Lesson 17: Homogeneous First-Order Differential Equations

ODE 1 - Prof. Adi Ditkowski

1 Recognition and Definition

Definition 1 (Homogeneous Function). *A function $f(x, y)$ is homogeneous of degree n if:*

$$f(tx, ty) = t^n f(x, y) \quad \textit{for all } t > 0$$

Definition 2 (Homogeneous Differential Equation). *A first-order ODE is homogeneous if it can be written as:*

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

or equivalently, if $M(x, y)dx + N(x, y)dy = 0$ where M and N are homogeneous functions of the same degree.

Quick Recognition Tests:

1. Check if all terms have the same total degree in x and y
2. Try to factor out powers to write as $F(y/x)$
3. Apply the scaling test: $f(tx, ty) = t^n f(x, y)$

2 The Substitution Method

The $v = y/x$ Substitution Algorithm:

1. Set $v = \frac{y}{x}$, so $y = vx$
2. Differentiate: $\frac{dy}{dx} = v + x\frac{dv}{dx}$ (Product Rule!)
3. Substitute into the original equation
4. Simplify to get: $x\frac{dv}{dx} = F(v) - v$
5. Separate variables: $\frac{dv}{F(v)-v} = \frac{dx}{x}$
6. Integrate both sides

7. Back-substitute $v = y/x$

Critical Points:

- The derivative $\frac{dy}{dx} = v + x\frac{dv}{dx}$ comes from the product rule
- Check for singular solutions where $F(v) - v = 0$
- The substitution fails along $x = 0$ (use $u = x/y$ instead if needed)

3 Detailed Examples

Example 1 (Standard Homogeneous). *Solve:* $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

Solution:

1. *Verify homogeneity:* $\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 = F(y/x)$ ✓
2. *Let* $v = y/x$, *then* $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$
3. *Substitute:* $v + x\frac{dv}{dx} = 1 + v + v^2$
4. *Simplify:* $x\frac{dv}{dx} = 1 + v^2$
5. *Separate:* $\frac{dv}{1+v^2} = \frac{dx}{x}$
6. *Integrate:* $\arctan(v) = \ln|x| + C$
7. *Back-substitute:* $\arctan\left(\frac{y}{x}\right) = \ln|x| + C$

Example 2 (Disguised Homogeneous). *Solve:* $(x-y)\frac{dy}{dx} = x + y$

Solution:

1. *Rewrite:* $\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+y/x}{1-y/x}$ *(homogeneous!)*
2. *Let* $v = y/x$: $v + x\frac{dv}{dx} = \frac{1+v}{1-v}$
3. *Simplify:* $x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$
4. *Separate:* $\frac{1-v}{1+v^2}dv = \frac{dx}{x}$
5. *Use partial fractions on the left side*
6. *Final solution involves* arctan *and* ln *terms*

4 Special Cases and Variations

Alternative Substitution: When the equation has more y terms, try $u = \frac{x}{y}$:

- $x = uy$ implies $\frac{dx}{dy} = u + y\frac{du}{dy}$
- The equation becomes separable in u and y

Prof. Ditkowski's Exam Patterns:

- Often combines homogeneous with initial conditions
- May ask to verify homogeneity before solving
- Likes equations of the form $(ax + by)dx + (cx + dy)dy = 0$
- Tests recognition with trigonometric terms like $\sin(y/x)$
- Partial credit for correct substitution setup

5 Geometric Interpretation

Solution curves of homogeneous equations have the property that they look similar under scaling from the origin. If (x, y) is on a solution curve, then (kx, ky) is on a geometrically similar curve.

6 Recognition Flowchart

The key steps for recognition are:

1. Check if the equation can be written as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$
2. Verify that all terms have the same total degree in x and y
3. Apply the scaling test: $f(tx, ty) = t^n f(x, y)$

If any test confirms homogeneity, proceed with the $v = y/x$ substitution.