Lesson 26: Practice Problems

Converting Higher-Order to First-Order Systems

Part A: Basic Conversions (6 problems)

1. Convert to a first-order system: $y'' + 4y = 0$
2. Convert with initial conditions: $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = -1$
3. Write the companion matrix for: $y''' + 2y'' - y' + 3y = 0$
4. Convert the fourth-order equation: $y^{(4)} - y = 0$
5. Convert with forcing: $y'' + \omega^2 y = \cos(\omega t)$
6. Convert the Airy equation: $y'' - ty = 0$

Part B: Companion Matrix Construction (5 problems)

7. Find the companion matrix for: $y^{(5)} + y''' - 2y' = 0$
8. Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, find the corresponding scalar ODE.
9. Construct the companion matrix for: $2y''' + 3y'' - y' + 4y = 0$ (note the leading coefficient)
10. Find the companion matrix eigenvalues for: $y'' + 2y' + 2y = 0$
11. Write the companion matrix for the Bessel equation: $t^2 y'' + ty' + (t^2 - n^2)y = 0$

Part C: Reverse Conversion (5 problems)

12. Convert the system $\begin{cases} x' = y \\ y' = -4x - 4y \end{cases}$ to a single second-order equation.
13. Given $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \mathbf{x}$, find the scalar equation.

14. Convert $\begin{cases} x' = y \\ y' = z \\ z' = 6x - 11y + 6z \end{cases}$ to a third-order equation.

15. The system $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}$ corresponds to which scalar ODE?

16. Convert the coupled system $\begin{cases} u' = v \\ v' = -u + 2v \end{cases}$ to find equations for both u and v.

Part D: Initial Condition Conversion (5 problems)

17. Convert IVP: $y''' + y' = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$

18. Transform: $y'' + 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

19. Convert the IVP: $t^2 y'' - 2ty' + 2y = 0$, $y(1) = 1$, $y'(1) = 2$ (for $t > 0$)

20. Set up the system for: $y^{(4)} + 4y'' + 4y = 0$ with $y(0) = y'(0) = y''(0) = 0$, $y'''(0) = 1$

21. Convert: $y'' + (1 - \frac{1}{4t^2})y = 0$, $y(\pi) = 0$, $y'(\pi) = 1$

Part E: Theoretical Problems (4 problems)

22. Prove that the characteristic polynomial of a companion matrix equals the characteristic equation of the original ODE.

23. Show that if $y(t)$ solves $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0$, then $\mathbf{x}(t) = [y, y', \ldots, y^{(n-1)}]^T$ solves $\mathbf{x}' = A\mathbf{x}$ where A is the companion matrix.

24. Prove that the companion matrix for a constant coefficient equation has n linearly independent eigenvectors if and only if all characteristic roots are distinct.

25. Show that the determinant of a companion matrix equals $(-1)^{n+1}a_0$.

Part F: Exam-Style Problems (5 problems)

26. (Prof. Ditkowski style) Given $y''' - 3y'' + 3y' - y = 0$:
    (a) Write the companion matrix
    (b) Find its eigenvalues
    (c) Explain the connection to the general solution

27. Convert the system and solve for eigenvalues:

$$y^{(4)} + 5y'' + 4y = 0$$

28. A mass-spring-damper system satisfies $m\ddot{x} + c\dot{x} + kx = F(t)$.
    (a) Convert to first-order form with $m = 1$, $c = 2$, $k = 5$
    (b) Write the companion matrix
    (c) Find conditions on c for complex eigenvalues

29. Given the companion matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & -\beta & -\gamma \end{bmatrix}$:
    (a) Write the scalar ODE
    (b) Find α, β, γ so that e^t, e^{2t}, e^{3t} are solutions
    (c) Verify using eigenvalues

30. (Comprehensive) The equation $y''' + py' + qy = 0$ has solutions e^t, e^{-t}, e^{2t}.
    (a) Find p and q
    (b) Write the companion matrix
    (c) Convert the IVP with $y(0) = 1$, $y'(0) = 0$, $y''(0) = 3$
    (d) Express the solution using the state vector

Solutions and Hints

Selected Solutions:

Problem 1: $\mathbf{x} = \begin{bmatrix} y \\ y' \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$

Problem 8: The scalar ODE is $y''' + 6y'' + 11y' + 6y = 0$

Problem 13: $x'' + 4x' + 4x = 0$ (repeated roots at $\lambda = -2$)

Problem 21: Characteristic polynomial: $\det(A - \lambda I) = (-\lambda)^{n-1}(-\lambda - a_{n-1}\lambda^{n-1} - \cdots - a_0)$

Problem 26: Eigenvalues are $1, 1, 1$ (triple root), indicating $y = (c_1 + c_2 t + c_3 t^2)e^t$

Key Insights:

- Always check dimensions when setting up state vectors
- The companion matrix structure is universal - memorize it!
- Initial conditions transform directly: $\mathbf{x}(t_0) = [y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0)]^T$
- Eigenvalues of companion matrix = roots of characteristic equation