ODE Lesson 32: Complex Eigenvalues - Real Solutions

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1 Complex Eigenvalues for Real Systems

Theorem 1 (Complex Conjugate Pairs)**.** *If A is a real $n \times n$ matrix and $\lambda = \alpha + i\beta$ is a complex eigenvalue with eigenvector $\mathbf{v} = \mathbf{p} + i\mathbf{q}$, then:*

- $\bar{\lambda} = \alpha - i\beta$ *is also an eigenvalue*
- $\bar{\mathbf{v}} = \mathbf{p} - i\mathbf{q}$ *is the corresponding eigenvector*

Euler's Formula - The Key Tool:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Therefore:

$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

Method 1 (Extracting Real Solutions)**.** *Given complex eigenvalue $\lambda = \alpha + i\beta$ with eigenvector $\mathbf{v} = \mathbf{p} + i\mathbf{q}$:*

Step 1: *Write the complex solution:*

$$\mathbf{x}_c(t) = e^{(\alpha+i\beta)t}(\mathbf{p} + i\mathbf{q})$$

Step 2: *Apply Euler's formula:*

$$\mathbf{x}_c(t) = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)][\mathbf{p} + i\mathbf{q}]$$

Step 3: *Expand and collect real and imaginary parts:*

$$\mathbf{x}_c(t) = e^{\alpha t}[(\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q}) \tag{1}$$
$$+ i(\sin(\beta t)\mathbf{p} + \cos(\beta t)\mathbf{q})] \tag{2}$$

Step 4: *Extract two real solutions:*

$$\mathbf{x}_1(t) = e^{\alpha t}[\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q}] \tag{3}$$
$$\mathbf{x}_2(t) = e^{\alpha t}[\sin(\beta t)\mathbf{p} + \cos(\beta t)\mathbf{q}] \tag{4}$$

Theorem 2 (General Real Solution)**.** *For a 2×2 system with complex eigenvalues $\lambda = \alpha \pm i\beta$:*

$$\mathbf{x}(t) = e^{\alpha t}[c_1(\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q}) + c_2(\sin(\beta t)\mathbf{p} + \cos(\beta t)\mathbf{q})]$$

where $\mathbf{p} = Re(\mathbf{v})$ and $\mathbf{q} = Im(\mathbf{v})$.

2 Standard Forms and Special Cases

Standard Matrix Form for Complex Eigenvalues: The matrix

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

has eigenvalues $\lambda = \alpha \pm i\beta$ with eigenvectors $\mathbf{v} = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$.

This represents:

- Rotation with angular velocity β
- Scaling with rate α
- Combined: logarithmic spiral

3 Complete Examples

Example 1 (Pure Rotation)**.** *Solve* $\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\mathbf{x}$ *with* $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution:

1. ***Eigenvalues:***

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -2 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 + 4 = 0$$

So $\lambda = \pm 2i$ *(pure imaginary* $\Rightarrow$ *center)*

2. ***Eigenvector for*** $\lambda = 2i$***:***

$$(A - 2iI)\mathbf{v} = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From row 1: $-2iv_1 - 2v_2 = 0 \Rightarrow v_2 = -iv_1$

Choose $v_1 = 1$*:* $\mathbf{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

So $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

3. ***Real Solutions:***

$$\mathbf{x}_1(t) = \cos(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(2t)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} \tag{5}$$

$$\mathbf{x}_2(t) = \sin(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos(2t)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix} \tag{6}$$

4. ***General Solution:***

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix}$$

5. ***Apply Initial Conditions:***

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

So $c_1 = 1$ and $c_2 = 0$.

6. ***Final Solution:***

$$\mathbf{x}(t) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$$

This is circular motion with radius 1 and period π!

Example 2 (Spiral Solution). *Solve* $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$

Solution: *This is already in standard form with $\alpha = 1$, $\beta = 2$.*

- *Eigenvalues:* $\lambda = 1 \pm 2i$
- *Since $\alpha = 1 > 0$: unstable spiral*
- *Angular frequency:* $\omega = 2$
- *Period of rotation:* $T = 2\pi/2 = \pi$

General solution:

$$\mathbf{x}(t) = e^t \left[c_1 \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix} \right]$$

Example 3 (3× 3 System with Complex Eigenvalues). *Consider* $\mathbf{x}' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \mathbf{x}$

Solution: *The eigenvalues are $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = -2$.*

For $\lambda = i$: eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$

Real solutions from complex pair:

$$\mathbf{x}_1(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} \sin t \\ -\cos t \\ 0 \end{pmatrix}$$

For $\lambda_3 = -2$: eigenvector $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

General solution:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \\ 0 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Phase Portrait Classification:

α	β	Behavior
$\alpha > 0$	$\beta \neq 0$	Unstable spiral (outward)
$\alpha < 0$	$\beta \neq 0$	Stable spiral (inward)
$\alpha = 0$	$\beta \neq 0$	Center (closed orbits)

The ratio $|\alpha/\beta|$ determines the "tightness" of the spiral:

- Small ratio: tight spiral (many rotations)
- Large ratio: loose spiral (few rotations)

Common Errors with Complex Eigenvalues:

- Forgetting to extract BOTH real and imaginary parts
- Using $e^{i\theta} = \sin\theta + i\cos\theta$ (wrong order!)
- Not including the $e^{\alpha t}$ scaling factor
- Trying to use complex eigenvectors directly in the real solution
- Missing the negative sign in conjugate eigenvalues

Prof. Ditkowski's exam patterns:

- Often uses matrices of form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$
- Asks about period of oscillation: $T = 2\pi/\beta$
- Tests understanding of spiral vs. center
- May ask for solution at specific times like $t = \pi/\beta$
- Loves IVPs that simplify nicely (e.g., $c_2 = 0$)

4 Connection to Linear Algebra

The matrix $A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ can be written as:

$$A = \alpha I + \beta J$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the $90°$ rotation matrix.

This decomposition shows:

- αI: uniform scaling
- βJ: rotation
- Combined: spiral motion

5 Quick Reference Table

Eigenvalues	Solution Form	Behavior
$\lambda = \pm i\beta$	$c_1 \cos(\beta t) + c_2 \sin(\beta t)$	Center
$\lambda = \alpha \pm i\beta$, $\alpha > 0$	$e^{\alpha t}[c_1 \cos(\beta t) + c_2 \sin(\beta t)]$	Unstable spiral
$\lambda = \alpha \pm i\beta$, $\alpha < 0$	$e^{\alpha t}[c_1 \cos(\beta t) + c_2 \sin(\beta t)]$	Stable spiral