Practice Problems: Lesson 10 - Qualitative Analysis

Master qualitative techniques without solving!

Part A: Basic Concepts (6 problems)

1. Draw the phase line for $\frac{dy}{dx} = y^2 - 4$ and classify all equilibria.
2. Using the derivative test, determine stability of equilibria for $\frac{dy}{dx} = \sin(y)$.
3. For $\frac{dy}{dx} = e^y - 1$, find the equilibrium and determine its stability without computing derivatives.
4. Explain why solutions to $\frac{dy}{dx} = y^2 + 1$ cannot have equilibria.
5. If $y_1(0) = 1$ and $y_2(0) = 2$ are solutions to $\frac{dy}{dx} = -y$, what is their relationship for all $x > 0$?
6. For $\frac{dy}{dx} = f(y)$ where f is strictly decreasing, prove all equilibria are stable.

Part B: Core Techniques (6 problems)

7. Analyze the complete phase portrait for $\frac{dy}{dx} = y^3 - y$.
8. For $\frac{dy}{dx} = (y-1)(y-2)(y-3)$, determine the long-term behavior of solutions with different initial conditions.
9. Find regions where solutions to $\frac{dy}{dx} = x^2 - y$ are concave up vs concave down.
10. Using comparison, prove that solutions to $\frac{dy}{dx} = -y + \sin(x)$ are bounded.
11. For $\frac{dy}{dx} = y - y^3$, find the inflection points of solution curves.
12. Determine all asymptotic behaviors for $\frac{dy}{dx} = y^2(1-y)$.

Part C: Applications (5 problems)

13. A population model follows $\frac{dP}{dt} = P(2-P)(P-1)$. Analyze long-term behavior for all initial populations.

14. For the Gompertz equation $\frac{dy}{dx} = y \ln(K/y)$ with $K > 0$, analyze stability without solving.

15. Show that $\frac{dy}{dx} = x + y^2$ has no periodic solutions.

16. Find a first integral for $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$.

17. For $\frac{dy}{dx} = -y + e^{-x}$, determine $\lim_{x\to\infty} y(x)$ without solving.

Part D: Advanced/Theoretical (5 problems)

18. Prove that if $H(x, y)$ is a first integral, then $\nabla H \perp (1, f(x, y))$ at each point.

19. Show that for $\frac{dy}{dx} = f(y)g(x)$ with $g(x) > 0$, stability is determined solely by f.

20. For what functions $f(y)$ does $\frac{dy}{dx} = f(y)$ have exactly three equilibria with alternating stability?

21. Prove that if all solutions to $\frac{dy}{dx} = f(x, y)$ are bounded, then f must change sign.

22. Characterize all ODEs of the form $\frac{dy}{dx} = f(x)g(y)$ that admit periodic solutions.

Part E: Exam-Style Questions (6 problems)

23. [**Prof. Ditkowski Special**] For $\frac{dy}{dx} = y^2 - xy$:
    a) Find all curves where solutions have horizontal tangents
    b) Determine regions of increasing/decreasing behavior
    c) Analyze concavity
    d) Describe all possible asymptotic behaviors

24. Given $\frac{dy}{dx} = (1 - y^2)(x^2 + 1)$:
    a) Find and classify all equilibria
    b) Prove all solutions are bounded
    c) Determine if periodic solutions exist
    d) Sketch the qualitative behavior

25. [**Multi-method**] For $\frac{dy}{dx} = y(1 - y)(2 - y)$:
    a) Draw the complete phase line
    b) Use linearization to confirm stability
    c) Find maximum growth rate locations
    d) Describe basin of attraction for each stable equilibrium

26. Without solving, prove that all solutions to $\frac{dy}{dx} = -x - y^3$ approach zero as $x \to \infty$.

27. [**Energy Method**] Show that $\frac{dy}{dx} = \frac{x^3 - xy^2}{x^2 y + y^3}$ has circular solution curves by finding an appropriate first integral.

28. A system satisfies $\frac{dy}{dx} = f(y)$ where f is continuous, $f(0) = f(1) = f(2) = 0$, $f'(0) > 0$, $f'(1) < 0$, $f'(2) > 0$.

    a) Sketch the phase line

    b) How many distinct asymptotic behaviors exist?

    c) What fraction of initial conditions lead to $y \to 1$?

    d) Can solutions oscillate?

Answer Key with Hints

Problem 1: Equilibria at $y = \pm 2$; $y = -2$ unstable, $y = 2$ stable
Problem 7: Three equilibria: $y = 0$ (unstable), $y = \pm 1$ (stable)
Problem 10: Use comparison with $\frac{dy}{dx} = -y \pm 1$
Problem 16: First integral: $H = x^2 y - y^3/3$
Problem 25: Use Lyapunov function $V = x^2 + y^2$
Problem 28c: Basin of $y = 1$ includes all $y \in (0, 2)$