

# 3D Systems and Higher Dimensions

## ODE 1 - Lesson 39

### 1 3D Linear Systems

Consider the 3D linear system:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \text{where } A \in \mathbb{R}^{3 \times 3}$$

### 2 Classification of 3D Equilibria

**Classification 1** (3D Equilibrium Types).

<i>Type</i>	<i>Eigenvalues</i>	<i>Stability</i>	<i>Behavior</i>
<i>Stable Node</i>	$\lambda_i \neq 0 (\text{all real})$	<i>Asymp. Stable</i>	<i>All trajectories converge</i>
<i>Unstable Node</i>	$\lambda_i \neq 0 (\text{all real})$	<i>Unstable</i>	<i>All trajectories diverge</i>
<i>Saddle (1-2)</i>	<i>1 pos., 2 neg.</i>	<i>Unstable</i>	<i>1D unstable, 2D stable manifold</i>
<i>Saddle (2-1)</i>	<i>2 pos., 1 neg.</i>	<i>Unstable</i>	<i>2D unstable, 1D stable manifold</i>
<i>Spiral-Node (S)</i>	<i>1 real neg., 2 complex (Re&lt;0)</i>	<i>Asymp. Stable</i>	<i>Spiral in plane, converge perpendicular</i>
<i>Spiral-Node (U)</i>	<i>1 real pos., 2 complex (Re&gt;0)</i>	<i>Unstable</i>	<i>Spiral out plane, diverge perpendicular</i>
<i>Spiral-Saddle</i>	<i>1 real, 2 complex (mixed signs)</i>	<i>Unstable</i>	<i>Spiral in one direction, escape another</i>
<i>Center-Line</i>	<i>1 real = 0, 2 pure imag.</i>	<i>Marginally Stable</i>	<i>Circles in plane, line of equilibria</i>

### 3 Invariant Manifolds

**Definition 1** (Stable, Unstable, and Center Manifolds). *For an equilibrium at the origin with eigenvalues  $\{\lambda_i\}$ : Dimensions :  $\dim(E^s) + \dim(E^u) + \dim(E^c) = 3$*

**Theorem 1** (Manifold Dynamics). • *Trajectories in  $E^s$  approach origin as  $t \rightarrow \infty$*

• *Trajectories in  $E^u$  approach origin as  $t \rightarrow -\infty$*

• *Trajectories in  $E^c$  may be periodic or stationary*

## 4 Visualization Techniques

### 4.1 Projection Methods

Standard Projections:

- **xy-projection:** View from above  $(x, y, 0)$
- **xz-projection:** View from side  $(x, 0, z)$
- **yz-projection:** View from front  $(0, y, z)$

Each projection shows 2D "shadow" of 3D dynamics

**Example 1** (Projection Analysis). *System with  $\lambda_1 = -1, \lambda_2, \lambda_3 = -0.5 \pm 2i$ :*

- *xy-projection: Spiral (from complex pair)*
- *xz-projection: Node-like (real eigenvalue dominates)*
- *yz-projection: Pure spiral (complex pair only)*

### 4.2 Poincaré Sections

**Definition 2** (Poincaré Section). A **Poincaré section** is a lower-dimensional slice through phase space. For a 3D system, choose a 2D surface  $\Sigma$  and record intersection points of trajectories with  $\Sigma$ .

Poincaré Map Reveals:

- Fixed point  $\rightarrow$  Periodic orbit
- Closed curve  $\rightarrow$  Quasi-periodic (torus)
- Strange pattern  $\rightarrow$  Chaos
- No pattern  $\rightarrow$  Transient behavior

## 5 Examples of 3D Systems

**Example 2** (Complete 3D Analysis). *Consider the matrix:*

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

**Step 1:** Eigenvalues are  $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 3$

**Step 2:** Classification: Saddle (2-1 type)

**Step 3:** Manifolds:

- $E^s = \text{span}\{e_1, e_2\}$  (xy-plane),  $\dim(E^s) = 2$   
 $E^u = \text{span}\{e_3\}$  (z-axis),  $\dim(E^u) = 1$   
**Step 4:** Behavior: Trajectories spiral toward xy-plane, then escape along z-axis

**Example 3** (Spiral-Node System).

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & -4 & -1 \end{pmatrix}$$

*Eigenvalues:*  $\lambda_1 = -3, \lambda_{2,3} = -1 \pm 4i$

*Classification:* Stable spiral-node - Exponential decay along x-axis (rate = 3) - Spiral decay in yz-plane (rate = 1, frequency = 4)

## 6 Higher Dimensions (n ≥ 3)

- **Classification in n-Dimensions:** For  $\dot{\mathbf{x}} = A\mathbf{x}$  with  $A \in \mathbb{R}^{n \times n}$ :
  - Count eigenvalues with  $\text{Re}(\lambda) < 0$ :  $n_s$  (stable directions) Count eigenvalues with  $\text{Re}(\lambda) > 0$ :  $n_u$  (unstable directions)
  - Count eigenvalues with  $\text{Re}(\lambda) = 0$ :  $n_c$  (center directions) Check:  $n_s + n_u + n_c = n$

**Classification 2** (Stability in Higher Dimensions). • **Asymptotically Stable:** All  $\text{Re}(\lambda_i) < 0$   
 • **Unstable:** At least one  $\text{Re}(\lambda_i) > 0$

- **Marginally Stable:** All  $\text{Re}(\lambda_i) \leq 0$ , some = 0

## 7 Numerical Visualization Tools

**Essential Plotting Techniques:**

1. **3D Trajectory Plots:** Direct visualization in 3D
2. **Multiple 2D Projections:** Show different viewpoints
3. **Animation:** Time evolution along trajectories
4. **Isosurfaces:** Level sets of conserved quantities
5. **Vector Field Slices:** Direction fields on 2D slices

## 8 Special Phenomena in 3D

**Definition 3** (Limit Cycle). A **limit cycle** is an isolated closed orbit. In 3D, nearby trajectories can spiral toward it (stable) or away (unstable).

**Definition 4** (Torus Attractor). Quasi-periodic motion on a 2D torus surface in 3D phase space. Occurs with two incommensurate frequencies.

**Definition 5** (Strange Attractor). A fractal set in phase space exhibiting sensitive dependence on initial conditions (chaos). Example: Lorenz attractor.

## 9 The Lorenz System (Preview)

**Example 4** (Lorenz Equations).

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$

For  $\sigma = 10$ ,  $b = 8/3$ ,  $r = 28$ : *Chaotic attractor!*  
*Equilibria:*

- *Origin: Always exists*
- $C_{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$  for  $r > 1$

## 10 Exam Strategy for 3D Problems

### Prof. Ditkowski's 3D Problem Approach:

1. State system dimension explicitly
2. Find ALL eigenvalues (check: product = det, sum = trace)
3. Classify equilibrium type precisely
4. State manifold dimensions: "2D stable, 1D unstable"
5. Describe behavior in words
6. Optional: Sketch ONE key projection
7. Never attempt full 3D drawing

### Common 3D Errors:

- Forgetting complex eigenvalues come in pairs
- Wrong manifold dimension count
- Trying to visualize 4D+ systems
- Missing the third eigenvalue
- Not checking  $\sum \lambda_i = \text{tr}(A)$