Lesson 20: Riccati to Second-Order Linear Transformation

ODE 1 - Prof. Adi Ditkowski

1 The Fundamental Transformation

Theorem 1 (Riccati to Second-Order Linear)**.** *The Riccati equation*

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2$$

can be transformed into the second-order linear equation

$$u'' + p(x)u' + r(x)u = 0$$

via the substitution $y = -\frac{1}{q_2} \cdot \frac{u'}{u}$*, where:*

$$p(x) = -q_1 - \frac{q_2'}{q_2} \tag{1}$$

$$r(x) = q_0 q_2 \tag{2}$$

Proof. Starting with $y = -\frac{1}{q_2} \cdot \frac{u'}{u}$:

$$\begin{aligned}
y' &= -\frac{1}{q_2} \cdot \frac{d}{dx}\left(\frac{u'}{u}\right) + \frac{q_2'}{q_2^2} \cdot \frac{u'}{u} && (3)\\
&= -\frac{1}{q_2}\left(\frac{u''}{u} - \frac{(u')^2}{u^2}\right) + \frac{q_2'}{q_2^2} \cdot \frac{u'}{u} && (4)\\
&= -\frac{u''}{q_2 u} + \frac{1}{q_2} \cdot \frac{(u')^2}{u^2} + \frac{q_2'}{q_2^2} \cdot \frac{u'}{u} && (5)\\
&= -\frac{u''}{q_2 u} + \frac{1}{q_2} \cdot q_2^2 y^2 + \frac{q_2'}{q_2^2} \cdot (-q_2 y) && (6)\\
&= -\frac{u''}{q_2 u} + q_2 y^2 - \frac{q_2'}{q_2} y && (7)
\end{aligned}$$

Substituting into the Riccati equation and simplifying yields the stated result. □ □

2 Alternative Transformation

The Exponential Integral Approach:

For the Riccati equation $y' = q_0 + q_1 y + q_2 y^2$, we can use:

$$v = \exp\left(\int y\,dx\right)$$

Then:

1. $\frac{v'}{v} = y$ implies $y = \frac{v'}{v}$
2. $y' = \frac{v''}{v} - \left(\frac{v'}{v}\right)^2 = \frac{v''}{v} - y^2$
3. Substituting into Riccati: $\frac{v''}{v} - y^2 = q_0 + q_1 y + q_2 y^2$
4. This gives: $v'' - q_1 v' - (1+q_2)(v')^2/v + q_0 v = 0$

This form is useful when $q_2 = -1$ as it simplifies to:

$$v'' - q_1 v' + q_0 v = 0$$

3 Standard Examples

Example 1 (Constant Coefficients). *Solve:* $y' = 1 + y^2$

Method 1: Using $y = -u'/u$

Here $q_0 = 1$*,* $q_1 = 0$*,* $q_2 = 1$*. Since* q_2 *is constant,* $q_2' = 0$*.*

The second-order equation becomes:

$$u'' + 0 \cdot u' + 1 \cdot u = 0 \implies u'' + u = 0$$

General solution: $u = c_1 \cos x + c_2 \sin x$

Therefore: $y = -\frac{u'}{u} = -\frac{-c_1 \sin x + c_2 \cos x}{c_1 \cos x + c_2 \sin x} = \frac{c_1 \sin x - c_2 \cos x}{c_1 \cos x + c_2 \sin x}$

Simplifying: $y = \tan(x - \phi)$ *where* $\tan\phi = c_2/c_1$

Example 2 (Euler-Type Transformation). *Solve:* $y' = \frac{a}{x^2} + b \cdot y^2$ *where* a, b *are constants.*

Using $y = -\frac{1}{b} \cdot \frac{u'}{u}$ *(note* $q_2 = b$*):*

The second-order equation becomes:

$$u'' + 0 \cdot u' + \frac{ab}{x^2} u = 0$$

or equivalently:

$$x^2 u'' + ab \cdot u = 0$$

This is an Euler equation. Let $u = x^m$*:*

$$m(m-1) + ab = 0 \implies m = \frac{1 \pm \sqrt{1-4ab}}{2}$$

The nature of solutions depends on the discriminant $1 - 4ab$*.*

4 Special Second-Order Forms

Common Transformations and Their Second-Order Forms:

Riccati Form	Second-Order Result
$y' = a + by^2$ (const)	$u'' + ab \cdot u = 0$
$y' = \frac{a}{x^2} + by^2$	$x^2u'' + ab \cdot u = 0$ (Euler)
$y' = ax^2 + by^2$	$u'' + abx^2u = 0$ (Airy-type)
$y' = q_0(x) - y^2$	$u'' - q_0(x)u = 0$
$y' = \frac{n(n+1)}{x^2} + y^2$	Modified Bessel equation

When the Method Fails:

- If $q_2(x) = 0$ at some points, the transformation is singular there
- The resulting second-order equation may not have elementary solutions
- Series solutions or special functions may be required
- Numerical methods might be necessary for general $q_i(x)$

5 Reverse Transformation

Theorem 2 (Second-Order to Riccati)**.** *Given the second-order linear equation:*

$$u'' + p(x)u' + q(x)u = 0$$

The substitution $y = -\frac{u'}{u}$ yields the Riccati equation:

$$y' = -q(x) - p(x)y - y^2$$

Why This Matters:

- Every linear second-order equation has an associated Riccati
- Solutions of one determine solutions of the other
- Qualitative properties transfer between the two forms
- Stability analysis can be done in either form
- Special functions defined by second-order equations give Riccati solutions

6 Solution Process Flowchart

Solution Process Algorithm:

1. Start with Riccati: $y' = q_0 + q_1 y + q_2 y^2$
2. **If** you know a particular solution y_p:
   - Use substitution $y = y_p + v$
   - Solve resulting Bernoulli equation
3. **If** no particular solution is known:
   - Use transformation $y = -\frac{u'}{q_2 u}$
   - Solve resulting second-order linear equation
4. Back-substitute to find general solution

7 Connection to Special Functions

Prof. Ditkowski's Favorite Transformations:

1. Riccati $\to$ Constant coefficient: Usually solvable with exponentials
2. Riccati $\to$ Euler equation: Power solutions
3. Riccati $\to$ Airy equation: Special functions required
4. Riccati $\to$ Bessel equation: Bessel functions needed
5. Simple forms where $q_2 = \pm 1$: Often give nice solutions