Practice Problems: Lesson 8 - Parameter-Dependent Problems

Master parametric analysis for the exam!

Part A: Basic Parameter Analysis

For each parametric ODE, identify critical parameter values:

1. $y' = \mu y + 1$, $y(0) = 0$
2. $y' = y^2 - \mu^2$, $y(0) = 0$
3. $y' = \frac{\mu y}{1+x^2}$, $y(0) = 1$
4. $y' = |y|^\mu$, $y(0) = 0$
5. $y' = \sin(\mu y)$, $y(0) = \pi/2$

Part B: Existence and Uniqueness

6. For $y' = \frac{y}{x+\mu}$, $y(0) = 1$:
   (a) For which μ does no solution exist?
   (b) For which μ does the solution blow up?
   (c) Find the general solution when it exists
7. Consider $y' = \mu\sqrt{|y|}$, $y(0) = 0$:
   (a) For which μ is the solution unique?
   (b) Describe all solutions when $\mu \neq 0$
   (c) What happens as $|\mu| \to \infty$?
8. For $y' = y^2 + \mu$:
   (a) Find all equilibria as functions of μ
   (b) For which μ do real equilibria exist?
   (c) When do all solutions blow up in finite time?

Part C: Bifurcation Analysis

9. Analyze the saddle-node bifurcation: $y' = \mu - y^2$
   (a) Find equilibria for all μ
   (b) Determine stability of each equilibrium
   (c) Sketch the bifurcation diagram
   (d) What happens at $\mu = 0$?

10. For the transcritical bifurcation: $y' = \mu y - y^2$
    (a) Find all equilibria
    (b) Determine where equilibria exchange stability
    (c) Sketch solutions for $\mu = -1, 0, 1$

11. Study the pitchfork: $y' = \mu y - y^3 + \epsilon y^2$
    (a) What happens when $\epsilon = 0$?
    (b) How does $\epsilon \neq 0$ break the symmetry?
    (c) Find the bifurcation point(s)

Part D: Global Existence

12. For $y' = -y^3 + \mu y$, determine:
    (a) All values of μ for which every solution exists globally
    (b) Values where some solutions blow up
    (c) The critical value of μ

13. Consider $y' = e^{\mu y} - 1$:
    (a) For which μ does an equilibrium exist?
    (b) When do solutions exist globally?
    (c) Find the equilibrium when it exists

14. For the Bernoulli equation $y' + y = \mu y^2$:
    (a) Transform to linear form
    (b) Find conditions on μ for global existence
    (c) What happens as $\mu \to 0$?

Part E: Riccati with Parameters

15. Analyze $y' = y^2 - 2\mu y + \mu^2 - 1$:
    (a) Simplify using substitution $z = y - \mu$
    (b) Find equilibria
    (c) Determine blow-up conditions
16. For $y' = \mu y^2 + y + 1$:
    (a) When does this have real equilibria?
    (b) Find the discriminant condition
    (c) For which μ do all solutions blow up?

Part F: Asymptotic Behavior

17. For $y' = y(1 - y/\mu)$ with $\mu > 0$ (logistic):
    (a) Find equilibria
    (b) Determine $\lim_{t\to\infty} y(t)$ for various $y(0)$
    (c) What happens as $\mu \to \infty$?
    (d) What happens as $\mu \to 0^+$?
18. Consider $y' = -y + \mu \sin(y)$:
    (a) For small $|\mu|$, how many equilibria exist?
    (b) Find the critical value where new equilibria appear
    (c) Describe the bifurcation type

Part G: Singular Perturbations

19. For $\epsilon y' = -y + \mu$ with $y(0) = y_0$:
    (a) Find the exact solution
    (b) What happens as $\epsilon \to 0^+$?
    (c) Identify the fast time scale
    (d) Sketch solutions for $\epsilon = 1, 0.1, 0.01$
20. Consider $\epsilon y'' + y' + y = \mu$:
    (a) What's the reduced problem when $\epsilon = 0$?
    (b) Find boundary layers if they exist
    (c) Compare orders of the full and reduced problems

Part H: Parameter Identification

21. Given observations that a solution to $y' = \mu y - y^3$ starts at $y(0) = 0.1$ and approaches 2 as $t \to \infty$:
    (a) What can you deduce about μ?
    (b) Find the exact value of μ
    (c) Are there other equilibria?
22. You know that $y' = y^2 + ay + b$ has exactly one equilibrium at $y = -1$:
    (a) Find the relationship between a and b
    (b) What's the discriminant condition?
    (c) Give specific values of a and b

Part I: Exam-Style Problems

23. Professor Ditkowski asks: "For the equation $y' = \mu \sin(y) - y$, determine all μ such that there exists a non-zero equilibrium."
    (a) Set up the equilibrium equation
    (b) Analyze graphically
    (c) Find the critical value of μ
    (d) How many equilibria exist for various μ?
24. Consider the competition model: $y' = y(1 - y)(\mu - y)$
    (a) Find all equilibria for arbitrary μ
    (b) Classify stability for each equilibrium
    (c) Sketch the phase line for $\mu = -1, 0.5, 2$
    (d) Identify all bifurcation points
25. For the delay equation approximation $y'(t) = \mu[y(t-1) - y(t)]$, approximate as $y' = -\mu(y - y_0 e^{-\mu})$:
    (a) Find equilibria as functions of μ
    (b) When do oscillations appear?
    (c) Analyze stability for small μ
    (d) What happens as $\mu \to \infty$?

Part J: Advanced Theory

26. Prove that for $y' = f(y;\mu)$ where f is smooth, if $\frac{\partial f}{\partial y}(y_0;\mu_0) = 0$ and $\frac{\partial^2 f}{\partial y^2}(y_0;\mu_0) \neq 0$, then a saddle-node bifurcation occurs.

27. Show that the solution to $y' = \mu y + g(x)$ depends continuously on μ for any continuous g.

28. For the general Riccati $y' = p(x;\mu) + q(x;\mu)y + r(x;\mu)y^2$:

    (a) When is global existence guaranteed?

    (b) How do zeros of $r(x;\mu)$ affect the solution?

    (c) Give conditions on μ for bounded solutions

Key Insights for Solutions

Remember:

- Critical parameters occur where: equilibria appear/vanish, Lipschitz fails, singularities arise
- Bifurcations change solution structure
- μ in denominator $\Rightarrow$ watch for singularities
- μ in exponent $\Rightarrow$ check Lipschitz
- Always sketch bifurcation diagrams!