Lesson 44: Practice Problems
Repeated Roots and $t^k Terms$

ODE 1 with Prof. Adi Ditkowski

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Part A: Identifying Multiplicities (5 problems)

1. Given the characteristic polynomial $(r-3)^2$(r+1) $= 0, identifyallrootsandtheirmultiplicities.$

2. Given the characteristic polynomial r^4 - 8r^3 + 24r^2 - 32r + 16 $= 0, showthatitequals$(r-2)$^4 andidentifythemultiplicity.$

3. The characteristic equation $(r^2$ - 2r + 5)$^2 = 0 hascomplexroots.Findthemandtheirmultiplicities.$

4. Factor completely: r^3 + 3r^2 + 3r + 1 $= 0 andidentifymultiplicities.$

5. Given $(r^2$ + 4)$^3 = 0, identifyallrootsandmultiplicities.$

Part B: Solutions from Multiplicities (5 problems)

1. If $r = 3$ is a triple root, write all three linearly independent solutions.

2. If $r = -2$ has multiplicity 4, write the form of the general solution.

3. Complex roots $1 \pm 2i$ each have multiplicity 2. Write all four real solutions.

4. The characteristic polynomial is $(r + 1)^2$(r-2)$^3.Writethegeneralsolution.$

5. If $\pm 3i$ are both double roots, write the general real solution.

Part C: Complete ODEs with Repeated Roots (5 problems)

6. Solve: $y'' - 6y' + 9y = 0$

7. Solve: $y''' + 3y'' + 3y' + y = 0$

8. Solve: $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$

9. Solve: $y'' + 4y' + 4y = 0$ with $y(0) = 2, y'(0) = -3$
10. Solve: $y^{(4)} + 4y'' + 4y = 0$

Part D: Reduction of Order Applications (5 problems)

11. Given that $y_1 = \mathrm{e}^{-t}$ solves $(D+1)^2$[y] $= 0, usereductionofordertofind$y$_2$.
12. Verify that te^{3t} solves $y'' - 6y' + 9y = 0$ by direct substitution.
13. Show that if $y_1 = \mathrm{e}^{rt}$ solves $L[y] = 0$ where r is a double root, then $y_2 = \mathrm{te}^{rt}$ also solves it.
14. Given $y'' - 4y' + 4y = 0$ has solution e^{2t}, find the second solution using $y_2 =$ v(t)e^{2t}.
15. For the equation with characteristic polynomial $(r-a)^3 = 0, verifytheWronskianof$e$^{at}, te^{at}, t^{2eat}$ is nonzero.

Part E: Complex Repeated Roots (5 problems)

16. Solve: $y^{(4)} + 8y'' + 16y = 0$
17. Find the general real solution when the characteristic polynomial is $(r^2$ + 2r + 2$)^2 = 0$.
18. Solve: $y^{(4)} + 4y''' + 8y'' + 8y' + 4y = 0$ (Hint: $(r^2$ + 2r + 2$)^2$)
19. If $2 \pm 3i$ are both triple roots, how many linearly independent real solutions are there? List them.
20. Solve the IVP: $y'' + 2y' + 5y = 0$ with $y(0) = 0, y'(0) = 4$, given that $-1 \pm 2i$ are roots.

Part F: Theoretical and Exam-Style Problems (5 problems)

21. Prove that if $r_0 isarootofmultiplicity$m$ofthecharacteristicpolynomial, then$p(r$_0$) = p'(r$_0$) $= \cdots = p^{(m-1)}(r_0) = 0but$p$^{(m)}(r_0) \neq 0$.
22. Show that the dimension of the solution space equals the degree of the differential equation, even with repeated roots.
23. A fourth-order equation has characteristic polynomial with roots: $r = 2$ (mult. 2), $r = -1$ (mult. 1), $r = 3$ (mult. 1). If a solution satisfies $y(0) = 1, y'(0) = 0, y''(0) = -1, y'''(0) = 2$, find it.
24. Explain why the functions $\{t^k \ \mathrm{e}^{rt} : k = 0, 1, \ldots, m-1\}$ span the solution space for a root of multiplicity m.

25. (Prof. Ditkowski style) Consider the family of equations $y''-2ay'+a^{2y}=0 parameterized by$a $\in \mathbb{R}$.

   (a) Show that for all a, the characteristic equation has a repeated root.

   (b) Find the general solution in terms of a.

   (c) For which values of a do all solutions remain bounded as $t \to \infty$?

   (d) Find the solution with $y(0) = 1, y'(0) = 0$.

Solutions

Part A: Identifying Multiplicities

1. Roots: $r = 3$ (multiplicity 2), $r = -1$ (multiplicity 1)

2. Expand $(r-2)^4$ = r^4 - 8r^3 + 24r^2 - 32r + 16✓
   Root :r = 2(*multiplicity*4)

3. (r^2 - 2r + 5)^2 = 0 ⇒ r^2 - 2r + 5 = 0
   r = 2 ±$\sqrt{4-20}_{\overline{2=1\pm 2i}}$
   Roots: $1 + 2i$ (mult. 2), $1 - 2i$ (mult. 2)

4. r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0
   Root :r = -1(*multiplicity*3)

5. (r^2 + 4)^3 = 0 ⇒ r^2 = -4 ⇒ $r = \pm 2i$
   Roots: $2i$ (mult. 3), $-2i$ (mult. 3)

Part B: Solutions from Multiplicities

6. $e^{3t}, te^{3t}, t^{2e3t}$

7. $y(t) = (c_1$ + c$_{2t}$ + c$_{3t}{}^2$ + $c_{4t}{}^3)e^{-2t}$

8. Real solutions: $e^t\cos(2t), te^t\cos(2t), e^t\sin(2t), te^t\sin(2t)$

9. $y(t) = (c_1$ + c$_{2t}$)e^{-t} + (c_3 + c$_{4t}$ + c$_{5t}{}^2$)e^{2t}

10. $y(t) = (c_1$ + c$_{2t}$)cos($3t$) + (c_3 + c$_{4t}$)sin($3t$)

Part C: Complete ODEs with Repeated Roots

11. Char. eq.: r^2 - 6r + 9 = (r-3)^2 = 0
    Root :r = 3(*mult.*2)
    Generalsolution :y(t) = (c$_1$ + c$_{2t}$)e^{3t}

12. Char. eq.: r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0
    Root :r = -1(*mult.*3)
    Generalsolution :y(t) = (c$_1$ + c$_{2t}$ + c$_{3t}{}^2$)e^{-t}

13. Char. eq.: $(r-1)^4 = 0$
    Root :r = 1(*mult.*4)
    Generalsolution :y(t) = (c$_1$ + c$_{2t}$ + c$_{3t}{}^2$ + $c_{4t}{}^3$)e^t

14. $(r+2)^2 = 0, so$y = (c$_1$ + c$_{2t}$)e^{-2t}
    $y(0) = c_1$ = 2
    y'(t) = c$_{2e}{}^{-2t}$ $-$ 2(c_1 + c$_{2t}$)e^{-2t}
    $y'(0) = c_2$ - 2c$_1$ = -3

$c_2 = 1$
Solution :$y(t) = (2 + t)e^{-2t}$

15. $r^4 + 4r^2 + 4 = (r^2 + 2)^2 = 0$
$Roots :\pm i\sqrt{2}$ (each mult. 2)
$y(t) = (c_1 + c_{2t})\cos(\sqrt{2}t) + (c_3 + c_{4t})\sin(\sqrt{2}t)$

Part D: Reduction of Order Applications

16. Let $y_2 = v(t)e^{-t}$
Substituting into $(D + 1)^2[y] = y'' + 2y' + y = 0$:
$After simplification :v'' = 0$
$Thus v = c_{1t} + c_2$
$Second solution :y_2 = te^{-t}$

17. $y = te^{3t}$
$y' = e^{3t} + 3te^{3t} = (1 + 3t)e^{3t}$
$y'' = 3e^{3t} + 3(1 + 3t)e^{3t} = (6 + 9t)e^{3t}$
Substitute: $(6 + 9t)e^{3t} - 6(1 + 3t)e^{3t} + 9te^{3t}$
$= e^{3t}[6 + 9t - 6 - 18t + 9t] = 0$ ✓

18. For double root r: $(D - r)^2[y] = 0$
$If y = te^{rt}$: $y' = e^{rt} + rte^{rt}$
$y'' = 2re^{rt} + r^{2tert}$
$(D - r)^2[y] = D[(D-r)[te^{rt}]] = D[e^{rt}] = re^{rt} - re^{rt} = 0$ ✓

19. $y_2 = v(t)e^{2t}$
$y_2' = (v' + 2v)e^{2t}$
$y_2'' = (v'' + 4v' + 4v)e^{2t}$
Substituting: $e^{2t}[v'' + 4v' + 4v - 4v' - 8v + 4v] = e^{2t} \cdot v'' = 0$
$v'' = 0 \Rightarrow v = c_{1t} + c_2$
$Second solution :y_2 = te^{2t}$

20. $W = \begin{vmatrix} e^{at} & te^{at} & t^{2eat} \\ ae^{at} & (1 + at)e^{at} & (2t + at^2)e^{at} \\ a^{2eat} & (2a + a^{2t})e^{at} & (2 + 4at + a^{2t2})e^{at} \end{vmatrix}$
After calculation: $W = 2e^{3at} \neq 0$

Part E: Complex Repeated Roots

21. $(r^2 + 4)^2 = 0 \Rightarrow r = \pm 2i$ (each mult. 2)
$y(t) = (c_1 + c_{2t})\cos(2t) + (c_3 + c_{4t})\sin(2t)$

22. $r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$
Each root has mult. 2
$y(t) = e^{-t}[(c_1 + c_{2t})\cos(t) + (c_3 + c_{4t})\sin(t)]$

23. Roots: $-1 \pm i$ (each mult. 2)
    $y(t) = e^{-t}[(c_1 + \mathrm{c}_{2t})\cos(t) + (c_3 + \mathrm{c}_{4t})\sin(t)]$

24. 6 linearly independent real solutions:
    $e^{2t}\cos(3t), te^{2t}\cos(3t), t^{2e2t}\cos(3t)$
    $e^{2t}\sin(3t), te^{2t}\sin(3t), t^{2e2t}\sin(3t)$

25. Simple roots (not repeated): $r = -1 \pm 2i$
    $y(t) = e^{-t}[c_1\cos(2t) + c_2\sin(2t)]$
    $y(0) = c_1 = 0$
    y'(t) = $\mathrm{e}^{-t}[-c_1\cos(2t) - c_2\sin(2t) - 2c_1\sin(2t) + 2c_2\cos(2t)]$
    $y'(0) = -c_1 + 2\mathrm{c}_2 = 4$
    $\mathrm{c}_2 = 2$
    $Solution$:y(t) = $2\mathrm{e}^{-t}\sin(2t)$

Part F: Theoretical and Exam-Style Problems

26. If $p(r) = (r - r_0)^m$ q(r)$where$q(r_0) $\neq 0$:
    $p'(r) = m(r - r_0)^{m-1}q(r) + (r - r_0)^m$ q'(r)
    Atr = r_0:p'(r_0) = $0 if$m ¿ 1
    $Continuedifferentiatingtoshow$p$^{(k)}(r_0) = 0for$k ¡ m
    Butp$^{(m)}(r_0)$ = m! $\cdot q(r_0) \neq 0$

27. Total number of solutions = sum of all multiplicities = degree of polynomial = order of ODE

28. Char. poly: $(r-2)^2$(r+1)(r-3) = 0
    y = (c_1 + c_{2t})$\mathrm{e}^{2t} + c_{3e}{}^{-t} + c_{4e}{}^{3t}$
    Apply initial conditions (solve $4{\times}4 system$) :
    $Finalsolution$:y(t) = (1 - 2$_{3t)e^{2t}+\frac{1}{4}e^{-t}-\frac{1}{4}e^{3t}}$

29. The operator $(D-r)^m hasan$m$-dimensionalkernel.Thefunctions$t^k e^{rt} for $k = 0, \ldots, m-1$ are linearly independent (nonzero Wronskian) and all satisfy $(D-r)^m$[y] = 0.

30. (a) Char. eq.: r^2 - 2ar + a^2 = (r-a)^2 = 0
    $Alwaysadoublerootat$r = a

(b) $y(t) = (c_1 + \mathrm{c}_{2t})\mathrm{e}^{at}$

(c) Bounded as $t \to \infty$ only if $a \leq 0$

(d) $y(0) = c_1 = 1$
    y'(t) = $\mathrm{c}_{2e}{}^{at} + a(c_1 + \mathrm{c}_{2t})\mathrm{e}^{at}$
    $y'(0) = c_2 + \mathrm{ac}_1 = 0$
    c_2 = -a
    $Solution$:y(t) = (1 - at)e^{at}