

Practice Problems - Lesson 38: Drawing Phase Portraits

ODE 1 Course

Part A: Basic Portrait Construction (Problems 1-6)

1. Draw complete phase portraits for:

- (a) $\dot{x} = x, \dot{y} = -y$ (saddle)
- (b) $\dot{x} = -2x, \dot{y} = -3y$ (stable node)
- (c) $\dot{x} = y, \dot{y} = -x$ (center)
- (d) $\dot{x} = x + y, \dot{y} = -x + y$ (spiral)

2. For the system $\dot{x} = 3x - 2y, \dot{y} = 2x - y$:

- (a) Find eigenvalues and eigenvectors
- (b) Draw eigenvector lines with correct arrows
- (c) Add at least 4 trajectories
- (d) Verify no trajectories cross

3. Given eigenvalues $\lambda_1 = 2, \lambda_2 = -3$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

- (a) Classify the equilibrium
- (b) Draw the phase portrait
- (c) Find the matrix A
- (d) Verify your portrait using nullclines

4. For $\dot{x} = x - y, \dot{y} = 2x - y$:

- (a) Find and draw both nullclines
- (b) Mark flow direction in each region
- (c) Find eigenvalues
- (d) Complete the phase portrait

5. Draw a phase portrait for:

$$A = \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix}$$

Show all five steps explicitly.

6. Create phase portraits with:
 - (a) Clockwise spiral, stable
 - (b) Counterclockwise center
 - (c) Improper node, unstable
 - (d) Saddle with horizontal stable manifold

Part B: Nullcline Method (Problems 7-11)

7. For $\dot{x} = y - x^2$, $\dot{y} = x - 2$ (nonlinear):
 - (a) Find nullclines
 - (b) Locate equilibria
 - (c) Determine flow in each region
 - (d) Linearize at each equilibrium
 - (e) Sketch the global portrait
8. Use nullclines to draw $\dot{x} = 2x - y + 1$, $\dot{y} = x - 2y$:
 - (a) Find where $\dot{x} = 0$ and $\dot{y} = 0$
 - (b) Find equilibrium (not at origin!)
 - (c) Shift coordinates to equilibrium
 - (d) Draw the portrait
9. For the competing species model:

$$\dot{x} = x(1 - x - 0.5y), \quad \dot{y} = y(1 - 0.5x - y)$$
 - (a) Find all four equilibria
 - (b) Draw nullclines (they're not straight!)
 - (c) Classify each equilibrium
 - (d) Sketch the first quadrant portrait
10. Given nullclines are the lines $x + y = 0$ and $x - 2y = 0$:
 - (a) Find possible systems with these nullclines
 - (b) Draw flow directions
 - (c) Complete a phase portrait
11. For $\dot{x} = \sin(y)$, $\dot{y} = x$:
 - (a) Sketch nullclines
 - (b) Find equilibria in $[-\pi, \pi] \times \mathbb{R}$
 - (c) Linearize at $(0, 0)$
 - (d) Draw local portrait near origin

Part C: Direction Fields (Problems 12-16)

12. Draw direction field arrows at the 8 points $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$ for:
- (a) $\dot{x} = y, \dot{y} = -x - y$
 - (b) $\dot{x} = x + 2y, \dot{y} = 3x + 2y$
 - (c) $\dot{x} = -y + x(x^2 + y^2), \dot{y} = x + y(x^2 + y^2)$
13. For the system $\dot{x} = ax + y, \dot{y} = x + ay$ with $a = 0.5$:
- (a) Compute direction vectors at 6 points
 - (b) Identify spiral rotation direction
 - (c) Estimate spiral pitch
 - (d) Draw complete portrait
14. Given the direction field: - At $(1, 0)$: arrow points to $(1, 1)$ - At $(0, 1)$: arrow points to $(-1, 1)$ - At $(-1, 0)$: arrow points to $(-1, -1)$ - At $(0, -1)$: arrow points to $(1, -1)$
- (a) What type of equilibrium is at origin?
 - (b) Find a compatible linear system
 - (c) Complete the portrait
15. Use isoclines to draw $\dot{x} = y, \dot{y} = -\sin(x)$:
- (a) Find curves where $dy/dx = 0, 1, -1, \infty$
 - (b) Mark trajectory slopes on each isocline
 - (c) Draw the phase portrait
16. For a system with spiral trajectories:
- (a) How many direction arrows minimum to determine rotation?
 - (b) Where should you place test points?
 - (c) Draw example with 3 arrows showing clockwise spiral

Part D: Special Cases (Problems 17-21)

17. Draw phase portraits for repeated eigenvalue cases:

- (a) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ (improper node)
- (b) $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (star node)
- (c) $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ (stable improper)

18. For the degenerate system $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$:

- (a) Show $\det(A) = 0$
- (b) Find the line of equilibria
- (c) Find the non-zero eigenvalue
- (d) Draw parallel trajectories

19. Draw accurate ellipses for the center:

$$\dot{x} = -2y, \quad \dot{y} = 0.5x$$

- (a) Find the period
- (b) Determine ellipse aspect ratio
- (c) Draw 3 nested closed orbits

20. For a stable spiral with eigenvalues $-1 \pm 3i$:

- (a) How many rotations before amplitude halves?
- (b) Draw showing at least 2 complete rotations
- (c) Mark points at $t = 0, \pi/3, 2\pi/3, \pi$

21. Draw the portrait for the shear flow:

$$\dot{x} = y, \quad \dot{y} = 0$$

- (a) Find eigenvalues
- (b) Identify all equilibria
- (c) Draw the unusual portrait
- (d) What physical system does this represent?

Part E: Portrait Verification (Problems 22-26)

22. Check these portraits for errors (each has one mistake):

- (a) Saddle with both manifolds attracting
- (b) Node with trajectories crossing
- (c) Spiral that changes rotation direction
- (d) Center with touching ellipses

23. Verify portrait consistency:

- (a) Do trajectories respect uniqueness?

- (b) Are arrow directions consistent?
 - (c) Do eigenvalue signs match portrait?
 - (d) Is tangency behavior correct for nodes?
24. For the portrait you drew in Problem 2:
- (a) Check using 3 additional test points
 - (b) Verify nullcline crossings
 - (c) Confirm no trajectory intersections
 - (d) Test symmetry if applicable
25. Quality assessment checklist:
- (a) Are at least 4 trajectories shown?
 - (b) Are arrows indicating flow present?
 - (c) Are axes and origin labeled?
 - (d) Is classification stated?
 - (e) Are special features marked?
26. Compare hand-drawn vs computer-generated portraits:
- (a) What details does computer miss?
 - (b) What does hand-drawing emphasize?
 - (c) How to combine both approaches?

Part F: Exam-Style Problems (Problems 27-30)

27. [Complete Portrait - Prof. Ditkowski Style] For the system:

$$\dot{x} = 4x - 5y, \quad \dot{y} = 2x - 2y$$

- (a) Find all equilibria
- (b) Calculate eigenvalues and eigenvectors
- (c) Classify the equilibrium type
- (d) Find and draw nullclines
- (e) Draw eigenvector directions with arrows
- (f) Test flow at points $(1, 0)$ and $(0, 1)$
- (g) Complete phase portrait with 6 trajectories
- (h) Verify using trace-determinant
- (i) State stability

28. **[Nonlinear with Multiple Equilibria]** Consider:

$$\dot{x} = y, \quad \dot{y} = x - x^3$$

- (a) Find all equilibria
- (b) Linearize at each equilibrium
- (c) Classify each equilibrium
- (d) Find the energy function $E(x, y)$
- (e) Use energy contours to draw portrait
- (f) Identify separatrices
- (g) Describe physical interpretation

29. **[Parameter-Dependent Portrait]** For the system depending on parameter μ :

$$\dot{x} = \mu x + y, \quad \dot{y} = -x + \mu y$$

- (a) Find eigenvalues in terms of μ
- (b) Determine portrait type for: - $\mu = -1$ - $\mu = 0$ - $\mu = 0.5$
- (c) Draw all three portraits
- (d) What happens as μ passes through 0?
- (e) Is this a bifurcation? What type?

30. **[Advanced Drawing Challenge]** Without computing eigenvalues explicitly, draw the portrait for:

$$A = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}$$

- (a) Use trace-determinant to classify
- (b) Use nullclines to structure portrait
- (c) Test 6 direction field points
- (d) Draw complete portrait
- (e) Then verify by computing eigenvalues
- (f) Compare your sketch to exact solution

Pro Tips for Drawing:

- Start with eigenvectors (the skeleton)
- Add nullclines (the grid)
- Sample direction field (the flow)
- Draw smooth curves following all three
- Check: no crossings, correct arrows, proper tangency

- Time limit: 5 minutes per portrait on exam!