

# Lesson 47: Euler-Cauchy Equations

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## 1 Introduction

**Definition 1** (Euler-Cauchy Equation). *An  $n$ -th order Euler-Cauchy equation (also called an equidimensional equation) has the form:*

$$t^n y^{(n)} + a_{n-1} t^{n-1} y^{(n-1)} + \cdots + a_1 t y' + a_0 y = 0$$

where  $a_0, a_1, \dots, a_{n-1}$  are constants and  $t \neq 0$ .

The defining characteristic: the power of  $t$  equals the order of the derivative in each term. This creates a scaling symmetry that makes these equations solvable by elementary methods.

## 2 Method 1: Power Function Ansatz

**Theorem 1** (Power Solutions). *For the Euler equation, solutions have the form  $y = t^r$  where  $r$  satisfies an algebraic characteristic equation.*

### 2.1 Second-Order Case

Consider the second-order Euler equation:

$$t^2 y'' + aty' + by = 0, \quad t > 0$$

**Method 1** (Direct Solution Method). 1. Assume  $y = t^r$  for some constant  $r$

2. Compute derivatives:

$$y = t^r \tag{1}$$

$$y' = r t^{r-1} \tag{2}$$

$$y'' = r(r-1) t^{r-2} \tag{3}$$

3. Substitute into the equation:

$$t^2 \cdot r(r-1) t^{r-2} + at \cdot r t^{r-1} + b \cdot t^r = 0$$

4. Simplify:

$$t^r[r(r-1) + ar + b] = 0$$

5. Since  $t^r \neq 0$  for  $t > 0$ , we get the characteristic equation:

$$r^2 + (a-1)r + b = 0$$

### Characteristic Equation for Second-Order Euler:

$$r(r-1) + ar + b = 0 \quad \text{or} \quad r^2 + (a-1)r + b = 0$$

## 2.2 Solution Forms Based on Roots

**Theorem 2** (Complete Solution Classification). *Let  $r_1, r_2$  be roots of the characteristic equation. Then:*

1. **Distinct real roots:**  $y = c_1 t^{r_1} + c_2 t^{r_2}$

2. **Repeated real root  $r$ :**  $y = (c_1 + c_2 \ln t) t^r$

3. **Complex roots  $r = \alpha \pm i\beta$ :**

$$y = t^\alpha [c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)]$$

*Proof of Repeated Root Case.* When  $r$  is a repeated root, we use reduction of order. Given  $y_1 = t^r$ , we seek  $y_2 = v(t) \cdot t^r$ .

The Euler equation in standard form:

$$y'' + \frac{a}{t}y' + \frac{b}{t^2}y = 0$$

Using the reduction formula with  $p(t) = a/t$ :

$$v = \int \frac{e^{-\int (a/t) dt}}{t^{2r}} dt = \int \frac{t^{-a}}{t^{2r}} dt$$

For a repeated root,  $r = (1-a)/2$ , so  $2r = 1-a$ . Thus:

$$v = \int \frac{t^{-a}}{t^{1-a}} dt = \int \frac{1}{t} dt = \ln t$$

Therefore,  $y_2 = t^r \ln t$ . □

## 3 Method 2: Logarithmic Transformation

**Theorem 3** (Transformation to Constant Coefficients). *The substitution  $x = \ln t$  (equivalently,  $t = e^x$ ) transforms an Euler equation into a constant coefficient equation.*

### 3.1 Derivative Transformations

**Lemma 1** (Change of Variables). *Under the substitution  $x = \ln t$ , if  $v(x) = y(t)$ :*

$$t \frac{dy}{dt} = \frac{dv}{dx} \quad (4)$$

$$t^2 \frac{d^2y}{dt^2} = \frac{d^2v}{dx^2} - \frac{dv}{dx} \quad (5)$$

$$t^3 \frac{d^3y}{dt^3} = \frac{d^3v}{dx^3} - 3 \frac{d^2v}{dx^2} + 2 \frac{dv}{dx} \quad (6)$$

*Proof for Second Derivative.* Using the chain rule:

$$\frac{dy}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{1}{t} \frac{dv}{dx}$$

For the second derivative:

$$\frac{d^2y}{dt^2} = \frac{d}{dt} \left( \frac{1}{t} \frac{dv}{dx} \right) = -\frac{1}{t^2} \frac{dv}{dx} + \frac{1}{t} \frac{d}{dt} \left( \frac{dv}{dx} \right)$$

Since  $\frac{d}{dt} \left( \frac{dv}{dx} \right) = \frac{d^2v}{dx^2} \cdot \frac{dx}{dt} = \frac{1}{t} \frac{d^2v}{dx^2}$ :

$$\frac{d^2y}{dt^2} = \frac{1}{t^2} \left( \frac{d^2v}{dx^2} - \frac{dv}{dx} \right)$$

□

### 3.2 The Transformed Equation

**Theorem 4** (Constant Coefficient Form). *The Euler equation  $t^2y'' + aty' + by = 0$  becomes:*

$$v'' + (a - 1)v' + bv = 0$$

where  $v(x) = y(e^x)$  and  $x = \ln t$ .

The characteristic polynomial of the transformed equation is identical to the characteristic equation from the direct method! This confirms the deep connection between the two approaches.

## 4 Higher-Order Euler Equations

**Theorem 5** (General  $n$ -th Order). *For the  $n$ -th order Euler equation, assume  $y = t^r$ . The characteristic equation is:*

$$\prod_{k=0}^{n-1} (r - k) + \sum_{j=1}^{n-1} a_j \prod_{k=0}^{j-1} (r - k) + a_0 = 0$$

**Example 1** (Third-Order). *For  $t^3y''' + at^2y'' + bty' + cy = 0$ :*

*Assuming  $y = t^r$ : -  $y' = rt^{r-1}$  -  $y'' = r(r-1)t^{r-2}$  -  $y''' = r(r-1)(r-2)t^{r-3}$*

*Characteristic equation:*

$$r(r-1)(r-2) + ar(r-1) + br + c = 0$$

## 5 Important Special Cases

### 5.1 The Cauchy-Euler Equation

**Definition 2** (Standard Cauchy-Euler). *The equation  $t^2y'' + ty' + (\lambda^2t^2 + \nu^2)y = 0$  appears in many applications, especially in Bessel function theory.*

### 5.2 Modified Euler Equations

**Example 2** (Shifted Center). *For equations centered at  $t = t_0$ :*

$$(t - t_0)^2y'' + a(t - t_0)y' + by = 0$$

*Use the substitution  $s = t - t_0$  first.*

## 6 Solution Strategies

**Method 2** (Complete Solution Process). 1. **Identify:** *Check if powers match derivatives*

2. **Choose domain:** *Usually  $t > 0$*

3. **Select method:**

- *Direct ( $y = t^r$ ) for straightforward cases*
- *Transform ( $x = \ln t$ ) for complex or higher-order*

4. **Find characteristic equation**

5. **Solve for roots**

6. **Build solution** based on root types:

- *Real distinct:  $t^{r_1}, t^{r_2}, \dots$*
- *Real repeated:  $t^r, t^r \ln t, t^r (\ln t)^2, \dots$*
- *Complex:  $t^\alpha \cos(\beta \ln t), t^\alpha \sin(\beta \ln t)$*

## 7 Applications

Euler equations arise naturally in:

1. **Radial symmetry:** Solutions to Laplace's equation in polar/spherical coordinates
2. **Self-similar solutions:** PDEs with scaling symmetry
3. **Power-law media:** Heat conduction in materials with power-law properties

4. **Financial mathematics:** Option pricing models
5. **Fractal geometry:** Equations on self-similar domains

## 8 Connection to Other Topics

### 8.1 Series Solutions

**Theorem 6** (Frobenius Connection). *The point  $t = 0$  is a regular singular point of the Euler equation. The indicial equation from the Frobenius method is exactly our characteristic equation.*

### 8.2 Relationship to Constant Coefficients

Constant Coefficient	Euler Equation
$y = e^{rt}$ ansatz	$y = t^r$ ansatz
Repeated root gives $te^{rt}$	Repeated root gives $t^r \ln t$
Complex roots give $e^{\alpha t} \sin(\beta t)$	Complex roots give $t^\alpha \sin(\beta \ln t)$
Linear time scale	Logarithmic time scale

## 9 Common Pitfalls

1. **Domain:** Solutions differ for  $t > 0$  and  $t < 0$
2. **Characteristic equation:** It's  $r(r-1) + ar + b$ , not  $r^2 + ar + b$
3. **Complex roots:** Arguments are  $\beta \ln t$ , not  $\beta t$
4. **At  $t = 0$ :** This is a singular point; solutions may not extend through it
5. **Initial conditions:** Often given at  $t = 1$  to avoid the singularity

Prof. Ditkowski often combines Euler equations with:

- Initial value problems at  $t = 1$
- Boundary value problems on  $[1, e]$
- Questions about solution behavior as  $t \rightarrow 0^+$  or  $t \rightarrow \infty$
- Transformation to constant coefficients

## 10 Summary Table

Equation	Characteristic Eq.	Solutions
$t^2y'' + aty' + by = 0$	$r^2 + (a - 1)r + b = 0$	Based on roots
Distinct real $r_1, r_2$	—	$c_1t^{r_1} + c_2t^{r_2}$
Repeated root $r$	—	$(c_1 + c_2 \ln t)t^r$
Complex $\alpha \pm i\beta$	—	$t^\alpha[c_1 \cos(\beta \ln t) + c_2 \sin(\beta \ln t)]$