Practice Problems - Lesson 36: Phase Space and Trajectories

ODE 1 Course

Part A: Phase Space Concepts (Problems 1-5)

1. Consider the system $\dot{x} = y$, $\dot{y} = -x$.
   (a) What is the dimension of the phase space?
   (b) Find all equilibrium points.
   (c) Show that $x^2 + y^2$ is constant along trajectories.
   (d) What does this tell you about the shape of trajectories?

2. For the equation $\ddot{x} + 2\dot{x} + 5x = 0$:
   (a) Convert to a first-order system.
   (b) Identify the phase space variables.
   (c) Find the equilibrium point(s) in phase space.

3. True or False (explain your answer):
   (a) Two different trajectories can pass through the same point.
   (b) A trajectory can cross itself.
   (c) Every bounded trajectory must be a closed orbit.
   (d) In 1D phase space, closed orbits are possible.

4. Given the direction field at point $(2, 3)$ is $(1, -2)$:
   (a) What is the instantaneous direction of motion?
   (b) What is the speed at this point?
   (c) Find the slope of the trajectory at this point.

5. For the system $\dot{x} = x(1 - x - y)$, $\dot{y} = y(1 - 2x - y)$:
   (a) Find all equilibrium points.
   (b) Verify your answers by substitution.

Part B: Nullclines and Direction Fields (Problems 6-10)

6. For the system $\dot{x} = x - y$, $\dot{y} = x + y$:
   (a) Find the x-nullcline.
   (b) Find the y-nullcline.
   (c) Where do the nullclines intersect?
   (d) What is special about this intersection point?

7. Consider $\dot{x} = y^2 - x$, $\dot{y} = x - 2y$:
   (a) Find both nullclines.
   (b) Sketch the nullclines on the same axes.
   (c) Identify all equilibrium points.

8. For the system $\dot{x} = \sin(y)$, $\dot{y} = \cos(x)$:
   (a) Find the equilibrium at the origin.
   (b) Find another equilibrium point.
   (c) Are there infinitely many equilibria? Why?

9. Given nullclines $x = 0$ and $y = x^2$:
   (a) What system could have these nullclines?
   (b) Find the equilibrium points.
   (c) Determine the flow direction in the region $x > 0, y > x^2$.

10. For $\dot{x} = y - x^2$, $\dot{y} = -x$:
    (a) Show that the origin is an equilibrium.
    (b) Find any other equilibria.
    (c) What are the nullclines?
    (d) Sketch the direction field at points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

Part C: Trajectories and Invariant Sets (Problems 11-16)

11. Show that the unit circle $x^2 + y^2 = 1$ is an invariant set for:

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2)$$

12. For the system $\dot{x} = y$, $\dot{y} = -\sin(x)$:

(a) Find equilibria in $-\pi \leq x \leq \pi$.

(b) Show that $E = \frac{1}{2}y^2 - \cos(x)$ is constant along trajectories.

(c) Use this to sketch different types of trajectories.

13. Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$:

    (a) Find the eigenvalues of A.

    (b) What type of trajectories do you expect?

    (c) Find a conserved quantity.

14. Prove that if a trajectory is bounded and the vector field is continuous, then:

    (a) The trajectory exists for all time.

    (b) The omega-limit set is non-empty.

15. For the system $\dot{r} = r(1-r)$, $\dot{\theta} = 1$ in polar coordinates:

    (a) Find all circular trajectories.

    (b) Describe the behavior as $t \to \infty$.

    (c) Is the unit circle attracting or repelling?

16. Given a gradient system $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$:

    (a) Show that V decreases along trajectories.

    (b) What does this imply about closed orbits?

    (c) Find equilibria in terms of V.

Part D: Special Trajectories (Problems 17-21)

17. For a 2D system, explain why:

    (a) A homoclinic orbit cannot exist in linear systems.

    (b) Closed orbits require at least one equilibrium inside.

    (c) Trajectories cannot spiral into a closed orbit from outside.

18. Consider the Van der Pol system $\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$:

    (a) Convert to first-order form.

    (b) Find the unique equilibrium.

    (c) For $\mu = 0$, describe all trajectories.

19. For the Hamiltonian system with $H(x, y) = \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$:

(a) Write the corresponding ODE system.
(b) Show that H is conserved.
(c) Find all equilibria.

20. Analyze the system $\dot{x} = y + x(x^2 + y^2)$, $\dot{y} = -x + y(x^2 + y^2)$:
    (a) Show the origin is an equilibrium.
    (b) Convert to polar coordinates.
    (c) Describe the trajectory behavior near the origin.

21. For the Lotka-Volterra system $\dot{x} = x(a - by)$, $\dot{y} = y(-c + dx)$:
    (a) Find all equilibria (assume $a, b, c, d > 0$).
    (b) Find a conserved quantity.
    (c) What does this imply about trajectory types?

Part E: Advanced Concepts (Problems 22-26)

22. Prove the Poincaré-Bendixson theorem consequence: If a trajectory in 2D is bounded and contains no equilibria, it must approach a closed orbit.

23. For the system on the cylinder ($x \mod 2\pi, y \in \mathbb{R}$):

$$\dot{x} = 1, \quad \dot{y} = \sin(x)$$

    (a) Are there any equilibria?
    (b) Describe all possible trajectories.
    (c) Are there any closed orbits?

24. Consider a 3D system with a 2D invariant plane:
    (a) How can you identify such a plane?
    (b) What does this mean for trajectories starting in the plane?
    (c) Give an example of such a system.

25. For the delayed logistic equation $\dot{x} = rx(1 - x(t - \tau))$:
    (a) What is the phase space?
    (b) Why is it infinite-dimensional?
    (c) Find the equilibria.

26. Analyze the reversible system (invariant under $t \to -t, y \to -y$):

$$\dot{x} = y, \quad \dot{y} = x - x^3$$

    (a) Find all equilibria.
    (b) Use reversibility to deduce trajectory properties.
    (c) Find the homoclinic orbits.

Part F: Exam-Style Problems (Problems 27-30)

27. [**Prof. Ditkowski Style**] Consider the system:

$$\dot{x} = 2x - y - x^2, \quad \dot{y} = x - 2y + xy$$

   (a) Find ALL equilibrium points.
   (b) Find the nullclines.
   (c) Determine the direction field at $(1, 1)$.
   (d) Sketch the phase portrait near each equilibrium.
   (e) Classify the stability of each equilibrium (intuitive).

28. [**Comprehensive**] For the pendulum equation $\ddot{\theta} + \sin(\theta) = 0$:

   (a) Convert to phase space form.
   (b) Find equilibria in $[0, 2\pi]$.
   (c) Derive the energy function.
   (d) Sketch the complete phase portrait.
   (e) Identify separatrices, closed orbits, and equilibrium types.
   (f) Explain the physical meaning of each trajectory type.

29. [**Theoretical**] Prove or disprove:

   (a) Every 2D system has at least one equilibrium.
   (b) If all trajectories are bounded, there exists a closed orbit.
   (c) A trajectory can have empty omega-limit set.
   (d) Two closed orbits cannot intersect.

30. [**Application**] A chemical reaction follows:

$$\dot{A} = -kAB + \ell, \quad \dot{B} = -kAB + m$$

   where $k, \ell, m > 0$ are constants.

   (a) Find equilibria.
   (b) Show that $A + B$ changes linearly with time.
   (c) Use this to reduce to a 1D problem.
   (d) Describe the long-term behavior.
   (e) Sketch the phase portrait.

Key Solution Strategies:

- Always find equilibria first
- Look for conserved quantities (energy, momentum, etc.)
- Use nullclines to organize phase space
- Check trajectory uniqueness carefully
- Connect mathematical results to physical meaning
- For Prof. Ditkowski: Show ALL algebraic steps