ODE Lesson 31: Repeated Eigenvalues and Jordan Forms

ODE 1 - Prof. Adi Ditkowski

1 The Defective Eigenvalue Problem

Definition 1 (Algebraic and Geometric Multiplicity). *For an eigenvalue λ:*

- ***Algebraic multiplicity***: *The multiplicity of λ as a root of the characteristic polynomial*
- ***Geometric multiplicity***: *The dimension of the eigenspace* $\ker(A - \lambda I)$
- ***Defect***: *Algebraic multiplicity minus geometric multiplicity*

Definition 2 (Generalized Eigenvector). *A **generalized eigenvector** of order k associated with eigenvalue λ is a vector $\mathbf{v}$ satisfying:*

$$(A - \lambda I)^k \mathbf{v} = \mathbf{0} \quad \textit{but} \quad (A - \lambda I)^{k-1} \mathbf{v} \neq \mathbf{0}$$

Jordan Block Structure: A Jordan block of size n for eigenvalue λ is:

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

Algorithm 1 (Jordan Chain Construction). *To find a Jordan chain for repeated eigenvalue λ:*

1. *Start with an eigenvector:* $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}$
2. *Find generalized eigenvector:* $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$
3. *Continue the chain:* $(A - \lambda I)\mathbf{v}_{k+1} = \mathbf{v}_k$
4. *Stop when you have enough vectors to match algebraic multiplicity*

Theorem 1 (Solution Form for Repeated Eigenvalues)**.** *For eigenvalue λ with a Jordan chain $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$:*

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 \tag{1}$$

$$\mathbf{x}_2(t) = e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2) \tag{2}$$

$$\mathbf{x}_3(t) = e^{\lambda t}\left(\frac{t^2}{2!}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right) \tag{3}$$

$$\vdots \tag{4}$$

$$\mathbf{x}_k(t) = e^{\lambda t}\sum_{j=0}^{k-1}\frac{t^j}{j!}\mathbf{v}_{k-j} \tag{5}$$

Why Polynomial Terms Appear:
The polynomial terms arise from the nilpotent part $(A-\lambda I)$. When we write $A = \lambda I + N$ where $N = A - \lambda I$ is nilpotent:

$$e^{At} = e^{\lambda t}e^{Nt} = e^{\lambda t}\left(I + Nt + \frac{N^2t^2}{2!} + \cdots\right)$$

The series terminates because $N^k = 0$ for some k!

2 Complete Examples

Example 1 (2×2 System with Double Eigenvalue)**.** *Solve* $\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}\mathbf{x}$ *with* $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Solution:

1. ***Find eigenvalues:*** $\det(A - \lambda I) = 0$ *gives* $\lambda = 3$ *(double)*

2. ***Find eigenvector:*** $(A - 3I)\mathbf{v}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\mathbf{v}_1 = \mathbf{0}$ *This gives* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

3. ***Find generalized eigenvector:*** *Solve* $(A - 3I)\mathbf{w} = \mathbf{v}_1$*:*

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

From the first equation: $w_2 = 1$*. From the second:* $-w_1 + 2(1) = 0$*, so* $w_1 = 2$*. Therefore* $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

4. ***General Solution:***

$$\mathbf{x}(t) = c_1e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{3t}\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

5. ***Apply Initial Conditions:*** $\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   This gives: $c_1 + 2c_2 = 2$ *and* $c_2 = 1$ *Therefore* $c_1 = 0$ *and* $c_2 = 1$

6. ***Final Solution:***

$$\mathbf{x}(t) = e^{3t} \left(t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = e^{3t} \begin{pmatrix} t+2 \\ 1 \end{pmatrix}$$

Critical Mistakes to Avoid:

- Do NOT solve $(A - \lambda I)^2 \mathbf{v} = \mathbf{0}$ directly for generalized eigenvectors
- The order matters in Jordan chains: always solve sequentially
- Check that $(A - \lambda I)\mathbf{v}_{k+1} = \mathbf{v}_k$ is satisfied
- The coefficient of t^k is $1/k!$, not just t^k

Physical Interpretation: The polynomial terms in t arise from the system's failure to diagonalize completely. The Jordan form captures the "almost diagonal" structure, and the polynomial terms represent the coupling between modes that can't be completely separated.

Prof. Ditkowski's exams typically include:

- One 2×2 system with repeated eigenvalues
- Finding both eigenvectors and generalized eigenvectors
- Initial value problems (most common)
- Questions about Jordan canonical form
- Verification of solutions