\pi/4) = 0$.

16. Solve $y' = \frac{1-x^2}{x^2} + \frac{2y}{x} - y^2$ knowing that it has a polynomial particular solution.

Part D: Advanced Problems (5 problems)

17. Consider the parametric family $y' = a + y^2$ where a is a constant.
    (a) Find particular solutions for $a = 1, 0, -1$
    (b) Solve each case completely
    (c) Discuss the qualitative behavior of solutions

18. The equation $y' = q(x) + y^2$ where $q(x)$ is continuous:
    (a) Show that if $q(x) = -f'(x)/f(x)$ for some $f(x) > 0$, then $y_p = f'(x)/(2f(x))$ is a solution
    (b) Apply this to $q(x) = -2x/(1 + x^2)$

19. Solve the Riccati equation arising in optimal control:
    $$y' = 1 - y^2$$
    with $y(0) = 0$.

20. For the equation $y' = x^{2n} + y^2$ where n is a positive integer:
    (a) Show there's no polynomial particular solution
    (b) Transform to a second-order linear equation
    (c) Find the solution for $n = 0$

21. Consider the Riccati equation with periodic coefficients:
    $$y' = \cos(2x) + 2\sin x \cdot y - y^2$$
    Given $y_p = \sin x$, find all periodic solutions.

Part E: Theoretical Problems (4 problems)

22. Prove that if a Riccati equation has three known particular solutions y_1, y_2, y_3, then the general solution can be written as:

$$\frac{y - y_1}{y - y_2} = C \cdot \frac{y_3 - y_1}{y_3 - y_2}$$

23. Show that the transformation $y = -u'/u$ converts the second-order linear equation $u'' + p(x)u' + q(x)u = 0$ into the Riccati equation:

$$y' = -q(x) - p(x)y - y^2$$

24. Prove that if y_1 and y_2 are two solutions of a Riccati equation, then:

$$\frac{d}{dx}\left(\frac{1}{y_1 - y_2}\right) = -q_1 - q_2(y_1 + y_2)$$

25. For the autonomous Riccati $y' = a + by + cy^2$:

    (a) Find conditions for existence of equilibrium points

    (b) Analyze stability of equilibria

    (c) Show that solutions either blow up in finite time or exist for all time

Part F: Exam-Style Complete Problems (5 problems)

26. [**Prof. Ditkowski Style**] Consider the equation: $y' = \frac{4}{x^2} - \frac{4y}{x} + y^2$

    (a) Verify that $y_p = 2/x$ is a particular solution

    (b) Use the substitution $y = y_p + v$ to transform to Bernoulli form

    (c) Solve the resulting Bernoulli equation

    (d) Find the general solution

    (e) Determine the solution satisfying $y(1) = 3$

    (f) Are there any singular solutions?

27. [**Comprehensive Problem**] For the equation $y' = 1 + xy - y^2$:

    (a) Show that no constant particular solution exists

    (b) Try $y_p = ax + b$ and find values of a and b

    (c) Solve the equation completely

    (d) Analyze behavior as $x \to \pm\infty$

28. [**Multiple Methods**] Given $y' = 2 - 3y + y^2$:

(a) Find two different particular solutions

(b) Use each to find the general solution

(c) Verify both give the same general solution

(d) Express the solution using partial fractions

29. [**Application to Projectile Motion**] The equation for the envelope of projectile trajectories:

$$y' = \frac{g}{2v_0^2}x + \sqrt{1 + \left(\frac{gx}{v_0^2}\right)^2} - \frac{g^2x}{2v_0^4}y^2$$

(a) Show this is approximately Riccati for small x

(b) Find the linear approximation

(c) Discuss physical interpretation

30. [**Challenge Problem**] Consider the family of Riccati equations:

$$y' = \frac{n(n+1)}{x^2} - \frac{2n}{x}y + y^2$$

where n is a positive integer.

(a) Show that $y_p = n/x$ is always a particular solution

(b) Find the general solution for arbitrary n

(c) What happens as $n \to \infty$?

(d) Connect to Legendre polynomials

Solutions and Hints

Selected Solutions:

Problem 1(a): Not Riccati (wrong sign on y^2 term). Would need $+y^2$.

Problem 6(a): Try constants: $0 = 2 + c - c^2$, so $c^2 - c - 2 = 0$. Thus $c = 2$ or $c = -1$.

Problem 7: With $y = \tan x + v$: $v' = 2\tan x \cdot v + v^2$. Using $w = 1/v$: $w' = -2\tan x \cdot w - 1$. Solution: $w = (\cos x)(C - x)$, so $y = \tan x + \frac{1}{(\cos x)(C-x)}$.

Problem 12: With $y = 2/x + v$: $v' = \frac{2v}{x} + v^2$ (Bernoulli with $n = 2$). Let $w = 1/v$: $w' = -\frac{2w}{x} - 1$. Solution: $w = \frac{C}{x^2} - \frac{x}{3}$.

Problem 25: For equilibria: $0 = a + by + cy^2$. Discriminant $\Delta = b^2 - 4ac$ determines number of equilibria. Stability: Check $f'(y^*) = b + 2cy^*$.

Key Transformation Formulas:

- Riccati: $y' = q_0 + q_1 y + q_2 y^2$
- If y_p known: $y = y_p + v$
- Bernoulli form: $v' = (q_1 + 2q_2 y_p)v + q_2 v^2$

- Linear form: $w' = -(q_1 + 2q_2 y_p)w - q_2$ where $w = 1/v$

Common Particular Solutions:

- Constants when q_0, q_1, q_2 are constants
- $y = a/x$ for equations with $1/x^2$ terms
- $y = \tan(ax)$ for $y' = a^2 + y^2$
- $y = \tanh(ax)$ for $y' = -a^2 + y^2$