ODE Lesson 28: Wronskian and Linear Independence

ODE 1 - Prof. Adi Ditkowski

1 The Wronskian Matrix and Determinant

Definition 1 (Wronskian Matrix)**.** *For n vector solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ of a system, the* ***Wronskian matrix*** *is:*

$$W(t) = [\mathbf{x}_1(t) \mid \mathbf{x}_2(t) \mid \cdots \mid \mathbf{x}_n(t)]$$

Definition 2 (Wronskian Determinant)**.** *The* ***Wronskian determinant*** *(or simply Wronskian) is:*

$$W(t) = \det(W(t))$$

For solutions of $\mathbf{x}' = A(t)\mathbf{x}$ with continuous $A(t)$:

- Either $W(t) = 0$ for all t (linearly dependent)
- Or $W(t) \neq 0$ for all t (linearly independent)

This is the **all-or-nothing property**.

2 Abel's Identity

Theorem 1 (Abel's Identity for Systems)**.** *If $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are solutions to $\mathbf{x}' = A(t)\mathbf{x}$, then:*

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} tr(A(s))\, ds\right)$$

where $tr(A)$ is the trace (sum of diagonal elements).

Abel's identity shows that:

- $W(t) = 0$ for some $t \Rightarrow W(t) = 0$ for all t
- $W(t) \neq 0$ for some $t \Rightarrow W(t) \neq 0$ for all t
- The growth/decay rate of $W(t)$ depends only on $\mathrm{tr}(A)$

3 Wronskian for Scalar Equations

Definition 3 (Scalar Wronskian)**.** *For n solutions $y_1(t), \ldots, y_n(t)$ of an nth-order linear ODE:*

$$W[y_1, \ldots, y_n](t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Example 1 (Second-Order Wronskian)**.** *For two solutions y_1, y_2 of a second-order equation:*

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

4 Linear Independence Criteria

Theorem 2 (Independence Characterization)**.** *Solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly independent on interval I if and only if $W(t) \neq 0$ for all $t \in I$.*

Proposition 1 (Fundamental Matrix Connection)**.** *A matrix $\Phi(t)$ is a fundamental matrix if and only if:*

1. $\Phi'(t) = A(t)\Phi(t)$
2. $\det(\Phi(t)) \neq 0$ *for all* t

5 Computational Techniques

For a 2×2 system with solutions $\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$:

$$W(t) = x_{11}x_{22} - x_{12}x_{21}$$

Example 2 (Using Abel's Identity)**.** *For the system $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x}$:*

1. $tr(A) = 2 + 3 = 5$
2. $W(t) = W(0) \cdot e^{5t}$
3. *If* $W(0) = 1$, *then* $W(t) = e^{5t}$

6 Special Cases and Properties

Lemma 1 (Constant Coefficient Systems)**.** *For x' = $AxwithconstantA$: W(t) = W(0) $\cdot e^{tr}$(A) $\cdot t$*

Common errors:

- Assuming $W(t) = 0$ at one point means dependent only at that point
- Wrong row ordering in scalar Wronskian (derivatives increase downward)
- Forgetting that $W(t)$ can be negative (it's the absolute value that matters for independence)
- Not using Abel's identity when it simplifies calculations

7 Wronskian Applications

Theorem 3 (Finding Differential Equations)**.** *If $y_1, \ldots, y_n$ are n linearly independent solutions, then y is also a solution if and only if:*

$$W[y, y_1, \ldots, y_n](t) = 0$$

Example 3 (Constructing ODEs)**.** *Given solutions e^t and e^{-2t}, find the differential equation:*
$\begin{vmatrix} y & e^t & e^{-2t} \\ y' & e^t & -2e^{-2t} \\ y'' & e^t & 4e^{-2t} \end{vmatrix}$ *= 0*

$$Expanding:$$

y" + y' - 2y = 0

Prof. Ditkowski's favorite Wronskian problems:

- Compute $W(t)$ for given solutions
- Use Abel's identity to find $W(t)$ without direct computation
- Test linear independence
- Find differential equation from solutions
- Verify all-or-nothing property

8 Connection to Linear Algebra

The Wronskian connects ODE theory to linear algebra:

- Linear independence of solutions $\Leftrightarrow$ Non-zero Wronskian
- Fundamental matrix $\Leftrightarrow$ Wronskian never zero
- Dimension of solution space = number of independent solutions

- Wronskian = Volume of parallelepiped in solution space

9 Quick Wronskian Tests

Proposition 2 (Obvious Dependencies). *Without calculation:*

- *If* $y_2 = c \cdot y_1$ *(proportional), then* $W = 0$
- *If solutions have different exponential growth rates,* $W \neq 0 For e^{\lambda_1} \ t, \ldots, e^{\lambda_n} \ t with distinct \lambda_i$*: always independent*