Phase Space and Trajectories: Geometric Theory of ODEs

ODE 1 - Lesson 36

1 Phase Space Fundamentals

Definition 1 (Phase Space)**.** *For a system of n first-order ODEs:*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t)$$

*where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, the **phase space** is the n-dimensional space with coordinates $(x_1, x_2, \ldots, x_n)$.*

Key Insight: Phase space represents all possible states of the system. Each point corresponds to a unique state, regardless of time.

Definition 2 (Trajectory/Orbit)**.** *A **trajectory** (or **orbit**) is the curve traced out in phase space by a solution $\mathbf{x}(t)$ as t varies. Formally, it is the set:*

$$\Gamma = \{\mathbf{x}(t) : t \in I\}$$

where I is the interval of existence.

Critical Distinction:

- **Solution:** $\mathbf{x}(t)$ - includes time parametrization
- **Trajectory:** The geometric curve - no time information

Multiple solutions can give the same trajectory (time-shifted solutions)!

2 Direction Fields and Flow

Definition 3 (Direction Field)**.** *The **direction field** (or **vector field**) of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ assigns to each point $\mathbf{x}$ the vector $\mathbf{f}(\mathbf{x})$, indicating the instantaneous direction and speed of motion.*

Theorem 1 (Non-Intersection of Trajectories)**.** *If $\mathbf{f}(\mathbf{x})$ satisfies the conditions for existence and uniqueness, then trajectories cannot intersect except at equilibrium points.*

Proof. Suppose two trajectories Γ_1 and Γ_2 intersect at point $\mathbf{x}_0$ which is not an equilibrium. By uniqueness, the solution starting at $\mathbf{x}_0$ is unique, so $\Gamma_1 = \Gamma_2$. Contradiction. □

3 Equilibrium Points and Invariant Sets

Definition 4 (Equilibrium Point). *A point* $\mathbf{x}^*$ *is an* ***equilibrium point*** *(or* ***critical point, fixed point****) if:*

$$\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$$

Finding Equilibria - Prof. Ditkowski's Method:

1. Set all derivatives equal to zero
2. Solve the resulting algebraic system
3. Check each solution carefully
4. State coordinates explicitly: "Equilibrium at $(x^*, y^*) = (a, b)$"

Definition 5 (Invariant Set). *A set* $S \subset \mathbb{R}^n$ *is* ***invariant*** *under the flow if:*

$$\mathbf{x}(0) \in S \implies \mathbf{x}(t) \in S \text{ for all } t$$

4 Special Types of Trajectories

Definition 6 (Closed Orbit). *A trajectory* Γ *is a* ***closed orbit*** *if it is homeomorphic to a circle and the solution is periodic:*

$$\exists T > 0 : \mathbf{x}(t + T) = \mathbf{x}(t) \text{ for all } t$$

Dimension Restriction: Closed orbits cannot exist in 1D phase space! In 1D, trajectories are confined to the real line and cannot loop back without violating uniqueness.

Definition 7 (Heteroclinic and Homoclinic Orbits).

- ***Heteroclinic orbit:*** *Connects two different equilibria*
- ***Homoclinic orbit:*** *Starts and ends at the same equilibrium*

5 Phase Portraits for 2D Systems

For a 2D autonomous system:

$$\frac{dx}{dt} = f(x, y) \tag{1}$$

$$\frac{dy}{dt} = g(x, y) \tag{2}$$

Construction Steps:

1. Find all equilibria: solve $f(x, y) = 0$ and $g(x, y) = 0$
2. Compute the direction field at selected points
3. Identify special trajectories (if any)
4. Sketch trajectories following the direction field
5. Add arrows indicating flow direction

6 Converting Higher-Order Equations

Example 1 (Second-Order to First-Order System)**.** *Convert $\ddot{x} + p(x)\dot{x} + q(x) = 0$ to phase space form:*

Let $x_1 = x$ and $x_2 = \dot{x}$. Then:

$$\dot{x}_1 = x_2 \tag{3}$$

$$\dot{x}_2 = -p(x_1)x_2 - q(x_1) \tag{4}$$

Phase space is the (x_1, x_2)-plane, often relabeled as $(x, \dot{x})$-plane.

7 Nullclines Method

Definition 8 (Nullclines)**.** *The **nullclines** are curves where one component of the vector field vanishes:*

- *x-**nullcline:** $f(x, y) = 0$ (vertical flow)*
- *y-**nullcline:** $g(x, y) = 0$ (horizontal flow)*

Using Nullclines:

- Equilibria occur at nullcline intersections
- Trajectories cross nullclines vertically or horizontally
- Nullclines divide phase space into regions with consistent flow direction

8 Exam-Critical Formulas

Must-Know for Prof. Ditkowski's Exam:

Concept	Formula/Property
Equilibrium condition	$\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$
Trajectory uniqueness	No intersections except at equilibria
Closed orbit period	$\mathbf{x}(t + T) = \mathbf{x}(t)$
Direction at point (x, y)	Vector $(f(x, y), g(x, y))$
Speed along trajectory	$\lvert\mathbf{f}(\mathbf{x})\rvert = \sqrt{f^2 + g^2}$