Lesson 35: Practice Problems - Duhamel's Principle

ODE 1 - Prof. Adi Ditkowski

Part A: Direct Application of Duhamel's Formula

1. Apply Duhamel's principle to solve: $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. Use Duhamel to find: $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
3. Solve using Duhamel: $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
4. Apply the formula to: $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
5. Find the solution: $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Part B: Systems with Step Functions

6. Solve: $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} H(t-2) \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
7. Find: $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ H(t-1) \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
8. Solve with piecewise forcing: $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 - H(t-1) \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
9. Compute: $\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} H(t-1) - H(t-2) \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
10. Find the response to multiple steps: $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} H(t) \\ H(t-1) \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Part C: Impulse Response Problems

11. Find the response to $\mathbf{f}(t) = \delta(t-1)\begin{pmatrix}1\\0\end{pmatrix}$ for $\mathbf{x}' = \begin{pmatrix}-1 & 0\\0 & -2\end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix}0\\1\end{pmatrix}$
12. Solve with double impulse: $\mathbf{x}' = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\mathbf{x} + \delta(t)\begin{pmatrix}1\\0\end{pmatrix} + \delta(t-\pi)\begin{pmatrix}0\\1\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}0\\0\end{pmatrix}$
13. Compute the impulse response matrix for $A = \begin{pmatrix}1 & 2\\0 & 3\end{pmatrix}$
14. Find the solution with initial impulse: $\mathbf{x}' = \begin{pmatrix}-2 & 1\\1 & -2\end{pmatrix}\mathbf{x}$, $\mathbf{x}(0^+) = \begin{pmatrix}1\\1\end{pmatrix}$ (impulse at $t = 0$)
15. Determine the long-term behavior after impulse: $\mathbf{x}' = \begin{pmatrix}-1 & 2\\0 & -3\end{pmatrix}\mathbf{x} + \delta(t-1)\begin{pmatrix}1\\1\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}0\\0\end{pmatrix}$

Part D: Complete IVPs with Various Forcings

16. Solve completely: $\mathbf{x}' = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}\mathbf{x} + \begin{pmatrix}e^{2t}\\0\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}1\\-1\end{pmatrix}$
17. Find: $\mathbf{x}' = \begin{pmatrix}0 & -2\\2 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}\cos(2t)\\\sin(2t)\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}1\\0\end{pmatrix}$ (resonance!)
18. Solve: $\mathbf{x}' = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}t\\e^{-t}\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}0\\2\end{pmatrix}$
19. Complete solution for: $\mathbf{x}' = \begin{pmatrix}-1 & 1\\0 & -1\end{pmatrix}\mathbf{x} + \begin{pmatrix}1\\1\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}1\\1\end{pmatrix}$
20. Find $\mathbf{x}(2)$ for: $\mathbf{x}' = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}e^{t}\\t\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}0\\0\end{pmatrix}$

Part E: 3$\times 3 Systems Using Duhamel$

21. Apply Duhamel to: $\mathbf{x}' = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}\mathbf{x} + \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}1\\0\\-1\end{pmatrix}$
22. Solve: $\mathbf{x}' = \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}\mathbf{x} + \begin{pmatrix}0\\0\\e^{t}\end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix}1\\0\\0\end{pmatrix}$

23. Find: $\mathbf{x}' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sin t \\ \cos t \\ 1 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

24. Compute steady-state for: $\mathbf{x}' = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

25. Solve with Jordan block: $\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Part F: Advanced Applications

26. Show that for constant forcing $\mathbf{f}(t) = \mathbf{c}$ and stable A (all eigenvalues have negative real parts), the steady-state solution is $\mathbf{x}_{ss} = -A^{-1}\mathbf{c}$.

27. Verify Duhamel gives the same result as variation of parameters for: $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{3t} \\ 0 \end{pmatrix}$

28. For periodic forcing $\mathbf{f}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ with $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, find the steady-state periodic solution.

29. Derive the response to a ramp input: $\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \cdot H(t) \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

30. **Challenge:** Prove that Duhamel's formula satisfies the differential equation and initial condition.

Solutions and Hints

Problem 1: $\mathbf{x}(t) = \begin{pmatrix} \frac{1}{2}(1 - e^{2t}) \\ \frac{1}{3}(1 - e^{3t}) \end{pmatrix}$

Problem 3: Use $e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, get $\mathbf{x}(t) = \begin{pmatrix} 1 + t^3/6 \\ t^2/2 \end{pmatrix}$

Problem 5: $e^{At} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, particular solution is $\begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}$

Problem 11: $\mathbf{x}(t) = \begin{pmatrix} H(t-1)e^{-(t-1)} \\ e^{-2t} + 0 \end{pmatrix}$

Problem 16: Repeated eigenvalue with resonance gives t^{2e2t} terms.

Problem 17: Resonance produces $t\cos(2t)$ and $t\sin(2t)$ terms.

Key Strategy: Always write Duhamel's formula first, then systematically compute e^{At}, apply to IC, set up convolution, and combine.

Verification: Differentiate your solution and verify it satisfies both the ODE and initial condition.