# Lesson 24: Special Integrating Factors - $\mu(xy)$ and Beyond

ODE 1 - Prof. Adi Ditkowski

## 1 Beyond Simple Integrating Factors

When neither $\mu(x)$ nor $\mu(y)$ exists, we look for integrating factors of special forms:

- $\mu(xy)$ - depends on the product $xy$
- $\mu(x^2 + y^2)$ - depends on radial distance
- $\mu = x^a y^b$ - power form
- $\mu = 1/(xM + yN)$ - for homogeneous equations

Each special form has its own existence test. Master the tests to quickly identify which form to use!

## 2 Case 3: Integrating Factor $\mu(xy)$

**Theorem 1** (Existence of $\mu(xy)$)**.** *An integrating factor depending only on $z = xy$ exists if and only if*

$$\frac{1}{xN - yM}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = F(xy)$$

*where $F$ is a function of $xy$ alone. The integrating factor is:*

$$\mu(xy) =$$

$e^{\int}$ *F(z) dz where* $z = xy$

*Proof Sketch.* If $\mu = \mu(z)$ where $z = xy$, then:

$$\frac{\partial \mu}{\partial x} = \frac{d\mu}{dz} \cdot y, \quad \frac{\partial \mu}{\partial y} = \frac{d\mu}{dz} \cdot x$$

Substituting into the exactness condition:

$$M \cdot \frac{d\mu}{dz} \cdot x - N \cdot \frac{d\mu}{dz} \cdot y = \mu(N_x - M_y)$$

$$\frac{1}{\mu}\frac{d\mu}{dz} = \frac{M_y - N_x}{xN - yM}$$

This requires the right side to be a function of $z = xy$ only. $\square$

**Testing for $\mu(xy)$:**

1. Compute $M_y - N_x$ (numerator)
2. Compute $xN - yM$ (denominator - note the pattern!)
3. Form the ratio $\frac{M_y - N_x}{xN - yM}$
4. Check if this can be expressed as $F(xy)$
5. If yes, solve $\frac{d\mu}{dz} = \mu \cdot F(z)$ where $z = xy$

**Example 1** ($\mu(xy)$ Application)**.** *Solve* $(2y^2 + 3xy)dx + (2xy + x^2)dy = 0$

***Step 1:*** *Check exactness:* $M_y = 4y + 3x$*,* $N_x = 2y + 2x$*. Not exact.*

***Step 2:*** *Test* $\mu(x)$*:* $\frac{M_y - N_x}{N} = \frac{2y+x}{2xy+x^2} = \frac{2y+x}{x(2y+x)} = \frac{1}{x}$

*Actually,* $\mu(x) = x$ *works here! But let's continue for illustration...*

***Step 3:*** *Test* $\mu(xy)$*:*

$$\frac{M_y - N_x}{xN - yM} = \frac{2y + x}{x(2xy + x^2) - y(2y^2 + 3xy)}$$

$$= \frac{2y + x}{2x^2y + x^3 - 2y^3 - 3xy^2}$$

*This expression is complex and not a clear function of* $xy$*.*

# 3 Case 4: Integrating Factor $\mu(x^2 + y^2)$

**Theorem 2** (Existence of $\mu(x^2 + y^2)$)**.** *An integrating factor depending only on* $r^2 = x^2 + y^2$ *exists if and only if*

$$\frac{1}{xM + yN}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = G(x^2 + y^2)$$

*where* $G$ *is a function of* $x^2 + y^2$ *alone.*

The denominator $xM + yN$ appears naturally in polar coordinate transformations. This test often succeeds for equations with circular symmetry.

**Example 2** (Radial Integrating Factor)**.** *Consider* $(x^2 + y^2 + x)dx + (x^2 + y^2 + y)dy = 0$

*Let's test for* $\mu(x^2 + y^2)$*:*

$$M_y - N_x = 2y - 2x$$

$$xM + yN = x(x^2 + y^2 + x) + y(x^2 + y^2 + y) = (x^2 + y^2)(x + y) + x^2 + y^2$$

$$= (x^2 + y^2)(x + y + 1)$$

*The ratio doesn't simplify to a function of $x^2 + y^2$ easily, but if we had $(x^2 + y^2)$ as a common factor throughout, it would work.*

# 4 Case 5: Power Form $\mu = x^a y^b$

**Method 1** (Finding $\mu = x^a y^b$)**.** *To find an integrating factor of the form $\mu = x^a y^b$:*

1. *Multiply the equation by $x^a y^b$*
2. *Apply the exactness condition*
3. *Compare powers of $x$ and $y$ on both sides*
4. *Solve the resulting system for $a$ and $b$*

**Example 3** (Power Form)**.** *Find $\mu = x^a y^b$ for $ydx + 2xdy = 0$*

*After multiplication: $x^a y^{b+1} dx + 2x^{a+1} y^b dy = 0$*

*Exactness requires:*

$$\frac{\partial}{\partial y}(x^a y^{b+1}) = \frac{\partial}{\partial x}(2x^{a+1} y^b)$$

$$(b + 1)x^a y^b = 2(a + 1)x^a y^b$$

$$b + 1 = 2(a + 1)$$

$$b = 2a + 1$$

*Choosing $a = -1$ gives $b = -1$, so $\mu = \frac{1}{xy}$ works.*

# 5 Case 6: Homogeneous Equations

**Theorem 3** (Integrating Factor for Homogeneous Equations)**.** *If $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree $n$, then*

$$\mu = \frac{1}{xM + yN}$$

*is an integrating factor (provided $xM + yN \neq 0$).*

*Proof Outline.* Using Euler's theorem for homogeneous functions of degree $n$:

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = nM$$

$$x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN$$

The exactness condition with $\mu = 1/(xM + yN)$ can be verified using these relations. $\square$

**Quick Recognition of Homogeneous Equations:**

- All terms have the same total degree in $x$ and $y$
- $M(tx, ty) = t^n M(x, y)$ for some $n$
- Common forms: rational functions where numerator and denominator have same degree

## 6 Strategy Flowchart

**Complete Integrating Factor Strategy:**

1. Test for exactness - if exact, solve directly
2. Test for $\mu(x)$: Is $(M_y - N_x)/N$ a function of $x$ only?
3. Test for $\mu(y)$: Is $(N_x - M_y)/M$ a function of $y$ only?
4. Check for homogeneity - if yes, use $\mu = 1/(xM + yN)$
5. Test for $\mu(xy)$: Is $(M_y - N_x)/(xN - yM)$ a function of $xy$?
6. Test for $\mu(x^2 + y^2)$: Is $(M_y - N_x)/(xM + yN)$ a function of $x^2 + y^2$?
7. Try $\mu = x^a y^b$ by comparing powers
8. Look for patterns or use inspection

## 7 Memory Aids and Patterns

**Denominator Patterns:**

| Form | Test Denominator | Mnemonic |
|---|---|---|
| $\mu(x)$ | $N$ | "N for x" |
| $\mu(y)$ | $M$ | "M for y" |
| $\mu(xy)$ | $xN - yM$ | "Cross product" |
| $\mu(x^2 + y^2)$ | $xM + yN$ | "Dot product" |
| Homogeneous | $xM + yN$ | "Euler's friend" |

## 8 Common Exam Patterns

Prof. Ditkowski often gives hints about the form:

- "Find an integrating factor of the form $x^a y^b$"

- "Show that the equation has an integrating factor depending on $xy$"
- "Find $\mu$ assuming it depends only on $x^2 + y^2$"

When you see these hints, skip the testing phase and work directly with the given form!