Practice Problems - Lesson 37: 2D Linear Classification

ODE 1 Course

Part A: Quick Classification (Problems 1-6)

1. Classify each system using ONLY trace and determinant:

   (a) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

   (b) $A = \begin{pmatrix} -2 & 4 \\ -1 & -2 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$

   (d) $A = \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}$

2. For each trace-determinant pair, determine the portrait type:

   (a) $\tau = 4, \Delta = 3$

   (b) $\tau = -2, \Delta = 5$

   (c) $\tau = 0, \Delta = 4$

   (d) $\tau = 3, \Delta = -2$

3. Find all values of k such that $A = \begin{pmatrix} k & 2 \\ 3 & 1 \end{pmatrix}$ gives:

   (a) A saddle point

   (b) A center

   (c) A stable node

   (d) A stable spiral

4. Determine the stability of the origin for:

   $$\dot{x} = -3x + 2y, \quad \dot{y} = -2x - 3y$$

   without computing eigenvalues.

5. Which of these systems have closed orbits?

(a) $\dot{x} = y, \dot{y} = -4x$
(b) $\dot{x} = y, \dot{y} = -x - y$
(c) $\dot{x} = -y, \dot{y} = x$
(d) $\dot{x} = x + y, \dot{y} = -x + y$

6. A system has eigenvalues $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$.
   (a) Classify the equilibrium type.
   (b) Find the trace and determinant.
   (c) Is the origin stable?
   (d) What is the rotation frequency?

Part B: Eigenvalue Analysis (Problems 7-12)

7. For $A = \begin{pmatrix} 4 & -5 \\ 2 & -2 \end{pmatrix}$:
   (a) Find eigenvalues and eigenvectors.
   (b) Classify the equilibrium.
   (c) Sketch the phase portrait.
   (d) Find equations for stable/unstable manifolds.

8. Consider $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$:
   (a) Show eigenvalues are complex.
   (b) Find the real and imaginary parts.
   (c) Determine rotation direction.
   (d) Calculate the period of near-circular orbits.

9. For the repeated eigenvalue case $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$:
   (a) Find the eigenvalue(s).
   (b) Find all eigenvectors.
   (c) Classify as star, improper, or proper node.
   (d) Sketch the portrait.

10. Given eigenvalues $\lambda_1 = -3, \lambda_2 = -1$ with eigenvectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$:
   (a) Reconstruct matrix A.

(b) Verify the classification.
(c) Which eigenvector determines long-term behavior?

11. A matrix has $\tau = -4$ and one eigenvalue is $\lambda_1 = -1$. Find:
    (a) The other eigenvalue.
    (b) The determinant.
    (c) The equilibrium type.

12. For what values of a does $A = \begin{pmatrix} a & 1 \\ -4 & a \end{pmatrix}$ have:
    (a) Real eigenvalues?
    (b) Complex eigenvalues?
    (c) Pure imaginary eigenvalues?

Part C: Portrait Sketching (Problems 13-18)

13. Sketch the phase portrait for each:
    (a) $\dot{x} = 2x + y, \dot{y} = x + 2y$ (proper node)
    (b) $\dot{x} = x + 3y, \dot{y} = -3x + y$ (spiral)
    (c) $\dot{x} = 2x - 5y, \dot{y} = x - 2y$ (saddle)
    (d) $\dot{x} = -y, \dot{y} = 4x$ (center)

14. Draw a phase portrait with:
    (a) Eigenvalues $\lambda_1 = 3, \lambda_2 = 1$
    (b) Horizontal eigenvector for λ_1
    (c) Vertical eigenvector for λ_2

15. Sketch an improper node for $A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$.

16. Draw a stable spiral that rotates clockwise with eigenvalues $-1 \pm 2i$.

17. Create a phase portrait where:
    (a) Trajectories approach origin
    (b) No rotation occurs
    (c) All trajectories are tangent to $y = x$ at origin

18. Sketch the degenerate case $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$.

Part D: Special Cases (Problems 19-23)

19. Analyze the star node $A = 3I$:
    (a) Find all eigenvectors.
    (b) Describe trajectory behavior.
    (c) Find the solution starting at $(1, 2)$.

20. For the skew-symmetric matrix $A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$:
    (a) Show all orbits are circles.
    (b) Find the period.
    (c) What physical system does this represent?

21. Consider the nilpotent matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$:
    (a) Find eigenvalues.
    (b) Describe the phase portrait.
    (c) Solve the system explicitly.

22. Analyze the shear transformation $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$:
    (a) Find equilibria.
    (b) Describe trajectory behavior.
    (c) Is the origin stable?

23. For a Hamiltonian system with $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$:
    (a) Show $\tau = 0$ always.
    (b) When is it a center vs. saddle?
    (c) Find the conserved quantity.

Part E: Parameter Studies (Problems 24-28)

24. Consider $A_\mu = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$ as μ varies:
    (a) Find eigenvalues as functions of μ.
    (b) Identify all bifurcation values.
    (c) Sketch portraits for $\mu = -1, 0, 1$.

(d) What happens as μ crosses zero?

25. For the damped oscillator $\ddot{x} + 2\zeta\dot{x} + x = 0$:

    (a) Convert to first-order system.

    (b) Classify for $\zeta = 0, 0.5, 1, 2$.

    (c) Find the critical damping value.

    (d) Relate to physical behavior.

26. Study $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ (symmetric):

    (a) Show eigenvalues are always real.

    (b) Find conditions for each portrait type.

    (c) Can this give a spiral? Why not?

27. For $A = \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$:

    (a) Find the transition value where portrait type changes.

    (b) Classify for $k = 0, 1, 2$.

    (c) Sketch the trace-det diagram path as k varies.

28. Consider the one-parameter family $A_\alpha = \begin{pmatrix} \alpha & \alpha - 1 \\ 1 - \alpha & \alpha \end{pmatrix}$:

    (a) Show $\det(A_\alpha)$ is constant.

    (b) Find all portrait types as α varies.

    (c) Identify bifurcation points.

Part F: Exam-Style Problems (Problems 29-30)

29. [**Prof. Ditkowski Comprehensive**] Given the system $\dot{x} = ax + 4y$, $\dot{y} = -x + ay$:

    (a) Find trace and determinant in terms of a.

    (b) Determine all values of a giving:

        - Saddle points
        - Centers
        - Stable spirals
        - Stable nodes

    (c) Sketch the bifurcation diagram.

    (d) For $a = -1$, find eigenvalues and eigenvectors.

(e) Draw the phase portrait for $a = -1$.

(f) Find the solution with initial condition $(1, 0)$ when $a = 0$.

30. **[Complete Analysis]** A mechanical system has equations:

$$\ddot{x} + (k+1)\dot{x} + kx = 0$$

where $k > 0$ is a parameter.

(a) Convert to a first-order system.

(b) Find the matrix A in terms of k.

(c) Compute trace, determinant, and discriminant.

(d) Show the origin is always stable.

(e) Find the critical value k^* where behavior changes.

(f) Classify the portrait for:

- $0 < k < k^*$
- $k = k^*$
- $k > k^*$

(g) Interpret physically for a mass-spring-damper.

(h) Sketch portraits for $k = 0.1, 0.25, 1$.

Key Strategies:

- Always compute $\det(A)$ first - negative means saddle!
- Use $\tau^2 - 4\Delta$ to distinguish nodes from spirals
- Remember: $\tau = \lambda_1 + \lambda_2$, $\Delta = \lambda_1 \cdot \lambda_2$
- For spirals: rotation direction from off-diagonal signs
- Centers only when $\tau = 0$ exactly
- Check your portrait satisfies uniqueness theorem