ODE Lesson 7: Non-Unique Solutions - Classic Examples and Warnings

ODE 1 - Prof. Adi Ditkowski

1 When Uniqueness Fails

Core Principle: Non-uniqueness occurs when the Lipschitz condition fails. This typically happens at points where $\frac{\partial f}{\partial y}$ is unbounded or discontinuous.

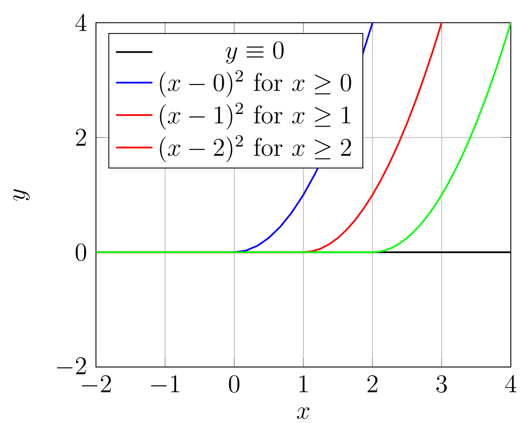
Definition 1 (Non-Unique Solutions)**.** *An IVP has* ***non-unique solutions*** *if multiple distinct functions satisfy both the differential equation and the initial conditions.*

2 The Classic Example

Example 1 (The Square Root Problem)**.** *Consider the IVP:* $y' = 2\sqrt{|y|}, \quad y(0) = 0$

This has infinitely many solutions:

- $y_1(x) \equiv 0$ *(trivial solution)*
- $y_2(x) = \begin{cases} 0 & \text{if } x \leq c \\ (x-c)^2 & \text{if } x > c \end{cases}$ *for any* $c \geq 0$
- $y_3(x) = \begin{cases} -(x-c)^2 & \text{if } x < c \\ 0 & \text{if } x \geq c \end{cases}$ *for any* $c \leq 0$



3 General Pattern for Non-Uniqueness

3.1 The Power Law Family

Theorem 1 (Power Law Non-Uniqueness). *The IVP $y' = k|y|^\alpha$ with $y(0) = 0$ where $0 < \alpha < 1$ has infinitely many solutions.*

$$\text{General solution family: } y(x) = \begin{cases} 0 & \text{if } |x| \le c \\ \pm\left[\frac{1-\alpha}{k}(|x| - c)\right]^{\frac{1}{1-\alpha}} & \text{if } |x| > c \end{cases}$$

Geometric Intuition: The smaller α, the "flatter" the slope field near $y = 0$, making it easier for solutions to "peel away" from the x-axis.

4 Lipschitz Failure Analysis

4.1 Why Square Roots Cause Problems

For $f(y) = |y|^\alpha$ with $0 < \alpha < 1$: $\frac{\partial f}{\partial y} = \alpha|y|^{\alpha-1} \to \infty$ as $y \to 0$

Function	Derivative	Lipschitz at 0?	Unique?
y	1	Yes	Yes
y^2	$2y$	Yes	Yes
$\sqrt{\lvert y\rvert}$	$\frac{1}{2\sqrt{\lvert y\rvert}}$	No	No
$\lvert y\rvert^{2/3}$	$\frac{2}{3}\lvert y\rvert^{-1/3}$	No	No
$\lvert y\rvert^{3/2}$	$\frac{3}{2}\lvert y\rvert^{1/2}$	Yes	Yes

5 The Peano Phenomenon

Definition 2 (Peano Phenomenon). *Solutions that can "branch" - starting from the same initial condition, the solution can follow different paths after some time.*

Example 2 (Branching Solutions). *Consider the implicit equation:* $(y')^2 = 4y$

From $y(0) = 0$, solutions can:

1. *Stay at $y = 0$ for interval $[0, a]$*
2. *Branch upward: $y = (x - a)^2$ for $x \ge a$*
3. *Branch downward: $y = -(x - a)^2$ for $x \le a$*

6 Construction Methods for Non-Unique Solutions

6.1 Method 1: Separation of Variables Pitfall

When separating variables in $\frac{dy}{dx} = g(y)h(x)$, if $g(y_0) = 0$, the constant solution $y \equiv y_0$ might be lost!

Example 3 (Lost Solutions). $y' = y(1-y)$ *Separating:* $\int \frac{dy}{y(1-y)} = \int dx$
This process misses the equilibrium solutions:

- $y \equiv 0$
- $y \equiv 1$

6.2 Method 2: Clairaut's Equation

Theorem 2 (Clairaut's Equation). *Equations of the form* $y = xy' + f(y')$ *have:*

- *General solution:* $y = cx + f(c)$ *(family of lines)*
- *Singular solution: The envelope of the family*

6.3 Method 3: Solution Patching

Patching Technique: If $f(x, y) = 0$ along curve C, solutions can:

1. Follow any solution until reaching C
2. "Pause" on C for arbitrary time
3. "Restart" with any solution leaving C

7 Physical Interpretations

Physical System	Non-Uniqueness Meaning
Dry friction	Object can start moving at any time
Water tank draining	Can't determine when draining started
Phase transitions	Multiple equilibrium states possible
Crystal growth	Nucleation can occur at various times
Chemical reactions	Reaction can initiate at different moments

8 Advanced Uniqueness Criteria

8.1 Osgood's Condition

Theorem 3 (Osgood's Criterion). *Even if f is not Lipschitz, uniqueness holds if:* $\int_0^\epsilon \frac{dy}{\omega(y)} = \infty$ *where ω is the modulus of continuity of f.*

Example 4 (Osgood Application). $f(y) = y \ln |y|$ *near* $y = 0$*:*

- *Not Lipschitz (derivative unbounded)*
- *But Osgood condition holds*
- *Therefore: unique solutions!*

9 Exam Strategy for Non-Uniqueness

When you find non-uniqueness:

1. State clearly that uniqueness fails
2. Identify where Lipschitz condition breaks (usually where $\partial f/\partial y \to \infty$)
3. Give at least two distinct solutions explicitly
4. Sketch the solution family if possible
5. Explain physical interpretation if applicable

10 Warning Signs

Red Flags for Non-Uniqueness:

- Powers less than 1: $y^{1/2}, y^{2/3}, |y|^{0.7}$
- Implicit equations where $\frac{\partial F}{\partial y'} = 0$
- Piecewise functions with "flat spots"
- $|y|$ or $\text{sign}(y)$ at $y = 0$
- Any cusp or corner in graph of $f(y)$

11 Common Exam Questions

Prof. Ditkowski's Favorites:

1. "Give an IVP with exactly n solutions"
2. "Can two solutions cross? When?"
3. "Find all solutions through $(0, 0)$"
4. "For what α does $y' = |y|^\alpha$ have unique solutions?"

12 Solution Crossing

Theorem 4 (Crossing Theorem)**.** *If f is Lipschitz, solutions cannot cross. If solutions cross at point (x_0, y_0), then f is not Lipschitz at (x_0, y_0).*

13 Memory Aid

PEANOS for Non-Uniqueness:
Powers less than one
Equilibria too attractive
Absolute values
Non-Lipschitz points
Osgood violations
Singularities