

Lesson 20: Riccati to Second-Order Linear Transformation

ODE 1 - Prof. Adi Ditkowski

1 The Fundamental Transformation

Theorem 1 (Riccati to Second-Order Linear). *The Riccati equation*

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2$$

can be transformed into the second-order linear equation

$$u'' + p(x)u' + r(x)u = 0$$

via the substitution

$$y = -\frac{1}{q_2} \cdot \frac{u'}{u}, \text{ where:}$$

$$p(x) = -q_1 - \frac{q_2'}{q_2} \tag{1}$$

$$r(x) = q_0 q_2 \tag{2}$$

Proof. Starting with $y = -\frac{1}{q_2} \cdot \frac{u'}{u}$:

$$y' = -\frac{1}{q_2} \cdot \frac{d}{dx} \left(\frac{u'}{u} \right) + \frac{q_2'}{q_2^2} \cdot \frac{u'}{u} \tag{3}$$

$$= -\frac{1}{q_2} \left(\frac{u''}{u} - \frac{(u')^2}{u^2} \right) + \frac{q_2'}{q_2^2} \cdot \frac{u'}{u} \tag{4}$$

$$= -\frac{u''}{q_2 u} + \frac{1}{q_2} \cdot \frac{(u')^2}{u^2} + \frac{q_2'}{q_2^2} \cdot \frac{u'}{u} \tag{5}$$

$$= -\frac{u''}{q_2 u} + \frac{1}{q_2} \cdot q_2^2 y^2 + \frac{q_2'}{q_2^2} \cdot (-q_2 y) \tag{6}$$

$$= -\frac{u''}{q_2 u} + q_2 y^2 - \frac{q_2'}{q_2} y \tag{7}$$

Substituting into the Riccati equation and simplifying yields the stated result. \square \square

2 Alternative Transformation

The Exponential Integral Approach:

For the Riccati equation $y' = q_0 + q_1 y + q_2 y^2$, we can use $v = \exp\left(\int y dx\right)$

Then:

1. $\frac{v'}{v} = y$ implies $y = \frac{v'}{v}$
2. $y' = \frac{v''}{v} - \left(\frac{v'}{v}\right)^2 = v'' \frac{1}{v-y}^2$ Substituting into Riccati: $v'' \frac{1}{v-y}^2 = q_0 + q_1 y + q_2 y^2$
3. This gives: $v'' - q_1 v' - (1 + q_2)(v')^2/v + q_0 v = 0$
This form is useful when $q_2 = -1$ as it simplifies to: $v'' - q_1 v' + q_0 v = 0$

3 Standard Examples

Example 1 (Constant Coefficients). Solve: $y' = 1 + y^2$

Method 1: Using $y = -u'/u$

Here $q_0 = 1, q_1 = 0, q_2 = 1$. Since q_2 is constant, $q_2' = 0$.

The second-order equation becomes:

$$u'' + 0 \cdot u' + 1 \cdot u = 0 \implies u'' + u = 0$$

General solution: $u = c_1 \cos x + c_2 \sin x$

Therefore: $y = -\frac{u'}{u} = -\frac{-c_1 \sin x + c_2 \cos x}{c_1 \cos x + c_2 \sin x} = \frac{c_1 \sin x - c_2 \cos x}{c_1 \cos x + c_2 \sin x}$

Simplifying: $y = \tan(x - \phi)$ where $\tan \phi = c_2/c_1$

Example 2 (Euler-Type Transformation). Solve: $y' = \frac{a}{x} + b \cdot y^2$ where a, b are constants.

Using $y = -\frac{1}{b} \cdot \frac{u'}{u}$ (note $q_2 = b$):

The second-order equation becomes:

$$u'' + 0 \cdot u' + \frac{ab}{x^2} u = 0$$

$$u'' + \frac{ab}{x^2} u = 0$$

or equivalently:

$$x^2 u'' + ab u = 0$$

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This is an Euler equation. Let $u = x^m$: $m(m-1) + ab = 0 \implies m = \frac{1 \pm \sqrt{1-4ab}}{2}$

The nature of solutions depends on the discriminant $1-4ab$.

4 Special Second-Order Forms

Common Transformations and Their Second-Order Forms:

Riccati Form	Second-Order Result
$y' = a + by^2 (const)$	$u'' + ab \cdot u = 0$
$y' = \frac{a^2}{x} + by^2$	$x^2 u'' + ab \cdot u = 0$ (Euler)
$y' = ax^2 + by^2$	$u'' + abx^2 u = 0$ (Airy - type)
$y' = q_0(x) - y^2$	$u'' - q_0(x)u = 0$
$y' = \frac{n(n+1)^2}{x} + y^2$	Modified Bessel equation

When the Method Fails:

- If $q_2(x) = 0$ at some points, the transformation is singular there. The resulting second-order equation may not have elementary solutions.
- Series solutions or special functions may be required.
- Numerical methods might be necessary for general $q_i(x)$.

5 Reverse Transformation

Theorem 2 (Second-Order to Riccati). *Given the second-order linear equation:*

$$u'' + p(x)u' + q(x)u = 0$$

The substitution $y = -\frac{u'}{u}$ yields the Riccati equation:

$$y' = -q(x) - p(x)y - y^2$$

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Why This Matters:

- Every linear second-order equation has an associated Riccati
- Solutions of one determine solutions of the other
- Qualitative properties transfer between the two forms
- Stability analysis can be done in either form
- Special functions defined by second-order equations give Riccati solutions

6 Solution Process Flowchart

Prof. Ditkowski's Favorite Transformations:

1. Riccati \rightarrow Constant coefficient: Usually solvable with exponentials
2. Riccati \rightarrow Euler equation: Power solutions
3. Riccati \rightarrow Airy equation: Special functions required
4. Riccati \rightarrow Bessel equation: Bessel functions needed
5. Simple forms where $q_2 = \pm 1$: Often give nice solutions