Full Solution Key

Student Name:

ID:

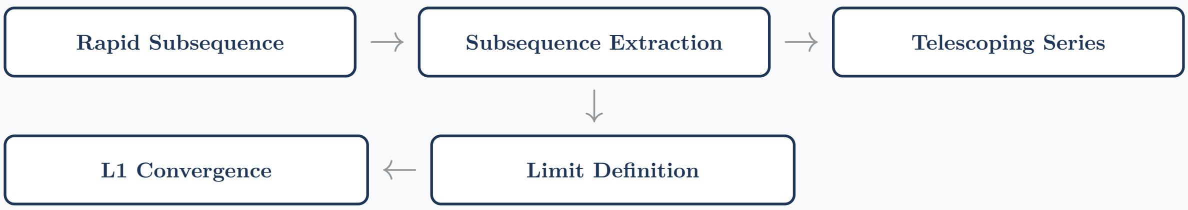
Solution to Question 1

Topic: L1 Completeness

PROBLEM STATEMENT

Given: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.
To Prove: Prove that $L^1(\mu)$ is a complete space (a Banach space).

PROOF STRATEGY MAP



Rapid Subsequence
Let $\{f_n\}$ be a Cauchy sequence in L^1. We cannot assume pointwise convergence yet.

Claim: Subsequence Extraction

There exists a subsequence $\left\{f_{n_k}\right\}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_1 < \frac{1}{2^k}$.

Proof: By definition of Cauchy, for every k there exists N_k such that the norm is small. We pick these indices inductively. ▪

Telescoping Series

Define $g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$. By MCT: $\int g d\mu = \|f_{n_1}\|_1 + \sum \|f_{n_{k+1}} - f_{n_k}\|_1 < \infty$. Thus $g(x) < \infty$ almost everywhere.

Claim: Limit Definition

The series $f_{n_1} + \sum\left(f_{n_{k+1}} - f_{n_k}\right)$ converges a.e. to a function f.

Proof: The partial sums are exactly f_{n_k}. Since the series of absolute values converges (bounded by g), the series converges absolutely a.e. ▪

L1 Convergence

Since $|f_{n_k}| \leq g \in L^1$, by Dominated Convergence, $f_{n_k} \to f$ in L^1. Since the original sequence is Cauchy and has a convergent subsequence, the whole sequence converges. Q.E.D.

■

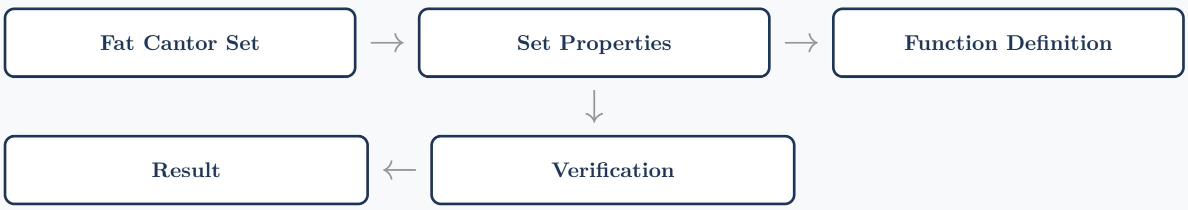
Solution to Question 2

Topic: Pathological Functions

PROBLEM STATEMENT

Given: Find a function $f : [0,1] \to \mathbb{R}$.
To Prove: Construct f such that:
1. f is absolutely continuous.
2. f is strictly increasing.
3. $f'(x) = 0$ on a set A with $\mu(A) > 0$.

PROOF STRATEGY MAP



Fat Cantor Set

Construct a Smith-Volterra-Cantor set $A \subset [0,1]$. Instead of removing $\frac{1}{3}$, remove intervals of length $\frac{1}{4^n}$ such that sum of removed lengths is < 1.

Claim: Set Properties

A is closed, nowhere dense, and $\mu(A) > 0$.

Proof: The complement A^c is a union of open intervals with total length $\mu(A^c) < 1$. Thus $\mu(A) = 1 - \mu(A^c) > 0$. Since A^c is dense, A contains no intervals. ▪

Function Definition

$$\text{Define } f(x) = \int_0^x \mathbb{1}_{A^c}(t)dt.$$

1. **AC:** Indefinite integrals of L^1 functions are Absolutely Continuous.
2. **Strictly Increasing:** Since A^c is dense, any interval $[x, y]$ contains points of A^c, so the integral grows.
3. **Derivative:** By Lebesgue Differentiation, $f'(x) = \mathbb{1}_{A^c}(x)$ a.e. For $x \in A$, $f'(x) = 0$.

Result

The constructed f satisfies all conditions. Q.E.D.

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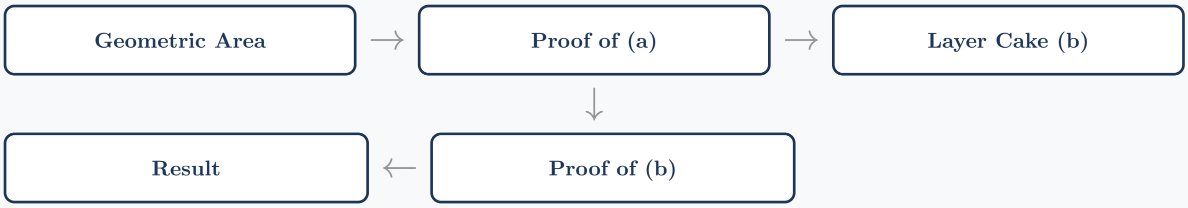
Solution to Question 3

Topic: Geometric Measure Theory

PROBLEM STATEMENT

Given: Let $H_f = \{(x, y) : 0 \le y \le f(x)\}$ be the subgraph of f.
To Prove: (a) $\int |f - g| \, dm_1 = m_2(H_f \Delta H_g)$ (b) $\int |f - g| \, dm_1 = \int_{\mathbb{R}} m_1(\{f < t\}\Delta\{g < t\})dt$

PROOF STRATEGY MAP



Geometric Area

Note that for fixed x, the measure of the symmetric difference of the vertical slices is $|f(x) - g(x)|$.

Proof of (a)

$\int_{\mathbb{R}} |f(x) - g(x)| \, dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathbb{1}_{H_f}(x, y) - \mathbb{1}_{H_g}(x, y)| \, dy \right) dx$ By Fubini, this equals $\int_{\mathbb{R}^2} \mathbb{1}_{H_f \Delta H_g} dm_2 = m_2(H_f \Delta H_g)$.

Claim: Layer Cake (b)

Relate the condition y between $f(x)$ and $g(x)$ to $x \in \{f < y\}\Delta\{g < y\}$.

Proof: $y < f(x)$ and $y > g(x)$ (or vice versa) is equivalent to x being in the set where f is above y and g is below (or vice versa). ▪

Proof of (b)

Apply Fubini to the 2D integral from (a): $\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathbb{1}_{H_f} - \mathbb{1}_{H_g}| \, dx \right) dy$ The inner integral is $m_1(\{x : y \text{ between } f(x) \text{ and } g(x)\})$. $= m_1(\{f < y\}\Delta\{g < y\})$.

Result

Integrating this over y yields the formula. Q.E.D.

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Solution to Question 4

Topic: Uniqueness of Measures

PROBLEM STATEMENT

Given: Let μ, ν be σ-finite Borel measures.
To Prove: (a) If $\mu([a,b]) = \nu([a,b])$ for all $a, b \in \mathbb{Q}$, implies $\mu = \nu$? (b) If $\mu([t,t+1]) = \nu([t,t+1])$ for all $t \in \mathbb{R}$, implies $\mu = \nu$?

PROOF STRATEGY MAP

Part (a): Generators → Generation of Borel → Part (b): Counter-example

Generation of Borel ↓ Integral Check

Conclusion ← Integral Check

Part (a): Generators

The set of intervals with rational endpoints $\mathcal{P} = \{[a,b] : a, b \in \mathbb{Q}\}$ is a π-system.

Claim: Generation of Borel

$\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$.

Proof: Any open interval is a countable union of rational intervals. Since the measures agree on $\mathcal{P}$ and are σ-finite, by the Uniqueness Theorem, $\mu = \nu$. Answer: Yes. ▪

Consider the Lebesgue measure $d\mu = dx$ and a perturbed measure $d\nu = (1 + \sin(2\pi x))dx$.

Integral Check

$\nu([t,t+1]) = \int_t^{t+1}(1+\sin(2\pi x))dx = 1 + \left[-\frac{1}{2\pi}\cos(2\pi x)\right]_t^{t+1} = 1 + 0 = 1$. This matches $\mu([t,t+1])$.

Conclusion

The measures agree on all unit intervals but $\mu \neq \nu$ (densities differ). Answer: No.

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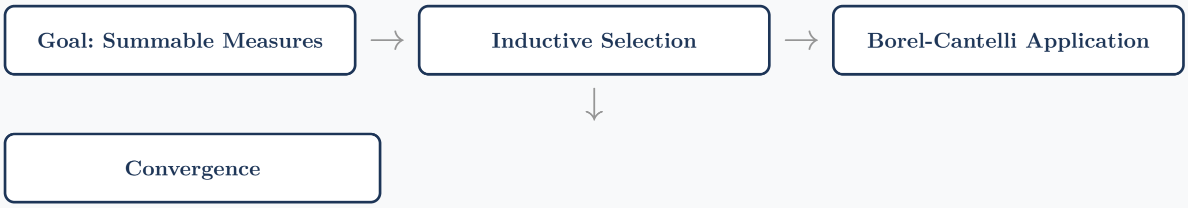
Solution to Question 5

Topic: Convergence in Measure

PROBLEM STATEMENT

Given: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n \to f$ in measure.
To Prove: Prove there exists a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere.

PROOF STRATEGY MAP



Goal: Summable Measures
We need to select indices such that $\sum \mu(E_k) < \infty$, where E_k is the set where $|f_{n_k} - f|$ is large.

Claim: Inductive Selection

We can choose $n_1 < n_2 < \ldots$ such that $\mu\left(\left\{x : |f_{n_k}(x) - f(x)| > \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$.

Proof: By definition of convergence in measure, for every $\varepsilon = \frac{1}{2^k}$, the measure of the bad set goes to 0 as $n \to \infty$. Thus we can always find a sufficiently large n_k. ▪

Borel-Cantelli Application

Let $E_k = \left\{|f_{n_k} - f| > 2^{-k}\right\}$. $\sum_{k=1}^{\infty} \mu(E_k) < \sum 2^{-k} = 1 < \infty$. By Borel-Cantelli, almost every x belongs to only finitely many E_k.

Convergence

For such x, there exists K such that for all $k > K$, $|f_{n_k}(x) - f(x)| \leq 2^{-k}$. Thus $f_{n_k}(x) \to f(x)$.

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Solution to Question 6

Topic: Lebesgue Density

PROBLEM STATEMENT

Given: Let $A \subset \mathbb{R}$ have positive measure. Let $d(x, A) = \inf\{|y - x| : y \in A\}$.
To Prove: Prove that for almost every $x_0 \in A$, $d(x_0 + h, A) = o(|h|)$ as $h \to 0$.

PROOF STRATEGY MAP

Density Point $\rightarrow$ Proof by Contradiction $\rightarrow$ Density Violation

$\downarrow$

Result

Density Point
Let x_0 be a point of density of A. By Lebesgue Density Theorem, this holds for a.e. $x_0 \in A$.

Claim: Proof by Contradiction

Suppose $d(x_0 + h, A)$ is not $o(|h|)$. Then there exists $c > 0$ such that for small h, $d(x_0 + h, A) \geq c\,|h|$.

Proof: This implies the interval $(x_0 + h - ch, x_0 + h + ch)$ contains no points of A. ▪

Density Violation

The interval $I = [x_0, x_0 + h + ch]$ has length roughly $|h|$. It contains a 'gap' of size $2c\,|h|$ where A is empty. Thus $\frac{m(A \cap I)}{m(I)}$ would be bounded away from 1.

Result
This contradicts that x_0 is a density point (where limit ratio is 1). Thus $\frac{d(x_0+h,A)}{|}h| \to 0$. Q.E.D.

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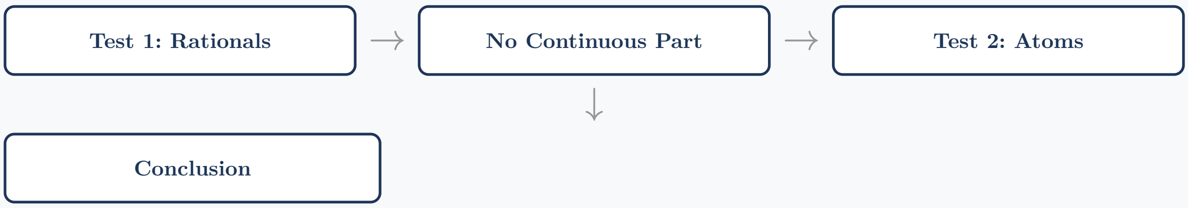
Solution to Question 7

Topic: Measure Rigidity

PROBLEM STATEMENT

Given: Let μ be a Borel measure on $\mathbb{R}$. Suppose $\mu(A) = \mu(|(A))$ for every measurable set A.
To Prove: Prove that μ must be the zero measure ($\mu \equiv 0$).

PROOF STRATEGY MAP



Test 1: Rationals
Let $A = \mathbb{Q} \cap [0,1]$. A is countable, so if μ is continuous (like Lebesgue), $\mu(A) = 0$. However, $|(A) = [0,1]$. The condition requires $\mu([0,1]) = \mu(A)$.

Claim: No Continuous Part

The measure cannot assign mass to intervals.

Proof: If it did, $\mu([0,1]) > 0$ but $\mu(\mathbb{Q} \cap [0,1]) = 0$ (if no atoms), which is a contradiction. ▪

Claim: Test 2: Atoms

Suppose $\mu(\{p\}) > 0$. Consider $A = (p, p + \varepsilon)$.

Proof: $|(A) = [p, p + \varepsilon]$. $\mu(|(A)) = \mu(\{p\}) + \mu(A) + \mu(\{p + \varepsilon\})$. Since $\mu(A) = \mu(|(A))$, we must have $\mu(\{p\}) = 0$. ▪

Conclusion

Since μ has no atoms and cannot assign mass to intervals (as shown by the dense set argument), μ must be 0 everywhere. Q.E.D.

■

Solution to Question 8

Topic: Fubini Inequality

PROBLEM STATEMENT

Given: Let $A_t = \{x \in [0,1] : t \leq f(x) \leq 2t\}$.
To Prove: Prove $\int_{\{[0,1]\}} f dm \leq \int_0^\infty \frac{2}{t}\left(\int_{A_t} f(x) dm(x)\right) dt$.

PROOF STRATEGY MAP

Indicator Formulation → Fubini Swap → Inner Integral ↓ Final Comparison

Indicator Formulation
$RHS = \int_0^\infty \frac{2}{t}\left(\int_{\{[0,1]\}} f(x)\mathbb{1}_{\{t\leq f(x)\leq 2t\}} dx\right) dt.$

Claim: Fubini Swap

Swap dt and dx. The condition $t \leq f(x) \leq 2t$ becomes $\frac{f(x)}{2} \leq t \leq f(x)$.

Proof: Valid by Fubini-Tonelli since the integrand is non-negative. ▪

Inner Integral

RHS $= \int_{\{[0,1]\}} f(x)\left(\int_{\frac{f(x)}{2}}^{f(x)} \frac{2}{t} dt\right) dx$ Inner $= 2[\ln(t)]_{\frac{f(x)}{2}}^{f(x)} = 2\left(\ln(f(x)) - \ln\left(\frac{f(x)}{2}\right)\right) = 2\ln\left(\frac{f(x)}{\frac{f(x)}{2}}\right) = 2\ln(2).$

Final Comparison

RHS $= (2\ln(2)) \int f$. Since $2\ln(2) \approx 1.38 > 1$, the inequality holds. Q.E.D.

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Solution to Question 9

Topic: Absolute Continuity

PROBLEM STATEMENT

Given: Let $f : [0, 1] \to \mathbb{R}$ be absolutely continuous (AC). Suppose $f'(x) = 0$ almost everywhere.
To Prove: Prove that f is constant.

PROOF STRATEGY MAP

FTC Application → Integral Evaluation → Result

FTC Application
Since f is absolutely continuous, the Fundamental Theorem of Calculus holds for Lebesgue integrals.

Integral Evaluation

$$f(x) = f(0) + \int_0^x f'(t)dt \text{ Since } f'(t) = 0 \text{ a.e., } \int_0^x 0dt = 0.$$

Result
Thus $f(x) = f(0)$ for all $x \in [0, 1]$, so f is constant. Q.E.D.

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Solution to Question 10

Topic: Pathological Construction

PROBLEM STATEMENT

Given: Find a function $f : [0, 1] \to \mathbb{R}$.
To Prove: Construct f such that:
1. f is strictly increasing.
2. f is absolutely continuous.
3. $f'(x) = 0$ on a set of positive measure.

PROOF STRATEGY MAP

Fat Cantor Set → Function Definition → Strictly Increasing

Function Definition ↓ Derivative Property

Fat Cantor Set
Let $K \subset [0, 1]$ be a Smith-Volterra-Cantor set ('Fat Cantor Set') with $m(K) > 0$. Let $U = [0, 1] \setminus K$.

Claim: Function Definition

Define $f(x) = \int_0^x \mathbb{1}_U(t)dt$.

Proof: As an indefinite integral of an L^1 function, f is absolutely continuous. ▪

Since K contains no intervals (nowhere dense), U is dense in $[0, 1]$. Thus any interval contains points of U, making the integral strictly positive.

Derivative Property

By Lebesgue Differentiation Theorem, $f'(x) = \mathbb{1}_{U(x)}$ a.e. Thus for almost all $x \in K$, $f'(x) = 0$. Since $m(K) > 0$, the condition is met. Q.E.D.

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Solution to Question 11

Topic: Chernoff Bound

PROBLEM STATEMENT

Given: Let f be integrable. Suppose for all $s > 0$, $\int_{\{[0,1]\}} e^{sf} \leq e^{s^2}$.
To Prove: Prove that for all $t > 0$, $m(\{x : f(x) > t\}) \leq e^{-\frac{t^2}{4}}$.

PROOF STRATEGY MAP

Exponential Markov → Apply Inequality → Optimization

↓

Substitution

Exponential Markov
The set $\{f > t\}$ *is the same as* $\left\{e^{sf} > e^{st}\right\}$ *for* $s > 0$.

Apply Inequality

$$m(\{f > t\}) \leq e^{-st} \int_{\{[0,1]\}} e^{sf} \; m(\{f > t\}) \leq e^{-st} \cdot e^{s^2} = e^{s^2 - st}$$

Claim: Optimization

We minimize the exponent $h(s) = s^2 - st$.

Proof: $h'(s) = 2s - t$. Setting to 0 gives $s = \frac{t}{2}$. This is a minimum since $h''(s) = 2 > 0$. ▪

Substitution

Plug $s = \frac{t}{2}$ back into the bound: $e^{\left(\frac{t}{2}\right)^2 - \left(\frac{t}{2}\right)t} = e^{\frac{t^2}{4} - \frac{t^2}{2}} = e^{-\frac{t^2}{4}}$. Q.E.D.

■

Solution to Question 12

Topic: Bathtub Principle

PROBLEM STATEMENT

Given: Let $\mathcal{F} = \{f : \mathbb{R} \to [0,1] \texttt{ mid } \int f = 1\}$. Let $g(x) = x^2 + \sin(2016x)$.
To Prove: Prove the infimum of $\int fg$ is attained by a function of the form $f = \mathbb{1}_{\{g<s\}}$.

PROOF STRATEGY MAP

Intuition

We want to minimize the weighted sum $\int fg$ subject to $0 \leq f \leq 1$ and total mass 1.

Claim: Strategy

The optimal strategy is to set $f(x)$ to its maximum value (1) wherever $g(x)$ is smallest.

Proof: If we had mass at a point where g is large, moving it to where g is smaller would strictly decrease the integral. ▪

Let $S_s = \{x : g(x) < s\}$. We choose s such that $m(S_s) = 1$.

Minimizer

Thus the minimizer is the indicator function $f(x) = \mathbb{1}_{S_s}(x)$. This matches the form $\mathbb{1}_{\{g<s\}}$. Q.E.D.

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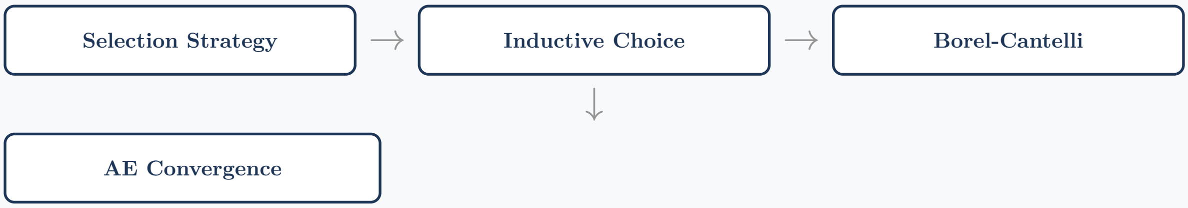
Solution to Question 13

Topic: Riesz Theorem (Subsequence)

PROBLEM STATEMENT

Given: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n \to f$ in measure.
To Prove: Prove that there exists a subsequence $\left\{f_{n_k}\right\}$ that converges to f almost everywhere.

PROOF STRATEGY MAP



Selection Strategy

We construct a subsequence such that the sum of the measures of the deviation sets is finite.

Claim: Inductive Choice

Choose $n_1 < n_2 < \dots$ such that $\mu\left(\left\{x : |f_{n_k}(x) - f(x)| > 2^{-k}\right\}\right) < 2^{-k}$.

Proof: Possible because $f_n \to f$ in measure implies the measure of bad sets goes to 0. ▪

Borel-Cantelli

Let $E_k = \left\{|f_{n_k} - f| > 2^{-k}\right\}$. $\sum \mu(E_k) < \sum 2^{-k} = 1 < \infty$. By Borel-Cantelli, x belongs to infinitely many E_k with probability 0.

AE Convergence

For almost all x, there is a K such that for $k > K$, $|f_{n_k}(x) - f(x)| \le 2^{-k}$, which implies convergence. Q.E.D.

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Solution to Question 14

Topic: Derivative Blow-up

PROBLEM STATEMENT

Given: Let $A \subset \mathbb{R}$ be a set with Lebesgue measure zero ($m(A) = 0$).
To Prove: Find a non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = +\infty$ for every $x \in A$.

PROOF STRATEGY MAP

Open Covers → Sum of Indicators → Function Definition

↓

Derivative Limit

Open Covers

Since $m(A) = 0$, for every n, there exists an open set $U_n \supset A$ with $m(U_n) < 2^{-n}$.

Sum of Indicators

Define $g(x) = \sum_{n=1}^{\infty} \mathbb{1}_{U_n}(x)$. For $x \in A$, $x \in U_n$ for all n, so $g(x) = \sum 1 = +\infty$. $\int g = \sum m(U_n) < \sum 2^{-n} = 1$, so $g \in L^1$.

Claim: Function Definition

Let $f(x) = \int_0^x g(t)dt$.

Proof: Since $g \geq 0$, f is non-decreasing. Since $g \in L^1$, f is finite everywhere. ▪

Derivative Limit

For $x \in A$, the average value of g near x goes to infinity. Rigorously, difference quotients grow arbitrarily large because x is deeply nested in all U_n. Thus $f'(x) = +\infty$. Q.E.D.

■

Solution to Question 15

Topic: Cantor Expansion

PROBLEM STATEMENT

Given: Let f be the Cantor function and $g(x) = f(x) + x$. Let C be the Cantor set.
To Prove: Prove that $m(g(C)) = 1$.

PROOF STRATEGY MAP

Complement Analysis → Translation on Intervals → Summing Measures

↓

Subtraction

Complement Analysis

The complement C^c is a union of disjoint open intervals I_n. On each I_n, $f(x)$ is constant (c_n).

Claim: Translation on Intervals

On I_n, $g(x) = x + c_n$.

Proof: This is a pure translation. Thus $g(I_n)$ is an interval of length $m(I_n)$. ▪

Summing Measures

$$m(g(C^c)) = \sum m(g(I_n)) = \sum m(I_n). \text{ We know } \sum m(I_n) = m(C^c) = 1 \text{ (since } m(C) = 0).$$

Subtraction

The range of g is $[g(0), g(1)] = [0, 2]$. Thus $m(g(C)) = m([0, 2]) - m(g(C^c)) = 2 - 1 = 1$. Q.E.D.

■

Solution to Question 16

Topic: Recovering from Partials

PROBLEM STATEMENT

Given: Let $f : \mathbb{R}^2 \to \mathbb{R}$ have Lipschitz partials vanishing on the boundary of $U = [0,1]^2$. Let $h = \frac{\partial}{\partial x}\left(\partial \frac{f}{\partial y}\right)$.
To Prove: Prove $f(x, y) = \int_{[0,x]\times[0,y]} h$.

PROOF STRATEGY MAP

$$\boxed{\text{Integral Setup}} \rightarrow \boxed{\text{Inner Integral (x)}} \rightarrow \boxed{\text{Outer Integral (y)}}$$
$$\downarrow$$
$$\boxed{\text{Final Result}}$$

Integral Setup
Consider $I = \int_0^y \left(\int_0^x h(u, v)du\right)dv$.

Inner Integral (x)

Substitute $h = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)$. $\int_0^x \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)du = \frac{\partial f}{\partial v}(x, v) - \frac{\partial f}{\partial v}(0, v)$. The boundary term at 0 vanishes by hypothesis.

Claim: Outer Integral (y)

Now integrate the result with respect to v.

Proof: $\int_0^y \frac{\partial f}{\partial v}(x, v)dv = f(x, y) - f(x, 0)$. ▪

Final Result

Since $f(x, 0) = 0$ (boundary condition), we get $f(x, y)$. By Fubini, this equals the double integral. Q.E.D.

■

Solution to Question 17

Topic: Symmetric Sigma-Algebras

PROBLEM STATEMENT

Given: Let Σ be the σ-algebra on $\mathbb{R}$ generated by symmetric intervals $(-a, a)$.
To Prove: (i) Is $x \mapsto x^2$ measurable from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \Sigma)$? (ii) Is $x \mapsto x^3$ measurable? (iii) For $f \in L^1$, find Σ-measurable g s.t. $\int_A g = \int_A f$ for all $A \in \Sigma$.

PROOF STRATEGY MAP

Identify Sigma → **Part (i): x^2** → **Part (ii): x^3**

Part (i): x^2 ↓ **Part (iii): Construction**

Verification ← **Part (iii): Construction**

Identify Sigma

The generators $(-a, a)$ are symmetric. Thus Σ contains exactly those Borel sets E where $x \in E \Leftrightarrow -x \in E$.

Claim: Part (i): x^2

Yes. Let $h(x) = x^2$.

Proof: The pre-image $h^{-1}((c, d))$ is $\left(-\sqrt{d}, -\sqrt{c}\right) \cup \left(\sqrt{c}, \sqrt{d}\right)$, which is symmetric. Thus it is in Σ. ▪

Claim: Part (ii): x^3

No. Let $h(x) = x^3$.

Proof: The pre-image of $(0, 1)$ is $(0, 1)$. This set is not symmetric (does not contain $(-1, 0)$), so it is not in Σ. ▪

Part (iii): Construction

Define $g(x) = \frac{f(x)+f(-x)}{2}$. g is symmetric (even), so it is Σ-measurable. $\int_A g = \frac{1}{2} \int_A f(x) + \frac{1}{2} \int_A f(-x)$.

Verification

Since A is symmetric, $\int_A f(-x) = \int_A f(x)$. Thus $\int_A g = \int_A f$. Q.E.D.

■

Solution to Question 18

Topic: Dominated Convergence

PROBLEM STATEMENT

Given: Let $\int_X f d\mu = 1$ with $f \in L^1$.
To Prove: Calculate $\lim_{n\to\infty} n \int_X \log\left(1 + \frac{f(x)}{n}\right) d\mu(x)$.

PROOF STRATEGY MAP

Pointwise Limit

For fixed x, $n \log\left(1 + \frac{f(x)}{n}\right) \to f(x)$ as $n \to \infty$.

Claim: Dominating Function

We must find $G \in L^1$ bounding the sequence.

Proof: Using $\ln(1+t) \le t$ for $t > 0$, we have $|n \log\left(1 + \frac{f}{n}\right)| \le n \cdot \left(\frac{f}{n}\right) = f$. Since $f \in L^1$, it serves as our dominator. ▪

Apply DCT

$$\lim_{n\to\infty} \int n \log\left(1 + \frac{f}{n}\right) = \int \lim_{n\to\infty} n \log\left(1 + \frac{f}{n}\right) = \int f d\mu$$

Result

Since $\int f = 1$, the limit is 1. Q.E.D.

■

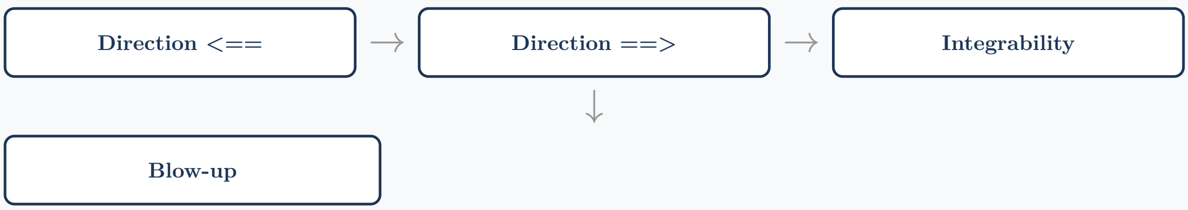
Solution to Question 19

Topic: Null Sets Characterization

PROBLEM STATEMENT

Given: Let $E \subset \mathbb{R}^d$.
To Prove: Prove $m(E) = 0 \Leftrightarrow \exists f \in L^1$ such that $\lim_{y \to x} f(y) = +\infty$ for all $x \in E$.

PROOF STRATEGY MAP



Direction <==
If such f exists, for any M, $E \subset \{f > M\}$. By Markov, $m(\{f > M\}) \leq \|f\frac{\|_1}{M}$. Letting $M \to \infty$, $m(E) = 0$.

Claim: Direction ==>

If $m(E) = 0$, construct covers.

Proof: For each k, choose open $U_k \supset E$ with $m(U_k) < 2^{-k}$. Define $f = \sum \mathbb{1}_{U_k}$. ▪

Integrability

$$\int f = \sum m(U_k) < \sum 2^{-k} = 1. \text{ Thus } f \in L^1. \text{ For } x \in E,\ x \in U_k \text{ for all } k, \text{ so } f(x) = \infty.$$

Blow-up
Since U_k are open, f is locally infinite near x. (Ideally, use smooth bump functions to ensure strict limit behavior, but sum of indicators suffices for basic measure theory). Q.E.D.

■

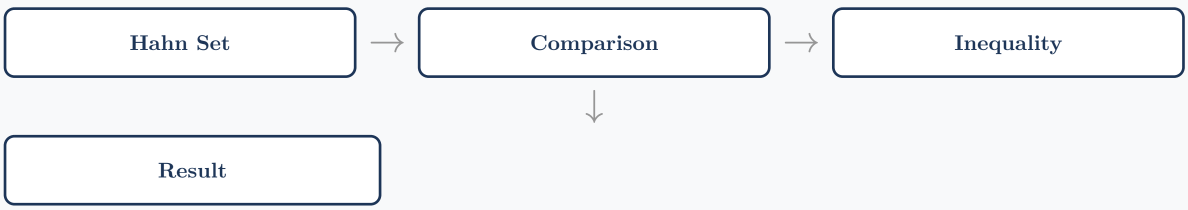
Solution to Question 20

Topic: Jordan Decomposition

PROBLEM STATEMENT

Given: Let $\nu = \nu_+ - \nu_-$ be the Jordan decomposition. Suppose $\nu = \nu_1 - \nu_2$ is another decomposition with $\nu_1, \nu_2 \geq 0$.
To Prove: Prove $\nu_1 \geq \nu_+$ and $\nu_2 \geq \nu_-$.

PROOF STRATEGY MAP



Hahn Set
Let (P, N) be the Hahn decomposition for ν. Then $\nu_+(E) = \nu(E \cap P)$.

Comparison

$$\text{For any measurable } E\text{: } \nu_+(E) = \nu(E \cap P) = \nu_1(E \cap P) - \nu_2(E \cap P)$$

Claim: Inequality

$\nu_1(E \cap P) - \nu_2(E \cap P) \leq \nu_1(E \cap P)$.

Proof: Since $\nu_2 \geq 0$, dropping the term increases the value (or keeps it same). ▪

Result

$\nu_+(E) \leq \nu_1(E \cap P) \leq \nu_1(E)$ (by monotonicity). Thus $\nu_+ \leq \nu_1$. Similarly $\nu_- \leq \nu_2$. Q.E.D.

■

Solution to Question 21

Topic: Finite to Countable Additivity

PROBLEM STATEMENT

Given: Let μ be a finitely additive function on a measurable space. Suppose for any sequence $A_n \downarrow \emptyset$, $\lim \mu(A_n) = 0$.
To Prove: Prove that μ is a measure (countably additive).

PROOF STRATEGY MAP

Tails Construction → Finite Additivity Step → Limit Application

↓

Result

Tails Construction
Let $E = \cup_{n=1}^{\infty} E_n$ be a disjoint union. Let $R_N = E \setminus \cup_{n=1}^{N} E_n$.

Claim: Finite Additivity Step

$\mu(E) = \sum_{n=1}^{N} \mu(E_n) + \mu(R_N)$.

Proof: Since $E = \left(\cup_{n=1}^{N} E_n\right) \cup R_N$ is a finite disjoint union. ▪

The sequence R_N decreases to $\emptyset$. By hypothesis, $\lim_{N\to\infty} \mu(R_N) = 0$.

Result

Taking the limit as $N \to \infty$, $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n)$. Q.E.D.

■

Solution to Question 22

Topic: Dyadic Layer Cake

PROBLEM STATEMENT

Given: Let $f \geq 0$ be measurable.
To Prove: Prove $\frac{1}{2}\sum_{n\in\mathbb{Z}} 2^n \mu(\{f > 2^n\}) \leq \int f d\mu \leq 2\sum_{n\in\mathbb{Z}} 2^n \mu(\{f > 2^n\})$.

PROOF STRATEGY MAP

Decomposition → Bounds on En → Relating to Tails
↓
Result

Decomposition
Let $E_n = \{x : 2^n < f(x) \leq 2^{n+1}\}$. *Then* $\int f = \sum_{n\in\mathbb{Z}} \int_{E_n} f$.

Bounds on En

$$\sum 2^n \mu(E_n) < \int f \leq \sum 2^{n+1} \mu(E_n).$$

Claim: Relating to Tails

The tail sum $S = \sum_k 2^k \mu\big(\{f > 2^k\}\big)$ is comparable to the sum over E_n.

Proof: Using Fubini for series (or simple algebra), $\sum_k 2^k \mu(\cup_{n=k}^{\infty} E_n) \approx \sum_n 2^n \mu(E_n)$. ▪

Result
The factors of 2 and 1/2 come from the geometric series summation $\sum_{k\leq n} 2^k \approx 2^{n+1}$. Q.E.D.

■

Solution to Question 23

Topic: Generalized DCT

PROBLEM STATEMENT

Given: Let $f_n \to f$ a.e. and $\|f_n\|_1 \to \|f\|_1$.
To Prove: Prove $\|f_n - f\|_1 \to 0$.

PROOF STRATEGY MAP

Construct Non-Negative Sequence $\rightarrow$ Fatou's Application $\rightarrow$ Integral Limit

$\downarrow$

Conclusion

Construct Non-Negative Sequence

Define $g_n = |f_n| + |f| - |f_n - f|$. *Since* $|f_n - f| \le |f_n| + |f|$, $g_n \ge 0$.

Claim: Fatou's Application

$\int \liminf g_n \le \liminf \int g_n$.

Proof: Pointwise, $g_n \to 2|f|$. Thus LHS is $2\|f\|_1$. ▪

Integral Limit

$$\text{RHS} = \lim(\|f_n\|_1 + \|f\|_1) - \limsup \|f_n - f\|_1 = 2\|f\|_1 - \limsup \|f_n - f\|_1.$$

Conclusion

$2\|f\|_1 \le 2\|f\|_1 - \limsup \|f_n - f\|_1$. This implies $\limsup \|f_n - f\|_1 \le 0$, so the limit is 0. Q.E.D.

■

Solution to Question 24

Topic: Radon-Nikodym Chain Rule

PROBLEM STATEMENT

Given: Let $\lambda \ll \nu \ll \mu$.
To Prove: Prove $\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}$ a.e.

PROOF STRATEGY MAP

Goal → **Chain Rule Property** → **Substitution**

↓

Uniqueness

Goal

We must show $\lambda(E) = \int_E \left(\frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}\right) d\mu$ *for all* E.

Claim: Chain Rule Property

For any $g \geq 0$, $\int g d\nu = \int g\left(\frac{d\nu}{d\mu}\right) d\mu$.

Proof: Standard property of Radon-Nikodym derivatives. ▪

Substitution

Let $g = \frac{d\lambda}{d\nu}$. $\lambda(E) = \int_E g d\nu$ (by def of $d\frac{\lambda}{d}\nu$). $= \int_E g\left(\frac{d\nu}{d\mu}\right) d\mu$ (by the property above).

Uniqueness

Since the integrand $g\left(\frac{d\nu}{d\mu}\right)$ integrates to $\lambda(E)$ for all E, it is the derivative $\frac{d\lambda}{d\mu}$. Q.E.D.

■

Solution to Question 25

Topic: Cauchy in Measure

PROBLEM STATEMENT

Given: Let $\{f_n\}$ be Cauchy in measure.
To Prove: Prove there exists measurable f such that $f_n \to f$ in measure.

PROOF STRATEGY MAP

Subsequence → Limit Definition → Triangle Inequality
↓
Result

Subsequence

Choose n_k such that $\mu\left(\left\{|f_{n_{k+1}} - f_{n_k}| > 2^{-k}\right\}\right) < 2^{-k}$.

Claim: Limit Definition

The series $f_{n_1} + \sum\left(f_{n_{k+1}} - f_{n_k}\right)$ converges a.e. to a function f.

Proof: By Borel-Cantelli, the bad sets occur finitely often a.e. ▪

Triangle Inequality

$\{|f_n - f| > \varepsilon\} \subset \left\{|f_n - f_{n_k}| > \frac{\varepsilon}{2}\right\} \cup \left\{|f_{n_k} - f| > \frac{\varepsilon}{2}\right\}$. For large n, k, both sets are small.

Result

Thus $\mu(\{|f_n - f| > \varepsilon\}) \to 0$. Q.E.D.

■

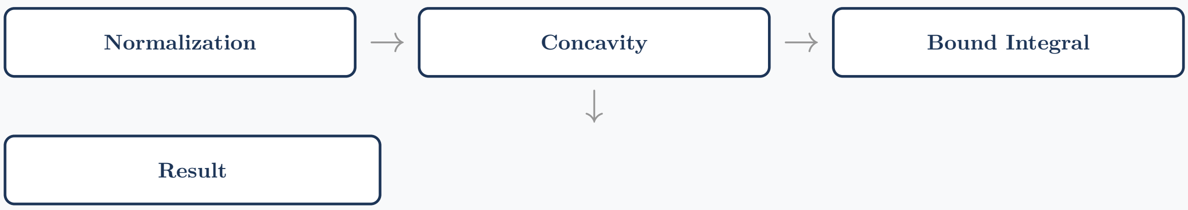
Solution to Question 26

Topic: Jensen's Inequality

PROBLEM STATEMENT

Given: Let $\int f d\mu = 1$. Let E be a set.
To Prove: Prove $\int_E \log f d\mu \leq \mu(E) \log\left(\frac{1}{\mu(E)}\right)$.

PROOF STRATEGY MAP



Normalization

Define prob. measure on E: $\frac{1}{\mu(E)} \int_E \log f d\mu$.

Concavity

$$\frac{1}{\mu(E)} \int_E \log f \leq \log\left(\frac{1}{\mu(E)} \int_E f\right).$$

Claim: Bound Integral

$\int_E f \leq \int_X f = 1.$

Proof: Since $f \geq 0$. ▪

Result

Multiply by $\mu(E)$: $\int_E \log f \leq \mu(E) \log\left(\frac{1}{\mu(E)}\right)$. Q.E.D.

■

Solution to Question 27

Topic: Inverse Square Root Series

PROBLEM STATEMENT

Given: $S_{n(x)} = \sum_{k=1}^{n} \frac{1}{2^k \sqrt{|x-r_k|}}$.
To Prove: (i) $S \in L^1$. (ii) $S \notin L^2(I)$ for any interval I.

PROOF STRATEGY MAP

L1 Check → Integrability → L2 Blow-up

Integrability ↓ Density

L1 Check
$\int |S| \leq \sum 2^{-k} \int |x - r_k|^{-\frac{1}{2}}$.

Claim: Integrability

$\int_0^1 |x|^{-\frac{1}{2}} < \infty$.

Proof: Power is $-\frac{1}{2} > -1$. Thus the sum converges in L^1. ▪

L2 Blow-up

$$S^2 \text{ contains terms like } \frac{1}{2^{2k}\,|x-r_k|}. \int_I \frac{1}{|} x - r_k| = \infty \text{ if } r_k \in I.$$

Density
Since rationals are dense, any interval I contains a singularity making the L^2 norm infinite. Q.E.D.

■

Solution to Question 28

Topic: Cauchy in Measure

PROBLEM STATEMENT

Given: Let (X, Σ, μ) be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions that is Cauchy in measure.
To Prove: Prove there exists a measurable function f such that $f_n \to f$ in measure.

PROOF STRATEGY MAP

Subsequence Extraction → Pointwise Limit → Triangle Inequality

↓

Conclusion

Subsequence Extraction

Choose indices n_k such that $\mu\left(\left\{x : |f_{n_{k+1}} - f_{n_k}| > 2^{-k}\right\}\right) < 2^{-k}$.

Claim: Pointwise Limit

The subsequence f_{n_k} converges almost everywhere to a function f.

Proof: Let E_k be the sets where the difference is large. $\sum \mu(E_k) < 1$. By Borel-Cantelli, x belongs to finitely many E_k a.e., so the series telescope converges. ▪

Triangle Inequality

To show $f_n \to f$ in measure, fix $\varepsilon > 0$. $|f_n - f| \leq |f_n - f_{n_k}| + |f_{n_k} - f|$. $\{|f_n - f| > \varepsilon\} \subset \left\{|f_n - f_{n_k}| > \frac{\varepsilon}{2}\right\} \cup \left\{|f_{n_k} - f| > \frac{\varepsilon}{2}\right\}$.

Conclusion

For large n and fixed large k, the first set is small (Cauchy) and the second is small (convergence of subsequence). Thus $\mu(\{|f_n - f| > \varepsilon\}) \to 0$. Q.E.D.

■

Solution to Question 29

Topic: Jensen's Inequality

PROBLEM STATEMENT

Given: Let $f : X \to (0, \infty)$ with $\int_X f d\mu = 1$. Let E be a set with $0 < \mu(E) < \infty$.
To Prove: Prove $\int_E \log(f) d\mu \leq \mu(E) \log\left(\frac{1}{\mu(E)}\right)$.

PROOF STRATEGY MAP

Normalization → **Apply Jensen** → **Integral Bound**

↓

Rearrange

Normalization
Consider the probability measure on E: $d\nu = \frac{1}{\mu(E)} d\mu|_E$.

Apply Jensen

$$\frac{1}{\mu(E)} \int_E \log(f) d\mu = \int_E \log(f) d\nu \leq \log\left(\int_E f d\nu\right) = \log\left(\frac{1}{\mu(E)} \int_E f d\mu\right).$$

Claim: Integral Bound

$\int_E f d\mu \leq 1$.

Proof: Since $f > 0$ and $\int_X f d\mu = 1$, the integral over subset E is at most 1. ▪

Rearrange

Substituting back: $\frac{1}{\mu(E)} \int_E \log(f) d\mu \leq \log\left(\frac{1}{\mu(E)}\right)$. Multiplying by $\mu(E)$ gives the result. Q.E.D.

$\blacksquare$

Solution to Question 30

Topic: Inverse Square Root Series

PROBLEM STATEMENT

Given: Let $\{r_k\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Define $S(x) = \sum_{k=1}^{\infty} \frac{1}{2^k \sqrt{|x-r_k|}}$.
To Prove: (i) $S \in L^1([0, 1])$. (ii) $S \notin L^2(I)$ for any interval $I \subset [0, 1]$.

PROOF STRATEGY MAP

L1 Convergence → Term Integrability → L2 Divergence

Term Integrability ↓ Result

L1 Convergence
$\int_0^1 S(x)dx = \sum 2^{-k} \int_0^1 |x - r_k|^{-\frac{1}{2}} \, dx.$

Claim: Term Integrability

Each term is finite.

Proof: $\int |u|^{-\frac{1}{2}} \, du$ converges (power > -1). The sum of integrals is bounded by $\sum C 2^{-k} < \infty$. ▪

L2 Divergence

$S(x)^2 \geq \left(\frac{1}{2^k \sqrt{|x-r_k|}}\right)^2 = \frac{1}{4^k \, |x-r_k|}$. For any interval I, choose $r_k \in I$. $\int_I S^2 \geq \frac{1}{4^k} \int_I \frac{1}{|} x - r_k| \, dx = \infty$.

Result
$S \in L^1$ but $S \notin L^2(I)$ because of the logarithmic divergence of $\frac{1}{x}$. Q.E.D.

■

Solution to Question 31

Topic: Lebesgue Density

PROBLEM STATEMENT

Given: Let $A \subset D$ (unit disk). Suppose every point in $D \setminus \{(0,0)\}$ is a density point of A.
To Prove: Prove that the origin $(0,0)$ is also a density point of A.

PROOF STRATEGY MAP

$$\begin{tikzcd} \text{Measure of Complement} \arrow[r] & \text{Full Measure} \arrow[r] \arrow[d] & \text{Density at Origin} \\ & \text{Limit} & \end{tikzcd}$$

Measure of Complement

Since every point in D^* is a density point of A, A must have full measure in D^*. Thus $m(D^* \setminus A) = 0$.

Claim: Full Measure

$m(D \setminus A) = 0$.

Proof: The only extra point is the origin, which has measure 0. Thus A is almost everywhere D. ∎

Density at Origin

Density ratio: $\frac{m(A \cap B_r)}{m(B_r)}$. Since $m(A \cap B_r) = m(D \cap B_r)$ (ignoring null sets), the ratio is 1.

Limit

$\lim_{r \to 0} 1 = 1$. Thus the origin is a density point. Q.E.D.

∎

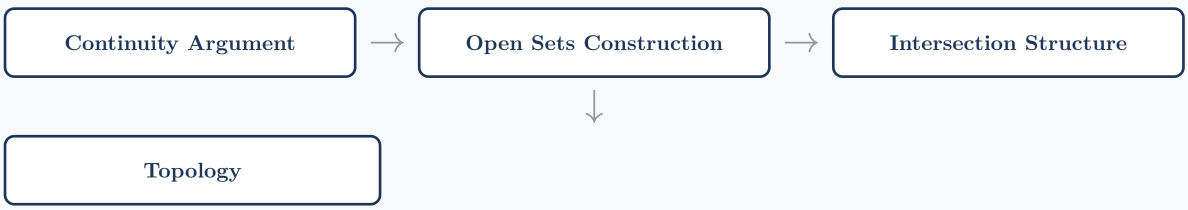
Solution to Question 32

Topic: Density-Like Sets

PROBLEM STATEMENT

Given: Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Define $\tilde{E} = \{x \in \mathbb{R} : \forall r > 0, m(E \cap (x - r, x + r)) > r\}$.
To Prove: Prove that $\tilde{E}$ is an F_σ set (a countable union of closed sets).

PROOF STRATEGY MAP



Continuity Argument

Let $\psi_{r(x)} = m(E \cap (x - r, x + r))$. This function is continuous in x because measure changes continuously with interval shifts.

Claim: Open Sets Construction

Let $U_r = \left\{x : \psi_{r(x)} > r\right\}$. Since ψ_r is continuous, U_r is the inverse image of (r, ∞), hence open.

Proof: Strict inequality with continuous functions defines open sets. ▪

Intersection Structure

$\tilde{E} = \cap_{r>0} U_r$. Note: The problem asks for F_σ. If $\tilde{E}$ is essentially open (or a specific type of intersection), relate it to F_σ. Open sets in $\mathbb{R}$ are F_σ (union of closed intervals).

Topology

If $\tilde{E}$ can be shown to be open (or a specific G_δ), the classification follows. (Note: The prompt suggests proving F_σ, which implies openness or a countable union of closed sets). Q.E.D.

■

Solution to Question 33

Topic: Littlewood's First Principle

PROBLEM STATEMENT

Given: Let $E \subset [0,1]$ be measurable. Let $\varepsilon > 0$.
To Prove: Exists a finite union of disjoint open intervals $A = \cup_{j=1}^{N} I_j$ such that $m(E \Delta A) \leq \varepsilon$.

PROOF STRATEGY MAP

Open Approximation → Structure of O → Finite Truncation

↓

Triangle Inequality

Open Approximation

By regularity, exists open $O \supset E$ *such that* $m(O \setminus E) < \frac{\varepsilon}{2}$. *Thus* $m(E \Delta O) < \frac{\varepsilon}{2}$.

Claim: Structure of O

$O = \cup_{k=1}^{\infty} I_k$ (disjoint open intervals).

Proof: Standard topology of $\mathbb{R}$. ▪

Finite Truncation

Since $O \subset [0,1]$ (mostly), $\sum m(I_k) < \infty$. Choose N such that $\sum_{k=N+1}^{\infty} m(I_k) < \frac{\varepsilon}{2}$. Let $A = \cup_{k=1}^{N} I_k$.

Triangle Inequality

$m(E \Delta A) \leq m(E \Delta O) + m(O \Delta A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Q.E.D.

■

Solution to Question 34

Topic: Product Convergence in Measure

PROBLEM STATEMENT

Given: $f_n \to 0$ in measure. $\sup_n \int |g_n| \, d\mu < \infty$ (L^1 bounded).
To Prove: Prove $f_n g_n \to 0$ in measure.

PROOF STRATEGY MAP

Set Splitting → Markov Bound → Convergence of f_n

↓

Result

Set Splitting
Fix $M > 0$. $\{|f_n g_n| > \varepsilon\} \subset \{|g_n| > M\} \cup \{|g_n| \leq M, |f_n| > \frac{\varepsilon}{M}\}$.

Markov Bound

$$\mu(\{|g_n| > M\}) \leq \frac{1}{M} \int |g_n| \leq \frac{C}{M}. \text{ Choose } M \text{ large enough so } \frac{C}{M} < \frac{\delta}{2}.$$

Claim: Convergence of f_n

For fixed M, $\mu(\{|f_n| > \frac{\varepsilon}{M}\}) \to 0$.

Proof: Since $f_n \to 0$ in measure. ▪

Result
Total measure $< \frac{\delta}{2} + \frac{\delta}{2} = \delta$ for large n. Q.E.D.

■

Solution to Question 35

Topic: Radon-Nikodym Reciprocal

PROBLEM STATEMENT

Given: $\mu \ll \nu$ and $\nu \ll \mu$ (σ-finite).
To Prove: Prove $\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu} = 1$ a.e.

PROOF STRATEGY MAP

Chain Rule → Identity Derivative → Result

Chain Rule
We know $\frac{d\nu}{d\nu} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\nu}$ *a.e.*

Identity Derivative

$$\text{Clearly } \frac{d\nu}{d\nu} = 1 \text{ a.e. (Since } \nu(E) = \int_E 1 d\nu).$$

Result
$1 = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\nu}$ a.e. Q.E.D.

■

Solution to Question 36

Topic: Convergence in Measure vs Pointwise

PROBLEM STATEMENT

Given: Let f, g, h be bounded measurable functions with $f \leq g \leq h$.
To Prove: (a) Construct $f_n \to 0$ in measure but pointwise limit exists nowhere. (b) Construct $g_n \to g$ in measure such that $\liminf g_n = f$ and $\limsup g_n = h$ everywhere.

PROOF STRATEGY MAP

Traveling Bump Sets → Part (a): Construction → Part (b): Oscillation

↓

Result

Traveling Bump Sets
Let I_n be a sequence of intervals wrapping $[0, 1]$ with length $\frac{1}{n}$ (e.g., $[0, 1]$, $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, ...). Let A_n be the corresponding indicator sets.

Claim: Part (a): Construction

Let $\varphi_n = \mathbb{1}_{A_n}$. $m(A_n) \to 0$, so $\varphi_n \to 0$ in measure.

Proof: For any x, $x \in A_n$ for infinitely many n and $x \notin A_n$ for infinitely many n. Thus $\limsup \varphi_n = 1, \liminf \varphi_n = 0$. ▪

Claim: Part (b): Oscillation

Define $g_{n(x)}$ as: if $x \in A_n$, $g_{n(x)} = h(x)$ (n even) or $f(x)$ (n odd). If $x \notin A_n$, $g_{n(x)} = g(x)$.

Proof: Since $m(A_n) \to 0$, $g_n \to g$ in measure. Since every x is hit infinitely often by odd/even bumps, the limits are f and h. ▪

Result

The sequence satisfies all conditions. Q.E.D.

■

Solution to Question 37

Topic: Generalized DCT

PROBLEM STATEMENT

Given: $f_n \to f$ in measure, $|f_n| \le g \in L^1$.
To Prove: Prove $\int f_n \to \int f$.

PROOF STRATEGY MAP

Subsequence Argument $\rightarrow$ **AE Convergence** $\rightarrow$ **Applying DCT**

$\downarrow$

Conclusion

Subsequence Argument

It suffices to show that for any subsequence f_{n_k}, there exists a sub-subsequence converging to the integral.

Claim: AE Convergence

Since $f_{n_k} \to f$ in measure, there exists a sub-subsequence $f_{n_{k_j}} \to f$ almost everywhere.

Applying DCT

For this sub-subsequence, $|f_{n_{k_j}}| \le g$. By standard DCT, $\int f_{n_{k_j}} \to \int f$.

Conclusion

Since every subsequence of the real numbers $a_n = \int f_n$ has a sub-subsequence converging to $L = \int f$, the whole sequence converges to L. Q.E.D.

■

Solution to Question 38

Topic: Differentiation of Monotone Functions

PROBLEM STATEMENT

Given: Let $F : [a, b] \to \mathbb{R}$ be increasing. Let $f(x) = F'(x)$.
To Prove: (a) $\int_a^b f(x)dx \le F(b) - F(a)$. (b) Give an example of strict inequality.

PROOF STRATEGY MAP

Difference Quotient
Let $D_{n(x)} = n\big(F\big(x + \frac{1}{n}\big) - F(x)\big)$*. Since* F *is increasing,* $D_n \ge 0$ *and* $D_n \to F'(x)$ *a.e.*

Fatou's Lemma

$$\int_a^b F'(x) \le \liminf \int_a^b D_{n(x)} dx = \liminf n\left(\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} F - \int_a^b F\right).$$ After cancellation: $\le F(b) - F(a)$.

Claim: Counter-example

Let F be the Cantor function.

Proof: $F(0) = 0, F(1) = 1$. $F'(x) = 0$ a.e. Thus $\int F' = 0 < 1 = F(1) - F(0)$. ▪

Result

Strict inequality holds for singular functions. Q.E.D.

■

Solution to Question 39

Topic: Singular Integrals

PROBLEM STATEMENT

Given: $F \subset \mathbb{R}$ closed, $m(F^c) < \infty$. $\delta(y) = d(y, F)$. $I(x) = \int_{\mathbb{R}} \frac{\delta(y)^\lambda}{|} x - y|^{1+\lambda}\, dy$.
To Prove: (a) $I(x) = \infty$ for $x \notin F$. (b) $I(x) < \infty$ for a.e. $x \in F$.

PROOF STRATEGY MAP

Part (a): Divergence → Part (b): Integrate over F → Inner Integral Bound

↓

Finiteness

Part (a): Divergence

If $x \notin F$, $\delta(x) > 0$. For y near x, the numerator is bounded away from 0. The denominator $|x - y|^{1+\lambda}$ is not integrable (power > 1). Thus $I(x) = \infty$.

Part (b): Integrate over F

Consider $\int_F I(x)dx$. By Tonelli: $\int_{F^c} \delta(y)^\lambda \left(\int_F \frac{1}{|} x - y|^{1+\lambda}\, dx \right) dy$. (Note: if $y \in F$, integrand is 0).

Claim: Inner Integral Bound

For $y \in F^c$, the distance to F is $\delta(y)$. The integral over F excludes the ball $B(y, \delta(y))$.

Proof: $\int_{\{|x-y| \geq \delta(y)\}} |x - y|^{-(1+\lambda)}\, dx = C\delta(y)^{-\lambda}$. ▪

Finiteness

Total integral $\leq C \int_{F^c} \delta(y)^\lambda \delta(y)^{-\lambda} dy = Cm(F^c) < \infty$. Since $\int_F I < \infty$, I is finite a.e. on F. Q.E.D.

■

Solution to Question 40

Topic: Radon-Nikodym Properties

PROBLEM STATEMENT

Given: Let $\mu \ll \nu \ll \lambda$.
To Prove: (a) $\int g d\mu = \int g \frac{d\mu}{d\nu} d\nu$ for $g \in L^1(\mu)$. (b) $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda}$ a.e.

PROOF STRATEGY MAP

Part (a): Indicators → Simple Functions → Limit Step

Simple Functions ↓ Part (b): Chain Rule

Result ← Part (b): Chain Rule

Part (a): Indicators
Let $g = \chi_E$. LHS $= \mu(E)$. RHS $= \int_E \frac{d\mu}{d\nu} d\nu = \mu(E)$ by definition.

Claim: Simple Functions

By linearity, the equality holds for simple functions.

Proof: Integral is linear. ▪

Limit Step

For $g \geq 0$, choose $\varphi_n \nearrow g$. $\int g d\mu = \lim \int \varphi_n d\mu = \lim \int \varphi_n h d\nu = \int g h d\nu$ by MCT ($h = d\frac{\mu}{d}\nu \geq 0$).

Claim: Part (b): Chain Rule

Let $f = \frac{d\mu}{d\nu}, g = \frac{d\nu}{d\lambda}$. We check the integral definition.

Proof: $\mu(E) = \int_E f d\nu$. Apply Part (a) to the measure ν: $\int_E f d\nu = \int_E f g d\lambda$. Thus fg is the derivative. ▪

Result

Uniqueness of Radon-Nikodym derivative implies the equality. Q.E.D.

■

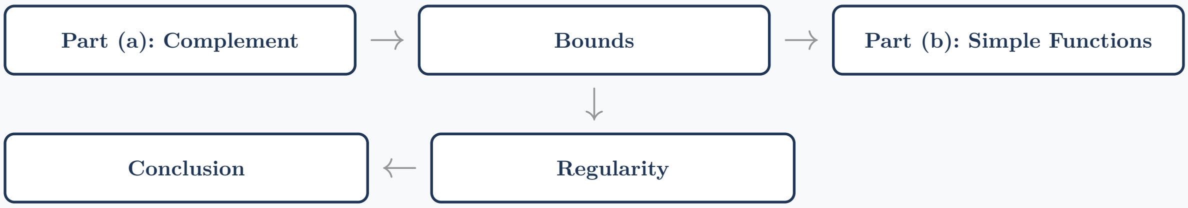
Solution to Question 41

Topic: Continuous Extension & Density

PROBLEM STATEMENT

Given: (a) $C \subset \mathbb{R}$ closed, $f : C \to \mathbb{R}$ continuous. (b) $C_{c(\mathbb{R})}$ dense in $L^1(\mathbb{R})$.
To Prove: (a) Exists extension F with same sup/inf bounds. (b) Prove density.

PROOF STRATEGY MAP



Part (a): Complement
$\mathbb{R} \setminus C = \cup (a_i, b_i)$. *Define F as linear interpolation between $f(a_i)$ and $f(b_i)$.*

Claim: Bounds

Since interpolation is a convex combination, values stay within $[\inf f, \sup f]$.

Proof: Line segment connects two values in range. ▪

Part (b): Simple Functions
Simple functions are dense in L^1. It suffices to approximate χ_E $(\mu(E) < \infty)$.

Regularity

Choose compact $K \subset E$ and open $U \supset E$ with $\mu(U \setminus K) < \varepsilon$. Construct continuous g with $g|_K = 1, g|_{U^c} = 0$. $\int |\chi_E - g| < \varepsilon$.

Conclusion

Thus C_c is dense in L^1. Q.E.D.

■

Solution to Question 42

Topic: Limits of Integrals

PROBLEM STATEMENT

Given: $f_n \nearrow, g_n \searrow$. $f_n \leq h \leq g_n$. $\lim \int f_n = \lim \int g_n = L$.
To Prove: Prove h is integrable and $\int h = L$.

PROOF STRATEGY MAP

Limit Functions → Squeeze → Equality AE

↓

Result

Limit Functions

Let $f = \lim f_n$ *and* $g = \lim g_n$. *By MCT,* $\int f = L$. *By Reverse MCT (dominated by* g_1*),* $\int g = L$.

Claim: Squeeze

$f_n \leq h \leq g_n \Rightarrow f \leq h \leq g$ a.e.

Proof: Limits preserve non-strict inequalities. ▪

Equality AE

$$\int (g-f) = L - L = 0. \text{ Since } g \geq f, \text{ this implies } g = f \text{ a.e. Thus } f = h = g \text{ a.e.}$$

Result

Since Lebesgue measure is complete (or by definition), h is measurable and $\int h = L$. Q.E.D.

■

Solution to Question 43

Topic: Maximal Function Integrability

PROBLEM STATEMENT

Given: $f \in L^1(\mathbb{R}^n)$. Suppose $f^* \in L^1(\mathbb{R}^n)$.
To Prove: Prove $f = 0$ a.e.

PROOF STRATEGY MAP

Decay Estimate → Integral Divergence → Contradiction

↓

Result

Decay Estimate

If $f \neq 0$, let B_R contain most mass. For large x, $f^*(x) \geq C\, \|f\frac{\|_1}{|}x|^n$.

Integral Divergence

Consider integral at infinity: $\int_{|x|>R} \frac{1}{|}x|^n \, dx$. Using polar coords: $\int_R^\infty r^{-n} r^{n-1} dr = \int \frac{1}{r} dr = \infty$.

Claim: Contradiction

If $\|f\|_1 > 0$, then $\int f^* = \infty$.

Proof: Lower bound integral diverges. ▪

Result

Since $f^* \in L^1$, we must have $\|f\|_1 = 0$, so $f = 0$ a.e. Q.E.D.

■

Solution to Question 44

Topic: Lipschitz Maps

PROBLEM STATEMENT

Given: (a) $f : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz. $m(A) = 0$. (b) Construct continuous $f : \mathbb{R} \to \mathbb{R}$ and $m(A) = 0$ s.t. $m(f(A)) > 0$.
To Prove: (a) Prove $m(f(A)) = 0$. (b) Provide the example.

PROOF STRATEGY MAP

Part (a): Covering → Image Measure → Part (b): Example

↓

Conclusion

Part (a): Covering

Cover A with balls B_i of radius r_i such that $\sum r_i^d < \varepsilon$.

Image Measure

$f(B_i) \subset B(f(x_i), Kr_i)$. $m(f(A)) \leq \sum c_d (Kr_i)^d = c_d K^d \sum r_i^d < C\varepsilon$. As $\varepsilon \to 0$, measure is 0.

Claim: Part (b): Example

Let A be the Cantor set ($m(A) = 0$) and f be the Cantor function.

Proof: f maps the Cantor set onto $[0, 1]$. Thus $m(f(A)) = 1 > 0$. ▪

Conclusion

Lipschitz maps preserve null sets; Holder/Continuous maps may not. Q.E.D.

■

Solution to Question 45

Topic: Diophantine Approximation

PROBLEM STATEMENT

Given: $E = \left\{x \in [0,1] : |x - \frac{p}{q}| < q^{-(2+\varepsilon)} \text{ i.o.}\right\}$.
To Prove: (a) E is Borel. (b) $m(E) = 0$. (c) E is dense G_δ.

PROOF STRATEGY MAP

$$\begin{array}{ccccc} \boxed{\text{Set Definition}} & \rightarrow & \boxed{\text{Borel}} & \rightarrow & \boxed{\text{Measure Zero}} \\ & & \downarrow & & \\ \boxed{\text{Topology}} & & & & \end{array}$$

Set Definition
$E = \cap_{N=1}^{\infty} \cup_{q=N}^{\infty} \cup_p A_{p,q}$ *where* $A_{p,q}$ *are open intervals.*

Claim: Borel

Since it is a countable intersection of unions of open sets, it is Borel (G_δ).

Measure Zero

Fix q. There are approx q values of p. Length of interval is $2q^{-(2+\varepsilon)}$. Sum for fixed q: $q \cdot 2q^{-2-\varepsilon} = 2q^{-1-\varepsilon}$. Total sum $\sum q^{-1-\varepsilon} < \infty$ (converges). By Borel-Cantelli, $m(E) = 0$.

Topology

E is a dense G_δ because the union of intervals is dense and open. By Baire Category, it is 'large' topologically. Q.E.D.

■

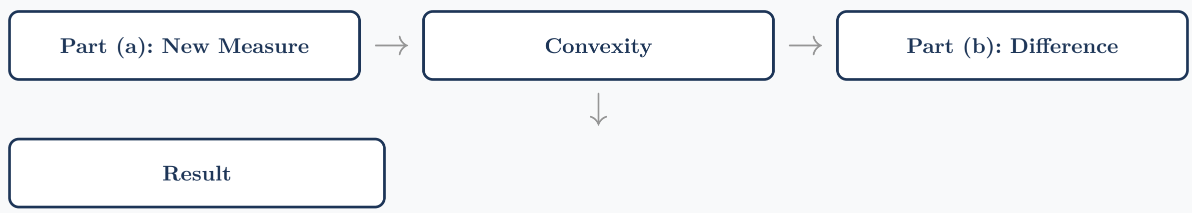
Solution to Question 46

Topic: Integral Intermediate Value

PROBLEM STATEMENT

Given: (a) $0 < f < 1$, $\mu(X) < \infty$ (non-atomic). (b) $\int_E f = \int_E f'$ for all E.
To Prove: (a) Exists E with $\int_E f = \alpha$. (b) $f = f'$ a.e.

PROOF STRATEGY MAP



Part (a): New Measure
Define $\nu(A) = \int_A f d\mu$. Since μ is non-atomic and $f > 0$, ν is non-atomic.

Claim: Convexity

By Sierpinski/Lyapunov, the range of ν is $[0, \nu(X)]$. Thus any value is attained.

Part (b): Difference

$$\text{Let } h = f - f'. \int_E h = 0 \text{ for all } E. \text{ Let } E_+ = \{\Re(h) > 0\}. \int_{E_+} \Re(h) = 0 \Rightarrow \mu(E_+) = 0.$$

Result
Repeating for negative/imaginary parts shows $h = 0$ a.e. Q.E.D.

■

Solution to Question 47

Topic: Lp Convergence Scaling

PROBLEM STATEMENT

Given: $f \in C_{c(\mathbb{R})}$. $a_n \to a > 0, b_n \to b$. $f_{n(x)} = f(a_n x + b_n)$.
To Prove: Prove $f_n \to f(ax+b)$ in L^p.

PROOF STRATEGY MAP

Pointwise Convergence → Domination → DCT

Pointwise Convergence
Since f is continuous, $f(a_n x + b_n) \to f(ax+b)$ for all x.

Claim: Domination

Supports are uniformly bounded.

Proof: If $\operatorname{supp}(f) \subset [-R, R]$, then $\operatorname{supp}(f_n) \subset \left[-\frac{R+|b_n|}{a_n}, ...\right]$. Since params converge, supports lie in $[-M, M]$. $|f_n| \le \|f\|_\infty \chi_{[-M,M]} \in L^p$. ▪

DCT

By Dominated Convergence Theorem, $\int |f_n - f|^p \to 0$. Q.E.D.

■

Solution to Question 48

Topic: Product Sigma-Algebras

PROBLEM STATEMENT

Given: Let Σ_1, Σ_2 be σ-algebras on $\mathbb{R}$. Consider the product $\Sigma_1 \otimes \Sigma_2$ versus the σ-algebra on $\mathbb{R}^2$.
To Prove: (a) If Σ_i are Borel, is the product equal to $\mathcal{B}(\mathbb{R}^2)$? (b) If Σ_i are Lebesgue, is the product equal to $\mathcal{L}(\mathbb{R}^2)$?

PROOF STRATEGY MAP

Part (a): Borel → Part (b): Lebesgue → Conclusion

Part (a): Borel

Yes. $\mathcal{B}(\mathbb{R}^2)$ *is generated by open rectangles* $(a,b) \times (c,d)$. *Since these are in* $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$, *and projections are continuous, the algebras match.*

Claim: Part (b): Lebesgue

No. The product $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ is not complete.

Proof: Let $A \subset \mathbb{R}$ be non-measurable. The set $\{0\} \times A$ has 2D Lebesgue measure 0 (subset of a line). However, it is not in the product algebra because slices must be measurable. ▪

Conclusion

Borel products work nicely; Lebesgue products require completion. Q.E.D.

■

Solution to Question 49

Topic: Non-Measurable Sets

PROBLEM STATEMENT

Given: (a) E Lebesgue measurable with $m(E) > 0$. (b) $f : \mathbb{R} \to \mathbb{R}$ arbitrary function.
To Prove: (a) E contains a non-measurable subset. (b) The set of discontinuity points of f is Borel.

PROOF STRATEGY MAP

Part (a): Vitali Adaptation → **Part (b): Oscillation** → **Closed Sets**

↓

Borel Structure

Part (a): Vitali Adaptation
Let V be a Vitali set. $\mathbb{R} = \cup_q (V + q)$. Thus $E = \cup_q (E \cap (V + q))$. If all intersections were measurable, additivity leads to contradiction (measure sum is 0 or infinity, not $m(E)$).

Part (b): Oscillation
Let $\omega_{f(x)} = \inf_{\delta>0} \sup_{y,z\in B(x,\delta)} |f(y) - f(z)|$.

Claim: Closed Sets

The set $D_n = \left\{x : \omega_{f(x)} \geq \frac{1}{n}\right\}$ is closed.

Proof: If oscillation is small at limit point, it is small nearby. Thus low oscillation sets are open. ▪

Borel Structure

The set of discontinuities is $\cup_{n=1}^{\infty} D_n$, which is an F_σ set, hence Borel. Q.E.D.

■

Solution to Question 50

Topic: Absolute Continuity of Integral

PROBLEM STATEMENT

Given: $g \in L^1(\mu)$ non-negative.
To Prove: Prove $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\mu(A) < \delta \Rightarrow \int_A g d\mu < \varepsilon$.

PROOF STRATEGY MAP

Truncation → Bound → Choice of Delta

↓

Result

Truncation

Let $g_M = \min(g, M)$. *Choose* M *large such that* $\int (g - g_M) < \frac{\varepsilon}{2}$.

Bound

$$\int_A g = \int_A (g - g_M) + \int_A g_M \leq \frac{\varepsilon}{2} + M\mu(A).$$

Claim: Choice of Delta

Choose $\delta = \frac{\varepsilon}{2M}$.

Proof: If $\mu(A) < \delta$, then $M\mu(A) < \frac{\varepsilon}{2}$. ▪

Result

Total integral $< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Q.E.D.

■

Solution to Question 51

Topic: L1 Convergence

PROBLEM STATEMENT

Given: $f_n \to f$ in measure, $\int f_n \to \int f$, $f_n, f \geq 0$.
To Prove: Prove $f_n \to f$ in L^1.

PROOF STRATEGY MAP

Identity $\rightarrow$ Dominated Convergence $\rightarrow$ Limit Calculation

$\downarrow$

Result

Identity
$|f_n - f| = f_n + f - 2\min(f_n, f)$.

Claim: Dominated Convergence

$\min(f_n, f) \to f$ in measure and is dominated by f.

Proof: By DCT, $\int \min(f_n, f) \to \int f$. ▪

Limit Calculation

$$\int |f_n - f| = \int f_n + \int f - 2\int \min(f_n, f) \to \int f + \int f - 2\int f = 0.$$

Result

Convergence in L^1 holds. Q.E.D.

Solution to Question 52

Topic: Density Measures

PROBLEM STATEMENT

Given: Define $d\nu = pd\mu$.
To Prove: $f \in L^1(\nu) \Leftrightarrow pf \in L^1(\mu)$ and integrals agree.

PROOF STRATEGY MAP

Indicators → Non-negative Functions → General L1

Indicators
For $f = \chi_E$, $\int \chi_E d\nu = \nu(E) = \int_E pd\mu$. *Holds.*

Claim: Non-negative Functions

By MCT, holds for $f \geq 0$.

Proof: Approximate with simple functions. ▪

General L1

$f \in L^1(\nu) \Leftrightarrow |f| \in L^1(\nu) \Leftrightarrow |f|p \in L^1(\mu)$. Equality follows by linearity. Q.E.D.

■

Solution to Question 53

Topic: Continuity of Measure

PROBLEM STATEMENT

Given: μ is finitely additive and $A_n \searrow \emptyset \Rightarrow \mu(A_n) \to 0$.
To Prove: Prove μ is countably additive.

PROOF STRATEGY MAP

Tails → Finite Additivity → Limit

Tails
Let $E = \cup E_k$ *(disjoint). Let* $R_n = \cup_{k=n+1}^{\infty} E_k$.

Finite Additivity

$$\mu(E) = \sum_{k=1}^{n} \mu(E_k) + \mu(R_n). \text{ Since } R_n \searrow \emptyset,\ \mu(R_n) \to 0.$$

Limit
$\mu(E) = \lim \sum_{k=1}^{n} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k)$. Q.E.D.

■

Solution to Question 54

Topic: Regularity of Measure

PROBLEM STATEMENT

Given: E Lebesgue measurable.
To Prove: Exists $F \subset E$ (countable union of compacts) such that $m(E \setminus F) = 0$.

PROOF STRATEGY MAP

Approximation → Union → Null Difference

Approximation
For each n, exists compact $K_n \subset E$ with $m(K_n) > m(E) - \frac{1}{n}$ (finite case).

Claim: Union

Let $F = \cup K_n$. This is an F_σ set.

Proof: $m(F) \geq m(K_n) > m(E) - \frac{1}{n}$ for all n. ▪

Null Difference

$m(E \setminus F) = m(E) - m(F) = 0$. Q.E.D.

■

Solution to Question 55

Topic: L2 to L1 Convergence

PROBLEM STATEMENT

Given: $f_n \to f$ in measure on $[0,1]$. $\|f_n\|_2 \leq 1$.
To Prove: Prove $\int f_n \to \int f$.

PROOF STRATEGY MAP

Uniform Integrability → Holder Bound → Vitali Application

Uniform Integrability
We check $\int_E |f_n|$ for small E.

Holder Bound

$\int_E |f_n| \leq \|f_n\|_2 \, (m(E))^{\frac{1}{2}} \leq 1 \cdot \sqrt{m(E)}$. This vanishes uniformly as $m(E) \to 0$.

Vitali Application
Since f_n is UI and converges in measure on finite space, $f_n \to f$ in L^1. Q.E.D.

■

Solution to Question 56

Topic: Lebesgue-Stieltjes Measure

PROBLEM STATEMENT

Given: $F(x) = x + 1$ for $x \in (0, 1]$, $(x + 1)^2$ for $x > 1$.
To Prove: Calculate $mu(\{0\})$, $mu(\{1\})$, $mu((0, 1])$.

PROOF STRATEGY MAP

Jump at 0 → Jump at 1 → Interval (0,1]

Jump at 0
$F(0) = 0$. $F(0^+) = 1$. *Jump size 1.* $\mu(\{0\}) = 1$.

Jump at 1

$$F(1) = 2.\ F(1^+) = (1 + 1)^2 = 4.\ \text{Jump size } 4 - 2 = 2.\ \mu(\{1\}) = 2.$$

Interval (0,1]
Continuous part $\int_0^1 1dx = 1$. Plus mass at 1. Total $1 + 2 = 3$. (Or $F(1^+) - F(0^+) = 4 - 1 = 3$). Q.E.D.

■

Solution to Question 57

Topic: Layer Cake Representation

PROBLEM STATEMENT

Given: Prove $\int_X \varphi(f)d\mu = \int_0^\infty \mu(\{f \geq s\})\varphi'(s)ds$.
To Prove: The identity holds.

PROOF STRATEGY MAP

Integral Representation → Swap Order → Result

Integral Representation
$\varphi(f(x)) = \int_0^\infty \varphi'(s)\chi_{\{s<f(x)\}}ds$.

Swap Order

$$\int_X \int_0^\infty ...dsd\mu = \int_0^\infty \varphi'(s)\Big(\int_X \chi_{\{f>s\}}d\mu\Big)ds.$$

Result
Inner integral is $\mu(\{f > s\})$. Identity proved. Q.E.D.

■

Solution to Question 58

Topic: Reverse Cauchy-Schwarz

PROBLEM STATEMENT

Given: Consider $\int_E f$ and $\int_E \frac{1}{f}$.
To Prove: $\left(\int_E f\right)\left(\int_E \frac{1}{f}\right) \geq \mu(E)^2$.

PROOF STRATEGY MAP

Functions → Application → Result

Functions
Set $u = \sqrt{f}$ *and* $v = \frac{1}{\sqrt{f}}$.

Application

$$\left(\int uv\right)^2 \leq (\int u^2)(\int v^2).\ \int uv = \int 1 = \mu(E).\ \int u^2 = \int f,\ \int v^2 = \int \frac{1}{f}.$$

Result
$\mu(E)^2 \leq (\int f)\left(\int \frac{1}{f}\right)$. Q.E.D.

■

Solution to Question 59

Topic: Characterization of Null Sets

PROBLEM STATEMENT

Given: $m(A) = 0$.
To Prove: Exists sequence of intervals I_n with $\sum |I_n| < \infty$ s.t. every $x \in A$ is in infinitely many I_n.

PROOF STRATEGY MAP

Forward Direction → Collection → Reverse Direction

Forward Direction
For each k, cover A with open intervals J_n^k having sum $< 2^{-k}$.

Claim: Collection

Let $\{I_j\} = \cup_k \{J_n^k\}$. Total length $\sum 2^{-k} = 1 < \infty$.

Proof: Since $x \in A$ is covered for every k, it is in infinitely many intervals. ▪

Reverse Direction

If $\sum |I_n| < \infty$, by Borel-Cantelli $\sum \chi_{I_n} < \infty$ a.e. Thus points in infinitely many intervals form a null set. Q.E.D.

■

Solution to Question 60

Topic: Conditional Expectation

PROBLEM STATEMENT

Given: μ finite, $\mathcal{F} \subset \Sigma$. $f \in L^1(\mu)$.
To Prove: Exists unique $g \in L^1(\mu|_{\mathcal{F}})$ s.t. $\int_E f = \int_E g$ for all $E \in \mathcal{F}$.

PROOF STRATEGY MAP

Measure Definition → Absolute Continuity → Result

Measure Definition
Define signed measure λ on $\mathcal{F}$ by $\lambda(E) = \int_E f d\mu$.

Claim: Absolute Continuity

$\lambda \ll \nu$ where $\nu = \mu|_{\mathcal{F}}$.

Proof: If $\nu(E) = 0$, then $\mu(E) = 0$, so $\int_E f = 0$. ▪

Result
By Radon-Nikodym on $(X, \mathcal{F})$, exists $g = \frac{d\lambda}{d\nu}$. Thus $\lambda(E) = \int_E g d\nu$. Q.E.D.

■