Lesson 20: Practice Problems - Riccati to Second-Order Transformation

ODE 1 - Prof. Adi Ditkowski

Part A: Basic Transformations (5 problems)

1. Transform each Riccati equation to second-order linear form:
   (a) $y' = 1 + y^2$
   (b) $y' = x^2 - y^2$
   (c) $y' = \frac{1}{x^2} + \frac{2y}{x} + y^2$
   (d) $y' = e^x + y^2$
   (e) $y' = 1 - 2y + y^2$
2. For the equation $y' = q_0(x) + y^2$, show that the transformation $y = -u'/u$ gives $u'' = q_0(x)u$.
3. Verify that if u_1 and u_2 solve the second-order equation, then $y = -\frac{u_1'u_2 - u_1u_2'}{u_1u_2}$ solves the associated Riccati.
4. Show that the transformation $y = \tan(\theta)$ converts $y' = a(x)(1 + y^2)$ into $\theta' = a(x)$.
5. Prove that if the Riccati has constant coefficients, the transformed second-order equation also has constant coefficients.

Part B: Complete Solutions via Transformation (6 problems)

6. Solve $y' = 1 - y^2$ by transforming to second-order form.
7. Find the general solution of $y' = \frac{2}{x^2} + y^2$ using the second-order approach.
8. Solve $y' = 4 + y^2$ and express the answer in terms of hyperbolic functions.
9. Transform and solve: $y' = \cos^2(x) + y^2$.
10. Find all solutions of $y' = e^{2x} - y^2$.
11. Solve the equation $y' = \frac{1-n^2}{x^2} + y^2$ where n is an integer.

Part C: Reverse Transformation (5 problems)

12. Given $u'' + u = 0$, find the associated Riccati equation and solve it.
13. Transform $u'' - 4u = 0$ to Riccati form and find all solutions.
14. Convert the Airy equation $u'' - xu = 0$ to its Riccati form.
15. Show that the Bessel equation $x^2u'' + xu' + (x^2 - n^2)u = 0$ corresponds to a specific Riccati equation.
16. Given $u'' + p(x)u' + q(x)u = 0$ with known solution u_1, find the Riccati solution.

Part D: Special Cases and Applications (5 problems)

17. The Riccati equation $y' = ax^{2n} + by^2$ where a, b are constants:
    (a) Transform to second-order form
    (b) Identify when elementary solutions exist
    (c) Solve for $n = 0, 1$
18. Consider $y' = \frac{A}{x^2} + \frac{B}{x}y + Cy^2$:
    (a) Show this transforms to an Euler equation
    (b) Find conditions on A, B, C for real solutions
    (c) Solve when $B = 0$, $C = 1$
19. The equation $y' = \sec^2(x) + y^2$:
    (a) Transform to second-order form
    (b) Explain why elementary solutions don't exist
    (c) Find series solution near $x = 0$
20. For the parametric family $y' = \lambda + y^2$:
    (a) Find the second-order form for each λ
    (b) Determine solution behavior as λ varies
    (c) Identify bifurcation at $\lambda = 0$
21. The Schwarzian derivative connection:
    (a) Show that $y' = -\frac{1}{2}S[f](x) + y^2$ where $S[f]$ is the Schwarzian
    (b) Find the second-order form
    (c) Discuss invariance properties

Part E: Theoretical Problems (4 problems)

22. Prove that the transformation $y = -\frac{1}{q_2}\frac{u'}{u}$ is invertible: given $y(x)$, we can recover $u(x)$ up to a constant multiple.

23. Show that if the Riccati equation has n particular solutions $y_1, \ldots, y_n$, the second-order equation has n corresponding solutions u_i with $y_i = -u_i'/u_i$.

24. Prove that the Wronskian $W(u_1, u_2) = u_1u_2' - u_1'u_2$ of two solutions of the second-order equation satisfies $W' = -p(x)W$.

25. Establish the connection: If y_1 and y_2 are two Riccati solutions, then $(y_1 - y_2)^{-1}$ satisfies a first-order linear equation.

Part F: Exam-Style Complete Problems (6 problems)

26. [**Prof. Ditkowski Style**] Consider the Riccati equation: $y' = \frac{4}{x^2} - y^2$

    (a) Transform to second-order linear form using $y = u'/u$

    (b) Identify the type of second-order equation obtained

    (c) Solve the second-order equation

    (d) Find the general solution of the original Riccati

    (e) Verify your solution satisfies the original equation

    (f) Find the solution with $y(1) = 2$

27. [**Multiple Methods**] For $y' = 1 + y^2$:

    (a) Solve using the known particular solution $y_p = \tan(x)$

    (b) Solve by transforming to second-order form

    (c) Verify both methods give the same general solution

    (d) Discuss the solution's periodicity and singularities

28. [**Comparison Problem**] Given the two equations:

    (i) $y' = 1 + y^2$

    (ii) $y' = 1 - y^2$

    (a) Transform both to second-order form

    (b) Solve both completely

    (c) Compare the qualitative behavior of solutions

    (d) Explain the difference using phase portraits

29. [**Application to Quantum Mechanics**] The radial Schrödinger equation can yield the Riccati:

$$y' = \frac{l(l+1)}{x^2} - k^2 + \frac{2m}{\hbar^2}V(x) + y^2$$

   (a) For $V(x) = 0$ (free particle), transform to second-order
   (b) Solve for $l = 0$
   (c) Discuss bound states vs scattering states

30. [**Comprehensive Problem**] Consider $y' = x^2 + y^2$:

   (a) Show no elementary particular solution exists
   (b) Transform to second-order form
   (c) Identify this as an Airy-type equation
   (d) Write the solution in terms of Airy functions
   (e) Analyze asymptotic behavior as $x \to \pm\infty$

31. [**Challenge: Connection to Painlevé**] The equation $y' = x + y^2$ is related to Painlevé II.

   (a) Transform to second-order form
   (b) Show the second-order equation has no elementary solutions
   (c) Prove solutions exist for all x
   (d) Find the asymptotic behavior as $x \to -\infty$
   (e) Explain why this is called a "Painlevé transcendent"

Solutions and Hints

Selected Solutions:

Problem 1(a): Using $y = -u'/u$ with $q_2 = 1$: Second-order form: $u'' + u = 0$ Solution: $u = c_1 \cos x + c_2 \sin x$ Riccati solution: $y = \tan(x - \phi)$

Problem 6: $y' = 1 - y^2$ transforms to $u'' - u = 0$. Solution: $u = c_1 e^x + c_2 e^{-x}$ Therefore: $y = -\frac{c_1 e^x - c_2 e^{-x}}{c_1 e^x + c_2 e^{-x}} = \tanh(x + C)$

Problem 7: $y' = \frac{2}{x^2} + y^2$ gives $x^2 u'' + 2u = 0$. This is Euler with $m(m-1) + 2 = 0$, so $m = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm i\sqrt{7}}{2}$. Solutions involve $x^{1/2}\cos(\frac{\sqrt{7}}{2}\ln x)$ and $x^{1/2}\sin(\frac{\sqrt{7}}{2}\ln x)$.

Problem 12: From $u'' + u = 0$, the Riccati is $y' = -1 - y^2$. This is $y' = 1 + y^2$ with $y \to iy$, giving $y = -\tan(x - C)$.

Problem 25: For $y' = \frac{4}{x^2} - y^2$: Second-order: $x^2 u'' - 4u = 0$ (Euler equation) With $u = x^m$: $m(m-1) = 4$, so $m = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}$

Key Transformation Formulas:

- Forward: $y = -\frac{1}{q_2} \cdot \frac{u'}{u}$

- Resulting 2nd-order: $u'' + p(x)u' + r(x)u = 0$
- Where: $p = -q_1 - q_2'/q_2$, $r = q_0 q_2$
- Reverse: Given $u'' + p(x)u' + q(x)u = 0$, get $y' = -q - py - y^2$

Common Second-Order Results:

- $y' = a + y^2 \to u'' + au = 0$
- $y' = a - y^2 \to u'' - au = 0$
- $y' = \frac{a}{x^2} + y^2 \to x^2u'' + au = 0$
- $y' = ax^n + y^2 \to u'' + ax^n u = 0$