Lesson 49: Practice Problems - Frobenius Method

ODE 1 - Prof. Adi Ditkowski

Part A: Classifying Singular Points (6 problems)

1. Classify all singular points (regular or irregular) for:

$$x^2(x-1)y'' + 2xy' + y = 0$$

2. For Bessel's equation $x^{2y"}$ + xy' + (x$^2-4)y = 0$, verify that $x = 0$ is a regular singular point.

3. Determine the nature of $x = 0$ for:

$$x$$

$^{3y"}$ + xy' + y = 0

4. Show that both $x = 0$ and $x = 1$ are regular singular points of:

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

(This is the hypergeometric equation)

5. Classify the singular point at $x = 0$ for:

$$x$$

$^{2y"}$ $+ \sin(x)y' + xy = 0$

6. Find and classify all singular points:

$$(x^2-1)$$

$^{2y"}$ + (x-1)y' + y = 0

Part B: Finding Indicial Equations (6 problems)

1. Find the indicial equation at $x = 0$ for:

$$xy'' + y' - y = 0$$

2. Determine the indicial equation for:

$$x$$

$^{2y"}$ + x(1+x)y' - y = 0

3. Find the indicial equation and its roots for:

$$2x$$

$^{2y"}$ + xy' - (1+x)y = 0

4. For the equation $x^{2y"}$ + 3xy' + (1-x)y = 0, $findtheindicialequationandidentifywhichcaseapplies.$
5. Find the indicial equation at $x = 1$ for:

$$(x-1)$$

$^{2y"}$ + (x-1)y' + y = 0

6. Determine the indicial equation for Laguerre's equation:

$$xy'' + (1-x)y' + ny = 0$$

Part C: Determining Solution Forms (5 problems)

7. For $x^{2y"}$ + xy' + (x$^2 - 1/4)y = 0$, find the roots of the indicial equation and state the form of the general solution.
8. Given the indicial roots $r_1 = 3$ and $r_2 = -2$, write the general form of the solution.
9. If the indicial equation has repeated root $r = 1/2$, write the form of both linearly independent solutions.
10. For Bessel's equation of order 2, explain why the second solution must contain $\ln(x)$.
11. Given roots $r_1 = 1$ and $r_2 = 0$, determine whether a logarithmic term is needed by checking the recurrence at $n = 1$.

Part D: Computing Frobenius Series (5 problems)

12. Find the first three non-zero terms of the Frobenius series solution for:

$$xy'' + 2y' + xy = 0$$

using the larger root.

13. For $x^{2y"}$ + xy' + (x$^2 - 1)y = 0$, find the recurrence relation for the root $r = 1$.
14. Solve using Frobenius method:

$$2xy'' + (1 + 2x)y' + y = 0$$

Find coefficients a_0, a_1, a_2 for the larger root.

15. For the equation $x^{2y"}$ + x$^{2y'}$ - 2y = 0:

    16. Find the indicial equation
    17. Find the recurrence relation for $r = 2$
    18. Compute the first four coefficients

Apply Frobenius method to find one solution of:

$$x$$

$^{2y"}$ + x(x+1)y' - y = 0

Part E: Special Cases and Logarithmic Solutions (3 problems)

19. Show that for $x^{2y"}$ + 3xy' + (1+x)y = $0 with repeated root$r = -1$, the second solution must containln(x)$.
20. For the equation with roots differing by an integer:

$$x$$

$^{2y"}$ + xy' - y = 0

Determine if both solutions can be pure Frobenius series or if logarithms are needed.

21. Verify that Euler's equation $x^{2y"}$ - xy' + y = $0 has solutions$y$_1 = x$ and $y_2 = x \ln(x)$.

Part F: Exam-Style Problems (5 problems)

22. [**10 points**] Consider the modified Bessel equation:

$$x$$

$^{2y"}$ + xy' - (x$^2 + n^2)y = 0 where$n = 2.

2 pts Show that $x = 0$ is a regular singular point

3 pts Find the indicial equation and its roots

2 pts Which case applies for the general solution?

3 pts Write the form of both linearly independent solutions

[9 points] For the equation:

$$x(x-1)y'' + 3y' + y = 0$$

3 pts Find and classify all singular points

3 pts Find the indicial equation at $x = 0$

3 pts Find the first three terms of the Frobenius series for the larger root

[8 points] Given:

$2x$

$^{2y"}$ + x(1+x)y' - 2y = 0

Find the indicial equation and roots

Set up the recurrence relation

Determine if the second solution requires logarithms

[10 points] *Comprehensive Problem*

x

$^{2y"}$ + x(1-x)y' - (1+3x)y = 0

Verify $x = 0$ is a regular singular point

Find the indicial equation and solve for r

Find the recurrence relation for the larger root

State the form of the general solution

[12 points] *Prof. Ditkowski Special - Hypergeometric Type* Consider: $x(1-x)y'' + [2-(3+x)]y' - y = 0$

3 pts Show both $x = 0$ and $x = 1$ are regular singular points

3 pts Find the indicial equation at $x = 0$

3 pts Find the indicial equation at $x = 1$

3 pts Around which point would you prefer to expand and why?

Solutions and Hints

Selected Solutions:

Problem 1: - $x = 0$: Check $xp(x) = 2x^2/(x(x-1))$ and x^{2q}(x) = x$^2/(x(x-1))$ at $x = 0 \rightarrow Regular -$x = 1: $Check$(x-1)p(x)and(x-1)^{2q}(x)atx = 1$\rightarrow$ $Regular$

Problem 7: Standard form: $y'' + (1/x)y' - (1/x)y = 0$ - $p_0 = 1$, $q_0 = 0$ - Indicial equation: $r(r-1) + r = 0$ $\rightarrow$r$^2 = 0$ $\rightarrow$r = $0(repeated)$

Problem 13: - Indicial equation: $r^2 - 1/4 = 0$ - Roots: $r_1 = 1/2$, $r_2 = -1/2$ (differ by 1) - Form: Check if pure Frobenius works for r_2 or needs log term

Problem 18: For larger root $r = 0$: - $y = a_0(1 - x^2/2 + x^4/24 - \cdots)$ - This gives the Bessel function $J_0(x)$ series

Problem 26: - Indicial roots: $r = 2, -1$ (differ by 3) - At $x = 0$: Regular singular point - Second solution likely needs no logarithm (check recurrence)

Key Insights: - Always check $r_1 - r_2$ first - Integer differences require careful analysis - Bessel-type equations are exam favorites - When $r = 0$ appears, one solution is a regular power series