Lesson 24: Special Integrating Factors - $\mu(xy)$ and Beyond

ODE 1 - Prof. Adi Ditkowski

1 Beyond Simple Integrating Factors

When neither $\mu(x)$ nor $\mu(y)$ exists, we need to consider more sophisticated integrating factors.

2 Integrating Factor of the Form $\mu(xy)$

Theorem 1 (Test for $\mu(xy)$)**.** *If $\frac{\partial M/\partial y - \partial N/\partial x}{xN - yM}$ depends only on the product xy, then there exists an integrating factor $\mu(xy)$ where $z = xy$.*

Finding $\mu(xy)$:

1. Calculate $R = \frac{M_y - N_x}{xN - yM}$
2. Check if R depends only on xy (substitute $z = xy$)
3. If yes, solve $\frac{d\mu}{dz} = \mu \cdot R(z)$
4. The integrating factor is $\mu(xy)$

3 Other Special Forms

Common Special Integrating Factors:

1. $\mu = x^a y^b$ (power form)
2. $\mu = (ax + by)^n$ (linear combination)
3. $\mu = e^{f(x,y)}$ (exponential form)
4. $\mu = f(x \pm y)$ (sum/difference)
5. $\mu = g(x/y)$ (ratio form)

4 Power Form: $\mu = x^a y^b$

Method 1 (Finding Powers)**.** *For $\mu = x^a y^b$, multiply the original equation and apply exactness:*

$$\frac{\partial}{\partial y}(x^a y^b M) = \frac{\partial}{\partial x}(x^a y^b N)$$

This gives:

$$x^a y^{b-1}(bM + yM_y) = x^{a-1} y^b (aN + xN_x)$$

Solve for a and b by comparing coefficients.

Example 1 (Power Form)**.** *Find an integrating factor for: $(y^2)dx + (xy)dy = 0$*

Try $\mu = x^a y^b$:

$$(x^a y^{b+2})dx + (x^{a+1} y^{b+1})dy = 0$$

For exactness: $\frac{\partial}{\partial y}(x^a y^{b+2}) = x^a(b+2)y^{b+1}$ $\frac{\partial}{\partial x}(x^{a+1} y^{b+1}) = (a+1)x^a y^{b+1}$
Setting equal: $b + 2 = a + 1$, so $a = b + 1$
Choose $b = -1$, then $a = 0$: $\mu = y^{-1} = \frac{1}{y}$
Result: $(y)dx + (x)dy = 0$, which gives $d(xy) = 0$, so $xy = C$.

5 Linear Combination: $\mu = (ax + by)^n$

This form is useful when the equation has homogeneous-like properties or when M and N have similar structures involving linear combinations of x and y.

6 Systematic Approach

Complete Strategy for Finding Integrating Factors:

1. Check if equation is already exact
2. Test for $\mu(x)$: $(M_y - N_x)/N$ function of x only
3. Test for $\mu(y)$: $(N_x - M_y)/M$ function of y only
4. Test for $\mu(xy)$: $(M_y - N_x)/(xN - yM)$ function of xy only
5. Try special forms:
   - $\mu = x^a y^b$ (solve system for a, b)
   - $\mu = (x \pm y)^n$
   - $\mu = e^{f(x,y)}$ for simple f
6. Use inspection/physical intuition

Common Challenges:

- Calculations become increasingly complex
- Multiple forms may work - choose the simplest
- Not all equations have elementary integrating factors
- Verification is crucial after finding μ

Prof. Ditkowski's Hints:

- Often provides hints: "Try $\mu = \ldots$"
- Looks for recognition of standard patterns
- Partial credit for systematic approach
- May give the integrating factor and ask you to solve
- Sometimes asks for verification rather than derivation

7 Special Cases Summary

Quick Reference Guide:

If you see...	**Try...**
M, N with same powers of x, y	$\mu = x^a y^b$
Linear terms dominate	$\mu = (ax + by)^n$
Exponential structure	$\mu = e^{f(x,y)}$
Symmetric in x, y	$\mu = f(xy)$ or $\mu = g(x \pm y)$
Rational functions	$\mu = \frac{x^a y^b}{(x^c + y^c)^n}$