Lesson 40: Practice Problems

Critical Points and Linearization

ODE 1 - Prof. Adi Ditkowski

Part A: Finding Critical Points (5 problems)

1. Find all critical points of the system:

$$\dot{x} = x - y, \quad \dot{y} = x^2 - 4$$

Solution Hint: Set both equal to zero: $y = x$ and $x^2 = 4$, giving two critical points.

2. Find all critical points of:

$$\dot{x} = y(1 - x^2), \quad \dot{y} = x - y$$

Solution Hint: Consider cases: either $y = 0$ or $x^2 = 1$. Check all combinations.

3. Determine the critical points of:

$$\dot{x} = \sin(x), \quad \dot{y} = y - \cos(x)$$

Solution Hint: From $\sin(x) = 0$, we get $x = n\pi$. Then find corresponding y values.

4. Find all equilibria of the predator-prey system:

$$\dot{x} = x(2 - x - y), \quad \dot{y} = y(x - 1)$$

Solution Hint: Factor first, then consider cases systematically.

5. For what values of a does the system have exactly one critical point?

$$\dot{x} = ax - y, \quad \dot{y} = x^2 + y^2 - 1$$

Solution Hint: Analyze when the system $ax - y = 0$ and $x^2 + y^2 = 1$ has unique solution.

Part B: Jacobian and Linearization (6 problems)

6. Compute the Jacobian matrix for:

$$\dot{x} = x^2 - y + 1, \quad \dot{y} = xy$$

Solution Hint: $J = \begin{pmatrix} 2x & -1 \\ y & x \end{pmatrix}$

7. Find the Jacobian at the origin for:

$$\dot{x} = \sin(x) + y, \quad \dot{y} = e^x - e^y$$

Solution Hint: Remember: $\frac{d}{dx}\sin(x)|_{x=0} = \cos(0) = 1$ and $\frac{d}{dx}e^x|_{x=0} = 1$

8. Linearize the pendulum equation at $(0, 0)$:

$$\dot{x} = y, \quad \dot{y} = -\sin(x) - 0.5y$$

Solution Hint: Use $\sin(x) \approx x$ near origin.

9. Find the linearization of the following system at $(1, 1)$:

$$\dot{x} = y - x^3, \quad \dot{y} = -x + y^3$$

Solution Hint: First verify $(1, 1)$ is a critical point, then compute $J(1, 1)$.

10. Compute the Jacobian for the Lotka-Volterra system:

$$\dot{x} = x(\alpha - \beta y), \quad \dot{y} = y(-\gamma + \delta x)$$

at the coexistence equilibrium. **Solution Hint:** First find the equilibrium: $x = \gamma/\delta$, $y = \alpha/\beta$.

11. For the system $\dot{x} = y + x^2$, $\dot{y} = -x + y^2$, find all critical points and compute the Jacobian at each. **Solution Hint:** From first equation: $y = -x^2$. Substitute into second.

Part C: Stability Classification (5 problems)

12. Classify the critical point $(0, 0)$ for:

$$\dot{x} = -2x + y, \quad \dot{y} = x - 3y$$

Solution Hint: Find eigenvalues of $J = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix}$

13. Determine the stability of all critical points:

$$\dot{x} = y, \quad \dot{y} = x - x^3$$

Solution Hint: Three critical points: $(0, 0)$, $(1, 0)$, $(-1, 0)$. Classify each.

14. For what values of μ is the origin stable?

$$\dot{x} = -x + \mu y, \quad \dot{y} = -\mu x - y$$

Solution Hint: Eigenvalues depend on μ. Find stability condition.

15. Classify the critical point $(1, 0)$ for:

$$\dot{x} = x(1 - x) - xy, \quad \dot{y} = y(x - 2)$$

Solution Hint: Compute $J(1, 0)$ carefully.

16. Determine the type and stability of the origin for:

$$\dot{x} = y + x^3, \quad \dot{y} = -2x - 3y + xy$$

Solution Hint: Nonlinear terms vanish at origin, so linearization is exact there.

Part D: Hartman-Grobman Applications (4 problems)

17. For which critical points does linearization determine the local behavior?

$$\dot{x} = y, \quad \dot{y} = \sin(x) - y$$

Solution Hint: Check if eigenvalues have non-zero real parts (hyperbolic condition).

18. Consider the system $\dot{x} = -y + x^2$, $\dot{y} = x + y^2$. At which critical point(s) might linearization fail to determine stability? **Solution Hint:** Find where eigenvalues are purely imaginary.

19. For the system:

$$\dot{x} = y^2 - x, \quad \dot{y} = x^2 - y$$

Identify all hyperbolic equilibria. **Solution Hint:** Check each critical point for zero real parts in eigenvalues.

20. Given that linearization at $(0, 0)$ yields eigenvalues $\lambda = \pm i$, what can you conclude about the nonlinear system's behavior near the origin? **Solution Hint:** Linearization inconclusive; could be center, spiral, or more complex.

Part E: Exam-Style Problems (5 problems)

21. [**Prof. Ditkowski Style**] Consider the competing species model:

$$\dot{x} = x(3 - x - 2y), \quad \dot{y} = y(2 - x - y)$$

(a) Find ALL critical points

(b) Compute the Jacobian at each critical point

(c) Classify each equilibrium and state its stability
(d) Which equilibria are biologically meaningful?

Solution Hint: Four critical points expected. Check axes and interior.

22. **[Comprehensive]** For the system:

$$\dot{x} = y - x^2, \quad \dot{y} = -x - y + 2$$

(a) Show that $(1, 1)$ is a critical point
(b) Find any other critical points
(c) Linearize at each critical point
(d) Sketch the expected behavior near each equilibrium

Solution Hint: Quadratic in x means at most 2 critical points.

23. **[Bifurcation Preview]** Consider:

$$\dot{x} = \mu x - y - x^3, \quad \dot{y} = x + \mu y - y^3$$

(a) Show $(0, 0)$ is always a critical point
(b) Find the linearization at the origin
(c) For what values of μ is the origin stable?
(d) What happens at $\mu = 0$?

Solution Hint: Eigenvalues are $\lambda = \mu \pm i$.

24. **[Physical System]** A nonlinear oscillator satisfies:

$$\ddot{x} + \dot{x} - x + x^3 = 0$$

(a) Rewrite as a first-order system
(b) Find all equilibria
(c) Determine stability of each equilibrium
(d) Interpret physically

Solution Hint: Let $y = \dot{x}$ to get system form.

25. **[Mixed Techniques]** For the gradient system:

$$\dot{x} = -\frac{\partial V}{\partial x}, \quad \dot{y} = -\frac{\partial V}{\partial y}$$

where $V(x, y) = x^4 + y^4 - 2x^2 - 2y^2 + xy$:

(a) Find all critical points
(b) Classify each using linearization
(c) What can you say about stability using V directly?

Solution Hint: Critical points occur where $\nabla V = 0$. Use second derivative test.

Part F: Additional Practice (5 problems)

26. Find and classify all equilibria of:

$$\dot{x} = x - x^3, \quad \dot{y} = -y$$

27. For the Hamiltonian system with $H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$:

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

Find and classify all critical points.

28. Consider the chemical reaction system:

$$\dot{x} = 1 - xy^2, \quad \dot{y} = xy^2 - y$$

Find the unique positive equilibrium and determine its stability.

29. For what values of a and b is the origin an asymptotically stable spiral?

$$\dot{x} = ax + y, \quad \dot{y} = -x + by$$

30. **[Challenge]** Consider the system:

$$\dot{x} = y + x(1 - x^2 - y^2), \quad \dot{y} = -x + y(1 - x^2 - y^2)$$

(a) Show that the origin is the only critical point inside $x^2 + y^2 < 1$

(b) Classify the origin using linearization

(c) Show that $x^2 + y^2 = 1$ is an invariant circle

(d) Describe the global behavior

Key Strategies for Prof. Ditkowski's Exam:

- Always find ALL critical points - check systematically
- Show Jacobian computation explicitly - no shortcuts
- State both type AND stability clearly
- When eigenvalues have zero real part, state "linearization inconclusive"
- For transcendental equations, graphical analysis is acceptable
- Double-check arithmetic - especially signs in the Jacobian