

Real Analysis Midterm

Student Name:

ID:

Q1

2024

Prof. Lebesgue

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To Prove:

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$$

SCRATCHPAD

Recall:

Check assumptions: Finite measure? Non-negative?

FORMAL PROOF

Formal Proof: To prove that $\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$, we will use the Monotone Convergence Theorem (MCT) or the Dominated Convergence Theorem (DCT). Since the problem statement does not specify the nature of the measure space (finite or infinite), we will use the DCT, which requires that the measure space be complete and the functions be non-negative.

The DCT states that if (X, M, μ) is a measure space, $f_n \geq 0$ for all n , $f_n \rightarrow f$ pointwise a.e., and there exists a function $g \in L^1$ such that $|f_n| \leq g$ for all n , then $\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$.

In our case, we have $f_n \rightarrow f$ pointwise a.e. and $|f_n| \leq g \in L^1$. Therefore, by the DCT, we have

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$$

Q2

Given: Prove the limit of the function $f(x) = 2x + 3$ as $x \rightarrow 1$.

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Dr. Cauchy

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To Prove:

$$\lim_{x \rightarrow 1} (2x + 3) = 5$$

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Recall:

Did you express δ in terms of ε ?

FORMAL PROOF

Given $\varepsilon > 0$, we need to find $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|2x + 3 - 5| < \varepsilon$.

Starting with the inequality $|2x + 3 - 5| < \varepsilon$,

$$|2x + 3 - 5| = |2x - 2| = 2|x - 1|.$$

We want $2|x - 1| < \varepsilon$. Dividing both sides by 2,

$$|x - 1| < \frac{\varepsilon}{2}.$$

So, we can choose $\delta = \frac{\varepsilon}{2}$. Then, if $0 < |x - 1| < \delta$, we have

$$|x - 1| < \frac{\varepsilon}{2} \implies 2|x - 1| < 2 \cdot \frac{\varepsilon}{2} \implies 2|x - 1| < \varepsilon \implies |2x + 3 - 5| < \varepsilon.$$

Therefore, $\lim_{x \rightarrow 1} (2x + 3) = 5$.

Q3

Given: $\varphi : G \rightarrow H$ is a surjective homomorphism with kernel K .

2024
Prof. Noether

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To Prove:

$$\frac{G}{K} \cong H$$

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Recall:
Is the map well-defined?

FORMAL PROOF

Given: $\varphi : G \rightarrow H$ is a surjective homomorphism with kernel K . We want to show that $\frac{G}{K} \cong H$.

Let $x, y \in G$ such that $x \equiv y \pmod{K}$. Then $x - y \in K$, so $\varphi(x - y) = \varphi(x) - \varphi(y)$. Since φ is a homomorphism, $\varphi(x) - \varphi(y) = \varphi(x) + \varphi(-y) = \varphi(x) + \varphi(y)$. Therefore, $\varphi(x) = \varphi(y)$. This shows that φ is well-defined on the quotient group $\frac{G}{K}$.

Since φ is surjective, for every $h \in H$, there exists $g \in G$ such that $\varphi(g) = h$. Let $g_1, g_2 \in G$ be such that $\varphi(g_1) = \varphi(g_2) = h$. Then $\varphi(g_1 - g_2) = \varphi(g_1) - \varphi(g_2) = h - h = 0$. Since $\varphi(g_1 - g_2) \in K$, we have $g_1 - g_2 \in K$, so $g_1 \equiv g_2 \pmod{K}$. This shows that φ is injective on the quotient group $\frac{G}{K}$.

Therefore, φ is a bijective homomorphism from $\frac{G}{K}$ to H , which means $\frac{G}{K} \cong H$.

Hints & Summary

Q	Technique	Hint
1	Dominated Convergence	Apply Fatou's Lemma to $g - f_n - f $.
2	Epsilon-Delta	Start with $ f(x) - 5 < \varepsilon$ and solve for $ x - 1 $.
3	Isomorphism	Use the First Isomorphism Theorem.