Lesson 44: Practice Problems
Repeated Roots and $t^k Terms$

ODE 1 with Prof. Adi Ditkowski

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Part A: Identifying Multiplicities (5 problems)

1. Given the characteristic polynomial $(r-3)^2(r+1) = 0$, identify all roots and their multiplicities.
2. Given the characteristic polynomial $r^4 - 8r^3 + 24r^2 - 32r + 16 = 0$, show that it equals $(r-2)^4$ and identify the multiplicity.
3. The characteristic equation $(r^2 - 2r + 5)^2 = 0$ has complex roots. Find them and their multiplicities.
4. Factor completely: $r^3 + 3r^2 + 3r + 1 = 0$ and identify multiplicities.
5. Given $(r^2 + 4)^3 = 0$, identify all roots and multiplicities.

Part B: Solutions from Multiplicities (5 problems)

6. If $r = 3$ is a triple root, write all three linearly independent solutions.
7. If $r = -2$ has multiplicity 4, write the form of the general solution.
8. Complex roots $1 \pm 2i$ each have multiplicity 2. Write all four real solutions.
9. The characteristic polynomial is $(r+1)^2(r-2)^3$. Write the general solution.
10. If $\pm 3i$ are both double roots, write the general real solution.

Part C: Complete ODEs with Repeated Roots (5 problems)

11. Solve: $y'' - 6y' + 9y = 0$
12. Solve: $y''' + 3y'' + 3y' + y = 0$

13. Solve: $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$
14. Solve: $y'' + 4y' + 4y = 0$ with $y(0) = 2, y'(0) = -3$
15. Solve: $y^{(4)} + 4y'' + 4y = 0$

Part D: Reduction of Order Applications (5 problems)

16. Given that $y_1 =$e$^{-t} solves$(D+1)$^2[y] = 0$, use reduction of order to find y_2.
17. Verify that te$^{3t} solves$y" - 6y' + 9y = $0 bydirectsubstitution.$
18. Show that if $y_1 =$e$^{rt} solves$L[y] $= 0 wherer is adoubleroot, then$y$_2 = te^{rt} alsosolvesit.$
19. Given $y'' - 4y' + 4y = 0$ has solution e^{2t}, find the second solution using $y_2 = v(t)$e^{2t}.
20. For the equation with characteristic polynomial $(r-a)^3 = 0$, verify the Wronskian of e^{at}, te^{at}, t^{2eat} is nonzero.

Part E: Complex Repeated Roots (5 problems)

21. Solve: $y^{(4)} + 8y'' + 16y = 0$
22. Find the general real solution when the characteristic polynomial is $(r^2 + 2r + 2)^2 = 0$.
23. Solve: $y^{(4)} + 4y''' + 8y'' + 8y' + 4y = 0$ (Hint: $(r^2 + 2r + 2)^2$)
24. If $2 \pm 3i$ are both triple roots, how many linearly independent real solutions are there? List them.
25. Solve the IVP: $y'' + 2y' + 5y = 0$ with $y(0) = 0, y'(0) = 4$, given that $-1 \pm 2i$ are roots.

Part F: Theoretical and Exam-Style Problems (5 problems)

26. Prove that if r_0 is a root of multiplicity m of the characteristic polynomial, then $p(r_0) = p'(r_0) = \cdots = p^{(m-1)}(r_0) = 0$ but $p^{(m)}(r_0) \neq 0$.
27. Show that the dimension of the solution space equals the degree of the differential equation, even with repeated roots.
28. A fourth-order equation has characteristic polynomial with roots: $r = 2$ (mult. 2), $r = -1$ (mult. 1), $r = 3$ (mult. 1). If a solution satisfies $y(0) = 1, y'(0) = 0, y''(0) = -1, y'''(0) = 2$, find it.
29. Explain why the functions $\{t^k$ e^{rt} : k = 0, 1, . . . , m-1$\} spanthesolutionspaceforarootofmultiplicity$m.

30. (Prof. Ditkowski style) Consider the family of equations $y'' - 2ay' + a^{2y} = 0 parameterized by$a $\in \mathbb{R}$.

   (a) Show that for all a, the characteristic equation has a repeated root.

   (b) Find the general solution in terms of a.

   (c) For which values of a do all solutions remain bounded as $t \to \infty$?

   (d) Find the solution with $y(0) = 1, y'(0) = 0$.

Solutions

Part A: Identifying Multiplicities

1. Roots: $r = 3$ (multiplicity 2), $r = -1$ (multiplicity 1)
2. Expand $(r-2)^4 = r^4 - 8r^3 + 24r^2 - 32r + 16$ ✓
   Root: $r = 2$ (multiplicity 4)
3. $(r^2 - 2r + 5)^2 = 0 \Rightarrow r^2 - 2r + 5 = 0$
   $r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$
   Roots: $1 + 2i$ (mult. 2), $1 - 2i$ (mult. 2)
4. $r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0$
   Root: $r = -1$ (multiplicity 3)
5. $(r^2 + 4)^3 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$
   Roots: $2i$ (mult. 3), $-2i$ (mult. 3)

Part B: Solutions from Multiplicities

6. e^{3t}, te^{3t}, t^{2e3t}
7. $y(t) = (c_1 + c_{2t} + c_{3t}{}^2 + c_{4t}{}^3)e^{-2t}$
8. Real solutions: $e^t\cos(2t), te^t\cos(2t), e^t\sin(2t), te^t\sin(2t)$
9. $y(t) = (c_1 + c_{2t})e^{-t} + (c_3 + c_{4t} + c_{5t}{}^2)e^{2t}$
10. $y(t) = (c_1 + c_{2t})\cos(3t) + (c_3 + c_{4t})\sin(3t)$

Part C: Complete ODEs with Repeated Roots

11. Char. eq.: $r^2 - 6r + 9 = (r-3)^2 = 0$
    Root: $r = 3$ (mult. 2)
    General solution: $y(t) = (c_1 + c_{2t})e^{3t}$
12. Char. eq.: $r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0$
    Root: $r = -1$ (mult. 3)
    General solution: $y(t) = (c_1 + c_{2t} + c_{3t}{}^2)e^{-t}$
13. Char. eq.: $(r-1)^4 = 0$
    Root: $r = 1$ (mult. 4)
    General solution: $y(t) = (c_1 + c_{2t} + c_{3t}{}^2 + c_{4t}{}^3)e^t$
14. $(r+2)^2 = 0$, so $y = (c_1 + c_{2t})e^{-2t}$
    $y(0) = c_1 = 2$
    $y'(t) = c_{2e}{}^{-2t} - 2(c_1 + c_{2t})e^{-2t}$
    $y'(0) = c_2 - 2c_1 = -3$

$c_2 = 1$
Solution: $y(t) = (2 + t)\mathrm{e}^{-2t}$

15. $r^4 + 4r^2 + 4 = (r^2 + 2)^2 = 0$
Roots: $\pm i\sqrt{2}$ (each mult. 2)
$y(t) = (c_1 + c_{2t})\cos(\sqrt{2}t) + (c_3 + c_{4t})\sin(\sqrt{2}t)$

Part D: Reduction of Order Applications

16. Let $y_2 = v(t)\mathrm{e}^{-t}$
$Substituting into$(D+1)$^2[y] = y'' + 2y' + y = 0$:
After simplification: $v'' = 0$
Thus $v = c_{1t}$ + c$_2$
Second solution: $y_2 = t\mathrm{e}^{-t}$

17. $y = t\mathrm{e}^{3t}$
y' = e^{3t} + 3te^{3t} = (1 + 3t)e^{3t}
y" = 3e^{3t} + 3(1 + 3t)e^{3t} = (6 + 9t)e^{3t}
$Substitute$:(6 + 9t)e^{3t} - 6(1 + 3t)e^{3t} + 9te^{3t}
= e^{3t}[6 + 9t - 6 - 18t + 9t] = 0✓

18. For double root r: $(D - r)^2[y] = 0$
If $y = te^{rt}$:y' = e^{rt} + rte^{rt}
y" = 2re^{rt} + r^{2tert}
$(D - r)^2[y] = D[(D - r)[te^{rt}]]$ = D[e^{rt}] = re^{rt} - re^{rt} = 0✓

19. $y_2 = v(t)\mathrm{e}^{2t}$
$\mathrm{y}_2' = (v' + 2v)\mathrm{e}^{2t}$
$\mathrm{y}_2'' = (v'' + 4v' + 4v)\mathrm{e}^{2t}$
$Substituting$:e$^{2t}[v'' + 4v' + 4v - 4v' - 8v + 4v]$ =e^{2t} $\cdot v'' = 0$
$v'' = 0 \Rightarrow v = c_{1t}$ + c$_2$
Second solution: $y_2 = t\mathrm{e}^{2t}$

20. $W = \begin{vmatrix} \mathrm{e}^{at} & t\mathrm{e}^{at} & t^{2eat} \\ a\mathrm{e}^{at} & (1 + at)\mathrm{e}^{at} & (2t + at^2)\mathrm{e}^{at} \\ a^{2eat} & (2a + a^{2t})e^{at} & (2 + 4at + a^{2t2})\mathrm{e}^{at} \end{vmatrix}$
After calculation: $W = 2\mathrm{e}^{3at} \neq 0$

Part E: Complex Repeated Roots

21. $(r^2 + 4)^2 = 0 \Rightarrow r = \pm 2i$ (each mult. 2)
$y(t) = (c_1 + c_{2t})\cos(2t) + (c_3 + c_{4t})\sin(2t)$

22. $r^2 + 2r + 2 = 0 \Rightarrow r = -1 \pm i$
Each root has mult. 2
$y(t)$ =e$^{-t}[(c_1$ + c$_{2t})\cos(t) + (c_3 + c_{4t})\sin(t)]$

23. Roots: $-1 \pm i$ (each mult. 2)
    $y(t) = \mathrm{e}^{-t}[(c_1 + \mathrm{c}_{2t})\cos(t) + (c_3 + c_{4t})\sin(t)]$

24. 6 linearly independent real solutions:
    $e^{2t}\cos(3t), te^{2t}\cos(3t)$, $\mathrm{t}^{2e2t}\cos(3t)$
    $e^{2t}\sin(3t), te^{2t}\sin(3t)$, $\mathrm{t}^{2e2t}\sin(3t)$

25. Simple roots (not repeated): $r = -1 \pm 2i$
    $y(t) = \mathrm{e}^{-t}[c_1\cos(2t) + \mathrm{c}_2\sin(2t)]$
    $y(0) = c_1 = 0$
    $y'(t) = \mathrm{e}^{-t}[-c_1\cos(2t)$ - $\mathrm{c}_2\sin(2t) - 2c_1\sin(2t) + 2c_2\cos(2t)]$
    $y'(0) = -c_1 + 2c_2 = 4$
    $c_2 = 2$
    Solution: $y(t) = 2\mathrm{e}^{-t}\sin(2t)$

Part F: Theoretical and Exam-Style Problems

21. If $p(r) = (r - r_0)^m$ q(r)$where$q(r$_0$) $\neq 0$:
    $p'(r) = m(r - r_0)^{m-1}q(r) + (r - r_0)^m$ q'(r)
    Atr = r$_0$: $p'(r_0) = 0$ if $m > 1$
    Continue differentiating to show $p^{(k)}(r_0) = 0$ for $k < m$
    But $p^{(m)}(r_0) = m! \cdot q(r_0) \neq 0$

22. Total number of solutions = sum of all multiplicities = degree of polynomial = order of ODE

23. Char. poly: $(r - 2)^2(r + 1)(r - 3) = 0$
    $y = (c_1 + c_{2t})e^{2t}$ + $\mathrm{c}_{3e}{}^{-t} + c_{4e}{}^{3t}$
    Apply initial conditions (solve 4×4$system$) :
    $Finalsolution$:y(t) = (1 - 2$_{\overline{3t)}}$e^{2t} + 1$_{\overline{4}}$e^{-t} - 1$_{\overline{4}}$e^{3t}

24. The operator $(D-r)^m hasan$m$-dimensionalkernel.Thefunctions$t^k $e^{rt} for$k = 0, . . . , m-1$arelinearlyindependent(nonzeroWronskian)andallsatisfy$(D-r)^m[y] = 0.

25. (a) Char. eq.: $r^2 - 2ar + a^2 = (r - a)^2 = 0$
        Always a double root at $r = a$
    (b) $y(t) = (c_1 + c_{2t})e^{at}$

(c) Bounded as $t \to \infty$ only if $a \leq 0$

(d) $y(0) = c_1 = 1$
    $y'(t) = c_{2e}{}^{at} + a(c_1 + c_{2t})e^{at}$
    y'(0) = $\mathrm{c}_2 + ac_1 = 0$
    $c_2 = -a$
    Solution: $y(t) = (1 - at)\mathrm{e}^{at}$