

# Lesson 21: Exact Equations - Theory and Recognition

ODE 1 - Prof. Adi Ditkowski

## 1 Introduction to Exact Equations

**Definition 1** (Exact Differential Equation). *A first-order differential equation of the form*

$$M(x, y)dx + N(x, y)dy = 0$$

*is called **exact** if there exists a function  $H(x, y)$  such that*

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy = M(x, y)dx + N(x, y)dy$$

When an equation is exact, its solution curves are the level curves  $H(x, y) = C$  of the potential function  $H$ .

## 2 The Exactness Criterion

**Theorem 1** (Test for Exactness). *Let  $M(x, y)$  and  $N(x, y)$  have continuous partial derivatives in a simply connected domain  $D$ . The equation  $M(x, y)dx + N(x, y)dy = 0$  is exact if and only if*

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

*Proof.* ( $\Rightarrow$ ) If the equation is exact, then  $M = \frac{\partial H}{\partial x}$  and  $N = \frac{\partial H}{\partial y}$  for some  $H$ . By Schwarz's theorem:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 H}{\partial y \partial x} = \frac{\partial^2 H}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

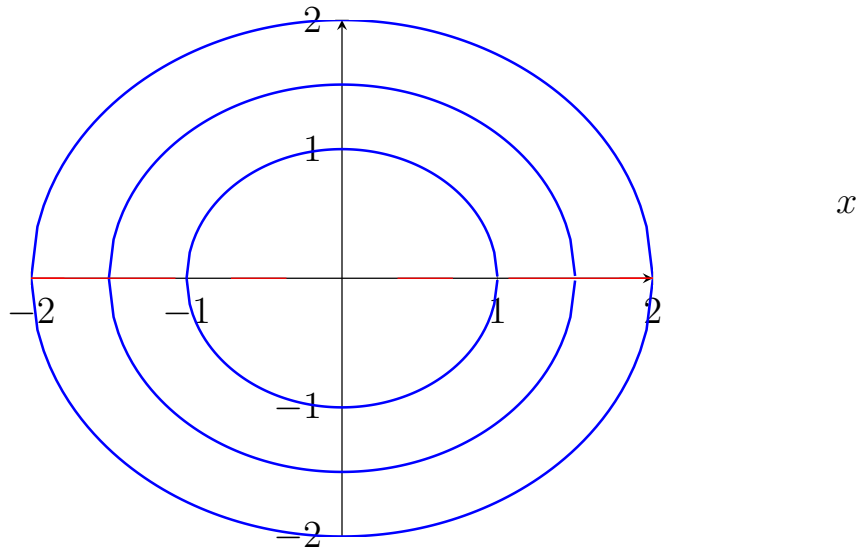
( $\Leftarrow$ ) The converse requires constructing  $H$  from the condition, shown in Lesson 22.  $\square$

## 3 Geometric Interpretation

The vector field  $\mathbf{F} = (M, N)$  is conservative (has a potential function) if and only if its curl vanishes:

$$\text{curl}(\mathbf{F}) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

Level Curves  $H(x, y) = C$  and Vector Field  $(M, N)$



## 4 Connection to Physics

In thermodynamics, exact differentials correspond to state functions:

- Internal Energy:  $dU = TdS - PdV$  (exact)
- Enthalpy:  $dH = TdS + VdP$  (exact)
- Work:  $\delta W = PdV$  (not exact - path dependent)
- Heat:  $\delta Q = TdS$  (not exact - path dependent)

## 5 Algorithm for Testing Exactness

**Step-by-Step Exactness Test:**

1. Write equation in standard form:  $M(x, y)dx + N(x, y)dy = 0$
2. Identify  $M(x, y)$  and  $N(x, y)$  explicitly
3. Compute  $\frac{\partial M}{\partial y}$  (show all steps)
4. Compute  $\frac{\partial N}{\partial x}$  (show all steps)
5. Compare the results:

- If equal  $\Rightarrow$  equation is exact
- If not equal  $\Rightarrow$  equation is not exact

6. State conclusion explicitly

## 6 Common Forms and Patterns

Recognize these patterns that often appear in Prof. Ditkowski's exams:

1. **Polynomial Forms:**  $(ax^ny^m + bx^py^q)dx + (cx^ry^s + dx^ty^u)dy = 0$
2. **Exponential Forms:**  $(ae^{x+y} + bx)dx + (ce^{x+y} + dy)dy = 0$  **Trigonometric:**  $(\cos(xy) \cdot y + f(x))dx + (\cos(xy) \cdot x + g(y))dy = 0$
3. **Mixed:**  $(x^2y + \sin x)dx + (x^3/3 + e^y)dy = 0$

## 7 Domain Considerations

The exactness condition guarantees existence of a potential function only in simply connected domains. Watch for:

- Punctured plane:  $\mathbb{R}^2 \setminus \{(0,0)\}$
- Domains with holes or discontinuities
- Multi-valued potential functions

## 8 Quick Reference

Memory Aid: "My Nexus"

$M_y$	Derivative of $M$ with respect to $y$
$N_x$	Derivative of $N$ with respect to $x$

If  $M_y = N_x$ , the equation is exact!