Lesson 29: Practice Problems

Liouville's Formula and Applications

Part A: Basic Liouville Calculations (6 problems)

1. Use Liouville's formula to find $W(t)$ for $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{x}$ with $W(0) = 5$.

2. Calculate the Wronskian at $t = 2$ for the system $\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x}$ if $W(0) = 1$.

3. Find $W(t)$ for solutions of $y''' - 3y'' + 2y' - y = 0$ with $W(0) = 2$.

4. Given $\mathbf{x}' = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{x}$, find $W(t)$ using Liouville.

5. For the harmonic oscillator $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \mathbf{x}$, verify that volume is preserved.

6. Calculate $W(\pi)$ for $\mathbf{x}' = \begin{bmatrix} \sin t & 0 \\ 0 & \cos t \end{bmatrix} \mathbf{x}$ with $W(0) = 3$.

Part B: Trace and Eigenvalue Connections (5 problems)

7. A 3×3 system has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = -4$. Find $W(t)/W(0)$.

8. If $\text{tr}(A) = -5$ for a constant matrix, and $W(1) = e^{-5}$, find $W(3)$.

9. The characteristic polynomial is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$. Find $\text{tr}(A)$ and describe $W(t)$ behavior.

10. Given eigenvalues $1 \pm 2i$, find the trace and Wronskian evolution for the 2×2 system.

11. A system has $W(t) = 3e^{-6t}$. If two eigenvalues are -1 and -2, find the third.

Part C: Stability Analysis (5 problems)

12. Determine stability of $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix} \mathbf{x}$ using Liouville's formula.

13. For what values of a is the system $\mathbf{x}' = \begin{bmatrix} a & 1 \\ -1 & -a \end{bmatrix} \mathbf{x}$ volume-preserving?

14. Analyze stability of $y'' + 3y' + 2y = 0$ using the trace of its companion matrix.

15. Given $\ddot{x} + b\dot{x} + 4x = 0$, find b values for which the Wronskian decays.

16. Determine long-term behavior of $W(t)$ for $\mathbf{x}' = \begin{bmatrix} t & 1 \\ 0 & -t \end{bmatrix} \mathbf{x}$.

Part D: Special Systems (5 problems)

17. Show that the system $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \mathbf{x}$ preserves volume.

18. For the Hamiltonian system with $H = \frac{1}{2}(p^2 + q^2)$, verify Liouville's theorem.

19. Find all 2×2 matrices A with $\mathrm{tr}(A) = 0$ and $\det(A) = 1$.

20. Prove that skew-symmetric matrices ($A^T = -A$) always have trace zero.

21. For the periodic system $\mathbf{x}' = \begin{bmatrix} \cos(2t) & 0 \\ 0 & -\cos(2t) \end{bmatrix} \mathbf{x}$, find $W(2\pi)/W(0)$.

Part E: Applications and Theory (4 problems)

22. Use Liouville to prove that if all eigenvalues have negative real parts, then $W(t) \to 0$ as $t \to \infty$.

23. Show that for the equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$, the Wronskian satisfies $W(t) = W(0)e^{-a_{n-1}t}$.

24. If two solutions have Wronskian $W_{12}(t) = e^{3t}$, what can you conclude about the trace of the system matrix?

25. Prove that similar matrices have the same trace, hence the same Wronskian evolution.

Part F: Exam-Style Problems (5 problems)

26. (Prof. Ditkowski style) Consider $\mathbf{x}' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}$:

    (a) Find $\text{tr}(A)$ and all eigenvalues

    (b) Use Liouville to find $W(t)$ with $W(0) = 1$

    (c) Verify using direct eigenvalue sum

    (d) Determine stability of the origin

    (e) Find $\lim_{t\to\infty} W(t)$

27. The damped oscillator $\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0$ has parameters $\omega = 2$, $\zeta = 0.5$.

    (a) Write as a first-order system

    (b) Find the trace

    (c) Calculate $W(t)/W(0)$

    (d) How long until the Wronskian decreases by factor of e?

28. Given that solutions to a third-order system have $W(t) = 5e^{-3t}$:

    (a) Find $\text{tr}(A)$

    (b) If $\lambda_1 = 2$ is an eigenvalue, and the other two are complex conjugates, find them

    (c) Write a possible matrix A

    (d) Analyze stability

29. For the time-dependent system $\mathbf{x}' = \begin{bmatrix} e^t & 0 \\ 0 & -e^t \end{bmatrix} \mathbf{x}$:

    (a) Compute $\text{tr}(A(t))$

    (b) Find $W(t)$ using Liouville with $W(0) = 2$

    (c) Determine if volume is preserved

    (d) Find $\lim_{t\to\infty} W(t)$

30. (Comprehensive) Consider the fourth-order equation $y^{(4)} - 2y''' - 3y'' + 4y' + 4y = 0$.

    (a) Convert to a system and find $\text{tr}(A)$

    (b) Use Liouville to express $W(t)$

    (c) Given that $\lambda_1 = 2$ and $\lambda_2 = -1$ are eigenvalues, find the others

    (d) Verify $\sum \lambda_i = \text{tr}(A)$

    (e) Determine the long-term behavior of solutions

    (f) Is the zero solution stable?

Solutions and Hints

Selected Solutions:

Problem 1: $\text{tr}(A) = 3 + (-2) = 1$, so $W(t) = 5e^t$
Problem 2: $\text{tr}(A) = 1 + 2 + (-3) = 0$, so $W(2) = W(0) = 1$
Problem 5: $\text{tr}(A) = 0 + 0 = 0$, volume preserved!
Problem 7: $W(t) = W(0) \cdot e^{(2-1-4)t} = W(0) \cdot e^{-3t}$
Problem 12: $\text{tr}(A) = -2 + (-4) = -6 < 0$, system is stable
Problem 13: Need $a + (-a) = 0$ for all a, so always volume-preserving
Problem 17: $\text{tr}(A) = 0 + 0 + 0 = 0$, confirms volume preservation
Problem 22: Average trace over period is zero, so $W(2\pi) = W(0)$
Problem 26: $\text{tr}(A) = 1 + (-1) + 2 = 2$, so $W(t) = e^{2t}$, unstable
Problem 29: $\text{tr}(A) = -3$, so $\lambda_2 + \lambda_3 = -5$. With conjugates $a \pm bi$: $2a = -5$, so $\lambda_{2,3} = -2.5 \pm bi$

Key Formulas:

- Liouville: $W(t) = W(t_0)e^{\int_{t_0}^{t} \text{tr}(A(s))ds}$
- Constant case: $W(t) = W(0)e^{\text{tr}(A)\cdot t}$
- Trace-eigenvalue: $\text{tr}(A) = \sum \lambda_i$
- Scalar nth-order: trace $= -a_{n-1}$
- Volume preserved $\Leftrightarrow \text{tr}(A) = 0$

Stability Quick Check:

- $\text{tr}(A) < 0 \Rightarrow W(t) \to 0$ (stable tendency)
- $\text{tr}(A) > 0 \Rightarrow W(t) \to \infty$ (unstable)
- $\text{tr}(A) = 0 \Rightarrow W(t) =$ constant (neutral)