Lesson 23: Practice Problems - Integrating Factors $\mu(x)$ and $\mu(y)$

ODE 1 - Prof. Adi Ditkowski

Part A: Testing for $\mu(x)$ (Problems 1-5)

1. Test if $\mu(x)$ exists and find it: $(y + xy^2)dx + (2x + x^2y)dy = 0$
2. Check for $\mu(x)$: $(2xy + y^3)dx + (x^2 + xy^2)dy = 0$
3. Find $\mu(x)$ if it exists: $(y^2 + xy)dx + (xy - x^2)dy = 0$
4. Test and find: $(3xy + 2y^2)dx + (x^2 + 2xy)dy = 0$
5. Determine $\mu(x)$: $(y \cos x + 2ye^x)dx + (\sin x + e^x)dy = 0$

Part B: Testing for $\mu(y)$ (Problems 6-10)

6. Test if $\mu(y)$ exists: $(2xy^3 + y)dx + (3x + x^2y^2)dy = 0$
7. Find $\mu(y)$: $(y + x^2y^3)dx + (x + 2xy^2)dy = 0$
8. Check for $\mu(y)$: $(x^2 + xy^2)dx + (2xy + y^3)dy = 0$
9. Test and determine: $(3x + 2xy)dx + (y + x^2)dy = 0$
10. Find $\mu(y)$: $(\frac{x}{y^2} + \frac{1}{y})dx + (\frac{1}{y} - \frac{x}{y^2})dy = 0$

Part C: Complete Solutions (Problems 11-15)

11. Solve completely: $(3xy + y^2)dx + (x^2 + xy)dy = 0$
12. Find general solution: $(y + 2xy^2)dx + (x + x^2y)dy = 0$
13. Solve: $(2y)dx + (3x)dy = 0$
14. Complete solution: $(x^2y + y)dx + (x^3 + x - y^2)dy = 0$
15. Solve with initial condition $y(1) = 2$: $(xy + y^2)dx + (x^2 + 2xy)dy = 0$

Part D: Both Types Exist (Problems 16-20)

16. Show both $\mu(x)$ and $\mu(y)$ exist, find both: $(2xy)dx + (x^2)dy = 0$
17. Find both integrating factors: $(y^2)dx + (2xy)dy = 0$
18. Compare solutions using different μ: $(3y)dx + (2x)dy = 0$
19. Choose the simpler μ and solve: $(xy^3)dx + (x^2y^2)dy = 0$
20. Both exist - solve using each: $(4xy)dx + (2x^2)dy = 0$

Part E: Mixed Recognition (Problems 21-25)

21. Determine the best approach: $(x^2 + y^2)dx + (2xy)dy = 0$
22. Check exactness first, then integrating factors: $(e^x + ye^x)dx + (xe^x + 1)dy = 0$
23. Full analysis: $(2x + 3y^2)dx + (6xy + 4y^3)dy = 0$
24. Complete classification: $(xy + y^3)dx + (x^2 + 3xy^2)dy = 0$
25. Strategy decision: $(\sin y + y\cos x)dx + (x\cos y + \sin x)dy = 0$

Solutions and Hints

Problem 1: Check $(M_y - N_x)/N = \frac{(1+2xy)-(2+2xy)}{2x+x^2y} = \frac{-1}{x(2+xy)}$ This is not a function of x alone, so $\mu(x)$ doesn't exist in this form.

Problem 3: $(M_y - N_x)/N = \frac{(2y+x)-(-2x)}{xy-x^2} = \frac{2y+3x}{x(y-x)}$ Not a function of x alone.

Problem 6: $(N_x - M_y)/M = \frac{(3+2xy^2)-(6xy^2+1)}{2xy^3+y} = \frac{2-4xy^2}{y(2xy^2+1)}$ Not a function of y alone.

Problem 11: From Example in theory: $\mu(x) = x$ gives solution $x^3y + \frac{x^2y^2}{2} = C$

Problem 13: Both $\mu(x) = x^{1/2}$ and $\mu(y) = y^{1/3}$ work. Using $\mu(y) = y^{1/3}$: Solution is $xy^{4/3} = C$

Problem 16: $\mu(x) = x$ (from $(M_y - N_x)/N = 2/x^2 \cdot x = 2/x$) $\mu(y) = y$ (from $(N_x - M_y)/M = (2x - 2x)/2xy = 0$... check calculation)

Key Test Formulas:

- For $\mu(x)$: $(M_y - N_x)/N$ must depend only on x
- For $\mu(y)$: $(N_x - M_y)/M$ must depend only on y
- If neither works, equation may not have $\mu(x)$ or $\mu(y)$