Lesson 45: Method of Undetermined Coefficients

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1 The Non-Homogeneous Linear ODE

Definition 1 (Non-Homogeneous Linear ODE). *An n-th order non-homogeneous linear ODE with constant coefficients has the form:*

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)$$

where $f(t) \not\equiv 0$ is called the forcing function or non-homogeneous term.

Theorem 1 (Structure of General Solution). *The general solution of $L[y] = f(t)$ is:*

$$y(t) = y_h(t) + y_p(t)$$

where:

- *$y_h(t)$ is the general solution of the homogeneous equation $L[y] = 0$*
- *$y_p(t)$ is any particular solution of $L[y] = f(t)$*

Proof. If $L[y_p] = f(t)$ and $L[y_h] = 0$, then:

$$L[y_h + y_p] = L[y_h] + L[y_p] = 0 + f(t) = f(t)$$

Conversely, if y_1 and y_2 both satisfy $L[y] = f(t)$, then:

$$L[y_1 - y_2] = L[y_1] - L[y_2] = f(t) - f(t) = 0$$

So $y_1 - y_2$ is a homogeneous solution. □

2 Suitable Forcing Functions

Definition 2 (UC-Suitable Functions). *A function $f(t)$ is suitable for the method of undetermined coefficients if it belongs to a finite-dimensional space that is closed under differentiation.*

The method works for linear combinations of:

1. Polynomials: $t^n, t^{n-1}, \ldots, t, 1$
2. Exponentials: e^{at} where $a \in \mathbb{C}$
3. Trigonometric: $\sin(bt), \cos(bt)$ where $b \in \mathbb{R}$
4. Products: $t^n \mathrm{e}^{at}$, $e^{at}\sin(bt)$, $e^{at}\cos(bt)$, $\mathrm{t}^n \mathrm{e}^{at}\sin(bt), etc.$

3 The Method: Non-Resonant Case

Method 1 (Undetermined Coefficients - Basic). *For $L[y] = f(t)$ where $f(t)$ is UC-suitable:*

1. *Solve the homogeneous equation $L[y] = 0$ to find y_h*
2. *Based on the form of $f(t)$, guess the form of y_p with undetermined coefficients*
3. *Compute the derivatives of y_p*
4. *Substitute into $L[y_p] = f(t)$*
5. *Equate coefficients of like terms to determine the unknowns*
6. *Write the general solution: $y = y_h + y_p$*

3.1 Guessing Rules for Common Functions

Forcing Function $f(t)$	**Trial Solution** $y_p(t)$
$P_n(t)$ (polynomial degree n)	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$
e^{at}	$A\mathrm{e}^{at}$
$\sin(bt)$ or $\cos(bt)$	$A\cos(bt) + B\sin(bt)$
$P_n(t)\mathrm{e}^{at}$	$(A_n t^n + \cdots + A_0)\mathrm{e}^{at}$
$e^{at}\sin(bt)$ or $e^{at}\cos(bt)$	$e^{at}[A\cos(bt) + B\sin(bt)]$
$P_n(t)\mathrm{e}^{at}\ \sin(bt)$	$e^{at}[(A_n t^n + \cdots + A_0)\cos(bt) +$ $(B_n t^n + \cdots + B_0)\sin(bt)]$

Always include both sin and cos terms when the forcing function contains either trigonometric function, as derivatives will produce both.

4 Resonance and the Modification Rule

Definition 3 (Resonance). *Resonance occurs when the trial solution y_p (or part of it) is a solution of the homogeneous equation $L[y] = 0$.*

Theorem 2 (Modification for Resonance). *If the standard trial solution is a homogeneous solution corresponding to a root r of multiplicity m, multiply the trial solution by t^m.*

Resonance physically corresponds to driving a system at its natural frequency, leading to unbounded growth in amplitude (the t^m factor).

4.1 Resonance Detection Algorithm

Method 2 (Checking for Resonance). *1. Find all roots of the characteristic equation (with multiplicities)*

2. For forcing function $f(t) = e^{at}g(t)$:

3. If a is not a characteristic root: no modification

4. If a is a simple root: multiply trial by t

5. If a is a root of multiplicity m: multiply trial by t^m

For $f(t) = e^{at}[\sin(bt) \;\; or \;\; \cos(bt)]$:

- *Check if $a \pm ib$ are characteristic roots*
- *Multiply by t^m where m is the multiplicity*

5 The Superposition Principle

Theorem 3 (Superposition). *If $f(t) = f_1(t) + f_2(t) + \cdots + f_k(t)$ and $L[y_{p_i}] = f_i(t)$ for each i, then:*

$$L[y_{p_1} + y_{p_2} + \cdots + y_{p_k}] = f_1(t) + f_2(t) + \cdots + f_k(t)$$

Therefore: $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$

Handle each term in $f(t)$ separately, then add the particular solutions. This often simplifies the algebra significantly.

6 Comprehensive Examples

Example 1 (Polynomial Forcing). *Solve: $y'' - 3y' + 2y = 6t^2 - 4t + 1$*

Solution:

1. *Homogeneous: $r^2 - 3r + 2 = 0 \Rightarrow r = 1, 2$*

   Thus: $y_h = c_1e^t + c_2e^{2t}$

2. *Trial: $y_p = At^2 + Bt + C$*

3. *Derivatives: $y_p' = 2At + B$, $y_p'' = 2A$*

4. *Substitute:*

$$2A - 3(2At + B) + 2(At^2 + Bt + C) = 6t^2 - 4t + 1$$
$$2At^2 + (-6A + 2B)t + (2A - 3B + 2C) = 6t^2 - 4t + 1$$

5. *Match coefficients:*

$$t^2 : \quad 2A = 6 \Rightarrow A = 3 \tag{1}$$
$$t^1 : \quad -6A + 2B = -4 \Rightarrow B = 7 \tag{2}$$
$$t^0 : \quad 2A - 3B + 2C = 1 \Rightarrow C = 8 \tag{3}$$

6. *Solution:* $y = c_1 e^t + c_2 e^{2t} + 3t^2 + 7t + 8$

Example 2 (Resonant Exponential). *Solve:* $y'' - 4y' + 4y = e^{2t}$

Solution:

1. *Homogeneous:* $(r-2)^2 = 0 \Rightarrow r = 2$ *(double root)*

   Thus: $y_h = (c_1 + c_2 t)e^{2t}$

2. *Standard trial* $Ae^{2t} appears in y_h$ *(resonance!)*

   Since 2 is a double root, multiply by t^2*:* $y_p = At^2 e^{2t}$

3. *Derivatives:*

$$y_p' \tag{4}$$
$$y_p'' = 2Ae^{2t}(1 + 4t + 2t^2) \tag{5}$$

4. *Substitute and simplify:* $2Ae^{2t} = e^{2t}$

5. *Thus* $A = 1/2$ *and* $y_p = \frac{t^2}{2}e^{2t}$

Example 3 (Resonant Trigonometric). *Solve:* $y'' + \omega^2 y = F_0 \cos(\omega t)$

Solution:

1. *Homogeneous:* $r^2 + \omega^2 = 0 \Rightarrow r = \pm i\omega$

   Thus: $y_h = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

2. *Resonance! Multiply by* t*:* $y_p = t[A\cos(\omega t) + B\sin(\omega t)]$

3. *After substitution:* $A = 0$, $B = \frac{F_0}{2\omega}$

4. *Solution exhibits linear growth:* $y_p = \frac{F_0 t}{2\omega}\sin(\omega t)$

7 Common Pitfalls and Tips

Common mistakes to avoid:

1. Forgetting lower-degree terms in polynomial trials
2. Omitting sine or cosine in trigonometric trials
3. Missing resonance when the forcing matches homogeneous solutions
4. Wrong multiplicity in resonance modification
5. Arithmetic errors in coefficient matching

Prof. Ditkowski's exam strategy:

1. Always find y_h first and write it clearly
2. Check for resonance before writing trial solution
3. For mixed forcing terms, use superposition
4. Verify one coefficient as a check
5. State the general solution explicitly

8 Complete Trial Solution Table

Forcing $f(t)$	**Char. Root?**	**Trial** y_p
$P_n(t)$	$r = 0$ not a root $r = 0$ mult. m	$A_n t^n + \cdots + A_0$ $t^m(A_n t^n + \cdots + A_0)$
e^{at}	$r = a$ not a root $r = a$ mult. m	$A\mathrm{e}^{at}$ $At^m\mathrm{e}^{at}$
$\sin(bt), \cos(bt)$	$r = \pm ib$ not roots $r = \pm ib$ mult. m	$A\cos(bt) + B\sin(bt)$ $t^m[A\cos(bt) + B\sin(bt)]$
$P_n(t)\mathrm{e}^{at}$	$r = a$ not a root $r = a$ mult. m	$(A_n t^n + \cdots + A_0)\mathrm{e}^{at}$ $t^m(A_n t^n + \cdots + A_0)\mathrm{e}^{at}$
$e^{at}\sin(bt)$ $e^{at}\cos(bt)$	$r = a \pm ib$ not roots $r = a \pm ib$ mult. m	$e^{at}[A\cos(bt) + B\sin(bt)]$ $t^m\mathrm{e}^{at}[A\cos(bt)$ + Bsin$(bt)]$