Full Solution Key

Student Name:

ID:

Solution to Question 1

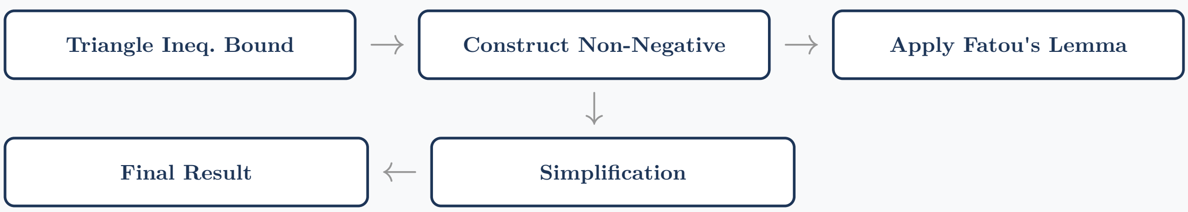
Topic: Measure Theory

PROBLEM STATEMENT

Given: Let (X, M, μ) be a measure space. Let $f_n \to f$ pointwise a.e. and $|f_n| \le g \in L^1$.
To Prove:

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

PROOF STRATEGY MAP



Triangle Ineq. Bound
Since $|f_n| \le g$ and $g \in L^1$, by the triangle inequality, we have $|f_n - f| \le |f_n| + |f| \le 2g$.

Claim: Construct Non-Negative

The sequence $h_n = 2g - |f_n - f|$ is non-negative.

Proof: Since $|f_n - f| \le 2g$, it follows that $2g - |f_n - f| \ge 0$. Thus Fatou's Lemma applies to h_n. ▪

Apply Fatou's Lemma

$$\int 2g \leq \liminf \int (2g - |f_n - f|)$$

$$\int 2g \leq \int 2g - \limsup \int |f_n - f|$$

Subtracting the finite integral $\int 2g$ from both sides (valid since $g \in L^1$):

Final Result

$$0 \leq -\limsup \int |f_n - f| \Rightarrow \limsup \int |f_n - f| \leq 0$$

Since the integral of absolute value is non-negative, the limit is 0. Q.E.D.

$\blacksquare$

Solution to Question 2

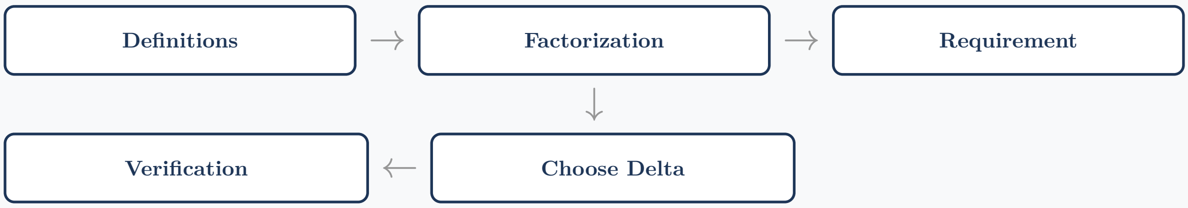
Topic: Real Analysis

PROBLEM STATEMENT

Given: Prove the limit of the function $f(x) = 2x + 3$ as $x \to 1$.
To Prove:

$$\lim_{x \to 1} (2x + 3) = 5$$

PROOF STRATEGY MAP



Definitions

Let $\varepsilon > 0$. We must find $\delta > 0$ such that $0 < |x - 1| < \delta \Rightarrow |f(x) - 5| < \varepsilon$.

Factorization

$$|(2x + 3) - 5| = |2x - 2| = 2|x - 1|$$

We want this quantity to be less than ε. Thus we need $2|x - 1| < \varepsilon$, or $|x - 1| < \frac{\varepsilon}{2}$.

Claim: Choose Delta

Choose $\delta = \frac{\varepsilon}{2}$.

Proof: This choice directly satisfies the inequality derived above. $\blacksquare$

Verification

If $|x - 1| < \delta$, then $|f(x) - 5| = 2|x - 1| < 2(\frac{\varepsilon}{2}) = \varepsilon$. Q.E.D.

$\blacksquare$

Solution to Question 3

Topic: Abstract Algebra

PROBLEM STATEMENT

Given: Let $\varphi : G \to H$ be a surjective homomorphism with kernel K.
To Prove:

$$\frac{G}{K} \cong H$$

PROOF STRATEGY MAP

Define Map $\rightarrow$ Well-Defined $\rightarrow$ Homomorphism

Well-Defined $\downarrow$

Isomorphism $\leftarrow$ Bijectivity

Define Map

Define the map $\psi : \frac{G}{K} \to H$ by $\psi(gK) = \varphi(g)$.

Claim: Well-Defined

The map ψ does not depend on the choice of representative g.

Proof: Suppose $g_1K = g_2K$. Then $g_2^{-1}g_1 \in K = \ker(\varphi)$. Thus $\varphi(g_2^{-1}g_1) = e_H$, which implies $\varphi(g_1) = \varphi(g_2)$. So $\psi(g_1K) = \psi(g_2K)$. ∎

Claim: Homomorphism

ψ is a homomorphism.

Proof: $\psi((xK)(yK)) = \psi(xyK) = \varphi(xy) = \varphi(x)\varphi(y) = \psi(xK)\psi(yK)$. ∎

Claim: Bijectivity

ψ is both injective and surjective.

Proof:

1. Surjective: Since φ is surjective, for any $h \in H$, exists g such that $\varphi(g) = h$. Thus $\psi(gK) = h$.
2. Injective: $\psi(gK) = e_H \Rightarrow \varphi(g) = e_H \Rightarrow g \in K \Rightarrow gK = K$ (the identity in $\frac{G}{K}$).

▪

Isomorphism

Since ψ is a bijective homomorphism, it is an isomorphism. $\frac{G}{K} \cong H$.

■

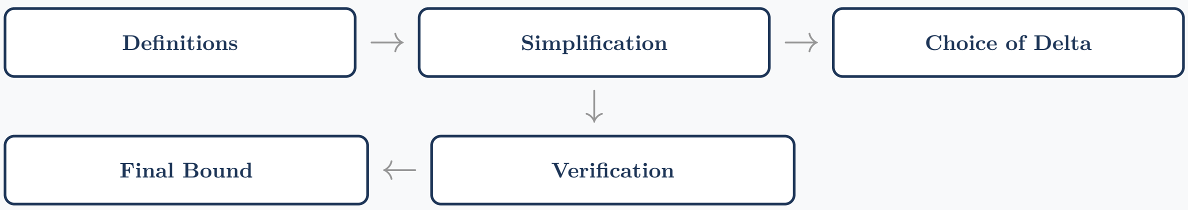
Solution to Question 4

Topic: Epsilon-Delta Limits

PROBLEM STATEMENT

Given: Let $f(x) = 2x + 3$.
To Prove: Prove that $\lim_{x\to 1} f(x) = 5$ using the definition of the limit.

PROOF STRATEGY MAP



Definitions
Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $0 < |x - 1| < \delta \Rightarrow |(2x + 3) - 5| < \varepsilon$.

Simplification

$$|(2x + 3) - 5| = |2x - 2| = 2\,|x - 1|$$

Claim: Choice of Delta

We choose $\delta = \frac{\varepsilon}{2}$.

Proof: Since $\varepsilon > 0$, it follows that $\delta > 0$. ▪

Assume $0 < |x - 1| < \delta$.

Final Bound
Then $|(2x + 3) - 5| = 2\,|x - 1| < 2\delta = 2(\frac{\varepsilon}{2}) = \varepsilon$. Q.E.D.

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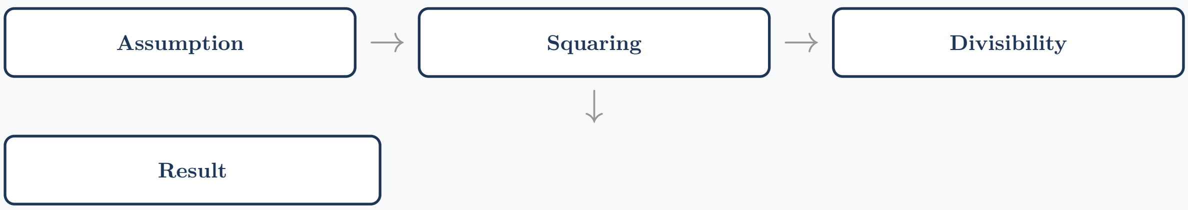
Solution to Question 5

Topic: Parity Proofs

PROBLEM STATEMENT

Given: Let n be an integer.
To Prove: Prove that if n is even, then n^2 is divisible by 4.

PROOF STRATEGY MAP



Assumption
Assume n is an even integer. By definition, there exists an integer k such that $n = 2k$.

Squaring

$$n^2 = (2k)^2 \quad n^2 = 4k^2$$

Claim: Divisibility

n^2 is a multiple of 4.

Proof: Since k is an integer, k^2 is an integer. Thus n^2 is 4 times an integer. ▪

Result

Therefore, $4 \mid n^2$. Q.E.D.

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Solution to Question 6

Topic: Subspaces

PROBLEM STATEMENT

Given: Let $T : V \to W$ be a linear transformation between vector spaces.
To Prove: Prove that $\ker(T)$ is a subspace of V.

PROOF STRATEGY MAP

Zero Vector → Closure under Addition → Closure under Scaling
↓
Conclusion

Zero Vector
Since T is linear, $T(0_V) = 0_W$. Thus $0_V \in \ker(T)$.

Claim: Closure under Addition

If $u, v \in \ker(T)$, then $u + v \in \ker(T)$.

Proof: We know $T(u) = 0$ and $T(v) = 0$. By linearity, $T(u + v) = T(u) + T(v) = 0 + 0 = 0$. ▪

Claim: Closure under Scaling

If $u \in \ker(T)$ and $c \in \mathbb{R}$, then $cu \in \ker(T)$.

Proof: By linearity, $T(cu) = cT(u) = c \cdot 0 = 0$. ▪

Conclusion

Since all conditions are met, $\ker(T)$ is a subspace of V. Q.E.D.

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Solution to Question 7

Topic: Set Operations

PROBLEM STATEMENT

Given: Let A and B be subsets of a universal set U.
To Prove: Prove $(A \cup B)^c \subseteq A^c \cap B^c$.

PROOF STRATEGY MAP

Element Selection → Definition of Complement → Logical Expansion

↓

Subset Result ← Intersection Definition

Element Selection
Let $x \in (A \cup B)^c$.

This means $x \notin (A \cup B)$.

Claim: Logical Expansion

$x \notin A$ and $x \notin B$.

Proof: By definition of union, if x is not in the union, it is in neither set. ▪

Since $x \in A^c$ and $x \in B^c$, by definition $x \in A^c \cap B^c$.

Subset Result
Thus $(A \cup B)^c \subseteq A^c \cap B^c$. Q.E.D.

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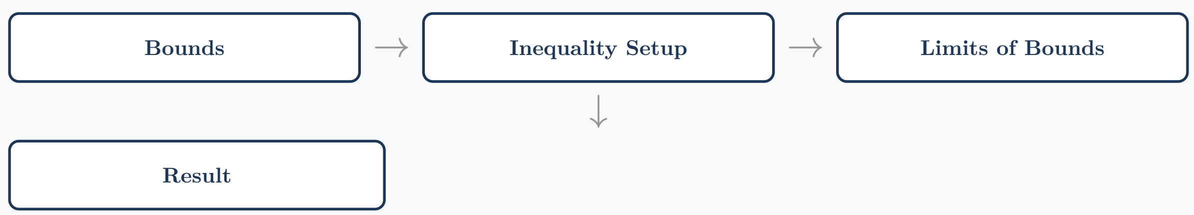
Solution to Question 8

Topic: Squeeze Theorem

PROBLEM STATEMENT

Given: Let $a_n = \frac{\sin(n)}{n}$ for $n \geq 1$.
To Prove: Prove $\lim_{n\to\infty} a_n = 0$.

PROOF STRATEGY MAP



Bounds

We know that for all real n, $-1 \leq \sin(n) \leq 1$.

Inequality Setup

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \; -\frac{1}{n} \leq a_n \leq \frac{1}{n}$$

Claim: Limits of Bounds

$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$ and $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$.

Proof: Standard limit of $\frac{1}{n}$. ▪

Result

By the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$. Q.E.D.

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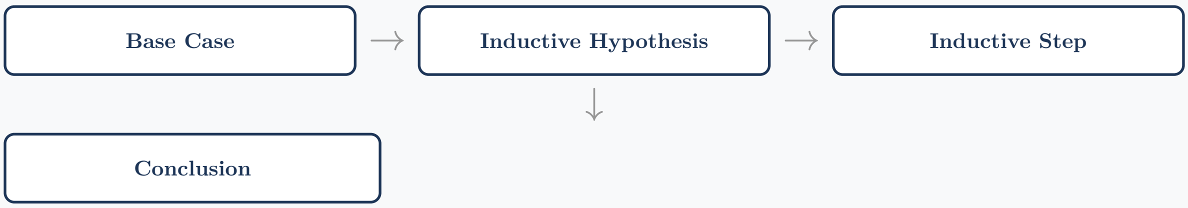
Solution to Question 9

Topic: Mathematical Induction

PROBLEM STATEMENT

Given: Let $S(n)$ be the statement $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
To Prove: Prove $S(n)$ is true for all $n \geq 1$.

PROOF STRATEGY MAP



Base Case

For $n = 1$, *LHS* $= 1$. *RHS* $= \frac{1(2)}{2} = 1$. *Thus* $S(1)$ *holds.*

Assume $S(k)$ is true: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Inductive Step

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2} + 1\right) = (k+1)\left(\frac{k+2}{2}\right) = \frac{(k+1)(k+2)}{2}$$

Conclusion

This matches the formula for $n = k+1$. By induction, the statement holds for all n. Q.E.D.

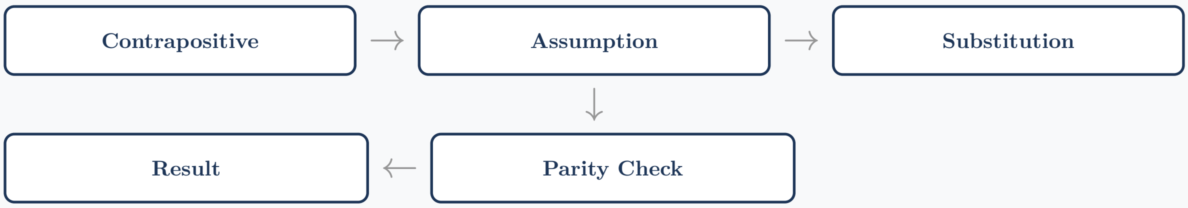
Solution to Question 10

Topic: Methods of Proof

PROBLEM STATEMENT

Given: Let n be an integer.
To Prove: Prove that if $3n + 2$ is odd, then n is odd.

PROOF STRATEGY MAP



Contrapositive

We will prove: If n is even, then $3n + 2$ is even.

Assume n is even. Then $n = 2k$ for some integer k.

Substitution

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

Claim: Parity Check

$2(3k + 1)$ is even.

Proof: Since $(3k + 1)$ is an integer, the expression is divisible by 2. ▪

Result

We proved the contrapositive, so the original statement is true. Q.E.D.

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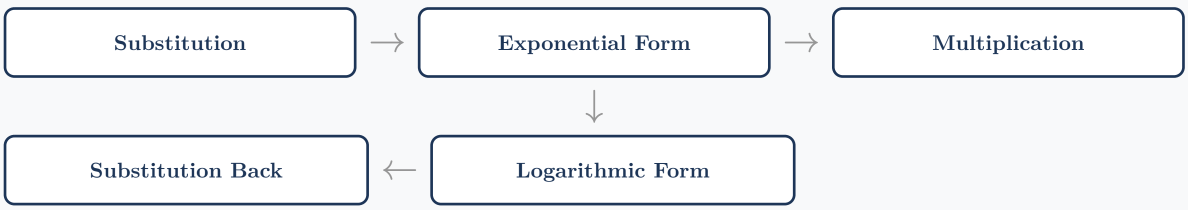
Solution to Question 11

Topic: Log Properties

PROBLEM STATEMENT

Given: Let $x, y > 0$.
To Prove: Prove $\log(xy) = \log(x) + \log(y)$ using exponent definitions.

PROOF STRATEGY MAP



Substitution

Let $u = \log_{b(x)}$ *and* $v = \log_{b(y)}$.

This implies $x = b^u$ and $y = b^v$.

Multiplication

$$xy = b^u \cdot b^v \ xy = b^{u+v}$$

Claim: Logarithmic Form

$\log_{b(xy)} = u + v$.

Proof: By the definition of the logarithm as the inverse of exponentiation. ▪

Substitution Back

Substituting u and v back, we get $\log_{b(xy)} = \log_{b(x)} + \log_{b(y)}$. Q.E.D.

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Solution to Question 12

Topic: Trigonometry

PROBLEM STATEMENT

Given: Consider a right triangle with hypotenuse 1 and angle θ.
To Prove: Prove $\sin^2(\theta) + \cos^2(\theta) = 1$.

PROOF STRATEGY MAP

Definitions → Trig Mapping → Pythagorean Theorem

Trig Mapping ↓ Substitution

Result ← Substitution

Definitions

Let the sides be a (adjacent), b (opposite), and $c = 1$ (hypotenuse).

Then $\cos(\theta) = \frac{a}{1} = a$ and $\sin(\theta) = \frac{b}{1} = b$.

Claim: Pythagorean Theorem

$a^2 + b^2 = c^2$.

Proof: Fundamental theorem of Euclidean geometry for right triangles. ▪

Substitution

$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1^2 \quad \cos^2(\theta) + \sin^2(\theta) = 1$$

Result
The identity holds. Q.E.D.

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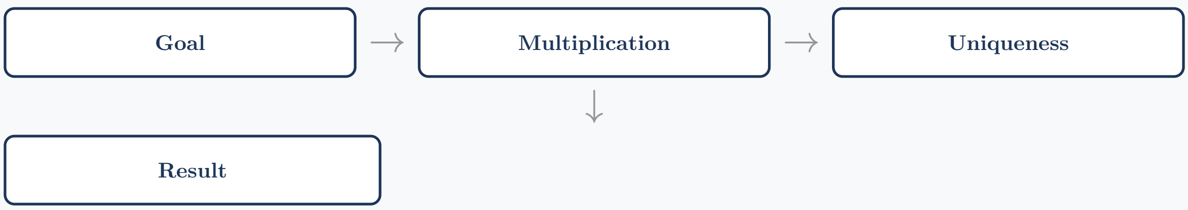
Solution to Question 13

Topic: Matrix Properties

PROBLEM STATEMENT

Given: Let A and B be invertible $n \times n$ matrices.
To Prove: Prove $(AB)^{-1} = B^{-1}A^{-1}$.

PROOF STRATEGY MAP



Goal
We must show that $(AB)(B^{-1}A^{-1}) = I$.

Multiplication

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I$$

Claim: Uniqueness

The inverse is unique.

Proof: Since the product is I, $(B^{-1}A^{-1})$ must be the inverse of AB. ▪

Result

Thus $(AB)^{-1} = B^{-1}A^{-1}$. Q.E.D.

■