Practice Problems: Lesson 6 - Mastering Lipschitz Conditions

Essential skill for uniqueness!

Part A: Basic Lipschitz Checks

Determine if each function is Lipschitz in y. If yes, find the constant L:

1. $f(x, y) = 5y + \cos(x)$ on $\mathbb{R}^2$
2. $f(x, y) = y^4$ on $|y| \leq 1$
3. $f(x, y) = \frac{1}{y}$ on $y \geq 1$
4. $f(x, y) = y \sin(y)$ on $\mathbb{R}$
5. $f(x, y) = \sqrt{y^2 + 1}$ on $\mathbb{R}$

Part B: Finding Optimal Constants

Find the smallest Lipschitz constant for:

6. $f(y) = y^2 - 4y + 3$ on $[0, 5]$
7. $f(y) = e^{-y^2}$ on $[-1, 1]$
8. $f(x, y) = x^2y^2$ on $|x| \leq 2$, $|y| \leq 3$
9. $f(y) = \frac{y}{1+y^2}$ on $\mathbb{R}$

Part C: Piecewise Functions

Check Lipschitz continuity for these piecewise functions:

10. $f(y) = \begin{cases} y^2 & y \geq 0 \\ -y^2 & y < 0 \end{cases}$ on $[-2, 2]$
11. $f(y) = \begin{cases} \sqrt{y} & y \geq 0 \\ 0 & y < 0 \end{cases}$ near $y = 0$
12. $f(y) = \begin{cases} y \sin(1/y) & y \neq 0 \\ 0 & y = 0 \end{cases}$ on $[-1, 1]$

Part D: Composition and Operations

13. If $g(y) = y + 1$ is Lipschitz with $L_g = 1$ and $h(y) = 2y$ is Lipschitz with $L_h = 2$:
    (a) Find the Lipschitz constant of $g + h$
    (b) Find the Lipschitz constant of $3g - h$
    (c) Is $g \cdot h$ Lipschitz on $[-1, 1]$?

14. Show that if f is Lipschitz with constant L and $|f| \leq M$, then f^2 is Lipschitz with constant $2ML$.

Part E: Non-Lipschitz Examples

Prove that these functions are NOT Lipschitz at the specified point:

15. $f(y) = y^{1/3}$ at $y = 0$

16. $f(y) = y \ln |y|$ at $y = 0$ (defined as $f(0) = 0$)

17. $f(y) = \begin{cases} \sin(1/y) & y \neq 0 \\ 0 & y = 0 \end{cases}$ at $y = 0$

Part F: Domain Dependence

18. For $f(y) = y^2$:
    (a) Find the Lipschitz constant on $[-1, 1]$
    (b) Find the Lipschitz constant on $[-5, 5]$
    (c) Find the Lipschitz constant on $[-M, M]$
    (d) Is it globally Lipschitz?

19. For $f(y) = e^y$:
    (a) Show it's Lipschitz on any bounded interval $[a, b]$
    (b) Find L on $[-2, 3]$
    (c) Why isn't it globally Lipschitz?

Part G: Applications to ODEs

20. For which values of α is the IVP $y' = |y|^\alpha$, $y(0) = 0$ guaranteed to have a unique solution?

21. Consider $y' = f(y)$ where f is continuously differentiable with $|f'(y)| \leq 10$ for all y:

(a) Why does every IVP have a unique solution?

(b) If two solutions start 0.01 apart, how far apart can they be after time $t = 1$?

22. The ODE $y' = y^2 \sin(1/y)$ for $y \neq 0$ and $y' = 0$ for $y = 0$:

    (a) Is the right-hand side continuous?

    (b) Is it Lipschitz near $y = 0$?

    (c) What does this imply about uniqueness?

Part H: Theoretical Problems

23. Prove the reverse triangle inequality: $||a| - |b|| \leq |a - b|$

24. Show that if f is differentiable with continuous derivative on $[a, b]$, then f is Lipschitz on $[a, b]$.

25. Give an example of:

    (a) A Lipschitz function that's not differentiable

    (b) A continuous function that's not Lipschitz

    (c) A Lipschitz function with discontinuous derivative

Part I: Exam-Style Questions

26. Professor Ditkowski asks: "For $f(y) = y^n$ on $[-2, 2]$, find the smallest n such that the Lipschitz constant exceeds 100."

27. Consider the family $f_\epsilon(y) = \sqrt{y^2 + \epsilon^2}$:

    (a) Show each f_ϵ is Lipschitz for $\epsilon > 0$

    (b) Find the Lipschitz constant as function of ϵ

    (c) What happens as $\epsilon \to 0$?

28. You're told that f satisfies: $|f(y_1) - f(y_2)| \leq K|y_1 - y_2|^\alpha$ for some $\alpha > 0$:

    (a) When is f Lipschitz?

    (b) When is f uniformly continuous?

    (c) Give examples for $\alpha = 0.5, 1, 2$

29. The "Lipschitz constant function" $L(M)$ = smallest L such that f is Lipschitz on $[-M, M]$:

    (a) Find $L(M)$ for $f(y) = y^3$

    (b) Find $L(M)$ for $f(y) = \sin(y^2)$

    (c) For which functions is $L(M)$ bounded as $M \to \infty$?

Solutions Guide

Part A Quick Answers: 1. Yes, $L = 5$ 2. Yes, $L = 4$ 3. Yes, $L = 1$ 4. No (unbounded derivative) 5. Yes, $L = 1$

Part B Hints: 6. Check $f'(y) = 2y - 4$; max at boundaries, $L = 6$ 7. Check critical points of $f'(y) = -2ye^{-y^2}$ 8. Use product rule carefully 9. Find max of $|f'(y)| = |1-y^2|/(1+y^2)^2$

Key Concepts: - Derivative bounded $\Rightarrow$ Lipschitz - Powers less than 1 fail at origin - Piecewise needs checking at transitions - Local Lipschitz $\neq$ Global Lipschitz