

# Lesson 19: Riccati Equations with Known Particular Solution

ODE 1 - Prof. Adi Ditkowski

## 1 Definition and Recognition

**Definition 1** (Riccati Equation). *A Riccati differential equation has the form:*

$$\frac{dy}{dx} = q_0(x) + q_1(x)y + q_2(x)y^2$$

where  $q_0(x)$ ,  $q_1(x)$ , and  $q_2(x)$  are continuous functions and  $q_2(x) \neq 0$ .

### Special Cases:

- If  $q_2(x) \equiv 0$ : Linear equation
- If  $q_0(x) \equiv 0$ : Bernoulli equation with  $n = 2$
- If  $q_1(x) \equiv 0$ : Separable after substitution
- General case: Requires particular solution or transformation to second-order

## 2 The Fundamental Transformation

**Theorem 1** (Riccati to Bernoulli Reduction). *If  $y_p(x)$  is a particular solution of the Riccati equation, then the substitution*

$$y = y_p + v$$

*transforms the Riccati equation into the Bernoulli equation:*

$$\frac{dv}{dx} = (q_1 + 2q_2y_p)v + q_2v^2$$

*Proof.* Given:  $y' = q_0 + q_1y + q_2y^2$  and  $y'_p = q_0 + q_1y_p + q_2y_p^2$

Substitute  $y = y_p + v$ :

$$y'_p + v' = q_0 + q_1(y_p + v) + q_2(y_p + v)^2 \tag{1}$$

$$y'_p + v' = q_0 + q_1y_p + q_1v + q_2y_p^2 + 2q_2y_pv + q_2v^2 \tag{2}$$

$$v' = q_1v + 2q_2y_pv + q_2v^2 \quad (\text{using } y'_p = q_0 + q_1y_p + q_2y_p^2) \tag{3}$$

$$v' = (q_1 + 2q_2y_p)v + q_2v^2 \quad \square \tag{4}$$

□

### Complete Solution Algorithm:

1. Find or verify particular solution  $y_p$
2. Substitute  $y = y_p + v$
3. Obtain Bernoulli equation:  $v' = (q_1 + 2q_2y_p)v + q_2v^2$
4. Use substitution  $w = v^{-1}$  (since  $n = 2$ )
5. Solve linear equation:  $w' = -(q_1 + 2q_2y_p)w - q_2$
6. Back-substitute:  $v = 1/w$ , then  $y = y_p + v$

## 3 Finding Particular Solutions

Method 1 (Inspection Techniques). *Common forms to try:*

1. **Constants:** Try  $y_p = c$  when coefficients allow
2. **Linear:** Try  $y_p = ax + b$  for polynomial coefficients
3. **Rational:** Try  $y_p = a/x$  or  $y_p = a/(x + b)$
4. **Exponential:** Try  $y_p = ae^{bx}$  for constant coefficients
5. **Trigonometric:** Try  $y_p = a \tan(bx)$  or  $a \cot(bx)$
6. **Special:**  $y_p = -q_1/(2q_2)$  when this ratio is constant

**Example 1** (Polynomial Particular Solution). Solve:  $y' = \frac{2}{x^2} - \frac{2y}{x} + y^2$

**Finding  $y_p$ :** Try  $y_p = \frac{a}{x}$

$$-\frac{a}{x^2} = \frac{2}{x^2} - \frac{2a}{x^2} + \frac{a^2}{x^2}$$
$$-a = 2 - 2a + a^2 \implies a^2 - a + 2 = 0$$

This gives  $a = 2$  or  $a = -1$ . Use  $y_p = \frac{2}{x}$ .

**Transformation:** Let  $y = \frac{2}{x} + v$

$$v' = -\frac{2v}{x} + \frac{4v}{x} + v^2 = \frac{2v}{x} + v^2$$

**Bernoulli to Linear:** Let  $w = v^{-1}$

$$w' = -\frac{2w}{x} - 1$$

**Solution:**  $w = \frac{C}{x^2} - \frac{x}{3}$

**Final Answer:**  $y = \frac{2}{x} + \frac{1}{C/x^2 - x/3}$

## 4 Special Riccati Forms

**Constant Coefficient Riccati:**  $y' = a + by + cy^2$

- If  $b^2 - 4ac > 0$ : Two constant particular solutions
- If  $b^2 - 4ac = 0$ : One constant particular solution
- If  $b^2 - 4ac < 0$ : No real constant solutions

For  $b^2 - 4ac < 0$ , try  $y_p = \alpha \tan(\beta x)$  where  $\beta = \sqrt{4ac - b^2}/(2c)$

**Example 2** (Trigonometric Particular Solution). *Solve:*  $y' = 1 + y^2$

**Observation:** This matches  $\frac{d}{dx}[\tan x] = \sec^2 x = 1 + \tan^2 x$

**Particular solution:**  $y_p = \tan x$

**General solution:** Let  $y = \tan x + v$

$$v' = 2 \tan x \cdot v + v^2$$

After solving the Bernoulli equation:

$$y = \tan x + \frac{\sin x}{C - \cos x}$$

## 5 Connection to Linear Second-Order

**Theorem 2** (Riccati-Linear Duality). *The Riccati equation  $y' = q_0 + q_1 y + q_2 y^2$  is equivalent to the second-order linear equation:*

$$u'' - (q_1 + \frac{q_2'}{q_2})u' + q_0 q_2 \cdot u = 0$$

via the transformation  $y = -\frac{1}{q_2} \cdot \frac{u'}{u}$

**Prof. Ditkowski's Patterns:**

- Often provides  $y_p$  or strong hints ("verify that...")
- Tests connection to Bernoulli reduction
- Likes rational particular solutions  $y_p = a/x$
- May ask for multiple particular solutions
- Tests the second-order connection
- Partial credit for correct transformation setup

## 6 Geometric Interpretation

**Lemma 1** (Cross-Ratio Property). *If  $y_1, y_2, y_3, y_4$  are four solutions of a Riccati equation, their cross-ratio:*

$$\frac{(y_1 - y_3)(y_2 - y_4)}{(y_1 - y_4)(y_2 - y_3)}$$

*is constant (independent of  $x$ ).*

### Common Pitfalls:

- Not verifying that  $y_p$  satisfies the equation
- Sign errors in the transformation to Bernoulli
- Forgetting that Bernoulli with  $n = 2$  needs  $w = v^{-1}$
- Missing singular solutions when  $v = 0$

## 7 Solution Structure

### General Solution Form:

$$y = y_p + \frac{1}{w(x)}$$

where  $w(x)$  satisfies the linear equation:

$$w' + (q_1 + 2q_2 y_p)w = -q_2$$

The general solution has the structure:

$$y = y_p + \frac{1}{\phi(x) + C\psi(x)}$$

where  $\phi$  and  $\psi$  depend on the particular solution chosen.