Lesson 47: Practice Problems
Euler-Cauchy Equations

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Part A: Basic Euler Equations (5 problems)

1. Solve: t^{2y}" - 2ty' + 2y = 0fort ¿ 0
2. Solve: t^{2y}" + 3ty' + y = 0fort ¿ 0
3. Solve: t^{2y}" - ty' - 3y = 0fort ¿ 0
4. Solve: t^{2y}" + 4ty' + 2y = 0fort ¿ 0
5. Solve: t^{2y}" - 3ty' + 4y = 0fort ¿ 0

Part B: Repeated Roots Cases (5 problems)

1. Solve: t^{2y}" - 3ty' + 4y = 0fort ¿ 0$(verifyrepeatedroot)$
2. Solve: t^{2y}" + 5ty' + 4y = 0fort ¿ 0
3. Solve: t^{2y}" - ty' + y = 0fort ¿ 0
4. Solve: 4t^{2y}" + 8ty' + y = 0fort ¿ 0
5. Solve: t^{2y}" + 3ty' + y = 0fort ¿ 0

Part C: Complex Roots Cases (5 problems)

1. Solve: t^{2y}" - ty' + 5y = 0fort ¿ 0
2. Solve: t^{2y}" + ty' + y = 0fort ¿ 0
3. Solve: t^{2y}" - 3ty' + 13y = 0fort ¿ 0
4. Solve: t^{2y}" + ty' + 4y = 0fort ¿ 0
5. Solve: t^{2y}" + 3ty' + 5y = 0fort ¿ 0

Part D: Using the Transform Method (5 problems)

1. Using $x = \ln t$, transform and solve: t^{2y}" - 4ty' + 6y = 0
2. Transform and solve: t^{2y}" + 2ty' - 2y = 0
3. Show that the transform method gives the same solution as the direct method for: t^{2y}" - ty' - 8y = 0
4. Use the transform to solve: t^{2y}" + 5ty' + 3y = 0
5. Transform the third-order equation: t^{3y}"' + 3t^{2y}" - 2ty' + 2y = 0

Part E: Initial Value Problems (5 problems)

1. Solve: t^{2y}" - 2ty' + 2y = 0,y(1) = 3,y'(1) = 5
2. Solve: t^{2y}" + 3ty' + y = 0,y(1) = 2,y'(1) = -1
3. Solve: t^{2y}" - ty' + y = 0,y(1) = 0,y'(1) = 1
4. Solve: t^{2y}" - ty' + 5y = 0,y(1) = 1,y'(1) = 2
5. Solve: t^{2y}" - 3ty' + 4y = 0,y(2) = 8,y'(2) = 12

Part F: Exam-Style Problems (5 problems)

1. (Prof. Ditkowski style) Consider the equation t^{2y}" + aty' + by = 0.

    2. For what values of a and b are all solutions bounded as $t \to \infty$?
    3. For what values do solutions oscillate on a logarithmic scale?
    4. Find conditions for polynomial solutions.

The equation t^{2y}" - $2\alpha ty' + \alpha(\alpha + 1)y = 0$ has $y_1 =$ t$^{\alpha}$ as a solution.

(a) Verify this directly.
(b) Find the second solution.
(c) What is special about this equation?

Transform the equation $(t + 1)^{2y}$" + 3(t+1)y' + y = $0 to an Euler equation and solve.$

For the equation t^{2y}" + ty' + (t^2 - ν^2)y = $0 (Bessel's equation of order \nu)$:

(a) Show this is NOT a pure Euler equation.
(b) Find the indicial equation at $t = 0$.
(c) What are the indices?

Consider the system of Euler equations:

$$\begin{cases} t^{2y''} \text{ - 2ty' + 2y} = \text{z} \\ t^{2z''} \text{ - 4tz' + 6z} = \text{y} \end{cases}$$

(a) Find the general solution for y and z.

(b) Determine if solutions can remain bounded as $t \to 0^+$.

Solutions

Part A: Basic Euler Equations

1. Characteristic equation: $r(r-1) - 2r + 2 = 0$
   r^2 - 3r + 2 = 0
   (r-1)(r-2) = 0
   Roots :r = 1, 2
   Generalsolution :y = $c_{1t} + c_{2t}^2$

2. Characteristic equation: $r(r-1) + 3r + 1 = 0$
   r^2 + 2r + 1 = 0
   $(r+1)^2 = 0$
   Repeatedroot :r = -1
   Generalsolution :y = $c_1 + c_2 \ln t_{\frac{}{t}}$

3. Characteristic equation: $r(r-1) - r - 3 = 0$
r^2 - 2r - 3 = 0
(r-3)(r+1) = 0
Roots :r = 3, -1
Generalsolution :y = $c_{1t}^3 + c_{2\frac{}{t}}$

Characteristic equation: $r(r-1) + 4r + 2 = 0$
r^2 + 3r + 2 = 0
(r+1)(r+2) = 0
Roots :r = -1, -2
Generalsolution :y = $c_{1\frac{}{t+c}2\frac{}{t}}^2$

Characteristic equation: $r(r-1) - 3r + 4 = 0$
r^2 - 4r + 4 = 0
$(r-2)^2 = 0$
Repeatedroot :r = 2
Generalsolution :y = $(c_1 + c_2 \ln t)t^2$

Part B: Repeated Roots Cases

3. Already solved in Part A, 5:
   $r = 2$ (double root)
   General solution: $y = (c_1 + c_2 \ln t)t^2$

4. Characteristic equation: r^2 + 4r + 4 = 0
$(r+2)^2 = 0$
Repeatedroot :r = -2
Generalsolution :y = $c_1 + c_2 \ln t_{\frac{}{t}}^2$

5. Characteristic equation: r^2 - 2r + 1 = 0
$(r-1)^2 = 0$

$Repeated root$:r = 1
$General solution$:y = (c_1 + $c_2 \ln t)t$

6. Divide by 4: $t^{2y"}$ + 2ty' + $1_{\overline{4y=0}}$
   Characteristic equation: r^2 + r + $1_{\overline{4=0}}$
   $(r + \frac{1}{2})^2 = 0$
   $Repeated root$:r = -$1_{\overline{2}}$
   General solution: $y = c_1$ + $c_2 \ln t_{\overline{\sqrt{t}}}$

   Characteristic equation: r^2 + 2r + 1 = 0
   $(r+1)^2 = 0$
   $Repeated root$:r = -1
   $General solution$:y = c_1 + $c_2 \ln t_{\overline{t}}$

Part C: Complex Roots Cases

3. Characteristic equation: r^2 - 2r + 5 = 0
   r = 2 $\pm\sqrt{4-20}_{\overline{2=1\pm 2i}}$
   General solution: $y = t[c_1 \cos(2\ln t) + c_2 \sin(2\ln t)]$

4. Characteristic equation: r^2 + 1 = 0
   r = $\pm i$
   General solution: $y = c_1 \cos(\ln t) + c_2 \sin(\ln t)$

5. Characteristic equation: r^2 - 4r + 13 = 0
   r = 4 $\pm\sqrt{16-52}_{\overline{2=2\pm 3i}}$
   General solution: $y = t^2[c_1 \cos(3\ln t) + c_2 \sin(3\ln t)]$

6. Characteristic equation: r^2 + 4 = 0
   r = $\pm 2i$
   General solution: $y = c_1 \cos(2\ln t) + c_2 \sin(2\ln t)$

7. Characteristic equation: r^2 + 2r + 5 = 0
   r = -2 $\pm\sqrt{4-20}_{\overline{2=-1\pm 2i}}$
   General solution: $y = \frac{1}{t}[c_1 \cos(2\ln t) + c_2 \sin(2\ln t)]$

Part D: Using the Transform Method

3. Let $x = \ln t$, $v(x) = y(t)$
   Transform: $v'' - 5v' + 6v = 0$
   Characteristic: λ^2 - $5\lambda + 6 = 0$
   $(\lambda - 2)(\lambda - 3) = 0$
   $v = c_{1e}{}^{2x} + c_{2e}{}^{3x}$
   Back-transform: $y = c_{1t}{}^2 + c_{2t}{}^3$

4. Transform: $v'' + v' - 2v = 0$
   Characteristic: λ^2 + $\lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$
$v = c_{1e}{}^{-2x} + c_{2e}{}^{x}$
Back-transform: $y = c_{1\frac{}{t}}{}^{2}$ + c_{2t}

Direct method: r^2 - 2r - 8 = 0 $\Rightarrow r = 4, -2$
Transform method: $v'' - 2v' - 8v = 0$
Same characteristic equation!
Solution: $y = c_{1t}{}^{4} + c_{2\frac{}{t}}{}^{2}$

Transform: $v'' + 4v' + 3v = 0$
$(\lambda + 1)(\lambda + 3) = 0$
$v = c_{1e}{}^{-x} + c_{2e}{}^{-3x}$
Back-transform: $y = c_{1\frac{}{t+c}2\frac{}{t}}{}^{3}$

Let $D = d/dx$. Transform gives:
(D^3 - 3D^2 + 2D) + 3(D^2 - D) - 2D + 2 = 0
D^3 - D - 2D + 2 = D^3 - 3D + 2 = 0
Characteristic :λ^3 - $3\lambda + 2 = 0$
$(\lambda - 1)^2(\lambda + 2) = 0$
$v = (c_1$ + c$_{2x}$)e^x + c$_{3e}{}^{-2x}$
$y = (c_1$ + c$_2\ln t)t + c_{3\frac{}{t}}{}^{2}$

Part E: Initial Value Problems

3. From Part A 1: $y = c_{1t}$ + c$_{2t}{}^{2}$
   $y(1) = c_1$ + c$_2$ = 3
   y'(t) = c$_1$ + 2c$_{2t}$
   y'(1) = c$_1$ + 2c$_2$ = 5
   Solving :c$_2$ = 2,c$_1$ = 1
   Solution :y = t + 2t^2

4. From Part A 2: $y = c_1$ + c$_2\ln t_{\frac{}{t}}$
   $y(1) = c_1 = 2$
   y'(t) = c$_2$ - c$_1$ - c$_2\ln t_{\frac{}{t}}{}^{2}$
   y'(1) = c$_2$ - c$_1$ = -1
   c$_2$ = 1
   Solution :y = 2 + $\ln t_{\frac{}{t}}$

   From Part B 8: $y = (c_1$ + c$_2\ln t)t$
   $y(1) = c_1 = 0$
   y'(t) = c$_1$ + c$_2$(1 + $\ln t$)
   $y'(1) = c_2 = 1$
   Solution :y = t$\ln t$

   From Part C 11: $y = t[c_1\cos(2\ln t) + c_2\sin(2\ln t)]$
   $y(1) = c_1 = 1$
   y'(t) = $[c_1\cos(2\ln t) + c_2\sin(2\ln t)] + t[-2c_1\sin(2\ln t)/t + 2c_2\cos(2\ln t)/t]$
   $y'(1) = 1 + 2c_2 = 2$

$c_2 = 1\frac{}{2}$
Solution: $y = t[\cos(2\ln t) + \frac{1}{2}\sin(2\ln t)]$

From Part A 5: $y = (c_1 + c_2 \ln t)t^2$
$y(2) = 4c_1 + 4c_2 \ln 2 = 8$
$y'(t) = 2(c_1 + c_2 \ln t)t + c_{2t}$
$y'(2) = 8c_1 + 8c_2 \ln 2 + 2c_2 = 12$
$From first: c_1 + c_2 \ln 2 = 2$
From second: $4c_1 + 4c_2 \ln 2 + c_2 = 6$
$c_2 = -2, c_1 = 2 + 2\ln 2$
Solution: $y = [2 + 2\ln 2 - 2\ln t]t^2 = 2t^2[1 + \ln(4/t)]$

Part F: Exam-Style Problems

3. (a) Bounded as $t \to \infty$: Need both roots to have negative real parts.
   From $r^2 + (a-1)r + b = 0$: $Need a ¿ 1 and b ¿ 0 (and (a-1)^2 ¿ 4b for real roots)$.

(b) Oscillation: Need complex roots, so $(a-1)^2$ ¡ 4b.

(c) Polynomial solutions: Need positive integer roots. For example, $r = n$ requires $n^2 + (a-1)n + b = 0$.

4. (a) $y_1' = \alpha t^{\alpha-1}$, $y_1'' = \alpha(\alpha-1)t^{\alpha-2}$
   Substitute: $\alpha(\alpha-1) - 2\alpha \cdot \alpha + \alpha(\alpha+1) = 0$ ✓

   (b) Characteristic equation: $r^2 - (2\alpha+1)r + \alpha(\alpha+1) = 0$
   $(r-\alpha)(r-(\alpha+1)) = 0$
   Second solution: $y_2 = t^{\alpha+1}$

   (c) This is the Euler equation whose solutions are consecutive powers of t.

5. Let $s = t + 1$, then the equation becomes:
   $s^{2y''} + 3sy' + y = 0 (standard Euler in s)$
   $Characteristic: r^2 + 2r + 1 = (r+1)^2 = 0$
   $y = c_1 + c_2 \ln(t+1)\frac{}{t+1}$

6. (a) The $t^2 in the last term makes this NOT a pure Euler equation.$

(b) For the indicial equation, consider $y = t^r(1 + a_{1t} + \dots)$
$Leading terms give: r(r-1) + r - \nu^2 = 0$
$r^2 - \nu^2 = 0$

(c) Indices: $r = \pm\nu$

7. (a) Decouple: Fourth-order equations result.
   For y: $(t^{2D2} - 2tD + 2)(t^{2D2} - 4tD + 6) - 1 = 0$
   This gives $t^{4y(4)} - 6t^{3y'''} + 15t^{2y''} - 15ty' + (12-1)y = 0$
   $Complex calculation yields general solutions.$

(b) As $t \to 0^+$: Solutions behave like $t^r where r are the characteristic roots. Bounded only if all$ Re(r) ≥ 0.