Lesson 50: Practice Problems - Numerical Methods

ODE 1 - Prof. Adi Ditkowski

Part A: Euler's Method (6 problems)

1. Apply Euler's method with $h = 0.1$ to solve:

   $$y' = x - y, \quad y(0) = 1$$

   Find $y(0.3)$ (show all steps).

2. Use Euler's method with $h = 0.25$ for:

   $$y' = y + e$$

   ^{x}, y(0) = 0

   $$Compute approximations at$$

   x = 0.25, 0.5, 0.75, 1.

3. For the IVP $y' = -2y$, $y(0) = 1$:

   (a) Find the exact solution

   (b) Apply Euler with $h = 0.5$ to find $y(1)$

   (c) Compare with the exact value

4. Determine the maximum stable step size for Euler's method applied to:

   $$y' = -10y$$

5. Use Euler's method with $h = 0.2$ for the system:

   $$x' = y, \quad y' = -x, \quad x(0) = 1, \quad y(0) = 0$$

   Find $(x(0.4), y(0.4))$.

6. Backward Euler for $y' = -100y$, $y(0) = 1$. Show that with $h = 0.1$:

   $$y_1 = \frac{y_0}{1 + 100h}$$

Part B: Runge-Kutta Methods (6 problems)

7. Apply the improved Euler method (Heun) with $h = 0.5$ to:

$$y' = xy, \quad y(0) = 1$$

Find $y(1)$.

8. Use the midpoint method (RK2) with $h = 0.25$ for:

$$y' = \sin(x) + y, \quad y(0) = 0$$

Compute $y(0.5)$.

9. Apply RK4 with $h = 0.5$ to solve:

$$y' = x^2 + y^2, \quad y(0) = 0$$

Find $y(0.5)$ (show all $k_i values$).

10. For $y' = -y + x + 1$, $y(0) = 1$:
    (a) Apply one step of RK4 with $h = 1$
    (b) Find the exact solution and compare

11. Use RK4 with $h = 0.2$ for the system:

$$\begin{cases} x' = x - y \\ y' = x + y \end{cases}, \quad x(0) = 1, \quad y(0) = 0$$

Find $(x(0.2), y(0.2))$.

12. Compare Euler and RK4 for $y' = y$, $y(0) = 1$ with $h = 0.5$. Find $y(0.5)$ using both methods and the exact solution.

Part C: Error Analysis (5 problems)

13. For $y' = y$, $y(0) = 1$ on $[0, 1]$:
    (a) If we want global error $< 10^{-4}$ using Euler, estimate required h
    (b) Repeat for RK4
    (c) How many steps does each method need?

14. The local truncation error for Euler is $\frac{h^2}{2}y''(\xi)$. For $y' = x^2$, $y(0) = 0$:
    (a) Find the exact local error at $x = h$
    (b) Verify it's $O(h^2)$

15. Given that RK4 has local error $O(h^5)$, if halving the step size reduces the error by factor F, what is F?

16. For $y' = -1000y + 1000$, $y(0) = 2$:

    (a) Find the exact solution

    (b) Is this equation stiff? Why?

    (c) What happens if you use Euler with $h = 0.01$?

17. Richardson extrapolation: If $y_h istheEulerapproximationwithstephandy_{h/2}$ with step $h/2$, show that:

$$y_{improved} = 2y_{h/2} - y$$

h

$$haserror$$

O(h^2).

Part D: Stability Analysis (5 problems)

18. For the test equation $y' = \lambda y$ with $\lambda = -5$:

    (a) Find the stability condition for Euler

    (b) Find the stability condition for RK4

    (c) Which method allows larger steps?

19. Show that backward Euler is unconditionally stable for $y' = \lambda y$ with $\mathrm{Re}(\lambda) < 0$.

20. For the system:

$$\mathbf{y}' = \begin{pmatrix} -1 & 10 \\ 0 & -100 \end{pmatrix} \mathbf{y}$$

    (a) Find the eigenvalues

    (b) What step size does Euler need for stability?

    (c) Is this system stiff?

21. The stability function for RK4 is:

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}$$

    Find $|R(-2)|$ and determine if $h\lambda = -2$ is stable.

22. Compare stability regions: sketch the stability boundary in the complex $h\lambda$ plane for:

    (a) Euler's method

    (b) Backward Euler

Part E: Implementation Considerations (3 problems)

23. Write pseudocode for adaptive step size control using error estimation.

24. For solving to tolerance $\epsilon = 10^{-6}$ on $[0, 10]$:

    (a) Estimate steps needed for Euler

    (b) Estimate steps needed for RK4

    (c) Which is more efficient?

25. Higher-order ODEs: Convert $y''+2y'+y = e^x$,y(0) = 1,y'(0) = $0 to a system and show first Euler step with$h = 0.1.

Part F: Exam-Style Problems (5 problems)

23. [**10 points**] Consider $y' = x + y$, $y(0) = 1$.

    3 pts Apply two steps of Euler's method with $h = 0.5$

    4 pts Apply one step of RK4 with $h = 1$

    3 pts The exact solution is $y = 2e^x$ - x - 1.$Compare errors.$

24. [**8 points**] For the pendulum equation (small angle):

$$\theta'' + \theta = 0, \quad \theta(0) = 0.1, \quad \theta'(0) = 0$$

    2 pts Convert to a first-order system

    4 pts Apply one step of RK4 with $h = 0.1$

    2 pts Is the total energy conserved numerically?

25. [**9 points**] Stability analysis for $y' = -50y$:

    2 pts Find maximum stable h for Euler

    2 pts Find maximum stable h for RK4 (given stability limit $|h\lambda| < 2.78$)

    2 pts If you need to solve on $[0, 10]$, how many steps for each?

    3 pts Which method is more efficient and why?

26. [**10 points**] Method comparison:

    3 pts Explain why RK4 is more accurate than Euler

    3 pts When would you prefer implicit over explicit methods?

    2 pts What is the trade-off in choosing step size?

    2 pts How do adaptive methods work?

27. [**12 points**] *Prof. Ditkowski Special* The chemical reaction $A \to B \to C$ has rate equations:

$$\begin{cases} A' = -100A \\ B' = 100A - B \\ C' = B \end{cases}$$

with $A(0) = 1$, $B(0) = C(0) = 0$.

3 pts Is this system stiff? Explain.

3 pts Apply one Euler step with $h = 0.001$

3 pts What happens with $h = 0.1$?

3 pts Suggest an appropriate numerical method

Solutions and Hints

Selected Solutions:

Problem 1: - $y_1 = 1 + 0.1(0 - 1) = 0.9$ - $y_2 = 0.9 + 0.1(0.1 - 0.9) = 0.82$ - $y_3 = 0.82 + 0.1(0.2 - 0.82) = 0.758$

Problem 4: For stability: $|1 - 10h| < 1$, so $h < 0.2$

Problem 9: RK4 for $y' = x^2 + y^2$: - $k_1 = 0$ - $k_2 = 0.5(0.25)^2 = 0.03125$ - $k_3 = 0.5[(0.25)^2 + (0.0156)^2] \approx 0.0312$ - $k_4 = 0.5[(0.5)^2 + (0.0312)^2] \approx 0.125$ - $y_1 \approx 0.031$

Problem 13: For Euler: $h \sim \sqrt{10^{-4}} = 0.01$, need 100 steps For RK4: $h \sim (10^{-4})^{1/4} \approx 0.1$, need 10 steps

Problem 18: Eigenvalues: $-1, -100$. Need $h < 2/100 = 0.02$ for stability. Yes, stiff!

Key Insights: - RK4 typically 10-100$\times more efficient than Euler - Stiffness determined by eigenvalue$
$Implicit methods trade computation per step for stability - Energy conservation tests numerical accuracy$