Practice Problems: Lesson 9 - Direction Fields and Isoclines

Master the visualization of ODEs!

Part A: Basic Concepts (6 problems)

1. For $\frac{dy}{dx} = x + y$, find the isoclines for slopes $c = 0, 1, -1, 2$.
2. Identify all equilibrium points for $\frac{dy}{dx} = (x - 1)(y + 2)$.
3. Sketch the direction field for $\frac{dy}{dx} = -y$ on the region $[-2, 2] \times [-2, 2]$.
4. Given a direction field, how can you identify points where solutions have horizontal tangents?
5. For $\frac{dy}{dx} = x^2$, explain why all isoclines are vertical lines.
6. True or False: A solution curve can cross an isocline at most once. Explain.

Part B: Core Techniques (6 problems)

7. Construct the direction field for $\frac{dy}{dx} = \frac{y}{x}$ (exclude $x = 0$). What are the isoclines?
8. For $\frac{dy}{dx} = xy$, find equations for all isoclines and sketch the nullcline.
9. Draw the direction field for $\frac{dy}{dx} = y - x^2$ and identify regions where solutions are increasing.
10. Given $\frac{dy}{dx} = \sin(x) - y$, find the isocline for slope 0 and sketch nearby solution behavior.
11. For $\frac{dy}{dx} = \frac{x}{y}$, describe the direction field along the axes. What's special about these points?
12. Sketch solution curves for $\frac{dy}{dx} = 2x$ passing through $(0, 0)$, $(1, 1)$, and $(-1, 2)$.

Part C: Applications (5 problems)

13. A direction field shows all arrows pointing toward the line $y = 2x + 1$. What can you conclude about long-term behavior?

14. For the logistic equation $\frac{dy}{dx} = y(1-y)$, analyze the direction field to determine stability of equilibria.

15. Given $\frac{dy}{dx} = (x-1)^2 + (y-1)^2 - 1$, describe the direction field on the circle centered at $(1, 1)$ with radius 1.

16. For $\frac{dy}{dx} = -x/y$, explain why solution curves must be circles centered at the origin.

17. A chemical reaction follows $\frac{dy}{dx} = k(a-y)(b-y)$ where $a > b > 0$. Analyze the direction field behavior.

Part D: Advanced/Theoretical (5 problems)

18. Prove that if $f(x, y)$ is continuous, solution curves cannot cross except at equilibrium points.

19. For $\frac{dy}{dx} = f(y)$ (autonomous equation), explain why all isoclines are horizontal lines.

20. Show that if a direction field has a line of symmetry, solution curves respect this symmetry.

21. Given two ODEs with direction fields that differ only in magnitude (not direction) of arrows, how do their solution curves relate?

22. Prove that near a saddle point, there exist exactly four special solution curves (separatrices).

Part E: Exam-Style Questions (6 problems)

23. [**Prof. Ditkowski Special**] Sketch the complete direction field for $\frac{dy}{dx} = x^2 - y^2$. Find all equilibria, draw five distinct isoclines, and sketch three solution curves showing different behaviors.

24. Given only the direction field (figure provided on exam), determine the ODE from:

    a) $\frac{dy}{dx} = x - y$
    b) $\frac{dy}{dx} = x + y$
    c) $\frac{dy}{dx} = xy$
    d) $\frac{dy}{dx} = x/y$

25. For $\frac{dy}{dx} = y^2 - x$, without solving:

a) Find all nullclines

b) Determine regions where solutions are concave up

c) Sketch the solution passing through $(1, 1)$

d) Describe behavior as $x \to \infty$

26. [**Multi-part**] Consider $\frac{dy}{dx} = (x-1)(y-1)(y+1)$:

    a) Find all equilibrium points

    b) Classify stability of each equilibrium using the direction field

    c) Identify all separatrices

    d) Sketch the complete phase portrait

27. A direction field shows spiraling arrows converging to a point. Which ODE could produce this?

    a) $\frac{dy}{dx} = -x - y$

    b) $\frac{dy}{dx} = x^2 + y^2$

    c) $\frac{dy}{dx} = xy$

    d) Cannot determine

28. [**Conceptual**] Explain how to determine from a direction field alone whether an ODE has periodic solutions. Apply your method to analyze $\frac{dy}{dx} = -x + y^3$.

Answer Key with Hints

Problem 1: $y = -x$ (slope 0), $y = -x + 1$ (slope 1), $y = -x - 1$ (slope -1), $y = -x + 2$ (slope 2)

Problem 2: Single equilibrium at $(1, -2)$

Problem 7: Isoclines are rays from origin: $y = cx$

Problem 13: Along $y = 2x + 1$, slope equals 0, so this is an attractor

Problem 19: Horizontal isoclines mean f doesn't depend on x explicitly

Problem 24: Use nullclines $y = \pm 1$ and $x = 1$ to divide plane into regions