Full Solution Key

Student Name:

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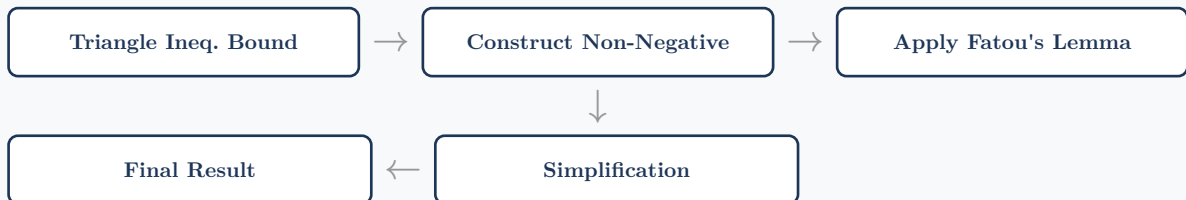## Solution to Question 1

Topic: Measure Theory

PROBLEM STATEMENT

Given: Let (X, M, μ) be a measure space. Let $f_n \to f$ pointwise a.e. and $|f_n| \le g \in L^1$.
To Prove:

$$\lim_{n \to \infty} \int f_n d\mu = \int f d\mu$$

PROOF STRATEGY MAP

| Triangle Ineq. Bound | $\rightarrow$ | Construct Non-Negative | $\rightarrow$ | Apply Fatou's Lemma |

$\downarrow$

| Final Result | $\leftarrow$ | Simplification |

Triangle Ineq. Bound
Since $|f_n| \le g$ and $g \in L^1$, by the triangle inequality, we have $|f_n - f| \le |f_n| + |f| \le 2g$.

Claim: Construct Non-Negative

The sequence $h_n = 2g - |f_n - f|$ is non-negative.

Proof: Since $|f_n - f| \le 2g$, it follows that $2g - |f_n - f| \ge 0$. Thus Fatou's Lemma applies to h_n. ∎

$$\int 2g \leq \liminf \int (2g - |f_n - f|)$$

$$\int 2g \leq \int 2g - \limsup \int |f_n - f|$$

Subtracting the finite integral $\int 2g$ from both sides (valid since $g \in L^1$):

Final Result

$$0 \leq -\limsup \int |f_n - f| \Rightarrow \limsup \int |f_n - f| \leq 0$$

Since the integral of absolute value is non-negative, the limit is 0. Q.E.D.
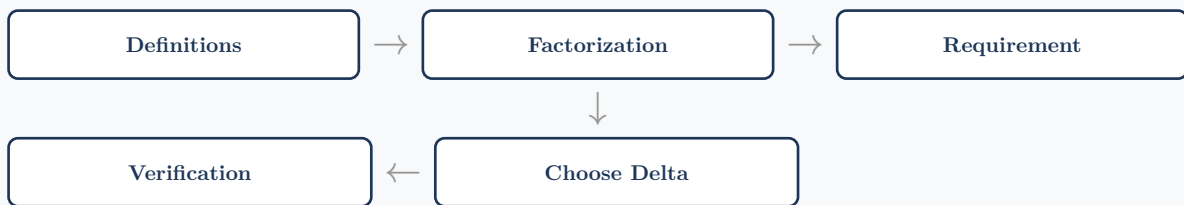
Solution to Question 2

Topic: Real Analysis

PROBLEM STATEMENT

Given: Prove the limit of the function $f(x) = 2x + 3$ as $x \to 1$.
To Prove:

$$\lim_{x \to 1} (2x + 3) = 5$$

PROOF STRATEGY MAP

| Definitions | $\to$ | Factorization | $\to$ | Requirement |

$\downarrow$

| Verification | $\leftarrow$ | Choose Delta |

Definitions
Let $\varepsilon > 0$. We must find $\delta > 0$ such that $0 < |x - 1| < \delta \Rightarrow |f(x) - 5| < \varepsilon$.

Factorization

$$|(2x + 3) - 5| = |2x - 2| = 2|x - 1|$$

We want this quantity to be less than ε. Thus we need $2|x - 1| < \varepsilon$, or $|x - 1| < \frac{\varepsilon}{2}$.

Claim: Choose Delta

Choose $\delta = \frac{\varepsilon}{2}$.

Proof: This choice directly satisfies the inequality derived above. ▪

Verification
If $|x - 1| < \delta$, then $|f(x) - 5| = 2|x - 1| < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$. Q.E.D.
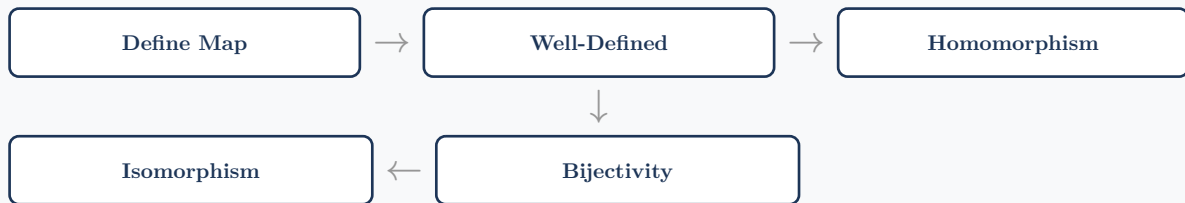
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Solution to Question 3

Topic: Abstract Algebra

PROBLEM STATEMENT

Given: Let $\varphi : G \to H$ be a surjective homomorphism with kernel K.
To Prove:

$$\frac{G}{K} \cong H$$

PROOF STRATEGY MAP

| Define Map | $\to$ | Well-Defined | $\to$ | Homomorphism |

$\downarrow$

| Isomorphism | $\leftarrow$ | Bijectivity |

Define Map
Define the map $\psi : \frac{G}{K} \to H$ by $\psi(gK) = \varphi(g)$.

Claim: Well-Defined

The map ψ does not depend on the choice of representative g.

Proof: Suppose $g_1 K = g_2 K$. Then $g_2^{-1} g_1 \in K = \ker(\varphi)$. Thus $\varphi(g_2^{-1} g_1) = e_H$, which implies $\varphi(g_1) = \varphi(g_2)$. So $\psi(g_1 K) = \psi(g_2 K)$. ∎

Claim: Homomorphism

ψ is a homomorphism.

Proof: $\psi((xK)(yK)) = \psi(xyK) = \varphi(xy) = \varphi(x)\varphi(y) = \psi(xK)\psi(yK)$. ∎

Claim: Bijectivity

ψ is both injective and surjective.

Proof:

1. Surjective: Since φ is surjective, for any $h \in H$, exists g such that $\varphi(g) = h$. Thus $\psi(gK) = h$.
2. Injective: $\psi(gK) = e_H \Rightarrow \varphi(g) = e_H \Rightarrow g \in K \Rightarrow gK = K$ (the identity in $\frac{G}{K}$).

$\blacksquare$

Isomorphism

Since ψ is a bijective homomorphism, it is an isomorphism. $\frac{G}{K} \cong H$.

$\blacksquare$