

ODE Lesson 36: Phase Space and Trajectories: Geometric Theory of ODEs

ODE 1 - Prof. Adi Ditkowski

1 Phase Space Fundamentals

Definition 1 (Phase Space). *For a system of n first-order ODEs:*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, the **phase space** is the n -dimensional space with coordinates (x_1, x_2, \dots, x_n) .

Key Insight: Phase space represents all possible states of the system. Each point corresponds to a unique state, regardless of time.

Definition 2 (Trajectory/Orbit). *A **trajectory** (or **orbit**) is the curve traced out in phase space by a solution $\mathbf{x}(t)$ as t varies. Formally, it is the set:*

$$\Gamma = \{\mathbf{x}(t) : t \in I\}$$

where I is the interval of existence.

Critical Distinction:

- **Solution:** $\mathbf{x}(t)$ - includes time parametrization
- **Trajectory:** The geometric curve - no time information

Multiple solutions can give the same trajectory (time-shifted solutions)!

2 Direction Fields and Flow

Definition 3 (Direction Field). *The **direction field** (or **vector field**) of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ assigns to each point \mathbf{x} the vector $\mathbf{f}(\mathbf{x})$, indicating the instantaneous direction and speed of motion.*

Theorem 1 (Non-Intersection of Trajectories). *If $\mathbf{f}(\mathbf{x})$ satisfies the conditions for existence and uniqueness, then trajectories cannot intersect except at equilibrium points.*

Proof. Suppose two trajectories Γ_1 and Γ_2 intersect at point \mathbf{x}_0 which is not an equilibrium. By uniqueness, the solution starting at \mathbf{x}_0 is unique, so $\Gamma_1 = \Gamma_2$. Contradiction. \square

3 Equilibrium Points and Invariant Sets

Definition 4 (Equilibrium Point). A point \mathbf{x}^* is an *equilibrium point* (or *critical point*, *fixed point*) if:

$$\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$$

Finding Equilibria - Prof. Ditkowski's Method:

1. Set all derivatives equal to zero
2. Solve the resulting algebraic system
3. Check each solution carefully
4. State coordinates explicitly: "Equilibrium at $(x^*, y^*) = (a, b)$ "

Definition 5 (Invariant Set). A set $S \subset \mathbb{R}^n$ is *invariant* under the flow if:

$$\mathbf{x}(0) \in S \implies \mathbf{x}(t) \in S \text{ for all } t$$

4 Special Types of Trajectories

Definition 6 (Closed Orbit). A trajectory Γ is a *closed orbit* if it is homeomorphic to a circle and the solution is periodic:

$$\exists T > 0 : \mathbf{x}(t + T) = \mathbf{x}(t) \text{ for all } t$$

Dimension Restriction: Closed orbits cannot exist in 1D phase space! In 1D, trajectories are confined to the real line and cannot loop back without violating uniqueness.

Definition 7 (Heteroclinic and Homoclinic Orbits). • **Heteroclinic orbit:** Connects two different equilibria

- **Homoclinic orbit:** Starts and ends at the same equilibrium

5 Phase Portraits for 2D Systems

For a 2D autonomous system:

$$\frac{dx}{dt} = f(x, y) \tag{1}$$

$$\frac{dy}{dt} = g(x, y) \tag{2}$$

Construction Steps:

1. Find all equilibria: solve $f(x, y) = 0$ and $g(x, y) = 0$
2. Compute the direction field at selected points
3. Identify special trajectories (if any)
4. Sketch trajectories following the direction field
5. Add arrows indicating flow direction

6 Converting Higher-Order Equations

Example 1 (Second-Order to First-Order System). *Convert $\ddot{x} + p(x)\dot{x} + q(x) = 0$ to phase space form:*

Let $x_1 = x$ and $x_2 = \dot{x}$. Then:

$$\dot{x}_1 = x_2 \quad (3)$$

$$\dot{x}_2 = -p(x_1)x_2 - q(x_1) \quad (4)$$

Phase space is the (x_1, x_2) -plane, often relabeled as (x, \dot{x}) -plane.

7 Nullclines Method

Definition 8 (Nullclines). *The **nullclines** are curves where one component of the vector field vanishes:*

- ***x-nullcline***: $f(x, y) = 0$ (*vertical flow*)
- ***y-nullcline***: $g(x, y) = 0$ (*horizontal flow*)

Using Nullclines:

- Equilibria occur at nullcline intersections
- Trajectories cross nullclines vertically or horizontally
- Nullclines divide phase space into regions with consistent flow direction

8 Exam-Critical Formulas

Must-Know for Prof. Ditkowski's Exam:

Concept	Formula/Property
Equilibrium condition	$\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$
Trajectory uniqueness	No intersections except at equilibria
Closed orbit period	$\mathbf{x}(t + T) = \mathbf{x}(t)$
Direction at point (x, y)	Vector $(f(x, y), g(x, y))$
Speed along trajectory	$ \mathbf{f}(\mathbf{x}) = \sqrt{f^2 + g^2}$