Practice Problems: Lesson 4 - Peano's Theorem

Master existence conditions!

Part A: Basic Peano Checks

For each IVP, determine if Peano's theorem guarantees existence of a solution:

1. $\frac{dy}{dx} = \sqrt{y}$, $y(0) = 1$
2. $\frac{dy}{dx} = \sqrt{y}$, $y(0) = -1$
3. $\frac{dy}{dx} = \frac{1}{x^2 + y^2}$, $y(0) = 0$
4. $\frac{dy}{dx} = \ln(xy)$, $y(1) = 2$
5. $\frac{dy}{dx} = \frac{\sin y}{x}$, $y(0) = \pi$

Part B: Finding the Guaranteed Interval

For these IVPs where Peano applies, find the guaranteed interval of existence h:

6. $\frac{dy}{dx} = x + y$, $y(0) = 0$, in rectangle $|x| \leq 1$, $|y| \leq 2$
7. $\frac{dy}{dx} = y^2$, $y(0) = 1$, in rectangle $|x| \leq 0.5$, $|y - 1| \leq 1$
8. $\frac{dy}{dx} = \cos(xy)$, $y(0) = 0$, in rectangle $|x| \leq 2$, $|y| \leq \pi$

Part C: Discontinuity Analysis

Identify all points where $f(x, y)$ is discontinuous, then determine where Peano fails:

9. $f(x, y) = \frac{y^2}{x^2 - 1}$
10. $f(x, y) = \tan\left(\frac{y}{x}\right)$

11. $f(x, y) = \dfrac{\ln(x + y)}{x - y}$

12. $f(x, y) = \sqrt{xy} + \dfrac{1}{\sin(y)}$

Part D: Existence Without Uniqueness

For each problem, verify that Peano guarantees existence but find multiple solutions:

13. $\dfrac{dy}{dx} = 2\sqrt{y}$, $y(0) = 0$

14. $\dfrac{dy}{dx} = |y|^{1/3}$, $y(0) = 0$

15. $\dfrac{dy}{dx} = \begin{cases} 2\sqrt{y} & y \geq 0 \\ 0 & y < 0 \end{cases}$, $y(0) = 0$

Part E: Local vs Global Existence

16. For $\dfrac{dy}{dx} = 1 + y^2$, $y(0) = 0$:
    (a) Show Peano applies locally
    (b) Find the actual solution
    (c) Determine the maximal interval of existence
    (d) Why doesn't the solution exist globally?

17. For $\dfrac{dy}{dx} = \dfrac{y}{1 - x}$, $y(0) = 1$:
    (a) Where does Peano guarantee existence?
    (b) Find the solution
    (c) What happens as $x \to 1$?

Part F: Theoretical Questions

18. True or False (justify):
    (a) If f is continuous, Peano guarantees a unique solution
    (b) If f is discontinuous at (x_0, y_0), no solution exists
    (c) Peano's theorem guarantees global existence
    (d) If Peano applies, at least one solution exists

19. Explain why Peano's theorem requires f to be continuous in a rectangle, not just at the initial point.

20. Give an example of an IVP where:

    (a) Peano applies and there's exactly one solution

    (b) Peano applies and there are exactly two solutions

    (c) Peano applies and there are infinitely many solutions

    (d) Peano doesn't apply but a solution still exists

Part G: Exam-Style Problems

21. Consider $\frac{dy}{dx} = f(x, y)$ where

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

    (a) Is f continuous at $(0, 0)$?

    (b) Does Peano apply for the IVP with $y(0) = 0$?

    (c) Find a solution if it exists

22. Professor Ditkowski asks: "For the equation $\frac{dy}{dx} = |y|^\alpha$ with $y(0) = 0$:"

    (a) For which values of $\alpha > 0$ does Peano guarantee existence?

    (b) For which values is the solution unique?

    (c) Find all solutions when $\alpha = 1/2$

23. The rectangle in Peano's theorem has dimensions $2a \times 2b$ centered at (x_0, y_0). If $M = \max |f|$ in this rectangle:

    (a) Why is the guaranteed interval $h = \min(a, b/M)$?

    (b) What happens if we make the rectangle larger?

    (c) Give an example where increasing a doesn't increase h

Part H: Advanced Applications

24. Consider the parametric family: $\frac{dy}{dx} = y^2 + \lambda$, $y(0) = 0$

    (a) For which λ does Peano guarantee existence?

    (b) For $\lambda = -1$, find the maximal interval of existence

(c) For $\lambda = 0$, what happens to the solution?

(d) For $\lambda = 1$, where does the solution blow up?

25. Peano's theorem in higher dimensions: For the system

$$\begin{cases} x' = y \\ y' = -\sin(x) \end{cases}$$

with $x(0) = 0$, $y(0) = 2$:

(a) Verify Peano applies

(b) What does this system represent physically?

(c) Is the solution periodic?

Solutions Guide

Part A Quick Answers: 1. Yes (continuous) 2. No ($\sqrt{y}$ undefined for $y < 0$) 3. No (undefined at origin) 4. Yes (continuous at $(1, 2)$) 5. No (division by zero at $x = 0$)

Part B Hints: 6. $M = 3$, so $h = \min(1, 2/3) = 2/3$ 7. $M = 4$, so $h = \min(0.5, 1/4) = 1/4$ 8. $M = 1$, so $h = \min(2, \pi) = \pi$

Key Concept: Peano gives existence but not uniqueness. Always check continuity first!