ODE Lesson 2: Order, Linearity, and Autonomy - Pattern Recognition

ODE 1 - Prof. Adi Ditkowski

1 Systematic Recognition Approach

The Three Questions:

1. What's the order?
2. Is it linear?
3. What special form does it have?

2 Order Determination - Advanced

2.1 Hidden Derivatives

Watch for disguised derivatives!

- If $p = y'$, then $p' = y''$
- Parametric forms: $\frac{dp}{dx} = \frac{d^2y}{dx^2}$
- Substitutions can change apparent order

Example 1 (Hidden Second-Order)**.** *Given:* $p = \frac{dy}{dx}$ *and the equation* $x\frac{dp}{dx} + p = p^2$
This is actually: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$
Order: Second (not first!)

3 Linearity Testing - The Superposition Principle

Method 1 (Superposition Test)**.** *An ODE is linear if and only if:*

If y_1 and y_2 are solutions, then $c_1y_1 + c_2y_2$ is also a solution.

3.1 Systematic Linearity Check

1. **Rearrange:** All y-terms left, non-y terms right
2. **Examine:** Left side must be $\sum a_i(x)y^{(i)}$
3. **Verify:** No products, powers, or functions of y

Example 2 (Tricky Linear Cases)**.**

$$x^2y + xy' + y'' = e^x \quad \checkmark \textit{ Linear (coefficients can be any function of } x\text{)} \tag{1}$$

$$e^y \cdot y' = x \quad \times \textit{ Nonlinear (exponential of } y\text{)} \tag{2}$$

$$\frac{y'}{y} = x \quad \times \textit{ Nonlinear (division by } y\text{)} \tag{3}$$

3.2 Common Linearity Mistakes

Expression	**Linear?**	**Why?**
x^2y	Yes	x^2 is just a coefficient
y^2	No	Power of y
yy'	No	Product of y and its derivative
$\sin(x)y$	Yes	$\sin(x)$ is a coefficient
$\sin(y)$	No	Nonlinear function of y
y/x	Yes	Same as $(1/x) \cdot y$
x/y	No	Division by y

4 Autonomy - Time Independence

Definition 1 (Autonomous System)**.** *An ODE is autonomous if it can be written without explicit appearance of the independent variable:*

$$\frac{dy}{dx} = f(y) \quad or \quad F(y, y', y'', \ldots) = 0$$

4.1 Physical Interpretation

- **Autonomous:** Laws don't change with time
- **Non-autonomous:** External time-dependent forcing

4.2 Key Property of Autonomous Equations

Time-Translation Invariance: If $y(x)$ solves an autonomous equation, then $y(x-c)$ is also a solution for any constant c.

Example 3 (Equilibrium Solutions)**.** *For autonomous $y' = f(y)$:*

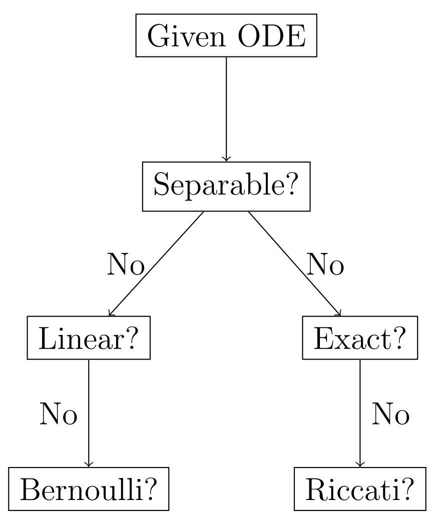
- *Set $f(y) = 0$ to find equilibria*
- *These are constant solutions*
- *Example: $y' = y(1-y)$ has equilibria at $y = 0$ and $y = 1$*

5 Special Forms - Quick Recognition Guide

5.1 First-Order Special Types

Type	**Standard Form**	**Recognition**
Separable	$y' = f(x)g(y)$	Variables multiply separately
Linear	$y' + P(x)y = Q(x)$	Linear in y and y'
Bernoulli	$y' + P(x)y = Q(x)y^n$	Almost linear, power of y
Riccati	$y' = q_0(x) + q_1(x)y + q_2(x)y^2$	Quadratic in y
Exact	$M(x,y)dx + N(x,y)dy = 0$	Check $\partial M/\partial y = \partial N/\partial x$
Homogeneous	$y' = F(y/x)$	Scaling property

5.2 Recognition Flowchart



6 Advanced Classification Examples

Example 4 (Multi-characteristic Equation)**.**

$$y' + \frac{y}{x} = \frac{y^2}{x}$$

Analysis:

- *Order: First*
- *Linear: No (contains y^2)*
- *Autonomous: No (explicit x)*
- *Special type: Riccati (rewrite as $y' = -y/x + y^2/x$)*
- *Alternative: Bernoulli with $n = 2$ after rearrangement*

7 Exam Trap Patterns

Common Exam Traps:

1. Equations that look like one type but are another
2. Hidden nonlinearities: $e^{y'} = x + y$ (nonlinear in y'!)
3. Degree tricks: $\sqrt{1 + (y')^2} = x$ has degree 2 after clearing
4. Almost-linear: $y' + y = y^{1.001}$ is Bernoulli, not linear!

8 Memory Devices

SLBR-EH: Some Lions Bring Real Excitement Home

- **S**eparable
- **L**inear
- **B**ernoulli
- **R**iccati
- **E**xact
- **H**omogeneous