Integrating Factor Technique: Deep Mathematical Analysis

ODE 1 - Prof. Adi Ditkowski

Lesson 15

1 The Mathematical Foundation

The integrating factor technique is more than a computational tool - it represents a fundamental principle in the theory of linear differential equations. This lesson explores the deep mathematical structure underlying this method.

1.1 The Fundamental Question

Given a linear first-order ODE:

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1}$$

We seek a function $\mu(t)$ such that multiplication transforms the equation into:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t) \tag{2}$$

Theorem 1 (Existence and Uniqueness of Integrating Factor)**.** *For any continuous function $p(t)$ on an interval I, there exists a unique (up to scalar multiplication) integrating factor $\mu(t) > 0$ given by:*

$$\mu(t) = \exp\left(\int_{t_0}^{t} p(s)ds\right) \tag{3}$$

where $t_0 \in I$ is arbitrary.

Proof. Requiring $\mu(t)y' + \mu(t)p(t)y = \frac{d}{dt}[\mu(t)y] = \mu'(t)y + \mu(t)y'$ yields:

$$\mu'(t)y + \mu(t)y' = \mu(t)y' + \mu(t)p(t)y \tag{4}$$

$$\mu'(t) = \mu(t)p(t) \tag{5}$$

$$\frac{d\mu}{dt} = p(t)\mu \tag{6}$$

This is a separable equation with solution:

$$\mu(t) = C\exp\left(\int p(t)dt\right) \tag{7}$$

Since any non-zero constant C yields a valid integrating factor, we choose $C = 1$ for simplicity. The uniqueness (up to scalar multiplication) follows from the uniqueness of solutions to the initial value problem $\mu' = p(t)\mu$, $\mu(t_0) = \mu_0$. □

2 Connection to Exact Differential Equations

The integrating factor transforms a non-exact equation into an exact one. This reveals the deep connection between linear equations and exact equations.

Proposition 1 (Exactness After Multiplication). *The equation $\mu(t)[y' + p(t)y - g(t)] = 0$ can be written as:*

$$M(t, y)dt + N(t, y)dy = 0 \tag{8}$$

where $M = \mu(t)[p(t)y - g(t)]$ and $N = \mu(t)$. This equation is exact, meaning:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{9}$$

Proof. Computing the partial derivatives:

$$\frac{\partial M}{\partial y} = \mu(t)p(t) \tag{10}$$

$$\frac{\partial N}{\partial t} = \mu'(t) = \mu(t)p(t) \tag{11}$$

The equality follows from the defining property of $\mu(t)$. □

3 Alternative Forms and Generalizations

3.1 Non-Standard Form Integrating Factors

Consider the general first-order linear equation:

$$a_1(t)y' + a_0(t)y = g(t) \tag{12}$$

Method 1 (Direct Integrating Factor). *Instead of converting to standard form, find $\mu(t)$ such that:*

$$\frac{d}{dt}[\mu(t)a_1(t)y] = \mu(t)g(t) \tag{13}$$

This requires:

$$\mu(t)a_1(t)y' + [\mu'(t)a_1(t) + \mu(t)a_1'(t)]y = \mu(t)a_1(t)y' + \mu(t)a_0(t)y \tag{14}$$

Leading to:

$$\mu'(t) = \mu(t)\left[\frac{a_0(t) - a_1'(t)}{a_1(t)}\right] \tag{15}$$

3.2 Integrating Factors Depending on y

While not applicable to linear equations, for completeness we note that some equations admit integrating factors $\mu(y)$:

Theorem 2 (Integrating Factor $\mu(y)$). *For the equation $M(t,y)dt + N(t,y)dy = 0$, an integrating factor $\mu(y)$ exists if and only if:*

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) = h(y) \tag{16}$$

is a function of y alone. Then $\mu(y) = \exp\left(\int h(y)dy\right)$.

4 The Operator Perspective

Definition 1 (Linear Differential Operator). *Define the operator $L : C^1(I) \to C(I)$ by:*

$$L[y] = y' + p(t)y \tag{17}$$

The integrating factor $\mu(t)$ transforms L into the operator μL, which can be written as:

$$\mu L[y] = \frac{d}{dt}[\mu y] \tag{18}$$

This is a composition of multiplication by μ followed by differentiation.

4.1 The Adjoint Connection

Theorem 3 (Adjoint Operator and Integrating Factor). *The adjoint operator L^* defined by:*

$$L^*[v] = -v' + p(t)v \tag{19}$$

has the property that $\mu(t)$ satisfies $L^*[\mu] = 0$, making it a solution to the homogeneous adjoint equation.

5 Discontinuous Coefficients

When $p(t)$ has discontinuities, the integrating factor may have different forms on different intervals.

Example 1 (Jump Discontinuity). *Consider:*

$$y' + p(t)y = 1, \quad p(t) = \begin{cases} 1 & t < 0 \\ 2 & t \geq 0 \end{cases} \tag{20}$$

The integrating factor is:

$$\mu(t) = \begin{cases} e^t & t < 0 \\ e^{2t} & t \geq 0 \end{cases} \tag{21}$$

Note that $\mu(t)$ has a jump discontinuity at $t = 0$, with $\mu(0^-) = 1$ and $\mu(0^+) = 1$.

6 Special Integration Techniques

6.1 Pattern Recognition

Key patterns for $\int p(t)dt$:

$$p(t) = \frac{f'(t)}{f(t)} \Rightarrow \int p(t)dt = \ln|f(t)| \tag{22}$$

$$p(t) = \tan(t) \Rightarrow \int p(t)dt = -\ln|\cos(t)| \tag{23}$$

$$p(t) = \cot(t) \Rightarrow \int p(t)dt = \ln|\sin(t)| \tag{24}$$

$$p(t) = \frac{n}{t} \Rightarrow \int p(t)dt = n\ln|t| \tag{25}$$

6.2 Reduction Formulas

For repeated integration by parts situations:

Lemma 1 (Reduction Formula for $\int t^n e^{at} dt$)**.**

$$\int t^n e^{at} dt = \frac{t^n e^{at}}{a} - \frac{n}{a}\int t^{n-1} e^{at} dt \tag{26}$$

7 The Fundamental Solution Connection

Theorem 4 (Reciprocal Relationship)**.** *If $y_h(t)$ is a non-zero solution to the homogeneous equation $y' + p(t)y = 0$, then:*

$$\mu(t) = \frac{1}{y_h(t)} \tag{27}$$

is an integrating factor for the non-homogeneous equation.

Proof. Since $y_h' + p(t)y_h = 0$, we have $y_h'/y_h = -p(t)$. Therefore:

$$y_h = Ce^{-\int p(t)dt} \Rightarrow \frac{1}{y_h} = \frac{1}{C}e^{\int p(t)dt} \tag{28}$$

which is indeed an integrating factor (the constant $1/C$ can be absorbed). □

8 Numerical Stability Considerations

Prof. Ditkowski may ask about the numerical implications of the integrating factor method.

Proposition 2 (Growth of Integrating Factor)**.** *If $p(t) > 0$ on $[a, b]$, then $\mu(t)$ grows exponentially. If $p(t) < 0$, then $\mu(t)$ decays exponentially. This affects numerical stability:*

- *Growing μ: May cause overflow but preserves relative accuracy*
- *Decaying μ: May cause underflow and loss of significant digits*

9 Historical Development

The integrating factor method was developed by Leonhard Euler in the 18th century. The key insight was recognizing that certain non-exact equations could be made exact through multiplication by an appropriate function.

Euler's contribution: Transform difficult problems into solvable ones through clever multiplication - a recurring theme in mathematics.

10 Extension to Systems

For the system $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{g}(t)$, the integrating factor becomes the matrix exponential or fundamental matrix:

$$\Phi(t) = \exp\left(\int_0^t A(s)ds\right) \tag{29}$$

when $A(t)$ commutes with its integral. Otherwise, we need the Peano-Baker series.

11 Summary: The Complete Picture

The integrating factor method connects to multiple areas of mathematics:

- Creates exact differential equations
- Relates to operator theory and adjoint operators
- Connected to fundamental solutions via reciprocal relationship
- Has numerical stability implications
- Generalizes to matrix systems