Lesson 50: Practice Problems - Numerical Methods

ODE 1 - Prof. Adi Ditkowski

Part A: Euler's Method (6 problems)

1. Apply Euler's method with $h = 0.1$ to solve:

$$y' = x - y, \quad y(0) = 1$$

Find $y(0.3)$ (show all steps).

2. Use Euler's method with $h = 0.25$ for:

$$y' = y + e$$

^{x}, y(0) = 0

$$Computeapproximationsat$$

x = 0.25, 0.5, 0.75, 1.

3. For the IVP $y' = -2y$, $y(0) = 1$:
   (a) Find the exact solution
   (b) Apply Euler with $h = 0.5$ to find $y(1)$
   (c) Compare with the exact value

4. Determine the maximum stable step size for Euler's method applied to:

$$y' = -10y$$

5. Use Euler's method with $h = 0.2$ for the system:

$$x' = y, \quad y' = -x, \quad x(0) = 1, \quad y(0) = 0$$

Find $(x(0.4), y(0.4))$.

6. Backward Euler for $y' = -100y$, $y(0) = 1$. Show that with $h = 0.1$:

$$y$$

$_1$ = y$_0 \frac{}{1+100h}$

Part B: Runge-Kutta Methods (6 problems)

1. Apply the improved Euler method (Heun) with $h = 0.5$ to:

$$y' = xy, \quad y(0) = 1$$

Find $y(1)$.

2. Use the midpoint method (RK2) with $h = 0.25$ for:

$$y' = \sin(x) + y, \quad y(0) = 0$$

Compute $y(0.5)$.

3. Apply RK4 with $h = 0.5$ to solve:

$$y' = x$$

^{2} + y^2, y(0) = 0

$$Find$$

y(0.5)$(showall$k$_i values)$.

4. For $y' = -y + x + 1$, $y(0) = 1$:
   (a) Apply one step of RK4 with $h = 1$
   (b) Find the exact solution and compare

5. Use RK4 with $h = 0.2$ for the system:

$$\begin{cases} x' = x - y \\ y' = x + y \end{cases}, \quad x(0) = 1, \quad y(0) = 0$$

Find $(x(0.2), y(0.2))$.

6. Compare Euler and RK4 for $y' = y$, $y(0) = 1$ with $h = 0.5$. Find $y(0.5)$ using both methods and the exact solution.

Part C: Error Analysis (5 problems)

7. For $y' = y$, $y(0) = 1$ on $[0, 1]$:
   (a) If we want global error $< 10^{-4}$ using Euler, estimate required h
   (b) Repeat for RK4
   (c) How many steps does each method need?

8. The local truncation error for Euler is $h^2 \overline{2y''(\xi)}$. For $y' = x^2$,y$(0) = 0$:
   Find the exact local error at $x = h$

Verify it's $O(h^2)$

Given that RK4 has local error $O(h^5), if halving the step size reduces the error by factor$F$, what is$F?

For $y' = -1000y + 1000$, $y(0) = 2$:

(a) Find the exact solution

(b) Is this equation stiff? Why?

(c) What happens if you use Euler with $h = 0.01$?

Richardson extrapolation: If $y_h is the Euler approximation with step$h and $y_{h/2}$ with step $h/2$, show that:

$$y_{improved} = 2y_{h/2} - y$$

h

$$has error$$

O(h²).

Part D: Stability Analysis (5 problems)

4. For the test equation $y' = \lambda y$ with $\lambda = -5$:

   (a) Find the stability condition for Euler

   (b) Find the stability condition for RK4

   (c) Which method allows larger steps?

5. Show that backward Euler is unconditionally stable for $y' = \lambda y$ with $\text{Re}(\lambda) < 0$.

6. For the system:

$$\mathbf{y}' = \begin{pmatrix} -1 & 10 \\ 0 & -100 \end{pmatrix} \mathbf{y}$$

   (a) Find the eigenvalues

   (b) What step size does Euler need for stability?

   (c) Is this system stiff?

7. The stability function for RK4 is:

$$R(z) = 1 + z + z$$

$^2 \frac{}{2+z}{}^3 \frac{}{6+z}{}^4 \frac{}{24} Find$—R(-2)—$and determine if$ h$\lambda = -2$ is stable.

Compare stability regions: sketch the stability boundary in the complex $h\lambda$ plane for:

(a) Euler's method

(b) Backward Euler

Part E: Implementation Considerations (3 problems)

3. Write pseudocode for adaptive step size control using error estimation.
4. For solving to tolerance $\epsilon = 10^{-6}$ on $[0, 10]$:
   (a) Estimate steps needed for Euler
   (b) Estimate steps needed for RK4
   (c) Which is more efficient?

5. Higher-order ODEs: Convert $y''+2y'+y = e^x$,y(0) = 1,y'(0) = $0 to a system and show first Euler step with$h = 0.1.

Part F: Exam-Style Problems (5 problems)

3. [**10 points**] Consider $y' = x + y$, $y(0) = 1$.
   3 pts Apply two steps of Euler's method with $h = 0.5$
   4 pts Apply one step of RK4 with $h = 1$
   3 pts The exact solution is $y = 2e^x$ - x - 1.$Compare errors$.
4. [**8 points**] For the pendulum equation (small angle):
$$\theta'' + \theta = 0, \quad \theta(0) = 0.1, \quad \theta'(0) = 0$$
   2 pts Convert to a first-order system
   4 pts Apply one step of RK4 with $h = 0.1$
   2 pts Is the total energy conserved numerically?
5. [**9 points**] Stability analysis for $y' = -50y$:
   2 pts Find maximum stable h for Euler
   2 pts Find maximum stable h for RK4 (given stability limit $|h\lambda| < 2.78$)
   2 pts If you need to solve on $[0, 10]$, how many steps for each?
   3 pts Which method is more efficient and why?
6. [**10 points**] Method comparison:
   3 pts Explain why RK4 is more accurate than Euler
   3 pts When would you prefer implicit over explicit methods?
   2 pts What is the trade-off in choosing step size?
   2 pts How do adaptive methods work?

7. **[12 points]** *Prof. Ditkowski Special* The chemical reaction $A \to B \to C$ has rate equations:

$$\begin{cases} A' = -100A \\ B' = 100A - B \\ C' = B \end{cases}$$

with $A(0) = 1$, $B(0) = C(0) = 0$.

3 pts Is this system stiff? Explain.

3 pts Apply one Euler step with $h = 0.001$

3 pts What happens with $h = 0.1$?

3 pts Suggest an appropriate numerical method

Solutions and Hints

Selected Solutions:

Problem 1: - $y_1 = 1 + 0.1(0 - 1) = 0.9 - y_2 = 0.9 + 0.1(0.1 - 0.9) = 0.82 - y_3 = 0.82 + 0.1(0.2 - 0.82) = 0.758$

Problem 4: For stability: $|1 - 10h| < 1$, so $h < 0.2$

Problem 9: RK4 for $y' = x^2 + y^2$: $-k_1 = 0 - k_2 = 0.5(0.25)^2 = 0.03125 - k_3 = 0.5[(0.25)^2 + (0.0156)^2] \approx 0.0312$ - $k_4 = 0.5[(0.5)^2 + (0.0312)^2] \approx 0.125$ - $y_1 \approx 0.031$

Problem 13: For Euler: $h \sim \sqrt{10^{-4}} = 0.01$, need 100 steps For RK4: $h \sim (10^{-4})^{1/4} \approx 0.1$, need 10 steps

Problem 18: Eigenvalues: $-1, -100$. Need $h < 2/100 = 0.02$ for stability. Yes, stiff!

Key Insights: - RK4 typically 10-100$\times moreefficientthanEuler - Stiffnessdeterminedbyeigenvaluerat$ $Implicitmethodstradecomputationperstepforstability - Energyconservationtestsnumericalaccuracy$