2D Linear Systems: Complete Classification

ODE 1 - Lesson 37

1 Classification Framework

Consider the 2D linear system:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Master Classification Parameters:

$$\text{Trace: } \tau = \text{tr}(A) = a + d \tag{1}$$
$$\text{Determinant: } \Delta = \det(A) = ad - bc \tag{2}$$
$$\text{Discriminant: } D = \tau^2 - 4\Delta \tag{3}$$

These three quantities completely determine the phase portrait!

2 Eigenvalue Analysis

The characteristic equation is:

$$\lambda$$

^{2} - $\tau\lambda + \Delta = 0$
with eigenvalues:

$$\lambda_{1,2} = \tau \pm \sqrt{\tau}$$

^{2} - $4\Delta_{\overline{2=\frac{\tau\pm\sqrt{D}}{2}}}$

Theorem 1 (Eigenvalue Classification)**.** *The nature of eigenvalues depends on the discriminant D:*

- $D > 0$*: Two distinct real eigenvalues*
- $D = 0$*: One repeated real eigenvalue*
- $D < 0$*: Two complex conjugate eigenvalues $\alpha \pm i\beta$*

3 Complete Classification Table

Type	Condition	Eigenvalues	Stability	Behavior
Saddle	$\Delta < 0$	λ_1 ¡ 0 ¡ λ_2	Unstable	Hyperbolic
Stable Node	$\Delta > 0, D > 0, \tau < 0$	λ_1, λ_2 ¡ 0	Asymp. Stable	Approach
Unstable Node	$\Delta > 0, D > 0, \tau > 0$	λ_1, λ_2 ¿ 0	Unstable	Repelling
Stable Spiral	$\Delta > 0, D < 0, \tau < 0$	$\alpha \pm i\beta, \alpha < 0$	Asymp. Stable	Spiral in
Unstable Spiral	$\Delta > 0, D < 0, \tau > 0$	$\alpha \pm i\beta, \alpha > 0$	Unstable	Spiral out
Center	$\Delta > 0, \tau = 0$	$\pm i\beta$	Neutrally Stable	Ellipses

4 Detailed Portrait Descriptions

4.1 Saddle Point

Classification 1 (Saddle)**.** ***Condition:*** $\det(A) < 0$
Eigenvalues: *Real with opposite signs*
Key Features:

- *Stable manifold:* $E^s = span\{v_1\} where \lambda_1$ ¡ 0$Unstable manifold : E^u = span\{v_2\} where \lambda_2$ ¿ 0

- *Trajectories approach along* $E^s, depart along E^u Four hyperbolic sectors$

- **Drawing a Saddle:**
  1. Draw stable eigenvector (arrows pointing in)
  2. Draw unstable eigenvector (arrows pointing out)
  3. Add hyperbolic trajectories in each sector
  4. These are the separatrices - label them!

4.2 Nodes

Classification 2 (Proper Node)**.** ***Condition:*** $\Delta > 0,\ D > 0,\ \lambda_1 \neq \lambda_2$
Behavior:

All trajectories tangent to slow eigenvector at origin

Fast eigenvector: $|\lambda_1 —$ ¿ $— \lambda_2 — determines initial direction Slow eigenvector : determines final approachd$

Classification 3 (Improper Node)**.** ***Condition:*** $D = 0$, *single eigenvector*
Special Case: $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ *(Jordan form)*
Behavior: *Trajectories curve, all tangent to eigenvector*

Classification 4 (Star Node)**.** ***Condition:*** $A = \lambda I$ *(scalar multiple of identity)*
Behavior: *Straight-line trajectories in all directions*

4.3 Spirals and Centers

Classification 5 (Spiral/Focus)**.** ***Condition:*** *$D < 0$ (complex eigenvalues)*
Eigenvalues: $\lambda = \alpha \pm i\beta$
Behavior:

- *Rotation frequency: $\omega = \beta$*
- *Decay/growth rate: α*
- *Clockwise if $\det(A) > 0$ and $b < 0$ (or $c > 0$)*
- *Counterclockwise if $\det(A) > 0$ and $b > 0$ (or $c < 0$)*

Quick Spiral Direction: Look at the off-diagonal elements of A:

- Upper-right positive $(b > 0) \rightarrow counterclockwiseUpper-rightnegative$(b ¡ 0)$\rightarrow$ *clockwise*

Classification 6 (Center)**.** ***Condition:*** *$\tau = 0$, $\Delta > 0$*
Eigenvalues: *Pure imaginary $\pm i\beta$*
Behavior: *Closed elliptical orbits, period $T = 2\pi/\beta$*

5 Degenerate Cases

- **When** $\det(A) = 0$**:**
  - One zero eigenvalue: Line of equilibria
  - Two zero eigenvalues: Every point is an equilibrium
  - Parallel trajectories if one eigenvalue is nonzero

6 The Trace-Determinant Diagram

7 Classification Algorithm

Prof. Ditkowski's Quick Classification:

1. Compute $\det(A)$
    - If $\det(A) < 0 \to$ **SADDLE**$(done!) If \det(A) = 0 \to Degenerate case$
- If $\det(A) > 0 \to$ *Continue...*
2. Compute $\tau = \mathrm{tr}(A)$
- If $\tau = 0 \to$ **CENTER**$If \tau \neq 0 \to$ *Continue...*
3. Compute $D = \tau^2$ - $4\det(A)$
    - If $D > 0 \to$ **NODE**$(stable if \tau < 0)$
    - If $D < 0 \to$ **SPIRAL**$(stable if \tau < 0)$
- If $D = 0 \to$ **IMPROPER/STAR NODE**

8 Examples Gallery

Example 1 (Complete Classification)**.** *Classify* $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$*:*

$$\det(A) = 4 - 9 = -5 < 0 \tag{4}$$

Therefore: ***SADDLE*** *(unstable)*

Verification: $\lambda = \frac{4 \pm \sqrt{16+20}}{2} = \frac{4 \pm 6}{2} = 5, -1$ ✓

Example 2 (Using Trace-Det Only)**.** *Classify* $A = \begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}$*:*

$$\tau = -6 < 0 \quad \textit{(stable)} \tag{5}$$
$$\Delta = 9 + 4 = 13 > 0 \quad \textit{(not a saddle)} \tag{6}$$
$$D = 36 - 52 = -16 < 0 \quad \textit{(complex eigenvalues)} \tag{7}$$

Therefore: ***STABLE SPIRAL***