Lesson 33: Practice Problems - Matrix Exponential

ODE 1 - Prof. Adi Ditkowski

Part A: Direct Computation

1. Compute e^{At} for $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$
2. Find e^{At} for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ using the series definition.
3. Calculate e^{At} for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
4. Verify that $\frac{d}{dt}e^{At} = Ae^{At}$ for $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

5. Show that $e^{A\cdot 0} = I$ for any $2{\times}2 matrix$A.

Part B: Diagonalization Method

6. Use diagonalization to find e^{At} for $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$
7. Compute e^{At} for $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$
8. Find e^{At} for $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$
9. Calculate e^{At} for $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$
10. Use diagonalization for $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}$

Part C: Jordan Form Method

11. Find e^{At} for the Jordan block $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

12. Compute e^{At} for $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

13. Calculate e^{At} for $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

14. Find e^{At} for $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

15. Compute e^{At} when $A = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$

Part D: Complex Eigenvalues

16. Find e^{At} for $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

17. Compute e^{At} for $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

18. Calculate e^{At} for $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

19. Find e^{At} for $A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$

20. Show that e^{At} is a rotation matrix when $A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$

Part E: Properties and Applications

21. Verify that $(e^{At})^{-1} = e^{-At}$ for $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

22. Show that $\det(e^{At}) = e^{\mathrm{tr}(A)t}$ for $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

23. Prove that if $A^2 = 0, then e^{At} = I + At$.

24. For $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, verify that $e^{A(s+t)} = e^{As}e^{At}$.

25. If A and B commute, show that $e^{A+B} = e^{A}e^{B}$ using $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

Part F: Solving IVPs with Matrix Exponential

26. Solve $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ using e^{At}.

27. Use the matrix exponential to solve $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

28. Find the solution to $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

29. Compute $\mathbf{x}(1)$ if $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x}$ and $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

30. **Challenge:** Show that the fundamental matrix $\Phi(t)$ for $\mathbf{x}' = A\mathbf{x}$ satisfies $\Phi(t) = e^{At}\Phi(0)$.

Solutions and Hints

Problem 1: $e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix}$

Problem 2: $A^2 = 0, so e^{At} = I + At = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

Problem 6: Eigenvalues are 4 and 2, eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Problem 11: $e^{At} = e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} \\ 0 & e^{3t} \end{pmatrix}$

Problem 13: $e^{At} = e^{2t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

Problem 16: $e^{At} = \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}$

Problem 26: First find $e^{At} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, then $\mathbf{x}(t) = e^{At}\mathbf{x}_0$

Key Strategy: Identify the matrix type first (diagonal, diagonalizable, Jordan, nilpotent, or complex eigenvalues), then apply the appropriate method.

Verification: Always check that $e^{A\cdot 0} = I$ and that your solution satisfies the differential equation.