Lesson 28: Practice Problems

Wronskian and Linear Independence

Part A: Basic Wronskian Calculations (6 problems)

1. Calculate the Wronskian of $y_1 = \mathrm{e}^{3t} and \mathrm{y}_2 = \mathrm{e}^{-t}$.
2. Find $W(t)$ for the solutions $\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \mathrm{e}^{2t} \\ 2\mathrm{e}^{2t} \end{bmatrix}$.
3. Compute the Wronskian of $y_1 = \cos(2t)$, $y_2 = \sin(2t)$.
4. Calculate $W[t, t^2]$ and determine linear independence.
5. Find the Wronskian of $\mathbf{x}_1 = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$.
6. Compute $W[e^t, \mathrm{te}^t] fortherepeatedrootcase$.

Part B: Abel's Identity Applications (5 problems)

1. Given $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \mathbf{x}$ with $W(0) = 2$, find $W(t)$ using Abel's identity.
2. For the system $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \mathbf{x}$, use Abel's identity to find $W(t)$ if $W(1) = \mathrm{e}^{-4}$.
3. The trace of $A(t) = \begin{bmatrix} t & 1 \\ 0 & -t \end{bmatrix}$ is zero. What does this imply about $W(t)$?
4. Find $W(2)$ if $W(0) = 3$ for solutions of $\mathbf{x}' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x}$.
5. Given $y'' + p(t)y' + q(t)y = 0$, express the Wronskian using Abel's identity.

Part C: Linear Independence Testing (5 problems)

6. Determine if $y_1 = \mathrm{e}^{2t}, \mathrm{y}_2 = \mathrm{e}^{-2t}, \mathrm{y}_3 = \cosh(2t)$ are linearly independent.

7. Test independence of $\mathbf{x}_1 = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathrm{x}_2 = e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathrm{x}_3 = e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

8. Are $y_1 = t^2$, $y_2 = t|t|$ linearly independent on $(-1, 1)$?

9. Check if $\sin t$, $\cos t$, $\sin(t + \pi/4)$ are linearly independent.

10. Determine independence of $e^t \cos t$, $e^t \sin t$ as solutions to a second-order equation.

Part D: Wronskian Theory (5 problems)

11. Prove that if $W(t_0) = 0$ for solutions of $\mathbf{x}' = A(t)\mathbf{x}$, then $W(t) = 0$ for all t.

12. Show that the Wronskian of a fundamental matrix equals its determinant.

13. If y_1 and y_2 are solutions with $W[y_1, y_2](0) = 0$ and $y_1(0) = 1$, $y_1'(0) = 2$, find $y_2(0)$ and $y_2'(0)$.

14. Prove that if the Wronskian of n functions is identically zero, and $n - 1$ of them are linearly independent, then all n cannot be solutions to the same nth-order linear homogeneous ODE.

15. Show that $W[cy_1, y_2] = c \cdot W[y_1, y_2]$ for any constant c.

Part E: Constructing Equations from Solutions (4 problems)

16. Find the second-order ODE having e^{2t} and e^{-3t} as solutions.

17. Construct the third-order equation with solutions 1, t, t^2.

18. Given solutions $e^t \cos t$ and $e^t \sin t$, find the differential equation.

19. Find the system matrix A if solutions have Wronskian $W(t) = \mathrm{e}^{5t} and \mathrm{tr}(\mathrm{A}) = 5$.

Part F: Exam-Style Problems (5 problems)

16. (Prof. Ditkowski style) For the system $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \mathbf{x}$:

    (a) Find two linearly independent solutions

(b) Calculate their Wronskian directly

(c) Verify using Abel's identity

(d) Explain why the Wronskian is never zero

17. Given that $y_1 = t^2$ and $y_2 = t^2 \ln t$ are solutions to $t^{2y"}$ - 3ty' + 4y $= 0 fort$ ¿ 0:

18. Compute $W[y_1, y_2](t)$

19. Verify they're linearly independent

20. Use Abel's identity to check your answer

21. Find the general solution

The Wronskian of three solutions satisfies $W(t) =$e^{-6t}.

Find tr(A) for the system matrix

If the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2$, find λ_3

Write a possible system matrix A

Consider solutions $\mathbf{x}_1(t) =$e$^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2(t) =$e$^{2t} \begin{bmatrix} \text{t} \\ 1 + 2t \end{bmatrix}$:

(a) Calculate $W(t)$

(b) Explain the presence of t in $\mathbf{x}_2$

(c) Find the system matrix A

(d) Verify Abel's identity

(Comprehensive) Given the differential equation $y''' - 6y'' + 11y' - 6y = 0$:

(a) Verify that e^t,e^{2t}, e^{3t} are solutions

(b) Calculate the Wronskian $W[e^t, e^{2t}, e^{3t}](t) Convert to a system and find$tr(A)

(c) Verify Abel's identity connects your answers

(d) Prove these solutions form a fundamental set

Solutions and Hints

Selected Solutions:

Problem 1: $W(t) =$e$^{3t}(-$e$^{-t})-$e$^{-t}(3$e$^{3t})= -4$e^{2t}

Problem 3: $W(t) = \cos(2t) \cdot 2\cos(2t) - \sin(2t) \cdot (-2\sin(2t)) = 2$

Problem 7: $\text{tr}(A) = 1 + 3 = 4$, so $W(t) = 2$e^{4t}

Problem 12: Dependent! $y_3 = \cosh(2t) = \frac{1}{2}(e^{2t} + e^{-2t}) = 1\frac{}{2y_1 + \frac{1}{2}y_2}$

Problem 17: If $W = 0$, then $y_2(0) = ky_1(0) = k$ and $y_2'(0) = ky_1'(0) = 2k$ for some k

Problem 21: The equation is $y'' + y' - 6y = 0$

Problem 26: $W(t) = t^2 \cdot \frac{2t \ln t + t}{t} - t^2 \ln t \cdot 2t = t^2$

Problem 30: Wronskian is $2\mathrm{e}^{6t}, and \mathrm{tr(A)} = 6 confirms Abel's identity$

Key Insights:

- Wronskian zero at one point $\Rightarrow$ zero everywhere
- Abel's identity: $W(t) = W(t_0)\mathrm{e}^{\int_{t_0} t}$ tr(A(s))ds$For constant$A:W(t) = W(0)$e^{\mathrm{tr}(A)}$ $\cdot t$
- Quick test: Different exponential rates $\Rightarrow$ independent
- Proportional solutions $\Rightarrow$ Wronskian zero