Lesson 25: Practice Problems - Orthogonal Trajectories

ODE 1 - Prof. Adi Ditkowski

Part A: Basic Orthogonal Trajectories (Problems 1-6)

1. Find the orthogonal trajectories of the family $y = cx^2$.
2. Find the orthogonal trajectories of $x^2 + y^2 = c$.
3. Find the orthogonal trajectories of $y = c\mathrm{e}^{-x}$.
4. Find the orthogonal trajectories of $xy = c$.
5. Find the orthogonal trajectories of $y^2 = cx$.
6. Find the orthogonal trajectories of $y = c \sin x$.

Part B: Eliminating Parameters (Problems 7-11)

7. For the family $x^2 + y^2 = 2cx$:
   (a) Eliminate c to find the differential equation
   (b) Find the orthogonal trajectories
8. For the family $y = c(x - 1)^2$:
   (a) Find the differential equation of the family
   (b) Determine the orthogonal trajectories
9. Given $(x - c)^2 + y^2 = c^2$, find orthogonal trajectories.
10. For $y^2 = c(x + c)$, eliminate c and find orthogonal curves.
11. The family $y = \tan(x + c)$ - find its orthogonal trajectories.

Part C: Self-Orthogonal Families (Problems 12-14)

12. Show that the family of rectangular hyperbolas $x^2 - y^2 = c$ is self-orthogonal.
13. Prove that confocal parabolas $y^2 = 4c(x + c)$ are self-orthogonal.
14. Find all self-orthogonal families of the form $ax^2 + by^2 = c$.

Part D: Physical Applications (Problems 15-19)

15. The equipotential lines in a 2D electric field are given by $x^2 - y^2 = c$. Find the electric field lines.

16. Temperature distribution in a plate: $T = x^2 + y^2 = c$. Find the heat flow lines.

17. Stream function: $\psi = xy = c$. Find the velocity potential lines.

18. Given electric field lines $y = ce^x, findtheequipotentials.$

19. In a magnetic field, the flux lines are circles $x^2 + y^2 = c^2$. What are the constant magnetic potential curves?

Part E: Complex Orthogonal Trajectories (Problems 20-24)

20. Find orthogonal trajectories of $y = c(1 + x^2)$.

21. Find orthogonal trajectories of the family of curves $r = c \sin\theta$ in polar coordinates.

22. For the family $e^x \cos y = c$:
    (a) Find the differential equation
    (b) Determine orthogonal trajectories

23. Find orthogonal trajectories of $x^3 + 3xy^2 = c$.

24. The family of curves is given implicitly by $x^{2y} + \mathrm{xy}^2 = c$. Find orthogonal trajectories.

Part F: Exam-Style Problems (Problems 25-30)

25. (Prof. Ditkowski 2023) Show that the families $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ are orthogonal, where a and b are parameters.

26. Given that two families of curves are orthogonal: Family 1: $y = f(x, c)$ Family 2: $y = g(x, k)$ Prove that at any intersection point, $\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} = -1$.

27. A family of curves satisfies the differential equation $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$.
    (a) Find the orthogonal differential equation
    (b) Show that both equations are exact
    (c) Solve both and verify orthogonality

28. In complex analysis, $f(z) = z^3$.
    (a) Find the families $u(x, y) = c$ and $v(x, y) = k$

(b) Verify they are orthogonal trajectories

(c) Sketch several curves from each family

29. Find all functions $F(x, y)$ such that the family $F(x, y) = c$ is self-orthogonal.

30. Two families of curves in polar coordinates are given by: Family 1: $r = c(1 + \cos\theta)$ (cardioids) Family 2: $r = k(1 - \cos\theta)$

    (a) Are these families orthogonal?

    (b) If not, find the orthogonal trajectories of Family 1

Solutions and Key Insights

Problem 1: $y = cx^2$ Differentiate: $\frac{dy}{dx} = 2cx$ Eliminate c: From $y = cx^2$, we get $c = \frac{y}{x^2}$ So $\frac{dy}{dx} = \frac{2y}{x}$ Orthogonal: $-\frac{dx}{dy} = \frac{2y}{x}$ or $\frac{dx}{dy} = -\frac{x}{2y}$ Solving: $2ydy = -xdx$ $y^2 = -\frac{x^2}{2} + C$ Result: $x^2 + 2y^2 = K$ (ellipses)

Problem 4: $xy = c$ Differentiate: $y + x\frac{dy}{dx} = 0$ So $\frac{dy}{dx} = -\frac{y}{x}$ Orthogonal: $-\frac{dx}{dy} = -\frac{y}{x}$ or $\frac{dx}{dy} = \frac{y}{x}$ This gives $xdx = ydy$ Result: $x^2 - y^2 = K$ (rectangular hyperbolas rotated 45°)

Problem 7: $x^2 + y^2 = 2cx$ Differentiate: $2x + 2y\frac{dy}{dx} = 2c$ From original: $c = \frac{x^2+y^2}{2x}$ Substitute: $2x + 2y\frac{dy}{dx} = \frac{x^2+y^2}{x}$ Simplify: $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ Orthogonal: $\frac{dx}{dy} = \frac{2xy}{x^2-y^2}$

Problem 15: Electric field lines are orthogonal to equipotentials. Given: $x^2 - y^2 = c$ Differentiate: $2x - 2y\frac{dy}{dx} = 0$ So $\frac{dy}{dx} = \frac{x}{y}$ Orthogonal (field lines): $\frac{dx}{dy} = -\frac{x}{y}$ Solving: $\frac{xdx}{ydy} = -1$, so $xdx = -ydy$ Result: $x^2 + y^2 = K$ (circles - radial field!)

Key Strategy for Problem 12: For self-orthogonal, show that the orthogonal differential equation, when solved, gives the same family with different parameter.

Warning for Problem 28: For $f(z) = z^3 = (x + iy)^3$, expand carefully: $u + iv = x^3 - 3xy^2 + i(3x^{2y}$ - y^3) So $u = x^3 - 3xy^2$ and $v = 3x^{2y}$ - y^3

Insight for Problem 25: These are circles passing through the origin. Use the fact that circles through origin with centers on perpendicular axes are orthogonal.