Hiding in Multilayer Networks—Supplementary Materials

Paper #6275

1 Proofs of Theoretical Results

Observation 1. *The problem of Multilayer Global Hiding is in P given the degree centrality measure. In fact, for a given problem instance either any A^* that connects $\widehat{v}$ with all contacts is a solution, or there are no solutions at all.*

Proof. Any valid solution to the problem A^* must connect the evader $\widehat{v}$ with all contacts. Therefore, after the addition of A^* the degree centrality of $\widehat{v}$ is $|F|$, while the degree centrality of every contact increases by 1. Hence, the degree centrality ranking in the network does not depend on the choice of layers in which $\widehat{v}$ gets connected with its contacts. □

Theorem 1. *The problem of Multilayer Global Hiding is NP-complete given the closeness centrality measure.*

Proof. The problem is trivially in NP, since after the addition of a given A^* the closeness centrality ranking can be computed in polynomial time.

Next, we prove that the problem is NP-hard. To this end, we show a reduction from the NP-complete problem of *Exact 3-Set Cover*. The decision version of this problem is defined by a set of subsets $S = \{S_1, \dots, S_m\}$ of universe $U = \{u_1, \dots, u_{3k}\}$, such that $\forall_i |S_i| = 3$. The goal is to determine whether there exist k pairwise disjoint elements of S the sum of which equals U.

Given an instance of the problem of *Exact 3-Set Cover*, let us construct a multilayer network, $M = (V_L, E_L, V', L)$, as follows (Figure 1 depicts an instance of this network):

- **The set of nodes V':** For every $u_i \in U$ we create a node u_i, as well as 3 nodes $w_{i,1}, w_{i,2}, w_{i,3}$. We will denote the set of all nodes u_i by U, and the set of all nodes $w_{i,j}$ by W. We also create the evader node $\widehat{v}$, the node v', and the following four sets of nodes:
  1. $A = \{a_1, \dots, a_m\}$;
  2. $B = \{b_1, \dots, b_{2k+2m}\}$;
  3. $B' = \{b'_1, \dots, b'_{k+2m+1}\}$;
  4. $B'' = \{b''_1, \dots, b''_{2k+m-1}\}$.
- **The set of layers L:** For every $S_i \in S$ we create a layer α_i. We also create an additional layer β.
- **The set of occurrences of nodes in layers V_L:** Node $u_j \in U$ appears in layer α_i if and only if $u_j \in S_i$. Node $w_{i,j} \in U$ appears only in layer α_i. The evader $\widehat{v}$, as well as all nodes in A appear in every layer α_i. Node v', as well as all nodes in B, B', and B'' appear only in layer β.
- **The set of edges E_L:** For every node that appears in multiple layers, we connect all occurrences of this node in a clique. For node u_j in layer α_i we connect it with node a_i. In every layer α_i we connect all nodes in A into a clique. Moreover, we connect every node b_i with node v', and connect every node b'_i with node b_i. Finally, we connect every node b''_i with node b'_i.

Now, consider the following instance of the problem of Multilayer Global Hiding, $(M, \widehat{v}, F, c, d)$, where:

- M is the multilayer network we just constructed;
- $\widehat{v}$ is the evader;
- $F = U \cup W$ is the set of contacts;
- c is the closeness centrality measure;
- $d = 1$.

Next, let us analyze the closeness centrality values of nodes in the network. Notice that every node $w_{i,j}$ appears only in a single layer α_i, hence $\widehat{v}$ has to connect with $w_{i,j}$ in layer α_i. Assume that the evader $\widehat{v}$ has connections with nodes in U in exactly x layers, *i.e.*, $x = |\{\alpha_i \in L : \exists u_j (\widehat{v}^{\alpha_i}, u_j^{\alpha_i}) \in A^*\}|$. We then have:

- $c_{clos}(\widehat{v}) = 3k + 3m + \frac{x}{2} + \frac{m-x}{3} \geq 3k + 3\frac{1}{3}m$ as $\widehat{v}$ is a neighbor of $3k$ nodes in U and $3m$ nodes in W, while for any $a_i \in A$ the distance between a_i and $\widehat{v}$ is 2 if $\widehat{v}$ is connected with any u_j in layer α_i and 3 otherwise;
- $c_{clos}(u_i) \leq 1 + m + \frac{3k-1}{2} + \frac{3m}{2} = 1\frac{1}{2}k + 2\frac{1}{2}m + \frac{1}{2} < c_{clos}(\widehat{v})$ as u_i is a neighbor of $\widehat{v}$ and at most m nodes in A, while the distance to all other nodes is at least 2;
- $c_{clos}(a_i) \leq 3+m-1+\frac{1}{2}+\frac{3k-3}{2}+\frac{3m}{3} = 1\frac{1}{2}k+2m+1 < c_{clos}(\widehat{v})$ as a_i is a neighbor of 3 nodes from U and all other $m-1$ nodes in A, while the distance to $\widehat{v}$ and all other nodes in U is 2, and the distance to all nodes is W is at least 3;
- $c_{clos}(w_{i,j}) < c_{clos}(\widehat{v})$ as for any other node v we have $\lambda(w_{i,j}, v) = \lambda(\widehat{v}, v) + 1$, since the shortest paths between $w_{i,j}$ and all other nodes go through $\widehat{v}$;
- $c_{clos}(v') = 2k + 2m + \frac{k+2m+1}{2} + \frac{2k+m-1}{3} = 3k + 3m + \frac{k}{2} + \frac{m-k}{3} + \frac{1}{6}$ as v' is a neighbor of all $2k + 2m$ nodes in

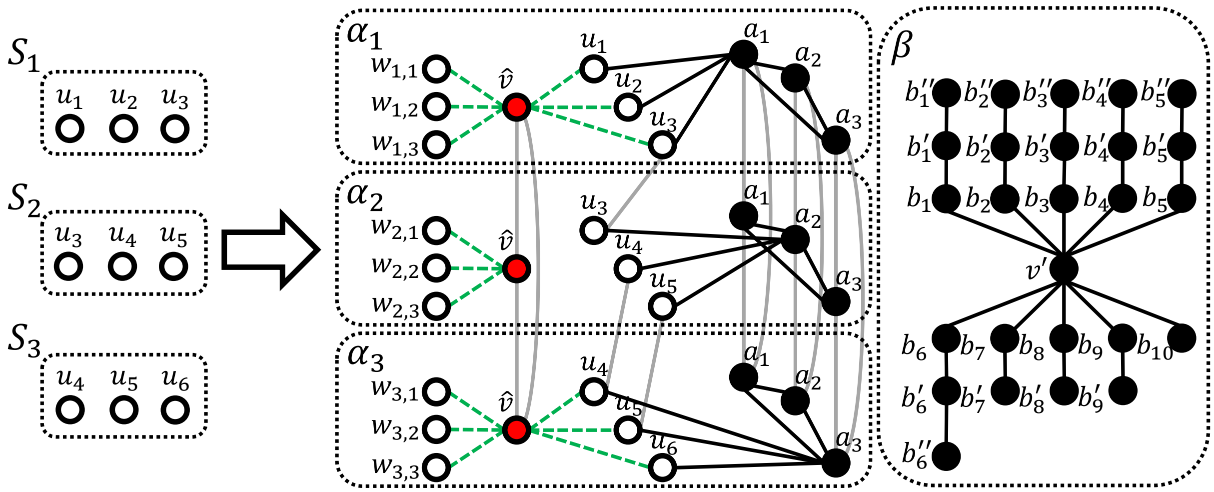

Figure 1: An illustration of the network used in the proof of Theorem 1. Edges connecting occurrences of the same node in different layers are highlighted in grey. The red node represents the evader, while the white nodes represent the contacts. Dashed (green) edges represent the solution to this problem instance.

B, the distance to all $k+2m+1$ nodes in B' is 2, while the distance to all $m-k+1$ nodes in B'' is 3.

We have shown that all nodes in A, U, and W have smaller closeness centrality than $\widehat{v}$. It is easy to check that v' has greater closeness centrality than all other nodes occurring in layer β. Hence, $\widehat{v}$ is hidden if and only if v' has greater closeness centrality than $\widehat{v}$. This is true when:

$$3k+3m+\frac{x}{2}+\frac{m-x}{3}<3k+3m+\frac{k}{2}+\frac{m-k}{3}+\frac{1}{6}$$

which can be simplified to $x<k+1$. Since both x and k are in $\mathbb{N}$ this is equivalent to $x \leq k$. Therefore, $\widehat{v}$ is hidden if and only if it has connections with nodes in U in at most k layers.

Now we will show that if there exists a solution to the given instance of the Exact 3-Set Cover problem, then there also exists a solution to the constructed instance of the Multilayer Global Hiding problem. Let S^* be an exact cover of U. In layer α_i we connect $\widehat{v}$ with all nodes from W that occurr in this layer. For every $S_i \in S^*$ we connect $\widehat{v}$ with $u_j \in S_i$ in layer α_i. This way, $\widehat{v}$ becomes connected to all $3k$ contacts from U, since all the sets in S^* are pairwise disjoint.

To complete the proof, we have to show that if there exists a solution A^* to the constructed instance of the Multilayer Global Hiding problem, then there also exists a solution to the given instance of the Exact 3-Set Cover problem. We have shown above that if $\widehat{v}$ is hidden, then it is connected to nodes in U in at most k layers from $\{\alpha_1,\ldots,\alpha_m\}$. However, since $\widehat{v}$ must be connected with all $3k$ nodes in U in order for A^* to be a correct solution, then $\{S_i : \exists u_j(\widehat{v}^{\alpha_i}, u_j^{\alpha_i}) \in A^*\}$ is a solution to the given instance of the Exact 3-Set Cover problem. This concludes the proof. □

Theorem 2. *The problem of Multilayer Global Hiding is NP-complete given the betweenness centrality measure.*

Proof. The problem is trivially in NP, since after the addition of a given A^* the betweenness centrality rankings can be computed in polynomial time.

Next, we prove that the problem is NP-hard. To this end, we show a reduction from the NP-complete problem of *Finding k-Clique*. The decision version of this problem is defined by a simple network, $G=(V,E)$, and a constant, $k \in \mathbb{N}$. The goal is to determine whether there exist k nodes in G that form a clique.

Given an instance of the problem of *Finding k-Clique*, defined by k and a simple network $G=(V,E)$, let us construct a multilayer network, $M=(V_L,E_L,V',L)$, as follows (Figure 2 depicts an instance of this network):

- **The set of nodes V':** For every node, $v_i \in V$, we create a node v_i. Additionally, we create the evader node $\widehat{v}$, node a, and the following three sets of nodes:
  1. $B=\{b_1,b_2\}$;
  2. $W=\{w_1,\ldots,w_k\}$;
  3. $C=\{c_1,\ldots,c_{n+k}\}$.
- **The set of layers L:** We create a layer α, a layer γ, as well as n layers $\beta_1,\ldots,\beta_n$.
- **The set of occurrences of nodes in layers V_L:** Node $\widehat{v}$ and node a appear in layer α and all layers $\{\beta_1,\ldots,\beta_n\}$. Each node v_i appears in layer α and β_i. Nodes in W appear in all layers $\{\beta_1,\ldots,\beta_n\}$. Nodes in B and C appear only in layer γ.
- **The set of edges E_L:** In layer α we create an edge between two nodes $v_i,v_j \in V$ if and only if this edge was present in G. In every layer where a appears we connect it with all occurring nodes from V and W. Finally, we connect every node c_i with both b_1 and b_2.

Now, consider the following instance of the problem of Multilayer Local Hiding, $(M,\widehat{v},F,c,(d^{\alpha})_{\alpha\in L})$, where:

- M is the multilayer network we just constructed;
- $\widehat{v}$ is the evader;
- $F=V\cup W$ is the set of contacts;

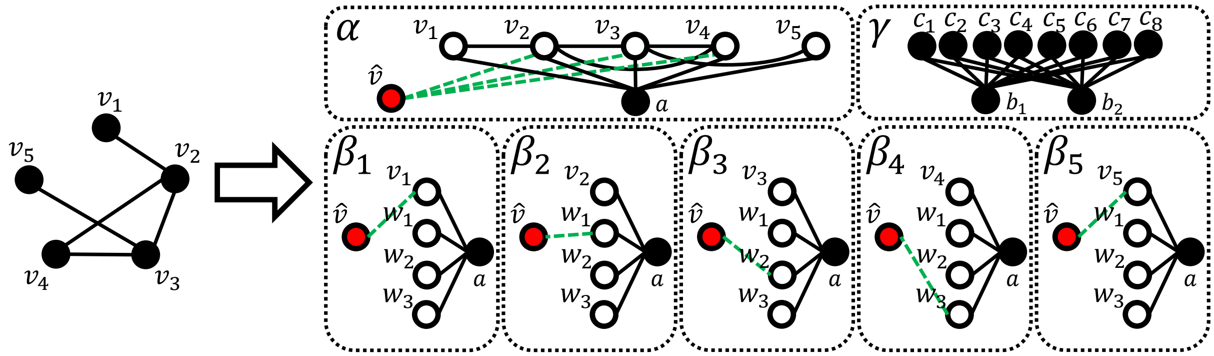

Figure 2: An illustration of the network used in the proof of Theorem 2. The red node represents the evader, while the white nodes represent the contacts. Dashed (green) edges represent the solution to this problem instance.

- c is the betweenness centrality measure;
- $d = 2n + 2k + 3$ is the safety margin.

Notice that, since $d = 2n + 2k + 3$, all other nodes must have greater betweenness centrality than the evader in order for $\widehat{v}$ to be hidden. Notice also that the betweenness centrality of every node c_i is $\frac{1}{n+k}$. Moreover, after adding A^* all nodes other than $\widehat{v}$ have non-zero betweenness centrality. If $\widehat{v}$ gets connected to at least two nodes from F that are not connected to each other, then $\widehat{v}$ controls one of at most $n+k-1$ shortest path between them (other paths can only go through nodes in $V \cup W \cup \{a\}$) and thus the betweenness centrality of $\widehat{v}$ is at least $\frac{1}{n+k-1}$. Therefore, in order to get hidden, $\widehat{v}$ cannot control any shortest paths in the network. This implies that, if $\widehat{v}$ is hidden then all nodes that are connected to $\widehat{v}$ in layer α must form a clique, and also implies that in every layer β, the evader $\widehat{v}$ can be connected to at most one node (otherwise $\widehat{v}$ controls one of the shortest paths between its two neighbors without an edges between them).

Now we will show that if there exists a solution to the given instance of the Finding k-Clique problem, then there also exists a solution to the constructed instance of the Multilayer Global Hiding problem. Let V^* be a group of k nodes forming a clique in G. Let us create A^* by connecting $\widehat{v}$ to nodes from V^* in layer α. Now, we connect every $v_i \in V \setminus V^*$ to $\widehat{v}$ in layer β_i. In the remaining layers from $\{\beta_1, \dots, \beta_n\}$ (corresponding to elements $v_i \in V^*$) we connect $\widehat{v}$ to all nodes in W. As argued above, for such A^*, the evader $\widehat{v}$ is hidden, hence A^* is a solution to the constructed instance of the Multilayer Global Hiding problem.

To complete the proof we have to show that if there exists a solution A^* to the constructed instance of the Multilayer Global Hiding problem, then there also exists a solution to the given instance of the Finding k-Clique problem. As argued above, in each layer β_i the evader $\widehat{v}$ can be connected to at most one node. Since all k nodes from W appear only in layers from $\{\beta_1, \dots, \beta_n\}$, the evader $\widehat{v}$ can be connected to at most $n - k$ nodes from V in layers from $\{\beta_1, \dots, \beta_n\}$. Therefore, $\widehat{v}$ has to have at least k neighbors from V in layer α As shown above, in order for $\widehat{v}$ to be hidden in α, all of its neighbors must form a clique. Hence, the neighbors of $\widehat{v}$ in layer α form a clique in G. This concludes the proof. □

Theorem 3. *The problem of Multilayer Local Hiding is NP-complete given the degree centrality measure.*

Proof. The problem is trivially in NP, since after the addition of a given A^* the degree centrality rankings for all layers can be computed in polynomial time.

Next, we prove that the problem is NP-hard. To this end, we show a reduction from the NP-complete problem of *Exact 3-Set Cover*. The decision version of this problem is defined by a set of subsets $S = \{S_1, \dots, S_m\}$ of universe $U = \{u_1, \dots, u_{3k}\}$, such that $\forall_i |S_i| = 3$. The goal is to determine whether there exist k pairwise disjoint elements of S the sum of which equals U.

Given an instance of the problem of *Exact 3-Set Cover*, let us construct a multilayer network, $M = (V_L, E_L, V', L)$, as follows (Figure 3 depicts an instance of this network):

- **The set of nodes V':** For every element, $u_i \in U$, we create a node u_i. We also create $2(m-k)$ nodes $w_1, \dots, w_{2(m-k)}$. Additionally, we create the evader node $\widehat{v}$ and three nodes a_1, a_2, a_3. We will denote the set of all nodes a_i as A, the set of all nodes u_i as U, and the set of all nodes w_i as W.
- **The set of layers L:** For every $S_i \in S$ we add a layer α_i.
- **The set of occurrences of nodes in layers V_L:** Node $u_j \in U$ appears in layer α_i if and only if $u_j \in S_i$. The evader $\widehat{v}$, as well as all nodes in A and W, appear in all layers.
- **The set of edges E_L:** In every layer we connect every node $u_j \in U$ occurring in this layer to every node in A.

Now, consider the following instance of the problem of Multilayer Local Hiding, $(M, \widehat{v}, F, c, (d^{\alpha})_{\alpha \in L})$, where:

- M is the multilayer network we just constructed;
- $\widehat{v}$ is the evader;
- $F = U \cup W$ is the set of contacts;
- c is the degree centrality measure;
- $d^{\alpha_i} = 3$ for every $\alpha_i \in L$.

Next, let us consider what are the sets of edges that can be added between the evader $\widehat{v}$ and the contacts F in each layer, so that the evader is hidden. In every layer α_i the nodes in A as well as the nodes $u_j \in S_i$ have degree 3, while all other

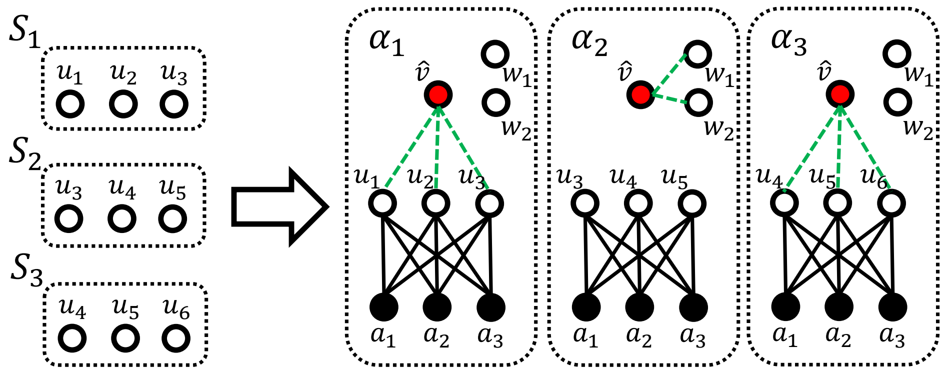

Figure 3: An illustration of the network used in the proof of Theorem 3. The red node represents the evader, while the white nodes represent the contacts. Dashed (green) edges represent the solution to this problem instance.

nodes have degree 0. We can connect $\widehat{v}$ to any two or less contacts and $\widehat{v}$ will still be hidden. If we connect the evader to three contacts, they have to be nodes in S_i (as these are the only nodes that potentially can have degree greater than 3). We cannot connect $\widehat{v}$ to more than three contacts and still have $\widehat{v}$ hidden.

Now we will show that if there exists a solution to the given instance of the Exact 3-Set Cover problem, then there also exists a solution to the constructed instance of the Multilayer Local Hiding problem. Let S^* be an exact cover of U. For every $S_i \in S^*$ we connect $\widehat{v}$ to every $u_j \in S_i$ in layer α_i. This way, $\widehat{v}$ becomes connected to all $3k$ contacts from U, since all the sets in S^* are pairwise disjoint. For every $S_i \notin S^*$ we connect $\widehat{v}$ to two nodes from W in layer α_i (since there are $m-k$ such layers, we can connect $\widehat{v}$ to all $2(m-k)$ contacts from W this way).

To complete the proof we have to show that if there exists a solution to the constructed instance of the Multilayer Local Hiding problem, then there also exists a solution to the given instance of the Exact 3-Set Cover problem. Let x be the number of layers from $\{\alpha_1,\dots,\alpha_m\}$ in which $\widehat{v}$ has at most two neighbors, and let $m-x$ be the number of layers from $\{\alpha_1,\dots,\alpha_m\}$ where $\widehat{v}$ has exactly three neighbors. Since $\widehat{v}$ has to be connected to all $3k+2(m-k)$ contacts, we have $2x+3(m-x) \geq 3k+2(m-k)$, which gives us $x \leq m-k$. However, since $\widehat{v}$ can connect to nodes from W in layer α_i if and only if it connects to at most two nodes in α_i, we also have $2x \geq 2(m-k)$. Hence, we have $x = m-k$, *i.e.*, $\widehat{v}$ is connected with all nodes from W in $m-k$ layers from $\{\alpha_1,\dots,\alpha_m\}$. Therefore, in the remaining k layers from $\{\alpha_1,\dots,\alpha_m\}$, the evader $\widehat{v}$ has to connect to all $3k$ nodes from U. Since the evader cannot connect to more than three nodes in any layer α_i, all these sets of neighbors from U have to be disjoint, thus forming the solution to the given instance of the Exact 3-Set Cover problem. This concludes the proof. □

Theorem 4. *The problem of Multilayer Local Hiding is NP-complete given the closeness centrality measure.*

Proof. The problem is trivially in NP, since after the addition of a given A^* the closeness centrality rankings for all layers can be computed in polynomial time.

Next, we prove that the problem is NP-hard. To this end, we show a reduction from the NP-complete problem of *Exact 3-Set Cover*. The decision version of this problem is defined by a set of subsets $S = \{S_1,\dots,S_m\}$ of universe $U = \{u_1,\dots,u_{3k}\}$, such that $\forall_i |S_i| = 3$. The goal is to determine whether there exist k pairwise disjoint elements of S the sum of which equals U.

Given an instance of the problem of *Exact 3-Set Cover*, let us construct a multilayer network, $M = (V_L, E_L, V', L)$, as follows (Figure 4 depicts an instance of this network):

- **The set of nodes V':** For every $u_i \in U$ we create a node u_i. In addition, we create the nodes $w_1,\dots,w_{2(m-k)}$ and $a_1,\dots,a_{2(m-k)}$. Finally, we create the evader node $\widehat{v}$ and 5 nodes $c_1,\dots,c_5$. We will denote the set of all nodes a_i by A, the set of all nodes c_i by C, the set of all nodes u_i by U, and the set of all nodes w_i by W.
- **The set of layers L:** For every $S_i \in S$ we add a layer α_i.
- **The set of occurrences of nodes in layers V_L:** Node $u_j \in U$ appears in layer α_i if and only if $u_j \in S_i$. The evader $\widehat{v}$, as well as all nodes in A, C, and W, appear in all layers.
- **The set of edges E_L:** In all layers we connect every node w_i with the node a_i, and we create edges $(c_1,c_2),(c_1,c_3),(c_1,c_4),(c_4,c_5)$.

Now, consider the following instance of the problem of Multilayer Local Hiding, $(M,\widehat{v},F,c,(d^\alpha)_{\alpha\in L})$, where:

- M is the multilayer network we just constructed;
- $\widehat{v}$ is the evader;
- $F = U \cup W$ is the set of contacts;
- c is the closeness centrality measure;
- $d^{\alpha_i} = 1$ for every $\alpha_i \in L$.

Next, let us consider what are the sets of edges that can be added between the evader $\widehat{v}$ and the contacts F in each layer, so that the evader is hidden. Notice that closeness centrality of the node c_1 is $3\frac{1}{2}$ and it is not affected by the edges added to $\widehat{v}$. Assume that we connect node $\widehat{v}$ with x nodes from U and y nodes from W. We then have the following (for easier comparison we express the centrality values as fractions with the common denominator 6):

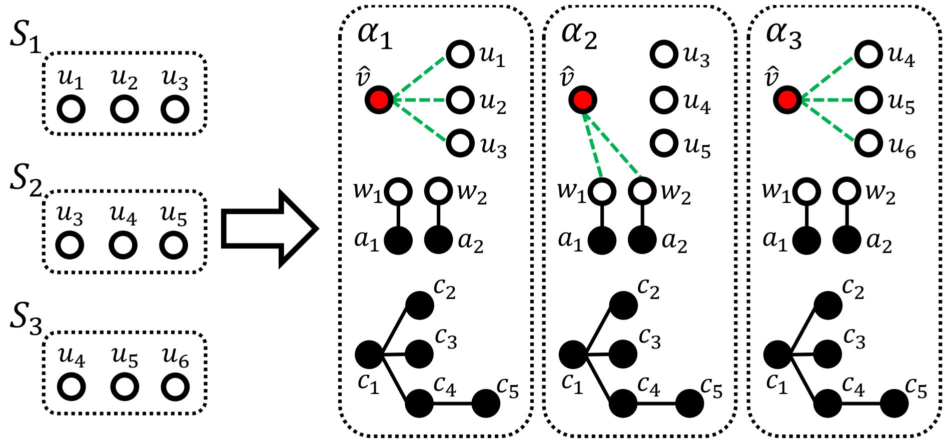

Figure 4: An illustration of the network used in the proof of Theorem 4. The red node represents the evader, while white the nodes represent the contacts. Dashed (green) edges represent the solution to this problem instance.

- $c_{clos}(\widehat{v}) = x + \frac{3y}{2} = \frac{6x+9y}{6}$;
- $c_{clos}(w_i) = \frac{x}{2} + \frac{5y}{6} + \frac{7}{6} = \frac{3x+5y+7}{6}$ if $w_i \in N(\widehat{v})$;
- $c_{clos}(c_1) = \frac{7}{2} = \frac{21}{6}$;

No other node can have greater closeness centrality than $\widehat{v}$. We can connect $\widehat{v}$ with at most two of any of the contacts, as node c_1 will still have greater closeness centrality. If we want to connect $\widehat{v}$ with three contacts, these contacts have to be nodes from U. If $x + y = 3$ and $y > 0$, or if $x + y > 3$, then the closeness centrality of $\widehat{v}$ is the highest in the network, meaning that $\widehat{v}$ is not hidden.

Now we will show that if there exists a solution to the given instance of the Exact 3-Set Cover problem, then there also exists a solution to the constructed instance of the Multilayer Local Hiding problem. Let S^* be an exact cover of U. For every $S_i \in S^*$ we connect $\widehat{v}$ to every $u_j \in S_i$ in layer α_i. This way, $\widehat{v}$ becomes connected to all $3k$ contacts from U, since all the sets in S^* are pairwise disjoint. For every $S_i \notin S^*$ we connect $\widehat{v}$ to two nodes from W in layer α_i (since there are $m - k$ such layers, we can connect $\widehat{v}$ to all $2(m - k)$ contacts from W this way).

To complete the proof, we have to show that if there exists a solution A^* to the constructed instance of the Multilayer Local Hiding problem, then there also exists a solution to the given instance of the Exact 3-Set Cover problem. Let z be the number of layers from $\{\alpha_1, \ldots, \alpha_m\}$ where $\widehat{v}$ has at most two neighbors, and let $z - x$ be the number of layers from $\{\alpha_1, \ldots, \alpha_m\}$ where $\widehat{v}$ has exactly three neighbors. Since we have to connect $\widehat{v}$ to all $3k + 2(m - k)$ contacts, we have $2z + 3(m - z) \geq 3k + 2(m - k)$, which gives us $z \leq m - k$. However, since $\widehat{v}$ can connect to nodes from W in layer α_i if and only if it connects to at most two nodes in α_i, we also have $2z \geq 2(m - k)$. Hence, we have $z = m - k$, *i.e.*, $\widehat{v}$ connects to all nodes from W in $m-k$ layers from $\{\alpha_1, \ldots, \alpha_m\}$. Therefore, in the remaining k layers from $\{\alpha_1, \ldots, \alpha_m\}$, the evader $\widehat{v}$ has to connect with all $3k$ nodes from U. Since the evader cannot connect to more than three nodes in any layer α_i, all these sets of neighbors from U have to be disjoint, thus forming a solution to the given instance of the Exact 3-Set Cover problem. This concludes the proof. □

2 Hardness of Approximation

In this section we investigate the hardness of approximation of hiding in multilayer networks.

First, we define the maximization versions of both problems. They take into consideration a situation when it is impossible to connect the evader with all contacts.

Definition 1 (Maximum Multilayer Global Hiding)**.** *This problem is defined by a tuple, $(M, \widehat{v}, F, c, d)$, where $M = (V_L, E_L, V, L)$ is a multilayer network, $\widehat{v} \in V$ is the evader, $F \subset V$ is the group of contacts, c is a centrality measure, and $d \in \mathbb{N}$ is a safety margin. The goal is then to identify a set of edges to be added to the network, $A^* \subseteq \{(\widehat{v}^\alpha, v^\alpha) : v \in F \wedge \widehat{v}^\alpha \in V_L \wedge v^\alpha \in V_L\}$, such that in the resulting network $\widehat{M} = (V_L, E_L \cup A^*, V, L)$ the evader is connected with as many contacts as possible, while there are at least d nodes with a centrality score greater than that of the evader.*

Definition 2 (Maximum Multilayer Local Hiding)**.** *This problem is defined by a tuple, $(M, \widehat{v}, F, c, (d^\alpha)_{\alpha \in L})$, where $M = (V_L, E_L, V, L)$ is a multilayer network, $\widehat{v} \in V$ is the evader, $F \subset V$ is the group of contacts, c is a centrality measure, and $d^\alpha \in \mathbb{N}$ is a safety margin for layer $\alpha \in L$. The goal is then to identify a set of edges to be added to the network, $A^* \subseteq \{(\widehat{v}^\alpha, v^\alpha) : v \in F \wedge \widehat{v}^\alpha \in V_L \wedge v^\alpha \in V_L\}$, such that in the resulting network $\widehat{M} = (V_L, E_L \cup A^*, V, L)$ the evader is connected with as many contacts as possible, while for each layer α the network G^α contains at least d^α nodes with a centrality score greater than that of the evader.*

Intuitively, the goal is to connect the evader with as many contacts as possible, while keeping the evader hidden. We now prove the inapproximability of both problems given the betweenness centrality.

Theorem 5. *Both Maximum Multilayer Global Hiding and Maximum Multilayer Local Hiding problems given the betweenness centrality cannot be approximated within $|F|^{1-\epsilon}$ for any $\epsilon > 0$, unless P=NP.*

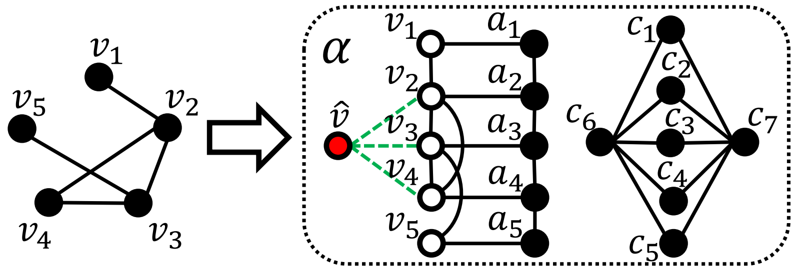

Figure 5: An illustration of the network used in the proof of Theorem 5. The red node represents the evader, while the white nodes represent the contacts. Dashed (green) edges represent the optimal solution to this problem instance.

Proof. In order to prove the theorem, we will use the result by Zuckerman [2006] that the *Maximum Clique* problem cannot be approximated within $|V|^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$. The Maximum Clique problem is defined by a simple network, $G = (V, E)$. The goal is to identify the maximum (in terms of size) group of nodes in G that form a clique.

First, we will show a function $f(G)$ that based on an instance of the problem of Maximum Clique, defined by a simple network $G = (V, E)$, constructs either an instance of the Maximum Multilayer Global Hiding or an instance of the Maximum Multilayer Local Hiding.

Let a multilayer network, $M = (V_L, E_L, V', L)$, be defined as follows (Figure 5 depicts an instance of this network):

- **The set of nodes V':** For every node, $v_i \in V$, we create a node v_i and a node a_i. Additionally, we create the evader node $\widehat{v}$ and the set of nodes $C = \{c_1, \dots, c_{n+2}\}$.
- **The set of layers L:** We create only a single layer α.
- **The set of occurrences of nodes in layers V_L:** All nodes occur in layer α.
- **The set of edges E_L:** In layer α we create an edge between two nodes $v_i, v_j \in V$ if and only if this edge was present in G. We also create an edge (v_i, a_i) for every v_i, and an edge between every pair a_i, a_{i+1}. Finally, for every node $c_i \in C : i \leq n$, we create edges (c_i, c_{n+1}) and (c_i, c_{n+2}).

  To complete the constructed instance of the problem let:

- $\widehat{v}$ be the evader;
- $F = V$ be the set of contacts;
- c be the betweenness centrality measure;
- $d = 3n + 2$ be the safety margin in the global version;
- $d^\alpha = 3n + 2$ be the safety margin in the local version.

Hence, the formula of the function f is $f(G) = (M, \widehat{v}, F, c, d)$ for the global version of the problem and $f(G) = (M, \widehat{v}, F, c, (d^\alpha)_{\alpha \in L})$ for the local version of the problem. Notice that since network M has only one layer, both problems are equivalent. In the following we will focus on the global version of the problem.

Let A^* be the solution to the constructed instance of the Maximum Multilayer Global Hiding problem. The function g computing corresponding solution to the instance G of the Maximum Clique problem is now $g(A^*) = \{v \in V : (\widehat{v}, v) \in A^*\}$, *i.e.*, the nodes forming the clique are the contacts that the evader is connected to.

Now, we will show that $g(A^*)$ is indeed a correct solution to G, *i.e.*, that the nodes form a clique. Notice that, since $d = 2n + 2k + 3$, all other nodes must have greater betweenness centrality than the evader in order for $\widehat{v}$ to be hidden. Notice also that the betweenness centrality of every node c_i for $i \leq n$ is $\frac{1}{n}$. Moreover, after adding A^* all nodes other than $\widehat{v}$ have non-zero betweenness centrality. If $\widehat{v}$ gets connected to at least two nodes from F that are not connected to each other, then $\widehat{v}$ controls one of at most $n - 1$ shortest path between them (other paths can only go through nodes in V) and thus the betweenness centrality of $\widehat{v}$ is at least $\frac{1}{n-1}$. Therefore, in order to get hidden, $\widehat{v}$ cannot control any shortest paths in the network. This implies that, if $\widehat{v}$ is hidden then all nodes that are connected to $\widehat{v}$ must form a clique.

Therefore, the optimal solution to the constructed instance of the Maximum Multilayer Global Hiding problem is returning nodes from V forming a clique of maximum size. Since the structure of connections between the nodes V is the same as in the network G, the optimal solution corresponds to the optimal solution to the given instance of the Maximum Clique problem.

Now, assume that there exists an approximation algorithm for the Maximum Multilayer Global Hiding problem with ratio $|F|^{1-\epsilon}$ for some $\epsilon > 0$. Let us use this algorithm to solve the constructed instance $f(G)$, acquiring solution A^*. and consider solution $g(A^*)$ to the given instance of the Maximum Clique problem. Since the size of the optimal solution is the same for both instances, we obtained an approximation algorithm that solves Maximum Clique problem to within $|V|^{1-\epsilon}$ for $\epsilon > 0$. However, Zuckerman [2006] shown that the Maximum Clique problem cannot be approximated within $|V|^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$. Therefore, such approximation algorithm for the Maximum Multilayer Global Hiding problem cannot exist, unless $P = NP$. This concludes the proof. $\square$

3 Simulation Results

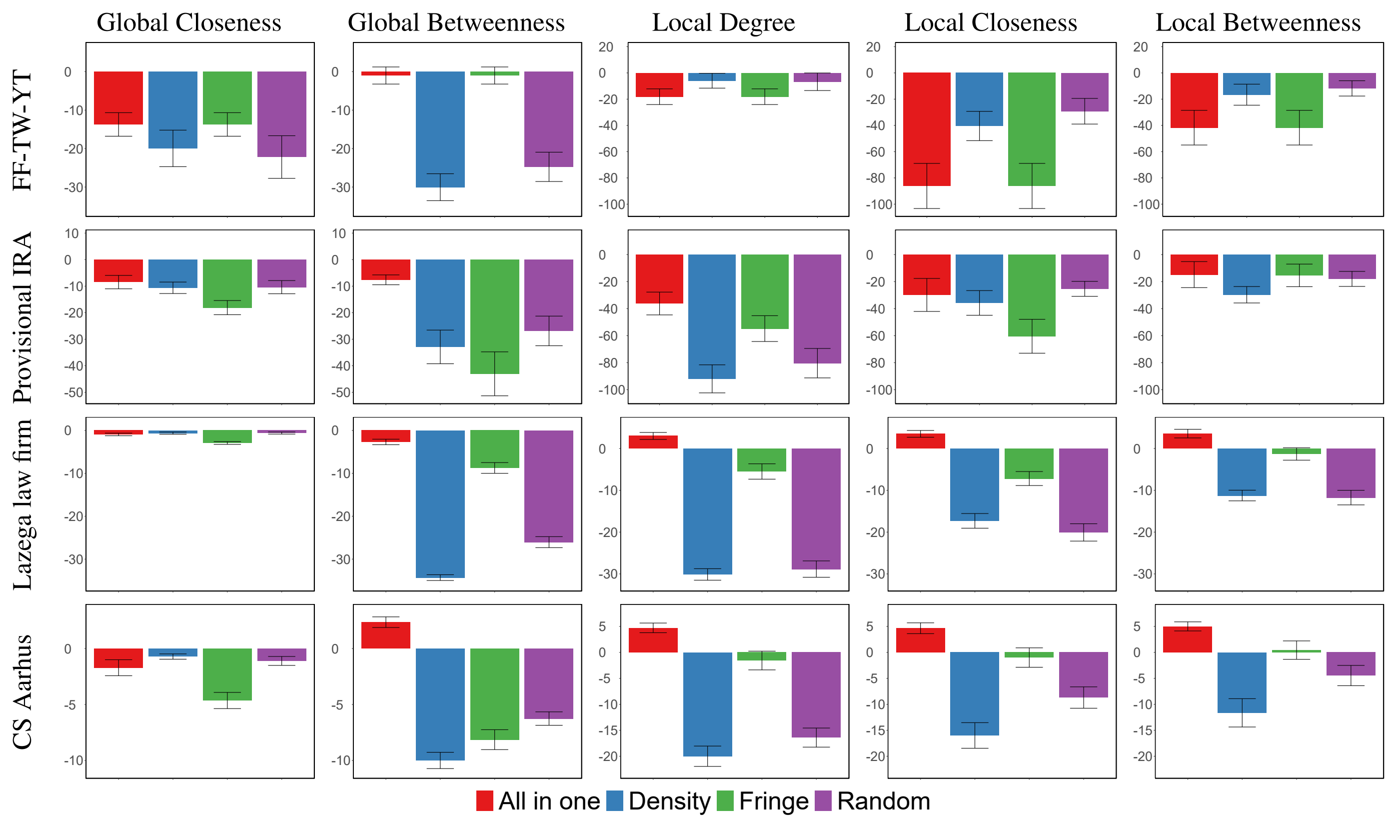

Figure 6: Given different centrality measures and different networks, the figure depicts the average change in centrality ranking of 10 different evaders as a result of execution of different hiding heuristics. For the randomly generate networks the experiment is repeated 100 times, with a new network generated each time. Error bars represent 95% confidence intervals.

References

[Zuckerman, 2006] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. In *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pages 681–690. ACM, 2006.