Theorem 0.1 (Lower bound for unsupervised domain adaptation). *It is possible to construct a distribution shift problem* (p^s, p^t) *with* $d(p^s, p^t) = \epsilon$ *such that for any unsupervised domain adaptation method, the excess misclassification error of the resultant classifier follows:*

$$R(p^t, f) - R(p^t, f^*_{p^t}) \geq \frac{\epsilon}{2}$$

Proof. Consider source distribution: $(x|y=0) \sim Bern(\frac{1}{2} - \frac{\epsilon}{2})$, $(x|y=1) \sim Bern(\frac{1}{2} + \frac{\epsilon}{2})$ Consider two target distributions:

- T1: $(x|y=0) \sim Bern(\frac{1}{2} - \frac{\epsilon}{2})$, $(x|y=1) \sim Bern(\frac{1}{2} + \frac{\epsilon}{2})$; $p^{t_1}(y=1) = \frac{1}{2}$ (i.e., T1 is same as source);
- T2: $(x|y=0) \sim Bern(\frac{1}{2} + \frac{\epsilon}{2})$, $(x|y=1) \sim Bern(\frac{1}{2} - \frac{\epsilon}{2})$; $p^{t_2}(y=1) = \frac{1}{2}$ (i.e., class conditional distributions are flipped).

We see that $p^{t_1}(x) = p^{t_2}(x)$ for all x and hence any UDA method would arrive at the same classifier f in both the scenarios. Additionally, we note that $d(p^s, p^{t_1}) = d(p^s, p^{t_1}) = \epsilon$.

If $\hat{y}_f(0) = 0$ then the model would acquire an excess error of 0 in T1 and $\frac{\epsilon}{2}$ in T2, whereas if $\hat{y}_f(0) = 1$ then it would acquire an excess error of $\frac{\epsilon}{2}$ in T1 and 0 in T2. Symmetric arguments can be made for $x = 1$ and this proves the result. □