Lesson 17: Practice Problems - Homogeneous Equations

ODE 1 - Prof. Adi Ditkowski

Part A: Recognition and Classification (5 problems)

1. Determine if each equation is homogeneous. If yes, identify the degree:
   (a) $\frac{dy}{dx} = \frac{x^2+3xy}{2x^2-y^2}$
   (b) $\frac{dy}{dx} = x + y$
   (c) $(x^2 + y^2)dx - 2xydy = 0$
   (d) $\frac{dy}{dx} = e^{y/x} + \frac{y}{x}$
   (e) $x\frac{dy}{dx} = y \ln\left(\frac{y}{x}\right)$
2. Show that if $M(x, y)$ and $N(x, y)$ are homogeneous of degree n, then $Mdx + Ndy = 0$ is a homogeneous equation.
3. Prove that $\frac{dy}{dx} = f\left(\frac{ax+by+c}{dx+ey+f}\right)$ is NOT homogeneous unless $c = f = 0$.
4. Verify that $(x \sin(y/x) - y \cos(y/x))dx + x \cos(y/x)dy = 0$ is homogeneous.
5. Transform $\frac{dy}{dx} = \frac{y^3+2x^2y}{x^3-xy^2}$ into the form $F(y/x)$.

Part B: Basic Substitution Problems (6 problems)

6. Solve using $v = y/x$ substitution:
$$\frac{dy}{dx} = \frac{y}{x} + 1$$
7. Solve: $(x + y)dx - (x - y)dy = 0$
8. Find the general solution:
$$x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}$$
9. Solve the IVP:
$$\frac{dy}{dx} = \frac{2y^2}{xy - x^2}, \quad y(1) = 2$$
10. Solve: $xy' = y + x \tan(y/x)$
11. Find all solutions:
$$(x^2 + 2y^2)\frac{dy}{dx} = xy$$

Part C: Complete Solution Process (5 problems)

12. Solve and verify your solution:

$$\left(y^2 - 2xy\right) dx + x^2 dy = 0$$

13. Find the solution curve passing through $(1, 1)$:

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

14. Solve using the appropriate substitution:

$$x^2 y' = y^2 + xy + x^2$$

15. Find the general solution:

$$(3x^2 + y^2)dx - 2xydy = 0$$

16. Solve the equation:

$$\frac{dy}{dx} = \frac{x + 2y}{2x - y}$$

Part D: Tricky Cases and Variations (5 problems)

17. Solve using $u = x/y$ instead of $v = y/x$:

$$x\frac{dy}{dx} = 2y + x \sec\left(\frac{x}{y}\right)$$

18. Find singular solutions if any:

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

19. Solve the equation with parameter a:

$$\frac{dy}{dx} = \frac{ax^2 + y^2}{2xy}$$

20. Transform to homogeneous and solve:

$$\frac{dy}{dx} = \frac{2x + 3y - 5}{x + 2y - 3}$$

(Hint: Use translation of coordinates)

21. Solve the implicit homogeneous equation:

$$x^3 + y^3 = 3xy(x + y)$$

Part E: Mixed Recognition Challenge (4 problems)

22. Identify which equations are homogeneous and solve only those:
    (a) $y' = (x^2 + y^2)/(2xy)$
    (b) $y' = x^2 + y^2$
    (c) $y' = (x - y)/(x + y)$
    (d) $y' + y/x = x^2$

23. Show that the substitution $y = vx$ transforms the homogeneous equation into a separable equation in v and x.

24. Find all homogeneous equations of the form:

$$\frac{dy}{dx} = \frac{ay + bx}{cy + dx}$$

25. Prove that if $y_1(x)$ is a solution to a homogeneous equation, then $y_2(x) = ky_1(kx)$ is also a solution for any constant k.

Part F: Exam-Style Complete Problems (5 problems)

26. [**Prof. Ditkowski Style**] Consider the equation: $(x^2 + y^2)dx + (x^2 - 2xy)dy = 0$
    (a) Verify that this is a homogeneous equation
    (b) Find the general solution
    (c) Find the particular solution satisfying $y(1) = 0$
    (d) Determine if there are any singular solutions

27. [**Comprehensive Problem**] For the equation $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$:
    (a) Show it's homogeneous
    (b) Solve using $v = y/x$
    (c) Find the solution through $(1, 1)$
    (d) Sketch the solution curves

28. [**Application Problem**] A curve has the property that at any point (x, y), the tangent line passes through the point $(x/2, 0)$. Find the equation of all such curves.

29. [**Theory and Computation**]
    (a) Prove that every homogeneous equation has solution curves that are similar under scaling from the origin
    (b) Solve: $x(x + y)dy = y(x - y)dx$

(c) Explain the geometric meaning of your solution

30. **[Challenge Problem]** Consider the family of equations: $\frac{dy}{dx} = F(y/x)$ where F is continuous.

    (a) Show that if $F(v) = v$ has solutions $v = v_i$, then $y = v_i x$ are solutions

    (b) For $F(v) = v^2 + v - 2$, find all straight-line solutions

    (c) Solve the complete equation and discuss the behavior near the straight-line solutions

Solutions and Hints

Selected Solutions:

Problem 1(a): Homogeneous of degree 0. Divide numerator and denominator by x^2.

Problem 6: Let $v = y/x$. After substitution: $v + x\frac{dv}{dx} = v + 1$, so $\frac{dv}{dx} = \frac{1}{x}$. Integrating: $v = \ln|x| + C$, thus $y = x\ln|x| + Cx$.

Problem 11: After substitution and separation: $\int \frac{vdv}{1+v^2} = \int \frac{dx}{x}$. This gives $\frac{1}{2}\ln(1+v^2) = \ln|x| + C$, or $(x^2 + y^2) = Ax^2$ where $A = e^{2C}$.

Problem 17: Check $F(v) - v = 0$: $v + \sqrt{1+v^2} - v = \sqrt{1+v^2} \neq 0$ for any real v. No singular solutions.

Problem 20: Translation: let $X = x - 1$, $Y = y - 1$ to eliminate constants.

Key Integration Formulas Needed:

- $\int \frac{dv}{1+v^2} = \arctan(v) + C$
- $\int \frac{vdv}{1+v^2} = \frac{1}{2}\ln(1+v^2) + C$
- $\int \frac{dv}{v^2-1} = \frac{1}{2}\ln\left|\frac{v-1}{v+1}\right| + C$