# ODE Lesson 39: 3D Systems and Higher Dimensions

ODE 1 - Prof. Adi Ditkowski

## 1 3D Linear Systems

Consider the 3D linear system:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \text{where } A \in \mathbb{R}^{3\times 3}$$

**Fundamental Principle:** The behavior is completely determined by the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $A$.

- Real eigenvalues $\rightarrow exponentialbehaviorComplexpair \rightarrow rotationinaplane$

- All three eigenvalues cannot be complex (odd dimension!)

## 2 Classification of 3D Equilibria

**Classification 1** (3D Equilibrium Types)**.**

| ***Type*** | ***Eigenvalues*** | ***Stability*** | ***Behavior*** |
|---|---|---|---|
| *Stable Node* | $\lambda_i$ *¡ 0(allreal)* | *Asymp. Stable* | *All trajectories converge* |
| *Unstable Node* | $\lambda_i$ *¿ 0(allreal)* | *Unstable* | *All trajectories diverge* |
| *Saddle (1-2)* | *1 pos., 2 neg.* | *Unstable* | *1D unstable, 2D stable manifold* |
| *Saddle (2-1)* | *2 pos., 1 neg.* | *Unstable* | *2D unstable, 1D stable manifold* |
| *Spiral-Node (S)* | *1 real neg., 2 complex (Re<0)* | *Asymp. Stable* | *Spiral in plane, converge perpendicular* |
| *Spiral-Node (U)* | *1 real pos., 2 complex (Re>0)* | *Unstable* | *Spiral out plane, diverge perpendicular* |
| *Spiral-Saddle* | *1 real, 2 complex (mixed signs)* | *Unstable* | *Spiral in one direction, escape another* |
| *Center-Line* | *1 real = 0, 2 pure imag.* | *Marginally Stable* | *Circles in plane, line of equilibria* |

## 3 Invariant Manifolds

**Definition 1** (Stable, Unstable, and Center Manifolds)**.** *For an equilibrium at the origin with eigenvalues* $\{\lambda_i\}$: *Dimensions* :$\dim(E^s) + \dim(E^u) + \dim(E^c) = 3$

**Theorem 1** (Manifold Dynamics). • *Trajectories in* $E^s approachoriginast \to \infty$

- *Trajectories in* $E^u approachoriginast \to -\infty$

- *Trajectories in* $E^c maybeperiodicorstationary$

# 4 Visualization Techniques

## 4.1 Projection Methods

**Standard Projections:**

- **xy-projection**: View from above $(x, y, 0)$
- **xz-projection**: View from side $(x, 0, z)$
- **yz-projection**: View from front $(0, y, z)$

Each projection shows 2D "shadow" of 3D dynamics

**Example 1** (Projection Analysis). *System with* $\lambda_1 = -1$, $\lambda_2, 3 = -0.5 \pm 2i$*:*

- *xy-projection: Spiral (from complex pair)*
- *xz-projection: Node-like (real eigenvalue dominates)*
- *yz-projection: Pure spiral (complex pair only)*

## 4.2 Poincaré Sections

**Definition 2** (Poincaré Section). *A* ***Poincaré section*** *is a lower-dimensional slice through phase space. For a 3D system, choose a 2D surface* $\Sigma$ *and record intersection points of trajectories with* $\Sigma$*.*

**Poincaré Map Reveals:**

- Fixed point $\to PeriodicorbitClosedcurve \to Quasi-periodic(torus)$

- Strange pattern $\to ChaosNopattern \to Transientbehavior$

# 5 Examples of 3D Systems

**Example 2** (Complete 3D Analysis). *Consider the matrix:*

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

***Step 1:*** *Eigenvalues are* $\lambda_1 = -2,\ \lambda_2 = -1,\ \lambda_3 = 3$
***Step 2:*** *Classification: Saddle (2-1 type)*
***Step 3:*** *Manifolds:*

- $E^s$ *= span*$\{e_1, e_2\}$ *(xy-plane),* $\dim(E^s)$ *= 2*$E^u$ *= span*$\{e_3\}$ *(z-axis),* $\dim(E^u)$ *= 1*
  ***Step 4:*** *Behavior: Trajectories spiral toward xy-plane, then escape along z-axis*

**Example 3** (Spiral-Node System)**.**

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & -4 & -1 \end{pmatrix}$$

*Eigenvalues:* $\lambda_1 = -3,\ \lambda_{2,3} = -1 \pm 4i$
*Classification: Stable spiral-node - Exponential decay along x-axis (rate = 3) - Spiral decay in yz-plane (rate = 1, frequency = 4)*

## 6 Higher Dimensions (n ¿ 3)

- **Classification in n-Dimensions:** For $\dot{\mathbf{x}} = A\mathbf{x}$ with $A \in \mathbb{R}^{n\times n}$:
  - Count eigenvalues with $\mathrm{Re}(\lambda) < 0$: $n_s (stabledirections) Counteigenvalueswith \mathrm{Re}(\lambda) > 0$: $n_u (unstabledirections)$
- Count eigenvalues with $\mathrm{Re}(\lambda) = 0$: $n_c (centerdirections) Check$ :$\mathrm{n}_s + \mathrm{n}_u + \mathrm{n}_c = \mathrm{n}$

**Classification 2** (Stability in Higher Dimensions)**.** • ***Asymptotically Stable****: All Re*$(\lambda_i)$ *¡ 0****Unstable*** : *AtleastoneRe*$(\lambda_i)$ *¿ 0*

- ***Marginally Stable****: All Re*$(\lambda_i) \le 0$*, some* $= 0$

## 7 Numerical Visualization Tools

**Essential Plotting Techniques:**

1. **3D Trajectory Plots**: Direct visualization in 3D
2. **Multiple 2D Projections**: Show different viewpoints
3. **Animation**: Time evolution along trajectories
4. **Isosurfaces**: Level sets of conserved quantities
5. **Vector Field Slices**: Direction fields on 2D slices

# 8 Special Phenomena in 3D

**Definition 3** (Limit Cycle)**.** *A **limit cycle** is an isolated closed orbit. In 3D, nearby trajectories can spiral toward it (stable) or away (unstable).*

**Definition 4** (Torus Attractor)**.** *Quasi-periodic motion on a 2D torus surface in 3D phase space. Occurs with two incommensurate frequencies.*

**Definition 5** (Strange Attractor)**.** *A fractal set in phase space exhibiting sensitive dependence on initial conditions (chaos). Example: Lorenz attractor.*

# 9 The Lorenz System (Preview)

**Example 4** (Lorenz Equations)**.**

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

*For* $\sigma = 10$*,* $b = 8/3$*,* $r = 28$*: Chaotic attractor!*
*Equilibria:*

- *Origin: Always exists*
- $C_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ *for* $r > 1$

# 10 Exam Strategy for 3D Problems

**Prof. Ditkowski's 3D Problem Approach:**

1. State system dimension explicitly
2. Find ALL eigenvalues (check: product = det, sum = trace)
3. Classify equilibrium type precisely
4. State manifold dimensions: "2D stable, 1D unstable"
5. Describe behavior in words
6. Optional: Sketch ONE key projection
7. Never attempt full 3D drawing

**Common 3D Errors:**

- Forgetting complex eigenvalues come in pairs
- Wrong manifold dimension count
- Trying to visualize 4D+ systems
- Missing the third eigenvalue
- Not checking $\sum \lambda_i = \text{tr}(A)$