

## Exam: General (KEY)

### Question 1

Calculus

2024 | Prof. Gauss

**Given:**

Let  $f(x) = x^2 e^x$ .

**To Prove:**

Calculate  $\int_0^1 f(x) dx$ .

*Hint: Use integration by parts twice.*

**Official Solution:**

**Step 1 (Calculation)**

Let  $u = x^2$  and  $dv = e^x dx$ .

**Final Answer (Result)**

$e - 2$ .

### Question 2

Combinatorics

2023 | Dr. Pascal

**Given:**

Expression  $(x + y)^4$ .

**To Prove:**

Expand using the Binomial Theorem.

*Hint: Coefficients are 1, 4, 6, 4, 1.*

**Official Solution:**

**Expansion (Result)**

$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ .

### Question 3

#### Complex Analysis

2024 | Prof. Euler

**Given:**

Consider  $z^3 = 1$ .

**To Prove:**

Find all roots of unity in the form  $a + bi$ .

**Hint:** Use De Moivre's Theorem:  $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ .

**Official Solution:****Roots (Result)**

$$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}.$$

### Question 4

#### Limits

2023 | Dr. Smith

**Given:**

Let  $x_n = \frac{3n+1}{n-2}$ .

**To Prove:**

Prove that  $\lim(x_n) = 3$ .

**Hint:** Use the definition of convergence.

**Official Solution:****Scratch Work (Proof)**

Find  $N$  such that  $|x_n - 3| < \varepsilon$ .

## Question 5

Limits

2022 | Dr. Smith

Given:

$$\text{Let } x_n = \frac{\sin(n)}{n}.$$

To Prove:

Prove that  $\lim(x_n) = 0$ .

**Hint:** Use the Squeeze Theorem with  $-\frac{1}{n} \leq x_n \leq \frac{1}{n}$ .

Official Solution:

**Bound (Proof)**

Since  $|\sin(n)| \leq 1$ , we have  $-1 \leq \sin(n) \leq 1$ .

## Question 6

Linear Algebra

2025 | Prof. Noether

Given:

$$\text{Let } A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

To Prove:

Find the eigenvalues of  $A$ .

**Hint:** Solve  $\det(A - \lambda I) = 0$ .

Official Solution:

**Characteristic Eq (Calculation)**

$$(2 - \lambda)^2 - 1 = 0.$$

**Question 7****Logic**

2025 | Dr. Boole

**Given:**Propositions  $P$  and  $Q$ .**To Prove:**Show that  $P \Rightarrow Q$  is equivalent to  $\neg Q \Rightarrow \neg P$ .**Hint:** Construct a truth table.**Official Solution:****Contrapositive (Proof)**

This is the Law of Contrapositive.

**Question 8****Series**

2023 | Dr. Cauchy

**Given:**Let  $S = \sum_{n=1}^{\infty} \frac{n!}{n^n}$ .**To Prove:**Determine if  $S$  converges.**Hint:** Use the Ratio Test.**Official Solution:****Ratio (Calculation)**

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{e} < 1.$$

## Question 9

Geometry

2024 | Prof. Euclid

**Given:**

Vectors  $(u) = (1, 2)$  and  $(v) = (3, -1)$ .

**To Prove:**

Calculate the angle between  $(u)$  and  $(v)$ .

**Hint:** Use  $(u) \cdot (v) = |(u)| |(v)| \cos(\theta)$ .

**Official Solution:**

**Dot Product (Calculation)**

$$1(3) + 2(-1) = 1.$$

## Question 10

Set Theory

2023 | Dr. Cantor

**Given:**

Let  $A$  and  $B$  be subsets of a universal set  $U$ .

**To Prove:**

Prove  $(A \cup B)^c = A^c \cap B^c$ .

**Hint:** Show mutual inclusion (LHS subset RHS and RHS subset LHS).

**Official Solution:**

**Forward Direction (Proof)**

Let  $x \in (A \cup B)^c$ . Then  $x \notin A \cup B$ .

## Question 11

**Topology**

2025 | Prof. Haus

**Given:**

Let  $X = \mathbb{R}$  with standard topology.

**To Prove:**

Prove that  $(0, 1)$  is an open set.

**Hint:** Find a ball  $B_{\varepsilon(x)}$  for every point.

**Official Solution:**

**Radius (Proof)**

For any  $x$ , choose  $\varepsilon = \min(x, 1 - x)$ .

**End of Examination**