Lesson 41: The Hartman-Grobman Theorem

When Linearization Works

ODE 1 - Prof. Adi Ditkowski

1 Hyperbolicity - The Key Condition

Definition 1 (Hyperbolic Equilibrium)**.** *A critical point (x_0, y_0) of the nonlinear system*

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

is called ***hyperbolic*** *if all eigenvalues of the Jacobian matrix $J(x_0, y_0)$ have non-zero real parts. That is, if λ_1, λ_2 are the eigenvalues, then:*

$$Re(\lambda_1) \neq 0 \quad and \quad Re(\lambda_2) \neq 0$$

Hyperbolic equilibria:

- Nodes (all eigenvalues real with same sign)
- Saddles (real eigenvalues with opposite signs)
- Spirals (complex eigenvalues with non-zero real part)

Non-hyperbolic equilibria:

- Centers (purely imaginary eigenvalues)
- Degenerate nodes (zero eigenvalue)
- Any case with $\text{Re}(\lambda) = 0$ for some eigenvalue

2 The Hartman-Grobman Theorem

Theorem 1 (Hartman-Grobman)**.** *Let (x_0, y_0) be a hyperbolic equilibrium point of the C^1 system:*

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$$

Then there exists a neighborhood U of (x_0, y_0) and a homeomorphism $h : U \to V$ (where V is a neighborhood of the origin) such that h maps trajectories of the nonlinear system to trajectories of the linearized system:

$$\dot{\xi} = J(x_0, y_0) \cdot \xi$$

preserving the direction of time.

The homeomorphism h provides a **topological conjugacy** between the nonlinear and linear systems. This means:

1. The qualitative behavior is identical
2. Stable/unstable manifolds correspond
3. The phase portrait structure is preserved
4. But geometric properties (angles, distances) may change

3 What Hartman-Grobman Tells Us

3.1 What IS Preserved

- **Stability type:** Stable remains stable, unstable remains unstable
- **Equilibrium type:** Nodes remain nodes, saddles remain saddles, spirals remain spirals
- **Invariant manifolds:** Stable and unstable manifolds exist with same dimensions
- **Local dynamics:** The direction of flow and separation of trajectories

3.2 What is NOT Preserved

- **Trajectory shape:** Straight lines may become curves
- **Time parametrization:** Speed along trajectories may change
- **Metric properties:** Distances and angles are not preserved
- **Special structures:** Hamiltonian or gradient structure may be lost

4 Applications and Examples

Example 1 (Hyperbolic Saddle)**.** *Consider the system:*

$$\dot{x} = x + y^2, \quad \dot{y} = -y + x^2$$

At the origin:

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = 1 > 0$, $\lambda_2 = -1 < 0$

Since both eigenvalues have non-zero real parts, the origin is hyperbolic. By Hartman-Grobman:

- *The origin is a saddle point for the nonlinear system*
- *There exists a 1D stable manifold (tangent to eigenvector for $\lambda = -1$)*
- *There exists a 1D unstable manifold (tangent to eigenvector for $\lambda = 1$)*
- *Near the origin, trajectories behave qualitatively like the linear saddle*

Example 2 (Non-hyperbolic Center)**.** *Consider:*

$$\dot{x} = -y + x(x^2 + y^2), \quad \dot{y} = x + y(x^2 + y^2)$$

At the origin:

$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues: $\lambda = \pm i$ (purely imaginary)
The origin is NOT hyperbolic. Hartman-Grobman does not apply! Indeed:

- *Linearization predicts a center (neutral stability)*
- *The actual nonlinear system has an unstable spiral!*
- *In polar coordinates: $\dot{r} = r^3 > 0$ for $r \neq 0$*

Common Exam Mistakes:

1. Applying Hartman-Grobman when eigenvalues are purely imaginary
2. Forgetting to verify hyperbolicity before concluding
3. Claiming exact trajectory shapes are preserved
4. Not recognizing when additional analysis is needed

5 The Center Problem

When linearization yields purely imaginary eigenvalues ($\lambda = \pm i\omega$), the nonlinear system near the equilibrium could be:

Possibility	**Determining Factor**
Center	Nonlinear terms preserve area/energy
Stable spiral	Nonlinear terms dissipate energy
Unstable spiral	Nonlinear terms add energy
More complex	Multiple timescale dynamics

When Prof. Ditkowski gives you a system with purely imaginary eigenvalues:

1. State clearly: "The equilibrium is non-hyperbolic"
2. Write: "Hartman-Grobman theorem does not apply"
3. Say: "Linearization alone cannot determine stability"
4. If asked for more, use Lyapunov functions or compute higher-order terms

6 Structural Stability

Definition 2 (Structural Stability). *A system is **structurally stable** near an equilibrium if small perturbations to the system preserve the qualitative dynamics.*

Theorem 2 (Consequence of Hartman-Grobman). *Hyperbolic equilibria are structurally stable. Small perturbations to f and g will:*

- *Slightly move the equilibrium location*
- *Slightly change eigenvalues (keeping signs of real parts)*
- *Preserve the topological type*

In real-world modeling:

- **Hyperbolic equilibria** are robust to modeling errors
- **Non-hyperbolic equilibria** are sensitive to perturbations
- This explains why centers are rarely observed in practice
- Bifurcations occur when equilibria lose hyperbolicity

7 Algorithm for Applying Hartman-Grobman

Step-by-Step Procedure:

1. Find the critical point (x_0, y_0)
2. Compute the Jacobian $J(x_0, y_0)$
3. Calculate eigenvalues λ_1, λ_2
4. Check hyperbolicity:
   - If $\text{Re}(\lambda_1) \neq 0$ AND $\text{Re}(\lambda_2) \neq 0$: HYPERBOLIC
   - Otherwise: NON-HYPERBOLIC

5. If hyperbolic:
   - State: "By Hartman-Grobman, linearization determines local behavior"
   - Classify using linear theory
   - Conclude about stability
6. If non-hyperbolic:
   - State: "Hartman-Grobman does not apply"
   - Note that additional analysis is required
   - Consider Lyapunov functions or normal forms

8 Connection to Bifurcation Theory

Bifurcations occur when a parameter change causes an equilibrium to lose hyperbolicity. Common scenarios:

- **Saddle-node:** Real eigenvalue crosses zero
- **Hopf:** Complex pair crosses imaginary axis
- **Pitchfork/Transcritical:** Zero eigenvalue appears

At bifurcation points, Hartman-Grobman fails and nonlinear terms determine the behavior!

9 Summary Table

Eigenvalues	Hyperbolic?	H-G Applies?	Conclusion
$\lambda_1, \lambda_2 < 0$	Yes	Yes	Stable node
$\lambda_1, \lambda_2 > 0$	Yes	Yes	Unstable node
$\lambda_1 < 0 < \lambda_2$	Yes	Yes	Saddle
$\alpha \pm i\beta$, $\alpha \neq 0$	Yes	Yes	Spiral (sign of α)
$\pm i\beta$	No	No	Inconclusive
$\lambda_1 = 0$, $\lambda_2 \neq 0$	No	No	Inconclusive