Lesson 23: Practice Problems - Integrating Factors $\mu(x)$ and $\mu(y)$

ODE 1 - Prof. Adi Ditkowski

Part A: Testing for $\mu(x)$ and $\mu(y)$ (Problems 1-6)

1. For $(3x + 2y)dx + xdy = 0$:
   (a) Show the equation is not exact
   (b) Test if $\mu(x)$ exists
   (c) Test if $\mu(y)$ exists
   (d) Find the integrating factor
2. For (y^2 + 2xy)dx + xydy = 0:
   3. Verify non-exactness
   4. Determine which type of integrating factor exists
   5. Find and apply the integrating factor

Test both $\mu(x)$ and $\mu(y)$ for: $(2y)dx + (3x + 4y^2)$dy = 0

Test both $\mu(x)$ and $\mu(y)$ for: $(xy + 1)dx + (x^2$ - 1)dy = 0

For $ydx - xdy = 0$, show that both $\mu(x) = 1/x^2 and \mu(y) = 1/y^2 work.$

Determine all possible integrating factors of the form $\mu(x)$ for: $(2y)dx + xdy = 0$

Part B: Finding and Using $\mu(x)$ (Problems 7-12)

7. Solve $(2xy + y^2)$dx + xdy = $0 by finding \mu(x)$
8. Solve $(3y + 2x)dx + xdy = 0$ using an integrating factor
9. Solve $(y + x^2)$dx + 2xdy = 0
10. Solve $(2y + 3x^{2y})$dx + xdy = 0
11. Find $\mu(x)$ and solve: (e^y + 2x)dx + xe^y dy = 0
12. Solve the initial value problem: $(y + x^3)$dx + 2xdy = 0,y(1) = 2

Part C: Finding and Using $\mu(y)$ (Problems 13-18)

13. Solve $ydx + (2x + 3y^2)$dy $= 0 by finding \mu(y)$
14. Solve $2ydx + (3x - y)dy = 0$ using an integrating factor
15. Solve (y^2 + 1)dx + xydy = 0
16. Find $\mu(y)$ and solve: $\sin y\, dx + (x \cos y + 1)dy = 0$
17. Solve ($2y^3$)dx + (3xy^2 - 1)dy = 0
18. Solve the IVP: $ydx + (3x - 2y^2)$dy = 0,y(0) = 1

Part D: Choice Between $\mu(x)$ and $\mu(y)$ (Problems 19-23)

19. For ($2xy^2$ + y)dx + xdy = 0:

    20. Show both $\mu(x)$ and $\mu(y)$ exist
    21. Find both integrating factors
    22. Solve using each and verify same solution

Find the simpler integrating factor and solve: ($3x^{2y}$ + 2y^2)dx + x^{3dy} = 0

Choose the appropriate integrating factor for: $(y \cos x + 1)dx + \sin x\, dy = 0$

For $(ax + by^2)$dx + ydy = 0, $find conditions on$ aand bfor :

$\mu(x)$ to exist

$\mu(y)$ to exist

Solve by choosing the simpler integrating factor: $(x^{2y3} + 2y)dx + xdy = 0$

Part E: Linear Equation Connection (Problems 24-26)

24. Show that $y' + \frac{2}{x}y = x^2 leads to \mu(x) = x^2 and solve.$
25. Convert to standard form and find integrating factor: $xy' - 2y = x^{3ex}$
26. Show that every linear equation $y' + P(x)y = Q(x)$ has $\mu(x) = e^{\int P(x)dx}$

Part F: Exam-Style Problems (Problems 27-32)

27. (Prof. Ditkowski 2023) Given that $\mu = x^n isanintegratingfactorfor$(2xy + y^3)dx + (x^2 + xy^2)dy = $0, findnandsolve.$

28. Show that ($3x^{2y}$ + y^2)dx + (x^3 + xy)dy = $0becomesexactwhenmultipliedby\mu = 1/xy$. Is this $\mu(x)$ or $\mu(y)$? Explain.

29. Find all integrating factors of the form $\mu = x^a$ $y^b for$:2ydx + xdy = 0

30. Given $(f(x) + 2y)dx + xdy = 0$ has $\mu(x) = x^2$:

    31. Find $f(x)$

    32. Solve the equation

For what value of k does $(ky + x^2)$dx + (2x + y^2)dy = $0have$:

An integrating factor $\mu(x)$?

An integrating factor $\mu(y)$?

A student claims that if an equation has both $\mu(x)$ and $\mu(y)$, then it must be exact. Prove or disprove with an example.

Solutions and Key Insights

Problem 1: (a) $M_y = 2$,$N_x = 1, notequal \to notexact(b)$($M_y$ - N_x)/N = (2-1)/x = 1/x→$\mu(x)$ exists! (c) (N_x - M_y)/M = (1-2)/(3x+2y)→ $containsboth$xandy$, no\mu(y)$ (d) $\mu(x) = e^{\int (1/x)dx} = x$

Problem 7: Test: (M_y - N_x)/N = (2x+2y-1)/x = (2x+2y-1)/x$Thiscontains$y$, sono\mu(x)$... Wait! Let's recheck: $M = 2xy + y^2$,N = x(M_y - N_x)/N = (2x + 2y - 0)/x = 2 + 2y/x$Stillhas$y$.Try\mu(y)$: (N_x - M_y)/M = (1 - 2x - 2y)/(2xy + y^2) = -1/y$So\mu(y) = e^{\int(-1/y)dy} = 1/y$

Problem 19: For ($2xy^2$ + y)dx + xdy = 0: (a)(M_y - N_x)/N = (4xy + 1 - 1)/x = 4y→ $No\mu(x)$ Actually, let me recalculate: M_y = 4xy + 1,N_x = 1(M_y - N_x)/N = 4xy/x = 4y→ $No\mu(x)$ (N_x - M_y)/M = (1 - 4xy - 1)/($2xy^2$ + y) = -4xy/(y(2xy + 1)) = -4x/(2xy + 1)$Hmm, thisiscomplex.Letmereconsidertheoriginalequation...$

Problem 24: $y' + (2/x)y = x^2 becomes$(2y/x - x^2)dx + dy = 0(M_y - N_x)/N = (2/x - 0)/1 = 2/x$\mu(x) = e^{\int(2/x)dx} = e^{2\ln|x|} = x^2 Multiply$:(2xy - x^4)dx + x^{2dy} = $0Nowexact!$H = x^{2y} - $x^5/5 Solution$:x^{2y} - x^5/5 = Cory = x^3/5 + C/x^2

Key Strategy: When both tests give functions of mixed variables, neither $\mu(x)$ nor $\mu(y)$ exists. Move to Lesson 24 for special forms!

Warning: Problem 32 - Counterexample: $2ydx + xdy = 0$ has both types but is not exact!