Lesson 22: Finding Potential Functions - Systematic Approach

ODE 1 - Prof. Adi Ditkowski

1 The Potential Function

Definition 1 (Potential Function)**.** *For an exact equation $M(x,y)dx + N(x,y)dy = 0$, the* ***potential function*** *$H(x,y)$ satisfies:*

$$\frac{\partial H}{\partial x} = M(x,y) \tag{1}$$

$$\frac{\partial H}{\partial y} = N(x,y) \tag{2}$$

The general solution is then given by $H(x,y) = C$.

The potential function is unique up to an additive constant. All methods for finding H yield the same result.

2 Method 1: Integration with Respect to x

Method 1 - Integrate M with respect to x:

1. Since $\frac{\partial H}{\partial x} = M(x,y)$, integrate:

$$H(x,y) = \int M(x,y)\,dx + g(y)$$

   where $g(y)$ is an arbitrary function of y alone.

2. Differentiate the result with respect to y:

$$\frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y)$$

3. Set this equal to $N(x,y)$:

$$\frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y) = N(x,y)$$

4. Solve for $g'(y)$:

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right]$$

5. Integrate to find $g(y)$:

$$g(y) = \int g'(y)\,dy$$

6. Write the complete potential function:

$$H(x,y) = \int M(x,y)\,dx + g(y)$$

Example 1 (Method 1 Application)**.** *Solve* $(3x^{2y} + y^3)dx + (x^3 + 3xy^2)dy = 0$

Step 1: *Verify exactness (assumed done):* $\frac{\partial M}{\partial y} = 3x^2 + 3y^2 = \frac{\partial N}{\partial x}$ ✓

Step 2: *Integrate* $M = 3x^{2y} + y^3$ *with respect to x:*

$$H = x$$

$^{3y} + xy^3 + g(y)$

Step 3: *Differentiate with respect to y:*

$$\frac{\partial H}{\partial y} = x^3 + 3xy^2 + g'(y)$$

Step 4: *Set equal to* $N = x^3 + 3xy^2$*:*

$$x^3 + 3xy^2 + g'(y) = x^3 + 3xy^2$$

Step 5: *Therefore* $g'(y) = 0$*, so* $g(y) = 0$ *(we can choose the constant to be 0).*

Solution: $H(x,y) = x^{3y} + xy^3$*, so the general solution is* $x^{3y} + xy^3 = C$*.*

3 Method 2: Integration with Respect to y

Method 2 - Integrate N with respect to y:

1. Since $\frac{\partial H}{\partial y} = N(x,y)$, integrate:

$$H(x,y) = \int N(x,y)\,dy + f(x)$$

where $f(x)$ is an arbitrary function of x alone.

2. Differentiate with respect to x:

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left[\int N(x,y)\,dy\right] + f'(x)$$

3. Set equal to $M(x, y)$ and solve for $f'(x)$:

$$f'(x) = M(x, y) - \frac{\partial}{\partial x}\left[\int N(x, y)\, dy\right]$$

4. Integrate to find $f(x)$ and write complete $H(x, y)$.

Choose Method 1 when M is simpler to integrate. Choose Method 2 when N is simpler. The choice can significantly reduce computation time on exams!

4 Method 3: Line Integral Approach

Method 3 - Path Integration:

Since the equation is exact, the line integral is path-independent:

$$H(x, y) = \int_{(x_0, y_0)}^{(x, y)} M\, dx + N\, dy$$

Common choice: Use path from $(0, 0) \to (x, 0) \to (x, y)$:

$$H(x, y) = \int_0^x M(t, 0)\, dt + \int_0^y N(x, s)\, ds$$

Prof. Ditkowski often asks: "Solve using two different methods and verify they give the same result." This tests your understanding that the potential function is unique.

5 Verification Process

Always verify your solution! Check that:

1. $\frac{\partial H}{\partial x} = M(x, y)$ ✓
2. $\frac{\partial H}{\partial y} = N(x, y)$ ✓

This catches errors and ensures partial credit.

6 Common Integration Patterns

Memorize these common potential functions:

If you see	**Think potential**
$y\,dx + x\,dy$	$H = xy$
$2xy\,dx + x^2\,dy$	$H = x^{2y}$
$\frac{y}{x^2}\,dx - \frac{1}{x}\,dy$	$H = -\frac{y}{x}$
$e^x \sin y\,dx + e^x \cos y\,dy$	$H = e^x \sin y$
$\frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$	$H = \sqrt{x^2+y^2}$

7 Initial Value Problems

Method 1 (Solving IVPs with Exact Equations)**.** *1. Find the potential function* $H(x, y)$

2. Use initial condition (x_0, y_0) *to find* C*:*

$$H(x_0, y_0) = C$$

3. Write particular solution: $H(x, y) = C$

Example 2 (IVP Example)**.** *Solve* $(2xy + 1)dx + (x^2 + 2y)dy = 0$ *with* $y(1) = 2$.

Solution: *From Method 1:* $H = x^{2y} + x + y^2$

Using $y(1) = 2$*:* $H(1, 2) = (1)^2(2) + 1 + (2)^2 = 7$

Particular solution: $x^{2y} + x + y^2 = 7$

8 Efficiency Tips

Strategic Integration Choices:

- If M contains ln, arctan, or complex expressions in $y \rightarrow UseMethod2If\mathrm{N}contains\ln,$ arctan, or complex expressions in $x \rightarrow UseMethod1$

- If both are complex but simplify when one variable is $0 \rightarrow UseMethod3Forpolynomials, choosebasedonlo$