Lesson 22: Finding Potential Functions - Systematic Approach

ODE 1 - Prof. Adi Ditkowski

1 The Potential Function

Definition 1 (Potential Function). *For an exact equation $M(x,y)dx + N(x,y)dy = 0$, the* ***potential function*** $H(x,y)$ *satisfies:*

$$\frac{\partial H}{\partial x} = M(x,y) \tag{1}$$

$$\frac{\partial H}{\partial y} = N(x,y) \tag{2}$$

The general solution is then given by $H(x,y) = C$.

2 Method 1: Integration with Respect to x

Method 1 - Integrate M with respect to x:

1. Since $\frac{\partial H}{\partial x} = M(x,y)$, integrate:

$$H(x,y) = \int M(x,y)\,dx + g(y)$$

   where $g(y)$ is an arbitrary function of y alone.

2. Differentiate the result with respect to y:

$$\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y)$$

3. Set this equal to $N(x,y)$ and solve for $g'(y)$:

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]$$

4. Integrate to find $g(y)$ and write the complete potential function.

3 Method 2: Integration with Respect to y

Method 2 - Integrate N with respect to y:

1. Since $\frac{\partial H}{\partial y} = N(x, y)$, integrate:

$$H(x, y) = \int N(x, y)\, dy + f(x)$$

   where $f(x)$ is an arbitrary function of x alone.

2. Differentiate with respect to x, set equal to $M(x, y)$, and solve for $f'(x)$.

3. Integrate to find $f(x)$ and write the complete potential function.

4 Method 3: Line Integral

Method 3 - Line Integral Approach: Choose a convenient base point (x_0, y_0) and integrate along any path to (x, y):

$$H(x, y) = \int_{(x_0,y_0)}^{(x,y)} M\, dx + N\, dy$$

Common choice: Use path $(0, 0) \to (x, 0) \to (x, y)$:

$$H(x, y) = \int_0^x M(t, 0)\, dt + \int_0^y N(x, s)\, ds$$

Method Selection Guidelines:

- Use Method 1 if $M(x, y)$ is easier to integrate
- Use Method 2 if $N(x, y)$ is easier to integrate
- Use Method 3 if both M and N simplify when one variable equals zero

5 Examples

Example 1 (Basic Potential Function)**.** *Solve* $(2xy + 3x^2)dx + (x^2 + 2y)dy = 0$.

Method 1: $H = \int(2xy + 3x^2)dx = x^2y + x^3 + g(y)$
$\frac{\partial H}{\partial y} = x^2 + g'(y) = x^2 + 2y$, *so* $g'(y) = 2y$, $g(y) = y^2$
$H(x, y) = x^2y + x^3 + y^2$
Solution: $x^2y + x^3 + y^2 = C$

Prof. Ditkowski's Exam Tips:

- Always verify your answer: check $\frac{\partial H}{\partial x} = M$ and $\frac{\partial H}{\partial y} = N$
- For initial value problems, substitute the given point to find C
- Show all integration steps clearly for partial credit
- State your final answer in the form $H(x, y) = C$