Appendix: New Pruning Rules for $P||C_{\max}$

1 PROOF OF THEOREM 1

Proof. By induction. For $i = n$ this property is clear.

For $i < n$, we assume that the statement holds true for $i + 1$. Given that $RET[i][u] = RET[i][u']$, we know that $RET[i+1][u] = RET[i+1][u']$ and $RET[i+1][u+w_i] = RET[i+1][u'+w_i]$. With our inductive assumption, this implies that

$$\begin{aligned}\phi(j_i, u) &= \phi(j_{i+1}, u) \cup \{j_i \cup x \mid x \in \phi(j_{i+1}, u + w_i)\} \\ &= \phi(j_{i+1}, u') \cup \{j_i \cup x \mid x \in \phi(j_{i+1}, u' + w_i)\} \\ &= \phi(j_i, u').\end{aligned}$$

Given that $RET[i][u] \neq RET[i][u']$, we know either $left(i, u) \neq left(i, u')$ or $right(i, u) \neq right(i, u')$. We assume w.l.o.g. that $u < u'$. If $left(i, u) \neq left(i, u')$, then by our inductive assumption, $\phi(j_{i+1}, u) \supsetneq \phi(j_{i+1}, u')$. We thus know that $\phi(j_i, u) \supsetneq \phi(j_i, u')$ since all other elements in either set must contain j_i. The case where $right(i, u) \neq right(i, u')$ proceeds analogously. Induction thus proves both directions in the above theorem. □

2 COMPARISON TO ILP

We present an experimental comparison of our best approach (R4+5) to a recent ILP-based approach [3].

We used three different benchmarks sets:

- `Lawrinenko`: 3500 benchmarks as described by Mrad and Souyah [3], originating from unpublished work by Lawrinenko et al.
- `Berndt`: 5000 benchmarks generated as described by Berndt et al. [1].
- `Frangioni`: 780 benchmarks as described by Frangioni et al. [2].

Results are provided in Fig. 1. Our approach outperforms the ILP approach on `Berndt` (2860 vs. 2418 solved) and `Frangioni` (379 vs. 256 solved), whereas the ILP approach performs substantially better on their own benchmark set `Lawrinenko` (3237 vs. 1579 solved). For sufficiently difficult instances, our approach (with our compressed representation of the RET) uses around *two orders of magnitude* less main memory, in particular for `Frangioni`, which features some instances with makespans ranging in the millions.

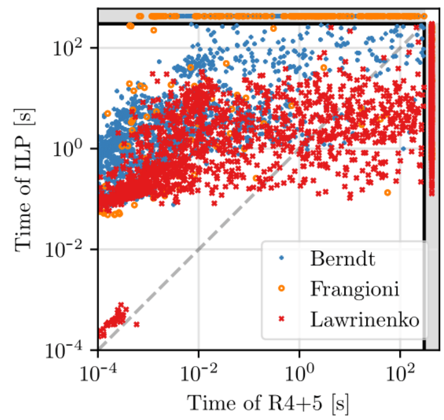

Figure 1: Per-instance scatter plot of our best approach (R4+5) versus the ILP-based approach by Mrad and Souayah.

3 DATA AVAILABILITY

We provide the full experimental data at https://github.com/anon943310/spaa24-pcmax-bnb. In the final version we will also refer to our (non anonymized) software.

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