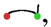
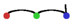
[**Ba**] relates strongly with our work; we plan to discuss it in (the revision's analogue of) §5.1, 3.2.

Locating [Ba] in our theory. [**Ba**]'s Thm3.1 computes order-η^2 weight displacements $\theta_T - \theta_0$ in the noiseless case $l_x = l$. The relevant diagrams are thus those with ≤ 2 edges and that contain no fuzzy outlines. Indeed, noiseless $\implies$ cumulants vanish $\implies$ any diagram that contains one or more fuzzy outlines has a uvalue (and rvalue) equal to zero. So a sum over diagrams is the same as a sum over fuzzless diagrams, i.e., over each diagram whose partition (Pg5Def1) is maximally fine.

Per §A.6, we use 'rootless' diagrams, e.g. , . These diagrams look different from ordinary ones because we are computing weight displacements $\Delta_l \triangleq \mathbb{E}[\theta_T - \theta_0]$, not test losses $\mathbb{E}[l(\theta_T)]$. Of course, in the noiseless case, those expectation symbols are redundant. Likewise, in the noiseless case Δ_l is a function only of η, T (and of the loss landscape l and the initialization θ_0); in particular, we may set E, B as convenient. Let's set $E = B = 1$.

GD's displacement. So, we seek rootless fuzzless diagrams width ≤ 2 edges. and are the only such. Let's use their uvalues as in Pg36Thm3 to compute $\Delta_l(T, \eta)$. We read off:

uvalue( ) $= G_\mu \eta^{\mu\nu} = hG$ uvalue( ) $= G_\mu \eta^{\mu\sigma} H_{\sigma\rho} \eta^{\rho\nu} = h^2(HG)$

The RHSs of the above concretize to the case that $\eta^{\mu\sigma}$ (in our directionality-aware theory a symmetric bilinear form that takes two covectors and outputs a scalar) is h times the standard dot product and that G, H are represented in standard ways as matrices. The diagrams embed (into an $E = B = 1$ grid that looks like the rightmost grid on Pg18) in T and in $\binom{T}{2}$ many ways, respectively.[1] The $\binom{T}{2}$ arises due to Pg19's time-ordering condition: has one embedding for every pair $0 \leq t < t' < T$, where t is the red node's column and t' is the green node's column.

These embeddings have trivial Aut groups (Pg28Exm5), so any fixed T has a grand total:

$$\Delta_l(T, h) = -hTGhT + (h^2(T^2 - T)/2)HG + o(\eta^2)$$

[Ba]'s regularizer. Since EulerMethod (EM) (simulation time h, k steps) is just GD with $\eta = h/k$, $T = k$, we can use $\Delta_{\tilde{l}}(k, h/k)$ to predict EM's behavior —and hence ODE's behavior— on a loss $\tilde{l}$. For k huge and η tiny (in a way that depends on k), $\Delta_{\tilde{l}}(k, h/k)$ is close to

$$\star(h) = -h\tilde{G} + (\tilde{H}\tilde{G})h^2/2$$

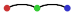
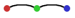
(I.e., $\tilde{l}$ analytic $\implies \forall \epsilon\ \exists k_0 \forall k > k_0\ \forall A > 0\ \exists h_0 > h_0 \forall h < h_0$: $\|\Delta_{\tilde{l}}(k, h/k) - \star(h)\| < Ah^2 + \epsilon$.)

To match $\star$ with ordinary GD's one-step displacement $\Delta_l(1, h) = -hG$, we just need $hG = h\tilde{G} - (\tilde{H}\tilde{G})h^2/2 + o(h^2)$; it's enough to set $G = \tilde{G} + (\tilde{H}\tilde{G})h/2$. Recognizing the RHS as a total derivative (as $\nabla(\tilde{G} \cdot \tilde{G}) = 2\tilde{H}\tilde{G}$), we see it's enough that $G = \nabla(\tilde{l} - (h/4)(\tilde{G} \cdot \tilde{G}))$ or:

$$\begin{aligned} l &= \tilde{l} - (h/4)(\tilde{G} \cdot \tilde{G}) \\ &= \tilde{l} - (h/4)(G \cdot G) + o(h^2) \end{aligned}$$

This shows how to turn a loss $\tilde{l}$ (on which we plan to run ODE), into a loss l such that running one GD step on l matches ODE on $\tilde{l}$ to leading non-trivial order. Or how to turn l into $\tilde{l}$. In either case, the key term is $(h/4)(G \cdot G)$ with the appropriate sign.

[1] An embedding of a rootless diagram (e.g.) assigns *every* node to a grid cell. Pg19 decrees that we assign only *non-root* nodes when computing $\mathbb{E}[l(\theta_T)]$; indeed, the root node represents the test-time factor l and thus corresponds to no training point or training step. By contrast, every factor of every term in $\mathbb{E}[\theta_T - \theta_0]$ corresponds to some training point n and training step t. So we assign *all* nodes to grid cells. We'll expand §A.6 to note as much.

Mistakes in our work. We've found several oversights in our paper. E.g.: our Cor3 fails to state that the computed loss difference between SGD and ODE is the leading-order difference *due to noise*, i.e., that scales with some higher cumulant such as C. Of course, even without noise, there is also a difference due to time discretization, given by 's embeddings into ODE's $T = kT_0$ grid minus its embeddings into SGD's $T = T_0$ grid. For large k, $(h/k)^2\binom{kT_0}{2} \approx h^2T_0^2/2$, so ODE suffers $(T_0^2/2 - \binom{T_0}{2})$uvalue() $= (h^2T_0/2)(GHG)$ more loss than SGD for noiseless loss landscapes.

And Tab1's caption should clarify the same point.