ODE Lesson 6: Lipschitz Conditions - Complete Checking Guide

ODE 1 - Prof. Adi Ditkowski

1 The Lipschitz Condition - Full Understanding

Definition 1 (Lipschitz Condition). *$f(x, y)$ is Lipschitz in y on domain D if:*

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in D$ and some constant $L \geq 0$.

Intuition: Lipschitz = "Speed limit on vertical change"

- The function can't change too rapidly in the y-direction
- Prevents vertical tangents or jumps
- Ensures controlled behavior

2 Method 1: The Derivative Test

Derivative Test (Most Common):

If $\frac{\partial f}{\partial y}$ exists and satisfies:

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq L \quad \text{for all } (x, y) \in D$$

Then f is Lipschitz in y with constant L.

2.1 Step-by-Step Algorithm

1. Compute $\frac{\partial f}{\partial y}$
2. Find the maximum of $\left|\frac{\partial f}{\partial y}\right|$ on your domain
3. This maximum is your Lipschitz constant L
4. If the maximum is infinite, f is not Lipschitz

Example 1 (Derivative Test Application). *Check if $f(x, y) = x^2y + e^x$ is Lipschitz on $|x| \leq 2$, $|y| \leq 3$:*

Solution:

$$\frac{\partial f}{\partial y} = x^2 \tag{1}$$

$$\left|\frac{\partial f}{\partial y}\right| = |x^2| = x^2 \tag{2}$$

$$\max_{|x|\leq 2} x^2 = 4 \tag{3}$$

Therefore, Lipschitz with $L = 4$. ✓

3 Method 2: Direct Estimation

Direct Method (When derivative doesn't exist everywhere):

Directly compute and bound:

$$|f(x, y_1) - f(x, y_2)|$$

Example 2 (Non-differentiable but Lipschitz). *Show $f(y) = |y|$ is Lipschitz:*

Solution:

$$||y_1| - |y_2|| \leq |y_1 - y_2|$$

(This is the reverse triangle inequality)

Therefore, Lipschitz with $L = 1$, even though $f'(0)$ doesn't exist! ✓

4 Method 3: Composition Rules

Theorem 1 (Building Lipschitz Functions).

1. *If g is Lipschitz with L_g and h is Lipschitz with L_h:*
   - *$g + h$ is Lipschitz with $L = L_g + L_h$*
   - *cg is Lipschitz with $L = |c|L_g$*
2. *If g is Lipschitz with L_g and bounded by M:*
   - *g^2 is Lipschitz with $L = 2ML_g$*
3. *If $g \circ h$ exists and both are Lipschitz:*
   - *$g \circ h$ is Lipschitz with $L = L_g \cdot L_h$*

Function	Domain	Lipschitz?	Constant L
$ay + b$	$\mathbb{R}$	Yes	$\lvert a\rvert$
y^2	$\lvert y\rvert \leq M$	Yes	$2M$
y^n $(n \geq 1)$	$\lvert y\rvert \leq M$	Yes	nM^{n-1}
$\sin(y)$, $\cos(y)$	$\mathbb{R}$	Yes	1
e^y	$y \leq M$	Yes	e^M
$\ln(y)$	$y \geq \epsilon > 0$	Yes	$1/\epsilon$
$\sqrt{y}$	$y \geq \epsilon > 0$	Yes	$1/(2\sqrt{\epsilon})$
$\lvert y\rvert$	$\mathbb{R}$	Yes	1
y^α $(0 < \alpha < 1)$	Near $y = 0$	No	∞
$1/y$	Near $y = 0$	No	∞

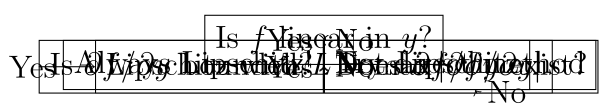
5 Lipschitz Check for Common Functions

6 The Non-Lipschitz Hall of Shame

Functions that are NOT Lipschitz at $y = 0$:

- $f(y) = |y|^\alpha$ for $0 < \alpha < 1$
- $f(y) = \sqrt{|y|}$ (special case of above)
- $f(y) = y^{2/3}$
- $f(y) = y \ln |y|$
- $f(y) = \text{sign}(y)$ (discontinuous)

7 Systematic Checking Flowchart



8 Local vs Global Lipschitz

Quick Classification:

- **Globally Lipschitz:** One L works everywhere
  - Linear functions: $f(y) = ay + b$
  - Bounded derivatives: $f(y) = \sin(y)$, $f(y) = \arctan(y)$
- **Locally Lipschitz:** Different L for different regions
  - Polynomials: $f(y) = y^n$ for $n \geq 2$
  - Exponentials: $f(y) = e^y$
  - Most smooth functions
- **Not Lipschitz:** At some points, no finite L
  - Powers less than 1: $f(y) = y^{1/2}$ at $y = 0$
  - Vertical tangents or cusps

9 Advanced Example: Piecewise Functions

Example 3 (Checking Piecewise Lipschitz). *Consider:*

$$f(y) = \begin{cases} y^2 & |y| \leq 1 \\ 2|y| - 1 & |y| > 1 \end{cases}$$

Check Lipschitz on $\mathbb{R}$:
Step 1: *Check each piece*

- *For $|y| \leq 1$: $f'(y) = 2y$, so $|f'(y)| \leq 2$*
- *For $|y| > 1$: $f'(y) = \pm 2$, so $|f'(y)| = 2$*

Step 2: *Check transition points $y = \pm 1$*

- *Left derivative at $y = 1$: $\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = 2$*
- *Right derivative at $y = 1$: $\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = 2$*
- *Derivatives match!*

Conclusion: *Lipschitz with $L = 2$ globally.* ✓

10 Computing Optimal Lipschitz Constants

Method 1 (Finding the Best L). *For $f(x, y)$ on domain D:*

1. *Compute $\frac{\partial f}{\partial y}$*
2. *Find critical points: solve $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0$*

3. *Evaluate* $\left|\frac{\partial f}{\partial y}\right|$ *at:*
    - *Critical points*
    - *Boundary of domain*
4. *Take the maximum value*

Example 4 (Optimal Constant)**.** *Find the best Lipschitz constant for* $f(y) = y^3 - 3y$ *on* $[-2, 2]$*:*

$$f'(y) = 3y^2 - 3 \tag{4}$$
$$f''(y) = 6y = 0 \Rightarrow y = 0 \tag{5}$$
$$|f'(0)| = 3 \tag{6}$$
$$|f'(\pm 2)| = |12 - 3| = 9 \tag{7}$$

Optimal $L = 9$*.* ✓

11 Exam Strategy Summary

Prof. Ditkowski's Lipschitz Checklist:

1. Linear in y? $\Rightarrow$ Always Lipschitz
2. Compute $\partial f/\partial y$
3. Check boundedness on your domain
4. Watch for y^α with $\alpha < 1$ at $y = 0$
5. For piecewise: check each piece AND transitions
6. State the Lipschitz constant explicitly

12 Memory Device

CHECK for Lipschitz:

- **C**ontinuous derivative helps
- **H**orizontal bounds needed
- **E**xponentials need bounded domain
- **C**orners might still work (like $|y|$)

- **K**eep away from vertical tangents