Sensitivity of Centrality Measures Against Strategic Manipulation Supplementary Materials

Paper #4212

1 Proofs

As mentioned in the main paper, we focus on the following question:

> *Given a centrality measure c, an evader $v^\dagger \in V$, and a class of edges $\zeta^\dagger \in \{\zeta_1^\dagger, \zeta_2^\dagger, \zeta_3^\dagger\}$, can the **addition** of an edge $e \in \zeta^\dagger$ to a network $(V, E) : e \notin E$ increase, or decrease, or leave unchanged, the ranking of $v^\dagger$ according to c? How about the **removal** of e from a network $(V, E) : e \in E$? Can it increase, or decrease, or leave unchanged, the ranking of $v^\dagger$ according to c?*

Table S1 summarizes our theoretical results, which address this question. Theorems 1 and 2 were presented in the main paper. In this appendix we present the remaining theorems. As mentioned in the main paper, the theorems focus on the effect of *adding* an edge to a network; as for the *removal* of an edge, the results immediately follow (see Proposition 1).

Theorem 3. *Let $v^\dagger$ be a node in V and let $\zeta^\dagger$ be a class in $\left\{\zeta_1^\dagger, \zeta_2^\dagger, \zeta_3^\dagger\right\}$. Furthermore, let c be a centrality measure in $\{c_{degr}, c_{clos}, c_{betw}\}$. There exists a network $G = (V, E)$ and an edge $e \in \zeta^\dagger : e \notin E$ such that the addition of e to G **does not change** the ranking of $v^\dagger$ according to c.*

Proof. Let $G = (V, E)$ be a network as depicted in Figure S1, with $k > 4$. In this network, we have:

- $c_{degr}(G, v^\dagger) = k$;
- $c_{degr}(G, x_i) = 1, \forall i \in \{1, \dots, k\}$;
- $c_{degr}(G, y_i) = 2, \forall i \in \{1, 2\}$;
- $c_{clos}(G, v^\dagger) = \frac{1}{k+4}$;
- $c_{clos}(G, x_i) \leq \frac{1}{2k+5}, \forall i \in \{1, \dots, k\}$;
- $c_{clos}(G, y_i) = \frac{1}{2k+3}, \forall i \in \{1, 2\}$;
- $c_{betw}(G, v^\dagger) = \frac{k^2+3k-2}{2}$;
- $c_{betw}(G, x_i) = 0, \forall i \in \{1, \dots, k\}$;
- $c_{betw}(G, y_i) = k + 1, \forall i \in \{1, 2\}$.

Hence, the node $v^\dagger$ is ranked first in G according to every centrality measure $c \in \{c_{degr}, c_{clos}, c_{betw}\}$.

Now, let G'_1 be the network that results from adding $(v^\dagger, x_1) \in \zeta_1^\dagger$ to G. Formally, $G'_1 = (V, E \cup \{(v^\dagger, x_1)\})$. In this network, we have:

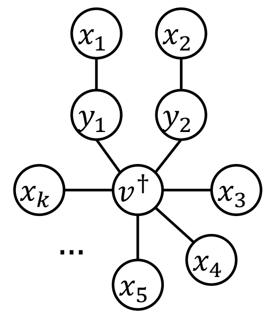

Figure S1: The network used in the proof of Theorem 3.

- $c_{degr}(G'_1, v^\dagger) = k + 1$;
- $c_{degr}(G'_1, x_i) \leq 2, \forall i \in \{1, \dots, k\}$;
- $c_{clos}(G'_1, v^\dagger) = \frac{1}{k+3}$;
- $c_{clos}(G'_1, x_i) \leq \frac{1}{2k+3}, \forall i \in \{1, \dots, k\}$;
- $c_{betw}(G'_1, y_1) = 0$.

The remaining centrality values do not change in comparison to G. As can be seen, the node $v^\dagger$ is still ranked first according to every $c \in \{c_{degr}, c_{clos}, c_{betw}\}$. We have shown that it is possible to add an edge $e \in \zeta_1^\dagger$ to a network without changing the ranking of $v^\dagger$ according to each of the centrality measures in $\{c_{degr}, c_{clos}, c_{betw}\}$.

Let G'_2 be the network that results from adding $(y_1, y_2) \in \zeta_2^\dagger$ to G. More formally, $G'_2 = (V, E \cup \{(y_1, y_2)\})$. In G'_2 we have:

- $c_{degr}(G'_2, y_i) = 3, \forall i \in \{1, 2\}$;
- $c_{clos}(G'_2, y_i) = \frac{1}{2k+1}, \forall i \in \{1, 2\}$;
- $c_{betw}(G'_2, v^\dagger) = \frac{k^2+3k-10}{2}$.

The remaining centrality values do not change compared to G. As can be seen, the node $v^\dagger$ is still ranked first according to every $c \in \{c_{degr}, c_{clos}, c_{betw}\}$. This implies that it is possible to add an edge $e \in \zeta_2^\dagger$ to a network without changing the ranking of $v^\dagger$ according to each of the centrality measures in $\{c_{degr}, c_{clos}, c_{betw}\}$.

Let G'_3 be the network that results from adding $(x_1, x_2) \in \zeta_3^\dagger$ to G. More formally, $G'_3 = (V, E \cup \{(x_1, x_2)\})$. In this network, we have:

	$e \in \zeta_1^\dagger$					
	Add e to a network $(V,E) : e \notin E$			Remove e from a network $(V,E) : e \in E$		
	Increase $v^\dagger$ ranking	No change in $v^\dagger$ ranking	Decrease $v^\dagger$ ranking	Increase $v^\dagger$ ranking	No change in $v^\dagger$ ranking	Decrease $v^\dagger$ ranking
Degree	✓ (Theorem 4)	✓ (Theorem 3)	✗ (Theorem 1)	✗ (Theorem 1)	✓ (Theorem 3)	✓ (Theorem 4)
Closeness	✓ (Theorem 5)	✓ (Theorem 3)	✓ (Theorem 6)	✓ (Theorem 6)	✓ (Theorem 3)	✓ (Theorem 5)
Betweenness	✓ (Theorem 8)	✓ (Theorem 3)	✓ (Theorem 11)	✓ (Theorem 11)	✓ (Theorem 3)	✓ (Theorem 8)

	$e \in \zeta_2^\dagger$					
	Add e to a network $(V,E) : e \notin E$			Remove e from a network $(V,E) : e \in E$		
	Increase $v^\dagger$ ranking	No change in $v^\dagger$ ranking	Decrease $v^\dagger$ ranking	Increase $v^\dagger$ ranking	No change in $v^\dagger$ ranking	Decrease $v^\dagger$ ranking
Degree	✗ (Theorem 1)	✓ (Theorem 3)	✓ (Theorem 4)	✓ (Theorem 4)	✓ (Theorem 3)	✗ (Theorem 1)
Closeness	✗ (Theorem 2)	✓ (Theorem 3)	✓ (Theorem 7)	✓ (Theorem 7)	✓ (Theorem 3)	✗ (Theorem 2)
Betweenness	✓ (Theorem 9)	✓ (Theorem 3)	✓ (Theorem 12)	✓ (Theorem 12)	✓ (Theorem 3)	✓ (Theorem 9)

	$e \in \zeta_3^\dagger$					
	Add e to a network $(V,E) : e \notin E$			Remove e from a network $(V,E) : e \in E$		
	Increase $v^\dagger$ ranking	No change in $v^\dagger$ ranking	Decrease $v^\dagger$ ranking	Increase $v^\dagger$ ranking	No change in $v^\dagger$ ranking	Decrease $v^\dagger$ ranking
Degree	✗ (Theorem 1)	✓ (Theorem 3)	✓ (Theorem 4)	✓ (Theorem 4)	✓ (Theorem 3)	✗ (Theorem 1)
Closeness	✓ (Theorem 5)	✓ (Theorem 3)	✓ (Theorem 7)	✓ (Theorem 7)	✓ (Theorem 3)	✓ (Theorem 5)
Betweenness	✓ (Theorem 10)	✓ (Theorem 3)	✓ (Theorem 12)	✓ (Theorem 12)	✓ (Theorem 3)	✓ (Theorem 10)

Table S1: **Summary of our theoretical results.** For any given evader $v^\dagger \in V$, we study three classes of edges: $\zeta_1^\dagger$, $\zeta_2^\dagger$ and $\zeta_3^\dagger$, and three centrality measures: Degree, Closeness and Betweenness. For every class and every measure, we study the effect of adding or removing an edge from that class on the centrality-based ranking of $v^\dagger$; this effect can either be an "*increase*", a "*decrease*", or "*no change*" in the ranking of $v^\dagger$. The "✓" and "✗" indicate whether such an effect is "*possible*", or "*impossible*", respectively.

- $c_{degr}(G'_3, x_i) \leq 2, \forall i \in \{1, \dots, k\}$;
- $c_{clos}(G'_3, y_i) = \frac{1}{2k+2}, \forall i \in \{1, 2\}$;
- $c_{betw}(G'_3, v^\dagger) = \frac{k^2+3k-8}{2}$;
- $c_{betw}(G'_3, x_i) \leq 1, \forall i \in \{1, \dots, k\}$;
- $c_{betw}(G'_3, y_i) = k-1, \forall i \in \{1, 2\}$.

The remaining centrality values do not change in comparison to G. Based on these values, the node $v^\dagger$ is still ranked first. We have shown that there exists a network G and an edge $e \in \zeta_3^\dagger$ such that the addition of e to G does not change the ranking of $v^\dagger$ according to each of the centrality measures in $\{c_{degr}, c_{clos}, c_{betw}\}$. ☐

Theorem 4. *Let $v^\dagger$ be a node in V. Then:*

- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_1^\dagger : e \notin G$ such that the addition of e to G **increases** the ranking of $v^\dagger$ according to **degree** centrality;*
- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_2^\dagger : e \notin G$ such that the addition of e to G **decrease** the ranking of $v^\dagger$ according to **degree** centrality;*
- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_3^\dagger : e \notin G$ such that the addition of e to G **decrease** the ranking of $v^\dagger$ according to **degree** centrality.*

Proof. Consider adding an edge $(v^\dagger, z) \in \zeta_1^\dagger$ to the network depicted in Figure S2(a). Before this addition, $v^\dagger$ is ranked second according to degree centrality, whereas after the addition it is ranked first (ex aequo with node y).

Now, consider adding an edge (x, y) to either the network depicted in Figure S2(b) (in which case $(x, y) \in \zeta_2^\dagger$) or to the network depicted in Figure S2(c) (in which case $(x, y) \in \zeta_3^\dagger$). In both cases, $v^\dagger$ is ranked first according to degree centrality (ex aequo with nodes x and y) before the addition of (x, y), but ranked third after this addition. ☐

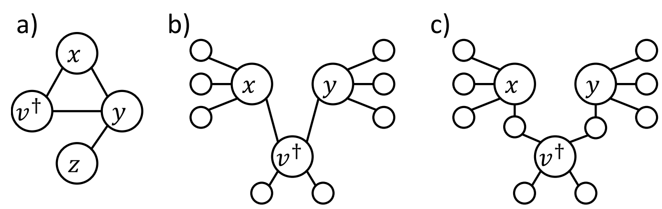

Figure S2: The networks used in the proof of Theorem 4.

Theorem 5. *Let $v^\dagger$ be a node in V. Then:*

- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_1^\dagger : e \notin G$ such that the addition of e to G **increases** the ranking of $v^\dagger$ according to **closeness** centrality;*
- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_3^\dagger : e \notin G$ such that the addition of e to G **increases** the ranking of $v^\dagger$ according to **closeness** centrality.*

Proof. Let $G = (V, E)$ be a network as depicted in Figure S3, with $k \geq 4$. In this network, the nodes x_k and x_{k+1} have the highest closeness centrality, followed by the nodes x_{k-1} and x_{k+2}, and then the nodes x_{k-2} and x_{k+3}, *etc.*. This implies that the closeness-based rankings of x_1 and x_2 are $2k-1$ and $2k-3$, respectively.

Now, consider the network $G' = (V, E \cup \{(x_1, x_{2k})\})$. Since G' is a ring, all nodes have the same closeness centrality, and thus the same (first) ranking. Then:

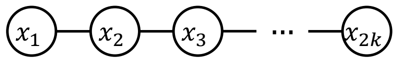

Figure S3: The network used in the proof of Theorems 5 and 10.

- If the evader is x_1, *i.e.*, if $v^\dagger = x_1$, then the added edge (x_1, x_{2k}) belongs to the class $\zeta_1^\dagger$;
- If the evader is x_2, *i.e.*, if $v^\dagger = x_2$, then the added edge (x_1, x_{2k}) belongs to the class $\zeta_3^\dagger$.

In either case, adding (x_1, x_{2k}) results in an increase in the closeness-based ranking of $v^\dagger$. □

Theorem 6. *Let $v^\dagger$ be a node in V. There exists a network $G = (V, E)$ and an edge $e \in \zeta_1^\dagger : e \notin G$ such that the addition of e to G **decreases** the ranking of $v^\dagger$ according to **closeness** centrality.*

Proof. Let $G = (V, E)$ be the network depicted in Figure S4. In this network we have:

- $c_{clos}(G, y) = \frac{1}{20}$;
- $c_{clos}(G, x) = \frac{1}{23}$;
- $c_{clos}(G, v^\dagger) = \frac{1}{28}$;
- $c_{clos}(G, z) = \frac{1}{29}$;
- $c_{clos}(G, b_i) = \frac{1}{31}, \forall i \in \{1, \dots, 5\}$;
- $c_{clos}(G, a_i) = \frac{1}{39}, \forall i \in \{1, 2, 3\}$;
- $c_{clos}(G, w) = \frac{1}{40}$.

As can be seen, the node $v^\dagger$ is ranked third in G according to closeness centrality.

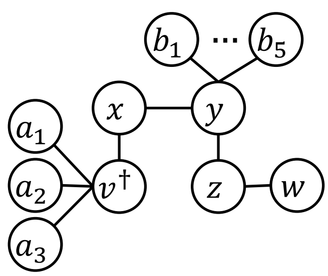

Figure S4: The network used in the proof of Theorem 6.

Now, let G' be the network that results from adding $(v^\dagger, z) \in \zeta_1^\dagger$ to G. More formally, $G' = (V, E \cup \{(v^\dagger, z)\})$. In this network, we have:

- $c_{clos}(G', y) = \frac{1}{20}$;
- $c_{clos}(G', z) = \frac{1}{21}$;
- $c_{clos}(G', x) = \frac{1}{23}$;
- $c_{clos}(G', v^\dagger) = \frac{1}{24}$;
- $c_{clos}(G', b_i) = \frac{1}{31}$;
- $c_{clos}(G', w) = \frac{1}{32}$;
- $c_{clos}(G', a_i) = \frac{1}{35}$.

As can be seen, the node $v^\dagger$ is ranked fourth in G' according to closeness centrality. We have shown that it is possible to decrease the closeness-based ranking of $v^\dagger$ by adding an edge $e \in \zeta_1^\dagger$. □

Theorem 7. *Let $v^\dagger$ be a node in V. Then:*

- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_2^\dagger : e \notin G$ such that the addition of e to G **decrease** the ranking of $v^\dagger$ according to **closeness** centrality;*
- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_3^\dagger : e \notin G$ such that the addition of e to G **decrease** the ranking of $v^\dagger$ according to **closeness** centrality.*

Proof. Consider the network $G = (V, E)$ depicted in Figure S5(a). In this network, we have:

- $c_{clos}(G, v^\dagger) = \frac{1}{4k+2}$;
- $c_{clos}(G, x) = c_{clos}(G, y) = \frac{1}{4k+3}$;
- $c_{clos}(G, a_i) = c_{clos}(G, b_i) = \frac{1}{5k+5}, \forall i \in \{1, \dots, k\}$.

Based on these values, $v^\dagger$ is ranked first in G according to closeness centrality.

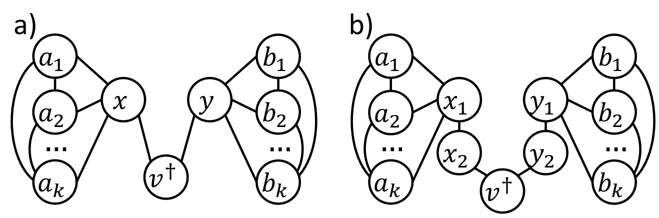

Figure S5: The networks used in the proofs of Theorems 7 and 12.

Now, let G' be the network that results from adding (x, y) (which is an edge in $\zeta_2^\dagger$) to the network G. More formally, $G' = (V, E \cup \{(x, y)\})$. In this network, we have:

- $c_{clos}(G', v^\dagger) = \frac{1}{4k+2}$;
- $c_{clos}(G', x) = c_{clos}(G', y) = \frac{1}{3k+2}$;
- $c_{clos}(G', a_i) = c_{clos}(G', b_i) = \frac{1}{4k+4}, \forall i \in \{1, \dots, k\}$.

As such, $v^\dagger$ is ranked third in G' according to closeness centrality. We have shown that it is possible to decrease the closeness-based ranking of $v^\dagger$ by adding an edge $e \in \zeta_2^\dagger$.

Analogically, in the network depicted in Figure S5(b), one can show that the closeness-based ranking of $v^\dagger$ decreases after adding (x_1, y_1), which is an edge in $\zeta_3^\dagger$. □

Theorem 8. *Let $v^\dagger$ be a node in V. There exists a network $G = (V, E)$ and an edge $e \in \zeta_1^\dagger : e \notin G$ such that the addition of e to G **increases** the ranking of $v^\dagger$ according to **betweenness** centrality.*

Proof. Let $G = (V, E)$ be the network depicted in Figure S6. Here, we have:

- $c_{betw}(G, x) = (2k + 3) + k(k + 4)$;

- $c_{betw}(G, y) = k(k + 4)$;
- $c_{betw}(G, z) = (k + 2)(k + 2)$;
- $c_{betw}(G, w) = (k + 3)(k + 1)$;
- $c_{betw}(G, v) = 0, \forall v \in \{v^{\dagger}, a_1, \dots, a_k, b_1, \dots, b_k\}$.

Based on this, $v^{\dagger}$ is ranked fifth in G according to betweenness centrality.

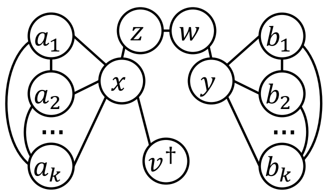

Figure S6: The network used in the proof of Theorem 8.

Now, let G' be the network that results from adding $(v^{\dagger}, y) \in \zeta_1^{\dagger}$ to G. More formally, $G' = (V, E \cup \{(v^{\dagger}, y)\})$. The, in G' we have:

- $c_{betw}(G', v^{\dagger}) = (k + 1)(k + 1)$;
- $c_{betw}(G', x) = c_{betw}(G', y) = k(k + 4) + 1$;
- $c_{betw}(G', z) = c_{betw}(G', w) = k + 1$.

The betweenness centrality of all remaining nodes remains unchanged in comparison to G. Hence, $v^{\dagger}$ is ranked third in G' according to betweenness centrality. We have shown that it is possible to increase the betweenness-based ranking of $v^{\dagger}$ by adding an edge $e \in \zeta_1^{\dagger}$. □

Theorem 9. *Let $v^{\dagger}$ be a node in V. There exists a network $G = (V, E)$ and an edge $e \in \zeta_2^{\dagger} : e \notin G$ such that the addition of e to G **increases** the ranking of $v^{\dagger}$ according to **betweenness** centrality.*

Proof. Let $G = (V, E)$ be the network depicted in Figure S7. In this network, we have:

- $c_{betw}(G, v^{\dagger}) = \frac{(k+1)^2}{3} + 2k + 5 + \frac{k+1}{2}$;
- $c_{betw}(G, x) = \frac{2(k+1)^2}{3} + \frac{3(k+1)}{2} + 1$;
- $c_{betw}(G, y) = \frac{2(k+1)^2}{3} + 2(k + 1) + 1$;
- $c_{betw}(G, z) = \frac{(k+1)^2}{3} + \frac{k+1}{2}$;
- $c_{betw}(G, w_i) = k(k + 6), \forall i \in \{1, 2\}$;
- $c_{betw}(G, v) = 0, \forall v \in \{c, a_1, \dots, a_k, b_1, \dots, b_k\}$.

Based on these values, $v^{\dagger}$ is ranked fifth in G according to betweenness centrality.

Now, let G' be the network that results from adding the edge $(y, w_2) \in \zeta_2^{\dagger}$ to G. Then, in G' we have:

- $c_{betw}(G', v^{\dagger}) = 2k + 5$;
- $c_{betw}(G', x) = \frac{k+1}{2} + 1$;
- $c_{betw}(G', y) = (k + 1)^2 + \frac{5(k+1)}{2} + 1$;
- $c_{betw}(G', z) = \frac{k+1}{2}$;

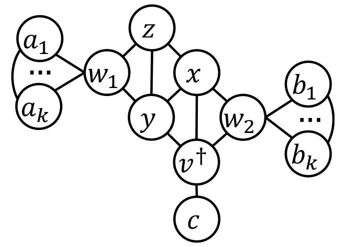

Figure S7: The network used in the proof of Theorem 9.

- $c_{betw}(G', w_i) = k(k + 6), \forall i \in \{1, 2\}$.

All remaining betweenness centralities remain unchanged compared to G'. Based on these values, $v^{\dagger}$ is ranked fourth in G' according to betweenness centrality ranking. We have shown that it is possible to increase the betweenness-based ranking of $v^{\dagger}$ by adding an edge $e \in \zeta_2^{\dagger}$. □

Theorem 10. *Let $v^{\dagger}$ be a node in V. There exists a network $G = (V, E)$ and an edge $e \in \zeta_3^{\dagger} : e \notin G$ such that the addition of e to G **increases** the ranking of $v^{\dagger}$ according to **betweenness** centrality.*

Proof. Let $G = (V, E)$ be a network as depicted in Figure S3, with $k \geq 4$. Here, nodes x_k and x_{k+1} have the highest betweenness centrality, followed by the nodes x_{k-1} and x_{k+2}, and then the nodes x_{k-2} and x_{k+3}, *etc.*. This implies that the betweenness-based rankings of x_1 and x_2 are $2k - 1$ and $2k - 3$, respectively.

Now, consider the network $G' = (V, E \cup \{(x_1, x_{2k})\})$. Since G' is a ring, all nodes have the same betweenness centrality, and thus the same (first) ranking. If the evader is x_2, *i.e.*, if $v^{\dagger} = x_2$, then the added edge (x_1, x_{2k}) belongs to the class $\zeta_3^{\dagger}$, and adding it increases the betweenness-based ranking of $v^{\dagger}$. □

Theorem 11. *Let $v^{\dagger}$ be a node in V. There exists a network $G = (V, E)$ and an edge $e \in \zeta_1^{\dagger} : e \notin G$ such that the addition of e to G **decreases** the ranking of $v^{\dagger}$ according to **betweenness** centrality.*

Proof. Let $G = (V, E)$ be the network depicted in Figure S8. In this network, we have:

- $c_{betw}(G, y) = 80$;
- $c_{betw}(G, x) = 55$;
- $c_{betw}(G, v^{\dagger}) = 30$;
- $c_{betw}(G, z) = 16$;
- $c_{betw}(G, v) = 0, \forall v \in V \setminus \{x, y, z, v^{\dagger}\}$.

Based on these values, the node $v^{\dagger}$ is ranked third in G according to betweenness centrality.

Now, let G' be the network that results from adding the edge $(v^{\dagger}, z) \in \zeta_1^{\dagger}$ to G. That is, $G' = (V, E \cup \{v^{\dagger}, z)\})$. In G' we have:

- $c_{betw}(G', y) = 72$;
- $c_{betw}(G', x) = 32.5$;

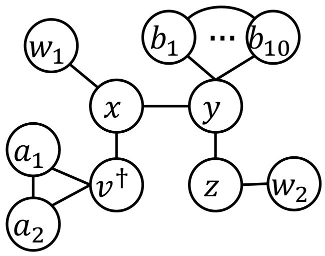

Figure S8: Network used in the proof of Theorem 11.

- $c_{betw}(G', z) = 32.5$;
- $c_{betw}(G', v^\dagger) = 32$;
- $c_{betw}(G, v) = 0, \forall v \in V \setminus \{x, y, z, v^\dagger\}$.

The node $v^\dagger$ is now ranked fourth in G' according to betweenness centrality. We have shown that it is possible to decrease the betweenness-based ranking of $v^\dagger$ by adding an edge $e \in \zeta_1^\dagger$. $\square$

Theorem 12. *Let $v^\dagger$ be a node in V. Then:*

- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_2^\dagger$: $e \notin G$ such that the addition of e to G **decreases** the ranking of $v^\dagger$ according to **betweenness** centrality;*
- *There exists a network $G = (V, E)$ and an edge $e \in \zeta_3^\dagger$: $e \notin G$ such that the addition of e to G **decreases** the ranking of $v^\dagger$ according to **betweenness** centrality.*

Proof. Let $G = (V, E)$ be the network depicted in Figure S5(a). In this network, we have:

- $c_{betw}(G, v^\dagger) = (k+1)^2$;
- $c_{betw}(G, x) = c_{betw}(G, y) = k(k+2)$;
- $c_{betw}(G, a_i) = c_{betw}(G, b_i) = 0$.

Based on these values, $v^\dagger$ is ranked first in G according to betweenness centrality.

Now, let G' be the network that results from adding the edge $(x, y) \in \zeta_2^\dagger$ to the network G. That is, $G' = (V, E \cup \{(x, y)\})$. In G' we have:

- $c_{betw}(G', v^\dagger) = 0$;
- $c_{betw}(G', x) = c_{betw}(G', y) = k(k+2)$;
- $c_{betw}(G', a_i) = c_{betw}(G', b_i) = 0$.

Here, $v^\dagger$ is ranked third in G' according to betweenness centrality. We have shown that it is possible to decrease the betweenness-based ranking of $v^\dagger$ by adding an edge $e \in \zeta_2^\dagger$.

Analogically, in the network depicted in Figure S2(b), one can show that the betweenness-based ranking of $v^\dagger$ decreases after adding (x_1, y_1), which is an edge in $\zeta_3^\dagger$). $\square$