ODE Lesson 5: Picard-Lindelöf Theorem - Existence and Uniqueness

ODE 1 - Prof. Adi Ditkowski

1 The Uniqueness Question

Motivation: Peano gives existence but not uniqueness. We need stronger conditions to guarantee a UNIQUE solution!

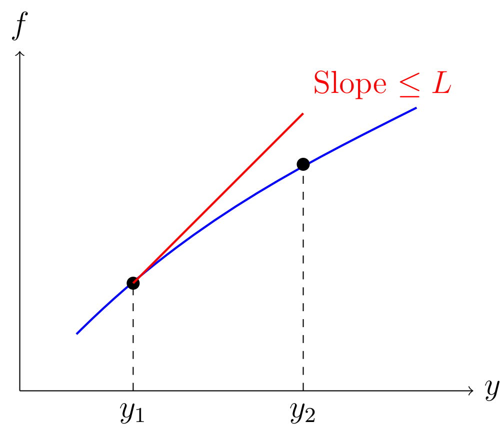
2 The Lipschitz Condition

Definition 1 (Lipschitz Continuity)**.** *A function $f(x, y)$ is **Lipschitz continuous in** y on a domain D if there exists a constant $L \geq 0$ such that:*

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$.

2.1 Geometric Interpretation



Lipschitz: bounded rate of change

3 Picard-Lindelöf Theorem

Picard-Lindelöf (Cauchy-Lipschitz) Theorem:

Consider the IVP: $\dfrac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$
If:

1. $f(x, y)$ is continuous in (x, y) on rectangle $R = \{|x - x_0| \leq a, |y - y_0| \leq b\}$
2. $f(x, y)$ is Lipschitz in y with constant L on R

Then there exists a **UNIQUE** solution $y(x)$ on the interval $|x - x_0| \leq h$ where:

$$h = \min\left(a, \frac{b}{M}\right), \quad M = \max_{(x,y)\in R} |f(x, y)|$$

4 Practical Lipschitz Test

Method 1 (Derivative Test for Lipschitz). *If $\frac{\partial f}{\partial y}$ exists and is bounded on R:*

$$\left|\frac{\partial f}{\partial y}\right| \leq L \quad \textit{for all } (x, y) \in R$$

Then f is Lipschitz in y with constant L.

4.1 Quick Check Algorithm

Steps to Verify Picard-Lindelöf:

1. Check continuity of $f(x, y)$ (as in Peano)
2. Compute $\frac{\partial f}{\partial y}$
3. Check if $\frac{\partial f}{\partial y}$ is bounded in your region
4. If yes to all $\Rightarrow$ Unique solution exists!

5 Examples: Lipschitz Analysis

Example 1 (Linear Case - Always Lipschitz). $\dfrac{dy}{dx} = 3x^2y + \sin(x)$, $y(0) = 1$

Analysis:

- $f(x, y) = 3x^2y + \sin(x)$
- $\frac{\partial f}{\partial y} = 3x^2$
- *On* $|x| \leq 1$*:* $|\frac{\partial f}{\partial y}| \leq 3$

- *Lipschitz with $L = 3 \Rightarrow$ Unique solution!*

Example 2 (Polynomial - Locally Lipschitz)**.** $\dfrac{dy}{dx} = y^2$, $y(0) = 1$

Analysis:

- $f(x, y) = y^2$
- $\frac{\partial f}{\partial y} = 2y$
- *On $|y - 1| \leq 2$: $|\frac{\partial f}{\partial y}| \leq 6$*
- *Locally Lipschitz with $L = 6 \Rightarrow$ Unique solution locally!*

Example 3 (Non-Lipschitz at Initial Point)**.** $\dfrac{dy}{dx} = \sqrt{y}$, $y(0) = 0$

Analysis:

- $f(x, y) = \sqrt{y}$
- $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}} \to \infty$ *as $y \to 0^+$*
- *NOT Lipschitz at $y = 0$!*
- *Solutions: $y = 0$ and $y = \frac{x^2}{4}$ for $x \geq 0$*
- *Non-unique as predicted!*

6 Picard Iteration Method

The proof of Picard-Lindelöf is constructive - it shows how to build the solution!

Method 2 (Picard Iteration)**.** *Starting with $y_0(x) = y_0$ (constant), iterate:*

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$$

The sequence $\{y_n(x)\}$ converges to the unique solution.

Example 4 (Picard Iteration for $y' = y$, $y(0) = 1$)**.**

$$y_0(x) = 1 \tag{1}$$

$$y_1(x) = 1 + \int_0^x 1\, dt = 1 + x \tag{2}$$

$$y_2(x) = 1 + \int_0^x (1 + t)\, dt = 1 + x + \frac{x^2}{2} \tag{3}$$

$$y_3(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \tag{4}$$

$$\vdots \tag{5}$$

$$y_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \to e^x \text{ as } n \to \infty \tag{6}$$

7 Global vs Local Lipschitz

Type	Definition	Example
Globally Lipschitz	One L works everywhere	$f(y) = \sin(y)$, $L = 1$
Locally Lipschitz	Different L for different regions	$f(y) = y^2$, L depends on bounds
Not Lipschitz	No finite L at some points	$f(y) = \sqrt{\lvert y \rvert}$ at $y = 0$

8 Comparison: Peano vs Picard-Lindelöf

Peano	Stronger $\longrightarrow$	Picard-Lindelöf
Continuity		Lipschitz
Existence only		Existence + Uniqueness

9 Continuous Dependence on Initial Conditions

Theorem 1 (Stability Result). *If f is Lipschitz with constant L, and we have two IVPs:*

$$y' = f(x, y), \quad y(x_0) = a \tag{7}$$

$$z' = f(x, z), \quad z(x_0) = b \tag{8}$$

Then: $|y(x) - z(x)| \le |a - b| e^{L|x - x_0|}$

This shows solutions depend continuously on initial conditions!

10 Special Cases Always Satisfying Picard-Lindelöf

Always Have Unique Solutions:

1. Linear ODEs: $y' + p(x)y = q(x)$ with continuous p, q
2. Equations with bounded $\partial f / \partial y$
3. f linear in y: $f(x, y) = a(x)y + b(x)$

11 Common Non-Lipschitz Functions

Watch Out For These at $y = 0$:

- $f(y) = |y|^\alpha$ for $0 < \alpha < 1$
- $f(y) = \sqrt{|y|}$

- $f(y) = y^{2/3}$
- $f(y) = y \ln |y|$

12 Memory Device

LIPSCHITZ: "Limited Increase Prevents Solutions Careening Haphazardly Into Total Zones"

13 Exam Strategy

Prof. Ditkowski's Picard-Lindelöf Questions:

1. "Verify uniqueness" $\Rightarrow$ Check Lipschitz condition
2. "Find the Lipschitz constant" $\Rightarrow$ Bound $|\partial f/\partial y|$
3. "Why is the solution unique?" $\Rightarrow$ Show Lipschitz holds
4. "Construct solution using Picard iteration" $\Rightarrow$ Do 3-4 iterations