Lesson 24: Practice Problems - Special Integrating Factors

ODE 1 - Prof. Adi Ditkowski

Part A: Testing for $\mu(xy)$ (Problems 1-5)

1. For $(y^2 + 2xy)dx + (x^2 - xy)dy = 0$:
   (a) Verify that neither $\mu(x)$ nor $\mu(y)$ exists
   (b) Test for $\mu(xy)$
   (c) Find the integrating factor if it exists
2. Test for $\mu(xy)$: $(3xy + y^2)dx + (3xy + x^2)dy = 0$
3. Determine if $\mu(xy)$ exists: $(2y + xy^2)dx + (2x - x^2y)dy = 0$
4. For $(y + x^2y^2)dx + (x + x^2y^2)dy = 0$, find $\mu(xy)$ if it exists.
5. Show that $(xy^2 - y)dx + (x^2y - x)dy = 0$ has $\mu = 1/(xy)$

Part B: Testing for $\mu(x^2 + y^2)$ (Problems 6-10)

6. Test for $\mu(x^2 + y^2)$: $(x^2 + y^2 + x)dx + ydy = 0$
7. For $(x + y)dx + (x - y)dy = 0$ on the punctured plane:
   (a) Show standard integrating factors don't exist
   (b) Find $\mu(x^2 + y^2)$
8. Verify that $\mu = 1/(x^2 + y^2)$ works for: $-ydx + xdy = 0$
9. Find $\mu(x^2 + y^2)$ for: $(x^3 + xy^2 + y)dx + (x^2y + y^3 - x)dy = 0$
10. Test whether $(2xy)dx + (x^2 + y^2)dy = 0$ has an integrating factor of the form $\mu(x^2 + y^2)$.

Part C: Power Form $\mu = x^a y^b$ (Problems 11-15)

11. Find values of a and b such that $\mu = x^a y^b$ is an integrating factor for: $ydx + 2xdy = 0$
12. Find $\mu = x^a y^b$ for: $(2y^2 + xy)dx + xydy = 0$
13. Determine all integrating factors of the form $x^a y^b$ for: $3ydx + 2xdy = 0$
14. For what values of a, b is $x^a y^b$ an integrating factor for: $(x^2 + y^2)dx + 2xydy = 0$?
15. Given that $(3x^2 y + ay^3)dx + (x^3 + 3bxy^2)dy = 0$ has an integrating factor $\mu = x^2 y$, find a and b.

Part D: Homogeneous Equations (Problems 16-20)

16. Verify homogeneity and find the integrating factor: $(x^2 + y^2)dx + 2xydy = 0$
17. Use the homogeneous property to solve: $(x^2 - 2y^2)dx + xydy = 0$
18. Show that $(x^3 + xy^2)dx + (x^2 y + y^3)dy = 0$ is homogeneous and find μ.
19. For the homogeneous equation $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$:
    (a) Find $\mu = 1/(xM + yN)$
    (b) Solve the equation
20. Explain why $\mu = 1/(xM + yN)$ works for all homogeneous equations of the same degree.

Part E: Mixed Special Forms (Problems 21-25)

21. For $(y^2 - xy + 1)dx + (xy - x^2 + 1)dy = 0$:
    (a) Test all standard forms ($\mu(x)$, $\mu(y)$, $\mu(xy)$)
    (b) Find an appropriate integrating factor
22. The equation $(f(y) + xy)dx + (g(x) + xy)dy = 0$ has $\mu(xy)$. What can you conclude about f and g?
23. Find the most appropriate integrating factor for: $(x^2 y^3 + y)dx + (x^3 y^2 + x)dy = 0$
24. Given $(P(x)y + Q(y))dx + (R(x) + S(y)x)dy = 0$ has $\mu = x^a y^b$, find relationships among P, Q, R, S.
25. For equations with integrating factor $\mu = e^{xy}$, what condition must $(M_y - N_x)/(xN - yM)$ satisfy?

Part F: Exam-Style Problems (Problems 26-30)

26. (Prof. Ditkowski 2022) Show that $(2xy^3 + y)dx + (3x^2y^2 + x)dy = 0$ becomes exact when multiplied by $\mu = xy$.
27. Find all integrating factors of the form $\mu = f(xy)$ for: $ydx - xdy = 0$
28. Given that an equation has both $\mu_1 = x$ and $\mu_2 = y$ as integrating factors:
    (a) What can you conclude about the equation?
    (b) Give an example of such an equation
29. For $(ax^2 + bxy + cy^2)dx + (dx^2 + exy + fy^2)dy = 0$ to have $\mu = 1/(x^2 + y^2)$, find the relationship between the coefficients.
30. A differential equation has the form $Mdx + Ndy = 0$ where M and N are polynomials of degree 3.
    (a) Under what conditions will it have $\mu = x^a y^b$?
    (b) Can it have $\mu(xy)$? Explain.
    (c) What about $\mu = 1/(xM + yN)$?

Solutions and Key Insights

Problem 1: (a) $\mu(x)$: $(M_y - N_x)/N = (2y + 2x - 2x + y)/(x^2 - xy) = 3y/(x^2 - xy)$ - has y, no $\mu(x)$ $\mu(y)$: $(N_x - M_y)/M = (2x - y - 2y - 2x)/(y^2 + 2xy) = -3y/(y^2 + 2xy) = -3/(y + 2x)$ - has x, no $\mu(y)$

(b) $\mu(xy)$: $(M_y - N_x)/(xN - yM) = 3y/(x(x^2 - xy) - y(y^2 + 2xy)) = 3y/(x^3 - x^2y - y^3 - 2xy^2)$

Problem 8: $-ydx + xdy = 0$ has $M_y = -1$, $N_x = 1$, so $(M_y - N_x) = -2$ $xM + yN = x(-y) + y(x) = 0$ - Division by zero issue! But note: $d(\arctan(y/x)) = \frac{xdy - ydx}{x^2+y^2}$, so $\mu = 1/(x^2 + y^2)$ works!

Problem 11: $ydx + 2xdy = 0$ with $\mu = x^a y^b$ gives: $x^a y^{b+1}dx + 2x^{a+1}y^b dy = 0$ Exactness: $(b + 1)x^a y^b = 2(a + 1)x^a y^b$ So $b + 1 = 2(a + 1)$, giving $b = 2a + 1$ Family of solutions: $\mu = x^a y^{2a+1}$ for any a

Problem 16: $(x^2 + y^2)dx + 2xydy = 0$ Both M and N are homogeneous of degree 2. $xM + yN = x(x^2 + y^2) + y(2xy) = x^3 + xy^2 + 2xy^2 = x^3 + 3xy^2$ So $\mu = 1/(x^3 + 3xy^2)$

Key Strategy: For Problem 23, notice the symmetry in x and y - this suggests $\mu = xy$ or $\mu = 1/(xy)$.

Warning: Problem 28 - If two different integrating factors exist, their ratio must be a function that makes the original equation exact when used as an integrating factor.