Lesson 23: Integrating Factors - $\mu(x)$ and $\mu(y)$ Cases

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1 The Concept of Integrating Factors

Definition 1 (Integrating Factor)**.** *An **integrating factor** $\mu(x, y)$ for the equation*

$$M(x, y)dx + N(x, y)dy = 0$$

is a function such that the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact.

Multiplying by an integrating factor doesn't change the solutions - it only changes the form of the equation. If $y = f(x)$ is a solution to the original equation, it remains a solution to the modified equation.

2 Condition for Exactness After Multiplication

For $\mu M dx + \mu N dy = 0$ to be exact, we need:

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

Expanding using the product rule:

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

Rearranging:

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

This is a partial differential equation for μ - generally very difficult to solve! We look for special cases where μ depends on only one variable.

3 Case 1: Integrating Factor $\mu(x)$

Theorem 1 (Existence of $\mu(x)$)**.** *An integrating factor depending only on x exists if and only if*

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$$

where $g(x)$ is a function of x alone. The integrating factor is then:

$$\mu(x) =$$

$e^{\int}$ *g(x) dx*

Proof. If $\mu = \mu(x)$, then $\frac{\partial \mu}{\partial y} = 0$. The exactness condition becomes:

$$-N\frac{d\mu}{dx} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

This is solvable only if the right side depends solely on x. □

4 Case 2: Integrating Factor $\mu(y)$

Theorem 2 (Existence of $\mu(y)$)**.** *An integrating factor depending only on y exists if and only if*

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = h(y)$$

where $h(y)$ is a function of y alone. The integrating factor is:

$$\mu(y) =$$

$e^{\int}$ *h(y) dy*

5 Complete Solution Process

Step-by-Step Solution with Integrating Factors:

1. Test for exactness (if exact, skip to step 6)

2. Check if $\mu(x)$ exists: Is $(M_y$ - N$_x)/$N$afunctionofxonly?Ifnot,checkif\mu(y)$ exists: Is $(N_x$ - M$_y)/$M$afunctionofyonly?$

3. Find the integrating factor using the appropriate formula

4. Multiply the original equation by μ
5. Verify the new equation is exact
6. Solve the exact equation using methods from Lesson 22

6 Important Examples

Example 1 (Standard $\mu(x)$ Case)**.** *Solve* $(2y + 3x^2)dx + xdy = 0$

Step 1: *Test exactness:* $M_y = 2, N_x = 1. Not exact!$

Step 2: *Check for* $\mu(x)$*:*

$$M$$

$_y$ - $N_x \frac{}{N=\frac{2-1}{x}=\frac{1}{x}}$ This is a function of x only!

Step 3: Find $\mu(x)$:

$$\mu(x) =$$

e$^{\int}$ 1$\frac{}{xdx=}$e$^{\ln|x|}$ = x

Step 4: Multiply by $\mu = x$:

$$(2xy + 3x^3)dx + x$$

^{2dy} = 0

Step 5: Verify exactness: M_y = 2x,N$_x$ = 2x.✓

Step 6: Find potential function:

$$H = \int x^2\,dy = x$$

^{2y} + f(x)

$$\frac{\partial H}{\partial x} = 2xy + f'(x) = 2xy + 3x^3$$

$$f'(x) = 3x^3 \Rightarrow f(x) = \frac{3x^4}{4}$$

Solution: x^{2y} + 3x$^4 \frac{}{4=C}$

Example 2 (Linear Equation Connection)**.** *The linear equation* $y' + P(x)y = Q(x)$ *can be written as:*

$$(Py - Q)dx + dy = 0$$

Check for $\mu(x)$*:*

$$M$$

$_y$ - $N_x \frac{}{N=\frac{P-0}{1}=P(x)}$

Therefore: $\mu(x)$ =e$^{\int}$ P(x)dx$-exactly the integrating factor from Block$5!

7 Common Patterns to Recognize

Quick Recognition Guide:

If you see	Try	Integrating Factor
$N = x^n$	$\mu(x)$	Often $\mu = x^k$
$M = y^n$	$\mu(y)$	Often $\mu = y^k$
Linear in y	$\mu(x)$	μ =e$^{\int}$ P(x)dx
Homogeneous	Either	Check both tests
$N = f(x)$ only	$\mu(x)$	Guaranteed to exist
$M = g(y)$ only	$\mu(y)$	Guaranteed to exist

8 Memory Aids

Mnemonic Devices:

- "$\mu(x)$: My Nexus over N" - (M_y - N$_x$)/N$forxdependence$"$\mu(y)$: Nexus My over M" - (N_x - M$_y$)/M$forydependence$

- Notice: Numerators are negatives of each other!
- The variable in μ matches what you divide by (sort of):
  - Divide by N (has x in deNominator) $\rightarrow \mu(x)$
  - Divide by M (has y sound in naMe) $\rightarrow \mu(y)$

9 Verification is Crucial

After finding an integrating factor, ALWAYS:

1. Multiply the original equation by μ
2. Verify the new equation is exact by checking $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$
3. Only then proceed to find the potential function

Skipping verification is a common source of errors!