Lesson 43: Characteristic Equation Method - All Cases

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1 Introduction

The characteristic equation method is the fundamental technique for solving linear ordinary differential equations with constant coefficients. This method transforms the differential equation into an algebraic problem, making it both computationally efficient and theoretically elegant.

Definition 1 (Linear ODE with Constant Coefficients). *An n-th order linear homogeneous ODE with constant coefficients has the form:*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ (or $\mathbb{C}$) are constants with $a_n \neq 0$.

2 The Characteristic Equation

Theorem 1 (Exponential Ansatz). *For the differential equation*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

if we assume a solution of the form $y = e^{rt}$ where $r \in \mathbb{C}$, then r must satisfy the characteristic equation:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Proof. Substituting $y = e^{rt}$ into the differential equation:

$$y = e^{rt} \tag{1}$$
$$y' = r e^{rt} \tag{2}$$
$$y'' = r^2 e^{rt} \tag{3}$$
$$\vdots \tag{4}$$
$$y^{(k)} = r^k e^{rt} \tag{5}$$

Therefore:

$$a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} = 0$$

Since $e^{rt} \neq 0$ for all t, we can divide by e^{rt}:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

$\square$

> The characteristic equation is a polynomial equation of degree n. By the Fundamental Theorem of Algebra, it has exactly n roots (counting multiplicities) in $\mathbb{C}$.

3 Solution Construction Based on Root Types

3.1 Case 1: Distinct Real Roots

Theorem 2 (Distinct Real Roots). *If the characteristic equation has n distinct real roots $r_1, r_2, \ldots, r_n$, then the general solution is:*

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

> The functions $\{e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}\}$ form a fundamental set of solutions. Their Wronskian is:
> $$W(t) = \prod_{i<j}(r_j - r_i) \cdot e^{(r_1 + r_2 + \cdots + r_n)t} \neq 0$$
> proving linear independence.

3.2 Case 2: Complex Conjugate Roots

Theorem 3 (Complex Roots for Real Equations). *If the differential equation has real coefficients and $r = \alpha + i\beta$ is a complex root, then $\bar{r} = \alpha - i\beta$ is also a root. The corresponding real-valued solutions are:*

$$y_1(t) = e^{\alpha t} \cos(\beta t), \quad y_2(t) = e^{\alpha t} \sin(\beta t)$$

Proof. From the complex solutions $e^{(\alpha + i\beta)t}$ and $e^{(\alpha - i\beta)t}$, we use Euler's formula:

$$e^{(\alpha \pm i\beta)t} = e^{\alpha t} e^{\pm i\beta t} = e^{\alpha t}[\cos(\beta t) \pm i\sin(\beta t)]$$

Taking linear combinations:

$$y_1 = \frac{1}{2}[e^{(\alpha + i\beta)t} + e^{(\alpha - i\beta)t}] = e^{\alpha t}\cos(\beta t) \tag{6}$$

$$y_2 = \frac{1}{2i}[e^{(\alpha + i\beta)t} - e^{(\alpha - i\beta)t}] = e^{\alpha t}\sin(\beta t) \tag{7}$$

$\square$

3.3 Case 3: Repeated Roots (Overview)

Theorem 4 (Repeated Roots). *If r is a root of multiplicity k, then the k linearly independent solutions are:*

$$e^{rt}, \, te^{rt}, \, t^2 e^{rt}, \, \ldots, \, t^{k-1} e^{rt}$$

The full proof of the repeated roots case requires reduction of order or the theory of generalized eigenvectors. We will cover this in detail in Lesson 44.

4 Complete Solution Algorithm

Method 1 (Characteristic Equation Method). *1. Write the characteristic equation by replacing $y^{(k)}$ with r^k.*

2. *Find all roots of the characteristic polynomial.*

3. *For each root, construct the corresponding solutions:*

   - *Real root r of multiplicity 1: e^{rt}*
   - *Real root r of multiplicity m: $e^{rt}, te^{rt}, \ldots, t^{m-1} e^{rt}$*
   - *Complex conjugate pair $\alpha \pm i\beta$ of multiplicity 1: $e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)$*
   - *Complex conjugate pair $\alpha \pm i\beta$ of multiplicity m:*

   $$e^{\alpha t} \cos(\beta t), te^{\alpha t} \cos(\beta t), \ldots, t^{m-1} e^{\alpha t} \cos(\beta t)$$

   $$e^{\alpha t} \sin(\beta t), te^{\alpha t} \sin(\beta t), \ldots, t^{m-1} e^{\alpha t} \sin(\beta t)$$

4. *The general solution is the linear combination of all these solutions.*

5 Worked Examples

Example 1 (Second-Order with Distinct Real Roots). *Solve: $y'' - 5y' + 6y = 0$*
   Solution:

1. *Characteristic equation: $r^2 - 5r + 6 = 0$*

2. *Factor: $(r-2)(r-3) = 0$*

3. *Roots: $r_1 = 2, r_2 = 3$*

4. *General solution: $y(t) = c_1 e^{2t} + c_2 e^{3t}$*

Example 2 (Second-Order with Complex Roots). *Solve: $y'' + 2y' + 5y = 0$*
   Solution:

1. *Characteristic equation: $r^2 + 2r + 5 = 0$*

2. *Using quadratic formula:* $r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

3. *Roots:* $r_1 = -1 + 2i, r_2 = -1 - 2i$

4. *General solution:* $y(t) = e^{-t}[c_1 \cos(2t) + c_2 \sin(2t)]$

Example 3 (Third-Order Mixed Roots)**.** *Solve:* $y''' - y'' - y' + y = 0$
   Solution:

1. *Characteristic equation:* $r^3 - r^2 - r + 1 = 0$

2. *Factor by grouping:* $r^2(r-1) - (r-1) = (r^2 - 1)(r-1) = (r-1)^2(r+1) = 0$

3. *Roots:* $r_1 = 1$ *(multiplicity 2),* $r_2 = -1$

4. *General solution:* $y(t) = (c_1 + c_2 t)e^t + c_3 e^{-t}$

6   Connection to Linear Systems

Consider the second-order equation $y'' + py' + qy = 0$. Converting to a system with $x_1 = y$ and $x_2 = y'$:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic polynomial of this companion matrix is:

$$\det \begin{pmatrix} -\lambda & 1 \\ -q & -p - \lambda \end{pmatrix} = \lambda^2 + p\lambda + q$$

This is exactly our characteristic equation! The roots are the eigenvalues of the companion matrix.

7   Initial Value Problems

Method 2 (Solving IVPs)**.** *For an n-th order IVP with initial conditions* $y(t_0) = y_0, y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}$:

1. *Find the general solution using the characteristic equation method.*

2. *Apply the initial conditions to get n linear equations in the n unknown constants.*

3. *Solve the linear system for the constants.*

Prof. Ditkowski often gives initial conditions at $t = 0$. This simplifies calculations since $e^0 = 1$, $\cos(0) = 1$, and $\sin(0) = 0$.

8 Common Errors to Avoid

1. **Sign errors:** The signs in the characteristic equation match those in the differential equation.

2. **Complex roots:** Always convert to real form for real differential equations.

3. **Multiplicities:** Count multiplicities carefully - $(r-2)^3 = 0$ means multiplicity 3, not 1.

4. **Quadratic formula:** Check discriminant sign to determine root type.

5. **Initial conditions:** Don't forget to differentiate when applying $y'(t_0) = y_1$.

9 Solution Forms Summary Table

Root Type	Multiplicity	Solution Terms
Real r	1	e^{rt}
Real r	k	$e^{rt}, te^{rt}, \ldots, t^{k-1}e^{rt}$
Complex $\alpha \pm i\beta$	1	$e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)$
Complex $\alpha \pm i\beta$	k	$t^j e^{\alpha t}\cos(\beta t), t^j e^{\alpha t}\sin(\beta t)$ for $j = 0, 1, \ldots, k-1$