Lesson 42: Practice Problems

Lyapunov Functions and Global Stability

ODE 1 - Prof. Adi Ditkowski

Part A: Verifying Lyapunov Functions (5 problems)

1. Verify that $V(x, y) = x^2 + 2y^2$ is a Lyapunov function for:

$$\dot{x} = -x + y^2, \quad \dot{y} = -x - y$$

What can you conclude about stability? **Solution Hint:** Compute $\dot{V} = 2x\dot{x} + 4y\dot{y}$. Check sign near origin.

2. Given $V(x, y) = x^4 + y^4$, determine if it's a valid Lyapunov function for:

$$\dot{x} = -x^3, \quad \dot{y} = -y^3$$

Solution Hint: Check all three conditions. Note $\dot{V} = -4(x^6 + y^6)$.

3. For the system $\dot{x} = y$, $\dot{y} = -x - y$, show that $V = x^2 + xy + y^2$ is a Lyapunov function. **Solution Hint:** First verify $V > 0$ using completing the square.

4. Consider $V(x, y) = \frac{1}{2}(x^2 + y^2) + x^2y^2$. For what systems could this be a Lyapunov function? **Solution Hint:** Work backwards: what f, g make $\dot{V} \leq 0$?

5. Verify whether $V = e^{x^2+y^2} - 1$ satisfies the conditions for a Lyapunov function near the origin for:

$$\dot{x} = -x, \quad \dot{y} = -y$$

Solution Hint: Check $V(0, 0) = 0$ and positivity. Compute $\dot{V}$ carefully.

Part B: Computing $\dot{V}$ (6 problems)

6. For $V = x^2 + y^2$ and the system:

$$\dot{x} = -x + x^2y, \quad \dot{y} = -2y + xy^2$$

compute $\dot{V}$ and determine stability. **Solution Hint:** $\dot{V} = 2x\dot{x} + 2y\dot{y}$. Factor carefully.

7. Given $V = ax^2 + cy^2$, find conditions on a, c for stability of:

$$\dot{x} = -x + y, \quad \dot{y} = -x - y$$

Solution Hint: Compute $\dot{V}$ in terms of a, c. When is it negative definite?

8. For the Van der Pol oscillator with $\mu > 0$:

$$\dot{x} = y, \quad \dot{y} = -x + \mu(1 - x^2)y$$

Show that $V = x^2 + y^2$ gives $\dot{V} = 2\mu(1 - x^2)y^2$. **Solution Hint:** Direct computation. Note sign depends on x^2 vs 1.

9. Compute $\dot{V}$ for $V = \frac{1}{2}y^2 + (1 - \cos x)$ along trajectories of:

$$\dot{x} = y, \quad \dot{y} = -\sin x$$

Solution Hint: This is the undamped pendulum. What does $\dot{V} = 0$ mean?

10. For $V = x^2 - 2xy + 2y^2$ and:

$$\dot{x} = -x + y, \quad \dot{y} = -x - y$$

compute $\dot{V}$ and interpret. **Solution Hint:** Careful with the cross term $-2xy$.

11. Given a general quadratic $V = ax^2 + bxy + cy^2$, derive the formula for $\dot{V}$ in terms of $f(x, y)$ and $g(x, y)$. **Solution Hint:** $\dot{V} = (2ax + by)f + (bx + 2cy)g$.

Part C: Constructing Lyapunov Functions (5 problems)

12. Find a Lyapunov function for:

$$\dot{x} = -x^3, \quad \dot{y} = -y^3$$

Solution Hint: Try $V = x^{2n} + y^{2n}$ for appropriate n.

13. Construct a Lyapunov function to prove stability of:

$$\dot{x} = -x + y^2, \quad \dot{y} = -y + x^2$$

when x, y are small. **Solution Hint:** Try $V = ax^2 + cy^2$ with $a \neq c$.

14. For the gradient system with potential $U = x^4 + y^4 - x^2 - y^2$:

$$\dot{x} = -\frac{\partial U}{\partial x}, \quad \dot{y} = -\frac{\partial U}{\partial y}$$

find a Lyapunov function for the equilibrium at origin. **Solution Hint:** Shift U so minimum is at origin.

15. Find a Lyapunov function for the damped oscillator:

$$\dot{x} = y, \quad \dot{y} = -\omega^2 x - 2\zeta\omega y$$

with $\omega > 0$ and $\zeta > 0$. **Solution Hint:** Try energy-like: $V = \frac{1}{2}(\omega^2 x^2 + y^2)$.

16. Construct a Lyapunov function for:

$$\dot{x} = y - x^3, \quad \dot{y} = -x - y^3$$

Solution Hint: Try $V = x^2 + y^2$ first. If that fails, try $V = ax^2 + y^2$.

Part D: LaSalle's Principle (4 problems)

17. For the system:
$$\dot{x} = -y^3, \quad \dot{y} = x^3$$
with $V = x^4 + y^4$:
    (a) Show $\dot{V} = 0$
    (b) Apply LaSalle's principle
    (c) What can you conclude about stability?

    Solution Hint: $E = \mathbb{R}^2$. Check invariant sets.

18. Consider:
$$\dot{x} = y, \quad \dot{y} = -x - y^3$$
with $V = x^2 + y^2$:
    (a) Compute $\dot{V}$
    (b) Find the set E where $\dot{V} = 0$
    (c) Determine the largest invariant set in E
    (d) State stability conclusion

    Solution Hint: $\dot{V} = -2y^4 \leq 0$. $E = \{y = 0\}$.

19. For the pendulum with friction:
$$\dot{x} = y, \quad \dot{y} = -\sin x - cy$$
Use energy $V = (1 - \cos x) + \frac{1}{2}y^2$ and LaSalle to prove asymptotic stability. **Solution Hint:** $\dot{V} = -cy^2$. On $\{y = 0\}$, only equilibria are invariant.

20. Apply LaSalle to:
$$\dot{x} = -x^3 + xy^2, \quad \dot{y} = -y^3 + x^2 y$$
with $V = x^2 + y^2$. **Solution Hint:** Find where $(x^2 - y^2)^2 = 0$.

Part E: Global Stability (5 problems)

21. Show that the origin is globally asymptotically stable for:
$$\dot{x} = -x - x^3, \quad \dot{y} = -y - y^3$$
Solution Hint: Use $V = x^2 + y^2$. Check radial unboundedness.

22. For the system:
$$\dot{x} = -x + y^2, \quad \dot{y} = -y + x^2$$
determine if the origin is globally asymptotically stable. **Solution Hint:** Check if other equilibria exist. Can't be global if there are others.

23. Prove global asymptotic stability for:

$$\dot{x} = -x^3 - xy^2, \quad \dot{y} = -x^2 y - y^3$$

Solution Hint: Try $V = x^2 + y^2$. Show $\dot{V} < 0$ except at origin.

24. Consider the scaled system:

$$\dot{x} = -f(x), \quad \dot{y} = -g(y)$$

where f, g are odd functions with $xf(x) > 0$ for $x \neq 0$. Prove global stability. **Solution Hint:** Use $V = \int_0^x f(s)ds + \int_0^y g(s)ds$.

25. Determine the basin of attraction for:

$$\dot{x} = -x + x^3, \quad \dot{y} = -y$$

using $V = x^2 + y^2$. **Solution Hint:** Find where $\dot{V} < 0$. Note $x^2 < 1$ needed.

Part F: Exam-Style Problems (5 problems)

26. [**Prof. Ditkowski Style**] Consider the system:

$$\dot{x} = -x^3 + 2xy^2, \quad \dot{y} = -2x^2 y - y^3$$

   (a) Verify that $V = x^2 + y^2$ is a Lyapunov function
   (b) Compute $\dot{V}$ explicitly
   (c) Is the origin locally or globally asymptotically stable?
   (d) Find the basin of attraction

Solution Hint: $\dot{V} = -2(x^2 + y^2)^2 < 0$. Global stability follows.

27. [**Comprehensive**] For the nonlinear oscillator:

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$$

   (a) Rewrite as a first-order system
   (b) Show the origin is the only equilibrium
   (c) Using $V = x^2 + y^2$, determine stability
   (d) What happens on the unit circle $x^2 + y^2 = 1$?

Solution Hint: This is a Van der Pol type equation. Check $\dot{V}$ sign.

28. [**Method Comparison**] Consider:

$$\dot{x} = y + x^3, \quad \dot{y} = -x + y^3$$

(a) Show linearization is inconclusive
(b) Try $V = x^2 + y^2$ as a Lyapunov function
(c) What does this tell us about stability?
(d) Find a different V that proves instability

Solution Hint: Linearization gives $\pm i$. First V gives $\dot{V} = 2(x^4 + y^4) > 0$.

29. [**Physical System**] A particle moves according to:

$$\dot{x} = y, \quad \dot{y} = -\nabla U(x)$$

where $U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$.

(a) Find all equilibria
(b) Classify each using linearization where possible
(c) Use energy methods to determine stability
(d) Sketch the phase portrait

Solution Hint: Three equilibria. Use $V = \frac{1}{2}y^2 + U(x)$ shifted appropriately.

30. [**Challenge**] Consider the coupled system:

$$\dot{x} = y - x^3, \quad \dot{y} = -x - y + y^2$$

(a) Find all equilibria
(b) For the origin, try to construct a Lyapunov function
(c) If $V = ax^2 + bxy + cy^2$, find conditions for stability
(d) Can you prove global results?

Solution Hint: Need to handle the y^2 term carefully. Try $V = x^2 + \alpha xy + \beta y^2$.

Part G: Additional Practice (5 problems)

31. Show that for any positive definite matrix P, $V = \mathbf{x}^T P \mathbf{x}$ is a valid Lyapunov function form.

32. For the Lorenz system restricted to the x-y plane:

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y$$

find a Lyapunov function when $0 < r < 1$.

33. Consider the "reversed" gradient system:

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

Show that H is constant along trajectories. What does this mean for stability?

34. Prove that if V is a Lyapunov function with $\dot{V} < -\alpha V$ for some $\alpha > 0$, then the origin is exponentially stable.

35. [**Research Connection**] For the system:

$$\dot{x} = -x + y^2, \quad \dot{y} = -y + x^2$$

   (a) Show there are exactly 3 equilibria

   (b) Find a common Lyapunov function for all three

   (c) What does this imply about basins of attraction?

Key Strategies for Prof. Ditkowski's Exam:

- If given V, always verify ALL three conditions
- Remember: $\dot{V} = \nabla V \cdot \mathbf{f}$, NOT $\partial V / \partial t$
- When $\dot{V} \leq 0$, always apply LaSalle
- For global stability, check for other equilibria first
- Standard forms: $V = x^2 + y^2$ or $V = ax^2 + cy^2$
- Energy/gradient systems have natural Lyapunov functions
- State conclusions explicitly: ”locally/globally asymptotically stable”