Lesson 34: Practice Problems - Variation of Parameters for Systems

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Part A: Finding Fundamental Matrices

1. Find the fundamental matrix for $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$

2. Compute $\Phi(t)$ and $\Phi^{-1}(t)$ for $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x}$

3. Find the fundamental matrix for $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}$

4. Given $\Phi(t) = \begin{pmatrix} e^t & \mathrm{e}^{2t} \\ e^t & 2\mathrm{e}^{2t} \end{pmatrix}$, find $\Phi^{-1}(t)$

5. Verify that $\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ is a fundamental matrix and find its inverse.

Part B: Basic Variation of Parameters

6. Solve $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathrm{e}^{3t} \\ 0 \end{pmatrix}$

7. Find a particular solution: $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ t \end{pmatrix}$

8. Solve $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

9. Find $\mathbf{x}_p for$x' $= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$x + $\begin{pmatrix} e^t \\ \mathrm{e}^{2t} \end{pmatrix}$

10. Solve $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \\ 0 \end{pmatrix}$

Part C: Systems with Trigonometric Forcing

11. Solve $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \sin t \\ 0 \end{pmatrix}$

12. Find a particular solution: $\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}$

13. Solve $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \cos t \\ 0 \end{pmatrix}$

14. Find the general solution: $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ \sin(2t) \end{pmatrix}$ (resonance!)

15. Solve $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathrm{e}^{-t} \cos(2t) \\ \mathrm{e}^{-t} \sin(2t) \end{pmatrix}$

Part D: Initial Value Problems

16. Solve the IVP: $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathrm{e}^{2t} \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

17. Find the solution: $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \\ t \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

18. Solve: $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

19. Find $\mathbf{x}(1)$ for: $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{2t} \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

20. Solve the IVP: $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Part E: $\mathbf{3} \times 3 Systems$

21. Find a particular solution: $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t \\ \mathrm{e}^{2t} \\ \mathrm{e}^{3t} \end{pmatrix}$

22. Solve: $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$

23. Find $\mathbf{x}_p for$:x' $= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ x + $\begin{pmatrix} \mathrm{e}^{2t} \\ 0 \\ 0 \end{pmatrix}$

24. Solve: $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ \cos t \\ \sin t \end{pmatrix}$

25. Find the general solution: $\mathbf{x}' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Part F: Special Methods and Applications

26. Use undetermined coefficients to solve: $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \mathrm{e}^{3t} \\ \mathrm{e}^{4t} \end{pmatrix}$

27. For the system $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ F\cos(\omega t) \end{pmatrix}$, find the resonant solution.

28. A coupled tank system: $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 \\ 5 \end{pmatrix}$ where $\mathbf{x}$ represents salt concentrations. Find the steady-state solution.

29. Verify that variation of parameters gives the correct particular solution for: $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

30. **Challenge:** Show that if $\mathbf{f}(t) = \mathbf{f}_0$ is constant and A is invertible, then $\mathbf{x}_p$ = -A$^{-1}\mathbf{f}_0$ is a particular solution.

Solutions and Hints

Problem 1: $\Phi(t)$ =e$^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

Problem 6: $\mathbf{x}_p = \begin{pmatrix} \frac{1}{2}\mathrm{e}^{3t} \\ 0 \end{pmatrix}$

Problem 7: Use $\Phi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, get $\mathbf{x}_p = \begin{pmatrix} t^3/6 \\ t^2/2 \end{pmatrix}$

Problem 11: Resonance occurs! Solution involves $t\cos t$ and $t\sin t$ terms.

Problem 16: Particular solution has $t\mathrm{e}^{2t} terms due to repeated eigenvalue.$

Problem 26: Try $\mathbf{x}_p = \begin{pmatrix} a\mathrm{e}^{3t} \\ b\mathrm{e}^{4t} \end{pmatrix} and solve for$a, b.

Problem 28: Steady-state: $\mathbf{x}_p = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

Key Strategy: Always find the fundamental matrix first. For constant coefficient systems, this is e^{At}. Then apply the variation formula systematically.

Verification: Always check that $\mathbf{x}_p$' = Ax$_p$ + f(t).