# Lesson 33: Practice Problems - Matrix Exponential

ODE 1 - Prof. Adi Ditkowski

## Part A: Direct Computation

1. Compute $e^{At}$ for $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

2. Find $e^{At}$ for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ using the series definition.

3. Calculate $e^{At}$ for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

4. Verify that $\frac{d}{dt}$e$^{At}$ = A$e^{At} for$A = $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

5. Show that $e^{A \cdot 0} = I$ for any $2 \times 2 matrix$A.

## Part B: Diagonalization Method

6. Use diagonalization to find $e^{At}$ for $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

7. Compute $e^{At}$ for $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

8. Find $e^{At}$ for $A = \begin{pmatrix} 5 & 3 \\ 1 & 3 \end{pmatrix}$

9. Calculate $e^{At}$ for $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

10. Use diagonalization for $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}$

## Part C: Jordan Form Method

11. Find $e^{At}$ for the Jordan block $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

12. Compute $e^{At}$ for $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

13. Calculate $e^{At}$ for $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

14. Find $e^{At}$ for $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

15. Compute $e^{At}$ when $A = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$

## Part D: Complex Eigenvalues

16. Find $e^{At}$ for $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

17. Compute $e^{At}$ for $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

18. Calculate $e^{At}$ for $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

19. Find $e^{At}$ for $A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$

20. Show that $e^{At}$ is a rotation matrix when $A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$

## Part E: Properties and Applications

21. Verify that (e$^{At})^{-1} = e^{-At} for$A $= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

22. Show that det(e$^{At}) = e^{\text{tr}(A)t} for$A $= \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

23. Prove that if $A^2 = 0$, then $e^{At} = I + At$.

24. For $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, verify that $e^{A(s+t)} = \mathrm{e}^{As}e^{At}$.

25. If $A$ and $B$ commute, show that $e^{A+B} = e^{A}e^{B}$ using $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$.

## Part F: Solving IVPs with Matrix Exponential

26. Solve $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ using $e^{At}$.

27. Use the matrix exponential to solve $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

28. Find the solution to $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

29. Compute $\mathbf{x}(1)$ if $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\mathbf{x}$ and $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

30. **Challenge:** Show that the fundamental matrix $\Phi(t)$ for $\mathbf{x}' = A\mathbf{x}$ satisfies $\Phi(t) = \mathrm{e}^{At}\Phi(0)$.

## Solutions and Hints

**Problem 1:** $e^{At} = \begin{pmatrix} \mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{-3t} \end{pmatrix}$

**Problem 2:** $A^2 = 0$, so $e^{At} = I + At = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

**Problem 6:** Eigenvalues are 4 and 2, eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

**Problem 11:** $e^{At} = \mathrm{e}^{3t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{3t} & t\mathrm{e}^{3t} \\ 0 & \mathrm{e}^{3t} \end{pmatrix}$

**Problem 13:** $e^{At} = \mathrm{e}^{2t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$

**Problem 16:** $e^{At} = \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}$

**Problem 26:** First find $e^{At} = \mathrm{e}^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, then $\mathbf{x}(t) = \mathrm{e}^{At}\mathbf{x}_0$

**Key Strategy:** Identify the matrix type first (diagonal, diagonalizable, Jordan, nilpotent, or complex eigenvalues), then apply the appropriate method.

**Verification:** Always check that $e^{A\cdot 0} = I$ and that your solution satisfies the differential equation.