Direct Integration: The Foundation of ODE Solutions

ODE 1 - Prof. Adi Ditkowski

Lesson 11

1 Introduction

Definition 1 (Directly Integrable ODE). *An ODE is **directly integrable** if it can be written in the form:*

$$\frac{d^n y}{dx^n} = f(x)$$

where $f(x)$ is a function of x only (not involving y or its derivatives).

Direct integration is the simplest solution method, requiring only antiderivatives. Yet it forms the foundation for all other ODE techniques!

2 First-Order Direct Integration

2.1 General Method

Method 1 (Solving $y' = f(x)$).

1. *Recognize the form:* $\frac{dy}{dx} = f(x)$
2. *Integrate both sides:* $y = \int f(x)\, dx + C$
3. *If initial condition $y(x_0) = y_0$ is given, determine C*
4. *Verify the solution by differentiation*

Example 1 (Basic Direct Integration). *Solve:* $\frac{dy}{dx} = 3x^2 - 2x + 1$ *with* $y(0) = 5$

Solution:

$$y = \int (3x^2 - 2x + 1)\, dx \tag{1}$$
$$= x^3 - x^2 + x + C \tag{2}$$

Using $y(0) = 5$:

$$5 = 0^3 - 0^2 + 0 + C \implies C = 5$$

Therefore: $y = x^3 - x^2 + x + 5$

2.2 Definite Integral Form

Theorem 1 (IVP Solution via Definite Integral). *The solution to* $\frac{dy}{dx} = f(x)$ *with* $y(x_0) = y_0$ *is:*

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$$

The definite integral form automatically incorporates the initial condition - no need to find C separately!

3 Higher-Order Direct Integration

3.1 General Pattern

Theorem 2 (n-th Order Direct Integration). *For* $\frac{d^n y}{dx^n} = f(x)$*, the general solution contains* n *arbitrary constants:*

$$y(x) = \int^{(n)} f(x)\,dx + C_1 x^{n-1} + C_2 x^{n-2} + \cdots + C_{n-1} x + C_n$$

where $\int^{(n)}$ *denotes n-fold integration.*

Example 2 (Second-Order). *Solve:* $y'' = \cos(x)$ *with* $y(0) = 1$, $y'(0) = 0$

First integration:

$$y' = \int \cos(x)\,dx = \sin(x) + C_1$$

Using $y'(0) = 0$*:* $0 = \sin(0) + C_1 \implies C_1 = 0$

Second integration:

$$y = \int \sin(x)\,dx = -\cos(x) + C_2$$

Using $y(0) = 1$*:* $1 = -\cos(0) + C_2 = -1 + C_2 \implies C_2 = 2$

Solution: $y = -\cos(x) + 2$

4 Special Cases and Common Integrals

4.1 Important Antiderivatives for ODEs

$f(x)$	$\int f(x)\,dx$	**Domain Note**
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$	All x
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	$x \neq 0$
e^{ax}	$\frac{1}{a}e^{ax} + C$	All x
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + C$	All x
$\cos(ax)$	$\frac{1}{a}\sin(ax) + C$	All x
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$	All x
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) + C$	$\lvert x\rvert < a$

Always include absolute values in $\ln|x|$ and check domain restrictions!

4.2 The Zero Derivative Case

Theorem 3 (Polynomial Solutions). *If $\frac{d^n y}{dx^n} = 0$, then y is a polynomial of degree at most $n-1$:*

$$y(x) = C_1 x^{n-1} + C_2 x^{n-2} + \cdots + C_{n-1} x + C_n$$

5 Solution Verification

Method 2 (Verification Checklist).

1. *Differentiate your solution n times*
2. *Substitute into the original ODE*
3. *Verify the equation holds identically*
4. *Check initial conditions (if given)*
5. *Verify domain of validity*

Prof. Ditkowski awards partial credit for solution verification even if your answer is incorrect. Always show this step!

6 Common Errors to Avoid

Critical mistakes that lose points:

- Forgetting $+C$ in indefinite integrals
- Missing absolute values: $\int \frac{1}{x} dx = \ln|x| + C$
- Wrong number of constants for higher-order equations
- Not checking domain restrictions
- Arithmetic errors in determining constants

7 Physical Applications

Example 3 (Free Fall Motion). *For an object in free fall: $\frac{d^2 y}{dt^2} = -g$ where y is height, t is time, g is gravitational acceleration.*

Solution:

$$v = \frac{dy}{dt} = -gt + v_0 \quad \textit{(velocity)} \tag{3}$$

$$y = -\frac{1}{2}gt^2 + v_0 t + y_0 \quad \textit{(position)} \tag{4}$$

The constants v_0 and y_0 represent initial velocity and position.