Lesson 47: Euler-Cauchy Equations

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1 Introduction

Definition 1 (Euler-Cauchy Equation)**.** *An n-th order Euler-Cauchy equation (also called an equidimensional equation) has the form:*

$$t$$

^{n} $y^{(n)} + a_{n-1}t^{n-1}y^{(n-1)} + \cdots + a_1 ty' + a_0 y = 0 where a_0, a_1, \ldots, a_{n-1}$ *are constants and* $t \neq 0$.

The defining characteristic: the power of t equals the order of the derivative in each term. This creates a scaling symmetry that makes these equations solvable by elementary methods.

2 Method 1: Power Function Ansatz

Theorem 1 (Power Solutions)**.** *For the Euler equation, solutions have the form* $y = t^r wherersatisfiesanal$

2.1 Second-Order Case

Consider the second-order Euler equation:

$$t^2y'' + aty' + by = 0, \quad t > 0$$

Method 1 (Direct Solution Method)**.** *1. Assume* $y = t^r forsomeconstantrComputederivatives:$

2. *Substitute into the equation:*

$$t^2 \cdot r(r-1)t^{r-2} + at \cdot rt^{r-1} + b \cdot t$$

^{r} *= 0*

$$Simplify:$$

t^{r} [r(r-1) + ar + b] = 0

3. *Since $t^r \neq 0$ for $t > 0$, we get the characteristic equation:*

$$r^2 + (a-1)r + b = 0$$

Characteristic Equation for Second-Order Euler:

$$\boxed{r(r-1) + ar + b = 0 \quad \text{or} \quad r^2 + (a-1)r + b = 0}$$

2.2 Solution Forms Based on Roots

Theorem 2 (Complete Solution Classification)**.** *Let r_1, r_2 be roots of the characteristic equation. Then:*

1. ***Distinct real roots**: $y = c_1 t^{r_1} + c_2 t^{r_2}$*

2. ***Repeated real root** r: $y = (c_1 + c_2 \ln t)t^r$ **Complex roots**$r = \alpha \pm i\beta$:*

$$y = t^\alpha [c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)]$$

3. *Proof of Repeated Root Case.* When r is a repeated root, we use reduction of order. Given $y_1 = t^r, weseek y_2 = v(t) \cdot t^r$.

The Euler equation in standard form:

$$y'' + \frac{a}{t}y' + \frac{b}{t^2}y = 0$$

Using the reduction formula with $p(t) = a/t$:

$$v = \int$$

e$^{-\int}$ (a/t) dt$\overline{}_{t^{2r}dt = \int \frac{t^{-a}}{t^{2r}}dt}$

For a repeated root, $r = (1-a)/2$, so $2r = 1 - a$. Thus:

$$v = \int \frac{t^{-a}}{t^{1-a}} dt = \int \frac{1}{t} dt = \ln t$$

Therefore, $y_2 = t^r \ln t$. □

3 Method 2: Logarithmic Transformation

Theorem 3 (Transformation to Constant Coefficients)**.** *The substitution $x = \ln t$ (equivalently, $t = e^x$)$transformsanEulerequationintoaconstantcoefficientequation$.*

3.1 Derivative Transformations

Lemma 1 (Change of Variables)**.** *Under the substitution $x = \ln t$, if $v(x) = y(t)$:*

$$t\frac{dy}{dt} = \frac{dv}{dx} \tag{1}$$

$$t^2\frac{d^{2y}}{dt^2} = \frac{d^{2v}}{dx^2} - \frac{dv}{dx} \tag{2}$$

$$t^3\frac{d^{3y}}{dt^3} = \frac{d^{3v}}{dx^3} - 3\frac{d^{2v}}{dx^2} + 2\frac{dv}{dx} \tag{3}$$

Proof for Second Derivative. Using the chain rule:

$$\frac{dy}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{1}{t}\frac{dv}{dx}$$

For the second derivative:

$$d$$

$^{2y}\frac{}{dt^2 = \frac{d}{dt}\left(\frac{1}{t}\frac{dv}{dx}\right) = -\frac{1}{t^2}\frac{dv}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dv}{dx}\right)}$

Since $\frac{d}{dt}\left(\frac{dv}{dx}\right) = d^{2v}\frac{}{dx^2 \cdot \frac{dx}{dt} = \frac{1}{t}d}{}^{2v}\frac{}{dx^2}$:

$$d$$

$^{2y}\frac{}{dt^2 = \frac{1}{t^2}(d}{}^{2v}\frac{}{dx^2 - \frac{dv}{dx}}$ □

3.2 The Transformed Equation

Theorem 4 (Constant Coefficient Form)**.** *The Euler equation t^{2y}" + aty' + by = 0becomes :$v''+$ $(a-1)v' + bv = 0$wherev(x) = y(e^x)andx = $\ln t$.*

The characteristic polynomial of the transformed equation is identical to the characteristic equation from the direct method! This confirms the deep connection between the two approaches.

4 Higher-Order Euler Equations

Theorem 5 (General n-th Order)**.** *For the n-th order Euler equation, assume $y = t^r$.Thecharacteristicequa*
$k) + \sum_{j=1}^{n-1} a_j \prod_{k=0}^{j-1}(r-k) + a_0 = 0$

Example 1 (Third-Order)**.** *For t^{3y}"' + at^{2y}" + bty' + cy = 0*:

Assuming $y = t^r$: $-y$' = rt^{r-1} - $y'' = r(r-1)t^{r-2}$ - $y''' = r(r-1)(r-2)t^{r-3}$
Characteristic equation:

$$r(r-1)(r-2) + ar(r-1) + br + c = 0$$

5 Important Special Cases

5.1 The Cauchy-Euler Equation

Definition 2 (Standard Cauchy-Euler). *The equation t^{2y}" + ty' + $(\lambda^2 t^2 + \nu^2)y = 0$ appears in many applications, especially in Bessel function theory.*

5.2 Modified Euler Equations

Example 2 (Shifted Center). *For equations centered at $t = t_0$:*

$$(t - t_0)^2 y'' + a(t - t_0)y' + by = 0$$

Use the substitution $s = t - t_0$ first.

6 Solution Strategies

Method 2 (Complete Solution Process). *1. **Identify**: Check if powers match derivatives*

*2. **Choose domain**: Usually $t > 0$*

*3. **Select method**:*

- *Direct $(y = t^r) forstraightforwardcasesTransform (x = \ln t)$ for complex or higher-order*

*4. **Find characteristic equation***

*5. **Solve for roots***

*6. **Build solution** based on root types:*

- *Real distinct: $t^{r_1}, t^{r_2}, \ldots$*
- *Real repeated: t^r, $t^r \ln t, t^r (\ln t)^2, \ldots$*
- *Complex: $t^\alpha \cos(\beta \ln t), t^\alpha \sin(\beta \ln t)$*

7 Applications

Euler equations arise naturally in:

1. **Radial symmetry**: Solutions to Laplace's equation in polar/spherical coordinates
2. **Self-similar solutions**: PDEs with scaling symmetry
3. **Power-law media**: Heat conduction in materials with power-law properties

4. **Financial mathematics**: Option pricing models
5. **Fractal geometry**: Equations on self-similar domains

8 Connection to Other Topics

8.1 Series Solutions

Theorem 6 (Frobenius Connection). *The point $t = 0$ is a regular singular point of the Euler equation. The indicial equation from the Frobenius method is exactly our characteristic equation.*

8.2 Relationship to Constant Coefficients

Constant Coefficient	**Euler Equation**
$y = e^{rt} ansatz$	$y = t^r ansatz$
Repeated root gives te^{rt}	Repeated root gives $t^r \ln t$
Complex roots give $e^{\alpha t} \sin(\beta t)$	Complex roots give $t^\alpha \sin(\beta \ln t)$
Linear time scale	Logarithmic time scale

9 Common Pitfalls

1. **Domain**: Solutions differ for $t > 0$ and $t < 0$
2. **Characteristic equation**: It's $r(r-1) + ar + b$, not $r^2 + ar + b$
3. **Complex roots**: Arguments are $\beta \ln t$, not βt
4. **At $t = 0$**: This is a singular point; solutions may not extend through it
5. **Initial conditions**: Often given at $t = 1$ to avoid the singularity

Prof. Ditkowski often combines Euler equations with:

- Initial value problems at $t = 1$
- Boundary value problems on $[1, e]$
- Questions about solution behavior as $t \to 0^+$ or $t \to \infty$
- Transformation to constant coefficients

10 Summary Table

Equation	**Characteristic Eq.**	**Solutions**
$t^{2y''}$ + aty' + by $= 0$	$r^2 + (a-1)r + b = 0$	Based on roots
Distinct real r_1, r_2	–	${c_{1t}}^{r_1} + {c_{2t}}^{r_2}$
Repeated root r	–	$(c_1 + c_2 \ln t)t^r$
Complex $\alpha \pm i\beta$	–	$t^\alpha[c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)]$