Lesson 48: Practice Problems - Power Series Solutions

ODE 1 - Prof. Adi Ditkowski

Part A: Identifying Ordinary Points (6 problems)

1. Classify all points as ordinary or singular for:

$$y'' + \frac{1}{x}y' + y = 0$$

2. For the equation $(x^2 - 4)y'' + xy' + y = 0$, identify all singular points and classify them.

3. Show that $x = 0$ is an ordinary point of:

$$y'' + e^x y' + \sin(x)y = 0$$

4. Determine the ordinary points of:

$$y'' + \frac{2x}{1-x^2}y' + \frac{1}{1-x^2}y = 0$$

5. For Bessel's equation $x^2y'' + xy' + (x^2 - n^2)y = 0$, explain why $x = 0$ is NOT an ordinary point.

6. Find all ordinary points in the complex plane for:

$$y'' + \frac{1}{x^2+1}y = 0$$

Part B: Setting Up Power Series (6 problems)

7. Set up (but don't solve) the power series solution around $x = 0$ for:

$$y'' - xy' + 2y = 0$$

8. Write the first four terms of y, y', and y'' if:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

9. Express xy' and x^2y'' in summation form if $y = \sum_{n=0}^{\infty} a_n x^n$.

10. For the series $y = \sum_{n=0}^{\infty} a_n(x-1)^n$, write y'' and $(x-1)y'$ in summation notation.

11. Show how to shift indices to combine:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

12. Rewrite with common powers of x:

$$\sum_{n=3}^{\infty} a_n x^{n-3} - 2\sum_{n=1}^{\infty} n a_n x^{n+1}$$

Part C: Finding Recurrence Relations (5 problems)

13. Find the recurrence relation for:
$$y'' + xy = 0$$

14. Derive the recurrence relation for:

$$y'' - 2xy' + 4y = 0$$

15. Find the recurrence for the Hermite equation:

$$y'' - 2xy' + 6y = 0$$

16. Determine the recurrence relation for:

$$(1+x)y'' + y' - y = 0$$

    around $x = 0$.

17. Find the three-term recurrence for:

$$y'' + xy' + y = 0$$

Part D: Computing Series Coefficients (5 problems)

18. Given $y'' - xy = 0$ with $y(0) = 1$, $y'(0) = 0$, find a_0 through a_6.

19. For $y'' + y = 0$ with initial conditions $y(0) = 1$, $y'(0) = 1$, compute the first 5 non-zero terms and identify the solution.

20. Solve $y'' - 2xy' + 4y = 0$ with $y(0) = 2$, $y'(0) = 0$. Find coefficients up to x^4.

21. For the Airy equation $y'' - xy = 0$, express a_6 and a_7 in terms of a_0 and a_1.

22. Given the recurrence $a_{n+2} = \frac{n-2}{(n+2)(n+1)} a_n$ with $a_0 = 1$, $a_1 = 0$, find which coefficients are zero.

Part E: Convergence Analysis (3 problems)

23. Find the radius of convergence for the series solution of:

$$y'' + \frac{1}{x^2 - 1} y = 0$$

around $x = 0$.

24. Determine the convergence region for:

$$(x^2 + 2x + 5)y'' + xy' + y = 0$$

around $x = 0$.

25. Explain why the series solution of $y'' + y = 0$ around any point converges everywhere.

Part F: Exam-Style Problems (5 problems)

26. [**10 points**] Consider the equation:

$$y'' - 2xy' + (\lambda - 1)y = 0$$

2 pts Verify that $x = 0$ is an ordinary point

4 pts Find the recurrence relation

3 pts Find the first four non-zero terms when $\lambda = 5$

1 pt What is the radius of convergence?

27. [**8 points**] For the modified Airy equation:

$$y'' + (x - 1)y = 0$$

3 pts Find the recurrence relation for solutions around $x = 1$

3 pts Express a_3, a_4, a_5 in terms of a_0 and a_1

2 pts Which pattern do you observe?

28. [**9 points**] Solve using power series:

$$y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

Find coefficients through x^6.

29. [**7 points**] The equation $y'' - 2xy' + 2ny = 0$ has polynomial solutions when $n \in \mathbb{N}$.

3 pts Find the recurrence relation

2 pts Show that $a_{n+1} = 0$ when n is a positive integer

2 pts Find the polynomial solution for $n = 3$

30. **[10 points]** *Challenge - Prof. Ditkowski Special* Consider: $(1-x^2)y'' - xy' + \alpha^2 y = 0$

2 pts Find all singular points

3 pts Derive the recurrence around $x = 0$

3 pts Show the series has only even or odd powers

2 pts Find radius of convergence

Solutions and Hints

Selected Solutions:

Problem 1: $x = 0$ is a singular point (p(x) = 1/x not analytic at 0)

Problem 13: Recurrence: $(n+2)(n+1)a_{n+2} + a_n = 0$, so $a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$

Problem 18: With $y(0) = 1, y'(0) = 0$: $a_0 = 1, a_1 = 0, a_2 = 0, a_3 = 1/6, a_4 = 0, a_5 = 0, a_6 = 1/180$

Problem 23: Radius = 1 (singular points at $x = \pm 1$)

Key Insight for Problem 28: When $(1-x^2)$ appears, expect Legendre-type behavior with $R = 1$.