Lesson 18: Bernoulli Differential Equations

ODE 1 - Prof. Adi Ditkowski

1 Definition and Recognition

Definition 1 (Bernoulli Equation)**.** *A Bernoulli differential equation has the form:*

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

where $p(x)$ and $q(x)$ are continuous functions and $n \neq 0, 1$.

Why $n \neq 0, 1$?

- If $n = 0$: $y' + p(x)y = q(x)$ is already linear
- If $n = 1$: $y' + p(x)y = q(x)y$ gives $y' = y(q(x) - p(x))$, which is separable
- For all other n: The equation is genuinely nonlinear and requires the Bernoulli transformation

2 The Bernoulli Transformation

The Power Substitution Method:

1. Given: $\frac{dy}{dx} + p(x)y = q(x)y^n$ with $n \neq 0, 1$
2. Substitute: $v = y^{1-n}$
3. Then: $y = v^{\frac{1}{1-n}}$ and $\frac{dy}{dx} = \frac{1}{1-n}v^{\frac{n}{1-n}} \cdot \frac{dv}{dx}$
4. Alternatively: $\frac{dv}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx}$
5. The transformed equation becomes:
$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$
6. This is linear in v! Solve using integrating factor method

7. Back-substitute: $y = v^{\frac{1}{1-n}}$

Critical Algebraic Steps: Starting from $v = y^{1-n}$:

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx} \tag{1}$$
$$\frac{dy}{dx} = \frac{y^n}{1-n}\frac{dv}{dx} \tag{2}$$
$$\tag{3}$$

Substituting into the original equation and multiplying by $(1-n)y^{-n}$ gives the linear form.

3 Detailed Derivation

Theorem 1 (Bernoulli to Linear Transformation)**.** *The substitution $v = y^{1-n}$ transforms the Bernoulli equation into a linear equation.*

Proof:

$$\text{Given: } y' + p(x)y = q(x)y^n \tag{4}$$
$$\text{Let: } v = y^{1-n} \tag{5}$$
$$\text{Then: } \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx} \tag{6}$$
$$\text{So: } \frac{dy}{dx} = \frac{y^n}{1-n}\frac{dv}{dx} \tag{7}$$
$$\text{Substitute: } \frac{y^n}{1-n}\frac{dv}{dx} + p(x)y = q(x)y^n \tag{8}$$
$$\text{Multiply by } \frac{1-n}{y^n}: \frac{dv}{dx} + (1-n)p(x)y^{1-n} = (1-n)q(x) \tag{9}$$
$$\text{Since } v = y^{1-n}: \frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x) \quad \square \tag{10}$$

4 Standard Examples

Example 1 (Basic Bernoulli)**.** *Solve: $y' + \frac{2y}{x} = x^2y^3$*

Solution:

1. *Identify: $p(x) = \frac{2}{x}$, $q(x) = x^2$, $n = 3$*
2. *Substitute: $v = y^{1-3} = y^{-2}$*
3. *Transform: $\frac{dv}{dx} + (1-3)\frac{2v}{x} = (1-3)x^2$*
4. *Simplify: $\frac{dv}{dx} - \frac{4v}{x} = -2x^2$*

5. *Integrating factor:* $\mu = e^{\int 4/x \, dx} = x^4$

6. *Multiply:* $x^4 \frac{dv}{dx} - 4x^3 v = -2x^6$

7. *Integrate:* $x^4 v = -\frac{2x^7}{7} + C$

8. *Solve for v:* $v = -\frac{2x^3}{7} + \frac{C}{x^4}$

9. *Back-substitute:* $y^{-2} = -\frac{2x^3}{7} + \frac{C}{x^4}$

10. *Final:* $y = \pm \left(-\frac{2x^3}{7} + \frac{C}{x^4}\right)^{-1/2}$

Example 2 (Logistic Equation)**.** *Solve:* $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ *where* $r, K > 0$

Solution:

1. *Expand:* $\frac{dP}{dt} = rP - \frac{r}{K}P^2$

2. *Rearrange:* $\frac{dP}{dt} - rP = -\frac{r}{K}P^2$ *(Bernoulli with* $n = 2$*)*

3. *Substitute:* $v = P^{1-2} = P^{-1} = \frac{1}{P}$

4. *Transform:* $\frac{dv}{dt} + rv = \frac{r}{K}$

5. *Solve linear equation:* $v = \frac{1}{K} + Ce^{-rt}$

6. *Back-substitute:* $P = \frac{1}{v} = \frac{K}{1+CKe^{-rt}}$

7. *With initial condition* $P(0) = P_0$*:* $P(t) = \frac{K}{1+\left(\frac{K-P_0}{P_0}\right)e^{-rt}}$

5 Special Cases and Variations

Common Values of n and Their Applications:

n	Application	Substitution
2	Logistic growth, Riccati connection	$v = y^{-1}$
3	Certain chemical reactions	$v = y^{-2}$
1/2	Fluid dynamics, heat transfer	$v = y^{1/2}$
-1	Inverse relationships	$v = y^2$
-2	Gravitational problems	$v = y^3$

Prof. Ditkowski's Exam Patterns:

- Often disguises Bernoulli equations - practice recognition
- Likes fractional powers: $n = 1/2, 3/2, -1/2$
- May combine with initial conditions

- Tests the connection to logistic growth
- Sometimes asks for qualitative behavior without full solution
- Partial credit for correct substitution identification

6 Recognition Flowchart

The key steps for recognizing and solving Bernoulli equations:

1. Check if the equation has form $y' + p(x)y = q(x)y^n$
2. If $n = 0$: Linear equation, use integrating factor
3. If $n = 1$: Separable equation
4. Otherwise: Apply Bernoulli substitution $v = y^{1-n}$

7 Solution Quality Check

Verification Steps:

1. Check that your transformed equation is truly linear
2. Verify the integrating factor calculation
3. Ensure back-substitution is algebraically correct
4. Test with initial conditions if given
5. Check for lost solutions when $y = 0$