Lesson 44: Repeated Roots - Why We Get t^kTerms

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1 Introduction: The Problem of Repeated Roots

When the characteristic equation of a linear ODE with constant coefficients has repeated roots, the naive approach of using only exponential solutions fails to generate enough linearly independent solutions.

Definition 1 (Multiplicity of a Root)**.** *If the characteristic polynomial $p(r)$ can be factored as $p(r) = (r - r_0)^m\ q(r)whereq(r_0) \neq 0$, then r_0 is a root of multiplicity m.*

Theorem 1 (Dimension of Solution Space)**.** *An n-th order linear homogeneous ODE has an n-dimensional solution space. Therefore, we need exactly n linearly independent solutions.*

2 Failure of Simple Exponentials

Example 1 (The Problem)**.** *Consider $y''-4y'+4y = 0$ with characteristic equation $(r-2)^2 = 0$.*

If we only use $y =$e$^{2t}, wehavejustonesolution.Butweneedtwolinearlyindependentsolutionsforthisseco$ $orderequation$.

When a root r has multiplicity $m > 1$, the solution e^{rt} alone cannot span the m-dimensional subspace of solutions corresponding to that root.

3 Derivation via Reduction of Order

Method 1 (Reduction of Order for Repeated Roots)**.** *Given one solution $y_1 =$e$^{rt}whererisarepeatedroot, we$ $v(t)y_1 = v(t)e^{rt}$*

Theorem 2 (Second Solution for Double Root)**.** *If r is a double root of the characteristic equation for $y''+py'+qy = 0$, and $y_1 =$e$^{rt}isonesolution, theny_2 = te^{rt}isasecond, linearlyindependentsolutic$*

Proof. Let $y_2 = v(t)$e$^{rt}.Then:$

Substituting into the differential equation:

$$v''$$

e^{rt} + 2rv'e^{rt} + r^{2vert} + $p(v'$e^{rt} + rve^{rt}) + qve^{rt} = 0

Factoring out e^{rt}:

$$v'' + v'(2r + p) + v(r^2 + pr + q) = 0$$

Since r is a double root of $\lambda^2 + p\lambda + q = 0$:

1. $r^2 + pr + q = 0$ (r is a root)

2. $2r + p = 0$ (derivative of characteristic polynomial at r equals 0)

Therefore: $v'' = 0$, giving $v = c_{1t}$ + c$_2$.
Taking $c_1 = 1, c_2 = 0$ yields $y_2 = t$e^{rt}. □

4 The Differential Operator Approach

The differential equation $(D - r)^m$[y] = 0*naturallyproducessolutions*t^k e^{rt}*for*k = 0, 1, . . . , m-1.

Theorem 3 (Operator Factorization)**.** *If the characteristic polynomial factors as* $p(\lambda) = (\lambda - r)^m$*, thenthedifferentialoperatorfactorsas* $: L = (D - r)^m$

where

$D = d_{\overline{dt}}$.

Lemma 1 (Kernel of $(D-r)^m$). *The kernel (null space) of the operator* $(D-r)^m$*is* $:ker((D-r)^m)$ *= span*{e^{rt}, *t*e^{rt}, *t*^{2ert}, . . . , $t^{m-1}e^{rt}$\$

Proof by Induction. Base case ($m = 1$): $(D - r)[y] = 0 \Rightarrow y = c$e^{rt}.
Inductive step: Assume true for $m = k$. For $m = k + 1$: If $(D - r)^{k+1}[y] = 0$, let $w = (D-r)[y]$. Then $(D-r)^k$[w] = 0*, so*w = p$_k$(t)e^{rt}*where*p$_k$*isapolynomialofdegree*$\leq k-1$.
Solving $(D - r)[y] = p_k$(t)e^{rt}: *Usingintegratingfactor*e^{-rt}: $\frac{d}{dt}$[e^{-rt}y] = p$_k$(t)
Integrating: $e^{-rt}y = P_{k+1}(t) + C$ where P_{k+1} has degree $\leq k$.
Therefore: $y = [P_{k+1}(t) + C]$e^{rt} = p$_{k+1}$(t)e^{rt}*where*deg(p_{k+1}) $\leq k$. □

5 The Limit Perspective

Theorem 4 (Repeated Roots as Limits)**.** *The solution te*^{rt}*canbeunderstoodasthelimitofsolutionsfornearl*

Heuristic Derivation. Consider two roots $r_1 = r$ and $r_2 = r + \epsilon$ with solutions:

$$y = c_1$$

e^{rt} + c$_2$e$^{(r+\epsilon)t}$

Rewriting:

$$y = c_1$$

e^{rt} + c$_2$e^{rt} e^{ϵ} t

For small ϵ: $e^{\epsilon t} \approx 1 + \epsilon t$
Thus:

$$y \approx (c_1 + c_2)$$

e^{rt} + c$_2\epsilon t$e^{rt}

As $\epsilon \to 0$, setting $A = c_1 + c_2$ and $B = \lim_{\epsilon \to 0} c_2 \epsilon$:

$$y = A$$

e^{rt} + Bte^{rt}

$\square$

6 Higher Multiplicities

Theorem 5 (General Multiplicity Case)**.** *If r is a root of multiplicity m, then the m linearly independent solutions are:*

$$e^{rt}, t$$

e^{rt}, t^{2ert}, . . . , $t^{m-1}e^{rt}$

For a root r of multiplicity m:

$$\text{Solutions} = \{t$$

^{k} e^{rt} : k = 0, 1, 2, . . . , m-1}

7 Complex Repeated Roots

Theorem 6 (Complex Repeated Roots)**.** *If $\alpha \pm i\beta$ are complex conjugate roots each of multiplicity m, the $2m$ real-valued linearly independent solutions are:*

$$t$$

^{k} e^{α} $t\cos(\beta t)$, t^{k} e^{α} $t\sin(\beta t)$ *for* $k = 0, 1, \ldots, m-1$

For complex repeated roots, both the sine AND cosine terms need all powers of t up to t^{m-1}.

8 The Wronskian for Repeated Roots

Theorem 7 (Linear Independence via Wronskian). *The functions* $\{t^k\ e^{rt} : k = 0,\ 1,\ \ldots,\ m\text{-}1\} are linearly independent$.

Wronskian Calculation. For simplicity, consider $m = 2$ with solutions e^{rt} and te^{rt}:

$$W(t) = \begin{vmatrix} \mathrm{e}^{rt} & t\mathrm{e}^{rt} \\ r\mathrm{e}^{rt} & (1 + rt)\mathrm{e}^{rt} \end{vmatrix}$$

$$W(t) =$$

$\mathrm{e}^{rt} \cdot (1 + rt)\mathrm{e}^{rt}$ - $t\mathrm{e}^{rt} \cdot r\mathrm{e}^{rt} = e^{2rt}(1$ + rt - rt$) = e^{2rt} \neq 0$

For general m, the Wronskian is:

$$W(t) =$$

$\mathrm{e}^{mrt} \cdot \prod_{0 \leq i < j < m}(j - i) \neq 0$ □

9 Connection to Jordan Normal Form

The appearance of $t^k terms in solutions corresponds exactly to the Jordan block structure of the companion m$

Theorem 8 (Jordan Form and Solutions). *For a system* $\mathbf{x}' = A\mathbf{x}$ *where* A *has a Jordan block:*

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

The matrix exponential is:

$$e^{Jt} =$$

$$e^{\lambda}\ t \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

10 Algorithm for Repeated Roots

Method 2 (Complete Solution Construction).

1. *Factor the characteristic polynomial completely.*
2. *For each distinct root* $r_i with multiplicity m_i$:

3. *If* $r_i isreal: Includesolutionst^k\ er_i\ tfork =$ *0, 1, . . . ,* m_i *- 1If* $fr_i = \alpha + i\beta$ *is complex (with conjugate* $\alpha - i\beta$*): Include solutions* $t^k\ e^\alpha\ t\cos(\beta t)$ *and* $t^k\ e^\alpha\ t\sin(\beta t)$ *for* $k = 0, 1, \ldots, m_i$ *- 1*

4. *Form the general solution as a linear combination of all these functions.*

11 Worked Examples

Example 2 (Triple Root)**.** *Solve:* $y''' - 3y'' + 3y' - y = 0$

Solution: *Characteristic equation:* $r^3 - 3r^2 + 3r - 1 = 0$
Recognize this as $(r - 1)^3 = 0$ *(expand to verify).*
Root: $r = 1$ *with multiplicity 3.*
Solutions: e^t, te^t, t^{2et}
General solution: $y(t) = (c_1 + c_2t + c_3t^2)e^t$

Example 3 (Complex Double Roots)**.** *Solve:* $y^{(4)} + 8y'' + 16y = 0$

Solution: *Characteristic equation:* $r^4 + 8r^2 + 16 = 0$
This is $(r^2 + 4)^2 = 0$*, giving* $(r - 2i)^2(r + 2i)^2 = 0$
Roots: $\pm 2i$*, each with multiplicity 2.*
Real solutions:

- *From* $2i$ *(mult. 2):* $\cos(2t), t\cos(2t), \sin(2t), t\sin(2t)$

General solution: $y(t) = (c_1 + c_2t)\cos(2t) + (c_3 + c_4t)\sin(2t)$

Prof. Ditkowski often presents the characteristic polynomial already factored. Don't waste time trying to factor what's already factored - just identify multiplicities and write solutions systematically.