ODE Lesson 10: Qualitative Analysis Without Solving

ODE 1 - Prof. Adi Ditkowski

1 Phase Line Analysis

Definition 1 (Phase Line). *For an autonomous ODE $\frac{dy}{dx} = f(y)$, the **phase line** is a one-dimensional representation showing equilibria and flow direction along the y-axis.*

Method 1 (Constructing Phase Lines).

1. *Find equilibria: solve $f(y) = 0$*
2. *Determine sign of $f(y)$ between equilibria*
3. *Draw arrows: up if $f(y) > 0$, down if $f(y) < 0$*
4. *Classify stability based on arrow directions*

2 Stability Analysis

Theorem 1 (Linear Stability Test). *For $\frac{dy}{dx} = f(y)$ with equilibrium y^*:*

- *If $f'(y^*) < 0$: asymptotically stable*
- *If $f'(y^*) > 0$: unstable*
- *If $f'(y^*) = 0$: test inconclusive (need higher derivatives)*

Example 1 (Logistic Equation). *For $\frac{dy}{dx} = ry(1 - y/K)$ with $r, K > 0$:*

- *Equilibria: $y = 0$ and $y = K$*
- *$f'(y) = r(1 - 2y/K)$*
- *At $y = 0$: $f'(0) = r > 0$ (unstable)*
- *At $y = K$: $f'(K) = -r < 0$ (stable)*

3 Comparison Theorems

Theorem 2 (Basic Comparison Principle). *If $y_1(x)$ and $y_2(x)$ are solutions to $\frac{dy}{dx} = f(x, y)$ where f satisfies uniqueness conditions, and $y_1(x_0) < y_2(x_0)$, then $y_1(x) < y_2(x)$ for all x in the interval of existence.*

Solutions cannot cross! This provides powerful bounds on solution behavior.

4 Monotonicity and Concavity

Method 2 (Analyzing Solution Shape). *For $\frac{dy}{dx} = f(x, y)$:*

1. ***Monotonicity**: Sign of $f(x, y)$ determines if y increases or decreases*
2. ***Concavity**: Compute $\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot f$*
3. *Positive second derivative $\Rightarrow$ concave up*
4. *Negative second derivative $\Rightarrow$ concave down*

5 Energy Methods and First Integrals

Definition 2 (First Integral). *A function $H(x, y)$ is a **first integral** of $\frac{dy}{dx} = f(x, y)$ if H is constant along solution curves.*

Theorem 3 (Exactness Criterion). *The ODE $M(x, y)dx + N(x, y)dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. When exact, solutions are level curves of $H(x, y)$ where $\frac{\partial H}{\partial x} = M$ and $\frac{\partial H}{\partial y} = N$.*

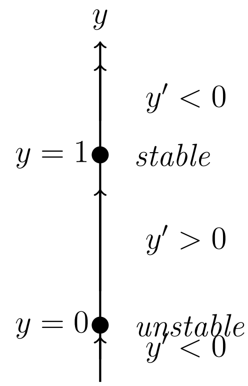
Energy methods reveal hidden conservation laws and geometric structure!

6 Asymptotic Behavior

Method 3 (Determining Long-term Behavior).

1. *Identify all equilibria*
2. *Analyze stability of each equilibrium*
3. *Look for invariant regions (where solutions cannot escape)*
4. *Apply comparison theorems for bounds*
5. *Check for periodic orbits (closed trajectories)*

7 Phase Line Examples



Example 2 (Phase Line for $y' = y(1 - y)$).

8 Periodic Solutions

Theorem 4 (Conditions for Periodicity).

- *Autonomous 1D equations cannot have periodic solutions*
- *For 2D systems, look for:*
  - *Centers in linear analysis*
  - *Conserved quantities (Hamiltonian systems)*
  - *Application of Poincaré-Bendixson theorem*

9 Exam Strategy

Prof. Ditkowski's favorite qualitative analysis questions:

1. Sketch phase line and determine all asymptotic behaviors
2. Prove solution boundedness without solving
3. Use comparison to relate unknown solutions to known ones
4. Find conserved quantities for given ODEs
5. Determine existence/non-existence of periodic solutions

10 Memory Aids

QUALITATIVE Analysis Steps:
Quick equilibrium check
Use stability tests
Analyze monotonicity

Look for conservation
Identify invariant regions
Track asymptotic behavior
Apply comparison theorems
Test for periodicity
Interpret physically
Verify with phase line
Examine concavity