Practice Problems - Lesson 38: Drawing Phase Portraits

ODE 1 Course

Part A: Basic Portrait Construction (Problems 1-6)

1. Draw complete phase portraits for:
   (a) $\dot{x} = x, \dot{y} = -y$ (saddle)
   (b) $\dot{x} = -2x, \dot{y} = -3y$ (stable node)
   (c) $\dot{x} = y, \dot{y} = -x$ (center)
   (d) $\dot{x} = x + y, \dot{y} = -x + y$ (spiral)

2. For the system $\dot{x} = 3x - 2y$, $\dot{y} = 2x - y$:
   (a) Find eigenvalues and eigenvectors
   (b) Draw eigenvector lines with correct arrows
   (c) Add at least 4 trajectories
   (d) Verify no trajectories cross

3. Given eigenvalues $\lambda_1 = 2$, $\lambda_2 = -3$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:
   (a) Classify the equilibrium
   (b) Draw the phase portrait
   (c) Find the matrix A
   (d) Verify your portrait using nullclines

4. For $\dot{x} = x - y$, $\dot{y} = 2x - y$:
   (a) Find and draw both nullclines
   (b) Mark flow direction in each region
   (c) Find eigenvalues
   (d) Complete the phase portrait

5. Draw a phase portrait for:
   $$A = \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix}$$
   Show all five steps explicitly.

6. Create phase portraits with:
   (a) Clockwise spiral, stable
   (b) Counterclockwise center
   (c) Improper node, unstable
   (d) Saddle with horizontal stable manifold

Part B: Nullcline Method (Problems 7-11)

7. For $\dot{x} = y - x^2$, $\dot{y} = x - 2$ (nonlinear):
   (a) Find nullclines
   (b) Locate equilibria
   (c) Determine flow in each region
   (d) Linearize at each equilibrium
   (e) Sketch the global portrait
8. Use nullclines to draw $\dot{x} = 2x - y + 1$, $\dot{y} = x - 2y$:
   (a) Find where $\dot{x} = 0$ and $\dot{y} = 0$
   (b) Find equilibrium (not at origin!)
   (c) Shift coordinates to equilibrium
   (d) Draw the portrait
9. For the competing species model:
   $$\dot{x} = x(1 - x - 0.5y), \quad \dot{y} = y(1 - 0.5x - y)$$
   (a) Find all four equilibria
   (b) Draw nullclines (they're not straight!)
   (c) Classify each equilibrium
   (d) Sketch the first quadrant portrait
10. Given nullclines are the lines $x + y = 0$ and $x - 2y = 0$:
   (a) Find possible systems with these nullclines
   (b) Draw flow directions
   (c) Complete a phase portrait
11. For $\dot{x} = \sin(y)$, $\dot{y} = x$:
   (a) Sketch nullclines
   (b) Find equilibria in $[-\pi, \pi] \times \mathbb{R}$
   (c) Linearize at $(0, 0)$
   (d) Draw local portrait near origin

Part C: Direction Fields (Problems 12-16)

12. Draw direction field arrows at the 8 points $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm 1, \pm 1)$ for:
    (a) $\dot{x} = y$, $\dot{y} = -x - y$
    (b) $\dot{x} = x + 2y$, $\dot{y} = 3x + 2y$
    (c) $\dot{x} = -y + x(x^2 + y^2)$, $\dot{y} = x + y(x^2 + y^2)$

13. For the system $\dot{x} = ax + y$, $\dot{y} = x + ay$ with $a = 0.5$:
    (a) Compute direction vectors at 6 points
    (b) Identify spiral rotation direction
    (c) Estimate spiral pitch
    (d) Draw complete portrait

14. Given the direction field: - At $(1, 0)$: arrow points to $(1, 1)$ - At $(0, 1)$: arrow points to $(-1, 1)$ - At $(-1, 0)$: arrow points to $(-1, -1)$ - At $(0, -1)$: arrow points to $(1, -1)$
    (a) What type of equilibrium is at origin?
    (b) Find a compatible linear system
    (c) Complete the portrait

15. Use isoclines to draw $\dot{x} = y$, $\dot{y} = -\sin(x)$:
    (a) Find curves where $dy/dx = 0, 1, -1, \infty$
    (b) Mark trajectory slopes on each isocline
    (c) Draw the phase portrait

16. For a system with spiral trajectories:
    (a) How many direction arrows minimum to determine rotation?
    (b) Where should you place test points?
    (c) Draw example with 3 arrows showing clockwise spiral

Part D: Special Cases (Problems 17-21)

17. Draw phase portraits for repeated eigenvalue cases:
    (a) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ (improper node)
    (b) $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (star node)
    (c) $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ (stable improper)

18. For the degenerate system $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$:

    (a) Show $\det(A) = 0$
    (b) Find the line of equilibria
    (c) Find the non-zero eigenvalue
    (d) Draw parallel trajectories

19. Draw accurate ellipses for the center:

$$\dot{x} = -2y, \quad \dot{y} = 0.5x$$

    (a) Find the period
    (b) Determine ellipse aspect ratio
    (c) Draw 3 nested closed orbits

20. For a stable spiral with eigenvalues $-1 \pm 3i$:

    (a) How many rotations before amplitude halves?
    (b) Draw showing at least 2 complete rotations
    (c) Mark points at $t = 0, \pi/3, 2\pi/3, \pi$

21. Draw the portrait for the shear flow:

$$\dot{x} = y, \quad \dot{y} = 0$$

    (a) Find eigenvalues
    (b) Identify all equilibria
    (c) Draw the unusual portrait
    (d) What physical system does this represent?

Part E: Portrait Verification (Problems 22-26)

22. Check these portraits for errors (each has one mistake):

    (a) Saddle with both manifolds attracting
    (b) Node with trajectories crossing
    (c) Spiral that changes rotation direction
    (d) Center with touching ellipses

23. Verify portrait consistency:

    (a) Do trajectories respect uniqueness?

(b) Are arrow directions consistent?

(c) Do eigenvalue signs match portrait?

(d) Is tangency behavior correct for nodes?

24. For the portrait you drew in Problem 2:

    (a) Check using 3 additional test points

    (b) Verify nullcline crossings

    (c) Confirm no trajectory intersections

    (d) Test symmetry if applicable

25. Quality assessment checklist:

    (a) Are at least 4 trajectories shown?

    (b) Are arrows indicating flow present?

    (c) Are axes and origin labeled?

    (d) Is classification stated?

    (e) Are special features marked?

26. Compare hand-drawn vs computer-generated portraits:

    (a) What details does computer miss?

    (b) What does hand-drawing emphasize?

    (c) How to combine both approaches?

Part F: Exam-Style Problems (Problems 27-30)

27. [**Complete Portrait - Prof. Ditkowski Style**] For the system:

$$\dot{x} = 4x - 5y, \quad \dot{y} = 2x - 2y$$

    (a) Find all equilibria

    (b) Calculate eigenvalues and eigenvectors

    (c) Classify the equilibrium type

    (d) Find and draw nullclines

    (e) Draw eigenvector directions with arrows

    (f) Test flow at points $(1, 0)$ and $(0, 1)$

    (g) Complete phase portrait with 6 trajectories

    (h) Verify using trace-determinant

    (i) State stability

28. [**Nonlinear with Multiple Equilibria**] Consider:

$$\dot{x} = y, \quad \dot{y} = x - x^3$$

(a) Find all equilibria

(b) Linearize at each equilibrium

(c) Classify each equilibrium

(d) Find the energy function $E(x, y)$

(e) Use energy contours to draw portrait

(f) Identify separatrices

(g) Describe physical interpretation

29. [**Parameter-Dependent Portrait**] For the system depending on parameter μ:

$$\dot{x} = \mu x + y, \quad \dot{y} = -x + \mu y$$

(a) Find eigenvalues in terms of μ

(b) Determine portrait type for: - $\mu = -1$ - $\mu = 0$ - $\mu = 0.5$

(c) Draw all three portraits

(d) What happens as μ passes through 0?

(e) Is this a bifurcation? What type?

30. [**Advanced Drawing Challenge**] Without computing eigenvalues explicitly, draw the portrait for:

$$A = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}$$

(a) Use trace-determinant to classify

(b) Use nullclines to structure portrait

(c) Test 6 direction field points

(d) Draw complete portrait

(e) Then verify by computing eigenvalues

(f) Compare your sketch to exact solution

Pro Tips for Drawing:

- Start with eigenvectors (the skeleton)
- Add nullclines (the grid)
- Sample direction field (the flow)
- Draw smooth curves following all three
- Check: no crossings, correct arrows, proper tangency

- Time limit: 5 minutes per portrait on exam!