Practice Problems - Lesson 39: 3D Systems and Higher Dimensions

ODE 1 Course

Part A: 3D Classification (Problems 1-6)

1. Classify the 3D equilibrium at the origin for:

   (a) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

   (d) $A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

2. For each set of eigenvalues, classify the 3D equilibrium:

   (a) $\lambda = -1, -2, -3$

   (b) $\lambda = 2, -1 \pm 3i$

   (c) $\lambda = 0, \pm 2i$

   (d) $\lambda = 1, 1, 1$ (repeated)

3. Find the dimensions of stable and unstable manifolds for:

   (a) $\lambda = -2, 1, 3$

   (b) $\lambda = -1, -2 \pm i$

   (c) $\lambda = 0, -1, 2$

   (d) $\lambda = \pm 2i, 3$

4. For the system with $A = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$:

(a) Find all eigenvalues and eigenvectors
(b) Classify the equilibrium
(c) Identify stable and unstable manifolds
(d) Describe the behavior near origin

5. Given a 3D system has one eigenvalue $\lambda_1 = -2$ and a complex pair with $\text{Re}(\lambda_{2,3}) = 1$:

(a) What type of equilibrium is this?
(b) What are the manifold dimensions?
(c) Is the origin stable?
(d) Describe trajectory behavior

6. For the block diagonal system:

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

(a) Find eigenvalues without full calculation
(b) Classify the equilibrium
(c) Which coordinate decouples?
(d) Describe motion in each subspace

Part B: Manifold Analysis (Problems 7-11)

7. For $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$:

(a) Find all eigenvalues
(b) Find eigenvectors
(c) Determine E^s and E^u explicitly
(d) Write parametric equations for each manifold

8. Given eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$, $\lambda_3 = 1$ with eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(a) Find the stable manifold equation
(b) Find the unstable manifold equation

(c) Describe trajectories starting in E^u

9. For a system with 2D stable manifold in the xy-plane:
   (a) What can you say about eigenvalues?
   (b) What happens to trajectories starting at $(0, 0, 1)$?
   (c) Sketch behavior in xz-projection

10. Consider the saddle with $\lambda = -2, -1, 3$:
    (a) What type of saddle is this?
    (b) If $v_3 = (1, 1, 1)^T$, describe escape direction
    (c) How fast do trajectories approach E^s?

11. For a spiral-saddle system:
    (a) What are possible eigenvalue configurations?
    (b) Describe motion in the spiral plane
    (c) What determines overall stability?

Part C: Projections (Problems 12-16)

12. For the system $\dot{x} = -x$, $\dot{y} = -2y + z$, $\dot{z} = -y - 2z$:
    (a) Find eigenvalues
    (b) Sketch xy-projection
    (c) Sketch xz-projection
    (d) Sketch yz-projection
    (e) Which projection best shows the spiral?

13. Given a 3D center-line with eigenvalues $0, \pm 3i$:
    (a) Which variable corresponds to the zero eigenvalue?
    (b) What do trajectories look like?
    (c) Draw the three projections

14. For a stable spiral-node with spiral in xy-plane:
    (a) What does xz-projection show?
    (b) What does yz-projection show?
    (c) Which projection shows spiral clearly?

15. Project the trajectory of:

$$\dot{x} = -x, \quad \dot{y} = y, \quad \dot{z} = -2z$$

starting at $(1, 1, 1)$ onto all three coordinate planes.

16. For the system with solution:

$$\mathbf{x}(t) = e^{-t} \begin{pmatrix} \cos(2t) \\ \sin(2t) \\ 1 \end{pmatrix}$$

   (a) Find the system matrix A
   (b) Draw xy-projection
   (c) Draw xz-projection
   (d) What type of equilibrium?

Part D: Poincaré Sections (Problems 17-20)

17. For periodic motion in 3D with period $T = 2\pi$:
   (a) What does Poincaré section show?
   (b) How many points for one trajectory?
   (c) What if period doubles?

18. Consider the plane $z = 0$ as Poincaré section for:

$$\dot{x} = y, \quad \dot{y} = -x, \quad \dot{z} = -z + x^2$$

   (a) When do trajectories cross $z = 0$?
   (b) What pattern emerges?
   (c) Is motion periodic?

19. For quasi-periodic motion on a torus:
   (a) What does Poincaré section look like?
   (b) How to distinguish from periodic?
   (c) What if frequencies are rational?

20. Design a Poincaré section to best reveal:
   (a) Periodic orbits
   (b) Chaotic trajectories
   (c) Quasi-periodic motion

Part E: Higher Dimensions (Problems 21-24)

21. For a 4D system with eigenvalues $-1, -2, 3, 4$:
    (a) Find dimensions of invariant manifolds
    (b) Is the origin stable?
    (c) What's the dominant behavior?

22. In 5D with eigenvalues $-1 \pm i, -2, 3 \pm 2i$:
    (a) Classify stability
    (b) Find $\dim(E^s)$ and $\dim(E^u)$
    (c) How many rotation planes?

23. For an n-dimensional system:
    (a) Maximum number of complex eigenvalues?
    (b) When is origin globally stable?
    (c) When do all trajectories escape?

24. Given a 10D system with 8 negative and 2 positive eigenvalues:
    (a) What are manifold dimensions?
    (b) Describe long-term behavior
    (c) What's the "most likely" outcome?

Part F: Special 3D Phenomena (Problems 25-27)

25. For the Rössler system:

$$\dot{x} = -y - z, \quad \dot{y} = x + ay, \quad \dot{z} = b + z(x - c)$$

    (a) Find equilibria when $a = b = 0.2$, $c = 5.7$
    (b) Linearize at equilibria
    (c) What type of attractor exists?

26. Consider a 3D system with a stable limit cycle:
    (a) Can this happen in a linear system?
    (b) What do nearby trajectories do?
    (c) How would you detect this numerically?

27. For motion on a 2-torus in 3D:
    (a) What are the two frequencies?
    (b) When is motion periodic?
    (c) When is trajectory dense on torus?

Part G: Exam-Style Problems (Problems 28-30)

28. [**Prof. Ditkowski Style - 3D Complete Analysis**] For the system:

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

   (a) Find all eigenvalues (show characteristic polynomial)
   (b) Verify using trace and determinant
   (c) Classify the equilibrium type
   (d) Find eigenvectors for all eigenvalues
   (e) Identify and state dimensions of E^s and E^u
   (f) Describe behavior in each invariant subspace
   (g) Sketch the xy-projection
   (h) Sketch the xz-projection
   (i) Which trajectories approach origin fastest?
   (j) Write the general solution

29. [**Higher-D Application**] A 6D linear system models coupled oscillators with eigenvalues:

$$\lambda_{1,2} = -0.1 \pm 2i, \quad \lambda_{3,4} = -0.2 \pm 3i, \quad \lambda_{5,6} = 0.1 \pm i$$

   (a) Is the system stable? Why?
   (b) How many oscillation frequencies?
   (c) Find dimensions of stable/unstable manifolds
   (d) Which oscillator damps fastest?
   (e) What happens as $t \to \infty$?
   (f) Describe the attractor type

30. [**Conceptual Understanding**]

   (a) Prove: A 3D linear system cannot have all complex eigenvalues
   (b) Show: If two eigenvalues are complex conjugates, the third must be real
   (c) Explain: Why can 4D systems have two pairs of complex eigenvalues?
   (d) True/False: A 3D saddle is always unstable
   (e) Can a 3D linear system have a limit cycle?
   (f) What's the minimum dimension for quasi-periodic motion?
   (g) In 3D, if $\det(A) < 0$, what can you conclude?
   (h) How many types of 3D saddles exist?

3D Visualization Strategy:

- Don't try to draw full 3D portraits
- Use projections to show key features
- Describe behavior verbally
- State manifold dimensions explicitly
- Use computer tools when possible
- Focus on eigenvalue analysis