Lesson 30: Practice Problems - Distinct Eigenvalues

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Part A: Eigenvalue/Eigenvector Computation

1. Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$
2. Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 5 & -2 \\ 3 & -2 \end{pmatrix}$
3. Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
4. Verify that $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$ and find the corresponding eigenvalue.
5. For the matrix $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$, find conditions on a and b for distinct real eigenvalues.

Part B: $\mathbf{2} \times 2 Systems with Distinct Real Eigenvalues$

6. Solve the system $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \mathbf{x}$
7. Solve the IVP: $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
8. Find the solution to $\mathbf{x}' = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \mathbf{x}$ that satisfies $x_1(0) = 2$, x$_2$(0) = -1
9. Determine all solutions of $\mathbf{x}' = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} \mathbf{x}$ that remain bounded as $t \to \infty$
10. Find the fundamental matrix for $\mathbf{x}' = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix} \mathbf{x}$

Part C: $\mathbf{3}\times 3 Systems with Distinct Eigenvalues$

11. Solve: $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}$

12. Solve: $\mathbf{x}' = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} \mathbf{x}$

13. Find the general solution: $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} \mathbf{x}$

14. Solve the IVP: $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

15. Find all equilibrium solutions and their stability for: $\mathbf{x}' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}$

Part D: Stability and Behavior Analysis

16. For what values of k is the origin a stable equilibrium for $\mathbf{x}' = \begin{pmatrix} -1 & k \\ 0 & -2 \end{pmatrix} \mathbf{x}$?

17. Classify the equilibrium at the origin for $\mathbf{x}' = \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

18. Find a system $\mathbf{x}' = A\mathbf{x}$ where all solutions approach the line $x_1 = \mathrm{x}_2 as \mathrm{t} \to \infty$

19. Determine the long-term behavior of solutions to $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} \mathbf{x}$

20. Show that if A has all negative eigenvalues, then $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$

Part E: Special Cases and Theory

21. Prove that if A is symmetric, all eigenvalues are real

22. Show that $\mathrm{tr}(A) = \sum \lambda_i and \det(A) = \prod \lambda_i$

23. If A has eigenvalues 2, 3, 5, what are the eigenvalues of $A^2? Of \mathrm{A}^{-1}$?

24. Construct a $2\times 2 matrix with eigenvalues \lambda_1 = 1,\ \lambda_2 =$ -$2 and eigenvectors$ $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

25. Find a $3\times 3 upper triangular matrix with eigenvalues 1, 2, 3 and determine its eigenvectors$

Part F: Application Problems

26. Two tanks contain salt solutions. Tank 1 has rate of change x_1' = -0.1x_1 + 0.05$x_2 and Tank 2 has$ x_2' = 0.1x_1 - 0.15$x_2. Find the salt amounts over time if initially$ $x_1(0) = 100$, $x_2(0) = 50$.

27. A predator-prey model gives $\mathbf{x}' = \begin{pmatrix}2 & -1\\1 & -1\end{pmatrix}\mathbf{x}$ where $x_1 is prey,$ x_2 $is predator population. Analyze the long-term behavior.$

28. Coupled springs lead to $\mathbf{x}' = \begin{pmatrix}0 & 1\\-2 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0 & 0\\0 & -3\end{pmatrix}\mathbf{x}$. Find the general solution.

29. An electrical circuit gives $\mathbf{x}' = \begin{pmatrix}-R/L & -1/L\\1/C & 0\end{pmatrix}\mathbf{x}$ with $R = 2, L = 1, C = 0.5$. Solve for the current and voltage.

30. **Challenge:** Show that the solution to $\mathbf{x}' = A\mathbf{x}$ can be written as $\mathbf{x}(t) = e^{At}\mathbf{x}_0 and verify this for a diagonal m$

Solutions and Hints

Problem 1: $\lambda_1 = 5$, $\lambda_2 = 2$;$v_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

Problem 7: First find $\lambda_1 = 3$, $\lambda_2 =$ -$1. General solution involves$ e^{3t} and e^{-t} terms.

Problem 11: Diagonal matrix - eigenvectors are standard basis vectors.

Problem 16: Stable for all k since both eigenvalues are negative regardless of k.

Problem 22: Use the fact that $\text{tr}(A) = \text{tr}(PDP^{-1}) = \text{tr}(D)$ where D is diagonal with eigenvalues.

Key Strategy: Always verify your eigenvalues by checking that $\det(A - \lambda I) = 0$ and eigenvectors by confirming $A\mathbf{v} = \lambda\mathbf{v}$.