Lesson 23: Integrating Factors - $\mu(x)$ and $\mu(y)$ Cases

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1 The Concept of Integrating Factors

Definition 1 (Integrating Factor)**.** *An **integrating factor** $\mu(x, y)$ for the equation*

$$M(x, y)dx + N(x, y)dy = 0$$

is a function such that the equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact.

Multiplying by an integrating factor doesn't change the solutions - it only changes the form of the equation. If $y = f(x)$ is a solution to the original equation, it remains a solution to the modified equation.

2 Condition for Exactness After Multiplication

For $\mu M dx + \mu N dy = 0$ to be exact, we need:

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

Expanding using the product rule:

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

Rearranging:

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

This is a partial differential equation for μ - generally very difficult to solve! We look for special cases where μ depends on only one variable.

3 Case 1: Integrating Factor $\mu(x)$

Theorem 1 (Existence of $\mu(x)$)**.** *An integrating factor depending only on x exists if and only if*

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$$

where $g(x)$ is a function of x alone. The integrating factor is then:

$$\mu(x) = e^{\int g(x)\,dx}$$

Proof. If $\mu = \mu(x)$, then $\frac{\partial \mu}{\partial y} = 0$. The exactness condition becomes:

$$-N\frac{d\mu}{dx} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

This is solvable only if the right side depends solely on x. □

Finding $\mu(x)$:

1. Compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$
2. Calculate $R(x, y) = \frac{M_y - N_x}{N}$
3. If $R = g(x)$ (function of x only), then:

$$\mu(x) = e^{\int g(x)\,dx}$$

4. If R contains y, then $\mu(x)$ doesn't exist

4 Case 2: Integrating Factor $\mu(y)$

Theorem 2 (Existence of $\mu(y)$)**.** *An integrating factor depending only on y exists if and only if*

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = h(y)$$

where $h(y)$ is a function of y alone. The integrating factor is:

$$\mu(y) = e^{\int h(y)\,dy}$$

Finding $\mu(y)$:

1. Compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$
2. Calculate $S(x, y) = \frac{N_x - M_y}{M}$
3. If $S = h(y)$ (function of y only), then:

$$\mu(y) = e^{\int h(y)\, dy}$$

4. If S contains x, then $\mu(y)$ doesn't exist

5 Complete Solution Process

Step-by-Step Solution with Integrating Factors:

1. Test for exactness (if exact, skip to step 6)
2. Check if $\mu(x)$ exists: Is $(M_y - N_x)/N$ a function of x only?
3. If not, check if $\mu(y)$ exists: Is $(N_x - M_y)/M$ a function of y only?
4. Find the integrating factor using the appropriate formula
5. Multiply the original equation by μ
6. Verify the new equation is exact
7. Solve the exact equation using methods from Lesson 22

6 Important Examples

Example 1 (Standard $\mu(x)$ Case)**.** *Solve* $(2y + 3x^2)dx + xdy = 0$

Step 1: *Test exactness:* $M_y = 2$, $N_x = 1$. *Not exact!*

Step 2: *Check for* $\mu(x)$*:*

$$\frac{M_y - N_x}{N} = \frac{2 - 1}{x} = \frac{1}{x}$$

This is a function of x *only!*

Step 3: *Find* $\mu(x)$*:*

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$$

Step 4: *Multiply by* $\mu = x$*:*

$$(2xy + 3x^3)dx + x^2 dy = 0$$

Step 5: *Verify exactness:* $M_y = 2x$, $N_x = 2x$. ✓

Step 6: *Find potential function:*

$$H = \int x^2\,dy = x^2y + f(x)$$

$$\frac{\partial H}{\partial x} = 2xy + f'(x) = 2xy + 3x^3$$

$$f'(x) = 3x^3 \Rightarrow f(x) = \frac{3x^4}{4}$$

Solution: $x^2y + \frac{3x^4}{4} = C$

Example 2 (Linear Equation Connection)**.** *The linear equation $y' + P(x)y = Q(x)$ can be written as:*

$$(Py - Q)dx + dy = 0$$

Check for $\mu(x)$:

$$\frac{M_y - N_x}{N} = \frac{P - 0}{1} = P(x)$$

Therefore: $\mu(x) = e^{\int P(x)dx}$ - exactly the integrating factor from Block 5!

7 Common Patterns to Recognize

Quick Recognition Guide:

If you see	Try	Integrating Factor
$N = x^n$	$\mu(x)$	Often $\mu = x^k$
$M = y^n$	$\mu(y)$	Often $\mu = y^k$
Linear in y	$\mu(x)$	$\mu = e^{\int P(x)dx}$
Homogeneous	Either	Check both tests
$N = f(x)$ only	$\mu(x)$	Guaranteed to exist
$M = g(y)$ only	$\mu(y)$	Guaranteed to exist

8 Memory Aids

Mnemonic Devices:

- "$\mu(x)$: My Nexus over N" - $(M_y - N_x)/N$ for x dependence
- "$\mu(y)$: Nexus My over M" - $(N_x - M_y)/M$ for y dependence
- Notice: Numerators are negatives of each other!
- The variable in μ matches what you divide by (sort of):
  - Divide by N (has x in deNominator) $\to \mu(x)$

- Divide by M (has y sound in naMe) $\to \mu(y)$

9 Verification is Crucial

After finding an integrating factor, ALWAYS:

1. Multiply the original equation by μ
2. Verify the new equation is exact by checking $\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$
3. Only then proceed to find the potential function

Skipping verification is a common source of errors!