Lesson 32: Practice Problems - Complex Eigenvalues

ODE 1 - Prof. Adi Ditkowski

Part A: Finding Complex Eigenvalues and Eigenvectors

1. Find the eigenvalues of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and classify the equilibrium.
2. Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & -5 \\ 1 & 2 \end{pmatrix}$.
3. Determine α and β if the eigenvalues are $\lambda = 3 \pm 4i$.
4. For what values of a does $A = \begin{pmatrix} a & -2 \\ 2 & a \end{pmatrix}$ have complex eigenvalues?
5. Find a $2{\times}2 matrix with eigenvalues \lambda = -1 \pm 3i$.

Part B: Extracting Real Solutions

6. Given eigenvalue $\lambda = 2i$ with eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, find two real solutions.
7. Extract real solutions from $\lambda = 1 + i$ with eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.
8. If $\mathbf{x}_c(\mathrm{t}) = \mathrm{e}^{(2+3i)t} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$, find the real and imaginary parts.
9. Convert the complex solution $\mathbf{x}(t) = e^{-t+2it} \begin{pmatrix} i \\ 1 \end{pmatrix}$ to real form.
10. Given $\lambda = 3 - 2i$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$, find the general real solution.

Part C: **2**$\times 2 Systems with Complex Eigenvalues$

11. Solve: $\mathbf{x}' = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} \mathbf{x}$

12. Solve the IVP: $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

13. Find the general solution: $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{x}$

14. Solve: $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & 3 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

15. Determine the solution: $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Part D: Stability and Behavior Analysis

16. Classify the stability of the origin for $\mathbf{x}' = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix} \mathbf{x}$

17. Find the period of oscillation for $\mathbf{x}' = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} \mathbf{x}$

18. Determine when the solution reaches maximum distance from origin: $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

19. For what values of k is the origin stable for $\mathbf{x}' = \begin{pmatrix} k & -4 \\ 1 & k \end{pmatrix} \mathbf{x}$?

20. Find a system where solutions spiral inward with period π.

Part E: **3**$\times 3 and Higher Dimensional Systems$

21. Solve: $\mathbf{x}' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$

22. Find the general solution: $\mathbf{x}' = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{x}$

23. Solve: $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

24. Analyze: $\mathbf{x}' = \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & -2 \\ -2 & 2 & -1 \end{pmatrix} \mathbf{x}$

25. Find all eigenvalues: $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 3 & 2 \end{pmatrix}$

Part F: Applications and Special Cases

26. A mass-spring system gives $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}$ where $x_1 isposition, x_2 isvelocity. Findthefrequencyofosc$

27. An RLC circuit yields $\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \mathbf{x}$. Determine if the circuit is underdamped, overdamped, or critically damped.

28. Two coupled oscillators give $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} \mathbf{x}$. Find the normal modes.

29. Show that if A is skew-symmetric ($A^T =$ -A)$, alleigenvaluesarepureimaginary.$

30. **Challenge:** Prove that for the system $\mathbf{x}' = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \mathbf{x}$, the solution curves are logarithmic spirals with equation $r = r_{0e}{}^{(\alpha/\beta)\theta}$ in polar coordinates.

Solutions and Hints

Problem 1: $\lambda = \pm i$, center (pure rotation)

Problem 6: $\mathbf{x}_1$(t) $= \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}$,$\mathrm{x}_2$(t) $= \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix}$

Problem 11: $\lambda = \pm 2i$, $\mathbf{x}(t) = c_1 \begin{pmatrix} \cos(2t) \\ \sin(2t)/2 \end{pmatrix} + \mathrm{c}_2 \begin{pmatrix} \sin(2t) \\ -\cos(2t)/2 \end{pmatrix}$

Problem 12: First find $\lambda = 1 \pm i$, then apply initial conditions.

Problem 17: Period $T = 2\pi/\beta = 2\pi/3$

Problem 19: Stable for $k < 0$

Problem 26: Natural frequency $\omega = 2$ rad/s

Problem 27: Underdamped (complex eigenvalues with negative real part)

Key Strategy: Always use Euler's formula to convert complex exponentials to real trig functions. Remember that complex eigenvalues come in conjugate pairs, and you only need to work with one of them.

Verification: Check that your real solutions satisfy the original differential equation!