# ODE Lesson 29: Liouville's Formula and Applications

ODE 1 - Prof. Adi Ditkowski

## 1 Liouville's Formula

**Theorem 1** (Liouville's Formula). *For the system $\mathbf{x}' = A(t)\mathbf{x}$ with continuous $A(t)$, the Wronskian of any $n$ solutions satisfies:*

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} tr(A(s))\, ds\right)$$

*where $tr(A)$ denotes the trace (sum of diagonal elements).*

Liouville's Formula reveals that:

- The trace alone determines Wronskian evolution
- Volume in phase space changes at rate $\mathrm{tr}(A(t))$
- System stability is linked to the sign of $\mathrm{tr}(A)$

## 2 Trace Properties and Computation

**Definition 1** (Matrix Trace). *For an $n \times n$ matrix $A = [a_{ij}]$:*

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

**Proposition 1** (Trace Properties).

1. *$tr(A + B) = tr(A) + tr(B)$ (linearity)*
2. *$tr(cA) = c \cdot tr(A)$*
3. *$tr(AB) = tr(BA)$ (cyclic property)*
4. *$tr(A) = \sum_{i=1}^{n} \lambda_i$ (sum of eigenvalues)*
5. *$tr(A^T) = tr(A)$*

## 3 Special Cases

**Corollary 1** (Constant Coefficient Systems). *For* $\mathbf{x}' = A\mathbf{x}$ *with constant* $A$*:* $W(t) = W(0) \cdot e^{tr}(A)\ \cdot t$

**Example 1** (Volume-Preserving Systems). *If* $tr(A(t)) = 0$ *for all* $t$*, then:*

$$W(t) = W(t_0) = constant$$

*Such systems preserve phase space volume (incompressible flow).*

Classification by trace:

- $\text{tr}(A) > 0$: Expanding system (unstable)
- $\text{tr}(A) < 0$: Contracting system (stable)
- $\text{tr}(A) = 0$: Volume-preserving (neutral)

## 4 Applications to Stability

**Theorem 2** (Stability via Trace). *For the system* $\mathbf{x}' = A\mathbf{x}$ *with constant* $A$*:*

1. *If* $tr(A) < 0$ *and* $\det(A) > 0$*, the origin is asymptotically stable*
2. *If* $tr(A) > 0$*, the origin is unstable*
3. *If* $tr(A) = 0$*, further analysis is needed*

**Example 2** (Second-Order Scalar Equation). *For* $y'' + py' + qy = 0$*, the companion matrix has trace* $-p. Thus: W(t) = W(0) \cdot e^{-pt}$

*Stability criterion: p ¿ 0 implies Wronskian decay.*

## 5 Geometric Interpretation

**Proposition 2** (Phase Space Volume). *The Wronskian* $W(t)$ *represents the volume of the parallelepiped formed by solution vectors in phase space. Liouville's formula describes how this volume evolves.*

For a 2D system:

- Solutions $\mathbf{x}_1(t), \mathbf{x}_2(t)$ form a parallelogram
- Area $= |W(t)|$
- Rate of area change $= \text{tr}(A(t)) \cdot |W(t)|$

## 6 Connection to Divergence

**Theorem 3** (Divergence Theorem Connection)**.** *For the vector field* $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$*:*

$$div(\mathbf{F}) = tr(A)$$

*This links Liouville's formula to the divergence theorem in vector calculus.*

## 7 Examples and Applications

**Example 3** (Hamiltonian System)**.** *For a Hamiltonian system:*

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$$

*The system matrix has trace zero, so phase space volume is preserved (Liouville's theorem in mechanics).*

**Example 4** (Damped Oscillator)**.** *For* $\ddot{x} + 2\zeta\omega\dot{x} + \omega^{2x} = 0$*:*

- *Companion matrix trace* $= -2\zeta\omega$
- *W(t) = W(0)* $\cdot e^{-2\zeta\omega t}$
- *Damping* $(\zeta > 0)$ *causes Wronskian decay*

Common errors:

- Confusing trace with determinant
- Forgetting the integral for time-dependent $A(t)$
- Missing that $\mathrm{tr}(A) = \sum \lambda_i$
- Not recognizing volume-preserving systems

## 8 Advanced Applications

**Proposition 3** (Periodic Systems)**.** *For* $T$*-periodic* $A(t)$*, stability depends on:*

$$\mu = \frac{1}{T} \int_0^T tr(A(s))\, ds$$

- $\mu < 0$*: Stable*
- $\mu > 0$*: Unstable*
- $\mu = 0$*: Marginally stable*

Prof. Ditkowski's favorite Liouville problems:

- Computing $W(t)$ using trace
- Identifying volume-preserving systems
- Stability analysis via trace
- Connection to eigenvalue sum
- Time-dependent trace integrals

# 9 Quick Reference

Essential formulas:

- General: W(t) = W($\mathrm{t_0}$)$\mathrm{e}^{\int_{t_0} t}$tr$(A(s))dsConstant : W(t) = W(0)\mathrm{e}^{\mathrm{tr}}$(A) $\cdot t$
- Scalar $n$th order: W(t) = W(0)$e^{-\int a_{n-1}}$(s)ds
- Trace-eigenvalue: $\mathrm{tr}(A) = \lambda_1 + \cdots + \lambda_n$