Lesson 25: Orthogonal Trajectories and Applications

ODE 1 - Prof. Adi Ditkowski

1 Definition and Geometric Interpretation

Definition 1 (Orthogonal Trajectories)**.** *Given a one-parameter family of curves $F(x, y, c) = 0$, the* ***orthogonal trajectories*** *are curves that intersect each member of the family at right angles (90°).*

At any point of intersection, the product of the slopes of two orthogonal curves equals -1:

$$m$$

$_1$ $\cdot m_2 =$ -1

This is the fundamental principle behind finding orthogonal trajectories.

2 The Systematic Method

Finding Orthogonal Trajectories:

1. Start with family $F(x, y, c) = 0$
2. Differentiate to eliminate parameter c:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

3. Express as differential equation: $\frac{dy}{dx} = f(x, y)$
4. Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ to get orthogonal equation:

$$-\frac{dx}{dy} = f(x, y)$$

5. Solve the new differential equation

6. The solution gives the orthogonal trajectories

You MUST eliminate the parameter c before forming the differential equation. Failing to do so is the most common error and leads to incorrect results!

3 Classic Examples

Example 1 (Straight Lines Through Origin)**.** *Find orthogonal trajectories of $y = cx$.*

Solution:

1. *Differentiate:* $\frac{dy}{dx} = c$
2. *Eliminate c: Since* $c = \frac{y}{x}$*, we have* $\frac{dy}{dx} = \frac{y}{x}$
3. *Orthogonal equation:* $-\frac{dx}{dy} = \frac{y}{x}$ *or* $\frac{dx}{dy} = -\frac{y}{x}$
4. *Rearrange:* $xdx = -ydy$

5. *Integrate:* $x^2 \frac{}{2=-y}^2 \frac{}{2+C}$

Simplify: x^2 + y^2 = K The orthogonal trajectories are circles centered at the origin!

Example 2 (Exponential Curves)**.** *Find orthogonal trajectories of $y = ce^{2x}$.*

Solution:

1. *Differentiate:* $\frac{dy}{dx} = 2ce^{2x}$
2. *Eliminate c: From original,* $c = ye^{-2x}$*, so* $\frac{dy}{dx} = 2y$
3. *Orthogonal equation:* $-\frac{dx}{dy} = 2y$ *or* $\frac{dx}{dy} = -2y$

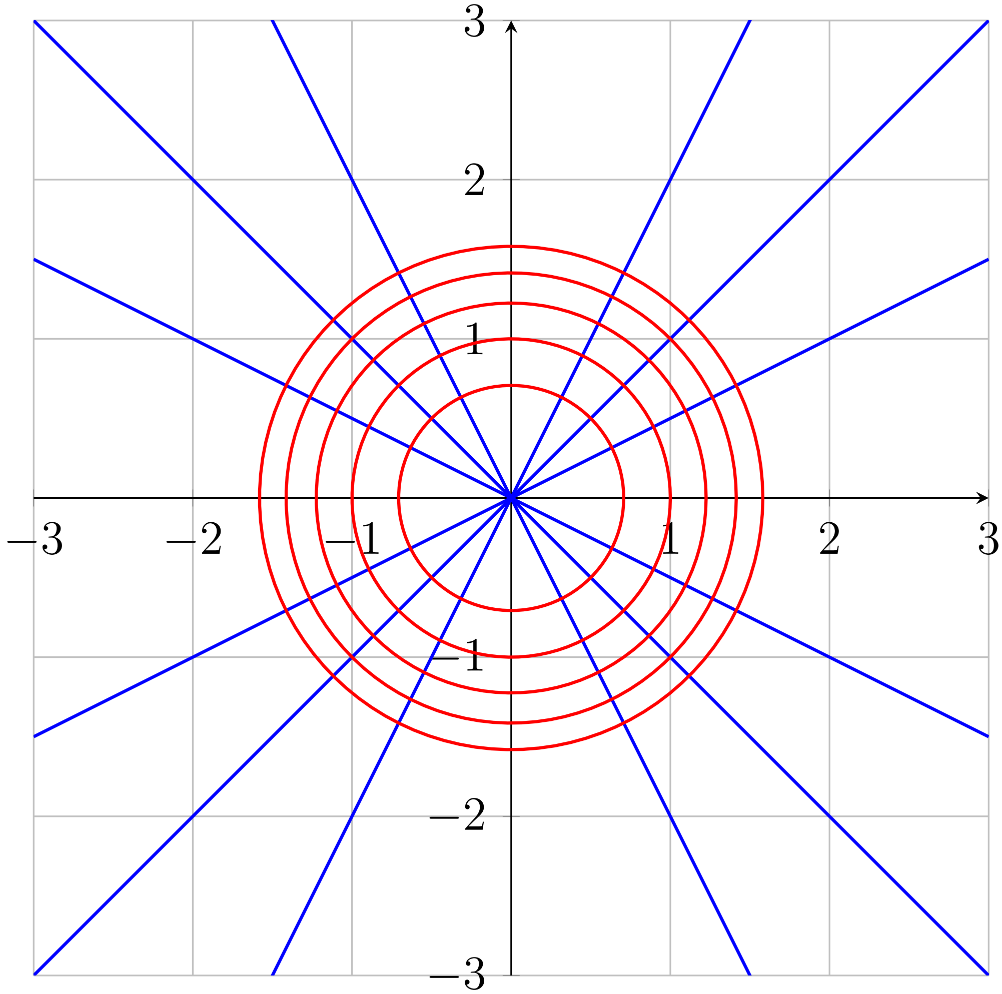
4. *Integrate:* $x = -y^2 + CResult : x + y^2 = K(parabolasopeningleft)$

4 Visualization of Orthogonal Families

5.

y

Orthogonal Families: Lines $y = cx$ and Circles x^2+y^2=K;



5 Self-Orthogonal Families

Definition 2 (Self-Orthogonal). *A family of curves is* ***self-orthogonal*** *if its orthogonal trajectories belong to the same family (with different parameter values).*

Example 3 (Rectangular Hyperbolas). *The family $xy = c$ is self-orthogonal.*

Proof:

1. *Differentiate:* $y + x\frac{dy}{dx} = 0$, *so* $\frac{dy}{dx} = -\frac{y}{x}$
2. *Orthogonal equation:* $-\frac{dx}{dy} = -\frac{y}{x}$ *or* $\frac{dx}{dy} = \frac{y}{x}$
3. *Rearrange:* $\frac{xdx}{ydy} = 1$, *so* $xdx = ydy$
4. *Integrate:* $x^2 \frac{}{2=y} {}^2 \frac{}{2+C}$

This gives x^2 - y^2 = K, or(x-y)(x+y) = K Actually, more careful analysis shows these are also rectangular hyperbolas, rotated by 45°.

6 Physical Applications

Orthogonal Trajectories in Physics:

Field	**Family 1**	**Family 2 (Orthogonal)**
Electrostatics	Electric field lines	Equipotential surfaces
Fluid Dynamics	Streamlines	Velocity potential lines
Heat Transfer	Heat flow lines	Isotherms (constant temperature)
Magnetism	Magnetic field lines	Constant flux surfaces
Stress Analysis	Principal stress trajectories	Shear stress trajectories

The orthogonality of field lines and equipotentials follows from the fact that the gradient (field) is perpendicular to level curves (equipotentials). This is why $\vec{E} = -\nabla V$.

7 Polar Coordinate Form

Theorem 1 (Orthogonal Trajectories in Polar Coordinates). *If a family of curves in polar coordinates satisfies the differential equation*

$$\frac{dr}{d\theta} = F(r, \theta)$$

then the orthogonal trajectories satisfy

$$r$$

${}^2 d\theta \frac{}{dr=F(r,\theta)}$

Example 4 (Cardioids). *Find orthogonal trajectories of $r = a(1+\cos\theta)$ (family of cardioids).*

The differential equation and solution involve more complex polar calculus, but the principle remains: multiply by $r^2 andinterchangetherolesofrand\theta$ derivatives.

8 Connection to Complex Analysis

Theorem 2 (Cauchy-Riemann and Orthogonality)**.** *If $f(z) = u(x, y) + iv(x, y)$ is analytic, then the families $u(x, y) = c_1 andv(x,y) = c_2 areorthogonaltrajectories.$*

Example 5 (Complex Function)**.** *For $f(z) = z^2$ = (x+iy)^2 = x^2 - y^2 + 2ixy:*

Real part: $u = x^2$ - y^2 = $c_1(hyperbolas) Imaginarypart$:v = 2xy = $c_2(hyperbolas)$

These families are orthogonal!

9 Common Exam Patterns

Prof. Ditkowski's favorite orthogonal trajectory problems:

1. Circles and radial lines: x^2 + y^2 = c^{2}y = kx$Parabolasandexponentials$:y^2 = cxy = ke^{2x}
3. Confocal ellipses and hyperbolas (advanced)
4. Temperature distribution and heat flow
5. Self-orthogonal families (always one question!)

10 Step-by-Step Strategy

Exam Problem Approach:

1. Identify the given family $F(x, y, c) = 0$
2. Differentiate with respect to x (implicit differentiation if needed)
3. Eliminate parameter c using the original equation
4. Write differential equation: $\frac{dy}{dx} = f(x, y)$
5. Form orthogonal equation: $\frac{dx}{dy} = -f(x, y)$
6. Solve (may require exact equations or integrating factors!)
7. Verify orthogonality by checking slopes at intersection points
8. Sketch both families showing perpendicular intersections