Lesson 28: Practice Problems

Wronskian and Linear Independence

Part A: Basic Wronskian Calculations (6 problems)

1. Calculate the Wronskian of $y_1 = e^{3t}$ and $y_2 = e^{-t}$.
2. Find $W(t)$ for the solutions $\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$.
3. Compute the Wronskian of $y_1 = \cos(2t)$, $y_2 = \sin(2t)$.
4. Calculate $W[t, t^2]$ and determine linear independence.
5. Find the Wronskian of $\mathbf{x}_1 = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$.
6. Compute $W[e^t, te^t]$ for the repeated root case.

Part B: Abel's Identity Applications (5 problems)

7. Given $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \mathbf{x}$ with $W(0) = 2$, find $W(t)$ using Abel's identity.
8. For the system $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \mathbf{x}$, use Abel's identity to find $W(t)$ if $W(1) = e^{-4}$.
9. The trace of $A(t) = \begin{bmatrix} t & 1 \\ 0 & -t \end{bmatrix}$ is zero. What does this imply about $W(t)$?
10. Find $W(2)$ if $W(0) = 3$ for solutions of $\mathbf{x}' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x}$.
11. Given $y'' + p(t)y' + q(t)y = 0$, express the Wronskian using Abel's identity.

Part C: Linear Independence Testing (5 problems)

12. Determine if $y_1 = e^{2t}$, $y_2 = e^{-2t}$, $y_3 = \cosh(2t)$ are linearly independent.

13. Test independence of $\mathbf{x}_1 = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{x}_3 = e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

14. Are $y_1 = t^2$, $y_2 = t|t|$ linearly independent on $(-1, 1)$?

15. Check if $\sin t$, $\cos t$, $\sin(t + \pi/4)$ are linearly independent.

16. Determine independence of $e^t \cos t$, $e^t \sin t$ as solutions to a second-order equation.

Part D: Wronskian Theory (5 problems)

17. Prove that if $W(t_0) = 0$ for solutions of $\mathbf{x}' = A(t)\mathbf{x}$, then $W(t) = 0$ for all t.

18. Show that the Wronskian of a fundamental matrix equals its determinant.

19. If y_1 and y_2 are solutions with $W[y_1, y_2](0) = 0$ and $y_1(0) = 1$, $y_1'(0) = 2$, find $y_2(0)$ and $y_2'(0)$.

20. Prove that if the Wronskian of n functions is identically zero, and $n - 1$ of them are linearly independent, then all n cannot be solutions to the same nth-order linear homogeneous ODE.

21. Show that $W[cy_1, y_2] = c \cdot W[y_1, y_2]$ for any constant c.

Part E: Constructing Equations from Solutions (4 problems)

22. Find the second-order ODE having e^{2t} and e^{-3t} as solutions.

23. Construct the third-order equation with solutions 1, t, t^2.

24. Given solutions $e^t \cos t$ and $e^t \sin t$, find the differential equation.

25. Find the system matrix A if solutions have Wronskian $W(t) = e^{5t}$ and $\mathrm{tr}(A) = 5$.

Part F: Exam-Style Problems (5 problems)

26. (Prof. Ditkowski style) For the system $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \mathbf{x}$:

    (a) Find two linearly independent solutions

(b) Calculate their Wronskian directly

(c) Verify using Abel's identity

(d) Explain why the Wronskian is never zero

27. Given that $y_1 = t^2$ and $y_2 = t^2 \ln t$ are solutions to $t^2y'' - 3ty' + 4y = 0$ for $t > 0$:

    (a) Compute $W[y_1, y_2](t)$

    (b) Verify they're linearly independent

    (c) Use Abel's identity to check your answer

    (d) Find the general solution

28. The Wronskian of three solutions satisfies $W(t) = e^{-6t}$.

    (a) Find $\text{tr}(A)$ for the system matrix

    (b) If the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2$, find λ_3

    (c) Write a possible system matrix A

29. Consider solutions $\mathbf{x}_1(t) = e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$ and $\mathbf{x}_2(t) = e^{2t}\begin{bmatrix}t\\1+2t\end{bmatrix}$:

    (a) Calculate $W(t)$

    (b) Explain the presence of t in $\mathbf{x}_2$

    (c) Find the system matrix A

    (d) Verify Abel's identity

30. (Comprehensive) Given the differential equation $y''' - 6y'' + 11y' - 6y = 0$:

    (a) Verify that e^t, e^{2t}, e^{3t} are solutions

    (b) Calculate the Wronskian $W[e^t, e^{2t}, e^{3t}](t)$

    (c) Convert to a system and find $\text{tr}(A)$

    (d) Verify Abel's identity connects your answers

    (e) Prove these solutions form a fundamental set

Solutions and Hints

Selected Solutions:

Problem 1: $W(t) = e^{3t}(-e^{-t}) - e^{-t}(3e^{3t}) = -4e^{2t}$
Problem 3: $W(t) = \cos(2t) \cdot 2\cos(2t) - \sin(2t) \cdot (-2\sin(2t)) = 2$
Problem 7: $\text{tr}(A) = 1 + 3 = 4$, so $W(t) = 2e^{4t}$
Problem 12: Dependent! $y_3 = \cosh(2t) = \frac{1}{2}(e^{2t} + e^{-2t}) = \frac{1}{2}y_1 + \frac{1}{2}y_2$
Problem 17: If $W = 0$, then $y_2(0) = ky_1(0) = k$ and $y_2'(0) = ky_1'(0) = 2k$ for some k
Problem 21: The equation is $y'' + y' - 6y = 0$

Problem 26: $W(t) = t^2 \cdot \frac{2t\ln t + t}{t} - t^2 \ln t \cdot 2t = t^2$
Problem 30: Wronskian is $2e^{6t}$, and $\text{tr}(A) = 6$ confirms Abel's identity
Key Insights:

- Wronskian zero at one point $\Rightarrow$ zero everywhere
- Abel's identity: $W(t) = W(t_0)e^{\int_{t_0}^{t} \text{tr}(A(s))ds}$
- For constant A: $W(t) = W(0)e^{\text{tr}(A)\cdot t}$
- Quick test: Different exponential rates $\Rightarrow$ independent
- Proportional solutions $\Rightarrow$ Wronskian zero