Linear First-Order ODEs: The Standard Method

ODE 1 - Prof. Adi Ditkowski

Lesson 14

1 Introduction and Motivation

Linear first-order ODEs form the backbone of differential equations theory. They appear in countless applications: population dynamics, RC circuits, mixing problems, radioactive decay, and Newton's law of cooling.

Definition 1 (Linear First-Order ODE)**.** *A **linear first-order ODE** has the form:*

$$a_1(t)\frac{dy}{dt} + a_0(t)y = g(t) \tag{1}$$

*where $a_1(t) \neq 0$ on the interval of interest. The **standard form** is:*

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2}$$

where $p(t) = a_0(t)/a_1(t)$ and the right-hand side has been divided by $a_1(t)$.

The equation is called **linear** because y and y' appear to the first power only, with no products like yy' or nonlinear terms like y^2, $\sin(y)$, etc.

2 Classification

Definition 2 (Homogeneous vs Non-homogeneous)**.**

- ***Homogeneous***: $y' + p(t)y = 0$ *(when $g(t) \equiv 0$)*
- ***Non-homogeneous***: $y' + p(t)y = g(t)$ *(when $g(t) \not\equiv 0$)*

3 Solution of Homogeneous Equation

For $y' + p(t)y = 0$, we can use separation of variables:

$$\frac{dy}{dt} = -p(t)y \tag{3}$$

$$\frac{dy}{y} = -p(t)dt \tag{4}$$

$$\ln|y| = -\int p(t)dt + C_1 \tag{5}$$

$$y = Ce^{-\int p(t)dt} \tag{6}$$

where $C = \pm e^{C_1}$ (or $C = 0$ if $y \equiv 0$).

Homogeneous Solution: $y_h = Ce^{-\int p(t)dt}$

4 The Integrating Factor Method

Theorem 1 (Integrating Factor Method)**.** *For the equation $y' + p(t)y = g(t)$, the integrating factor*

$$\mu(t) = e^{\int p(t)dt}$$

transforms the equation into an exact derivative:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

Proof. Starting with $y' + p(t)y = g(t)$, multiply both sides by $\mu(t)$:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

We want the left side to equal $\frac{d}{dt}[\mu(t)y]$. Computing this derivative:

$$\frac{d}{dt}[\mu(t)y] = \mu'(t)y + \mu(t)y'$$

For these to be equal:

$$\mu'(t)y + \mu(t)y' = \mu(t)y' + \mu(t)p(t)y$$

This requires $\mu'(t) = \mu(t)p(t)$, or $\frac{\mu'(t)}{\mu(t)} = p(t)$.
Integrating: $\ln|\mu(t)| = \int p(t)dt$, giving $\mu(t) = e^{\int p(t)dt}$. □

5 Complete Solution Algorithm

Method 1 (Solving Linear First-Order ODEs)**.**

1. *Convert to standard form: $y' + p(t)y = g(t)$*
2. *Compute integrating factor: $\mu(t) = e^{\int p(t)dt}$*

3. *Multiply equation by $\mu(t)$: $\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$*

4. *Recognize: $\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$*

5. *Integrate: $\mu(t)y = \int \mu(t)g(t)dt + C$*

6. *Solve for y: $y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right]$*

Common errors:

- Adding a constant when computing $\mu(t)$ (don't!)
- Forgetting absolute values in logarithms
- Not checking continuity of $p(t)$ and $g(t)$
- Missing the arbitrary constant in the final integration

6 Examples

Example 1 (Constant Coefficients)**.** *Solve $y' + 2y = e^{3t}$ with $y(0) = 1$.*

Solution:

- *$p(t) = 2$, so $\mu(t) = e^{2t}$*
- *Multiplying: $e^{2t}y' + 2e^{2t}y = e^{5t}$*
- *This is $\frac{d}{dt}[e^{2t}y] = e^{5t}$*
- *Integrating: $e^{2t}y = \frac{1}{5}e^{5t} + C$*
- *Therefore: $y = \frac{1}{5}e^{3t} + Ce^{-2t}$*
- *Using $y(0) = 1$: $1 = \frac{1}{5} + C$, so $C = \frac{4}{5}$*
- *Final answer: $y = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$*

Example 2 (Variable Coefficients)**.** *Solve $ty' + 2y = t^2$ for $t > 0$.*

Solution:

- *Standard form: $y' + \frac{2}{t}y = t$*
- *$p(t) = \frac{2}{t}$, so $\mu(t) = e^{2\ln t} = t^2$*
- *Multiplying: $t^2y' + 2ty = t^3$*
- *This is $\frac{d}{dt}[t^2y] = t^3$*
- *Integrating: $t^2y = \frac{t^4}{4} + C$*
- *Therefore: $y = \frac{t^2}{4} + \frac{C}{t^2}$*

7 Solution Structure

Theorem 2 (Superposition Principle)**.** *The general solution of $y' + p(t)y = g(t)$ can be written as:*

$$y = y_h + y_p$$

where y_h is the general solution of the homogeneous equation and y_p is any particular solution of the non-homogeneous equation.

This structure appears in ALL linear ODEs, regardless of order. Understanding it here prepares you for second-order and higher-order linear equations.

8 Existence and Uniqueness

Theorem 3 (Existence and Uniqueness for Linear First-Order)**.** *If $p(t)$ and $g(t)$ are continuous on an interval I containing t_0, then for any initial condition $y(t_0) = y_0$, there exists a unique solution defined on all of I.*

Prof. Ditkowski often asks about solution intervals. Remember:

- Solutions exist wherever $p(t)$ and $g(t)$ are continuous
- Discontinuities create natural boundaries for solution domains
- Always state the interval of validity for your solution

9 Connection to Exact Equations

After multiplying by $\mu(t)$, the equation becomes:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

This is exact with $M = \mu(t)p(t)y - \mu(t)g(t)$ and $N = \mu(t)$.

10 Physical Applications

10.1 RC Circuit

For a resistor-capacitor circuit with voltage source $V(t)$:

$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{V(t)}{R}$$

10.2 Mixing Problem

For a tank with volume V, inflow rate r_{in}, outflow rate r_{out}, and input concentration $c_{in}(t)$:

$$\frac{dy}{dt} + \frac{r_{out}}{V} y = \frac{r_{in}}{V} c_{in}(t)$$

10.3 Newton's Law of Cooling

For temperature $T(t)$ in ambient temperature T_a:

$$\frac{dT}{dt} + kT = kT_a$$

11 Summary Flowchart

Given: $y' + p(t)y = g(t)$

↓

Compute $\mu = e^{\int p(t)dt}$

↓

Multiply by μ

↓

$\frac{d}{dt}[\mu y] = \mu g$

↓

$\mu y = \int \mu g \, dt + C$

↓

$y = \frac{1}{\mu}[\int \mu g \, dt + C]$