ODE Lesson 33: Matrix Exponential - Computing e^{At}

ODE 1 - Prof. Adi Ditkowski

1 Definition and Basic Properties

Definition 1 (Matrix Exponential). *For an $n \times n$ matrix A, the matrix exponential is defined by the convergent series:*

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

Theorem 1 (Fundamental Properties). *The matrix exponential satisfies:*

1. $e^{A\cdot 0} = I$ *(initial condition)*
2. $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ *(derivative property)*
3. $e^{A(s+t)} = e^{As}e^{At}$ *(semigroup property)*
4. $(e^{At})^{-1} = e^{-At}$ *(inverse property)*
5. $\det(e^{At}) = e^{tr(A)t}$ *(determinant formula)*
6. *If $AB = BA$, then $e^{A+B} = e^A e^B$ (commutativity requirement)*

Fundamental Solution Property: The unique solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

2 Method 1: Diagonalization

Method 1 (Diagonalization Method). *If $A = PDP^{-1}$ where $D = diag(\lambda_1, \ldots, \lambda_n)$:*

$$e^{At} = Pe^{Dt}P^{-1}$$

where

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}$$

Example 1 (Diagonalizable 2×2). *Compute e^{At} for $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$*

Solution:

1. *Eigenvalues:* $\det(A - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = 0$
   $(\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = -1$
2. *Eigenvectors: For $\lambda_1 = 3$:* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *For $\lambda_2 = -1$:* $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
3. *Matrices:* $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
4. *Result:*

$$e^{At} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1}$$

$$= \frac{1}{2} \begin{pmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} + e^{-t} \end{pmatrix} \tag{2}$$

3 Method 2: Jordan Form

Method 2 (Jordan Form Method). *For a Jordan block $J_n(\lambda)$ of size n:*

$$e^{J_n(\lambda)t} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \frac{t^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example 2 (2×2 Jordan Block). *Compute e^{At} for $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$*

Solution: *This is a Jordan block with $\lambda = 2$. Using the formula:*

$$e^{At} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$$

Verification: $e^{A \cdot 0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ ✓

4 Method 3: Nilpotent Matrices

Nilpotent Matrix Property: If $N^k = 0$ for some k, then:

$$e^{Nt} = I + Nt + \frac{N^2t^2}{2!} + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)!}$$

The series terminates!

Example 3 (3×3 Nilpotent). *Compute* e^{At} *for* $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A^3 = 0$

Therefore:

$$e^{At} = I + At + \frac{A^2t^2}{2} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

5 Method 4: Cayley-Hamilton

Theorem 2 (Cayley-Hamilton Application). *For a* 2×2 *matrix with characteristic polynomial* $p(\lambda) = \lambda^2 - tr(A)\lambda + \det(A)$*:*

- *If* $\lambda_1 \neq \lambda_2$*:* $e^{At} = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} A + \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} I$
- *If* $\lambda_1 = \lambda_2 = \lambda$*:* $e^{At} = e^{\lambda t}[I + t(A - \lambda I)]$

6 Method 5: Complex Eigenvalues

Example 4 (Complex Eigenvalues). *Compute* e^{At} *for* $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Solution: *Eigenvalues:* $\lambda = \pm i$

Using the formula for complex eigenvalues:

$$e^{At} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

This is a rotation matrix!

Computational Strategies:

- Diagonal matrix: Use e^{Dt} directly
- Diagonalizable: Use $Pe^{Dt}P^{-1}$
- Jordan blocks: Use Jordan exponential formula
- Nilpotent: Truncate the series
- 2×2: Use direct formulas

- Complex eigenvalues: Get rotation matrices

Common Computational Errors:

- Matrix multiplication is NOT commutative
- $e^{A+B} \neq e^A e^B$ unless $AB = BA$
- Don't forget factorials in the series
- Jordan block exponentials have specific patterns
- Always verify $e^{A\cdot 0} = I$

Prof. Ditkowski's typical problems:

- Compute e^{At} for 2×2 diagonal matrices
- Find e^{At} for 2×2 Jordan blocks
- Use diagonalization for 2×2 systems
- Verify properties of matrix exponential
- Apply e^{At} to solve IVPs

7 Quick Reference Table

Matrix Type	**Formula for e^{At}**
Diagonal $D = \text{diag}(\lambda_i)$	$\text{diag}(e^{\lambda_i t})$
Diagonalizable $A = PDP^{-1}$	$Pe^{Dt}P^{-1}$
2×2 Jordan block $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$
Nilpotent $N^k = 0$	$\sum_{j=0}^{k-1} \frac{N^j t^j}{j!}$
Rotation $\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$	$\begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$