ODE Lesson 1: What is an ODE? Classification and Terminology

ODE 1 - Prof. Adi Ditkowski

1 Core Concepts

Definition 1 (Ordinary Differential Equation)**.** *An **ordinary differential equation (ODE)** is an equation containing:*

- *An unknown function* $y(x)$
- *One or more of its derivatives:* $y', y'', y''', \ldots$
- *The independent variable* x

General form: $F(x, y, y', y'', \ldots, y^{(n)}) = 0$

Key Insight: We're solving for an entire function, not just a number!

2 Classification System

2.1 Order of an ODE

Definition 2 (Order)**.** *The **order** of an ODE is the highest derivative that appears in the equation.*

Example 1.

$$\frac{dy}{dx} + 2y = e^x \quad \textit{(First-order)} \tag{1}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \quad \textit{(Second-order)} \tag{2}$$

$$\left(\frac{d^3y}{dx^3}\right)^2 + y = \sin x \quad \textit{(Third-order)} \tag{3}$$

2.2 Degree of an ODE

Definition 3 (Degree)**.** *The **degree** is the power of the highest derivative after clearing fractions and radicals, if the equation is polynomial in derivatives.*

Warning: Degree is undefined for non-polynomial expressions like $\sin(y')$ or $e^{y'}$.

Example 2.

$$\left(\frac{dy}{dx}\right)^2 + y = 0 \quad \textit{(Degree 2)} \tag{4}$$

$$\sqrt{\frac{dy}{dx}} + y = 0 \quad \rightarrow \quad \frac{dy}{dx} + y^2 = 0 \quad \textit{(Degree 1 after clearing radical)} \tag{5}$$

$$\sin\left(\frac{dy}{dx}\right) + y = 0 \quad \textit{(Degree undefined)} \tag{6}$$

2.3 Linearity

Definition 4 (Linear ODE)**.** *An ODE is **linear** if it can be written as:*

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where:

- *No products of y and its derivatives*
- *No powers of y or its derivatives (except first power)*
- *No nonlinear functions of y (like $\sin(y)$, e^y, $\sqrt{y}$)*

Quick Test for Linearity:

1. Write all terms with y on the left, others on the right
2. Check if left side is a linear combination of $y, y', y'', \ldots$
3. Coefficients can depend on x arbitrarily

Example 3 (Linear vs Nonlinear)**.** ***Linear:***

- $x^2y'' + xy' + (x^2 - n^2)y = 0$ *(Bessel's equation)*
- $e^x y' + (\sin x)y = \cos x$
- $y''' - 3y' + 2y = 0$

Nonlinear:

- $yy' = x$ *(product of y and y')*
- $(y')^2 + y = 0$ *(squared derivative)*
- $y' = e^y$ *(exponential of y)*
- $y'' + \sin(y) = 0$ *(nonlinear pendulum)*

2.4 Autonomous Equations

Definition 5 (Autonomous ODE)**.** *An ODE is **autonomous** if the independent variable x does not appear explicitly. Form: $y' = f(y)$ or $F(y, y', y'', \ldots) = 0$*

Physical Meaning: The laws governing the system don't change with time.

Example 4. ***Autonomous:***

- $y' = y^2 - y$ *(logistic with harvesting)*
- $y'' + \sin(y) = 0$ *(nonlinear pendulum)*
- $y' = y(1-y)(y-2)$ *(triple equilibrium)*

Non-autonomous:

- $y' = xy$ *(explicit x)*
- $y'' + \cos(x)y = 0$ *(time-varying frequency)*
- $y' = y + e^{-x}$ *(time-dependent forcing)*

3 Important Properties

Theorem 1 (Number of Arbitrary Constants)**.** *The general solution of an n-th order ODE contains exactly n arbitrary constants.*

Memory Aid: LINEAR = "Line up Each term with Add and multiply by Regular coefficients"

4 Special Named Equations

Name	**Equation**	**Type**
Separable	$y' = f(x)g(y)$	1st order, usually nonlinear
Bernoulli	$y' + P(x)y = Q(x)y^n$	1st order, nonlinear if $n \neq 0, 1$
Riccati	$y' = q_0(x) + q_1(x)y + q_2(x)y^2$	1st order, nonlinear
Bessel	$x^2y'' + xy' + (x^2 - n^2)y = 0$	2nd order, linear
Legendre	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$	2nd order, linear

5 Classification Algorithm

Exam Strategy: Complete Classification

1. State the **order** (highest derivative)
2. Determine **linearity** (check for products/powers of y)
3. Check if **autonomous** (no explicit x?)
4. Identify **special type** if applicable
5. State the **degree** if polynomial in derivatives