Lesson 16: Practice Problems - Variation of Constants

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Part A: Basic Variation of Constants (5 Problems)

Problem 1. *Solve* $y' - 2y = e^{4t}$ *using variation of constants.*

Problem 2. *Solve* $y' + y = \sin(2t)$ *using variation of constants.*

Problem 3. *Solve* $y' + 3y = t^2$ *using variation of constants.*

Problem 4. *Solve* $y' - \frac{1}{t}y = t^3$ *for* $t > 0$ *using variation of constants.*

Problem 5. *Solve* $y' + 4y = e^{-4t}$ *using variation of constants. Note the resonance!*

Part B: Non-Standard Forcing Functions (6 Problems)

Problem 6. *Solve* $y' - y = e^t \ln(t)$ *for* $t > 0$.

Problem 7. *Solve* $y' + 2y = \frac{\sin(t)}{t}$ *for* $t > 0$.

Problem 8. *Solve* $y' + \frac{1}{t}y = \cos(\ln(t))$ *for* $t > 0$.

Problem 9. *Solve* $y' - 3y = e^{3t}t^2$ *(resonance with polynomial).*

Problem 10. *Solve* $y' + y = \frac{1}{1+e^t}$.

Problem 11. *Solve* $y' + ty = te^{-t^2/2}$.

Part C: Initial Value Problems (5 Problems)

Problem 12. *Solve* $y' - 2y = e^t$, $y(0) = 3$.

Problem 13. *Solve* $y' + \frac{2}{t}y = t^2$, $y(1) = 2$ *for* $t > 0$.

Problem 14. *Solve* $y' + 3y = e^{-3t}\cos(t)$, $y(0) = 0$.

Problem 15. *Solve* $y' - y = \frac{1}{1+e^{-t}}$, $y(0) = 1$.

Problem 16. *Solve* $y' + (\tan t)y = \sec t$, $y(0) = 1$ *for* $-\pi/2 < t < \pi/2$.

Part D: Comparison with Other Methods (5 Problems)

Problem 17. *Solve $y' + 2y = 3e^{-2t}$ using: a) Integrating factor method b) Variation of constants c) Show the solutions are identical*

Problem 18. *For $y' - y = e^t$: a) Explain why undetermined coefficients fails b) Solve using variation of constants c) Interpret the resonance physically*

Problem 19. *Solve $y' + p(t)y = g(t)$ where:*

$$p(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 2 & t \geq 1 \end{cases}, \quad g(t) = 1$$

Use variation of constants on each interval.

Problem 20. *Show that for $y' + p(t)y = g(t)$, the variation of constants formula and integrating factor formula give the same result.*

Problem 21. *Solve $ty' - y = t^2 e^t$ for $t > 0$: a) First convert to standard form b) Apply variation of constants c) Verify the solution*

Part E: Theoretical Problems (4 Problems)

Problem 22. *Prove that if y_1 and y_2 are solutions to $y' + p(t)y = g_1(t)$ and $y' + p(t)y = g_2(t)$ respectively, then $y_1 + y_2$ solves $y' + p(t)y = g_1(t) + g_2(t)$.*

Problem 23. *Show that the variation of constants formula:*

$$y_p = y_h(t) \int \frac{g(t)}{y_h(t)} dt$$

can be written as:

$$y_p(t) = \int_{t_0}^{t} \frac{y_h(t)}{y_h(s)} g(s) ds$$

Interpret this as a Green's function representation.

Problem 24. *Prove that for the equation $y' + p(t)y = g(t)$, if $p(t)$ and $g(t)$ are continuous on $[a, b]$, then the solution exists and is unique on the entire interval.*

Problem 25. *Derive the variation of constants formula starting from the integrating factor approach. Show all steps.*

Part F: Exam-Style Problems (5 Problems)

Problem 26 (15 points). *Consider the equation* $y' + \frac{2t}{1+t^2}y = \frac{1}{1+t^2}$.

1. *Find the homogeneous solution*
2. *Use variation of constants to find the general solution*
3. *Find the solution satisfying* $y(0) = 1$
4. *Determine the behavior as* $t \to \infty$

Problem 27 (20 points). *The current* $i(t)$ *in an RL circuit satisfies:*

$$L\frac{di}{dt} + Ri = V(t)$$

where $L = 1$ *H,* $R = 2\ \Omega$*, and* $V(t) = 10e^{-2t}\sin(3t)$ *V.*

1. *Convert to standard form*
2. *Find the steady-state current using variation of constants*
3. *If* $i(0) = 0$*, find the complete solution*
4. *Identify transient and steady-state components*

Problem 28 (15 points). *For the equation* $y' - \frac{n}{t}y = t^m$ *where* n, m *are constants and* $t > 0$*:*

1. *Find the general solution using variation of constants*
2. *For what values of* n *and* m *does the solution remain bounded as* $t \to \infty$*?*
3. *Find the particular solution that remains finite at* $t = 0$ *when possible*

Problem 29 (20 points). *Consider* $y' + y = f(t)$ *where:*

$$f(t) = \begin{cases} 1 & 0 \leq t < \pi \\ \sin(t) & t \geq \pi \end{cases}$$

1. *Solve on* $[0, \pi)$ *with* $y(0) = 0$
2. *Use continuity at* $t = \pi$ *to find initial condition for* $t \geq \pi$
3. *Solve on* $[\pi, \infty)$
4. *Sketch the solution*

Problem 30 (15 points - Theoretical).

1. *State the variation of constants formula for* $y' + p(t)y = g(t)$
2. *Prove that this formula gives a particular solution*
3. *Show that if* $g(t) = g_1(t) + g_2(t)$*, then* $y_p = y_{p1} + y_{p2}$
4. *Explain why variation of constants works for any continuous* $g(t)$

Solution Hints:

- Problem 5: Resonance occurs when forcing matches natural frequency
- Problem 10: Use substitution $u = 1 + e^t$
- Problem 11: The coefficient t suggests $y_h = Ce^{-t^2/2}$
- Problem 20: Convert using $p(t) = 2t/(1 + t^2)$, notice $\int p(t)dt = \ln(1 + t^2)$
- Problem 24: Use piecewise approach with continuity condition