Lesson 21: Practice Problems - Exact Equations Recognition

ODE 1 - Prof. Adi Ditkowski

Part A: Basic Exactness Testing (Problems 1-5)

1. Test for exactness: $(2x + 3y)dx + (3x + 4y)dy = 0$
2. Test for exactness: $(y^2 + 2xy)dx + (2xy - x^2)dy = 0$
3. Test for exactness: (e^x sin y + 2x)dx + (e^x cos y + 2y)dy = 0
4. Test for exactness: $(3x^2y^2 + 2x)dx + (2x^3y + 3y^2)dy = 0$
5. Test for exactness: $\frac{2xy-y^2}{x^2}dx + \frac{1-\frac{y}{x}}{1}dy = 0$

Part B: Standard Form Conversion (Problems 6-10)

6. Convert to standard form and test: $\frac{dy}{dx} = -\frac{2xy+1}{x^2+2y}$
7. Convert and test: $(x + y)dy = (x - y)dx$
8. Convert and test: $xdy - ydx = x^2dx$
9. Convert and test: $\frac{dy}{dx} = \frac{y\cos x - \sin x\cos x}{-\sin x + y\sin x}$
10. Convert and test: $y' = \frac{3x^2+y^2}{2xy-x^2}$

Part C: Polynomial Exact Equations (Problems 11-15)

11. Test for exactness: $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$
12. Test for exactness: $(4x^3y - 2xy^3)dx + (x^4 - 3x^2y^2 + 2y)dy = 0$
13. Test for exactness: $(2x^3 - xy^2 + 2y)dx + (x^2y - 2xy + 3)dy = 0$
14. Test for exactness: $(6xy^2 + 4x^3)dx + (6x^2y - 3y^2)dy = 0$
15. Determine the value of k that makes the equation exact: $(kxy+y^3)dx+(x^2+3xy^2)dy = 0$

Part D: Transcendental Functions (Problems 16-20)

16. Test: $(\mathrm{e}^{2x} + \mathrm{y}\cos(xy))dx + (2\mathrm{y}\mathrm{e}^{y} + \mathrm{x}\cos(xy))dy = 0$
17. Test: $(\ln y + 2x)dx + \left(\frac{x}{y} + \ln x\right) dy = 0$
18. Test: $(y\mathrm{e}^{xy} + \sin x)dx + (x\mathrm{e}^{xy} + 2\mathrm{y})\mathrm{dy} = 0$
19. Test: $\left(\frac{1}{x} + \frac{1}{y}\right) dx + \left(\frac{x}{y^2} - \frac{1}{y}\right) dy = 0$
20. Test: $(\tan y + 2xy)dx + (x\sec^2 y + x^2 + 1)dy = 0$

Part E: Domain Issues (Problems 21-23)

21. Consider $\frac{-ydx+xdy}{x^2+y^2} = 0$. Test for exactness and discuss domain.
22. Test exactness of $\frac{xdx+ydy}{\sqrt{x^2+y^2}} = 0$ in appropriate domains.
23. For what values of (x, y) is $(x-1)^{-1}dx + (y+2)^{-1}dy = 0$ exact?

Part F: Exam-Style Problems (Problems 24-28)

24. (Prof. Ditkowski, 2023) Test for exactness and explain physical meaning: $(2xy + y^2\cos x)dx + (x^2 + 2y\sin x)dy = 0$
25. Find all values of a and b such that $(ax^2y + 2xy^2)dx + (x^3 + bx^2y)dy = 0$ is exact.
26. Show that if $M(x, y) = f(x)g(y)$ and $N(x, y) = p(x)q(y)$, the equation $Mdx+Ndy = 0$ is exact if and only if $f'(x)q(y) = g'(y)p(x)$.
27. Given that $(P(x) + Q(y))dx + (R(x) + S(y))dy = 0$ is exact, what can you conclude about P, Q, R, S?
28. Prove that if both $M_1dx + N_1dy = 0$ and $M_2dx + N_2dy = 0$ are exact, then $(aM_1 + bM_2)dx + (aN_1 + bN_2)dy = 0$ is exact for any constants a, b.

Solutions and Hints

Problem 1: $M = 2x + 3y$, $N = 3x + 4y$. $\frac{\partial M}{\partial y} = 3$, $\frac{\partial N}{\partial x} = 3$. Equal $\Rightarrow$ Exact!

Problem 2: $M = y^2 + 2xy$, $N = 2xy - x^2$. $\frac{\partial M}{\partial y} = 2y + 2x$, $\frac{\partial N}{\partial x} = 2y - 2x$. Not equal $\Rightarrow$ Not exact.

Problem 6: Multiply by denominator: $(2xy + 1)dx + (x^2 + 2y)dy = 0$. $M = 2xy + 1$, $N = x^2 + 2y$. $\frac{\partial M}{\partial y} = 2x$, $\frac{\partial N}{\partial x} = 2x$. Equal $\Rightarrow$ Exact!

Problem 15: For exactness: $\frac{\partial M}{\partial y} = kx + 3y^2 = \frac{\partial N}{\partial x} = 2x + 3y^2$. Therefore $k = 2$.

Problem 21: $M = -y/(x^2 + y^2)$, $N = x/(x^2 + y^2)$. $\frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}$, $\frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$. Exact everywhere except origin! But no single-valued potential on punctured plane.

Key Insight for Problem 27: $Q'(y) = 0$ and $R'(x) = 0$.