Lesson 18: Practice Problems - Bernoulli Equations

ODE 1 - Prof. Adi Ditkowski

Part A: Recognition and Classification (5 problems)

1. Identify which equations are Bernoulli and determine $p(x)$, $q(x)$, and n:
   (a) $y' + xy = xy^3$
   (b) $xy' - 2y = x^3\sqrt{y}$
   (c) $y' + y^2 = x$
   (d) $\frac{dy}{dx} = y(1-y)$
   (e) $ty' + 2ty = t^3y^{-1}$

2. Show that the equation $y' = ay + by^2 + c$ can be transformed into Bernoulli form only if $c = 0$.

3. Prove that if y_1 is a solution to the Bernoulli equation with $n = 2$, then $y_2 = y_1/(1 + Cy_1)$ is also a solution for any constant C.

4. Transform each equation to standard Bernoulli form:
   (a) $x^2y' = xy + y^3$
   (b) $\frac{dy}{dx} = \frac{y}{x} + x^2y^{-2}$

5. Determine the appropriate substitution v for each value of n:
   (a) $n = 3$
   (b) $n = 1/2$
   (c) $n = -1$
   (d) $n = 5/3$

Part B: Basic Substitution Problems (6 problems)

6. Solve: $y' - y = -y^2$

7. Find the general solution: $\frac{dy}{dx} + \frac{y}{x} = x^2y^2$

8. Solve the IVP: $y' + 2y = y^3e^{-2x}$, $y(0) = 1$

9. Solve: $xy' + y = x^2y^4$
10. Find all solutions: $\frac{dy}{dt} = 2ty - y^3$
11. Solve: $y' - \frac{3y}{x} = \frac{x^2}{y^2}$ (Hint: $n = -2$)

Part C: Complete Solution Process (5 problems)

12. Solve and verify: $(1 + x^2)y' + 2xy = (1 + x^2)^2y^3$
13. Find the solution satisfying $y(1) = 2$:
$$xy' - y = x^3y^{1/2}$$
14. Solve the logistic equation with harvesting:
$$\frac{dP}{dt} = P(2 - P) - h$$
where h is a constant harvesting rate.
15. Solve: $2xy' + y = xy^{-1}$
16. Find the general solution:
$$\cos x \cdot y' + y \sin x = y^2 \cos^3 x$$

Part D: Tricky Cases and Variations (5 problems)

17. Solve the generalized logistic equation:
$$\frac{dy}{dt} = ry^\alpha \left(1 - \left(\frac{y}{K}\right)^\beta\right)$$
where $\alpha = 1$ and $\beta = 1$.
18. Transform and solve:
$$y' = \frac{1}{x}(y - x^2y^3)$$
19. Find all solutions including singular ones:
$$y' + p(x)y = q(x)y^2$$
where $p(x) = 2/x$ and $q(x) = 1/x^2$.
20. Solve the Bernoulli equation with periodic coefficient:
$$y' + \sin x \cdot y = \cos x \cdot y^2$$
21. Consider the equation $y' = y^2 - 2xy^{3/2} + x^2y$.
    (a) Show this can be written in Bernoulli form
    (b) Find the appropriate substitution
    (c) Solve the equation

Part E: Mixed Recognition Challenge (4 problems)

22. Classify each equation and solve only the Bernoulli ones:
    (a) $y' + y/x = y^2 \ln x$
    (b) $y' + y = e^x$
    (c) $xy' = y + x^2 y^{-1}$
    (d) $y' + y \tan x = \sin x$

23. Show that the Riccati equation $y' = q_0(x) + q_1(x)y + q_2(x)y^2$ becomes Bernoulli if one particular solution y_p is known.

24. Prove that every autonomous Bernoulli equation $y' = ay + by^n$ can be solved by separation of variables.

25. Find conditions on $f(x)$ such that $y' + f(x)y = f(x)y^n$ has polynomial solutions.

Part F: Exam-Style Complete Problems (6 problems)

26. [**Prof. Ditkowski Style**] Consider the equation: $xy' - 2y = x^4 y^{1/2}$
    (a) Identify the type of equation and write in standard form
    (b) State the appropriate substitution
    (c) Transform to a linear equation
    (d) Solve the linear equation
    (e) Find the general solution in terms of y
    (f) Find the particular solution with $y(1) = 4$

27. [**Application Problem**] A population model with Allee effect is given by:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{A} - 1\right)$$

    where $0 < A < K$.
    (a) Show this is not in Bernoulli form
    (b) Find equilibrium points
    (c) Analyze stability of equilibria
    (d) Sketch solution curves

28. [**Comprehensive Problem**] For the equation $y' + \frac{n}{x}y = x^n y^{1-n}$ where n is a positive integer:
    (a) Verify this is Bernoulli

(b) Show that $y = x^n$ is a solution

(c) Find the general solution

(d) Discuss the behavior as $x \to 0^+$ and $x \to \infty$

29. [**Theory and Computation**] Consider $y' - \frac{y}{x} = x^\alpha y^\beta$

    (a) For what values of α and β is this Bernoulli?

    (b) Solve for $\alpha = 2$, $\beta = 2$

    (c) Find conditions on α and β for bounded solutions

30. [**Chemical Reaction Model**] The reaction rate equation:

$$\frac{dc}{dt} = -kc^2 + \frac{k'}{V}$$

where c is concentration, V is volume, and k, k' are rate constants.

    (a) Show this is Bernoulli with $n = 2$

    (b) Solve with $c(0) = c_0$

    (c) Find the steady-state concentration

    (d) Determine conditions for finite-time blow-up

31. [**Challenge Problem**] The generalized Bernoulli equation:

$$y' + p(x)y = q(x)y^n + r(x)y^m$$

    (a) Show that if $m = 2n - 1$, a single substitution reduces this to linear form

    (b) Solve the special case: $y' - y = y^2 - y^3$

    (c) Discuss the connection to Riccati equations

Solutions and Hints

Selected Solutions:

Problem 1(a): Bernoulli with $p(x) = x$, $q(x) = x$, $n = 3$.

Problem 6: $n = 2$, so $v = y^{-1}$. After substitution: $v' + v = -1$. Solution: $v = -1 + Ce^{-x}$, thus $y = \frac{1}{-1+Ce^{-x}}$.

Problem 7: With $n = 2$, use $v = y^{-1}$. The transformed equation is $v' - v/x = -x^2$. Integrating factor: $\mu = e^{\int -1/x\, dx} = 1/x$. Solution: $v = x(x^2/4 + C)$, so $y = \frac{1}{x(x^2/4+C)}$.

Problem 11: Use $v = y^{1/2}$. The transformed equation becomes: $v' + \frac{v}{2x} = \frac{x^2}{2}$. Solution involves $v = x^2 + Cx^{-1/2}$.

Problem 25: Standard form: $y' - \frac{2y}{x} = x^3 y^{1/2}$. Here $n = 1/2$, so $v = y^{1/2}$. Transformed: $v' - \frac{v}{x} = \frac{x^3}{2}$.

Key Formulas:

- Bernoulli: $y' + p(x)y = q(x)y^n$
- Substitution: $v = y^{1-n}$
- Linear form: $v' + (1-n)p(x)v = (1-n)q(x)$
- Special case $n = 2$: Logistic-type equations

Common Integration Results:

- $\int xe^x dx = (x-1)e^x + C$
- $\int x^n e^{ax} dx$ requires integration by parts
- $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan(x/a) + C$