

# Proofs for “Expertise and information: an epistemic logic perspective”

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## Appendix A Proofs

*Proof of Lemma 7* We first show that  $R$  is reflexive and has the *Euclidean property* ( $xRy$  and  $xRz$  implies  $yRz$ ). For reflexivity, let  $\Gamma \in X_L$ . Suppose  $A\varphi \in \Gamma$ . By  $(T_A)$  and closure of maximally consistent sets under modus ponens,  $\varphi \in \Gamma$ . Hence  $\Gamma R\Gamma$ .

For the Euclidean property, suppose  $\Gamma R\Delta$  and  $\Gamma R\Lambda$ . We show  $\Delta R\Lambda$  by contraposition. Suppose  $\varphi \notin \Lambda$ . Since  $\Gamma R\Lambda$ , this means  $A\varphi \notin \Gamma$ . Hence  $\neg A\varphi \in \Gamma$ , and by  $(5_A)$  we get  $A\neg A\varphi \in \Gamma$ . Now  $\Gamma R\Delta$  gives  $\neg A\varphi \in \Delta$ , so  $A\varphi \notin \Delta$ .

To conclude we need to show  $R$  is symmetric and transitive. For symmetry, suppose  $\Gamma R\Delta$ . By reflexivity,  $\Gamma R\Gamma$ . The Euclidean property therefore gives  $\Delta R\Gamma$ . For transitivity, suppose  $\Gamma R\Delta$  and  $\Delta R\Lambda$ . By symmetry,  $\Delta R\Gamma$ . The Euclidean property again gives  $\Gamma R\Lambda$ .  $\square$

*Proof of Lemma 8* Part (1) was shown in the main text. For (2), note that by (1) we have

$$\begin{aligned} A(\varphi \rightarrow \psi) \in \Sigma &\iff |\varphi \rightarrow \psi|_\Sigma = X_\Sigma \\ &\iff \forall \Gamma \in X_\Sigma : \varphi \rightarrow \psi \in \Gamma \end{aligned}$$

Suppose  $A(\varphi \rightarrow \psi) \in \Sigma$ . Take  $\Gamma \in |\varphi|_\Sigma$ . Then we have  $\varphi, \varphi \rightarrow \psi \in \Gamma$ , so  $\psi \in \Gamma$ . This shows  $|\varphi|_\Sigma \subseteq |\psi|_\Sigma$ . Conversely, suppose  $|\varphi|_\Sigma \subseteq |\psi|_\Sigma$ . Take  $\Gamma \in X_\Sigma$ . If  $\varphi \notin \Gamma$  then  $\neg\varphi \in \Gamma$ , so  $\neg\varphi \vee \psi \in \Gamma$  and thus  $\varphi \rightarrow \psi \in \Gamma$ . If  $\varphi \in \Gamma$  then  $\Gamma \in |\varphi|_\Sigma \subseteq |\psi|_\Sigma$ , so  $\psi \in \Gamma$ . Thus  $\varphi \rightarrow \psi \in \Gamma$  in this case too. Hence  $A(\varphi \rightarrow \psi) \in \Sigma$ .

For (3) first note that  $A(\alpha \wedge \beta) \in \Sigma$  iff both  $A\alpha \in \Sigma$  and  $A\beta \in \Sigma$ . This can be shown using  $(K_A)$ ,  $(MP)$  and instances of the propositional tautologies  $(p \wedge q) \rightarrow p$  (for the left-to-right implication) and  $p \rightarrow q \rightarrow (p \wedge q)$  (for the right-to-left implication). Recalling that  $\varphi \leftrightarrow \psi$  is an abbreviation for  $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ , we get

$$\begin{aligned} A(\varphi \leftrightarrow \psi) \in \Sigma &\iff A(\varphi \rightarrow \psi) \in \Sigma \text{ and } A(\psi \rightarrow \varphi) \in \Sigma \\ &\iff |\varphi|_\Sigma \subseteq |\psi|_\Sigma \text{ and } |\psi|_\Sigma \subseteq |\varphi|_\Sigma \\ &\iff |\varphi|_\Sigma = |\psi|_\Sigma \end{aligned}$$

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as required. □

*Proof of Theorem 5* Soundness was shown in the main text. For completeness, we adopt a roughly similar approach to the general case. Let consistency, maximal consistency and other standard notions and notation be defined as before, but now for  $L_{\text{int}}$  instead of  $L$ . Let  $X_{L_{\text{int}}}$  be the set of maximally  $L_{\text{int}}$ -consistent sets. Define the relation  $R$  on  $X_{L_{\text{int}}}$  in exactly the same way. Since  $L_{\text{int}}$  extends  $L$ ,  $R$  is again an equivalence relation, and we have the analogues of Lemma 8 and Corollary 1.

This time, however, the construction of the canonical model for a given  $\Sigma \in X_{L_{\text{int}}}$  is much more straightforward. The set of states is simply  $X_{\Sigma}$ , i.e. the equivalence class of  $\Sigma$  in  $R$ . Overriding earlier terminology, say  $A \subseteq X_{\Sigma}$  is *S-closed* iff  $|\varphi|_{\Sigma} \subseteq A$  implies  $|\mathcal{S}\varphi|_{\Sigma} \subseteq A$  for all  $\varphi \in \mathcal{L}$ . Then set

$$P_{\Sigma} = \{A \subseteq X_{\Sigma} \mid A \text{ is S-closed}\}.$$

Finally, set  $V_{\Sigma}(p) = |p|_{\Sigma}$ , and write  $M_{\Sigma} = (X_{\Sigma}, P_{\Sigma}, V_{\Sigma})$ .

First, we have  $M_{\Sigma} \in \mathbb{M}_{\text{int}}$ , i.e. intersections of S-closed sets are S-closed. Indeed, suppose  $\{A_i\}_{i \in I}$  is a collection of S-closed sets, and suppose  $|\varphi|_{\Sigma} \subseteq \bigcap_{i \in I} A_i$ . Then  $|\varphi|_{\Sigma} \subseteq A_i$  for each  $i$ , so S-closure gives  $|\mathcal{S}\varphi|_{\Sigma} \subseteq A_i$ . Hence  $|\mathcal{S}\varphi|_{\Sigma} \subseteq \bigcap_{i \in I} A_i$ .

Importantly, we have the truth lemma for  $M_{\Sigma}$ : for all  $\Gamma \in X_{\Sigma}$  and  $\varphi \in \mathcal{L}$ ,

$$M_{\Sigma}, \Gamma \models \varphi \iff \varphi \in \Gamma,$$

i.e.  $\|\varphi\|_{M_{\Sigma}} = |\varphi|_{\Sigma}$ .

As usual, the proof is by induction on formulas. The case for atomic propositions follows from the definition of  $V_{\Sigma}$ , the cases for conjunctions and negations hold by properties of maximally consistent sets, and the case for  $\mathbf{A}\varphi$  holds by an argument identical to the one used in the general case (Lemma 9). The only interesting cases are therefore for  $\mathbf{E}\varphi$  and  $\mathcal{S}\varphi$  formulas:

(E): First suppose  $\mathbf{E}\varphi \in \Gamma$ . We claim  $|\varphi|_{\Sigma}$  is S-closed. This will give  $\|\varphi\|_{M_{\Sigma}} \in P_{\Sigma}$  by the induction hypothesis and definition of  $P_{\Sigma}$ , and therefore  $M_{\Sigma}, \Gamma \models \mathbf{E}\varphi$ .

So, suppose  $|\psi|_{\Sigma} \subseteq |\varphi|_{\Sigma}$ . Then  $\mathbf{A}(\psi \rightarrow \varphi) \in \Sigma$ . Let  $\Delta \in |\mathcal{S}\psi|_{\Sigma}$ . Since  $\Delta, \Gamma, \Sigma \in X_{\Sigma}$ , we have  $\mathbf{E}\varphi \in \Delta$  and  $\mathbf{A}(\psi \rightarrow \varphi) \in \Delta$  too. By  $(W_E)$ ,  $\mathcal{S}\psi \wedge \mathbf{E}\varphi \rightarrow \varphi \in \Delta$ . But  $\mathcal{S}\psi \in \Delta$ , so  $\mathcal{S}\psi \wedge \mathbf{E}\varphi \in \Delta$  and thus  $\varphi \in \Delta$ , i.e.  $\Delta \in |\varphi|_{\Sigma}$ . This shows  $|\mathcal{S}\psi|_{\Sigma} \subseteq |\varphi|_{\Sigma}$ , so  $|\varphi|_{\Sigma}$  is S-closed as required.

Now suppose  $M_{\Sigma}, \Gamma \models \mathbf{E}\varphi$ . Then, by the induction hypothesis,  $|\varphi|_{\Sigma}$  is S-closed. Since  $|\varphi|_{\Sigma} \subseteq |\varphi|_{\Sigma}$  clearly holds, we get  $|\mathcal{S}\varphi|_{\Sigma} \subseteq |\varphi|_{\Sigma}$ . This implies  $\mathbf{A}(\mathcal{S}\varphi \rightarrow \varphi) \in \Sigma$ , and  $(\text{Red}_E)$  gives  $\mathbf{E}\varphi \in \Sigma$ . Since  $\Gamma \in X_{\Sigma}$ , we get  $\mathbf{E}\varphi \in \Gamma$  as required.

(S): Suppose  $\mathcal{S}\varphi \in \Gamma$ . Take any  $A \in P_{\Sigma}$  such that  $\|\varphi\|_{M_{\Sigma}} \subseteq A$ . By the induction hypothesis,  $|\varphi|_{\Sigma} \subseteq A$ . By S-closure of  $A$ ,  $|\mathcal{S}\varphi|_{\Sigma} \subseteq A$ . Hence  $\Gamma \in |\mathcal{S}\varphi|_{\Sigma} \subseteq A$ . This shows  $M_{\Sigma}, \Gamma \models \mathcal{S}\varphi$ .

For the other direction we show the contrapositive. Suppose  $\mathcal{S}\varphi \notin \Gamma$ . First, we claim  $|\mathcal{S}\varphi|_{\Sigma}$  is S-closed. Indeed, suppose  $|\psi|_{\Sigma} \subseteq |\mathcal{S}\varphi|_{\Sigma}$ . Then  $\mathbf{A}(\psi \rightarrow \mathcal{S}\varphi) \in \Sigma$ . Take any  $\Delta \in |\mathcal{S}\psi|_{\Sigma}$ . Since  $\Delta \in X_{\Sigma}$ ,  $\mathbf{A}(\psi \rightarrow \mathcal{S}\varphi) \in \Delta$  also. By  $(W_S)$ ,  $\mathcal{S}\psi \rightarrow \mathcal{S}\mathcal{S}\varphi \in \Delta$ . Now  $\mathcal{S}\psi \in \Delta$  implies  $\mathcal{S}\mathcal{S}\varphi \in \Delta$ , and  $(4_S)$  gives  $\mathcal{S}\varphi \in \Delta$ , i.e.  $\Delta \in |\mathcal{S}\varphi|_{\Sigma}$ . This shows  $|\mathcal{S}\psi|_{\Sigma} \subseteq |\mathcal{S}\varphi|_{\Sigma}$ , and thus  $|\mathcal{S}\varphi|_{\Sigma}$  is S-closed.

Hence  $|\mathcal{S}\varphi|_{\Sigma}$  is a set in  $P_{\Sigma}$  not containing  $\Gamma$ . Moreover,  $\|\varphi\|_{M_{\Sigma}} \subseteq |\mathcal{S}\varphi|_{\Sigma}$  by the induction hypothesis and  $(T_S)$ . Hence  $M_{\Sigma}, \Gamma \not\models \mathcal{S}\varphi$ .

Strong completeness now follows. If  $\Gamma \not\models_{L_{\text{int}}} \varphi$ , then  $\Gamma \cup \{\neg\varphi\}$  is consistent, so by Lindenbaum’s Lemma there is  $\Sigma \in X_{L_{\text{int}}}$  with  $\Sigma \supseteq \Gamma \cup \{\neg\varphi\}$ . Considering the

model  $M_\Sigma \in \mathbb{M}_{\text{int}}$ , we have  $M_\Sigma, \Sigma \models \Gamma$  and  $M_\Sigma, \Sigma \not\models \varphi$  by the truth lemma. Hence  $\Gamma \not\models_{\mathbb{M}_{\text{int}}} \varphi$ .  $\square$

*Proof of Theorem 6* Soundness was shown in the main text. For completeness, we go via relational semantics using the translation  $t : \mathcal{L} \rightarrow \mathcal{L}_{\text{KA}}$  and Theorem 3. First, let  $\mathbb{L}_{\text{S4A}}$  be the logic of  $\mathcal{L}_{\text{KA}}$  formulas formed by the axioms and inference rules shown in Table A1. It is well known that  $\mathbb{L}_{\text{S4A}}$  is strongly complete with respect to  $\mathbb{M}_{\text{S4}}^*$  (Blackburn, De Rijke, & Venema, 2002, Theorem 7.2).

**Table A1** Axioms and inference rules for  $\mathbb{L}_{\text{S4A}}$ .

$\text{K}(\varphi \rightarrow \psi) \rightarrow (\text{K}\varphi \rightarrow \text{K}\psi)$	(K <sub>K</sub> )
$\text{K}\varphi \rightarrow \varphi$	(T <sub>K</sub> )
$\text{K}\varphi \rightarrow \text{KK}\varphi$	(4 <sub>K</sub> )
$\text{A}(\varphi \rightarrow \psi) \rightarrow (\text{A}\varphi \rightarrow \text{A}\psi)$	(K <sub>A</sub> )
$\text{A}\varphi \rightarrow \varphi$	(T <sub>A</sub> )
$\neg\text{A}\varphi \rightarrow \text{A}\neg\text{A}\varphi$	(5 <sub>A</sub> )
$\text{A}\varphi \rightarrow \text{K}\varphi$	(Inc <sub>K</sub> )
From $\varphi$ infer $\text{A}\varphi$	(Nec <sub>A</sub> )
From $\varphi \rightarrow \psi$ and $\varphi$ infer $\psi$	(MP)

Now, define a translation  $u : \mathcal{L}_{\text{KA}} \rightarrow \mathcal{L}$  as follows:

$$\begin{aligned}
 u(p) &= p \\
 u(\varphi \wedge \psi) &= u(\varphi) \wedge u(\psi) \\
 u(\neg\varphi) &= \neg u(\varphi) \\
 u(\text{K}\varphi) &= \neg\text{S}\neg u(\varphi) \\
 u(\text{A}\varphi) &= \text{A}u(\varphi).
 \end{aligned}$$

Recall the translation  $t : \mathcal{L} \rightarrow \mathcal{L}_{\text{KA}}$  from Section 4. While  $u$  is not the inverse of  $t$  (for instance, there is no  $\psi \in \mathcal{L}_{\text{KA}}$  with  $u(\psi) = \text{E}p$ ), for any  $\varphi \in \mathcal{L}$  we have that  $\varphi$  is  $\mathbb{L}_{\text{top}}$ -provably equivalent to  $u(t(\varphi))$ .

**Claim 1** *Let  $\varphi \in \mathcal{L}$ . Then  $\vdash_{\mathbb{L}_{\text{top}}} \varphi \leftrightarrow u(t(\varphi))$ .*

*Proof* By induction on  $\mathcal{L}$  formulas. The cases of atomic propositions and propositional connectives are straightforward. For the other cases, first note that the “replacement of equivalents” rule is derivable in  $\mathbb{L}$  (and thus in  $\mathbb{L}_{\text{top}}$ ) for  $\text{S}$ ,  $\text{E}$  and  $\text{A}$ :

$$\text{From } \varphi \leftrightarrow \psi \text{ infer } \bigcirc \varphi \leftrightarrow \bigcirc \psi \quad (\bigcirc \in \{\text{S}, \text{E}, \text{A}\}).$$

For  $\text{S}$  this follows from (Nec<sub>A</sub>) and (W<sub>S</sub>); for  $\text{E}$  from (Nec<sub>A</sub>) and (RE<sub>E</sub>), and for  $\text{A}$  from (Nec<sub>A</sub>) and (K<sub>A</sub>). Now for the inductive step, suppose  $\vdash_{\mathbb{L}_{\text{top}}} \varphi \leftrightarrow u(t(\varphi))$ .

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- S: Note that

$$u(t(S\varphi)) = u(\neg K\neg t(\varphi)) = \neg\neg S\neg\neg u(t(\varphi)).$$

By the inductive hypothesis, propositional logic and replacement of equivalents,  $\vdash_{L_{top}} S\varphi \leftrightarrow u(t(S\varphi))$ .

- E: We have

$$\begin{aligned} u(t(E\varphi)) &= u(A(\neg t(\varphi) \rightarrow K\neg t(\varphi))) \\ &= Au(\neg t(\varphi) \rightarrow K\neg t(\varphi)) \\ &= A(u(\neg t(\varphi)) \rightarrow u(K\neg t(\varphi))) \\ &= A(\neg u(t(\varphi)) \rightarrow \neg S\neg u(\neg t(\varphi))) \\ &= A(\neg u(t(\varphi)) \rightarrow \neg S\neg\neg u(t(\varphi))). \end{aligned}$$

Taking the contrapositive of the implication, and using replacement of equivalents together with the inductive hypothesis, we get

$$\vdash_{L_{top}} u(t(E\varphi)) \leftrightarrow A(S\varphi \rightarrow \varphi).$$

But we have already seen that  $\vdash_{L_{int}} E\varphi \leftrightarrow A(S\varphi \rightarrow \varphi)$ ; since  $L_{top}$  extends  $L_{int}$ , we get  $\vdash_{L_{top}} E\varphi \leftrightarrow u(t(E\varphi))$ .

- A: This case is straightforward by the inductive hypothesis and replacement of equivalents, since  $u(t(A\varphi)) = Au(t(\varphi))$ . □

Next we show that if  $\varphi \in \mathcal{L}_{KA}$  is a theorem of  $L_{S4A}$ , then  $u(\varphi)$  is a theorem of  $L_{top}$ .

**Claim 2** *Let  $\varphi \in \mathcal{L}_{KA}$ . Then  $\vdash_{L_{S4A}} \varphi$  implies  $\vdash_{L_{top}} u(\varphi)$ .*

*Proof* By induction on the length of  $L_{S4A}$  proofs. The base case consists of showing that if  $\varphi$  is an instance of an  $L_{S4A}$  axiom or a substitution instance of a propositional tautology, then  $\vdash_{L_{top}} u(\varphi)$ . The case for instances of tautologies is straightforward, since  $u$  does not affect the structure of a propositional formula. We take the axioms of  $L_{S4A}$  in turn.

- $(K_K)$ : We have

$$\begin{aligned} u(K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)) \\ &= \neg S\neg(u(\varphi) \rightarrow u(\psi)) \rightarrow (\neg S\neg u(\varphi) \rightarrow \neg S\neg u(\psi)) \\ &= \hat{S}(u(\varphi) \rightarrow u(\psi)) \rightarrow (\hat{S}u(\varphi) \rightarrow \hat{S}u(\psi)) \end{aligned}$$

which is an instance of  $(K_S)$ .

- $(T_K)$ : We have

$$u(K\varphi \rightarrow \varphi) = \neg S\neg u(\varphi) \rightarrow u(\varphi)$$

Taking the contrapositive, this is  $L_{top}$ -provably equivalent to  $\neg u(\varphi) \rightarrow S\neg u(\varphi)$ , which is an instance of  $(T_S)$ .

- $(4_K)$ : We have

$$u(K\varphi \rightarrow KK\varphi) = \neg S\neg u(\varphi) \rightarrow \neg S\neg\neg S\neg u(\varphi)$$

This is provably equivalent to  $SS\neg u(\varphi) \rightarrow S\neg u(\varphi)$ , which is an instance of  $(4_S)$ .

- $(K_A)$ : We have

$$u(A(\varphi \rightarrow \psi) \rightarrow (A\varphi \rightarrow A\psi)) = A(u(\varphi) \rightarrow u(\psi)) \rightarrow (Au(\varphi) \rightarrow Au(\psi))$$

which is an instance of  $(K_A)$  in  $L_{top}$ .

- (T<sub>A</sub>): We have

$$u(A\varphi \rightarrow \varphi) = Au(\varphi) \rightarrow u(\varphi)$$

which is an instance of (T<sub>A</sub>) in  $\mathbf{L}_{\text{top}}$ .

- (5<sub>A</sub>): We have

$$u(\neg A\varphi \rightarrow A\neg A\varphi) = \neg Au(\varphi) \rightarrow A\neg Au(\varphi)$$

which is an instance of (5<sub>A</sub>) in  $\mathbf{L}_{\text{top}}$ .

- (Inc<sub>K</sub>): We have

$$u(A\varphi \rightarrow K\varphi) = Au(\varphi) \rightarrow \neg S\neg u(\varphi) = Au(\varphi) \rightarrow \hat{S}u(\varphi)$$

which is an instance of (Inc).

For the inductive step, we show that for each inference rule  $\frac{\psi_1, \dots, \psi_n}{\varphi}$ , if  $\vdash_{\mathbf{L}_{\text{top}}} u(\psi_i)$  for each  $i$  then  $\vdash_{\mathbf{L}_{\text{top}}} u(\varphi)$ .

- (Nec<sub>A</sub>): If  $\vdash_{\mathbf{L}_{\text{top}}} u(\varphi)$ , then from (Nec<sub>A</sub>) in  $\mathbf{L}_{\text{top}}$  we get  $\vdash_{\mathbf{L}_{\text{top}}} Au(\varphi)$ . But  $Au(\varphi) = u(A\varphi)$ , so we are done.
- (MP): Similarly, this clear from (MP) for  $\mathbf{L}_{\text{top}}$  and the fact that  $u(\varphi \rightarrow \psi) = u(\varphi) \rightarrow u(\psi)$ .

□

Claims 1 and 2 easily imply the following.

**Claim 3** *Let  $\varphi \in \mathcal{L}$ . Then  $\vdash_{\mathbf{L}_{\text{S4A}}} t(\varphi)$  implies  $\vdash_{\mathbf{L}_{\text{top}}} \varphi$ .*

*Proof* Suppose  $\vdash_{\mathbf{L}_{\text{S4A}}} t(\varphi)$ . By Claim 2,  $\vdash_{\mathbf{L}_{\text{top}}} u(t(\varphi))$ . By Claim 1,  $\vdash_{\mathbf{L}_{\text{top}}} \varphi \leftrightarrow u(t(\varphi))$ . By (MP),  $\vdash_{\mathbf{L}_{\text{top}}} \varphi$ .

□

We can now show strong completeness. Suppose  $\Gamma \subseteq \mathcal{L}$ ,  $\varphi \in \mathcal{L}$  and  $\Gamma \models_{\mathbf{M}_{\text{top}}} \varphi$ . We claim  $t(\Gamma) \models_{\mathbf{M}_{\text{S4}}^*} t(\varphi)$ . Indeed, if  $M^* \in \mathbf{M}_{\text{S4}}^*$  and  $x$  is a state in  $M^*$  with  $M^*, x \models t(\psi)$  for all  $\psi \in \Gamma$ , then with  $f$  as in Theorem 3 we have  $f^{-1}(M^*), x \models \psi$  for all  $\psi \in \Gamma$ . Since  $f^{-1}(M^*) \in \mathbf{M}_{\text{int}} \cap \mathbf{M}_{\text{unions}} \subseteq \mathbf{M}_{\text{top}}$ ,  $\Gamma \models_{\mathbf{M}_{\text{top}}} \varphi$  gives  $f^{-1}(M^*), x \models \varphi$ , and thus  $M^*, x \models t(\varphi)$ .

By (strong) completeness of  $\mathbf{L}_{\text{S4A}}$  for  $\mathbf{M}_{\text{S4}}^*$ , we get  $t(\Gamma) \vdash_{\mathbf{L}_{\text{S4A}}} t(\varphi)$ . That is, there are  $\psi_0, \dots, \psi_n \in \Gamma$  such that  $\vdash_{\mathbf{L}_{\text{S4A}}} t(\psi_0) \wedge \dots \wedge t(\psi_n) \rightarrow t(\varphi)$ . Since  $t$  passes over conjunctions and implications, this means  $\vdash_{\mathbf{L}_{\text{S4A}}} t(\psi_0 \wedge \dots \wedge \psi_n \rightarrow \varphi)$ . By Claim 3,  $\vdash_{\mathbf{L}_{\text{top}}} \psi_0 \wedge \dots \wedge \psi_n \rightarrow \varphi$ . Hence  $\Gamma \vdash_{\mathbf{L}_{\text{top}}} \varphi$ , and we are done.

□

*Proof of Theorem 7* For soundness, we need to check that (5<sub>S</sub>) is valid on  $\mathbf{M}_{\text{int}} \cap \mathbf{M}_{\text{compl}}$ . Let  $M = (X, P, V)$  be closed under intersections and complements, and suppose  $M, x \models S\neg S\varphi$ . Note that  $\|S\varphi\|_M = \bigcap \{A \in P \mid \|\varphi\|_M \subseteq A\}$  is an intersection from  $P$ , so  $\|S\varphi\|_M \in P$ . By closure under complements,  $\|\neg S\varphi\|_M \in P$  too. Hence  $M, x \models S\neg S\varphi \wedge E\neg S\varphi$ . By Proposition 1 (4), we get  $M, x \models \neg S\varphi$ .

The completeness proof goes in exactly the same way as Theorem 6. Letting  $\mathbf{L}_{\text{S5A}}$  be the extension of  $\mathbf{L}_{\text{S4A}}$  with the (5<sub>K</sub>) axiom  $\neg K\varphi \rightarrow K\neg K\varphi$ , it can be shown that  $\mathbf{L}_{\text{S5A}}$  is strongly complete with respect to  $\mathbf{M}_{\text{S5}}^*$ . With  $u$  as in the proof of Theorem 6, we have that  $\vdash_{\mathbf{L}_{\text{S5A}}} \varphi$  implies  $\vdash_{\mathbf{L}_{\text{int-compl}}} u(\varphi)$ , for  $\varphi \in \mathcal{L}_{\text{KA}}$  (the only new part to

check there is that  $u(\neg K\varphi \rightarrow K\neg K\varphi)$  is a theorem of  $\mathbf{L}_{\text{int-compl}}$ , but this follows from (5<sub>S</sub>). The remainder of the proof goes through as before, this time appealing to the bijection  $g : \mathbb{M}_{\text{int}} \cap \mathbb{M}_{\text{compl}} \rightarrow \mathbb{M}_{\text{SS}}^*$ .  $\square$

*Proof of Proposition 5* Let  $M^* = (X, \{R_j\}_{j \in \mathcal{J}}, V)$  be a multi-source relational model. Since  $K_J^{\text{com}}\psi \rightarrow K_J^{\text{sh}}\psi$  is valid for any  $\psi$ , the left-to-right implication of the above equivalence is straightforward.

For the right-to-left implication, suppose  $M^*, x \models A(\neg\varphi \rightarrow K_J^{\text{sh}}\neg\varphi)$ . We show by induction that  $M^*, x \models A(\neg\varphi \rightarrow K_J^n\neg\varphi)$  for all  $n \in \mathbb{N}$ , from which the result follows.

The base case  $n = 1$  is given, since  $K_J^1\neg\varphi = K_J^{\text{sh}}\neg\varphi$ . For the inductive step, suppose  $M^*, x \models A(\neg\varphi \rightarrow K_J^n\neg\varphi)$ . Take  $y \in X$  such that  $M^*, y \models \neg\varphi$ . Let  $j \in J$ . Take  $z \in X$  such that  $yR_jz$ . From the initial assumption we have  $M^*, y \models K_J^{\text{sh}}\neg\varphi$ , so  $M^*, y \models K_j\neg\varphi$  and thus  $M^*, z \models \neg\varphi$ . By the inductive hypothesis,  $M^*, z \models K_J^n\neg\varphi$ . This shows that  $M^*, y \models K_jK_J^n\neg\varphi$  for all  $j \in J$ , and thus  $M^*, y \models K_J^{n+1}\neg\varphi$ . Hence  $M^*, x \models A(\neg\varphi \rightarrow K_J^{n+1}\neg\varphi)$  as required.  $\square$

## References

Blackburn, P., De Rijke, M., Venema, Y. (2002). *Modal logic* (Vol. 53). Cambridge University Press.