Lesson 23: Practice Problems - Integrating Factors $\mu(x)$ and $\mu(y)$

ODE 1 - Prof. Adi Ditkowski

Part A: Testing for $\mu(x)$ and $\mu(y)$ (Problems 1-6)

1. For $(3x + 2y)dx + xdy = 0$:
   (a) Show the equation is not exact
   (b) Test if $\mu(x)$ exists
   (c) Test if $\mu(y)$ exists
   (d) Find the integrating factor
2. For $(y^2 + 2xy)dx + xydy = 0$:
   (a) Verify non-exactness
   (b) Determine which type of integrating factor exists
   (c) Find and apply the integrating factor
3. Test both $\mu(x)$ and $\mu(y)$ for: $(2y)dx + (3x + 4y^2)dy = 0$
4. Test both $\mu(x)$ and $\mu(y)$ for: $(xy + 1)dx + (x^2 - 1)dy = 0$
5. For $ydx - xdy = 0$, show that both $\mu(x) = 1/x^2$ and $\mu(y) = 1/y^2$ work.
6. Determine all possible integrating factors of the form $\mu(x)$ for: $(2y)dx + xdy = 0$

Part B: Finding and Using $\mu(x)$ (Problems 7-12)

7. Solve $(2xy + y^2)dx + xdy = 0$ by finding $\mu(x)$
8. Solve $(3y + 2x)dx + xdy = 0$ using an integrating factor
9. Solve $(y + x^2)dx + 2xdy = 0$
10. Solve $(2y + 3x^{2y})$dx + xdy = 0
11. Find $\mu(x)$ and solve: $(e^y$ + 2x)dx + xe^y dy = 0
12. Solve the initial value problem: $(y + x^3)dx + 2xdy = 0$, $y(1) = 2$

Part C: Finding and Using $\mu(y)$ (Problems 13-18)

13. Solve $ydx + (2x + 3y^2)dy = 0$ by finding $\mu(y)$
14. Solve $2ydx + (3x - y)dy = 0$ using an integrating factor
15. Solve $(y^2 + 1)dx + xydy = 0$
16. Find $\mu(y)$ and solve: $\sin y\, dx + (x \cos y + 1)dy = 0$
17. Solve $(2y^3)dx + (3xy^2 - 1)dy = 0$
18. Solve the IVP: $ydx + (3x - 2y^2)dy = 0$, $y(0) = 1$

Part D: Choice Between $\mu(x)$ and $\mu(y)$ (Problems 19-23)

19. For $(2xy^2 + y)dx + xdy = 0$:
    (a) Show both $\mu(x)$ and $\mu(y)$ exist
    (b) Find both integrating factors
    (c) Solve using each and verify same solution
20. Find the simpler integrating factor and solve: $(3x^{2y}$ + 2y$^2)dx + x^{3dy} = 0$
21. Choose the appropriate integrating factor for: $(y \cos x + 1)dx + \sin x\, dy = 0$
22. For $(ax + by^2)dx + ydy = 0$, find conditions on a and b for:
    (a) $\mu(x)$ to exist
    (b) $\mu(y)$ to exist
23. Solve by choosing the simpler integrating factor: $(x^{2y3} + 2y)dx + xdy = 0$

Part E: Linear Equation Connection (Problems 24-26)

24. Show that $y' + \frac{2}{x}y = x^2$ leads to $\mu(x) = x^2$ and solve.
25. Convert to standard form and find integrating factor: $xy' - 2y = x^{3ex}$
26. Show that every linear equation $y' + P(x)y = Q(x)$ has $\mu(x) =$e$^{\int}$ P(x)dx

Part F: Exam-Style Problems (Problems 27-32)

27. (Prof. Ditkowski 2023) Given that $\mu = x^n isanintegratingfactorfor$(2xy + y^3)$dx +$ $(x^2 + xy^2)dy = 0$, find n and solve.

28. Show that $(3x^{2y}$ + y^2)dx+$(x^3$+$xy)dy = 0$ becomes exact when multiplied by $\mu = 1/xy$. Is this $\mu(x)$ or $\mu(y)$? Explain.

29. Find all integrating factors of the form $\mu = x^a$ y$^b for$:2ydx + xdy = 0

30. Given $(f(x) + 2y)dx + xdy = 0$ has $\mu(x) = x^2$:

    (a) Find $f(x)$

    (b) Solve the equation

31. For what value of k does $(ky + x^2)dx + (2x + y^2)dy = 0$ have:

    (a) An integrating factor $\mu(x)$?

    (b) An integrating factor $\mu(y)$?

32. A student claims that if an equation has both $\mu(x)$ and $\mu(y)$, then it must be exact. Prove or disprove with an example.

Solutions and Key Insights

Problem 1: (a) $M_y = 2$,N$_x = 1, notequal \to notexact(b)$(M$_y$ - N$_x$)/N = (2-1)/x = 1/x→$\mu(x)$ exists! (c) $(N_x$ - M$_y$)/M = (1-2)/(3x+2y)→ $containsboth$xandy$, no\mu(y)$ (d) $\mu(x) =$e$^{\int(1/x)dx}$ = x

Problem 7: Test: $(M_y$ - N$_x$)/N = (2x+2y-1)/x = (2x+2y-1)/x$Thiscontains$y$, sono\mu(x)$... Wait! Let's recheck: $M = 2xy + y^2$, $N = x$ $(M_y$ - N$_x$)/N = (2x + 2y - 0)/x = 2 + 2y/x$Stillhas$y$.Try\mu(y)$: $(N_x$ - M$_y$)/M = (1 - 2x - 2y)/(2xy + y^2) $= -1/y$ So $\mu(y) =$e$^{\int(-1/y)dy}$ = 1/y

Problem 19: For $(2xy^2 + y)dx + xdy = 0$: (a) $(M_y$ - N$_x$)/N = (4xy + 1 - 1)/x = 4y→ $No\mu(x)$ Actually, let me recalculate: M_y = 4xy + 1,N$_x$ = 1(M$_y$ - N$_x$)/N = 4xy/x = 4y→ $No\mu(x)$ $(N_x$ - M$_y$)/M = (1 - 4xy - 1)/(2xy$^2 + y) = -4xy/(y(2xy+1)) = -4x/(2xy+1)$ Hmm, this is complex. Let me reconsider the original equation...

Problem 24: $y' + (2/x)y = x^2$ becomes $(2y/x - x^2)dx + dy = 0$ $(M_y$ - N$_x$)/N = (2/x - 0)/1 = 2/x$\mu(x) =$e$^{\int(2/x)dx} = e^{2\ln|x|} =$ x^2 Multiply: $(2xy - x^4)dx + x^{2dy} = 0 Nowexact!$H = x^{2y} - x^5/5 Solution: x^{2y} - x^5/5 $= C$ or $y = x^3/5 + C/x^2$

Key Strategy: When both tests give functions of mixed variables, neither $\mu(x)$ nor $\mu(y)$ exists. Move to Lesson 24 for special forms!

Warning: Problem 32 - Counterexample: $2ydx + xdy = 0$ has both types but is not exact!