Real Analysis Exam

Student Name:

ID:

Question 1 — L1 Completeness

2010 | Prof. Boaz Klartag

Given: **Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.**

To Prove: Prove that $L^1(\mu)$ is a complete space (a Banach space).

SCRATCHPAD

FORMAL PROOF

Given: Find a function $f : [0,1] \to \mathbb{R}$.

To Prove: Construct f such that:
1. f is absolutely continuous.
2. f is strictly increasing.
3. $f'(x) = 0$ on a set A with $\mu(A) > 0$.

SCRATCHPAD

FORMAL PROOF

Given: **Let $H_f = \{(x,y) : 0 \le y \le f(x)\}$ be the subgraph of f.**

To Prove: (a) $\int |f-g| \, dm_1 = m_2\big(H_f \Delta H_g\big)$ (b) $\int |f-g| \, dm_1 = \int_{\mathbb{R}} m_1(\{f<t\}\Delta\{g<t\})dt$

SCRATCHPAD

FORMAL PROOF

Given: **Let μ, ν be σ-finite Borel measures.**

To Prove: (a) If $\mu([a,b]) = \nu([a,b])$ for all $a, b \in \mathbb{Q}$, implies $\mu = \nu$? (b) If $\mu([t, t+1]) = \nu([t, t+1])$ for all $t \in \mathbb{R}$, implies $\mu = \nu$?

SCRATCHPAD

FORMAL PROOF

Question 5 Convergence in Measure *2010 | Prof. Boaz Klartag*

Given: **Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n \to f$ in measure.**

To Prove: Prove there exists a subsequence $\left\{f_{n_k}\right\}$ that converges to f almost everywhere.

SCRATCHPAD

FORMAL PROOF

Question 6 Lebesgue Density *2010 | Prof. Boaz Klartag*

Given: Let $A \subset \mathbb{R}$ have positive measure. Let $d(x, A) = \inf\{|y - x| : y \in A\}$.

To Prove: Prove that for almost every $x_0 \in A$, $d(x_0 + h, A) = o(|h|)$ as $h \to 0$.

SCRATCHPAD

FORMAL PROOF

Question 7 Measure Rigidity *2010 | Prof. Boaz Klartag*

Given: **Let μ be a Borel measure on $\mathbb{R}$. Suppose $\mu(A) = \mu(|(A))$ for every measurable set A.**

To Prove: Prove that μ must be the zero measure ($\mu \equiv 0$).

SCRATCHPAD

FORMAL PROOF

Question 8 Fubini Inequality *2010 | Prof. Boaz Klartag*

Given: **Let** $A_t = \{x \in [0,1] : t \leq f(x) \leq 2t\}$.

To Prove: Prove $\int_{\{[0,1]\}} f dm \leq \int_0^\infty \frac{2}{t}\left(\int_{A_t} f(x) dm(x)\right) dt$.

SCRATCHPAD

FORMAL PROOF

Given: **Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous (AC). Suppose $f'(x) = 0$ almost everywhere.**

To Prove: Prove that f is constant.

SCRATCHPAD

FORMAL PROOF

Question 10 Pathological Construction *2016 | Prof. Boaz Klartag*

Given: Find a function $f : [0, 1] \to \mathbb{R}$.

To Prove: Construct f such that:

1. f is strictly increasing.
2. f is absolutely continuous.
3. $f'(x) = 0$ on a set of positive measure.

SCRATCHPAD

FORMAL PROOF

Question 11 — Chernoff Bound — *2016 | Prof. Boaz Klartag*

Given: Let f be integrable. Suppose for all $s > 0$, $\int_{\{[0,1]\}} e^{sf} \le e^{s^2}$.

To Prove: Prove that for all $t > 0$, $m(\{x : f(x) > t\}) \le e^{-\frac{t^2}{4}}$.

SCRATCHPAD

FORMAL PROOF

Question 12 Bathtub Principle *2016 | Prof. Boaz Klartag*

Given: Let $\mathcal{F} = \{f : \mathbb{R} \to [0, 1] \texttt{ mid} \int f = 1\}$**. Let** $g(x) = x^2 + \sin(2016x)$**.**

To Prove: Prove the infimum of $\int fg$ is attained by a function of the form $f = \mathbb{1}_{\{g<s\}}$.

SCRATCHPAD

FORMAL PROOF

Question 13 Riesz Theorem (Subsequence) *2016 | Prof. Boaz Klartag*

Given: Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n \to f$ in measure.

To Prove: Prove that there exists a subsequence $\left\{f_{n_k}\right\}$ that converges to f almost everywhere.

SCRATCHPAD

FORMAL PROOF

Question 14 Derivative Blow-up *2016 | Prof. Boaz Klartag*

Given: Let $A \subset \mathbb{R}$ be a set with Lebesgue measure zero $(m(A) = 0)$.

To Prove: Find a non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = +\infty$ for every $x \in A$.

SCRATCHPAD

FORMAL PROOF

Question 15 — Cantor Expansion

2016 | Prof. Boaz Klartag

Given: **Let f be the Cantor function and $g(x) = f(x) + x$. Let C be the Cantor set.**

To Prove: Prove that $m(g(C)) = 1$.

SCRATCHPAD

FORMAL PROOF

Question 16 Recovering from Partials *2016 | Prof. Boaz Klartag*

Given: **Let $f : \mathbb{R}^2 \to \mathbb{R}$ have Lipschitz partials vanishing on the boundary of $U = [0,1]^2$. Let $h = \frac{\partial}{\partial x}\left(\partial \frac{f}{\partial y}\right)$.**

To Prove: Prove $f(x,y) = \int_{[0,x]\times[0,y]} h$.

SCRATCHPAD

FORMAL PROOF

Question 17 Symmetric Sigma-Algebras *2018 | Prof. Mikhail Sodin*

Given: **Let Σ be the σ-algebra on $\mathbb{R}$ generated by symmetric intervals $(-a, a)$.**

To Prove: (i) Is $x \mapsto x^2$ measurable from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \Sigma)$? (ii) Is $x \mapsto x^3$ measurable? (iii) For $f \in L^1$, find Σ-measurable g s.t. $\int_A g = \int_A f$ for all $A \in \Sigma$.

SCRATCHPAD

FORMAL PROOF

Question 18 Dominated Convergence *2018 | Prof. Mikhail Sodin*

Given: Let $\int_X f d\mu = 1$ with $f \in L^1$.

To Prove: Calculate $\lim_{n\to\infty} n \int_X \log\left(1 + \frac{f(x)}{n}\right) d\mu(x)$.

SCRATCHPAD

FORMAL PROOF

Given: **Let** $E \subset \mathbb{R}^d$**.**

To Prove: Prove $m(E) = 0 \Leftrightarrow \exists f \in L^1$ such that $\lim_{y \to x} f(y) = +\infty$ for all $x \in E$.

SCRATCHPAD

FORMAL PROOF

Given: **Let $\nu = \nu_+ - \nu_-$ be the Jordan decomposition. Suppose $\nu = \nu_1 - \nu_2$ is another decomposition with $\nu_1, \nu_2 \geq 0$.**

To Prove: Prove $\nu_1 \geq \nu_+$ and $\nu_2 \geq \nu_-$.

SCRATCHPAD

FORMAL PROOF

Given: Let μ be a finitely additive function on a measurable space. Suppose for any sequence $A_n \downarrow \emptyset$, $\lim \mu(A_n) = 0$.

To Prove: Prove that μ is a measure (countably additive).

SCRATCHPAD

FORMAL PROOF

Question 22 — Dyadic Layer Cake — *2018 | Prof. Mikhail Sodin*

Given: **Let $f \geq 0$ be measurable.**

To Prove: Prove $\frac{1}{2}\sum_{n\in\mathbb{Z}} 2^n \mu(\{f > 2^n\}) \leq \int f d\mu \leq 2\sum_{n\in\mathbb{Z}} 2^n \mu(\{f > 2^n\})$.

SCRATCHPAD

FORMAL PROOF

Question 23 Generalized DCT *2018 | Prof. Mikhail Sodin*

Given: Let $f_n \to f$ a.e. and $\|f_n\|_1 \to \|f\|_1$.

To Prove: Prove $\|f_n - f\|_1 \to 0$.

SCRATCHPAD

FORMAL PROOF

Question 24 Radon-Nikodym Chain Rule *2018 | Prof. Mikhail Sodin*

Given: **Let** $\lambda \ll \nu \ll \mu$**.**

To Prove: Prove $\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}$ a.e.

SCRATCHPAD

FORMAL PROOF

Question 25 Cauchy in Measure *2018 | Prof. Mikhail Sodin*

Given: Let $\{f_n\}$ be Cauchy in measure.

To Prove: Prove there exists measurable f such that $f_n \to f$ in measure.

SCRATCHPAD

FORMAL PROOF

Given: **Let $\int f d\mu = 1$. Let E be a set.**

To Prove: Prove $\int_E \log f d\mu \le \mu(E) \log\left(\frac{1}{\mu(E)}\right)$.

SCRATCHPAD

FORMAL PROOF

Question 27 Inverse Square Root Series *2018 | Prof. Mikhail Sodin*

Given: $S_{n(x)} = \sum_{k=1}^{n} \frac{1}{2^k \sqrt{|x-r_k|}}$.

To Prove: (i) $S \in L^1$. (ii) $S \notin L^2(I)$ for any interval I.

SCRATCHPAD

FORMAL PROOF

Question 28 Cauchy in Measure *2018 | Prof. Mikhail Sodin*

Given: **Let (X, Σ, μ) be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions that is Cauchy in measure.**

To Prove: Prove there exists a measurable function f such that $f_n \to f$ in measure.

SCRATCHPAD

FORMAL PROOF

Given: **Let** $f : X \to (0, \infty)$ **with** $\int_X f d\mu = 1$**. Let** E **be a set with** $0 < \mu(E) < \infty$**.**

To Prove: Prove $\int_E \log(f) d\mu \leq \mu(E) \log\left(\frac{1}{\mu(E)}\right)$.

SCRATCHPAD

FORMAL PROOF

Question 30 — Inverse Square Root Series — *2018 | Prof. Mikhail Sodin*

Given: Let $\{r_k\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Define $S(x) = \sum_{k=1}^{\infty} \frac{1}{2^k \sqrt{|x-r_k|}}$.

To Prove: (i) $S \in L^1([0,1])$. (ii) $S \notin L^2(I)$ for any interval $I \subset [0,1]$.

SCRATCHPAD

FORMAL PROOF

Question 31 — Lebesgue Density — *2018 | Prof. Mikhail Sodin*

Given: **Let $A \subset D$ (unit disk). Suppose every point in $D \setminus \{(0,0)\}$ is a density point of A.**

To Prove: Prove that the origin $(0,0)$ is also a density point of A.

SCRATCHPAD

FORMAL PROOF

Question 32 Density-Like Sets *2019 | Prof. Mikhail Sodin*

Given: Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Define $\tilde{E} = \{x \in \mathbb{R} : \forall r > 0, m(E \cap (x - r, x + r)) > r\}$.

To Prove: Prove that $\tilde{E}$ is an F_σ set (a countable union of closed sets).

SCRATCHPAD

FORMAL PROOF

Question 33 Littlewood's First Principle *2019 | Prof. Mikhail Sodin*

Given: **Let $E \subset [0,1]$ be measurable. Let $\varepsilon > 0$.**

To Prove: Exists a finite union of disjoint open intervals $A = \cup_{j=1}^{N} I_j$ such that $m(E \Delta A) \leq \varepsilon$.

SCRATCHPAD

FORMAL PROOF

Question 34 Product Convergence in Measure *2019 | Prof. Mikhail Sodin*

Given: $f_n \to 0$ **in measure.** $\sup_n \int |g_n| \, d\mu < \infty$ **(L^1 bounded).**

To Prove: Prove $f_n g_n \to 0$ in measure.

SCRATCHPAD

FORMAL PROOF

Given: $\mu \ll \nu$ **and** $\nu \ll \mu$ **(**σ**-finite).**

To Prove: Prove $\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu} = 1$ a.e.

SCRATCHPAD

FORMAL PROOF

Given: Let f, g, h be bounded measurable functions with $f \leq g \leq h$.

To Prove: (a) Construct $f_n \to 0$ in measure but pointwise limit exists nowhere. (b) Construct $g_n \to g$ in measure such that $\liminf g_n = f$ and $\limsup g_n = h$ everywhere.

SCRATCHPAD

FORMAL PROOF

Question 37 | Generalized DCT | *2019 | Mr. Adi Dickstein*

Given: $f_n \to f$ in measure, $|f_n| \le g \in L^1$.

To Prove: Prove $\int f_n \to \int f$.

SCRATCHPAD

FORMAL PROOF

Given: Let $F : [a, b] \to \mathbb{R}$ be increasing. Let $f(x) = F'(x)$.

To Prove: (a) $\int_a^b f(x)dx \le F(b) - F(a)$. (b) Give an example of strict inequality.

SCRATCHPAD

FORMAL PROOF

Question 39 — Singular Integrals — *2019 | Mr. Adi Dickstein*

Given: $F \subset \mathbb{R}$ **closed,** $m(F^c) < \infty$**.** $\delta(y) = d(y, F)$**.** $I(x) = \int_{\mathbb{R}} \frac{\delta(y)^\lambda}{|} x - y|^{1+\lambda}\, dy$**.**

To Prove: (a) $I(x) = \infty$ for $x \notin F$. (b) $I(x) < \infty$ for a.e. $x \in F$.

SCRATCHPAD

FORMAL PROOF

Question 40 Radon-Nikodym Properties *2019 | Mr. Adi Dickstein*

Given: Let $\mu \ll \nu \ll \lambda$.

To Prove: (a) $\int g d\mu = \int g \frac{d\mu}{d\nu} d\nu$ for $g \in L^1(\mu)$. (b) $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = \frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\lambda}$ a.e.

SCRATCHPAD

FORMAL PROOF

Question 41 Continuous Extension & Density *2019 | Mr. Adi Dickstein*

Given: (a) $C \subset \mathbb{R}$ closed, $f : C \to \mathbb{R}$ continuous. (b) $C_{c(\mathbb{R})}$ dense in $L^1(\mathbb{R})$.

To Prove: (a) Exists extension F with same sup/inf bounds. (b) Prove density.

SCRATCHPAD

FORMAL PROOF

Given: $f_n \nearrow, g_n \searrow$. $f_n \leq h \leq g_n$. $\lim \int f_n = \lim \int g_n = L$.

To Prove: Prove h is integrable and $\int h = L$.

SCRATCHPAD

FORMAL PROOF

Question 43 Maximal Function Integrability *2019 | Mr. Adi Dickstein*

Given: $f \in L^1(\mathbb{R}^n)$. **Suppose** $f^* \in L^1(\mathbb{R}^n)$.

To Prove: Prove $f = 0$ a.e.

SCRATCHPAD

FORMAL PROOF

Given: **(a)** $f : \mathbb{R}^d \to \mathbb{R}^d$ **is Lipschitz.** $m(A) = 0$**. (b) Construct continuous** $f : \mathbb{R} \to \mathbb{R}$ **and** $m(A) = 0$ **s.t.** $m(f(A)) > 0$**.**

To Prove: (a) Prove $m(f(A)) = 0$. (b) Provide the example.

SCRATCHPAD

FORMAL PROOF

Question 45 Diophantine Approximation *2020 | Prof. Barak Weiss*

Given: $E = \left\{x \in [0,1] : |x - \frac{p}{q}| < q^{-(2+\varepsilon)} \text{ i.o.}\right\}$.

To Prove: (a) E is Borel. (b) $m(E) = 0$. (c) E is dense G_δ.

SCRATCHPAD

FORMAL PROOF

Question 46 Integral Intermediate Value *2020 | Prof. Barak Weiss*

Given: **(a)** $0 < f < 1$, $\mu(X) < \infty$ **(non-atomic). (b)** $\int_E f = \int_E f'$ **for all** E**.**

To Prove: (a) Exists E with $\int_E f = \alpha$. (b) $f = f'$ a.e.

SCRATCHPAD

FORMAL PROOF

Question 47 Lp Convergence Scaling *2020 | Prof. Barak Weiss*

Given: $f \in C_{c(\mathbb{R})}$. $a_n \to a > 0, b_n \to b$. $f_{n(x)} = f(a_n x + b_n)$.

To Prove: Prove $f_n \to f(ax + b)$ in L^p.

SCRATCHPAD

FORMAL PROOF

Question 48	Product Sigma-Algebras	*2020 / Prof. Barak Weiss*

Given: Let Σ_1, Σ_2 be σ-algebras on $\mathbb{R}$. Consider the product $\Sigma_1 \otimes \Sigma_2$ versus the σ-algebra on $\mathbb{R}^2$.

To Prove: (a) If Σ_i are Borel, is the product equal to $\mathcal{B}(\mathbb{R}^2)$? (b) If Σ_i are Lebesgue, is the product equal to $\mathcal{L}(\mathbb{R}^2)$?

SCRATCHPAD

FORMAL PROOF

Given: **(a)** E **Lebesgue measurable with** $m(E) > 0$**. (b)** $f : \mathbb{R} \to \mathbb{R}$ **arbitrary function.**

To Prove: (a) E contains a non-measurable subset. (b) The set of discontinuity points of f is Borel.

SCRATCHPAD

FORMAL PROOF

Question 50 Absolute Continuity of Integral *2020 | Prof. Barak Weiss*

Given: $g \in L^1(\mu)$ **non-negative.**

To Prove: Prove $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\mu(A) < \delta \Rightarrow \int_A g d\mu < \varepsilon$.

SCRATCHPAD

FORMAL PROOF

Question 51 L1 Convergence *2020 | Prof. Barak Weiss*

Given: $f_n \to f$ in measure, $\int f_n \to \int f$, $f_n, f \geq 0$.

To Prove: Prove $f_n \to f$ in L^1.

SCRATCHPAD

FORMAL PROOF

Given: **Define** $d\nu = pd\mu$**.**

To Prove: $f \in L^1(\nu) \Leftrightarrow pf \in L^1(\mu)$ and integrals agree.

SCRATCHPAD

FORMAL PROOF

Given: μ **is finitely additive and** $A_n \searrow \emptyset \Rightarrow \mu(A_n) \to 0$**.**

To Prove: Prove μ is countably additive.

SCRATCHPAD

FORMAL PROOF

Given: E **Lebesgue measurable.**

To Prove: Exists $F \subset E$ (countable union of compacts) such that $m(E \setminus F) = 0$.

SCRATCHPAD

FORMAL PROOF

Question 55 L2 to L1 Convergence *2021 | Prof. Yefim Gluskin*

Given: $f_n \to f$ **in measure on** $[0,1]$**.** $\|f_n\|_2 \leq 1$**.**

To Prove: Prove $\int f_n \to \int f$.

SCRATCHPAD

FORMAL PROOF

Given: $F(x) = x + 1$ **for** $x \in (0, 1]$**,** $(x + 1)^2$ **for** $x > 1$**.**

To Prove: Calculate $mu(\{0\})$, $mu(\{1\})$, $mu((0, 1])$.

SCRATCHPAD

FORMAL PROOF

Question 57 Layer Cake Representation *2021 | Prof. Yefim Gluskin*

Given: Prove $\int_X \varphi(f)d\mu = \int_0^\infty \mu(\{f \geq s\})\varphi'(s)ds$**.**

To Prove: The identity holds.

SCRATCHPAD

FORMAL PROOF

Given: **Consider $\int_E f$ and $\int_E \frac{1}{f}$.**

To Prove: $\left(\int_E f\right)\left(\int_E \frac{1}{f}\right) \geq \mu(E)^2$.

SCRATCHPAD

FORMAL PROOF

Question 59 — Characterization of Null Sets — *2022 | Dr. Alon Nishry*

Given: $m(A) = 0$.

To Prove: Exists sequence of intervals I_n with $\sum |I_n| < \infty$ s.t. every $x \in A$ is in infinitely many I_n.

SCRATCHPAD

FORMAL PROOF

Question 60 Conditional Expectation *2022 | Dr. Alon Nishry*

Given: μ **finite,** $\mathcal{F} \subset \Sigma$**.** $f \in L^1(\mu)$**.**

To Prove: Exists unique $g \in L^1(\mu|_{\mathcal{F}})$ s.t. $\int_E f = \int_E g$ for all $E \in \mathcal{F}$.

SCRATCHPAD

FORMAL PROOF