ODE Lesson 26: Converting Higher-Order to First-Order Systems

ODE 1 - Prof. Adi Ditkowski

1 Fundamental Concept

Definition 1 (State Vector Representation). *For an nth-order differential equation, the* ***state vector*** *is defined as:*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

where each component represents a successive derivative of the solution $y(t)$.

Any nth-order linear ODE:

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = f(t)$$

can be converted to a first-order system:

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t)$$

2 The Companion Matrix

Definition 2 (Companion Matrix). *For the equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$, the companion matrix is:*

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

The companion matrix has a special structure:

- First $n-1$ rows: ones on the superdiagonal, zeros elsewhere
- Last row: negative coefficients in order from a_0 to a_{n-1}

3 Conversion Algorithm

Method 1 (Standard Conversion Procedure)**.** *1. Write the equation in standard form:* $y^{(n)} = -a_{n-1}y^{(n-1)} - \cdots - a_0 y + f(t)$

2. Define state variables: $x_1 = y, x_2 = y', \ldots, x_n = y^{(n-1)}$

3. Write the system of equations:

$$x_1' = x_2 \tag{1}$$
$$x_2' = x_3 \tag{2}$$
$$\vdots \tag{3}$$
$$x_{n-1}' = x_n \tag{4}$$
$$x_n' = -a_{n-1}x_n - \cdots - a_1 x_2 - a_0 x_1 + f(t) \tag{5}$$

4. Express in matrix form: $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$

4 Examples

Example 1 (Second-Order Conversion)**.** *Convert* $y''+3y'+2y = e^{-t}$ *with* $y(0) = 1$, $y'(0) = 0$.
Solution: *Set* $x_1 = y$, $x_2 = y'$. *Then:*

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

Initial condition: $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Example 2 (Third-Order Conversion)**.** *Convert* $y''' - 2y'' + y' - y = 0$.
Solution: *State vector:* $\mathbf{x} = [y, y', y'']^T$
Companion matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Common errors:

- Incorrect signs in the last row (must be negative of coefficients)
- Wrong ordering of coefficients (should be a_0 to a_{n-1})
- Forgetting to include non-homogeneous term in correct position

5 Reverse Conversion

Theorem 1 (System to Scalar Conversion)**.** *Given a system* $\mathbf{x}' = A\mathbf{x}$ *where* A *is a companion matrix, the first component* $x_1(t)$ *satisfies the scalar equation:*

$$x_1^{(n)} + a_{n-1}x_1^{(n-1)} + \cdots + a_0 x_1 = 0$$

To convert back:

1. Use $x_1' = x_2$, $x_2' = x_3$, etc.
2. Express all x_i in terms of derivatives of x_1
3. Substitute into the last equation
4. Simplify to get scalar ODE

6 Properties and Connections

The eigenvalues of the companion matrix are exactly the roots of the characteristic polynomial:

$$\det(A - \lambda I) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0)$$

Prof. Ditkowski often tests:

- Quick companion matrix construction (memorize the pattern!)
- Converting initial conditions correctly
- Recognizing when conversion simplifies a problem
- Connection between eigenvalues and characteristic roots

7 Special Cases

Example 3 (Variable Coefficients)**.** *For* $t^2y'' + ty' - y = 0$ *(Euler equation):*

1. *First divide by* t^2*:* $y'' + \frac{1}{t}y' - \frac{1}{t^2}y = 0$
2. *Then convert:* $A(t) = \begin{bmatrix} 0 & 1 \\ \frac{1}{t^2} & -\frac{1}{t} \end{bmatrix}$