Lesson 42: Lyapunov Functions and Global Stability

ODE 1 - Prof. Adi Ditkowski

Nonlinear Systems Analysis

1 Lyapunov Stability Concepts

Definition 1 (Lyapunov Stability). *The equilibrium point* $\mathbf{x} = \mathbf{0}$ *of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *is:*

- ***Lyapunov stable*** *if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that:*

$$||\mathbf{x}(0)|| < \delta \implies ||\mathbf{x}(t)|| < \epsilon \text{ for all } t \geq 0$$

- ***Asymptotically stable*** *if it is Lyapunov stable AND there exists* $\delta_0 > 0$ *such that:*

$$||\mathbf{x}(0)|| < \delta_0 \implies \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$$

- ***Globally asymptotically stable*** *if it is asymptotically stable for ALL initial conditions:*

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0} \text{ for all } \mathbf{x}(0) \in \mathbb{R}^2$$

Intuitive Understanding:

- Lyapunov stable: "Stay nearby forever"
- Asymptotically stable: "Stay nearby and converge"
- Globally asymptotically stable: "Converge from anywhere"

2 Lyapunov Functions

Definition 2 (Lyapunov Function). *A continuously differentiable function* $V : \mathbb{R}^2 \to \mathbb{R}$ *is a* ***Lyapunov function*** *for the system* $\dot{x} = f(x, y)$*,* $\dot{y} = g(x, y)$ *at the origin if:*

1. $V(0, 0) = 0$
2. $V(x, y) > 0$ *for all* $(x, y) \neq (0, 0)$ *in some neighborhood*
3. $\dot{V}(x, y) \leq 0$ *along trajectories in that neighborhood*

where the derivative along trajectories is:

$$\dot{V} = \frac{\partial V}{\partial x} f(x,y) + \frac{\partial V}{\partial y} g(x,y) = \nabla V \cdot \mathbf{f}$$

Critical Distinction:

$$\dot{V} \neq \frac{\partial V}{\partial t}$$

Since V doesn't explicitly depend on time, $\frac{\partial V}{\partial t} = 0$. Instead:

$$\dot{V} = \frac{dV}{dt}\bigg|_{\text{along trajectories}} = \nabla V \cdot \mathbf{f}$$

3 Main Stability Theorems

Theorem 1 (Lyapunov's Direct Method)**.** *For the equilibrium at the origin:*

1. *If there exists a Lyapunov function with $\dot{V} \leq 0$, then the origin is **Lyapunov stable***
2. *If additionally $\dot{V} < 0$ for $(x,y) \neq (0,0)$, then the origin is **asymptotically stable***
3. *If additionally $V(x,y) \to \infty$ as $||(x,y)|| \to \infty$ (radial unboundedness), then the origin is **globally asymptotically stable***

Theorem 2 (Instability Theorem)**.** *If there exists a function V with $V(0,0) = 0$ and:*

- *V can take positive values arbitrarily close to the origin*
- *$\dot{V} > 0$ where $V > 0$*

*then the origin is **unstable**.*

4 LaSalle's Invariance Principle

Theorem 3 (LaSalle's Principle)**.** *Let V be a Lyapunov function with $\dot{V} \leq 0$. Define:*

$$E = \{(x,y) : \dot{V}(x,y) = 0\}$$

Let M be the largest invariant set contained in E. Then all trajectories starting in a neighborhood of the origin approach M as $t \to \infty$.

LaSalle's principle often upgrades Lyapunov stability to asymptotic stability! If the only invariant set in E is the origin itself, then we have asymptotic stability even when $\dot{V} \leq 0$ (not strictly negative).

5 Construction Methods

Standard Approaches for Finding Lyapunov Functions:

1. **Quadratic Forms:** Try $V = ax^2 + cy^2$ with $a, c > 0$
2. **General Quadratic:** Try $V = ax^2 + bxy + cy^2$ with:
$$a > 0, \quad c > 0, \quad ac - b^2/4 > 0$$
3. **Energy Methods:** For mechanical systems, use:
$$V = \text{kinetic energy} + \text{potential energy}$$
4. **Gradient Systems:** If $\dot{\mathbf{x}} = -\nabla U(\mathbf{x})$, use $V = U$
5. **First Integrals:** If $H(x, y)$ is conserved, modifications of H often work
6. **Trial and Error:** Combine terms based on system structure

6 Complete Examples

Example 1 (Quadratic Lyapunov Function)**.** *Consider the system:*

$$\dot{x} = -x^3 + xy^2, \quad \dot{y} = -y^3 + x^2y$$

Try $V(x, y) = x^2 + y^2$:
Check conditions:

1. $V(0, 0) = 0$ ✓
2. $V(x, y) = x^2 + y^2 > 0$ *for* $(x, y) \neq (0, 0)$ ✓
3. *Compute* $\dot{V}$*:*

$$\begin{aligned} \dot{V} &= 2x\dot{x} + 2y\dot{y} && (1)\\ &= 2x(-x^3 + xy^2) + 2y(-y^3 + x^2y) && (2)\\ &= -2x^4 + 2x^2y^2 - 2y^4 + 2x^2y^2 && (3)\\ &= -2(x^4 + y^4) + 4x^2y^2 && (4)\\ &= -2(x^4 + y^4 - 2x^2y^2) && (5)\\ &= -2(x^2 - y^2)^2 \leq 0 && (6) \end{aligned}$$

So the origin is Lyapunov stable. For asymptotic stability, apply LaSalle: $\dot{V} = 0$ *only when* $x^2 = y^2$*, i.e.,* $x = \pm y$*.*

On the line $x = y$*:* $\dot{x} = 0$*,* $\dot{y} = 0$ *only at origin. On the line* $x = -y$*:* $\dot{x} = -2x^3$*,* $\dot{y} = 2y^3$*, so trajectories leave unless at origin.*

Therefore, $M = \{(0, 0)\}$ *and the origin is asymptotically stable.*

Example 2 (Energy Function for Pendulum). *Damped pendulum:* $\ddot{\theta} + c\dot{\theta} + \sin\theta = 0$

As a system: $\dot{x} = y$, $\dot{y} = -\sin x - cy$ *(where* $x = \theta$, $y = \dot{\theta}$*)*

Energy function: $V(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$

Analysis:

1. $V(0, 0) = 0$ ✓
2. *For small* x: $1 - \cos x \approx \frac{x^2}{2} > 0$, *so* $V > 0$ *near origin* ✓
3. $\dot{V} = y\dot{y} + \sin x \cdot \dot{x} = y(-\sin x - cy) + \sin x \cdot y = -cy^2 \leq 0$

By LaSalle: $\dot{V} = 0$ *when* $y = 0$. *The invariant set on* $\{y = 0\}$ *is just the equilibria. Near the origin, this is only* $(0, 0)$, *giving asymptotic stability.*

7 Basin of Attraction

Definition 3 (Basin of Attraction). *The basin of attraction of an asymptotically stable equilibrium* $\mathbf{x}^*$ *is:*

$$\mathcal{B}(\mathbf{x}^*) = \{\mathbf{x}_0 : \lim_{t\to\infty} \phi(t, \mathbf{x}_0) = \mathbf{x}^*\}$$

where $\phi(t, \mathbf{x}_0)$ *is the solution starting at* $\mathbf{x}_0$.

Method 1 (Estimating Basins with Lyapunov Functions). *If* V *is a Lyapunov function with* $\dot{V} < 0$ *for* $\mathbf{x} \neq \mathbf{0}$, *and* $\Omega_c = \{\mathbf{x} : V(\mathbf{x}) \leq c\}$ *is bounded and contains only the origin as an equilibrium, then* $\Omega_c \subseteq \mathcal{B}(\mathbf{0})$.

8 Comparison: Linearization vs Lyapunov

Aspect	**Linearization**	**Lyapunov**
Scope	Local only	Local or global
Applicability	Hyperbolic equilibria	Any equilibrium
Information	Qualitative behavior	Stability + basin
Construction	Automatic (Jacobian)	Requires creativity
Limitations	Fails at centers	May not exist/find

Prof. Ditkowski's Exam Strategy:

1. If given V, just verify conditions and compute $\dot{V}$
2. If asked to find V:
   - Start with $V = x^2 + y^2$
   - Try $V = ax^2 + cy^2$ with suitable a, c
   - Look for physical energy if mechanical system
   - Check if gradient system

3. Always apply LaSalle when $\dot{V} \leq 0$ (not strictly negative)

4. State stability conclusion explicitly

9 Summary Algorithm

Lyapunov Analysis Procedure:

1. Identify/construct candidate $V(x, y)$

2. Verify $V(0, 0) = 0$

3. Check $V(x, y) > 0$ for $(x, y) \neq (0, 0)$ nearby

4. Compute $\dot{V} = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial y} g$

5. Analyze sign of $\dot{V}$:

   - $\dot{V} \leq 0$: Lyapunov stable
   - $\dot{V} < 0$ (except origin): Asymptotically stable
   - $\dot{V} > 0$ somewhere: Check instability theorem

6. If $\dot{V} \leq 0$, apply LaSalle's principle

7. Check radial unboundedness for global stability