# Proof of Eq. (9)

Eq. (8) is updated to encompass the FDI attack as

$$\widetilde{\mathrm{ACE}_i} = \alpha_i \cdot \boldsymbol{\psi}_i^\intercal \left(\widetilde{\Delta \mathbf{p}_T} + \mathbf{T}\widetilde{\mathbf{a}}\right) + \beta_i \cdot \widetilde{\Delta\omega_i}.$$

By replacing $\widetilde{\mathrm{ACE}_i}$ in Eq. (4) with the above equation, we have

$$\widetilde{\Delta p_{Mi}^Y} = -\frac{G_i^Y(s)T_i^Y(s)}{R_i^Y}\widetilde{\Delta\omega_i} - \frac{G_i^Y(s)T_i^Y(s)K_i}{s}\left(\alpha_i \cdot \boldsymbol{\psi}_i^\intercal \left(\widetilde{\Delta \mathbf{p}_T} + \mathbf{T}\widetilde{\mathbf{a}}\right) + \beta_i \cdot \widetilde{\Delta\omega_i}\right). \tag{15}$$

Let $\mathbf{I}$ denote the $|\mathbb{A}| \times |\mathbb{A}|$ identity matrix. We define

$$\begin{aligned}
n_i &= \frac{1}{s} + \frac{\alpha_i K_i G_i^Y(s)T_i^Y(s)}{s^2},\\
q_i &= -M_i s - D_i - \frac{G_i^Y(s)T_i^Y(s)}{R_i^Y} - \frac{G_i^N(s)T_i^N(s)}{R_i^N} - \frac{\beta_i K_i G_i^Y(s)T_i^Y(s)}{s},\\
\mathbf{Q} &= \mathrm{diag}\left(q_1, q_2, \ldots, q_{|\mathbb{A}|}\right),\\
\widetilde{\Delta\boldsymbol{\omega}} &= \left[\widetilde{\Delta\omega_1}, \widetilde{\Delta\omega_2}, \ldots, \widetilde{\Delta\omega_{|\mathbb{A}|}}\right]^\intercal,\\
\boldsymbol{\Lambda} &= \mathrm{diag}\left(sn_1 - 1, sn_2 - 1, \ldots, sn_{|\mathbb{A}|} - 1\right),\\
\boldsymbol{\Gamma} &= \boldsymbol{\Lambda} + \mathbf{I},\\
\boldsymbol{\Psi} &= [\boldsymbol{\psi}_1^\intercal; \ldots; \boldsymbol{\psi}_{|\mathbb{A}|}^\intercal].
\end{aligned}$$

By replacing $\widetilde{\Delta p_{Mi}^N}$, $\widetilde{\Delta p_{Mi}^Y}$, $\widetilde{p_{Ei}}$ in Eq. (5) with Eq. (3), Eq. (15), and Eq. (6), we have

$$q_i\widetilde{\Delta\omega_i} = \widetilde{\Delta p_i} + sn_i\boldsymbol{\psi}_i^\intercal\widetilde{\Delta\mathbf{p}_T} + (sn_i - 1)\boldsymbol{\psi}_i^\intercal\mathbf{T}\widetilde{\mathbf{a}}, \tag{16}$$

By combing Eq. (16) for $1 \le i \le |\mathbb{A}|$, we have

$$\mathbf{Q}\widetilde{\Delta\boldsymbol{\omega}} = \widetilde{\Delta\mathbf{p}} + \boldsymbol{\Gamma}\boldsymbol{\Psi}\widetilde{\Delta\mathbf{p}_T} + \boldsymbol{\Lambda}\boldsymbol{\Psi}\mathbf{T}\widetilde{\mathbf{a}}. \tag{17}$$

From Eq. (7), each element of $\Delta\mathbf{p}_T$ is a linear combination of some grid frequency deviations. Thus, there exists a matrix $\mathbf{B}$ such that

$$\widetilde{\Delta\mathbf{p}_T} = \mathbf{B}\widetilde{\Delta\boldsymbol{\omega}}. \tag{18}$$

For instance, for the three-area grid in Fig. 1,

$$\mathbf{B} = \begin{bmatrix} \frac{T_{12}}{s} & -\frac{T_{12}}{s} & 0 \\ 0 & \frac{T_{23}}{s} & -\frac{T_{23}}{s} \\ -\frac{T_{31}}{s} & 0 & \frac{T_{31}}{s} \end{bmatrix}.$$

By replacing $\widetilde{\Delta\mathbf{p}_T}$ in Eq. (17) with Eq. (18), we can solve $\widetilde{\Delta\boldsymbol{\omega}}$ as

$$\widetilde{\Delta\boldsymbol{\omega}} = \boldsymbol{\Phi}^{-1}\widetilde{\Delta\mathbf{p}} + \boldsymbol{\Phi}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Psi}\mathbf{T}\widetilde{\mathbf{a}},$$

where $\boldsymbol{\Phi} = \mathbf{Q} - \boldsymbol{\Gamma}\boldsymbol{\Psi}\mathbf{B}$. Thus, the average grid frequency deviation, denoted by $\widetilde{\Delta\omega}$ is given by

$$\widetilde{\Delta\omega} = \boldsymbol{\theta}^\intercal\boldsymbol{\Phi}^{-1}\widetilde{\Delta\mathbf{p}} + \boldsymbol{\theta}^\intercal\boldsymbol{\Phi}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Psi}\mathbf{T}\widetilde{\mathbf{a}},$$

where $\boldsymbol{\theta} = \frac{1}{|\mathbb{A}|} \cdot [1, 1, \ldots, 1]^\intercal$.