Efficient Computation of Signature-Restricted Views for Semantic Web Ontologies

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ABSTRACT

Uniform Interpolation (UI) is an advanced reasoning service used to narrow down an ontology to a restricted view. This new ontology, known as a uniform interpolant, will only consist of the "relevant names", yet it will retain their original meanings. UI is immensely promising due to its applicability across various domains where custom views of ontologies are essential. Nonetheless, to unlock its full potential, we need optimized techniques to generate these tailored views. Previous studies suggest that creating uniform interpolants for $\mathcal{EL}$-ontologies is notably challenging. In some instances, it is not even feasible to compute a uniform interpolant. When feasible, the size of the uniform interpolant can be up to triple exponentially larger than the source ontology. Despite these challenges, our paper introduces an improved "forgetting" technique specifically designed for computing uniform interpolants of $\mathcal{ELI}$-ontologies. We demonstrate that, with good normalization and inference strategies, such uniform interpolants can be efficiently computed, just as swiftly as computing "modules". A comprehensive evaluation with a prototypical implementation of the method shows superb success rates over two popular benchmark datasets, demonstrating a clear computational advantage over state-of-the-art approaches.

CCS CONCEPTS

• **Theory of computation → Description logics**; **Automated reasoning**; • **Computing methodologies → Ontology engineering**.

KEYWORDS

Ontologies, Module Extraction, Uniform Interpolation, Forgetting

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1 INTRODUCTION

The increasing availability of machine-processable web data has put the desideratum of semantic interoperability on the top of the World Wide Web's agenda — a requirement to enable the exchange of data with precise, unambiguous, shared meaning across distributed web applications. Although it is not yet a reality in the Web of today, much effort and progress have been made towards achieving this vision — *the Semantic Web* [42]. The key idea is to add descriptions about the web data (aka metadata), linking each data element to a controlled vocabulary that provides a common reference point for aggregating and comparing data about a particular subject domain. These descriptions can rely on logical statements relating data to some terms within a given ontology [13, 43].

An ontology fixes a controlled vocabulary of names (aka *signature*) relevant to a subject domain and specifies constraints among the names by logical statements (aka *axioms*) [43]. However, due to the intrinsic heterogeneity of web resources, ontologies designed for the semantic web tend to exhibit large-scale and encompass knowledge spanning a broad range of topics. This sheer scale and comprehensiveness may hinder the reusability of ontologies in real-world web applications. This is mainly attributed to the challenges associated with the management and manipulation of large and complex ontologies, which can be unwieldy and incur considerable computational costs when engaged in the reasoning process.

One possible strategy to address these challenges is to extract a "module" from an ontology that retains the functionality of the original ontology within a specific context while achieving a substantial reduction in size. This is a desirable strategy for the flexible reuse of ontologies for several reasons. First, many ontology maintenance tasks can be done locally by simply adjusting the specific module in question. Next, the single components or respective modules can be reused in other contexts more easily. Further, from a more technical perspective, reasoning tasks can be done more efficiently under certain circumstances, as only a small module might be relevant for specific deductions, or the reasoning itself can be distributed to several machines separately handling the modules. A general module $\mathcal{M}$ is defined as [11]:

Condition I: a syntactic subset of a given ontology $\mathcal{O}$;
Condition II: preserves all logical entailments w.r.t. a specific sub-signature Σ of $\mathcal{O}$.

This means that $\mathcal{M}$ and $\mathcal{O}$ align in their logical entailments within Σ, and $\mathcal{M}$ can therefore be reused in other contexts within Σ as a substitute for $\mathcal{O}$. To satisfy both of the above conditions, $\mathcal{M}$ often needs to incorporate names outside of Σ. Take, for example, $\mathcal{O} = \{A \sqsubseteq B, B \sqsubseteq C, D \sqsubseteq E\}$ and $\Sigma = \{A, C\}$. A module $\mathcal{M}$ of $\mathcal{O}$ w.r.t. Σ is $\{A \sqsubseteq B, B \sqsubseteq C\}$. While this module preserves all logical entailments over $\{A, C\}$, it must include B — a name not in Σ — to achieve this preservation. Such constraints can limit the reusability of ontologies in specific real-world scenarios. For instance, in domains like medicine or the military, ontologies might hold sensitive data that should remain undisclosed when these ontologies are made public, distributed, or shared. This is also pertinent in industrial settings where proprietary details need stringent protection. A possible solution to ensure confidentiality is to limit the exposure of names considered sensitive. One strategy to manage this concealed information is to disseminate a fragment of the ontology that only includes names particular users have permission to view. This becomes especially vital when ontology proprietors wish to share their data with other users or the general public, but aim to disclose only non-sensitive details. Directly repurposing a module from an ontology may not be suitable in this context since it cannot assure the absolute concealment of specific names; the module might still include names that fall outside the designated signature.

In situations where one needs to preserve the functionality of the source ontology but only wishes to utilize a specific subset of names, it is more advantageous to have a signature-restricted view of the original ontology. This paper considers creating signature-restricted views using a uniform interpolation approach. Essentially, Uniform Interpolation (UI) is an advanced reasoning procedure that aims to narrow down an ontology to a smaller signature. When provided with an ontology that utilizes a specific sub-signature, Σ, representing the "relevant names" associated with certain topics, UI computes a new ontology, known as a *uniform interpolant*, that only employs the names in Σ while maintaining the same semantics of the Σ-names. However, a uniform interpolant does not just take a part of the original ontology; it might include axioms not found in the original. Think of it as a condensed version of a module. To ensure the semantics of the Σ-names remain intact, numerous new axioms will be derived from the original ontology. As a result, uniform interpolants can contain substantially more axioms than the source ontology. In fact, research indicates that the UI process is more computationally challenging than modularization [4, 31, 46].

Nevertheless, creating signature-restricted views of ontologies is still very much need since it may be used in a variety of applications where suitable views of ontologies need to be computed, such as debugging and repair [36, 45], merging and alignment [26, 35, 47], versioning [14, 15, 40], semantic difference [18, 19, 27, 52], abduction and explanation generation [6, 23] and interactive ontology revision [34]. However, this potential can only be fully realized if a highly optimized method (and its corresponding implementation) for computing such views exists.

In this paper, we present a highly optimized method for computing uniform interpolants of $\mathcal{ELI}$-ontologies. The method is based on a "forgetting procedure" that computes uniform interpolants by singly eliminating names from the original ontology that do not belong in Σ. Nikitina and Rudolph [33] show that computing uniform interpolants of $\mathcal{EL}$-ontologies is computationally extremely hard — a finite uniform interpolant does not always exist, and if it exists, then there exists one of at most triple exponential size in terms of the input ontology, and that, in the worst case, no shorter uniform interpolant exists. We show however in this paper that: (i) this result should not constitute a fundamental technical obstacle for UI in practice, and (ii) with good normalization and inference strategies, uniform interpolants can be computed as fast as computing modules. A comprehensive evaluation with a prototypical implementation of the method shows superb success rates over two popular benchmark datasets, demonstrating a clear computational advantage over state-of-the-art tools.

A long version of this paper including all missing proofs and additional illustrative examples, as well as the source code for the prototypical UI implementation alongside the test datasets, are *anonymously* distributed for review at https://github.com/anonymous-ai-researcher/www2024.

1.1 Related Work

Forgetting is an inherently difficult (non-standard) reasoning problem concerned with eliminating from an ontology a set of concept and role names in its signature, namely the *forgetting signature*, in such a way that all logical entailments are preserved up to the remaining signature; it is much harder than standard reasoning (satisfiability testing), and very few logics are known to be complete for forgetting. Foundational studies have shown that: (i) forgetting solutions do not always exist for the DL $\mathcal{EL}$ or $\mathcal{ALC}$ [17, 18, 31], (ii) deciding the existence of forgetting solutions is ExpTime-complete for $\mathcal{EL}$ [28] and 2ExpTime-complete for $\mathcal{ALC}$ [31], and (iii) forgetting solutions can be triple exponential in size w.r.t. the input ontologies for $\mathcal{EL}$ and $\mathcal{ALC}$ [31, 33].

Although forgetting presents a hard computational challenge, there is general consensus on its tremendous potential for ontology-based knowledge processing, and ongoing efforts have been dedicated to the development and automation of practical methods for computing solutions of forgetting. A few such methods have thus been developed and automated for various DLs.

Presently, the only practical methods for forgetting are Lethe and Fame. Lethe [21] employs the classic *resolution* calculus [3, 7] for single name elimination, and handles ontologies in $\mathcal{ALC}$ and several its extensions. Based on a monotonicity property called *Ackermann's Lemma* [1], Fame [53] addresses a stronger notion of forgetting called model-theoretic forgetting [48], and accommodates ontologies up to $\mathcal{ALCOIH}$. Nui and [50] are another two resolution-based approaches tailored for respectively $\mathcal{EL}$- and $\mathcal{SHQ}$-ontologies, but neither remains accessible at the moment. Hence, in this paper, we consider Lethe and Fame as our baselines.

2 PRELIMINARIES

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be pairwise disjoint and countably infinite sets of *concept* and *role* names, respectively. *Roles* in $\mathcal{ELI}$ are a role name $\mathsf{r} \in \mathsf{N_R}$ or its inverse r^-. *Concept descriptions* (or *concepts* for short) in $\mathcal{ELI}$ have one of the following forms:

$$\top \mid \mathsf{A} \mid C \sqcap D \mid \exists \mathsf{r}.C \mid \exists \mathsf{r}^-.C,$$

where $\mathsf{A} \in \mathsf{N_C}$, $\mathsf{r} \in \mathsf{N_R}$, and C and D range over concepts. We use r^- to denote s if $\mathsf{r} = s^-$ for $\mathsf{s} \in \mathsf{N_R}$ and identify $(\mathsf{r}^-)^-$ with r.

An $\mathcal{ELI}$-ontology $\mathcal{O}$ is a finite set of *axioms* of the form $C \sqsubseteq D$ (*general concept inclusion*, or GCI), where C and D are concepts. We use $C \equiv D$ as an abbreviation for the GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$.

Let $\mathcal{S} \in \mathsf{N_C} \cup \mathsf{N_R}$ be a designated concept name or role name. A concept (GCI) is called an $\mathcal{S}$-*concept* ($\mathcal{S}$-*GCI*) if it contains $\mathcal{S}$. An occurrence of $\mathcal{S}$ is said to be *positive* (*negative*) in an $\mathcal{S}$-GCI if it occurs at the right-hand (left-hand) side of the GCI.

The semantics of $\mathcal{ELI}$ is defined in terms of an *interpretation* $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set, known as the *domain of the interpretation*, and $\cdot^{\mathcal{I}}$ is the *interpretation function* that maps every concept name $A \in \mathsf{N_C}$ to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and every role name $r \in \mathsf{N_R}$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation function $\cdot^{\mathcal{I}}$ is inductively extended to concepts as follows:

$$\top^{\mathcal{I}} = \Delta^{\mathcal{I}} \qquad (C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$$
$$(\exists r.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y.(x,y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$$
$$(r^-)^{\mathcal{I}} = \{(y,x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x,y) \in r^{\mathcal{I}}\}$$

Let $\mathcal{I}$ be an interpretation. A GCI $C \sqsubseteq D$ is *true* in $\mathcal{I}$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. $\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$, written $\mathcal{I} \models \mathcal{O}$, iff every GCI in $\mathcal{O}$ is *true* in $\mathcal{I}$. A GCI $C \sqsubseteq D$ is a *logical entailment* of $\mathcal{O}$, written $\mathcal{O} \models C \sqsubseteq D$, iff $C \sqsubseteq D$ is true in every model $\mathcal{I}$ of $\mathcal{O}$.

A *signature* $\Sigma \subseteq \mathsf{N_C} \cup \mathsf{N_R}$ is a finite set of concept names and role names. We denote by $\mathrm{sig}_\mathsf{C}(X)$ and $\mathrm{sig}_\mathsf{R}(X)$ the sets of respectively the concept names and role names present in X, where X can be any syntactic objects including concepts, roles, GCIs, and ontologies. We further define $\mathrm{sig}(X) = \mathrm{sig}_\mathsf{C}(X) \cup \mathrm{sig}_\mathsf{R}(X)$.

Definition 1 (**Forgetting**). *Let $\mathcal{O}$ be an $\mathcal{ELI}$-ontology and $\mathcal{S} \in sig(\mathcal{O})$ be a concept/role name, referred to as the* pivot. *An $\mathcal{ELI}$-ontology $\mathcal{V}$ is a* result of forgetting $\{\mathcal{S}\}$ from $\mathcal{O}$ *if the following conditions hold:*
(i) *$sig(\mathcal{V}) \subseteq sig(\mathcal{O}) \backslash \{\mathcal{S}\}$, and*
(ii) *for any $\mathcal{ELI}$-GCI $C \sqsubseteq D$ with $sig(C \sqsubseteq D) \subseteq sig(\mathcal{O}) \backslash \{\mathcal{S}\}$, $\mathcal{V} \models C \sqsubseteq D$ iff $\mathcal{O} \models C \sqsubseteq D$.*
More generally, let $\mathcal{F} \subseteq sig(\mathcal{O})$ be a finite set of concept and role names, referred to as the forgetting signature. *An $\mathcal{ELI}$-ontology $\mathcal{V}$ is a* result of forgetting $\mathcal{F}$ from $\mathcal{O}$ *if the following conditions hold:*
(i) *$sig(\mathcal{V}) \subseteq sig(\mathcal{O}) \backslash \mathcal{F}$, and*
(ii) *for any $\mathcal{ELI}$-GCI $C \sqsubseteq D$ with $sig(C \sqsubseteq D) \subseteq sig(\mathcal{O}) \backslash \mathcal{F}$, $\mathcal{V} \models C \sqsubseteq D$ iff $\mathcal{O} \models C \sqsubseteq D$.*

The process of "forgetting" distills an ontology $\mathcal{O}$ into a more refined perspective, $\mathcal{V}$, which is anchored sorely on a sub-signature Σ of $\mathcal{O}$. Here, Σ is defined as $\Sigma \subseteq \mathrm{sig}(\mathcal{O}) \backslash \mathcal{F}$. Notably, when considering Σ, $\mathcal{V}$ mirrors the behavior of $\mathcal{O}$ within the $\mathcal{ELI}$ framework, implying that both ontologies yield the same $\mathcal{ELI}$-entailments w.r.t. Σ. Forgetting can also be defined in terms of *inseparability* [17, 30] and *conservative extensions* [9, 29]: an $\mathcal{ELI}$-ontology $\mathcal{V}$ is the result of forgetting $\mathcal{F}$ from an $\mathcal{ELI}$-ontology $\mathcal{O}$ for $\mathcal{F} = \mathrm{sig}(\mathcal{O}) \backslash \Sigma$ iff $\mathcal{V} \equiv_\Sigma^{\mathcal{ELI}} \mathcal{O}$, and $\mathcal{O}$ is an $\mathcal{ELI}$-conservative extension of $\mathcal{V}$ iff $\mathcal{V} \subseteq \mathcal{O}$ and $\mathcal{V} \equiv_\Sigma^{\mathcal{ELI}} \mathcal{O}$. The results of forgetting are unique up to logical equivalence, i.e., should both $\mathcal{V}_1$ and $\mathcal{V}_2$ result from the forgetting of $\mathcal{F}$ from $\mathcal{O}$, they would be logically indistinguishable, though their manifest representations may differ.

3 NORMALIZATION OF $\mathcal{ELI}$-ONTOLOGIES

Our method computes the result of forgetting $\mathcal{F}$ from $\mathcal{O}$ by iteratively forgetting single names in $\mathcal{F}$. The calculus for single-name elimination works on specialized normal forms of $\mathcal{ELI}$-ontologies.

3.1 A-Normal Form (A-NF)

Definition 2 (A-**Normal Form**). *We say that a GCI is in* A-normal form *(or* A-NF *for short) if it has one of the following forms, where (i) $r, s \in N_R$, and (ii) B, C, D, E, and F are concepts that do not*

	A-NF		***A-NF***
I	$C \sqsubseteq A$	*IV*	$A \sqcap E \sqsubseteq F$
II	$C \sqsubseteq \exists r.(A \sqcap D)$	*V*	$\exists s.(A \sqcap E) \sqcap F \sqsubseteq B$
III	$C \sqsubseteq \exists r^-.(A \sqcap D)$	*VI*	$\exists s^-.(A \sqcap E) \sqcap F \sqsubseteq B$

contain A. *An $\mathcal{ELI}$-ontology $\mathcal{O}$ is in* A-NF *if every* A*-GCI in $\mathcal{O}$ is in A-NF.*

One can transform a given $\mathcal{ELI}$-ontology $\mathcal{O}$ into a normalized one by exhaustively applying the following five normalization rules (NR1 – NR5) to the A-GCIs in $\mathcal{O}$ that have yet to be in A-NF (X, Y, Y_1 and Y_2 are $\mathcal{ELI}$-concepts).

- NR1 For each instance of a GCI $X \sqsubseteq Y_1 \sqcap Y_2$, if either Y_1 or Y_2 contains A, replace it by $X \sqsubseteq Y_1$ and $X \sqsubseteq Y_2$;
- NR2 For each instance of $X \sqsubseteq Y$, if A occurs more than once in it and a concept of the form $\exists R.C$ is present at the surface level of the left-hand side X, where R is a role and C is a concept that contains A, replace C by a fresh definer $Z \in \mathsf{N_C}$ and add $C \sqsubseteq Z$ to $\mathcal{O}$;
- NR3 For each instance of $X \sqsubseteq Y$, if A occurs more than once in it and a concept of the form $\exists R.C$ is present at the surface level of the right-hand side Y, where R is a role and C is a concept that contains A, replace C by a fresh definer $Z \in \mathsf{N_C}$ and add $C \sqsubseteq Z$ to $\mathcal{O}$;
- NR4 For each instance of $X \sqsubseteq Y$, if A occurs exactly once in it and a concept of the form $\exists R.C$ is present at the surface level of the left-hand side X, where R is a role — and provided that C contains A but is not in the form of $\mathsf{A} \sqcap E$ as specified by A-NF V or VI, replace C by a fresh definer $Z \in \mathsf{N_C}$ and add $C \sqsubseteq Z$ to $\mathcal{O}$;
- NR5 For each instance of $X \sqsubseteq Y$, if A occurs exactly once in it and a concept of the form $\exists R.C$ is present at the surface level of the right-hand side Y, where R is a role — and provided that C contains A but is not in the form of $\mathsf{A} \sqcap D$ as specified by A-NF II or III, replace C by a fresh definer $Z \in \mathsf{N_C}$ and add $C \sqsubseteq Z$ to $\mathcal{O}$;

The so-called *definers* [25], denoted as Z in the above context, are newly-introduced concept names that serve as "abbreviations" for compound concepts when applying the normalization rules.

Lemma 1. *Let $\mathcal{O}$ be an arbitrary $\mathcal{ELI}$-ontology. Then $\mathcal{O}$ can be transformed into A-normal form $\mathcal{O}'$ by a linear number of applications of the above normalization rules NR1 – NR5. In addition, the size of the resulting ontology $\mathcal{O}'$ is linear in the size of $\mathcal{O}$.*

Lemma 2. *Let $\mathcal{O}$ be an arbitrary $\mathcal{ELI}$-ontology and $\mathcal{O}'$ the normalized one obtained from $\mathcal{O}$ using the normalization rules NR1–NR5. Then we have*

$$\mathcal{O} \models C \sqsubseteq D \text{ iff } \mathcal{O}' \models C \sqsubseteq D,$$

for any $\mathcal{ELI}$-GCI $C \sqsubseteq D$ with $sig(C \sqsubseteq D) \subseteq sig(\mathcal{O})$.

Lemma 1 states the termination and completeness of the normalization and Lemma 2 states its soundness.

3.2 R-Normal Form (R-NF)

Definition 3 (R-**Normal Form**). *We say that a GCI is in* R-normal form *(or* R-NF*) if it has one of the following forms, where (i)*

	R-NF		***R-NF***
I	$C \sqsubseteq \exists r.D$	*III*	$E \sqcap \exists r.F \sqsubseteq B$
II	$C \sqsubseteq \exists r^-.D$	*IV*	$E \sqcap \exists r^-.F \sqsubseteq B$

$r \in N_R$, and (ii) B, C, D, E, and F are concepts that do not contain r. *An $\mathcal{ELI}$-ontology $\mathcal{O}$ is in* R-NF *if every* r*-GCI in $\mathcal{O}$ is in R-NF.*

One can compute the R-NF of a given $\mathcal{ELI}$-ontology $\mathcal{O}$ using a slightly adjusted approach for A-NF transformation.

4 DEFINER INTRODUCTION STRATEGY

Compared to the SOTA methods LETHE and FAME, our forgetting method introduces a novel normal form specification and exploits a non-traditional, yet notably cost-effective, definer introduction strategy for normalization, a factor that significantly bolsters the method's efficiency.

In closely examining LETHE and FAME, we delve into the intricacies of how they employ definers to facilitate the normalization of ontologies intended for the subsequent application of their respective forgetting rules. LETHE and FAME work on clauses of the form $L_1 \sqcup \ldots \sqcup L_k$, where each L_i ($1 \leq i \leq k$) is a TBox literal, defined as:

$$\mathsf{A} \mid \neg\mathsf{A} \mid \exists\mathsf{r}.Z \mid \exists\mathsf{r}^{-}.Z \mid \forall\mathsf{r}.Z \mid \forall\mathsf{r}^{-}.Z,$$

where $\mathsf{r} \in \mathsf{N_R}$ and $\mathsf{A}, Z \in \mathsf{N_C}$. A salient observation is that LETHE mandates every Z (essentially, any subconcept immediately below a $\exists$- or $\forall$-restriction) to be a definer at any stage of the forgetting process. In contrast, our method allows for a more flexible specification of Z. Such differentiation profoundly impacts the general applicability of the forgetting rules these methods employ, and by extension, has implications for their inferential efficiency.

For a deeper algorithmic understanding of LETHE's definer introduction strategy, we first fix some notations. By $\mathsf{sig_D}(\mathcal{O})$ we denote the set of definers introduced in $\mathcal{O}$, by $\mathsf{Sub}^{\forall}_{\exists}(\mathcal{O})$ the set of all subconcepts of the form $\exists\mathsf{r}^{(-)}.X$ or $\forall\mathsf{r}^{(-)}.X$ in $\mathcal{O}$, where $\mathsf{r} \in \mathsf{N_R}$ and X is an arbitrary concept, and by $\mathsf{Sub}_X(\mathcal{O})$ the set of all subconcepts X present in $\mathcal{O}$ with $\exists\mathsf{r}^{(-)}.X \in \mathsf{Sub}^{\forall}_{\exists}(\mathcal{O})$ or $\forall\mathsf{r}^{(-)}.X \in \mathsf{Sub}^{\forall}_{\exists}(\mathcal{O})$.

In the LETHE framework, which exploits a definer reuse strategy (i.e., LETHE consistently uses a definer to refer to identical subconcepts), an injective function f can be defined over $\mathsf{sig_D}(\mathcal{O})$, namely $f : \mathsf{sig_D}(\mathcal{O}) \to \mathsf{Sub}_X(\mathcal{O})$. f also exhibits surjectivity, given LETHE's exhaustive manner to introduce defines — LETHE mandates every subconcept immediately below an $\exists$- or $\forall$-restriction to be a definer. On the other hand, within our framework, f is defined as non-surjective. However, for both methods, the number of definers — denoted as $|\,\mathsf{sig_D}(\mathcal{O})|$ — relevant to the normalization of $\mathcal{O}$, is bounded by $\mathrm{O}(n)$. Here, n corresponds to the count of $\exists$- and $\forall$-restrictions present in $\mathcal{O}$. This implies a linear growth in the introduction of definers.

In the LETHE framework, a saturation-based reasoning procedure is employed to forget single concept names from $\mathcal{O}$. This is achieved by deriving new entailments and adding them to $\mathcal{O}$ using a generalized resolution calculus called Res, as described in [7]. Notably, Res works on clausal representation. LETHE computes the causal normal form of $\mathcal{O}$ using a two-stage approach. The first stage, termed as the *pre-resolution stage*, witnesses LETHE's inaugural computation of $\mathcal{O}$'s normal form to fire up Res. As previously discussed, this stage introduces definers in a *linear* and *static* manner.

Transitioning to the second stage, termed as the *intra-resolution stage*, LETHE exhaustively applies the inference rules of Res to $\mathcal{O}$ until $\mathcal{O}$ reaches saturation at $\mathsf{Res}(\mathcal{O})$. By definition, $\mathcal{O}$ is said to be *saturated* w.r.t. the pivot A iff every entailment α of an inference with $\mathsf{A} \notin \mathsf{sig}(\alpha)$ is already in $\mathcal{O}$, i.e., $\alpha \in \mathcal{O}$, or is redundant w.r.t. $\mathcal{O}$, i.e., $\mathcal{O}\backslash\{\alpha\} \models \alpha$. Here, new logical entailments are derived from the existing ones. To give an example, when we apply the $\forall\exists$-*role propagation* rule of Res to the clauses $C_1 \sqcup \forall r.D_1$ and $C_2 \sqcup \exists r.D_2$, it results in a new entailment $C_1 \sqcup C_2 \sqcup \exists r.(D_1 \sqcap D_2)$. To normalize this clause, LETHE introduces a fresh definer $D_{12} \in \mathsf{N_C}$ to replace $D1 \sqcap D_2$, and then adds $\neg D_{12} \sqcup (D_1 \sqcap D_2)$ to $\mathcal{O}$. As Res iterates over $\mathcal{O}$, definers are *dynamically* introduced in this way.

Transitioning to the second *intra-resolution stage*, LETHE exhaustively applies Res's inference rules on $\mathcal{O}$ until saturation is achieved at $\mathsf{Res}(\mathcal{O})$, where new entailments are propagated from existing ones. For example, applying the $\forall\exists$-*role propagation* rule to $C_1 \sqcup \forall r.D_1$ and $C_2 \sqcup \exists r.D_2$ yields a new entailment $C_1 \sqcup C_2 \sqcup \exists r.(D_1 \sqcap D_2)$, for which LETHE has to introduce a fresh definer $D_{12} \in \mathsf{N_C}$ to replace $D_1 \sqcap D_2$ for normalization, and then adds $\neg D_{12} \sqcup (D_1 \sqcap D_2)$ to $\mathcal{O}$ — definers are *dynamically* introduced as Res iterates over $\mathcal{O}$. Following this process, an additional injective yet non-surjective function f' emerges over $\mathsf{sig_D}(\mathsf{Res}(\mathcal{O}))$, defined as $f : \mathsf{sig_D}(\mathsf{Res}(\mathcal{O})) \to \mathsf{Sub}_X(\mathsf{Res}(\mathcal{O}))$, and the size of the codomain $|\mathsf{Sub}_X(\mathsf{Res}(\mathcal{O}))| = 2^{|\mathsf{Sub_D}(\mathcal{O})|}$. The number of the definers for the normalization of $\mathsf{Res}(O)$ is bounded by $\mathrm{O}(2^n)$, where n is the number of $\exists$- and $\forall$-restrictions in $\mathcal{O}$. Thus, LETHE introduces definers at an exponential rate during the intra-resolution stage. In contrast, our forgetting method restricts its normalization endeavors within the pre-resolution stage, indicating a linear trajectory in the introduction of definers during its entire forgetting span.

Definers are extraneous to the desired signature and thus should be excluded from the result of forgetting $\mathcal{F}$ from $\mathcal{O}$. Consequently, in the worst case, LETHE will be tasked with discarding as many as $(2^n) + |\mathcal{F}|$ names and executing Res for $\mathrm{O}(2^n) + |\mathcal{F}|$ iterations to compute the forgetting result. In contrast, our method introduces a maximum of n definers, and in the worst case, only needs to activate the forgetting calculus (described next) $n + |\mathcal{F}|$ times.

5 THE FORGETTING METHOD

5.1 Calculus for Forgetting A

Our forgetting method exploits a two-step calculus to forget a single concept name A from $\mathcal{O}$:

Step I: computes the A-normal form of $\mathcal{O}$ as described in the previous section;
Step II: forgets A by exhaustively applying the inference rules in Figure 1.

While LETHE and FAME work on the clausal representation of $\mathcal{O}$, our method directly addresses GCIs.

At the core of the exhaustive application of the inference rules is an effort to reveal all implicit logical entailments w.r.t. $\mathsf{sig}(\mathcal{O})\backslash\{\mathsf{A}\}$, and subsequently incorporate these new entailments into $\mathcal{O}$. This process continues until $\mathcal{O}$ becomes saturated w.r.t. A. Upon reaching this state of saturation, all A-GCIs can be safely removed from $\mathcal{O}$, resulting in the forgetting of A from $\mathcal{O}$ eventually.

The process of revealing implicit entailments from $\mathcal{O}$ involves the combination of positive occurrences (GCIs taking A-NF I, II, and III) with negative occurrences (GCIs taking A-NF IV, V, and VI) of A (i.e., resolving on A). This results in nine distinct combination scenarios, labeled as IR1 — IR9, respectively, as depicted in Figure 1. Through the exhaustive application of these inference rules until saturation, followed by the removal of all A-GCIs from the saturated $\mathcal{O}$, the result is a refined ontology, denoted as $\mathcal{O}^{-\mathsf{A}}$, devoid of any traces of A.

IR1. $C \sqsubseteq \mathsf{A}, \mathsf{A} \sqcap E \sqsubseteq F \implies C \sqcap E \sqsubseteq F$

IR2. $C \sqsubseteq \mathsf{A}, \exists s.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq G \implies \exists s.(C \sqcap E) \sqcap F \sqsubseteq G$

IR3. $C \sqsubseteq \mathsf{A}, \exists s^-.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq G \implies \exists s^-.(C \sqcap E) \sqcap F \sqsubseteq G$

IR4. $C \sqsubseteq \exists r.(\mathsf{A} \sqcap D), \mathsf{A} \sqcap E_1 \sqsubseteq F_1, \ldots, \mathsf{A} \sqcap E_n \sqsubseteq F_n$
$\implies C \sqsubseteq \exists r.(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D)$
provided that: $\mathcal{O} \models \mathsf{A} \sqcap D \sqsubseteq E_1 \sqcap \cdots \sqcap E_n$
$C \sqsubseteq \exists r.(\mathsf{A} \sqcap D), \mathsf{A} \sqcap E \sqsubseteq F \implies C \sqsubseteq \exists r.D$
provided that: $\mathcal{O} \not\models \mathsf{A} \sqcap D \sqsubseteq E$

IR5. $C \sqsubseteq \exists r.(\mathsf{A} \sqcap D), \exists s.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq B$
$\implies C \sqsubseteq \exists r.D, C \sqcap F \sqsubseteq B$
provided that: $\mathcal{O} \models \mathsf{A} \sqcap D \sqsubseteq E$ and $\mathcal{O} \models r \sqsubseteq s$
$C \sqsubseteq \exists r.(\mathsf{A} \sqcap D), \exists s.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq B \implies C \sqsubseteq \exists r.D$
provided that: $\mathcal{O} \not\models \mathsf{A} \sqcap D \sqsubseteq E$ or $\mathcal{O} \not\models r \sqsubseteq s$

IR6. $C \sqsubseteq \exists r.(\mathsf{A} \sqcap D), \exists s^-.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq B$
$\implies C \sqsubseteq \exists r.D, C \sqcap F \sqsubseteq B$
provided that: $\mathcal{O} \models \mathsf{A} \sqcap D \sqsubseteq E$ and $\mathcal{O} \models r \sqsubseteq s^-$
$C \sqsubseteq \exists r.(\mathsf{A} \sqcap D), \exists s^-.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq B \implies C \sqsubseteq \exists r.D$
provided that: $\mathcal{O} \not\models \mathsf{A} \sqcap D \sqsubseteq E$ or $\mathcal{O} \not\models r \sqsubseteq s^-$

IR7. $C \sqsubseteq \exists r^-.(\mathsf{A} \sqcap D), \mathsf{A} \sqcap E_1 \sqsubseteq F_1, \ldots, \mathsf{A} \sqcap E_n \sqsubseteq F_n$
$\implies C \sqsubseteq \exists r^-.(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D)$
provided that: $\mathcal{O} \models \mathsf{A} \sqcap D \sqsubseteq E_1 \sqcap \cdots \sqcap E_n$
$C \sqsubseteq \exists r^-.(\mathsf{A} \sqcap D), \mathsf{A} \sqcap E \sqsubseteq F \implies C \sqsubseteq \exists r^-.D$
provided that: $\mathcal{O} \not\models \mathsf{A} \sqcap D \sqsubseteq E$

IR8. $C \sqsubseteq \exists r^-.(\mathsf{A} \sqcap D), \exists s.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq B$
$\implies C \sqsubseteq \exists r^-.D, C \sqcap F \sqsubseteq B$
provided that: $\mathcal{O} \models \mathsf{A} \sqcap D \sqsubseteq E$ and $\mathcal{O} \models r^- \sqsubseteq s$
$C \sqsubseteq \exists r^-.(\mathsf{A} \sqcap D), \exists s.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq B \implies C \sqsubseteq \exists r.D$
provided that: $\mathcal{O} \not\models \mathsf{A} \sqcap D \sqsubseteq E$ or $\mathcal{O} \not\models r^- \sqsubseteq s$

IR9. $C \sqsubseteq \exists r^-.(\mathsf{A} \sqcap D), \exists s^-.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq B$
$\implies C \sqsubseteq \exists r^-.D, C \sqcap F \sqsubseteq B$
provided that: $\mathcal{O} \models \mathsf{A} \sqcap D \sqsubseteq E$ and $\mathcal{O} \models r^- \sqsubseteq s^-$
$C \sqsubseteq \exists r^-.(\mathsf{A} \sqcap D), \exists s^-.(\mathsf{A} \sqcap E) \sqcap F \sqsubseteq B \implies C \sqsubseteq \exists r^-.D$
provided that: $\mathcal{O} \not\models \mathsf{A} \sqcap D \sqsubseteq E$ or $\mathcal{O} \not\models r^- \sqsubseteq s^-$

Figure 1: Inference rules for forgetting A

Example 1. Consider the following $\mathcal{ELI}$-ontology $\mathcal{O}$:

$$\{1.\ \mathsf{E} \sqsubseteq \exists \mathsf{r}^-.(\mathsf{F} \sqcap \exists \mathsf{t.A}),\ 2.\ \exists \mathsf{t.A} \sqcap \exists \mathsf{t}^-.\mathsf{E} \sqsubseteq \mathsf{D}\}$$

Let $\mathcal{F} = \{\mathsf{A}\}$. The first step is to compute the A-NF of $\mathcal{O}$ by applying the normalization rules as described earlier, where $Z_1 \in \mathsf{N_D}$ is a fresh definer:

$$\{3.\ \mathsf{E} \sqsubseteq \exists \mathsf{r}^-.Z_1,\ 5.\ Z_1 \sqsubseteq \mathsf{F},\ 6.\ Z_1 \sqsubseteq \exists \mathsf{t.A},\ 2.\ \exists \mathsf{t.A} \sqcap \exists \mathsf{t}^-.\mathsf{E} \sqsubseteq \mathsf{D}\}$$

The above ontology is now in A-normal form. The second step is to apply the inference rules in Figure 1. Applying Rule IR5 to GCIs 2 and 6 gives:

$$\{3.\ \mathsf{E} \sqsubseteq \exists \mathsf{r}^-.Z_1,\ 5.\ Z_1 \sqsubseteq \mathsf{F},\ 7.\ Z_1 \sqcap \exists \mathsf{t}^-.\mathsf{E} \sqsubseteq \mathsf{D},\ 8.\ Z_1 \sqsubseteq \exists t.\top\}$$

Definers are treated as regular concept names, and are eliminated once all $\mathcal{F}$-names have been eliminated. Applying Rule IR7 to GCIs 3 and 7 gives $\{9.\ \mathsf{E} \sqsubseteq \exists r^-.\top\}$. Applying Rule IR7 to GCIs 3, 5 and 8 gives $\{10.\ E \sqsubseteq \exists r^-.(F \sqcap \exists t.\top)\}$. GCI 9 is redundant w.r.t. GCI 10 and thus removed. Our method implements a set of straightforward simplifications. In this case, $\{10.\ E \sqsubseteq \exists r^-.(F \sqcap \exists t.\top)\}$ is a Σ-uniform interpolant of $\mathcal{O}$, where $\Sigma = sig(\mathcal{O}) \backslash \mathcal{F}$.

An external DL reasoner is utilized to check the side conditions of the inference rules. It is known that checking subsumption in $\mathcal{ELI}$ is ExpTime-complete [2].

Lemma 3. *Let $\mathcal{O}$ be an $\mathcal{ELI}$-ontology in A-NF, and $\mathcal{O}^{-A}$ an ontology obtained from forgetting $\{\mathsf{A}\}$ from $\mathcal{O}$ using the inference rules in Figure 1, then we have:*

$$\mathcal{O} \models C \sqsubseteq D \text{ iff } \mathcal{O}^{-A} \models C \sqsubseteq D,$$

for any $\mathcal{ELI}$-GCI $C \sqsubseteq D$ with $sig(C \sqsubseteq D) \subseteq sig(O) \backslash \{A\}$.

Lemma 3 establishes the partial soundness of the calculus. Specifically, the derived ontology $\mathcal{O}^{-\mathsf{A}}$ fulfills the second condition necessary for it to be the result of forgetting $\{\mathsf{A}\}$ from $\mathcal{O}$. However, $\mathcal{O}^{-\mathsf{A}}$ may include definers which fall outside the scope of sig$(\mathcal{O})\backslash\{\mathsf{A}\}$, potentially failing to fulfill the first condition. A discussion on this will follow shortly.

5.2 Calculus for Forgetting r

The calculus for role forgetting parallels that for concept forgetting. Specifically, the calculus proceeds in two steps — Step (1) computes the r-NF of $\mathcal{O}$ as described in the previous section, and Step (2) forgets r by exhaustive application of the inference rules in Figure 2 to the normalized $\mathcal{O}$.

IR10. $C \sqsubseteq \exists \mathsf{r}.D, F \sqcap \exists \mathsf{r}.E \sqsubseteq G \implies F \sqcap C \sqsubseteq G$
provided that: $\mathcal{O} \models \exists \mathsf{r}.D \sqsubseteq \exists \mathsf{r}.E$

IR11. $C \sqsubseteq \exists \mathsf{r}.D, F \sqcap \exists \mathsf{r}^-.E \sqsubseteq G \implies F \sqcap C \sqsubseteq G$
provided that: $\mathcal{O} \models \exists \mathsf{r}.D \sqsubseteq \exists \mathsf{r}^-.E$

IR12. $C \sqsubseteq \exists \mathsf{r}^-.D, F \sqcap \exists \mathsf{r}.E \sqsubseteq G \implies F \sqcap C \sqsubseteq G$
provided that: $\mathcal{O} \models \exists \mathsf{r}^-.D \sqsubseteq \exists \mathsf{r}.E$

IR13. $C \sqsubseteq \exists \mathsf{r}^-.D, F \sqcap \exists \mathsf{r}^-.E \sqsubseteq G \implies F \sqcap C \sqsubseteq G$
provided that: $\mathcal{O} \models \exists \mathsf{r}.D \sqsubseteq \exists \mathsf{r}^-.E$

Figure 2: Inference rule for forgetting r

The process of eliciting implicit entailments from $\mathcal{O}$ involves the combination of positive occurrences (GCIs taking r-NF I and II) with negative occurrences (GCIs taking r-NF III and IV) of r (i.e., resolving upon r). This results in four distinct combination scenarios, labeled as IR10, IR11, IR12, and IR13, as depicted in Figure 2. By the exhaustive application of these inference rules, followed by the removal of all r-GCIs, the outcome is a refined ontology, denoted as $\mathcal{O}^{-\mathsf{r}}$, devoid of any traces of r. Likewise, an auxiliary DL reasoner is employed during the forgetting process to check the side conditions of the inference rules.

5.3 Properties of the Method

Forgetting is not always successful for $\mathcal{ELI}$ when cyclic dependencies exist over the intended names to be forgotten [19]. In these situations, the inference process might fall into an endless loop, causing the forgetting process to never terminate. Take an example where we aim to forget A from an $\mathcal{ELI}$-ontology $\{\mathsf{A} \sqsubseteq \exists r^-.\mathsf{A}\}$, which exhibits cyclic behavior over A. The result is $\{\mathsf{D}_1 \sqsubseteq \exists r^-.\mathsf{D}_1\}$, with D1 $\in \mathsf{N}_\mathsf{D}$ as a fresh definer. If one tried to forget D_1 from this result, it would yield a GCI of the same structure, specifically $\{\mathsf{D}_2 \sqsubseteq \exists r^-.\mathsf{D}_2\}$, with D2 $\in \mathsf{N}_\mathsf{D}$ as a fresher definer. This would result in an endless introduction of definers. Our method guarantees the termination of the forgetting process by giving up forgetting D_1. Instead, it keeps the initial definer D_1 in the resulting ontology, declaring an unsuccessful forgetting attempt and highlighting the method's inherent limitation. While cyclic situations might be tackled using fixpoints [5], as shown by Lethe, mainstream reasoning tools and the OWL API do not support fixpoints. Hence, to ensure our method remains practical, we opt out of including them in our target language, choosing practicality over completeness.

Theorem 1. *Given any $\mathcal{ELI}$-ontology $\mathcal{O}$ and any forgetting signature $\mathcal{F} \subseteq sig(\mathcal{O})$ as input, our forgetting method always terminates and returns an $\mathcal{ELI}$-ontology $\mathcal{V}$. If $\mathcal{V}$ does not contain any definers, then it is a result of forgetting $\mathcal{F}$ from $\mathcal{O}$.*

6 EXPERIMENTS

We have developed a prototype of our forgetting method in Java using the OWL API Version 5.1.7[1] To assess its practicality, we juxtaposed its performance with the state-of-the-art forgetting method, Lethe [22] using two large corpora of real-world ontologies.[2] The first corpus was created from a snapshot of the Oxford ISG Library,[3] aggregating diverse ontologies from a myriad of sources. The second corpus was derived from a March 2017 snapshot of the NCBO BioPortal [32], which features biomedical ontologies.

From the Oxford ISG snapshot, we cherry-picked 488 ontologies where the logical axiom (GCI) count did not exceed 10,000. We then excluded those ontologies lacking ∃-restrictions or inverse roles, or exhibiting cyclic dependencies. This left us with 177 ontologies. To refine further, we distilled the remaining ontologies down to their $\mathcal{ELI}$-fragments by omitting GCIs not expressible within $\mathcal{ELI}$. This process resulted in a 7.4% reduction in total GCIs.

To provide granular insights into the performance of our method across differently-sized Oxford-ISG ontologies, we partitioned these selections into three distinct categories:

- PART I: 115 ontologies with $10 \leq |\text{Onto}| < 1000$;
- PART II: 51 ontologies with $1000 \leq |\text{Onto}| < 4999$;
- PART III: 11 ontologies with $5000 \leq |\text{Onto}| < 10000$.

Implementing the same strategy for the BioPortal case, we amassed a collection of 76 ontologies and categorized them as:

- PART I: 38 ontologies with $10 \leq |\text{Onto}| < 1000$.
- PART II: 28 ontologies with $1000 \leq |\text{Onto}| < 4999$.
- PART III: 10 ontologies with $5000 \leq |\text{Onto}| < 10000$.

[1] http://owlcs.github.io/owlapi/
[2] Note that a comparative analysis with Nui and Fame was precluded due to accessibility issues during the period of our study (they remained inaccessible as of October 8, 2023).
[3] http://krr-nas.cs.ox.ac.uk/ontologies/lib/

Table 1: Experimental results over Oxford-ISG and BioPortal (Time: Time Consumption, Mem: Memory Consumption, SR: Success Rate, TR: Timeout Rate, RER: Runtime Error Rate)

Oxford	%	Part	Time (sec.)	Mem (MB)	SR	TR	RER
Lethe	0.1	I	4.55	37.06	92.07	4.45	3.48
		II	9.92	58.76	86.57	11.19	2.24
		III	14.72	88.54	77.73	22.27	0.00
	0.3	I	12.76	52.24	86.98	9.54	3.48
		II	29.50	75.16	74.88	22.88	2.24
		III	41.75	123.11	67.91	32.09	0.00
	0.5	I	14.81	71.36	79.80	16.72	3.48
		II	43.41	134.65	70.16	27.60	2.24
		III	74.22	189.13	63.45	36.55	0.00
Proto	0.1	I	0.18	24.76	100	0.00	0.00
		II	0.49	38.78	100	0.00	0.00
		III	0.87	59.45	100	0.00	0.00
	0.3	I	0.30	35.72	100	0.00	0.00
		II	0.77	51.11	100	0.00	0.00
		III	1.12	86.82	100	0.00	0.00
	0.5	I	0.81	48.36	100	0.00	0.00
		II	1.41	91.65	100	0.00	0.00
		III	1.62	130.50	100	0.00	0.00
BioPortal	**%**	**Part**	**Time (sec.)**	**Mem (MB)**	**SR**	**TR**	**RER**
Lethe	0.1	I	5.11	39.96	92.00	5.37	2.63
		II	11.23	59.04	85.14	11.29	3.57
		III	15.01	95.83	74.30	25.70	0.00
	0.3	I	14.19	53.26	83.29	14.08	2.63
		II	32.16	88.33	73.11	23.32	3.57
		III	46.15	133.20	65.50	34.50	0.00
	0.5	I	14.34	76.48	77.24	20.13	2.63
		II	45.98	140.11	69.00	27.43	3.57
		III	81.81	187.93	60.60	39.40	0.00
Proto	0.1	I	0.17	21.45	100	0.00	0.00
		II	0.45	34.11	100	0.00	0.00
		III	0.85	52.16	100	0.00	0.00
	0.3	I	0.32	31.34	100	0.00	0.00
		II	0.69	47.66	100	0.00	0.00
		III	1.06	78.38	100	0.00	0.00
	0.5	I	0.77	44.16	100	0.00	0.00
		II	1.36	88.67	100	0.00	0.00
		III	1.55	120.94	100	0.00	0.00

A comprehensive breakdown of the refined ontologies from both sources can be found in the long version of this paper.

We designed three sets of experiments, targeting the forgetting of either 10%, 30%, or 50% of the concept and role names present within the signature of each ontology. These configurations align with well-established practices in the evaluation of forgetting methods, as evidenced in literature sources such as [22, 24, 49, 51]. For the selection of $\mathcal{F}$, we employed a shuffling algorithm to ensure a randomized choice. The experimental set-up involved a laptop equipped with an Intel Core i7-9750H processor, boasting 6 cores that peak at 2.70 GHz, and bolstered by 12 GB of DDR4-1600 MHz RAM. To ensure consistent performance metrics, we imposed constraints: a maximum run time of 300 seconds and an upper heap space limit of 9GB. We deemed a forgetting experiment successful if it met the following criteria:

(1) successful elimination of all names specified in $\mathcal{F}$.
(2) absence of any definers in the forgetting outputs, should they have been introduced during the process.
(3) completion within the stipulated 300-second window.
(4) operation within the set 9GB space limit.

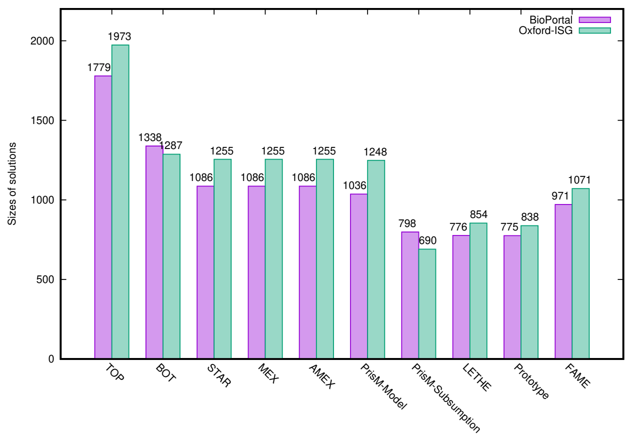

Figure 3: Average |Onto| in output ontologies

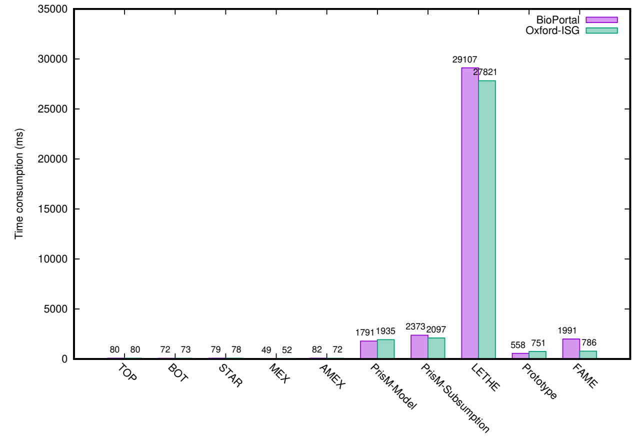

Figure 5: Computation time consumption

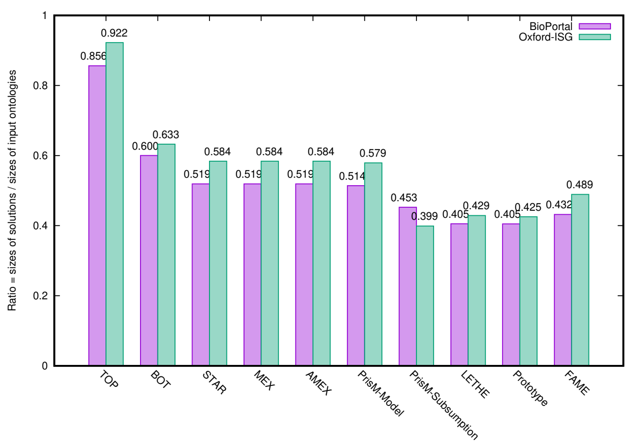

Figure 4: Average ratio of |Onto|: input vs. output

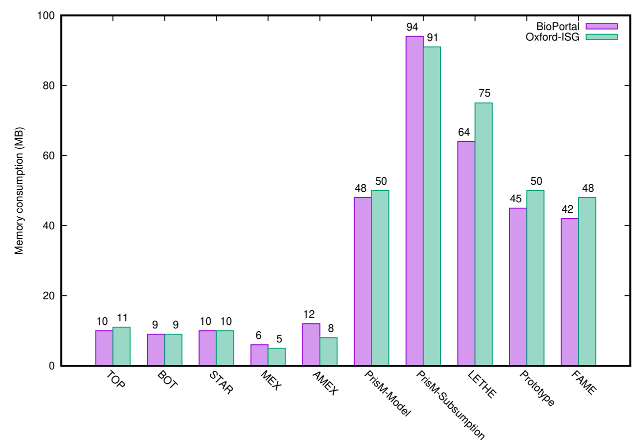

Figure 6: Memory consumption

We repeated the experiments 100 times for each test case and took the average to validate our results.

The results of our experiments are shown in Table 1. A notable observation is that our prototype registered a success rate of 100% across all evaluation tracks. A large portion of LETHE's failures was due to the timeout. Our logging of GPU and memory usage for each forgetting task indicated that LETHE generally demanded more computational resources compared to our prototype. Additionally, there was a stark contrast in processing times: our prototype consistently outperformed LETHE, being approximately 52 times faster on the Oxford-ISG dataset and 37 times faster on BioPortal. This marked increase in speed could be attributed to the distinct normalization approaches used by the two tools. A deeper dive into these results will follow later. Interestingly, LETHE incurred runtime errors when dealing with certain ontologies, but our prototype operated without any hitches, most likely because of the incompatibility between the OWL API version LETHE used and those ontologies.

To provide readers with a deeper insight into the inherent properties of our method and the nature of its forgetting results, we further conducted a comprehensive comparison with various types of modularization methods prevalent in the field, focusing on metrics such as result size, computation time, and memory consumption. A number of modularization approaches exist for creating views, each with unique properties and complexities. Among them, the MEX tool [20] extracts minimal modules from acyclic $\mathcal{ELI}$ ontologies with polynomial time complexity, while tools like AMEX [8] and PrisM [38] target other types of ontologies, with PrisM offering six inseparability notions. Locality-based methods and their extensions [12], such as TOP, BOT, and STAR, provide additional modularization strategies. With only MEX ensuring minimal modules, other tools generally approximate them.

Surprisingly, as a forgetting method, our prototype's output did not reflect the theoretical projections of an exponential size increase compared to the input ontologies. On the contrary, the forgetting results showcased impressive compactness in comparison to the

Table 2: Definers introduced during forgetting (Oxford)

Onto Code	Lethe (0.1)	Proto (0.1)	Lethe (0.3)	Proto (0.3)	Lethe (0.5)	Proto (0.5)
00646	2072	506	1329	0	2045	35
00645	1686	427	1005	0	1666	39
00522	884	359	2207	0	4233	266
00669	2628	1853	2960	638	2596	0
00696	1200	104	2992	133	6207	500
00523	4199	1361	2248	0	3173	20
00544	1349	8	5827	0	5885	0
00578	157	8	602	255	484	0
00356	1719	914	1719	228	1289	0
00367	23	5	24	0	31	0
00464	125	412	113	0	229	57
00513	14	29	37	4	28	0
00451	465	0	1156	0	2257	109
00445	27	0	120	5	88	0
00690	308	0	838	112	1276	175
00519	8	0	46	0	71	0
00527	97	0	501	57	371	0
00452	690	0	3072	257	2379	0
00650	111	0	353	0	663	49
00640	77	0	220	0	306	0
00495	417	0	1255	0	1772	0
00457	20	0	33	0	80	2
00494	446	0	1222	0	2335	133
00694	823	0	4580	480	5202	515
00469	5	0	10	0	46	0
00468	2	0	0	0	3	0
00497	1576	0	4223	0	6185	0
00520	39	0	73	6	67	0
00547	351	0	1952	327	1592	0
00433	47	0	141	0	221	20
00546	346	0	1950	255	1494	0
00591	10	0	66	0	67	0
00593	28	0	116	0	152	0
00357	331	0	1794	616	1794	285
00627	101	0	385	2	508	0
00545	1099	0	6108	0	6201	0
00592	41	0	70	0	104	0
00596	94	0	176	0	177	0
00423	129	0	338	0	458	0
00594	23	0	169	0	131	0
00770	499	0	2258	197	1968	24
00412	194	0	919	168	919	45
00413	188	0	550	0	776	0
00639	73	0	158	0	261	0
00605	33	0	75	6	69	0
00571	6	0	23	0	31	0
00411	37	0	137	0	172	0
00606	35	0	48	0	84	5
00570	38	4	19	0	32	0
00548	21	0	57	2	69	2
00366	8	0	19	5	34	2
00563	14	0	39	0	75	0
00629	111	0	419	3	471	9
00359	95	0	303	0	405	0
00403	467	0	1200	0	2479	404
00402	268	0	1407	294	1217	0
00358	37	2	90	0	85	0
00600	19	0	169	0	104	0
00006	211	0	488	0	1075	52
00562	10	0	32	0	36	0
00589	37	0	112	0	112	0
00505	9	0	3	0	9	0
00667	203	0	1181	270	890	0
00458	12	0	48	0	75	3
00498	1662	0	8316	1578	8316	754
00649	112	0	364	0	663	35
00514	27	0	35	0	36	0
00689	235	0	858	70	989	99
00515	168	0	649	35	773	27

Table 3: Definers introduced during forgetting (BioPortal)

Onto Name	Lethe (0.1)	Proto (0.1)	Lethe (0.3)	Proto (0.3)	Lethe (0.5)	Proto (0.5)
DUO	1	0	3	0	3	0
ARO	1304	0	2559	0	3162	5
PCAO	15	0	24	0	45	0
AMPHX	190	0	831	0	1334	656
FAO	4	0	15	0	21	0
DMTO	287	0	1307	0	1244	0
HIO	6	0	38	0	47	0
HSAPDV	75	0	475	388	421	16
LMHA	70	0	254	0	486	51
PMDO	2	0	12	0	16	0
MMUSDV	122	29	92	0	173	3
HANCESTRO	44	0	118	0	178	0
EMAPA	4088	0	9295	0	18035	2302
PREO	5	0	17	0	22	0
AEO	18	0	46	0	101	0
EOL	1	0	1	0	1	0
LHN	30	4	34	3	34	0
ORDO	15892	0	35407	78	40911	1898
ORNASEQ	3	0	2	0	5	0
COGAT	78	0	272	0	376	0

input ontologies, surpassing even the results produced by modularization techniques; these modules are mere syntactic subsets of the input ontologies; see Figures 3 and 4.

Regarding time consumption (see Figure 5), our prototype, when performing the same forgetting tasks, outpaced Lethe by what one might hyperbolically describe as "light years". Its speed was on par with the modularization methods; however, it is well-known that the computational complexity of forgetting is in general notably greater than that of modularization [4, 46].

Regarding memory consumption (see Figure 6), forgetting typically required more memory during computation than modularization. Yet, when comparing different forgetting methods, Fame and our prototype distinctly stood out, requiring only 66% to 70% of the memory that Lethe demanded, showcasing their efficiency.

Tables 2 and 3 present a detailed account of the definers introduced by Lethe and our prototype across a number of forgetting tasks. It is evident that, for any given task, our prototype introduced a substantially fewer number of definers compared to Lethe. In fact, in the Oxford-ISG settings, Lethe necessitated the introduction of definers in 65.0% of the forgetting tasks, whereas, for our prototype, this figure stood at 19.1%. In the BioPortal cases, these figures dropped to 26.3% for Lethe and a mere 5.2% for our prototype.

7 CONCLUSION AND FUTURE WORK

This paper presents a highly optimized forgetting method to produce signature-restricted views of acyclic $\mathcal{ELI}$-ontologies. Its enhanced efficiency results from a refined approach that reduces the number of definers needed for normalization. Despite the inherent computational challenges of the task, empirical evaluation demonstrates its algorithmic ascendancy over state-of-the-art tools.

Our immediate next step for future work is to enhance our current method to accommodate ABoxes. In addition, we also consider an adaptation of the method to more expressive DLs, such as $\mathcal{ALC}$ and its major decidable extensions [39, 44].

REFERENCES

[1] Wilhelm Ackermann. 1935. Untersuchungen úber das Eliminationsproblem der mathematischen Logik. *Math. Ann.* 110, 1 (1935), 390–413.

[2] Franz Baader, Ian Horrocks, Carsten Lutz, and Ulrike Sattler. 2017. *An Introduction to Description Logic.* Cambridge University Press.

[3] Leo Bachmair, Harald Ganzinger, David A. McAllester, and Christopher Lynch. 2001. Resolution Theorem Proving. See [37], 19–99.

[4] Elena Botoeva, Boris Konev, Carsten Lutz, Vladislav Ryzhikov, Frank Wolter, and Michael Zakharyaschev. 2016. Inseparability and Conservative Extensions of Description Logic Ontologies: A Survey. In *Proc. RW'16 (Lecture Notes in Computer Science, Vol. 9885).* Springer, 27–89.

[5] Diego Calvanese, Giuseppe De Giacomo, and Maurizio Lenzerini. 1999. Reasoning in Expressive Description Logics with Fixpoints based on Automata on Infinite Trees. In *Proc. IJCAI'99.* Morgan Kaufmann, 84–89.

[6] Warren Del-Pinto and Renate A. Schmidt. 2019. ABox Abduction via Forgetting in $\mathcal{ALC}$. In *Proc. AAAI'19.* AAAI Press, 2768–2775.

[7] Christian G. Fermüller, Alexander Leitsch, Ullrich Hustadt, and Tanel Tammet. 2001. Resolution Decision Procedures. See [37], 1791–1849.

[8] William Gatens, Boris Konev, and Frank Wolter. 2014. Lower and Upper Approximations for Depleting Modules of Description Logic Ontologies. In *Proc. ECAI'14 (Frontiers in Artificial Intelligence and Applications, Vol. 263).* IOS Press, 345–350.

[9] Silvio Ghilardi, Carsten Lutz, and Frank Wolter. 2006. Did I Damage My Ontology? A Case for Conservative Extensions in Description Logics. In *Proc. KR'06.* AAAI Press, 187–197.

[10] Birte Glimm, Ian Horrocks, Boris Motik, Giorgos Stoilos, and Zhe Wang. 2014. HermiT: An OWL 2 Reasoner. *J. Autom. Reasoning* 53, 3 (2014), 245–269.

[11] Bernardo Cuenca Grau, Ian Horrocks, Yevgeny Kazakov, and Ulrike Sattler. 2007. Just the right amount: extracting modules from ontologies. In *Proc. WWW'07.* ACM, 717–726.

[12] Bernardo Cuenca Grau, Ian Horrocks, Yevgeny Kazakov, and Ulrike Sattler. 2008. Modular Reuse of Ontologies: Theory and Practice. *J. Artif. Intell. Res.* 31 (2008), 273–318.

[13] Ian Horrocks. 2008. Ontologies and the semantic web. *Commun. ACM* 51, 12 (2008), 58–67.

[14] Michel C. A. Klein and Dieter Fensel. 2001. Ontology versioning on the Semantic Web. In *Proc. SWWS'01.* 75–91.

[15] Michel C. A. Klein, Dieter Fensel, Atanas Kiryakov, and Damyan Ognyanov. 2002. Ontology Versioning and Change Detection on the Web. In *Proc. EKAW'02 (Lecture Notes in Computer Science, Vol. 2473).* Springer, 197–212.

[16] Boris Konev, Michel Ludwig, Dirk Walther, and Frank Wolter. 2012. The Logical Difference for the Lightweight Description Logic $\mathcal{EL}$. *J. Artif. Intell. Res.* 44 (2012), 633–708.

[17] Boris Konev, Carsten Lutz, Dirk Walther, and Frank Wolter. 2013. Model-theoretic inseparability and modularity of description logic ontologies. *Artif. Intell.* 203 (2013), 66–103.

[18] Boris Konev, Dirk Walther, and Frank Wolter. 2008. The Logical Difference Problem for Description Logic Terminologies. In *Proc. IJCAR'14 (Lecture Notes in Computer Science, Vol. 5195).* Springer, 259–274.

[19] Boris Konev, Dirk Walther, and Frank Wolter. 2009. Forgetting and Uniform Interpolation in Large-Scale Description Logic Terminologies. In *Proc. IJCAI'09.* IJCAI/AAAI Press, 830–835.

[20] Roman Kontchakov, Frank Wolter, and Michael Zakharyaschev. 2010. Logic-based ontology comparison and module extraction, with an application to DL-Lite. *Artif. Intell.* 174, 15 (2010), 1093–1141.

[21] Patrick Koopmann. 2015. *Practical Uniform Interpolation for Expressive Description Logics.* Ph. D. Dissertation. The University of Manchester, UK.

[22] Patrick Koopmann. 2020. LETHE: Forgetting and Uniform Interpolation for Expressive Description Logics. *Künstliche Intell.* 34, 3 (2020), 381–387.

[23] Patrick Koopmann, Warren Del-Pinto, Sophie Tourret, and Renate A. Schmidt. 2020. Signature-Based Abduction for Expressive Description Logics. In *Proc. KR'20.* 592–602.

[24] Patrick Koopmann and Renate A. Schmidt. 2013. Implementation and Evaluation of Forgetting in $\mathcal{ALC}$-Ontologies. In *Proc. WoMo'13*, Vol. 1081. CEUR-WS.org, 1–12.

[25] Patrick Koopmann and Renate A. Schmidt. 2013. Uniform Interpolation of $\mathcal{ALC}$-Ontologies Using Fixpoints. In *Proc. FroCos'13 (Lecture Notes in Computer Science, Vol. 8152).* Springer, 87–102.

[26] Patrick Lambrix and He Tan. 2008. Ontology Alignment and Merging. In *Anatomy Ontologies for Bioinformatics, Principles and Practice.* Computational Biology, Vol. 6. Springer, 133–149.

[27] Michel Ludwig and Boris Konev. 2014. Practical Uniform Interpolation and Forgetting for $\mathcal{ALC}$ TBoxes with Applications to Logical Difference. In *Proc. KR'14.* AAAI Press, 318–327.

[28] Carsten Lutz, Inanç Seylan, and Frank Wolter. 2012. An Automata-Theoretic Approach to Uniform Interpolation and Approximation in the Description Logic EL. In *Proc. KR'12.* AAAI Press, 286–296.

[29] Carsten Lutz, Dirk Walther, and Frank Wolter. 2007. Conservative Extensions in Expressive Description Logics. In *Proc. IJCAI'07.* AAAI/IJCAI Press, 453–458.

[30] Carsten Lutz and Frank Wolter. 2010. Deciding inseparability and conservative extensions in the description logic $\mathcal{EL}$. *J. Symb. Comput.* 45, 2 (2010), 194–228.

[31] Carsten Lutz and Frank Wolter. 2011. Foundations for Uniform Interpolation and Forgetting in Expressive Description Logics. In *Proc. IJCAI'11.* IJCAI/AAAI Press, 989–995.

[32] Nicolas Matentzoglu and Bijan Parsia. 2017. BioPortal Snapshot 30.03.2017. https://doi.org/10.5281/zenodo.439510

[33] Nadeschda Nikitina and Sebastian Rudolph. 2014. (Non-)Succinctness of uniform interpolants of general terminologies in the description logic $\mathcal{EL}$. *Artif. Intell.* 215 (2014), 120–140.

[34] Nadeschda Nikitina, Sebastian Rudolph, and Birte Glimm. 2011. Reasoning-Supported Interactive Revision of Knowledge Bases. In *Proc. IJCAI'11.* IJCAI/AAAI, 1027–1032.

[35] Natalya Fridman Noy and Mark A. Musen. 2000. PROMPT: Algorithm and Tool for Automated Ontology Merging and Alignment. In *Proc. AAAI/IAAI'00.* AAAI Press/The MIT Press, 450–455.

[36] Márcio Moretto Ribeiro and Renata Wassermann. 2009. Base Revision for Ontology Debugging. *J. Log. Comput.* 19, 5 (2009), 721–743.

[37] John Alan Robinson and Andrei Voronkov (Eds.). 2001. *Handbook of Automated Reasoning (in 2 volumes).* Elsevier and MIT Press.

[38] Ana Armas Romero, Mark Kaminski, Bernardo Cuenca Grau, and Ian Horrocks. 2016. Module Extraction in Expressive Ontology Languages via Datalog Reasoning. *J. Artif. Intell. Res.* 55 (2016), 499–564.

[39] Manfred Schmidt-Schauß and Gert Smolka. 1991. Attributive Concept Descriptions with Complements. *Artif. Intell.* 48, 1 (1991), 1–26.

[40] Dan Schrimpsher, Zhiqiang Wu, Anthony M. Orme, and Letha H. Etzkorn. 2010. Dynamic ontology version control. In *Proc. ACMse'10.* ACM, 25.

[41] Stefan Schulz, Ronald Cornet, and Kent A. Spackman. 2011. Consolidating SNOMED CT's ontological commitment. *Appl. Ontology* 6, 1 (2011), 1–11.

[42] Nigel Shadbolt, Tim Berners-Lee, and Wendy Hall. 2006. The Semantic Web Revisited. *IEEE Intell. Syst.* 21, 3 (2006), 96–101.

[43] Steffen Staab and Rudi Studer (Eds.). 2009. *Handbook on Ontologies.* Springer.

[44] Stephan Tobies. 2001. *Complexity results and practical algorithms for logics in knowledge representation.* Ph. D. Dissertation. RWTH Aachen University, Germany.

[45] Nicolas Troquard, Roberto Confalonieri, Pietro Galliani, Rafael Peñaloza, Daniele Porello, and Oliver Kutz. 2018. Repairing Ontologies via Axiom Weakening. In *Proc. AAAI'18.* AAAI Press, 1981–1988.

[46] Chiara Del Vescovo, Matthew Horridge, Bijan Parsia, Uli Sattler, Thomas Schneider, and Haoruo Zhao. 2020. Modular Structures and Atomic Decomposition in Ontologies. *J. Artif. Intell. Res.* 69 (2020), 963–1021.

[47] Kewen Wang, Grigoris Antoniou, Rodney Topor, and Abdul Sattar. 2005. Merging and Aligning Ontologies in dl-Programs. In *Proc. RuleML'05 (Lecture Notes in Computer Science, Vol. 3791).* Springer, 160–171.

[48] Zhe Wang, Kewen Wang, Rodney W. Topor, and Jeff Z. Pan. 2010. Forgetting for knowledge bases in DL-Lite. *Ann. Math. Artif. Intell.* 58, 1-2 (2010), 117–151.

[49] Xuan Wu, Wenxing Deng, Chang Lu, Hao Feng, and Yizheng Zhao. 2020. UI-FAME: A High-Performance Forgetting System for Creating Views of Ontologies. In *Proc. CIKM'20.* ACM, 3473–3476.

[50] Yue Xiang, Xuan Wu, Chang Lu, and Yizheng Zhao. 2022. Creating Signature-Based Views for Description Logic Ontologies with Transitivity and Qualified Number Restrictions. In *Proc. WWW'22.* ACM, 808–817.

[51] Hui Yang, Patrick Koopmann, Yue Ma, and Nicole Bidoit. 2023. Efficient Computation of General Modules for ALC Ontologies. In *Proc. IJCAI'23.* ijcai.org, 3356–3364.

[52] Yizheng Zhao, Ghadah Alghamdi, Schmidt Renate A., Hao Feng, Giorgos Stoilos, Damir Juric, and Mohammad Khodadadi. 2019. Tracking Logical Difference in Large-Scale Ontologies: A Forgetting-Based Approach. In *Proc. AAAI'19.* AAAI Press, 3116–3124.

[53] Yizheng Zhao and Renate A. Schmidt. 2018. FAME: An Automated Tool for Semantic Forgetting in Expressive Description Logics. In *Proc. IJCAR'18 (Lecture Notes in Computer Science, Vol. 10900).* Springer, 19–27.

A APPENDIX

A.1 Missing Proofs

We define the *frequency* $\mathsf{fq}(\mathsf{A}, C)$ of A in an A-concept C inductively as follows:

- $\mathsf{fq}(\mathsf{A}, \mathsf{A}) = 1$,
- $\mathsf{fq}(\mathsf{A}, \exists R.E) = \mathsf{fq}(\mathsf{A}, E)$,
- $\mathsf{fq}(\mathsf{A}, E \sqcap F) = \mathsf{fq}(\mathsf{A}, E) + \mathsf{fq}(\mathsf{A}, F)$.

We define the *frequency* $\mathsf{fq}(\mathsf{A}, X \sqsubseteq Y)$ of A in a GCI $X \sqsubseteq Y$:

- $\mathsf{fq}(\mathsf{A}, X \sqsubseteq Y) = \mathsf{fq}(\mathsf{A}, X) + \mathsf{fq}(\mathsf{A}, Y)$.

Let A^* be a designated occurrence of A in C. We define the *depth* $\mathsf{dp}(\mathsf{A}^*, C)$ of A^* in C inductively as follows:

- $\mathsf{dp}(\mathsf{A}^*, C) = 0$, if C is of the form $\mathsf{A}^* \sqcap D$, where D is an arbitrary concept,
- $\mathsf{dp}(\mathsf{A}^*, C) = \mathsf{dp}(\mathsf{A}^*, E) + 1$, if C is of the form $D \sqcap \exists R.E$, where $r \in \mathsf{N_R}$, E is a concept that contains A^*, and D is an arbitrary concept,

i.e., $\mathsf{dp}(\mathsf{A}^*, C)$ counts the number of $\exists$-restrictions guarding A^* in C. We define the *depth* $\mathsf{dp}(\mathsf{A}, C)$ of A in C as the sum of the *depth* of all occurrences of A in C. We define the depth $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y)$ of A in a GCI $X \sqsubseteq Y$:

- $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) = \mathsf{dp}(\mathsf{A}, X) + \mathsf{dp}(\mathsf{A}, Y)$.

Proposition 1. *Any* A-*GCI* $X \sqsubseteq Y$ *with* $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) = 0$ *is in* A-*normal form.*

Proof. $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) = 0$ indicates that A is not guarded by any $\exists$-restriction in $X \sqsubseteq Y$, ensuring that A appears only at the surface level of the concepts X and Y. This leads to three possible cases (a simpler representation of $X \sqsubseteq Y$ is assumed):

- $\mathsf{A} \notin \mathsf{sig_C}(X)$ and $\mathsf{A} \in \mathsf{sig_C}(Y)$ (Form I)
- $\mathsf{A} \in \mathsf{sig_C}(X)$ and $\mathsf{A} \notin \mathsf{sig_C}(Y)$ (Form III)
- $\mathsf{A} \in \mathsf{sig_C}(X)$ and $\mathsf{A} \in \mathsf{sig_C}(Y)$

The last case can be generalized as $\mathsf{A} \sqcap C \sqsubseteq \mathsf{A} \sqcap D$, with C, D being A-free concepts. This is equivalent to $\mathsf{A} \sqcap C \sqsubseteq \mathsf{A}$ and $\mathsf{A} \sqcap C \sqsubseteq D$; the former is a tautology and the latter aligns with Form III. □

Proposition 2. *Any* A-*GCI* $X \sqsubseteq Y$ *with* $\mathsf{fq}(\mathsf{A}, X \sqsubseteq Y) = 1$ *and* $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) = 1$ *is in* A-*normal form.*

Proof. $\mathsf{fq}(\mathsf{A}, X \sqsubseteq Y) = 1$ ensures a single occurrence of A in $X \sqsubseteq Y$ and $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) = 1$ indicates that A is guarded by a single $\exists$-restriction. This leads to two possible cases:

- $\mathsf{A} \notin \mathsf{sig_C}(X)$ and $\mathsf{A} \in \mathsf{sig_C}(Y)$ (Form II)
- $\mathsf{A} \in \mathsf{sig_C}(X)$ and $\mathsf{A} \notin \mathsf{sig_C}(Y)$ (Form IV)

Both cases are already in A-normal form. □

Proposition 3. *For any* A-*GCI* $X \sqsubseteq Y$, $\mathsf{fq}(\mathsf{A}, X \sqsubseteq Y) - \mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) \leq 1$.

Proof. For any A-GCI $X \sqsubseteq Y$ with a depth $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) = m$ ($m \geq 0$), A appears in $X \sqsubseteq Y$ at most m+1 times. As per Proposition 1, A can occur unguarded (not within an $\exists$-restriction) only once in $X \sqsubseteq Y$. Given A's depth of m in $X \sqsubseteq Y$, there can be at most m guarded occurrences of A. Thus, A appears in $X \sqsubseteq Y$ a maximum of $m + 1$ times. □

We define $\mathsf{NLZ}_1(\mathcal{O})$ as the set derived from $\mathcal{O}$ by applying one of the rules NR1 – NR5 to $\mathcal{O}$, indicating one round of normalization for $\mathcal{O}$. Similarly, $\mathsf{NLZ}_k(\mathcal{O})$ is defined for any $k \geq 0$, with $\mathsf{NLZ}_0(\mathcal{O})$ representing the original $\mathcal{O}$. We further define $\mathsf{dp}(\mathsf{A}, \mathcal{O})$ as the sum of $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y)$, for every A-GCI $X \sqsubseteq Y$ in $\mathcal{O}$.

Proposition 4. $\mathsf{dp}(\mathsf{A}, \mathsf{NLZ}_{k-1}(\mathcal{O})) - \mathsf{dp}(\mathsf{A}, \mathsf{NLZ}_k(\mathcal{O})) = 1$, *where (i)* $k \geq 1$ *and (ii)* $\mathsf{NLZ}_{k-1}(\mathcal{O})$ *is not in* A-*normal form.*

Proof. Condition (ii) ensures that $\mathsf{NLZ}_{k-1}(\mathcal{O}) \neq \mathsf{NLZ}_k(\mathcal{O})$ and activates the applicability of NR2 – NR5. We prove that applying any of these rules decreases the depth of A in the resulting $\mathcal{O}$ by 1. We treat NR2 in detail. The rules NR3 – NR5 can be proved similarly. Consider $\mathsf{NLZ}_{k-1}(\mathcal{O})$ as a set with an A-GCI $X \sqsubseteq Y$ not in A-normal form, where $\mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) = m$ and $\mathsf{dp}(\mathsf{A}, \mathsf{NLZ}_{k-1}(\mathcal{O})) = m + x$ for $x \geq 0$. Rule NR2 implies that $m \geq 1$. Assume $\mathsf{dp}(\mathsf{A}, C) = n$, and Rule NR2 indicates $n \geq 1$. Consequently, $\mathsf{dp}(\mathsf{A}, \exists r.C) = n + 1$ and $m = n + 1 + y$ ($y \geq 0$). Replacing C with a new definer Z eliminates $X \sqsubseteq Y$ from $\mathsf{NLZ}_k(\mathcal{O})$ and introduces two GCIs: $(X \sqsubseteq Y)^C_Z$ and $C \sqsubseteq Z$. Here, $(X \sqsubseteq Y)^C_Z$ denotes the GCI obtained from $X \sqsubseteq Y$ by replacing C with Z.

Since $\mathsf{dp}(\mathsf{A}, (X \sqsubseteq Y)^C_Z) = \mathsf{dp}(\mathsf{A}, (X \sqsubseteq Y)) - \mathsf{dp}(\mathsf{A}, \exists r.C) = n + 1 + y - (n + 1) = y$ and $\mathsf{dp}(\mathsf{A}, C \sqsubseteq Z) = \mathsf{dp}(\mathsf{A}, C) + \mathsf{dp}(\mathsf{A}, Z) = n$, $\mathsf{dp}(\mathsf{A}, \mathsf{NLZ}_k(\mathcal{O})) = \mathsf{dp}(\mathsf{A}, \mathsf{NLZ}_{k-1}(\mathcal{O})) - \mathsf{dp}(\mathsf{A}, X \sqsubseteq Y) + y + n = x + y + n$. Given that $\mathsf{dp}(\mathsf{A}, \mathsf{NLZ}_{k-1}(\mathcal{O})) = m + x = x + y + n + 1$, $\mathsf{dp}(\mathsf{A}, \mathsf{NLZ}_{k-1}(\mathcal{O})) - \mathsf{dp}(\mathsf{A}, \mathsf{NLZ}_k(\mathcal{O})) = 1$. □

Lemma 1. *Let* $\mathcal{O}$ *be an arbitrary* $\mathcal{ELI}$*-ontology. Then* $\mathcal{O}$ *can be transformed into A-normal form* $\mathcal{O}'$ *by a linear number of applications of the above normalization rules NR1 – NR5. In addition, the size of the resulting ontology* $\mathcal{O}'$ *is linear in the size of* $\mathcal{O}$.

Proof. The A-normal form ensures that every A-GCI contains exactly one occurrence of A; it covers all elementary structures of an A-GCI and specifies where the A appears in these structures. Specifically, an A-concept is either the atomic concept A (the base case) or is constructed with $\exists$ and $\sqcap$ (the induction cases). In each case, A appears as a conjunct in an explicit or implicit conjunction. For example, the atomic concept A is a simpler representation of $\mathsf{A} \sqcap D$, with $D \equiv \top$. Since $\mathsf{A} \sqcap D$ appears under $\exists$-restrictions or without any, we identify two basic forms of an A-concept: $\mathsf{A} \sqcap D$ and $\exists \mathsf{r}.(\mathsf{A} \sqcap D) \sqcap F$. These can be on either side of a GCI. Form I and III generalize the scenarios where $\mathsf{A} \sqcap D$ is on the right and left sides of a GCI, respectively, while Form II (IV) covers cases with $\exists \mathsf{r}.(\mathsf{A} \sqcap D) \sqcap F$ on the right and left sides.

Proposition 4 states that any unnormalized A-GCI $X \sqsubseteq Y$ can be reduced to a set $\mathsf{NLZ}(X \sqsubseteq Y)$ of GCIs in finite steps with each GCI α in $\mathsf{NLZ}(X \sqsubseteq Y)$ having $\mathsf{dp}(\mathsf{A}, \alpha) \leq 1$. If $\mathsf{dp}(\mathsf{A}, \alpha) = 0$, α is either a non-A-GCI, or in A-normal form (as per Proposition 1). If $\mathsf{dp}(\mathsf{A}, \alpha) = 1$, α may not be in A-normal form because it might contain two occurrences of A (as per Proposition 3); for instance, $\mathsf{dp}(\mathsf{A}, \mathsf{A} \sqcap \exists r.\mathsf{A} \sqsubseteq Y) = 1$ but $\mathsf{fq}(\mathsf{A}, \mathsf{A} \sqcap \exists r.\mathsf{A} \sqsubseteq Y) = 2$. These cases, with an unguarded A and an A guarded by a single $\exists$-restriction, meet NR2 – NR3's syntactic requirements, and can be normalized in one step using NR2 – NR3. Since every definer replaces a subconcept immediately under an $\exists$-restriction, the number of the definers for $\mathcal{O}$'s normalization and the newly-added GCIs is bounded by $\mathrm{O}(n)$, where n is the count of $\exists$-restrictions in $\mathcal{O}$. This proves termination and completeness of the normalization procedure. $\square$

Definition 4 (Conservative Extension). *Given two general $\mathcal{ELI}$-ontologies $\mathcal{O}$ and $\mathcal{O}'$, we say that $\mathcal{O}$ is a* conservative extension *of $\mathcal{O}'$ if*

1. *$sig(\mathcal{O}') \subseteq sig(\mathcal{O})$,*
2. *every model of $\mathcal{O}$ is a model of $\mathcal{O}'$, and*
3. *for every model $\mathcal{I}'$ of $\mathcal{O}'$, there exists a model $\mathcal{I}$ of $\mathcal{O}$ such that the extensions of concept and role names from $sig(\mathcal{O}')$ coincide in $\mathcal{I}$ and $\mathcal{I}'$, i.e.,*
   - *$A^{\mathcal{I}} = A^{\mathcal{I}'}$ for all concept names $A \in sig(\mathcal{O}')$, and*
   - *$r^{\mathcal{I}} = r^{\mathcal{I}'}$ for all role names $r \in sig(\mathcal{O}')$.*

It is easy to see that the notion of a conservative extension is *transitive*, i.e., if $\mathcal{O}$ is a conservative extension of $\mathcal{O}'$ and $\mathcal{O}'$ is a conservative extension of $\mathcal{O}''$, then $\mathcal{O}$ is a conservative extension of $\mathcal{O}''$. In addition, the notion of conservative extension preserves *subsumption* in the following sense: if $\mathcal{O}$ is a conservative extension of $\mathcal{O}'$, then subsumption w.r.t. $\mathcal{O}'$ coincides with subsumption w.r.t. $\mathcal{O}$ for all GCIs built using only names from $\mathrm{sig}(\mathcal{O}')$.

Proposition 5. *Let $\mathcal{O}$ and $\mathcal{O}'$ be general $\mathcal{ELI}$ ontologies such that $\mathcal{O}$ is a conservative extension of $\mathcal{O}'$, and C, D are $\mathcal{ELI}$ concepts containing only concept and role names from $\mathcal{O}'$. Then we have*

$$\mathcal{O}' \models C \sqsubseteq D \text{ iff } \mathcal{O} \models C \sqsubseteq D.$$

Proof. First, assume that $\mathcal{O} \not\models C \sqsubseteq D$. Then there is a model $\mathcal{I}$ of $\mathcal{O}$ such that $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$. Since $\mathcal{I}$ is also a model of $\mathcal{O}'$, this implies $\mathcal{O}' \not\models C \sqsubseteq D$. Next, assume that $\mathcal{O}' \not\models C \sqsubseteq D$. Then there is a model $\mathcal{I}'$ of $\mathcal{O}'$ such that $C^{\mathcal{I}'} \not\subseteq D^{\mathcal{I}'}$. Let $\mathcal{I}$ be a model of $\mathcal{O}$ such that the extensions of concept and role names from $\mathrm{sig}(\mathcal{O}')$ coincide in $\mathcal{I}$ and $\mathcal{I}'$. Since C and D contain only concept and role names from $\mathrm{sig}(\mathcal{O}')$, we have $C^{\mathcal{I}} = C^{\mathcal{I}'} \not\subseteq D^{\mathcal{I}'} = D^{\mathcal{I}}$, and $\mathcal{O} \not\models C \sqsubseteq D$. $\square$

Lemma 2. *Let $\mathcal{O}$ be a general $\mathcal{ELI}$-ontology and $\mathcal{O}'$ the normalized one obtained from $\mathcal{O}$ using the normalization rules NR1 – NR5. Then we have*

$$\mathcal{O} \models C \sqsubseteq D \text{ iff } \mathcal{O}' \models C \sqsubseteq D,$$

for any $\mathcal{ELI}$-GCI $C \sqsubseteq D$ with $sig(C \sqsubseteq D) \subseteq sig(\mathcal{O})$.

Proof. Our proof is based on the notion of conservative extension defined above [9, 29, 30]. Specifically, we show that the ontology $\mathcal{O}'$ obtained from $\mathcal{O}$ by applying one of the normalization rules is a conservative extension of $\mathcal{O}$. We treat NR2 in detail. The rules NR3, NR4 and NR5 can be proved similarly. This statement holds trivially for NR1 since in that case $\mathcal{O}$ and $\mathcal{O}'$ have the same signature and are obviously equivalent.

Regarding NR2, assume that $\mathcal{O}'$ is obtained from $\mathcal{O}$ by replacing the GCI $X \sqsubseteq Y$ (NR2 generalizes the cases where $X = \exists R.C \sqcap D$) with two GCIs $\exists R.Z \sqcap D \sqsubseteq Y$ and $C \sqsubseteq Z$, where $Z \in \mathsf{N_C}$ is a fresh definer, i.e., $Z \notin \mathrm{sig}(\mathcal{O})$. Obviously, $\mathrm{sig}(\mathcal{O}') = \mathrm{sig}(\mathcal{O}) \cup \{Z\}$, and therefore $\mathrm{sig}(\mathcal{O}) \subseteq \mathrm{sig}(\mathcal{O}')$, satisfying Condition (1) of Definition 4. Next, assume that $\mathcal{I}'$ is a model of $\mathcal{O}'$. Then we have $(\exists R.Z)^{\mathcal{I}'} \cap D^{\mathcal{I}'} \subseteq Y^{\mathcal{I}'}$ and $C^{\mathcal{I}'} \subseteq Z^{\mathcal{I}'}$. This implies $(\exists R.C)^{\mathcal{I}'} \cap D^{\mathcal{I}'} \subseteq (\exists R.Z)^{\mathcal{I}'} \cap D^{\mathcal{I}'} \subseteq Y^{\mathcal{I}'}$, and thus $\mathcal{I}'$ is a model of $\mathcal{O}$. Finally, assume that $\mathcal{I}$ is a model of $\mathcal{O}$. Let $\mathcal{I}'$ be the interpretation that coincides with $\mathcal{I}$ on all concept and role names with the exception of Z. For Z, we define the extension in $\mathcal{I}'$ as $Z^{\mathcal{I}'} = C^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{O}$, we have $(\exists R.C)^{\mathcal{I}} \cap D^{\mathcal{I}} \subseteq Y^{\mathcal{I}}$. In addition, since Z does not occur in $\exists R.C$, D, or Y, we have $C^{\mathcal{I}} = C^{\mathcal{I}'}$, $(\exists R.C)^{\mathcal{I}} = (\exists R.C)^{\mathcal{I}'}$, $D^{\mathcal{I}} = D^{\mathcal{I}'}$ and $Y^{\mathcal{I}} = Y^{\mathcal{I}'}$. This yields $C^{\mathcal{I}'} = C^{\mathcal{I}} = Z^{\mathcal{I}'}$ and $(\exists R.Z)^{\mathcal{I}'} \cap D^{\mathcal{I}'} = (\exists R.C)^{\mathcal{I}} \cap D^{\mathcal{I}} \subseteq Y^{\mathcal{I}} = Y^{\mathcal{I}'}$, showing that $\mathcal{I}'$ is a model of $\mathcal{O}'$. Because of transitivity, Lemma 2 is an immediate consequence of Proposition 5. $\square$

Lemma 3. *Let $\mathcal{O}$ be an $\mathcal{ELI}$-ontology in A-NF, and $\mathcal{O}^{-A}$ an ontology obtained from forgetting $\{\mathsf{A}\}$ from $\mathcal{O}$ using the inference rules in Figure 1, then we have:*

$$\mathcal{O} \models C \sqsubseteq D \text{ iff } \mathcal{O}^{-A} \models C \sqsubseteq D,$$

for any $\mathcal{ELI}$-GCI $C \sqsubseteq D$ with $sig(C \sqsubseteq D) \subseteq sig(\mathcal{O})\backslash\{A\}$.

Proof. The rules IR1 – IR3 essentially reverse the normalization process, thereby preserving subsumption directly. We focus on proving Rules IR4 – IR9, with a detailed examination of Rule IR4. The other rules can be proved similarly. Regarding IR4, we show that the GCIs on the left side (the *premises*, denoted by $\mathcal{O}$) of the $\Longrightarrow$ symbol are a conservative extension of those on the right side (the *conclusion*, denoted by $\mathcal{O}'$). Obviously, $\mathrm{sig}(\mathcal{O}) = \mathrm{sig}(\mathcal{O}') \cup \{\mathsf{A}\}$, and therefore $\mathrm{sig}(\mathcal{O}') \subseteq \mathrm{sig}(\mathcal{O})$, satisfying Condition (1) of Definition 4.
Case I (i.e., provided that: $\mathcal{O} \models \mathsf{A} \sqcap D \sqsubseteq E_1 \sqcap \ldots \sqcap E_n$):

Assume that $\mathcal{I}$ is a model of $\mathcal{O}$. Then we have the following:

$$C^{\mathcal{I}} \subseteq (\exists r.(\mathsf{A} \sqcap D))^{\mathcal{I}} \tag{1}$$

$$(\mathsf{A} \sqcap E_1)^{\mathcal{I}} \subseteq {F_1}^{\mathcal{I}}, \ldots, (\mathsf{A} \sqcap E_n)^{\mathcal{I}} \subseteq {F_n}^{\mathcal{I}} \tag{2}$$

$$(\mathsf{A} \sqcap D)^{\mathcal{I}} \subseteq (E_1 \sqcap \ldots \sqcap E_n)^{\mathcal{I}} \tag{3}$$

As Inclusion (2) implies $(\mathsf{A} \sqcap E_1)^{\mathcal{I}} \cap \ldots \cap (\mathsf{A} \sqcap E_n)^{\mathcal{I}} \subseteq {F_1}^{\mathcal{I}} \cap \ldots \cap {F_n}^{\mathcal{I}}$, we have $(\mathsf{A} \sqcap E_1 \sqcap \ldots \sqcap E_n)^{\mathcal{I}} \subseteq (F_1 \sqcap \ldots \sqcap F_n)^{\mathcal{I}}$ and further $(\mathsf{A} \sqcap E_1 \sqcap \ldots \sqcap E_n)^{\mathcal{I}} \subseteq (E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n)^{\mathcal{I}}$. As Inclusion (3) implies $(\mathsf{A} \sqcap D)^{\mathcal{I}} \subseteq (\mathsf{A} \sqcap E_1 \sqcap \ldots \sqcap E_n)^{\mathcal{I}}$, we have $(\mathsf{A} \sqcap D)^{\mathcal{I}} \subseteq (E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n)^{\mathcal{I}}$ and further $(\mathsf{A} \sqcap D)^{\mathcal{I}} \subseteq (E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D)^{\mathcal{I}}$. Together with Inclusion (1), we have the following:

$$C^{\mathcal{I}} \subseteq (\exists r.(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D))^{\mathcal{I}} \tag{4}$$

This satisfies Condition (2) of Definition 4.

Assume that $\mathcal{I}'$ is a model of $\mathcal{O}'$. Let $\mathcal{I}$ be the interpretation that coincides with $\mathcal{I}'$ on all concept and role names with the exception of A. For A, we define the extension in $\mathcal{I}$ as $\mathsf{A}^{\mathcal{I}} = (E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n)^{\mathcal{I}'}$. Since $\mathcal{I}'$ is a model of $\mathcal{O}'$, we have $C^{\mathcal{I}'} \subseteq (\exists r.(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D))^{\mathcal{I}'}$. In addition, since A does not occur in C or $\exists r.(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D)$, we have $C^{\mathcal{I}'} = C^{\mathcal{I}}$, $(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n)^{\mathcal{I}'} = (E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n)^{\mathcal{I}} = \mathsf{A}^{\mathcal{I}}$, $(\exists r.(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D))^{\mathcal{I}'} = (\exists r.(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D))^{\mathcal{I}} = (\exists r.(\mathsf{A} \sqcap D))^{\mathcal{I}}$, and thus, we have

$$C^{\mathcal{I}} \subseteq (\exists r.(\mathsf{A} \sqcap D))^{\mathcal{I}} \tag{5}$$

Since $(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap E_1)^{\mathcal{I}} \subseteq {F_1}^{\mathcal{I}}$, we have

$$(\mathsf{A} \sqcap E_1)^{\mathcal{I}} \subseteq {F_1}^{\mathcal{I}} \tag{6}$$

In the same way, we can prove the other GCIs in (2). Since $(E_1 \sqcap \ldots \sqcap E_n \sqcap F_1 \sqcap \ldots \sqcap F_n \sqcap D)^{\mathcal{I}} \subseteq (E_1 \sqcap \ldots \sqcap E_n)^{\mathcal{I}}$, we have

$$(\mathsf{A} \sqcap D)^{\mathcal{I}} \subseteq (E_1 \sqcap \ldots \sqcap E_n)^{\mathcal{I}} \tag{7}$$

Inclusions (5), (6), and (7) collectively satisfy Condition (3) of Definition 4.

Case II (i.e., provided that: $\mathcal{O} \not\models \mathsf{A} \sqcap D \sqsubseteq E$):

Assume that $\mathcal{I}$ is a model of $\mathcal{O}$. Then we have the following:

$$C^{\mathcal{I}} \subseteq (\exists r.(\mathsf{A} \sqcap D))^{\mathcal{I}} \tag{8}$$

$$(\mathsf{A} \sqcap E)^{\mathcal{I}} \subseteq F^{\mathcal{I}} \tag{9}$$

Directly, we have

$$C^{\mathcal{I}} \subseteq (\exists r.D)^{\mathcal{I}} \tag{10}$$

Assume that $\mathcal{I}'$ is a model of $\mathcal{O}'$. Let $\mathcal{I}$ be the interpretation that coincides with $\mathcal{I}'$ on all concept and role names with the exception of A, E and F. For A, we define the extension in $\mathcal{I}$ as $\mathsf{A}^{\mathcal{I}} = \top^{\mathcal{I}'}$. Since $\mathcal{I}'$ is a model of $\mathcal{O}'$, we have $C^{\mathcal{I}'} \subseteq (\exists r.D)^{\mathcal{I}'}$. In addition, since A does not occur in C or $\exists r.D$, we have $C^{\mathcal{I}'} = C^{\mathcal{I}}$, $\top^{\mathcal{I}'} = \top^{\mathcal{I}} = \mathsf{A}^{\mathcal{I}}$, $(\exists r.D)^{\mathcal{I}'} = (\exists r.(\top \sqcap D))^{\mathcal{I}'} = (\exists r.(\top \sqcap D))^{\mathcal{I}} = (\exists r.(\mathsf{A} \sqcap D))^{\mathcal{I}}$, and thus, we have

$$C^{\mathcal{I}} \subseteq (\mathsf{A} \sqcap \exists r.D)^{\mathcal{I}} \tag{11}$$

For E and F, we define their extensions in $\mathcal{I}$ as $E^{\mathcal{I}} = F^{\mathcal{I}} = \top^{\mathcal{I}'}$. Then we have

$$(\mathsf{A} \sqcap E)^{\mathcal{I}} \subseteq F^{\mathcal{I}} \tag{12}$$

Thus, $\mathcal{O}$ is a conservative extension of $\mathcal{O}'$. Because of the transitivity of conservative extension, $\mathcal{O}$ in Lemma 3 is a conservative extension of $\mathcal{O}^{-\mathsf{A}}$. According to Proposition 5, Lemma 3 holds. □

Likewise, Lemma 4 establishes the partial soundness of the calculus. Specifically, the derived ontology $\mathcal{O}^{-\mathsf{r}}$ fulfills the second condition necessary for it to be the result of forgetting $\{\mathsf{r}\}$ from $\mathcal{O}$. However, $\mathcal{O}^{-\mathsf{r}}$ may include definers which fall outside the scope of $\mathrm{sig}(\mathcal{O}) \backslash \{\mathsf{r}\}$, potentially failing to fulfill the first condition.

Lemma 4. *Let $\mathcal{O}$ be an $\mathcal{ELI}$-ontology in r-NF, and $\mathcal{O}^{-r}$ an ontology obtained from forgetting $\{\mathsf{r}\}$ from $\mathcal{O}$ using the inference rules in Figure 2, then we have:*

$$\mathcal{O} \models C \sqsubseteq D \textit{ iff } \mathcal{O}^{-r} \models C \sqsubseteq D,$$

for any $\mathcal{ELI}$-GCI $C \sqsubseteq D$ with $sig(C \sqsubseteq D) \subseteq sig(\mathcal{O}) \backslash \{r\}$.

Proof. We prove Rules IR10 – IR13, with a detailed examination of Rule IR10. The other rules can be proved similarly. Regarding IR10, we show that the GCIs on the left side (the *premises*, denoted by $\mathcal{O}$) of the $\Longrightarrow$ symbol are a conservative extension of those on the right side (the *conclusion*, denoted by $\mathcal{O}'$). Obviously, $\mathrm{sig}(\mathcal{O}) = \mathrm{sig}(\mathcal{O}') \cup \{\mathsf{r}\}$, and therefore $\mathrm{sig}(\mathcal{O}') \subseteq \mathrm{sig}(\mathcal{O})$, satisfying Condition (1) of Definition 4.

Assume that $\mathcal{I}$ is a model of $\mathcal{O}$. Then we have the following:

$$C^{\mathcal{I}} \subseteq (\exists \mathsf{r}.D)^{\mathcal{I}} \tag{13}$$

$$(F \sqcap \exists \mathsf{r}.E)^{\mathcal{I}} \subseteq G^{\mathcal{I}} \tag{14}$$

$$(\exists \mathsf{r}.D)^{\mathcal{I}} \subseteq (\exists \mathsf{r}.E)^{\mathcal{I}} \tag{15}$$

Inclusions (13) and (15) jointly imply $C^{\mathcal{I}} \subseteq (\exists \mathsf{r}.E)^{\mathcal{I}}$, and when combined with (14), they further imply $(F \sqcap C)^{\mathcal{I}} \subseteq G^{\mathcal{I}}$.

Assume that $\mathcal{I}'$ is a model of $\mathcal{O}'$. Let $\mathcal{I}$ be the interpretation that coincides with $\mathcal{I}'$ on all concept and role names with the exception of r, D, and E. For the exceptions, we define their extensions in $\mathcal{I}$ as $(\exists \mathsf{r}.D)^{\mathcal{I}} = (\exists \mathsf{r}.E)^{\mathcal{I}} = C^{\mathcal{I}'}$. Then we have $(\exists \mathsf{r}.D)^{\mathcal{I}} \subseteq (\exists \mathsf{r}.E)^{\mathcal{I}}$ Since $\mathcal{I}'$ is a model of $\mathcal{O}'$, we have $(F \sqcap C)^{\mathcal{I}'} \subseteq G^{\mathcal{I}'}$. In addition, since r does not occur in C, D, E, F, and G, we have $C^{\mathcal{I}'} = C^{\mathcal{I}}$, $F^{\mathcal{I}'} = F^{\mathcal{I}}$ and $G^{\mathcal{I}'} = G^{\mathcal{I}}$. Thus we have $C^{\mathcal{I}'} = C^{\mathcal{I}} \subseteq (\exists \mathsf{r}.D)^{\mathcal{I}}$ and $(F^{\mathcal{I}} \sqcap \exists \mathsf{r}.E)^{\mathcal{I}} \subseteq G^{\mathcal{I}}$

Thus, $\mathcal{O}$ is a conservative extension of $\mathcal{O}'$. Because of the transitivity of conservative extension, $\mathcal{O}$ in Lemma 4 is a conservative extension of $\mathcal{O}^{-\mathsf{r}}$. According to Proposition 5, Lemma 4 holds. □

Theorem 1. *Given any $\mathcal{ELI}$-ontology $\mathcal{O}$ and any forgetting signature $\mathcal{F} \subseteq sig(\mathcal{O})$ as input, our forgetting method always terminates and returns an $\mathcal{ELI}$-ontology $\mathcal{V}$. If $\mathcal{V}$ does not contain any definers, then it is a result of forgetting $\mathcal{F}$ from $\mathcal{O}$.*

Proof. Note that the normalization and inference rules do not introduce new cycles. For cases where cyclic behavior originally exhibits over the names in $\mathcal{F}$, the method terminates upon detecting a cycle. In acyclic cases, termination of the method follows from Lemma 1 and the termination of the forgetting calculi. The method's soundness is ensured by Lemma 2 and its counterpart lemma for r-normalization, along with Lemmas 3 and 4. □

A.2 Case Study — Computing UI-based Semantic Difference in SNOMED CT

Real-world ontologies are collaboratively created and managed by domain experts, data scientists, and knowledge engineers, often with input from specialized scientists. Maintaining the quality of ontology content is vital for their effective use. Additionally, ontologies undergo continuous changes during their lifecycle to respond to different change requirements, where several problems emanate: capturing change requirements, change representation, change validation, change traceability and analysis, change propagation to dependant logical statements, etc.

Quality Assurance (QA) in the ontology evolution introduces additional complexity, especially concerning semantic changes. Validating changes made to evolving ontologies is crucial to maintaining semantic consistency with existing standards. Any semantic inconsistencies must be identified, addressed, and re-validated by the consortium before releasing a new version. However, the workload and complexity of these tasks can overwhelm a standard ontology QA team, necessitating automated methodologies and tools to assist in this process.

The *semantic difference* $\mathrm{Diff}(\mathcal{O}_1, \mathcal{O}_2)$ between two versions $\mathcal{O}_1$ and $\mathcal{O}_2$ of an ontology are the axioms entailed by $\mathcal{O}_2$ but not $\mathcal{O}_1$ (or vice versa), reflecting the information gain and loss between them. A meaningful notion of semantic difference should be defined upon the common signature of two ontologies (or subsets thereof). This signature unification can be done via UI.

Definition 5 (**UI-based Semantic Difference**). *Let Σ be a subset of the common signature of $\mathcal{O}_1$ and $\mathcal{O}_2$. The* UI-based semantic difference *between $\mathcal{O}_1$ and $\mathcal{O}_2$ is the set UI-Diff$_\Sigma(\mathcal{O}_1, \mathcal{O}_2)$ of all $\mathcal{ELH}$-axioms α such that: (i) sig$(\alpha) \subseteq \Sigma$, (ii) $\alpha \in \mathcal{V}_2$, and*

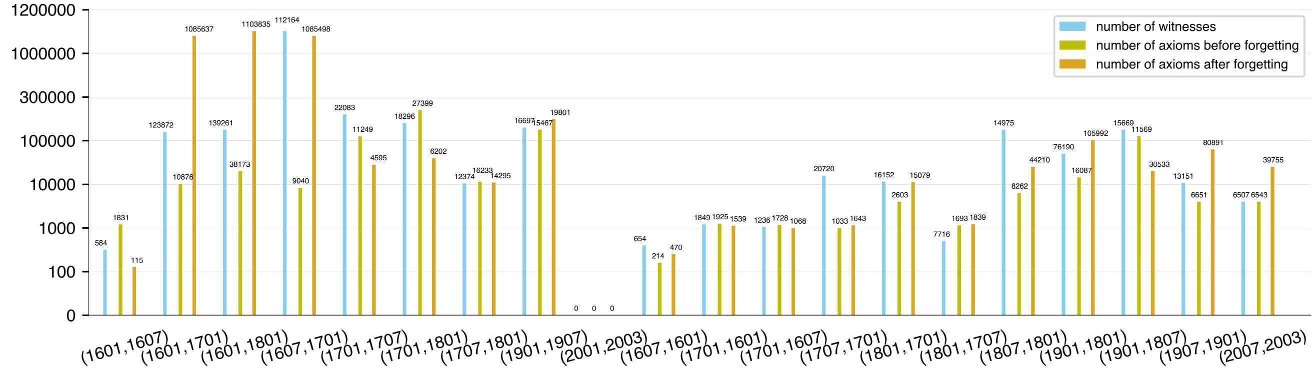

Figure 7: Information gain and information loss within different releases of SNOMED CT

Table 4: Metrics of the forgetting tasks on SNOMED CT

Tasks	$\lvert\mathcal{F}_C\rvert$	Time (sec.)	Tasks	$\lvert\mathcal{F}_C\rvert$	Time (sec.)
(1601,1607)	2599	20	(1701,1601)	1921	300
(1601,1701)	7695	1094	(1701,1607)	1795	288
(1601,1801)	23.4K	1875	(1707,1701)	614	97
(1607,1701)	5115	1005	(1801,1701)	1378	166
(1701,1707)	12.3K	302	(1801,1707)	779	84
(1701,1801)	17.2K	777	(1807,1801)	6955	324
(1707,1801)	4980	231	(1901,1801)	9389	1423
(1901,1907)	4423	212	(1901,1807)	3034	520
(2001,2003)	0	0	(1907,1901)	3142	599
(1607,1601)	145	153	(2007,2003)	2074	501

(iii) $O_1 \not\models \alpha$, where $\mathcal{V}_2$ is a Σ-uniform interpolant of O_2. An axiom α that satisfies the above conditions is called a UI-witness *of UI-Diff$_\Sigma(O_1, O_2)$.*

Konev et. al. [16, 27] noted that UI-Diff$_\Sigma(O_1, O_2)$ could be computed using the following two-step algorithm:

Step (1): compute a Σ-uniform interpolant $\mathcal{V}_2$ of O_2, where $\Sigma \subseteq \mathrm{sig}(O_1) \cap \mathrm{sig}(O_2)$, and then
Step (2): collect the axioms $\alpha \in \mathcal{V}_2$ not entailed by O_1.

SNOMEDCT[41] is currently the world's most comprehensive clinical healthcare terminology, widely integrated into web systems. SNOMED International[4] owns and maintains SNOMED CT, issuing releases of its International Edition at the end of January and July each year, where revisions have been made to meet users' needs and to reflect the results of quality assurance activities. With more than 350,000 concept codes and a similar number of logical axioms, SNOMED CT is an order of magnitude larger than the next largest DL-based ontology of which we have been aware. In this section, we studied how our UI method performs in practice for the task of tracking semantic evolutionary changes in 15 consecutive international releases of SNOMED CT, as well as 5 non-consecutive international releases. We used SNOMED CT to challenge our UI tool in order to see its viability as back-end technology in SNOMED's knowledge base interface for their main concerns of quality control of health data. To this end, we developed a UI-Diff pipeline that employed our UI prototype to perform **Step (1)** and the DL reasoner HermiT [10] to perform **Step (2)**.

Table 4 summarizes the forgetting task metrics for each UI-Diff comparison case, with year-month notations (e.g., 1601 for January 2016). $\lvert\mathcal{F}_C\rvert$ represents the number of concept names to forget. All tasks were completed within a reasonable time, a remarkable feat given the size of $\mathcal{F}_C$. Lethe failed (got stuck) in all these tasks.

A.3 Statistics of Oxford-ISG and BioPortal

Table 5: Statistics of Oxford-ISG and BioPortal

Oxford		min	max	med	mean	90 percentile
I	$\lvert N_C\rvert$	0	1582	86	191	545
	$\lvert N_R\rvert$	0	332	10	29	80
	$\lvert$Onto$\rvert$	10	990	162	262	658
II	$\lvert N_C\rvert$	200	5877	1665	1769	2801
	$\lvert N_R\rvert$	0	887	11	34	61
	$\lvert$Onto$\rvert$	1008	4976	2282	2416	3937
III	$\lvert N_C\rvert$	1162	9809	4042	5067	8758
	$\lvert N_R\rvert$	1	158	4	23	158
	$\lvert$Onto$\rvert$	5112	9783	7277	7195	9179
BioPortal		**min**	**max**	**med**	**mean**	**90 percentile**
I	$\lvert N_C\rvert$	0	784	127	192	214
	$\lvert N_R\rvert$	0	122	5	15	17
	$\lvert$Onto$\rvert$	10	794	283	312	346
II	$\lvert N_C\rvert$	5	4530	1185	1459	1591
	$\lvert N_R\rvert$	0	131	12	30	33
	$\lvert$Onto$\rvert$	1023	4880	2401	2619	2782
III	$\lvert N_C\rvert$	432	8340	4363	4387	4806
	$\lvert N_R\rvert$	0	135	17	30	34
	$\lvert$Onto$\rvert$	5457	8339	6934	6912	7109

A.4 Illustrative Examples

Example 2. Consider the following ontology O:

$$\{1.\ \mathsf{A} \sqsubseteq \mathsf{B} \sqcap \exists \mathsf{r}.(\mathsf{C} \sqcap \exists \mathsf{s}^-.\exists \mathsf{r}.\mathsf{A}) \quad 2.\ \mathsf{A} \sqcap \exists \mathsf{t}.\mathsf{A} \sqsubseteq \mathsf{D}\}$$

Let $\mathcal{F} = \{\mathsf{A}\}$. We compute A-NF of O by applying exhaustively the normalization rules as described above, where $Z_1, Z_2, Z_3 \in \mathsf{N_D}$ are

[4] https://www.snomed.org/

fresh definers:
Applying Rule NF1 to Axiom 1 gives:

$$\{3.\ \mathrm{A} \sqsubseteq \mathrm{B} \quad 4.\ \mathrm{A} \sqsubseteq \exists \mathrm{r}.(\mathrm{C} \sqcap \exists \mathrm{s}^-.\exists \mathrm{r}.\mathrm{A}) \quad 2.\ \mathrm{A} \sqcap \exists \mathrm{t}.\mathrm{A} \sqsubseteq \mathrm{D}\}$$

Applying Rule NF5 to Axiom 4 gives:

$$\{3.\ \mathrm{A} \sqsubseteq \mathrm{B} \quad 5.\ \mathrm{A} \sqsubseteq \exists r.Z_1 \quad 6.\ Z_1 \sqsubseteq \mathrm{C} \sqcap \exists s^-.\exists r.\mathrm{A}$$
$$2.\ \mathrm{A} \sqcap \exists \mathrm{t}.\mathrm{A} \sqsubseteq \mathrm{D}\}$$

Applying Rule NF1 to Axiom 6 gives:

$$\{3.\ \mathrm{A} \sqsubseteq \mathrm{B} \quad 5.\ \mathrm{A} \sqsubseteq \exists \mathrm{r}.Z_1 \quad 7.\ Z_1 \sqsubseteq \mathrm{C}$$
$$8.\ Z_1 \sqsubseteq \exists s^-.\exists r.\mathrm{A} \quad 2.\ \mathrm{A} \sqcap \exists \mathrm{t}.\mathrm{A} \sqsubseteq \mathrm{D}\}$$

Applying Rule NF5 to Axiom 8 gives:

$$\{3.\ \mathrm{A} \sqsubseteq \mathrm{B} \quad 5.\ \mathrm{A} \sqsubseteq \exists \mathrm{r}.Z_1 \quad 7.\ Z_1 \sqsubseteq \mathrm{C}$$
$$9.\ Z_1 \sqsubseteq \exists s^-.Z_2 \quad 10.\ Z_2 \sqsubseteq \exists r.\mathrm{A} \quad 2.\ \mathrm{A} \sqcap \exists \mathrm{t}.\mathrm{A} \sqsubseteq \mathrm{D}\}$$

Applying Rule NF2 to Axiom 2 gives:

$$\{3.\ \mathrm{A} \sqsubseteq \mathrm{B} \quad 5.\ \mathrm{A} \sqsubseteq \exists \mathrm{r}.Z_1 \quad 7.\ Z_1 \sqsubseteq \mathrm{C}$$
$$9.\ Z_1 \sqsubseteq \exists s^-.Z_2 \quad 10.\ Z_2 \sqsubseteq \exists r.\mathrm{A}$$
$$11.\ \mathrm{A} \sqcap \exists t.Z_3 \sqsubseteq \mathrm{D} \quad 12.\ \mathrm{A} \sqsubseteq Z_3\ \}$$

GCIs 3 and 12 are in A-NF IV, GCIs 5 and 11 are in A-NF V, GCI 10 is in A-NF IV, and the resulting $O' = \{3, 5, 7, 9, 10, 11, 12\}$ is in A-NF.