Lesson 16: Variation of Constants for First-Order ODEs

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1 Introduction and Motivation

Definition 1 (Variation of Constants Method). *For the linear first-order ODE*

$$y' + p(t)y = g(t)$$

the method of variation of constants seeks a particular solution by allowing the constant in the homogeneous solution to vary with time.

The variation of constants method works for ANY continuous forcing function $g(t)$, unlike undetermined coefficients which requires specific forms.

2 Theoretical Development

2.1 The Fundamental Idea

Theorem 1 (Variation of Constants Formula). *Given the linear ODE $y' + p(t)y = g(t)$ with homogeneous solution $y_h = Ce^{-\int p(t)dt}$, a particular solution is:*

$$y_p = y_h(t) \int \frac{g(t)}{y_h(t)} dt$$

Proof. Assume $y = C(t) \cdot y_h(t)$ where y_h solves $y_h' + p(t)y_h = 0$.

Differentiating:

$$y' = C'(t)y_h(t) + C(t)y_h'(t)$$

Substituting into the original equation:

$$C'(t)y_h(t) + C(t)y_h'(t) + p(t)C(t)y_h(t) = g(t)$$

Since $y_h' + p(t)y_h = 0$, we have $y_h' = -p(t)y_h$:

$$C'(t)y_h(t) + C(t)(-p(t)y_h(t)) + p(t)C(t)y_h(t) = g(t)$$

$$C'(t)y_h(t) = g(t)$$

Therefore:

$$C'(t) = \frac{g(t)}{y_h(t)}$$

$$C(t) = \int \frac{g(t)}{y_h(t)} dt + K$$

The general solution is:

$$y = y_h(t) \left(\int \frac{g(t)}{y_h(t)} dt + K \right) = y_p + K y_h$$

$\square$

2.2 Connection to Integrating Factor

The variation of constants formula is equivalent to the integrating factor method:

$$\mu(t) = e^{\int p(t)dt}, \quad y_h = \frac{C}{\mu(t)}$$

$$y_p = \frac{1}{\mu(t)} \int \mu(t) g(t) dt$$

3 Solution Algorithm

Method 1 (Step-by-Step Procedure)**.**

1. ***Identify*** *the equation in standard form:* $y' + p(t)y = g(t)$
2. ***Solve*** *the homogeneous equation:* $y_h = Ce^{-\int p(t)dt}$
3. ***Set up*** *the variation:* $y = C(t) \cdot y_h(t)$
4. ***Differentiate*** *and substitute to find:* $C'(t) = \frac{g(t)}{y_h(t)}$
5. ***Integrate*** *to find* $C(t)$*:* $C(t) = \int \frac{g(t)}{y_h(t)} dt + K$
6. ***Construct*** *the general solution:* $y = C(t) \cdot y_h(t)$
7. ***Apply*** *initial conditions if given*

4 Worked Examples

Example 1 (Exponential Forcing)**.** *Solve* $y' - 3y = e^{5t}$ *with* $y(0) = 2$.

Solution:

1. *Homogeneous solution:* $y_h = Ce^{3t}$

2. *Variation setup:* $y = C(t)e^{3t}$

3. *Finding* $C'(t)$*:*
$$C'(t)e^{3t} = e^{5t} \Rightarrow C'(t) = e^{2t}$$

4. *Integrating:*
$$C(t) = \frac{1}{2}e^{2t} + K$$

5. *General solution:*
$$y = \left(\frac{1}{2}e^{2t} + K\right)e^{3t} = \frac{1}{2}e^{5t} + Ke^{3t}$$

6. *Apply initial condition:*
$$2 = \frac{1}{2} + K \Rightarrow K = \frac{3}{2}$$

7. *Final solution:*
$$y = \frac{1}{2}e^{5t} + \frac{3}{2}e^{3t}$$

Example 2 (Non-Standard Forcing)**.** *Solve* $y' + \frac{2}{t}y = t\ln(t)$ *for* $t > 0$.

Solution:

1. *Homogeneous solution:* $y_h = \frac{C}{t^2}$

2. *Variation:* $y = \frac{C(t)}{t^2}$

3. *Finding* $C'(t)$*:*
$$\frac{C'(t)}{t^2} = t\ln(t) \Rightarrow C'(t) = t^3\ln(t)$$

4. *Integration by parts:*
$$C(t) = \int t^3\ln(t)dt = \frac{t^4\ln(t)}{4} - \frac{t^4}{16} + K$$

5. *General solution:*
$$y = \frac{t^2\ln(t)}{4} - \frac{t^2}{16} + \frac{K}{t^2}$$

5 Advantages and Applications

Advantages over Undetermined Coefficients:

- Works for any continuous $g(t)$
- No need to guess solution form
- Systematic procedure always succeeds
- Extends naturally to higher-order equations

- Provides Green's function interpretation

Common Errors:

- Incorrect homogeneous solution
- Forgetting to simplify before integration
- Missing the constant of integration
- Not including full general solution
- Sign errors in exponential arguments

6 Physical Interpretation

The variation of constants represents:

- **RC Circuit:** Time-varying charge accumulation
- **Population Model:** Variable immigration/emigration
- **Heat Transfer:** Time-dependent source term
- **Mechanical System:** External forcing modulation

The integral $\int \frac{g(t)}{y_h(t)} dt$ accumulates the forcing effect, weighted inversely by the system's natural response.

7 Connection to Green's Functions

Definition 2 (Green's Function Preview)**.** *The solution can be written as:*

$$y(t) = y_h(t)y(t_0)/y_h(t_0) + \int_{t_0}^{t} G(t,s)g(s)ds$$

where $G(t,s) = y_h(t)/y_h(s)$ for $s \leq t$ is the Green's function.

8 Exam Strategy

Prof. Ditkowski's Exam Focus:

1. Always show the homogeneous solution first
2. Explicitly write $C'(t) = g(t)/y_h(t)$

3. Simplify integrands before integrating
4. State the general solution as $y = y_p + y_h$
5. Verify solution by substitution if time permits
6. Use variation for non-standard forcing functions

Key Formulas to Memorize:

$$y_h = Ce^{-\int p(t)dt} \tag{1}$$

$$C'(t) = \frac{g(t)}{y_h(t)} \tag{2}$$

$$y_p = y_h(t) \int \frac{g(t)}{y_h(t)} dt \tag{3}$$

$$y = y_p + Cy_h \tag{4}$$