A APPENDIX

A.1 Proofs

Proposition 1. *Unifying $\mathcal{F}_1$ and $\mathcal{F}_2$ preserves valid configurations, i.e., $[\![\mathcal{F}_1]\!] = [\![\hat{\mathcal{F}}_1]\!]$ and $[\![\mathcal{F}_2]\!] = [\![\hat{\mathcal{F}}_2]\!]$ holds.*

Proof. The proofs for $\mathcal{F}_1$ and $\mathcal{F}_2$ are analogous, so we only show the case for $\mathcal{F}_1$.
(1) $[\![\mathcal{F}_1]\!] \subseteq [\![\hat{\mathcal{F}}_1]\!]$: Let $\theta \in [\![\mathcal{F}_1]\!]$. Then we know $\theta \subseteq \mathfrak{F}_1$ and thus $\theta \subseteq \hat{\mathfrak{F}}$. Further, we know $\theta \models \phi$ for all $\phi \in C_1$, and it remains to show $\theta \models \neg X$ for $X \in \mathfrak{F}_2 \backslash \mathfrak{F}_1$. Clearly, $X \notin \theta$ holds, as $\theta \subseteq \mathfrak{F}_1$.
(2) $[\![\hat{\mathcal{F}}_1]\!] \subseteq [\![\mathcal{F}_1]\!]$: Let $\hat{\theta} \in [\![\hat{\mathcal{F}}_1]\!]$. We know $\hat{\theta} \subseteq \hat{\mathfrak{F}} = \mathfrak{F}_1 \cup \mathfrak{F}_2$ and need to show $\hat{\theta} \subseteq \mathfrak{F}_1$, i.e., there is no $X \in \mathfrak{F}_2 \backslash \mathfrak{F}_1$ s.t. $X \in \hat{\theta}$. This holds as we know $\hat{\theta} \models \neg X$ for $X \in \mathfrak{F}_2 \backslash \mathfrak{F}_1$. □

Proposition 2.

(1) Every uniform n-sampler is also expectation-uniform.
(2) For every expectation-uniform n-sampler $\mathcal{V}$ over Ω, we have $\mathbf{E}[\mathcal{V}(\theta)] = n/|\Omega|$ for all $\theta \in \Omega$.

Proof. (1) Let $\mathcal{V}$ be a uniform n-sampler for Ω. Then there exists a uniform single sampler $\mathcal{U}$ and n-sampler $\mathcal{U}^n_\Omega$ and for all $\theta \in \Omega$:

$$\mathbf{E}[\mathcal{V}(\theta)] = \sum_{k=0}^{n} k \cdot \Pr[\mathcal{V}(\theta)=k] = \sum_{k=0}^{n} k \cdot \Pr[\mathcal{U}^n_\Omega(\theta)=k] \\ = \mathbf{E}[\mathcal{U}^n_\Omega(\theta)].$$

(2) Since $\mathcal{U}^n_\Omega$ returns n independent samples and the probability of each sample is given by $1/|\Omega|$ (due to uniformity of $\mathcal{U}$), the random variable $\mathcal{U}^n_\Omega(\theta)$ follows a binomial distribution with parameters n and $p = 1/|\Omega|$. The expected value of such a distribution is known to be np, therefore $\mathbf{E}[\mathcal{U}^n_\Omega(\theta)] = n/|\Omega|$. □

Proposition 3. *For Ω a sample space strictly partitioned into subspaces Ω_1 and Ω_2 with respective uniform samplers $\mathcal{U}_1$ and $\mathcal{U}_2$, the following single sampler $\mathcal{U}$ for Ω is uniform:*

- *With probability $|\Omega_1|/|\Omega|$ return a sample from $\mathcal{U}_1$.*
- *With probability $|\Omega_2|/|\Omega| = 1 - |\Omega_1|/|\Omega|$ return a sample from $\mathcal{U}_2$.*

Proof. We need to show $\Pr[\mathcal{U}=\theta] = 1/|\Omega|$ for all $\theta \in \Omega$. Since $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 disjoint, we consider two cases: For $\theta \in \Omega_1$, we have

$$\Pr[\mathcal{U}=\theta] = \frac{|\Omega_1|}{|\Omega|} \cdot \Pr[\mathcal{U}_1=\theta] = \frac{|\Omega_1|}{|\Omega|} \cdot \frac{1}{|\Omega_1|} = \frac{1}{|\Omega|}.$$

The case for $\theta \in \Omega_2$ is analogous by noting that $|\Omega| = |\Omega_1| + |\Omega_2|$ and thus $1 - |\Omega_1|/|\Omega| = |\Omega_2|/|\Omega|$. □

Proposition 4. *The sampler implemented by Algorithm 1 is uniform.*

Proof. Consider the sampler $\hat{\mathcal{U}}$ given by Algorithm 3 that generates a single sample for $\mathcal{F}'$. As $\tilde{\mathcal{U}}^1_{\text{old}}$ and $\mathcal{U}^1_{\text{new}}$ are both uniform and $[\![\mathcal{F}']\!]$ is strictly partitioned into $[\![\mathcal{F}'_{\text{old}}]\!]$ and $[\![\mathcal{F}'_{\text{new}}]\!]$, it follows from Proposition 3 that $\hat{\mathcal{U}}$ is uniform. Since drawing n times from a Bernoulli distribution with parameter r and counting the outcomes is equivalent to drawing from a binomial distribution with parameters n and r, it follows that Algorithm 1 is equivalent to sampling n times from Algorithm 3, i.e., the n-sampler $\hat{\mathcal{U}}^n_{[\![\mathcal{F}']\!]}$. □

Proposition 5. *The sampler implemented by Algorithm 2 is expectation-uniform.*

Algorithm 3: $\hat{\mathcal{U}}$: Generate a single sample for $\mathcal{F}'$

$r \leftarrow \frac{|[\![\mathcal{F}'_{\text{old}}]\!]|}{|[\![\mathcal{F}']\!]|}$
if `Bernoulli`$(r) = 1$ **then**
  $\theta \leftarrow \tilde{\mathcal{U}}^1_{\text{old}}$
else
  $\theta \leftarrow \mathcal{U}^1_{\text{new}}$
return θ

Proof. Let $S_{\mathcal{F}'} = S'_{\text{old}} \cup S'_{\text{new}}$ denote the multiset returned by Algorithm 2. Following Proposition 2, we show that for all configurations $\theta \in [\![\mathcal{F}']\!]$, we have $\mathbf{E}[S_{\mathcal{F}'}(\theta)] = n/|[\![\mathcal{F}']\!]|$. (1) Let $\theta \in [\![\mathcal{F}'_{\text{old}}]\!]$. Then

$$\mathbf{E}[S_{\mathcal{F}'}(\theta)] = \mathbf{E}[S'_{\text{old}}(\theta)] = \frac{\lfloor n \cdot r \rfloor + x}{\left|[\![\mathcal{F}'_{\text{old}}]\!]\right|}.$$

As $x = n \cdot r - \lfloor n \cdot r \rfloor$ and $r = |[\![\mathcal{F}'_{\text{old}}]\!]|/|[\![\mathcal{F}']\!]|$, we have

$$\mathbf{E}[S_{\mathcal{F}'}(\theta)] = \frac{n \cdot \frac{|[\![\mathcal{F}'_{\text{old}}]\!]|}{|[\![\mathcal{F}']\!]|}}{\left|[\![\mathcal{F}'_{\text{old}}]\!]\right|} = \frac{n}{\left|[\![\mathcal{F}']\!]\right|}.$$

(2) Let $\theta \in [\![\mathcal{F}'_{\text{new}}]\!]$. Then

$$\begin{aligned}\mathbf{E}[S_{\mathcal{F}'}(\theta)] &= \mathbf{E}[S'_{\text{new}}(\theta)] \\ &= \frac{\lfloor n \cdot (1-r) \rfloor + (1-x)}{\left|[\![\mathcal{F}'_{\text{new}}]\!]\right|} \\ &= \frac{\lfloor n - n \cdot r \rfloor + 1 - n \cdot r + \lfloor n \cdot r \rfloor}{\left|[\![\mathcal{F}'_{\text{new}}]\!]\right|}.\end{aligned}$$

Note that $\lfloor n - n \cdot r \rfloor = n + \lfloor -n \cdot r \rfloor$ since $n \in \mathbb{N}$ and $\lfloor -n \cdot r \rfloor + \lfloor n \cdot r \rfloor = -1$ since $n \cdot r \notin \mathbb{Z}$. With $|[\![\mathcal{F}']\!]| = |[\![\mathcal{F}'_{\text{old}}]\!]| + |[\![\mathcal{F}'_{\text{new}}]\!]|$, we have

$$\begin{aligned}\mathbf{E}[S_{\mathcal{F}'}(\theta)] &= \frac{n - n \cdot r}{\left|[\![\mathcal{F}'_{\text{new}}]\!]\right|} = \frac{n \cdot \left(1 - \frac{|[\![\mathcal{F}'_{\text{old}}]\!]|}{|[\![\mathcal{F}']\!]|}\right)}{\left|[\![\mathcal{F}'_{\text{new}}]\!]\right|} \\ &= \frac{n \cdot \frac{|[\![\mathcal{F}'_{\text{new}}]\!]|}{|[\![\mathcal{F}']\!]|}}{\left|[\![\mathcal{F}'_{\text{new}}]\!]\right|} = \frac{n}{\left|[\![\mathcal{F}']\!]\right|}.\end{aligned}$$

□