Lesson 31: Practice Problems - Repeated Eigenvalues and Jordan Forms

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Part A: Identifying Defective Eigenvalues

1. Determine the algebraic and geometric multiplicities of each eigenvalue for: $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

2. Find the defect of each eigenvalue for: $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

3. Show that $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is defective and find its Jordan form.

4. Determine which of these matrices are diagonalizable: a) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

5. For $A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$, find the size of the largest Jordan block.

Part B: Finding Generalized Eigenvectors

6. Find all eigenvectors and generalized eigenvectors for: $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

7. Construct a complete Jordan chain for: $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

8. Find the Jordan basis for: $A = \begin{pmatrix} 5 & 4 \\ -1 & 1 \end{pmatrix}$

9. Determine all generalized eigenvectors for: $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

10. Find a matrix P such that $P^{-1}AP$ is in Jordan form for: $A = \begin{pmatrix} 4 & 1 \\ -4 & 0 \end{pmatrix}$

Part C: $\mathbf{2}\times 2 Systems with Repeated Eigenvalues$

11. Solve: $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}$

12. Solve the IVP: $\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

13. Find the general solution: $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x}$

14. Solve: $\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

15. Find all solutions that remain bounded as $t \to \infty$ for: $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}$

Part D: $\mathbf{3}\times 3 Systems with Jordan Blocks$

16. Solve: $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}$

17. Find the general solution: $\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}$

18. Solve the IVP: $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

19. Find the solution: $\mathbf{x}' = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x}$ with $x_1(0) = 1$, $\mathrm{x}_2(0) = 0$, $\mathrm{x}_3(0) = 0$

20. Determine the Jordan form and solve: $\mathbf{x}' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$

Part E: Mixed Eigenvalue Problems

21. Solve: $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$ (repeated and distinct eigenvalues)

22. Find the general solution: $\mathbf{x}' = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \mathbf{x}$

23. Solve: $\mathbf{x}' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

24. Analyze stability for: $\mathbf{x}' = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} \mathbf{x}$

25. Find the fundamental matrix: $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}$

Part F: Theory and Applications

26. Prove that if λ is a repeated eigenvalue with full defect $n-1$, then the solutions contain terms up to $t^{n-1}e^{\lambda t}$.

27. Show that for a $2\times 2 matrix with repeated eigenvalue \lambda$, the trace equals 2λ and the determinant equals λ^2.

28. A coupled system has matrix $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Find the time when the solution with $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ reaches its maximum norm.

29. For the radioactive decay chain with matrix $A = \begin{pmatrix} -k & 0 & 0 \\ k & -k & 0 \\ 0 & k & -k \end{pmatrix}$, solve for the amounts of each isotope over time.

30. **Challenge:** Prove that e^{Jt} for a Jordan block J can be computed as: $e^{Jt} = e^{\lambda t} \sum_{k=0}^{n-1} t^k \frac{}{k!N}^k where$N = J - λI.

Solutions and Hints

Problem 1: Algebraic multiplicity = 2, geometric multiplicity = 1, defect = 1.

Problem 6: Eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, Generalized$:v$_2$ = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Problem 11: $\mathbf{x}(t) = e^{2t}[c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ + c$_2$(t$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ + $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$)]

Problem 16: Use the standard basis vectors as the Jordan chain.

Problem 24: All solutions decay to zero since $\lambda = -2 < 0$ despite the t and $t^2 terms.$

Problem 28: Maximum occurs at $t = 1$ (derivative of te^{-t} equals zero).

Key Strategy: Always check the defect first! If geometric multiplicity equals algebraic multiplicity, use standard eigenvector methods. Otherwise, build Jordan chains systematically.

Verification: For generalized eigenvector $\mathbf{v}_2 satisfying$(A-$\lambda I)\mathbf{v}_2 =$ v$_1, verify that$x(t) = e$^{\lambda t}(t\mathbf{v}_1 +$ v$_2) satisfies$x' = Ax.