Practice Problems: Direct Integration

Lesson 11 Exercises

August 23, 2025

Part A: Recognition and Classification (6 problems)

1. Which of the following ODEs can be solved by direct integration?
   (a) $y' = x^2 + y$
   (b) $y' = e^x \cos(x)$
   (c) $y' = \frac{y}{x}$
   (d) $y'' = \sin(x) - 2$

2. For the ODE $y^{(4)} = 0$, what is the general form of the solution?

3. How many arbitrary constants appear in the general solution of $\frac{d^5y}{dx^5} = e^x$?

4. True or False: If $y'' = f(x)$ has solution $y = g(x)$, then $y = g(x) + Ax + B$ is also a solution for any constants A, B.

5. What is the minimum number of initial conditions needed to uniquely determine the solution of $y''' = x^2$?

6. If $dy/dx = f(x)$ and $f(x)$ is odd, what can you say about the solution $y(x)$ passing through the origin?

Part B: Basic Integration Problems (6 problems)

7. Solve: $\frac{dy}{dx} = 3x^2 - 4x + 1$

8. Solve: $\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x^2}$ for $x > 0$

9. Find the general solution: $y' = e^{2x} + \sin(3x)$

10. Solve: $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ for $|x| < 1$

11. Find $y(x)$ if $y' = \sec^2(x)$ and $y(0) = 2$

12. Solve: $\frac{dy}{dx} = |x|$ (consider both $x \geq 0$ and $x < 0$)

Part C: Higher-Order Direct Integration (6 problems)

13. Solve: $y'' = 6x$ with $y(0) = 1$ and $y'(0) = -1$

14. Find the general solution: $\frac{d^3y}{dx^3} = 24$

15. Solve: $y'' = \cos(x)$ with $y(0) = 0$ and $y(\pi) = 0$

16. Find $y(x)$ if $y''' = e^x$ with $y(0) = y'(0) = y''(0) = 1$

17. Solve: $\frac{d^4y}{dx^4} = 0$ with $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$, $y'''(0) = 0$

18. A particle moves with constant jerk (rate of change of acceleration) $j = 2$ m/s³. If at $t = 0$ the particle is at rest at the origin with zero acceleration, find its position at time t.

Part D: Initial Value Problems with Definite Integrals (5 problems)

19. Using the definite integral method, solve: $y' = x^2$ with $y(1) = 3$

20. Express the solution to $y' = \sin(x^2)$ with $y(0) = 1$ as a definite integral.

21. Solve: $y' = \frac{1}{1+x^2}$ with $y(0) = \pi/4$

22. Find $y(2)$ if $y' = e^{-x^2}$ and $y(0) = 0$ (express as an integral if needed).

23. Show that the solution to $y' = f(x)$ with $y(a) = A$ and $y(b) = B$ must satisfy: $B - A = \int_a^b f(x)\,dx$

Part E: Mixed Exam-Style Questions (7 problems)

24. [**Prof. Ditkowski Special**] Consider $y'' = x\sin(x)$.

    (a) Find the general solution

    (b) Find the particular solution with $y(0) = 1$ and $y'(\pi) = 0$

    (c) Verify your solution

25. [**Conceptual**] If y_1 is a solution to $y'' = f(x)$ and y_2 is a solution to $y'' = g(x)$, what ODE does $y_1 + y_2$ solve?

26. [**Multi-part**] A projectile is launched vertically with $y'' = -10$ m/s² (taking up as positive).

    (a) Find the general solution for height $y(t)$

    (b) If launched from ground level with initial velocity 30 m/s, find $y(t)$

(c) When does it reach maximum height?

(d) What is the maximum height?

27. [**Verification Focus**] A student claims that $y = x^3/3 + \ln|x| + 2x + 5$ solves $y' = x^2 + 1/x + 2$ with $y(1) = 5 + 7/3$. Verify or disprove this claim.

28. [**Domain Awareness**] Solve $y' = \frac{1}{x\ln(x)}$ for $x > 1$ with $y(e) = 0$. State the domain of your solution.

29. [**Integration Challenge**] Find all functions $y(x)$ such that $y''' = y$ and $y(0) = y'(0) = y''(0) = 0$.

30. [**Prof. Ditkowski Comprehensive**] Consider the ODE: $\frac{d^2y}{dx^2} = \frac{2}{(1+x^2)^2}$

    (a) Show that $y' = \frac{x}{1+x^2} + C_1$ is the first integral

    (b) Find the general solution

    (c) Find the solution satisfying $y(0) = 0$ and $\lim_{x\to\infty} y'(x) = 1$

    (d) Sketch the solution curve

Answer Key with Essential Hints

1. Only (b) and (d) can be solved by direct integration
2. $y = C_1x^3 + C_2x^2 + C_3x + C_4$ (cubic polynomial)
5. Three initial conditions needed
7. $y = x^3 - 2x^2 + x + C$
9. $y = \frac{1}{2}e^{2x} - \frac{1}{3}\cos(3x) + C$
13. $y = x^3 - x + 1$
14. $y = 4x^3 + C_1x^2 + C_2x + C_3$
18. $y = \frac{t^3}{3}$ (position function)
19. $y = \frac{x^3}{3} - \frac{1}{3} + 3 = \frac{x^3}{3} + \frac{8}{3}$
22. $y(2) = \int_0^2 e^{-t^2} dt$ (cannot be expressed in elementary functions)
24. Use integration by parts: $\int x\sin(x)dx = -x\cos(x) + \sin(x) + C$
26. Max height at $t = 3$ seconds, height $= 45$ meters
28. Solution: $y = \ln|\ln(x)|$, domain: $x > 1$, $x \neq e$
30. First show that $\int \frac{2dx}{(1+x^2)^2} = \frac{x}{1+x^2} + C$ using substitution