Efficient Privacy-preserving Vertical Logistic Regression under Two Security Definitions

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Abstract. Privacy-preserving vertical logistic regression allows multiple data owners to collaboratively build a logistic regression (LR) model over vertically partitioned data without disclosing sensitive information. It has been implemented under two security definitions, namely infinite solution security and provable security, which offer distinct advantages in terms of efficiency and security, respectively. In this paper, we focus on solving potential security concerns under infinite solution security and optimizing the efficiency of model training under two security definitions. We first present an efficient vertical LR protocol with infinite solution security ($ISS\text{-}VLR$). In $ISS\text{-}VLR$, we establish two rigorous security constraints to avoid the potential risk of privacy leakage in existing protocols during model training and inference, respectively. Moreover, we propose a random mask-based privacy-preserving stochastic gradient descent ($RM\text{-}SGD$), which utilizes random masks to efficiently perform model updates without leaking intermediate results. Then, we present an efficient vertical LR protocol with provable security ($PS\text{-}VLR$). In $PS\text{-}VLR$, we adopt matrix triples to achieve model training and inference satisfying provable security in a communication-efficient manner and propose a rotation-free Single-Instruction Multiple-Data (SIMD) based matrix-vector multiplication method ($RS\text{-}MVM$) to accelerate matrix triple generation. Experimental results show that $ISS\text{-}VLR$ and $PS\text{-}VLR$ can be more efficient than state-of-the-art protocols under infinite solution security and provable security, respectively.

Keywords: Logistic regression · Privacy-preserving machine learning · Vertically partitioned data.

1 Introduction

Logistic regression (LR) is a popular machine learning model in data analysis and data mining. It is generally trained iteratively by stochastic gradient descent (SGD), where each iteration contains two stages, namely forward computation and backward updating. To train an LR model with high accuracy, it is crucial to have sufficient high-quality data. In many cases, data are distributed vertically among multiple data owners, especially in the business-to-business context. That is, all data owners hold disjoint features but overlap on a subset of the samples. However, due to the presence of sensitive information [32], data owners are reluctant to share the data, which is not conducive to building useful models. To

address this issue, various privacy-preserving vertical LR protocols are proposed to facilitate collaborative LR model building among multiple data owners over vertically partitioned data without compromising privacy.

Secure multi-party computation (MPC) allows multiple involved parties to collaboratively evaluate a function without exposing the input data and intermediate results. It has been employed as a foundational technique to build privacy-preserving vertical LR protocols. Nevertheless, MPC incurs large computational and communication overheads, leading to obstacles in practical applications. To improve efficiency, a part of the protocols [15,33,34,31,32] choose to expose several intermediate results during forward computation. For such exposure, the non-violation of privacy is guaranteed by making it possible for an adversary to deduce an infinite amount of possible sensitive information using exposed information. We call this privacy guarantee as infinite solution security and refer to these protocols as vertical LR protocols with infinite solution security. For other protocols [11,7,10,22,23,9] where all intermediate results are protected using MPC, we refer to them as vertical LR protocols with provable security. These two categories of protocols can offer greater efficiency and stronger security, respectively. In practical applications, based on the actual requirements for security and efficiency, selecting the appropriate category of protocols facilitates gaining the desired trade-off between privacy and utility. However, both categories of protocols still have drawbacks in terms of privacy or efficiency.

Existing protocols with infinite solution security [15,33,34,31,32] have set a security constraint to provide infinite solution security for the exposure of intermediate results in forward computation. However, after carefully reviewing the training process, we found that existing studies have neglected additional information that may be available to involved parties when establishing security constraints, leading to the invalidation of these security constraints. For existing protocols with provable security [11,7,10,22,23,9], most of them employ secret sharing (SS) to perform model training and inference, while facing the large communication overhead introduced by SS. Although the matrix triple technique [28,6] can alleviate this problem, it only supports two-party settings and is generated by performing homomorphic ciphertext computations with high computational overhead.

To address the above problems, we first present an efficient **V**ertical **LR** protocol with **I**nfinite **S**olution **S**ecurity, which is referred to as $ISS\text{-}VLR$. In $ISS\text{-}VLR$, taking into account the additional information that can be available to involved parties, we establish rigorous security constraints to ensure that the exposure of intermediate results satisfies infinite solution security during forward computation. Moreover, considering that forward computation is carried out with infinite solution security, we argue that providing infinite solution security, rather than provable security, for backward updating is sufficient and can significantly improve training efficiency. Therefore, we propose a **R**andom **M**ask-based privacy-preserving **SGD**, which is referred to as $RM\text{-}SGD$. In $RM\text{-}SGD$, we utilize random masks to guarantee the confidentiality of intermediate results from all parties and exploit the homomorphism property of random masks to

enable efficient backward updating. As a result, $ISS\text{-}VLR$ can achieve training efficiency comparable to non-private LR.

Then, we present an efficient **V**ertical **LR** protocol with **P**rovable **S**ecurity, which is referred to as $PS\text{-}VLR$. In $PS\text{-}VLR$, we extend the matrix triple technique to multi-party settings to reduce the communication overhead introduced by SS. Moreover, since homomorphic matrix-vector multiplication is the core operation for matrix triple generation, we propose a **R**otation-free **S**IMD-based **M**atrix-**V**ector **M**ultiplication method, called $RS\text{-}MVM$, to improve the generation efficiency. In $RS\text{-}MVM$, we exploit the parallel computing capability of Single-Instruction Multiple-Data (SIMD) operations and decompose the homomorphic matrix-vector multiplication into multiple SIMD-based homomorphic multiplications and additions. As a result, we can simultaneously generate multiple matrix triples during a single execution.

In summary, we make the following contributions:

- We present $ISS\text{-}VLR$, a vertical LR protocol with infinite solution security. In $ISS\text{-}VLR$, we establish rigorous security constraints to safeguard the privacy during forward computation and propose $RM\text{-}SGD$ to perform efficient and privacy-preserving backward updating.
- We present $PS\text{-}VLR$, a vertical LR protocol with provable security. In $PS\text{-}VLR$, we use matrix triples to reduce the communication overhead introduced by SS and propose $RS\text{-}MVM$ to improve the efficiency of matrix triple generation.
- We conduct the experiments using both real and synthetic datasets. The results show that $ISS\text{-}VLR$ and $PS\text{-}VLR$ can achieve higher efficiency than state-of-the-art vertical LR protocols under infinite solution security and provable security, respectively.

Roadmap. Section 2 provides the preliminaries. Section 3 and Section 4 describe the details of $ISS\text{-}VLR$ and $PS\text{-}VLR$, respectively. Section 5 shows our experimental results. Section 6 reviews the related work. Section 7 concludes this paper.

2 Preliminaries

In this section, we briefly present our system and threat models, as well as necessary background knowledge.

2.1 System Model

We focus on building a logistic regression (LR) model over vertically partitioned data in a p-party setting ($p > 2$) without relying on additional server assistance. The parties are categorized into two types: one active party and many passive parties. Each party P_i holds a local dataset $D_i = (ID_i, X_i)$, where ID_i is an individual identifier and X_i is the feature set. Only the active party holds the

labels $Y = \{y_i\}_{i=1}^n$, where $y_i \in \{0, 1\}$ and n is the number of samples. For any two local datasets D_i and D_j, there are disjoint features and a common subset of samples. To allow collaborative training, we assume that all parties have aligned their common samples $ID = ID_1 \cap \cdots \cap ID_p$ via private set intersection [3] prior to model training.

2.2 Threat Model and Security Definition

We consider the setting where an adversary is semi-honest and can corrupt up to p-1 of the p parties. Within this setting, the adversary adheres to the protocol and aims to extract sensitive information from the messages received by the corrupted parties. The corruption strategy is static, where the set of corrupted clients remains unchanged during the execution of protocols. Our proposed protocols are built on two different security definitions, namely infinite solution security and provable security. In particular, the infinite solution security [32,15] is defined in Definition 1. For the provable security, we follow the definition of that for static semi-honest adversaries in [17], as shown in Definition 2.

Definition 1. *(Infinite Solution Security) Let $\Pi(X, I, Y)$ be a multi-party protocol, where X denotes the input data of all parties, I denotes the intermediate results generated during the execution of Π, and Y denotes the final output of Π. We say that Π satisfies infinite solution security if the system of equations constructed from any $Z \subseteq \{I, Y\}$ has an infinite number of solutions.*

Definition 2. *(Provable Security) Let $\mathcal{F} = \{f_1, \ldots, f_p\}$ be an ideal functionality and Π be a p-party protocol for computing $\mathcal{F}$. Given input data $\bar{x} = (x_1, \ldots, x_p)$, where x_i is from P_i, the view of P_i during the execution of Π is denoted by $\text{view}_i^{\Pi}(\bar{x})$, and the outputs of all parties during the execution of Π are denoted by $\text{output}^{\Pi}(\bar{x})$. We say that Π securely computes $\mathcal{F}$ if 1) for any $I = \{i_1, \ldots, i_t\} \subseteq [p] \overset{def}{=} \{1, \ldots, p\}$, there exists a probabilistic polynomial-time simulator $\mathcal{S}$ such that*

$$\begin{aligned} &\mathcal{S}\left(I, \left(x_{i_1}, \ldots, x_{i_t}\right), \left(f_{i_1}(\bar{x}), \ldots, f_{i_t}(\bar{x})\right)\right) \\ &\equiv \left(\text{view}_{i_1}^{\Pi}(\bar{x}), \ldots, \text{view}_{i_t}^{\Pi}(\bar{x})\right) \end{aligned};$$

2) the outputs of the ideal functionality and the protocol satisfy $\mathcal{F}(\bar{x}) \equiv \text{output}^{\Pi}(\bar{x})$.

2.3 Logistic Regression

LR is an efficient linear model using the logistic function for binary classification problems. The standard logistic function is a sigmoid function, which takes any input x and outputs a value between zero and one. Given a dataset $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n$, the training objective of LR is to find a set of parameters $\theta \in \mathbb{R}^{m \times 1}$ that minimize the empirical loss function as follows:

$$\min_{\theta} f(\theta) := \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}\left(\sigma(\theta^{\top} x_i), y_i\right) + \lambda \sum_{j=1}^{m} g\left(\theta_j\right), \tag{1}$$

where $x_i \in \mathbb{R}^{m\times 1}$ is the feature vector of the i-th sample, $y_i \in \{0,1\}$ is the true label of the i-th sample, $\theta^\top x_i = \sum_{j=1}^{m} \theta_j x_{i,j}$ is the linear function, $\mathcal{L}$ denotes the loss function, $\sum_{j=1}^{m} g(\theta_j)$ is the regularization term, and λ is a constant. Stochastic gradient descent (SGD) is widely used to train the LR model, which iteratively updates the model parameters in the following manner:

$$\theta_j^{t+1} = \theta_j^t - \alpha \nabla \mathcal{L}\left(\theta_j^t\right), \tag{2}$$

where θ_j^t is the j-th parameter of the t-th iteration, α is the learning rate, and $\nabla\mathcal{L}$ is the gradient of $\mathcal{L}$. In general, the log loss is used to be the loss function, whose gradient is represented as $\nabla \mathcal{L}\left(\theta_j^t\right) = \frac{1}{n}\sum_{i=1}^{n} r_i^t x_{i,j}$, where $r_i^t = \hat{y}_i^t - y_i$ is the residual of the i-th sample in the t-iteration.

2.4 Secret Sharing

Secret sharing (SS) is a popular MPC technique. An SS scheme involves a dealer who breaks a secret value into shares and distributes these shares to a group of parties. SPD$\mathbb{Z}_{2^k}$ [9,10] is a commonly used SS framework, which securely computes an arithmetic circuit over an integer ring $\mathbb{Z}_{2^k}$. It has two kinds of SS schemes, including arithmetic SS and binary SS, and supports the conversion between these two schemes. In the following text, given an input x, its arithmetic SS is denoted as $\langle x\rangle^A$, and its binary SS is denoted as $\langle x\rangle^B$. The frequently used primitives in SPD$\mathbb{Z}_{2^k}$ are shown as follows:

(1) **Addition** $\langle z\rangle^A = \langle x\rangle^A + \langle y\rangle^A$. P_i computes $\langle z\rangle_i^A \equiv \langle x\rangle_i^A + \langle y\rangle_i^A (mod\ 2^k)$ locally. If a certain party wants to reconstruct a secret z, the remaining parties send their shared values of z to that party, and the secret $z \equiv \sum_i \langle z\rangle_i^A (mod\ 2^k)$ is constructed.

(2) **Multiplication** $\langle z\rangle^A = \langle x\rangle^A \cdot \langle y\rangle^A$. The multiplication of two shared values relies on Beaver triples [2], which is generated by oblivious transfer [22,9] or homomorphic encryption [11,23] in the offline phase. Given a triple ($\langle u\rangle^A$, $\langle v\rangle^A$, $\langle w\rangle^A$), where u, v are random values in $\mathbb{Z}_{2^k}$ and $w \equiv uv (mod\ 2^k)$, the multiplication between two shared values can be performed with one round of intersection in the online phase.

(3) **The Most Significant Bit Extraction** Π_{MSB} . The parties compute the shared values of the most significant bit of x. This primitive can be used to perform a secure comparison of two signed integers. Specifically, given $\langle x\rangle^A$ and $\langle y\rangle^A$, we first calculate $\langle x-y\rangle^A$ and then extract the most significant bit of $\langle x-y\rangle^A$ as the result of the comparison.

2.5 Threshold Fully Homomorphic Encryption

Threshold fully homomorphic encryption (Threshold FHE) allows multiple parties to evaluate arithmetic circuits directly on encrypted data, as opposed to having to decrypt the data first. In its scheme, each party holds a distinct secret key and engages in interactions to generate a unified public key. The unified

public key is used for encryption, and the secret keys are used for distributed decryption. Threshold CKKS [8] is a commonly used threshold FHE scheme, which supports addition and a pre-determined number of multiplications on encrypted data. Threshold CKKS [8,1] consists of the following primitives:

(1) **Key generation**. Each party first generates its own secret key sk_i and public key pk_i. Then, all parties exchange their public key for computing a unified public key $pk = (a, pk_1 + pk_2 + ... + pk_n)$, where a is a uniform ring element.

(2) **Encryption**. A given complex number x is encoded to an integral polynomial $\mathcal{P} \leftarrow Ecd(x; \Delta)$, where Δ is a scaling factor, and encrypted into a ciphertext $[\![x]\!] \leftarrow Enc_{pk}(\mathcal{P})$.

(3) **Decryption**. A ciphertext $[\![x]\!]$ is decrypted into an integral polynomial $\mathcal{P}' \leftarrow Dec_{sk}([\![x]\!])$, and decoded to the corresponding complex number $x \leftarrow Dcd(\mathcal{P}'; \Delta)$.

(4) **Operations**. Given two ciphertexts $[\![x]\!]$ and $[\![y]\!]$, there are two types of arithmetic operations, **Addititon**: $[\![x]\!] + [\![y]\!] = [\![x + y]\!]$, and **Multiplication**: $[\![x]\!] \cdot [\![y]\!] = [\![x \cdot y]\!]$. To improve computational efficiency, FHE supports the Single-Instruction Multiple-Data (SIMD) operations, which pack a vector of N elements into a ciphertext by assigning each element to a slot within the ciphertext, and perform a single homomorphic operation on all elements of this vector in parallel. In particular, the number of slots is half of the degree of integral polynomials.

3 ISS-VLR

In this section, we elaborate on $ISS\text{-}VLR$ for model training and inference. Due to space limitations, the protocol flow is given in Appendix B, while the analysis of efficiency and security is given in Appendix C.

3.1 Overview

We present an efficient **V**ertical **LR** protocol with **I**nfinite **S**olution **S**ecurity ($ISS\text{-}VLR$), whose training and inference processes are described as follows:

Training: During forward computation, each party first computes the local linear-functional outputs. Then, the active party gathers the local linear-functional outputs from other parties and evaluates the sigmoid function to obtain the model outputs. To make the exposure of local linear-functional outputs satisfy infinite solution security, we establish a security constraint on the number of epochs (See details in Section 3.2). For backward updating, we propose a **R**andom **M**asks-based privacy-preserving **SGD** ($RM\text{-}SGD$) to securely and efficiently perform the gradient calculation and model parameter update (See details in Section 3.3). When model training is stopped, each party obtains the masked model parameters that correspond to their input features.

Inference: The model inference process is the same as the process of forward computation in model training. We establish a security constraint on the number of continuous features for providing infinite solution security (See details in Section 3.2).

3.2 Security Constraints in Forward Computation

During forward computation, the active party can obtain linear-functional outputs in plaintext form. To ensure such exposure satisfies infinite solution security, existing studies [15,33,34,31,32] have derived a security constraint, under which the system of equations constructed by the linear-functional outputs has infinitely many solutions. However, these studies ignore two pieces of information accessible to the active party engaged in collaborative modeling, which might invalidate the stated security constraints. On the one hand, the active party may request the value ranges of input features from the passive parties to assess whether these input features are suitable for collaborative modeling. On the other hand, since the model parameters are updated iteratively during the training process, the active party is aware of the linear correlation of model parameters across multiple iterations. These two pieces of information are essentially additional restrictions on the variables in the system of equations constructed by the linear-functional outputs. Therefore, by leveraging the information mentioned above, the active party receiving the linear-functional outputs may solve the unique solution of this system of equations.

To address this issue, we establish rigorous security constraints by formalizing these two pieces of information as equations and adding these equations into the system of equations constructed by the linear-functional outputs. Specifically, for model training, we first formalize the linear-functional outputs from the i-th passive party P_i in all iterations as a system of nonlinear equations $\mathcal{S}$ containing the model parameters and the input features. We then formalize the value ranges of input features as equations and add these equations into $\mathcal{S}$. Without loss of generality, suppose there are discrete and continuous features, the value range of the corresponding unknown variable x is a set or an interval, respectively. The set F_d can be formalized as $\prod_{a \in F_d}(x - a) = 0$. For the interval, it can take one of four forms: $x \geq a$, $x > a$, $x \leq a$, and $x < a$, where a is the endpoint. By introducing an artificial variable y, these forms can be formalized as $y^2 = x - a$, $e^y = x - a$, $y^2 = a - x$, and $e^y = a - x$, respectively. It is worth noting that the value ranges of other variables in $\mathcal{S}$, such as model parameters, are set by default to the real domain. Next, we formalize the correlation of model parameters across multiple iterations as equations. Starting with the correlation between the model parameters of two successive iterations, it can be formalized as Equation (2). By analogy, the correlation between $\theta_{i,j}^t$ and the initial model parameters $\theta_{i,j}^1$ can be formalized as follows:

$$\theta_{i,j}^t = \theta_{i,j}^1 - \frac{\alpha}{b} \sum_{l=1}^{t-1} \sum_{k=1}^{b} r_{B_l[k]} x_{B_l[k],j}, \tag{3}$$

where $r_{B_l[k]}$ is the residual of the k-th sample in the l-th batch B_l and α is the public learning rate. By adding Equation (3) into $\mathcal{S}$, all the model parameters, except for the initial model parameters, can be eliminated. We call the system of equations consisting of $\mathcal{S}$ and these additional equations as $\mathcal{S}'$.

By restricting the number of variables to be greater than the number of equations in $\mathcal{S}'$, we establish a security constraint on the number of epochs

e, such that e is less than the number of continuous features $m_{c,i}$, as shown in Theorem 1. Under this security constraint, $\mathcal{S}'$ is underdetermined and has infinitely many solutions. Due to space limitations, the proof of Theorem 1 is given in Appendix A. Note that, this security constraint only guarantees that the exposure of the linear-functional outputs from P_i satisfies infinite solution security. In order for the exposure of all local linear-functional outputs to satisfy infinite solution security, we require that e is less than the minimum value of $\{m_{c,i}\}_{i\in I_p}$, where I_p is the set of passive parties.

Theorem 1. *If the number of epochs e satisfies $e < m_{c,i}$, where $m_{c,i}$ is the number of continuous features held by P_i, $\mathcal{S}'$ is underdetermined.*

For model inference, the process of constructing a security constraint is close to that of model training. The only difference is that there is no need to consider the correlation of the model parameters. Due to space limitations, we directly give the security constraint, which requires all passive parties to hold at least two continuous features.

3.3 Backward Updating with Random Masks

During backward updating, existing protocols [31,32,15] employ public-key cryptography techniques, e.g., homomorphic encryption, resulting in significant computational overhead. Therefore, we propose a **R**andom **M**ask-based privacy-preserving **SGD** (*RM-SGD*), which guarantees infinite solution security for backward updating by adding or multiplying a non-zero random mask. *RM-SGD* mainly exploits the following homomorphic properties of random masks: 1) **Addition**: If a secret x is added to a random mask ε, we can perform the addition operation between $x+\varepsilon$ and an unmasked data y as ε can be eliminated from $x+y+\varepsilon$ by subtraction; 2) **Multiplication**: If a secret x is multiplied by a random mask ε, we can perform the multiplication operation between $x\cdot\varepsilon$ and an unmasked data y as ε can be eliminated from $x\cdot y\cdot\varepsilon$ by division; 3) **Inner Product**: If a secret vector v is multiplied by a random mask ε, we can perform the inner product operation between $v\cdot\varepsilon$ and an unmasked vector w as ε can be eliminated from $v\cdot w\cdot\varepsilon$ by division. Specifically, during the backward updating stage of each iteration, *RM-SGD* works as follows:

Step 1. Calculating Gradients. The active party employs the true labels and model outputs to calculate the residuals and further calculate the gradients corresponding to its input features. To enable the passive parties to perform the gradient calculation, involving an inner product, without exposing the residuals, the active party multiplies all the residuals by an identical random mask. Specifically, the active party P_1 first randomly generates a non-zero mask $\sigma^t \in \mathbb{R}$ and multiples the set of all residuals r^t by σ^t. Then, P_1 sends the masked residuals $\sigma^t r^t$ to the passive parties. Finally, each passive party P_i can compute the masked gradients $\mathcal{G}_i^t = \sigma^t G_i^t$, where G_i^t is the set of the gradients. In particular, in the binary classification task, the residuals generally take values within $(0, 1)$, and the situation where the multiplicative mask is invalid can be avoided.

Step 2. Updating Model Parameters. To enable passive parties to update model parameters using $\mathcal{G}_i^t$, which involves a subtraction operation, the passive parties collaborate with the active party to convert the multiplicative mask in $\mathcal{G}_i^t$ to an additive mask without exposing the gradients to the active party. Specifically, each passive party P_i first generates a random matrix mask $K_i^t \in \mathbb{R}^{m_i \times m_i}$, multiples $\mathcal{G}_i^t$ by K_i^t, and sends $K_i^t \mathcal{G}_i^t$ to P_1, where m_i is the number of features held by P_i. Then, P_1 divides $K_i^t \mathcal{G}_i^t$ by σ^t to obtain $K_i^t G_i^t$ and adds a new random vector mask $\mu_i^t \in \mathbb{R}^{m_i \times 1}$ to $K_i^t G_i^t$. Finally, after receiving $K_i^t G_i^t + \mu_i^t$, P_i updates θ_i^t as follows:

$$K_i^t \theta_i^{t+1} - \mu_i^t = K_i^t (\theta_i^t - G_i^t) - \mu_i^t. \tag{4}$$

Step 3. Post-processing of Model Parameters. The model parameters are not only used to calculate local linear-functional outputs, which involves a matrix-vector multiplication operation, but also required to be hidden from passive parties, as the model parameters can be used to invert the gradients. Therefore, the active party eliminates the additive mask in the model parameters and re-masks them by a multiplicative mask. Specifically, each passive party P_i sends $K_i^t \theta_i^{t+1} - \mu_i^t$ to P_1. Then, P_1 eliminates μ_i^t, multiplies $K_i^t \theta_i^{t+1}$ by a random mask $\varphi_i^{t+1} \in \mathbb{R}$, and sends it to P_i. After eliminating K_i^t, P_i obtains $\varphi_i^{t+1} \theta_i^{t+1}$. Note that, before adding μ_i^{t+1} to $K_i^{t+1} G_i^{t+1}$ during the next iteration, P_1 multiplies $K_i^{t+1} G_i^{t+1}$ by φ_i^{t+1} for updating $\varphi_i^{t+1} \theta_i^{t+1}$.

4 PS-VLR

In this section, we elaborate on $PS\text{-}VLR$ for model training and inference. Due to space limitations, the protocol flow is given in Appendix B, while the analysis of efficiency and security is given in Appendix C.

4.1 Overview

We present an efficient **V**ertical **LR** protocol with **P**rovable **S**ecurity ($PS\text{-}VLR$), whose training and inference processes are described as follows:

Training: All parties jointly perform both forward computation and backward updating by employing SS, where matrix triples are used to facilitate online matrix multiplication with low communication overhead and generated offline using threshold FHE (See details in Section 4.2). Moreover, since homomorphic matrix-vector multiplication is the efficiency bottleneck of matrix triple generation, we propose a **R**otation-free **S**IMD-based **M**atrix-**V**ector **M**ultiplication method ($RS\text{-}MVM$) to accelerate this operation (See details in Section 4.3). When model training is stopped, the model parameters are stored among all parties in an SS form.

Inference: All parties jointly compute the linear-functional outputs and approximately evaluate the sigmoid function to obtain the secretly shared model outputs, followed by reconstructing the model outputs.

4.2 Multi-party Matrix Triple

The core operation of LR training is matrix multiplication, such as linear function evaluation in forward computation and gradient calculation in backward updating. If SS-based matrix multiplication is implemented by using Beaver triple [2], it introduces a huge communication overhead. To minimize this overhead, SecureML [28], an SS framework for two-party settings, proposes a matrix triple, which can be substituted for the Beaver triple [2] to perform matrix multiplication with less communication overhead. Therefore, we extend the matrix triple to the multi-party setting.

Specifically, suppose that $\langle XY\rangle^A = \langle X\rangle^A \cdot \langle Y\rangle^A$ is to be computed, where $\langle X\rangle^A$ and $\langle Y\rangle^A$ are secretly shared matrices. Given a secretly shared matrix triple $\{\langle U\rangle^A, \langle V\rangle^A, \langle Z\rangle^A\}$, where $Z = UV$, all parties first collaboratively compute $\langle E\rangle^A = \langle X\rangle^A - \langle U\rangle^A$ and $\langle F\rangle^A = \langle Y\rangle^A - \langle V\rangle^A$, followed by opening $\langle E\rangle^A$ and $\langle F\rangle^A$. Then, each party P_i locally computes $\langle XY\rangle_i^A = -\mathcal{I}(i=1)\cdot E\cdot F + \langle X\rangle_i^A\cdot F + E\cdot\langle Y\rangle_i^A + \langle Z\rangle_i^A$, where $\mathcal{I}(i=1)$ indicates whether P_i is the active party. The matrix triple can be generated offline by using threshold FHE. All parties first randomly generate the secretly shared first two components of a matrix triple, i.e., $\langle U\rangle^A$ and $\langle V\rangle^A$. Then, all parties jointly reconstruct the encryptions of U and V, denoted as $[\![U]\!]$ and $[\![V]\!]$. Next, P_1 calculates $[\![Z]\!] = [\![U]\!]\cdot[\![V]\!]$, which involves a homomorphic matrix multiplication. After converting $[\![Z]\!]$ into secretly shared values, the secretly shared matrix triple $(\langle U\rangle^A, \langle V\rangle^A, \langle Z\rangle^A)$ is obtained.

4.3 Rotation-free Matrix-Vector Multiplication

The most time-consuming step in the matrix triple generation is the computation of $[\![Z]\!]$. Since U and V in the matrix triple serving LR training are respectively a matrix and a vector, an efficient homomorphic matrix-vector multiplication is crucial for achieving high efficiency in matrix triple generation. SIMD operations can perform a single homomorphic operation on multiple encrypted data simultaneously, and thus effectively improve the computational efficiency of homomorphic operations. However, existing homomorphic matrix-vector multiplication methods [16,21,20] cannot simultaneously utilize the benefits of SIMD operations and avoid the introduction of expensive rotation operations.

Therefore, we propose a **R**otation-free **S**IMD-based **M**atrix-**V**ector **M**ultiplication method (*RS-MVM*). Its main idea is to decompose matrix-vector multiplication into the product of each column of the matrix and each element of the vector, and the summation of all the products. These products and the summation can be efficiently computed by SIMD-based multiplication and addition, respectively. As a result, *RS-MVM* utilizes only SIMD-based multiplications and additions in a single matrix-vector multiplication without involving rotation operations. Moreover, if the number of the matrix's rows is smaller than the number of slots in a ciphertext, *RS-MVM* has the ability to perform multiple matrix-vector multiplications in parallel during a single execution by concatenating the involved matrices and vectors during encryption, thereby maximizing the

parallel computing capabilities of SIMD operations. Specifically, given a matrix $A \in \mathbb{R}^{N_r \times N_c}$ and a vector $B \in \mathbb{R}^{N_c \times 1}$, *RS-MVM* consists of two steps:

Step 1. Vector Transformation. Each element of B is first copied N_r times to obtain a replicated vector of dimension N_r. Then, all the replicated vectors are combined into a replicated matrix B' of dimension (N_r, N_c), in which the values of all elements in the i-th column are the same and equal to the value of the i-th element in B for $i \in [1, N_c]$.

Step 2. SIMD-based Computation. A and B' are first encrypted in a manner where each column is packed into a ciphertext to obtain $[\![A]\!]$ and $[\![B']\!]$. Then, the result of the multiplication between A and B can be obtained by computing $[\![A]\!] \circ [\![B']\!] = \sum_{j=1}^{N_c} [\![A]\!][:, j] \odot [\![B]\!][:, j]$, where $[\![A]\!][:, j]$ and $[\![B]\!][:, j]$ respectively are the j-th column of $[\![A]\!]$ and $[\![B]\!]$, and $\odot$ denotes the component-wise product. In particular, the component-wise product is calculated by the SIMD-based multiplication operation, and all the products are aggregated by the SIMD-based addition operation.

If N_r is smaller than the number of slots in a ciphertext N_s, *RS-MVM* can perform parallel computation of multiple matrix-vector multiplications $A_1 \cdot B_1, \ldots, A_k \cdot B_k$, where $k \leq N_s/N_r$. Specifically, all the vectors $\{B_i\}_{i=1}^k$ are first transformed into the replicated matrices $\{B_i'\}_{i=1}^k$. Then, the sets $\{A_i\}_{i=1}^k$ and $\{B_i'\}_{i=1}^k$ are concatenated into A^* and B^* along the row dimension, respectively. Next, A^* and B^* are encrypted and used to compute $[\![C]\!] = [\![A^*]\!] \circ [\![B^*]\!]$. After decrypting $[\![C]\!]$, the vector C can be split into k sub-vectors, and the i-th sub-vector is the result of $A_i \cdot B_i$. Otherwise, if $N_r > N_s$, the matrix A is sliced into $\zeta = \lceil \frac{N_r}{N_s} \rceil$ sub-matrices $A_1, \cdots, A_\zeta$ by rows, and the task of $A \cdot B$ is converted to subtasks of $A_1 \cdot B$, ..., $A_\zeta \cdot B$. Note that, since N_r is generally equal to either the batch size or the input feature dimension for model training and inference, we can leverage the parallel computing capability of *RS-MVM* to efficiently generate matrix triples.

5 Experiments

5.1 Experimental Setup

Datasets. We use four real datasets, including MNIST [24] (60000 samples with 784 features), Adult [14] (45222 samples with 104 features), Bank [14] (45211 samples with 51 features), and Credit-card [14] (30000 samples with 91 features). In particular, for each real dataset, we first uniformly split the features into p partitions, which are held by p parties, and assign one party to hold the label set of datasets. By default, p is set to 3. Then, we perform one-hot encoding on the discrete features and max-min normalization on the continuous features.

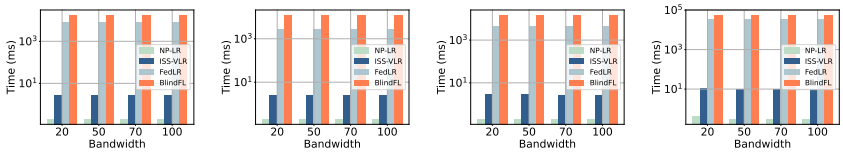
Hyper-parameters. The main hyper-parameters for training LR models using SGD include batch size, learning rate, and number of epochs. In our experiments, we configure the batch size, learning rate, and number of parties as 64, 0.001, and 10, respectively.

We conduct experiments on many machines equipped with Intel Ice Lake 32 cores CPU and 32GB of RAM. The software stack used in the experiments

will be described in each subsection, as different experiments employ different softwares.

5.2 Performance of ISS-VLR

Since the random mask used by $ISS\text{-}VLR$ does not cause computational errors, $ISS\text{-}VLR$ can obtain the same model accuracy as the non-private LR as long as the model can converge within the maximum number of epochs specified by the security constraint. As a result, we only evaluate the efficiency in this subsection and will discuss the impact of security constraints on model accuracy in Appendix E. To illustrate the efficiency of $ISS\text{-}VLR$, we compare it with three baselines in terms of the running time for training one iteration. The first baseline is NP-LR, a vertical LR protocol that does not account for privacy preservation. The other two baselines, FedLR [32] and BlindFL [15], are protocols from existing studies, both claimed to ensure infinite solution security. In particular, similar to $ISS\text{-}VLR$, both FedLR [32] and BlindLR [15] choose to expose linear-functional outputs and ensure infinite solution security by imposing specific security constraints. However, they differ in that the Paillier cryptosystem is employed to protect intermediate results during backward updating. In this experiment, all the protocols are implemented using Python, and the Paillier cryptosystem is achieved by Python-Paillier [13] with a key size of 2048 bits. Figure 1 presents the comparison results under different bandwidths. As expected, we can observe that $ISS\text{-}VLR$ is $10\times$ slower than NP-LR, while it is about $1000\times$ faster than FedLR [32] and BlindFL [15], respectively. This reason lies in that $ISS\text{-}VLR$ provides infinite solution security during backward updating and does not rely on any public-key cryptography technique. Additionally, we also verify the scalability of $ISS\text{-}VLR$ on the number of parties p. Due to space limitation, we show this experimental result in Appendix D.



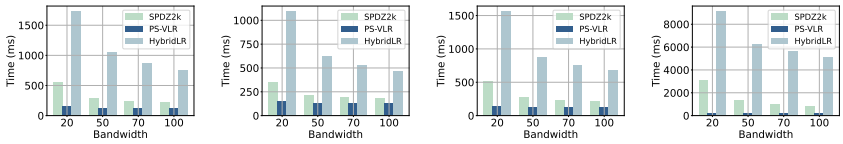
(a) Adult (b) Bank (c) Credit-card (d) MNIST

Fig. 1: Comparison of $ISS\text{-}VLR$ and baselines on the running time (ms) of model training for one iteration. The horizontal axis is the bandwidth (Mbps).

5.3 Performance of PS-VLR

To illustrate the efficiency of $PS\text{-}VLR$, we compare $PS\text{-}VLR$ with the following two baselines. The first one is the SS-based vertical LR protocol, which is implemented directly by the SPD$\mathbb{Z}_{2^k}$ framework [10] with Beaver triples and referred to as SPDZ2k. The second one is the hybrid vertical LR protocol, which is the extension of the multi-party setting of CAESAR [4], the state-of-the-art two-party

protocol based on HE and SS, and referred to as HybridLR. For comparing fairness, the running time of the evaluation of the sigmoid function is not recorded, and the matrix triples and Beaver triples are both generated by threshold FHE. In particular, we use the threshold CKKS scheme in the PALISADE library [1] and set its three key parameters, namely multiplicativeDepth, scalingFactorBits, and batchSize, to 1, 50, and 4096, respectively. In addition, to support floating-point operations by SS, we convert the inputs from floating-point to fixed-point with a 128-bit length and a 12-bit decimal part. All the protocols evaluated in this subsection are implemented using C++. Figure 2 and Table 1 respectively show the comparison results on the running time of model training and offline triple generation under different bandwidths. We can observe that $PS\text{-}VLR$ has less running time than SPDZ2k [10] in both model training and offline triple generation, which is consistent with our efficiency analysis described in Appendix C.2. We also observe that $PS\text{-}VLR$ and SPDZ2k [10] take less time to perform model training than HybridLR. This is because HybridLR requires performing homomorphic computations and transformations between SS and HE during training, which incurs significant computational and communication overheads. Nevertheless, for the total running time of model training and offline triple generation, $PS\text{-}VLR$ may be worse than HybridLR, especially when bandwidth is low. Note that, to further evaluate the efficiency of $PS\text{-}VLR$, we additionally conduct the following experiments: 1) we compare $PS\text{-}VLR$ with HybridLR in terms of the overall training time, covering sharing of input data, model training, and offline triple generation; 2) we evaluate the effect of various feature sparsities on the training efficiency of $PS\text{-}VLR$ and HybridLR; 3) we verify the scalability of $PS\text{-}VLR$ on the number of parties; 4) we validate the effectiveness of $RS\text{-}MVM$. Due to space limitations, these experimental results are shown in Appendix D.



(a) Adult (b) Bank (c) Credit-card (d) MNIST

Fig. 2: Comparison of $PS\text{-}VLR$ and baselines on the running time (ms) of model training for one iteration. The horizontal axis is the bandwidth (Mbps).

6 Related Work

In this section, we give literature reviews on privacy-preserving vertical LR protocols with infinite solution security and provable security, respectively.

Protocols with Infinite Solution Security. Existing protocols with infinite solution security [15,33,34,31,32] expose linear-functional outputs during forward computation to improve the computational efficiency. Zhang et.al [33] and Fu

Table 1: Comparison of $PS\text{-}VLR$ and SPDZ2k on the running time (s) of offline triple generation for one iteration

	Adult				Bank				Credit-card				MINST			
Bandwidth(Mbps)	20	50	70	100	20	50	70	100	20	50	70	100	20	50	70	100
SPDZ2k	6.1	2.5	1.8	1.3	2.9	1.2	0.9	0.6	5.3	2.2	1.6	1.1	45.6	18.9	13.8	9.9
PS-VLR	2.2	0.9	0.7	0.5	1.1	0.5	0.3	0.2	1.9	0.8	0.6	0.4	17.9	7.5	5.5	3.9

et.al [15] demonstrate that the exposure of linear-functional outputs satisfies infinite solution security by matrix decomposition. Yang et.al [32], Zhang et.al [34], and Xu et.al [31] treat the linear-functional output as a system of equations and establish a security constraint to ensure that this system has infinitely many solutions. During backward updating, previous works [34,33,31,32] reveal the gradient to part of the involved parties, which poses a risk of privacy leakage. For example, the active party can exploit the gradient to reconstruct the features held by passive parties [36,25,12]. To avoid this risk, Fu et.al [15] adopt SS and HE to securely compute gradients and update model parameters, while ensuring the non-disclosure of intermediate results.

Protocols with Provable Security. Existing protocols with provable security mainly employ HE and SS to prevent the disclosure of any intermediate result. Most HE-based protocols are applied in centralized LR modeling [5,18,19,16], which relies on a central server with large computing resources. For the SS-based protocols, one part of them [28,27] sets up many non-colluding servers and requires data owners to secretly share their input data with these servers for building models. Other protocols [11,7,10,22,23,9] secretly share the data among the data owners, who jointly perform LR model training and inference. SPDZ [11] and SPD$\mathbb{Z}_{2^k}$ [10] are the commonly used frameworks and design many primitives to support arbitrary collaborative computation. Recently, Chen et.al [4] propose CAESAR, a hybrid two-party protocol, which respectively employs HE and SS to perform secure matrix multiplication and the remaining computations during model training.

7 Conclusion

In this paper, we present two privacy-preserving vertical LR protocols, namely $ISS\text{-}VLR$ and $PS\text{-}VLR$, which guarantee infinite solution security and provable security, respectively. In $ISS\text{-}VLR$, we first establish rigorous security constraints to provide privacy safeguards during forward computation. Then, we propose $RM\text{-}SGD$ to securely and efficiently perform backward updating. In $PS\text{-}VLR$, we employ matrix triples to reduce communication overhead introduced by SS and propose $RS\text{-}MVM$ to achieve efficient generation of matrix triples. Experimental results demonstrate that our protocols can achieve higher efficiency than state-of-the-art protocols.

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A Proof of Theorem 1

Let n be the number of samples, m_i be the number of features held by P_i, $m_{d,i}$ be the number of discrete features held by P_i, $F_{c,i}$ be the set of continuous features held by P_i, ψ be a function used to count the number of intervals in $F_{c,i}$, $m_{c,i}$ be the size of $F_{c,i}$, and e be the number of epochs. We start by counting the number of unknown variables and equations in $\mathcal{S}'$. Suppose the training-related hyper-parameters, including the learning rate, batch size, and number of iterations, are public and constant, the variables present in $\mathcal{S}'$ contain the input features, initial model parameters, residuals, and artificial variables introduced by considering the value ranges of input features. For the active party who receives the local linear-functional outputs, only the residuals are known among the above variables as the active party is allowed to obtain the residuals during training. Therefore, the unknown variables in $\mathcal{S}'$ include the input features, the initial model parameters, and the artificial variables. The total number of unknown variables is $nm_i + m_i + n\psi(F_{c,i})$. For the equations in $\mathcal{S}'$, they contain not only the equations formed by the linear-functional outputs but also the equations introduced by considering the value ranges of input features. The total number of equations is $ne + nm_{d,i} + n\psi(F_{c,i})$. To ensure that $\mathcal{S}'$ is underdetermined, it is necessary to have a greater number of unknown variables than the number of equations. Consequently, we can establish a constraint on the number of epochs, such that $e < m_i - m_{d,i} + \frac{m_i}{n} = m_{c,i} + \frac{m_i}{n}$. In practice, n is generally much larger than m_i, resulting in $\frac{m_i}{n}$ approaching 0, and $e < m_{c,i}$. In other words, if the number of epochs is less than the number of continuous features, $\mathcal{S}'$ is underdetermined.

B Protocol Details

B.1 ISS-VLR

In this subsection, we describe the details of model training and model inference in *ISS-VLR*.

Protocol 1 shows the details of model training in *ISS-VLR*. In Protocol 1, each party starts by initializing the model parameters. Afterward, all parties engage in iterative training. For the t-th iteration, each passive party P_i first computes the local linear-functional outputs $\varphi_i^t \theta_i^t X_i^t$ and sends them to the active party P_1 (Lines 3-9). Then, after eliminating the random mask φ_i^t from $\varphi_i^t \theta_i^t X_i^t$ $(i \in [2, p])$, P_1 aggregates all local linear-functional outputs into the global linear-functional outputs and evaluates the sigmoid function to obtain the model outputs (Line 10). Next, P_1 computes the residuals and updates the model parameters (Lines 11-12). To enable other parties to update the model parameters without exposing the gradients and residuals, P_1 masks the residuals and interacts with other passive parties to compute the gradients and update the model parameters with random masks (Lines 13-23). When the number of iterations reaches the upper limit T, the model training is complete, and each party P_i obtains the masked model parameters $\varphi_i^T \theta_i^T$. For the details of model

inference in $ISS\text{-}VLR$, they follow the same process as forward computation during model training, i.e., Lines 3-10 in Protocol 1. Note that, the extension of $ISS\text{-}VLR$ to neural networks is discussed in Appendix F.

Protocol 1: Model Training in $ISS\text{-}VLR$

Input: $\{X_i\}_{i=1}^{p}$: input features from each party; $\{y_i\}_{i=1}^{n}$: input labels from active party; T: the number of iterations; α: learning rate; b: batch size

Output: $\varphi_i^T \theta_i^T$: model parameters for P_i

1 // P_1 is active party
2 **for** $t \in [1, T]$ **do**
3 **for** $i \in [2, p]$ **do**
4 **if** $t = 1$ **then**
5 P_i computes $\theta_i^t X_i^t$ and sends it to P_1
6 **else**
7 P_i computes $\varphi_i^t \theta_i^t X_i^t$ and sends it to P_1
8 **end**
9 **end**
10 P_1 aggregates $\theta^t X^t = \sum_{i=1}^{p} \theta_i^t X_i^t$ and evaluates the sigmoid function to obtain $\hat{Y}^t$
11 P_1 computes $r^t = \hat{Y}^t - Y^t$
12 P_1 computes the gradient $\frac{\partial L}{\partial \theta_1^t} = \frac{r^t X_1^t}{b}$ and updates the parameters $\theta_1^{t+1} = \theta_1^t - \alpha \frac{\partial L}{\partial \theta_1^t}$
13 P_1 randomly generates σ^t and sends $\sigma^t r^t$ to passive parties
14 **for** $i \in [2, p]$ **do**
15 P_i computes $\mathcal{G}_i^t = \sigma^t G_i^t = \frac{\sigma^t r^t X_i^t}{b}$
16 P_i randomly generates K_i^t and sends $K_i^t \mathcal{G}_i^t$ to P_1
17 P_1 computes $K_i^t G_i^t = \frac{K_i^t \mathcal{G}_i^t}{\sigma^t}$
18 P_1 randomly generates μ_i^t and sends $\alpha \varphi_i^t K_i^t G_i^t + \mu_i^t$ to P_i
19 P_i computes $\varphi_i^t K_i^t \theta_i^{t+1} - \mu_i^t$ and sends it to P_1
20 P_1 eliminates φ_i^t, μ_i^t and gets $K_i^t \theta_i^{t+1}$
21 P_1 randomly generates φ_i^{t+1} and sends $\varphi_i^{t+1} K_i^t \theta_i^{t+1}$ to P_i
22 P_i eliminates K_i^t and obtains $\varphi_i^{t+1} \theta_i^{t+1}$
23 **end**
24 **end**

B.2 PS-VLR

In this subsection, we describe the details of model training, matrix triple generation, and model inference in $PS\text{-}VLR$.

Protocol 2 shows the details of model training in $PS\text{-}VLR$. Specifically, to begin with, all parties jointly initialize the secretly shared values of model parameters, compute the secretly shared values of masked input features $\langle E^t \rangle^A$,

and reconstruct E^t for $t \in [1, T]$ (Lines 2-3). Afterward, all parties engage in iterative training. For the t-th iteration, during forward computation, all parties first jointly compute the secretly shared values of masked model parameters $\langle F_1^t \rangle^A$ and reconstruct F_1^t (Lines 5-6). Then, each party P_i computes the i-th shared values of linear-functional outputs $\langle \theta^t X^t \rangle_i^A$ (Line 7). Finally, all parties collaborate to approximately evaluate the sigmoid function and obtain the secretly shared model outputs (Line 8). During backward updating, all parties first jointly compute the secretly shared residuals $\langle r^t \rangle^A$ (Line 9). Then, all parties jointly mask $\langle r^t \rangle^A$ to obtain $\langle F_2^t \rangle^A$, reconstruct F_2^t, and compute the secretly shared gradients (Lines 10-12). Finally, all parties jointly update the model parameters (Line 13). When the number of iterations reaches the upper limit T, the model training is stopped, and the secretly shared model parameters $\langle \theta^T \rangle^A$ are output.

Protocol 2: Model Training in $PS\text{-}VLR$

Input: $\{\langle X \rangle^A\}$: secretly shared input features; $\langle Y \rangle^A$: secretly shared input labels; T: the number of iterations; α: learning rate; b: batch size; $\{\langle U^t \rangle^A, \langle V_1^t \rangle^A, \langle Z_1^t \rangle^A, \langle V_2^t \rangle^A, \langle Z_2^t \rangle^A\}_{t=1}^T$: secretly shared matrix triples

Output: $\langle \theta^T \rangle^A$: secretly shared model parameters

1 // P_1 is active party
2 P_i computes $\langle E^t \rangle_i^A = \langle X^t \rangle_i^A - \langle U^t \rangle_i^A$ for $i \in [1, p]$, $t \in [1, T]$
3 All parties jointly reconstruct E^t for $t \in [1, T]$
4 **for** $t \in [1, T]$ **do**
5 P_i computes $\langle F_1^t \rangle_i^A = \langle \theta^t \rangle_i^A - \langle V_1^t \rangle_i^A$ for $i \in [1, p]$
6 All parties jointly reconstruct F_1^t
7 P_i computes $\langle \theta^t X^t \rangle_i^A = -\mathcal{I}(i = 1) \cdot E^t \cdot F_1^t + \langle X^t \rangle_i^A \cdot F_1^t + E^t \cdot \langle \theta^t \rangle_i^A + \langle Z_1^t \rangle_i^A$, where $\mathcal{I}(\cdot)$ is indicator function, for $i \in [1, p]$
8 All parties jointly and approximately evaluate the sigmoid function to obtain $\langle \hat{Y}^t \rangle^A$
9 P_i computes $\langle r^t \rangle_i^A = \langle \hat{Y}^t \rangle_i^A - \langle Y^t \rangle_i^A$ for $i \in [1, p]$
10 P_i computes $\langle F_2^t \rangle_i^A = \langle r^t \rangle_i^A - \langle V_2^t \rangle_i^A$ for $i \in [1, p]$
11 All parties jointly reconstruct F_2^t
12 P_i computes $\langle \frac{\partial L}{\partial \theta^t} \rangle_i^A = \frac{1}{b}(-\mathcal{I}(i = 1) \cdot (E^t)^\top \cdot F_2^t + \langle (X^t)^\top \rangle_i^A \cdot F_2^t + (E^t)^\top \cdot \langle r^t \rangle_i^A + \langle Z_2^t \rangle_i^A)$, where $\mathcal{I}(\cdot)$ is indicator function, for $i \in [1, p]$
13 P_i computes $\langle \theta^{t+1} \rangle_i^A = \langle \theta^t \rangle_i^A - \alpha \cdot \langle \frac{\partial L}{\partial \theta^t} \rangle_i^A$ for $i \in [1, p]$
14 **end**

Protocol 3 shows the details of generating k matrix triples simultaneously in $PS\text{-}VLR$, where k is smaller than $\frac{N_s}{N_r}$, N_s is the number of slots in a ciphertext, and N_r is the row dimension of matrices in matrix triples. Specifically, each party P_i first randomly generates $\langle U_j \rangle_i^A \in \mathbb{R}^{N_r \times N_c}$ and $\langle V_j \rangle_i^A \in \mathbb{R}^{N_c \times 1}$, the i-th shared values of the matrix U_j and vector V_j, and transforms $\langle V_j \rangle_i^A$ into $\langle V_j' \rangle_i^A$ for

$j \in [1, k]$ (Lines 3-4). Then, P_i concatenates all the $\langle U_j \rangle_i^A$ and $\langle V_j' \rangle_i^A$ into $\langle U^* \rangle_i^A$ and $\langle V^* \rangle_i^A$ along the row dimension, respectively (Line 5). P_i next encrypts $\langle U^* \rangle_i^A$ and $\langle V^* \rangle_i^A$, and sends them to P_1 (Line 6). Subsequently, P_1 reconstructs $[\![U^*]\!]$ and $[\![V^*]\!]$, and computes $[\![Z^*]\!] = [\![U^*]\!] \circ [\![V^*]\!]$ (Lines 8-9). To convert $[\![Z^*]\!]$ into secretly shared values, P_1 collects the random vector $c_i \in \mathbb{R}^{N_r \times 1}$ generated by P_i for $i \in [2, p]$, computes $[\![Z^* - \sum_{i=2}^{p} c_i]\!]$, and cooperates with other parties to decrypt it (Lines 10-11). After that, $Z^* - \sum_{i=2}^{p} c_i$, c_2, $\cdots$, and c_p are set as the secretly shared values of Z^* (Line 12). Finally, P_i splits $\langle Z^* \rangle_i^A$ into k sub-vectors, and the j-th sub-vector is the i-th shared values of the third component Z_j for the j-th matrix triple (U_j, V_j, Z_j) (Line 13).

Protocol 3: Generation of k Matrix Triples

Input: N_r: the number of rows; N_c: the number of columns; k: the number of matrix triples; pk: public key; $\{sk_i\}_{i=1}^{p}$: secret keys from each party

Output: $\left\{(\langle U_j \rangle^A, \langle V_j \rangle^A, \langle Z_j \rangle^A)\right\}_{j=1}^{k}$: secretly shared matrix triples

1 // P_1 is active party
2 **for** $i \in [1, p]$ **do**
3 P_i randomly generates $\langle U_j \rangle_i^A$ and $\langle V_j \rangle_i^A$ for $j \in [1, k]$
4 P_i transforms $\langle V_j \rangle_i^A$ into $\langle V_j' \rangle_i^A$ for $j \in [1, k]$
5 P_i concatenates $\left\{\langle U_j \rangle_i^A\right\}_{j=1}^{k}$ and $\left\{\langle V_j' \rangle_i^A\right\}_{j=1}^{k}$ into $\langle U^* \rangle_i^A$ and $\langle V^* \rangle_i^A$ along the row dimension, respectively
6 P_i packs each column of $\langle U^* \rangle_i^A$ and $\langle V^* \rangle_i^A$ into a ciphertext and sends them to P_1
7 **end**
8 P_1 computes $[\![U^*[:, j]]\!] = \sum_{i=1}^{p} [\![\langle U^*[:, j] \rangle_i^A]\!]$ and $[\![V^*[:, j]]\!] = \sum_{i=1}^{p} [\![\langle V^*[:, j] \rangle_i^A]\!]$ for $j \in [1, N_c]$
9 P_1 computes $[\![Z^*]\!] = \sum_{j=1}^{N_c} [\![U^*[:, j]]\!] \odot [\![V^*[:, j]]\!]$
10 P_i randomly generates c_i and sends $[\![c_i]\!]$ to P_1 for $i \in [2, p]$
11 P_1 computes $[\![Z^* - \sum_{i=2}^{p} c_i]\!]$ and cooperates with other parties to decrypt $Z^* - \sum_{i=2}^{p} c_i$
12 P_1 sets $\langle Z^* \rangle_1^A = Z^* - \sum_{i=2}^{p} c_i$ and P_i sets $\langle Z^* \rangle_i^A = c_i$ for $i \in [2, p]$
13 P_i splits the $\langle Z^* \rangle_i^A$ into k sub-vectors $\left\{\langle Z_j \rangle_i^A\right\}_{j=1}^{k}$ for $i \in [1, p]$.

For details of model inference in *PS-VLR*, they follow a similar process to forward computation during model training, that is, lines 2-8 in Protocol 2. Specifically, all parties first jointly compute and reconstruct the masked input features E and the masked model parameters F. Then, all parties jointly compute the linear-functional outputs and approximately evaluate the sigmoid function to obtain the secretly shared model outputs. Finally, P_1 gathers the shared values of model outputs from other parties and reconstructs the model outputs. Note that, the extension of *PS-VLR* to neural networks is discussed in Appendix F.

C Efficiency and Security Analysis

C.1 ISS-VLR

In this subsection, we analyze the efficiency and security of Protocol 1, respectively.

Efficiency Let m be the number of features, b be the batch size, and p be the number of parties. In Protocol 1, since public-key cryptography techniques are not employed, the computation performed locally by each party is inexpensive. The dominating term in the execution overhead of Protocol 1 is the frequent communication between the active party and passive parties. Therefore, we only analyze the communication complexity of Protocol 1. Specifically, during forward computation, each passive party sends $O(b)$ local linear-functional outputs to the active party. When performing backward updating, the active party interacts with each passive party in three rounds, which requires the transmission of $O(b+4m)$ intermediate results. Overall, the total communication complexity of each iteration is $O((2b+4m)(p-1))$.

Security Following the definition of infinite solution security, we must guarantee that the system of equations constructed by any intermediate result in Protocol 1 has infinitely many solutions. We have the following theorem:

Theorem 2. *If the number of epochs e satisfies $e < m_c^*$, where $m_c^* = min(\{m_{c,i} \mid i \in I_p\})$, $m_{c,i}$ is the number of continuous features held by P_i, and I_p is the index set of passive parties, we say that Protocol 1 satisfies infinite solution security against a semi-honest adversary who can statically corrupt up to p-1 out of p parties.*

Proof. We start by declaring the private information in Protocol 1, which includes the feature set, label set, residuals, gradients, and model parameters. Then, we discuss whether the intermediate results received by the active party P_1 and the passive party P_i ($i \in [2, p]$) satisfy infinite solution security, respectively.

Active party. Let n be the number of samples, m_i be the number of features held by P_i, $m_{c,i}$ be the number of continuous features held by P_i, b be the batch size, T be the number of iterations, and e be the number of epochs. The intermediate results that P_1 can receive from P_i during training include $\left\{\theta_i^t X_i^t, K_i^t G_i^t, K_i^t \theta_i^{t+1}\right\}_{t=1}^{T}$. Since the systems of equations constructed by the above intermediate results contain common unknown variables, we can solve them jointly. As shown in Lemma 1, we require that either $m_i \geq \frac{1+\sqrt{1+4b}}{2}$ or $e < \frac{(nm_{c,i}+m_i)b}{(b-m_i^2+m_i)n}$ to make the joint system of equations have infinitely many solutions. Meanwhile, according to Theorem 1, e is also required to satisfy $e < m_{c,i}$ to make the system of equations constructed by $\{\theta_i^t X_i^t\}_{t=1}^{T}$ have infinitely many solutions. Therefore, when $0 < m_i < \frac{1+\sqrt{1+4b}}{2}$, we should determine the

quantitative relationship between $u = \frac{(nm_{c,i}+m_i)b}{(b-m_i^2+m_i)n}$ and $v = m_{c,i}$ for getting the least upper bound of e. To achieve this goal, we calculate $u - v$ and determine the sign of it. Specifically, $u - v$ is expressed as follows:

$$u - v = \frac{(nm_{c,i} + m_i)b}{(b - m_i^2 + m_i)n} - m_{c,i} = \frac{nm_{c,i}(m_i^2 - m_i) + m_i b}{(b - m_i^2 + m_i)n}. \tag{5}$$

We can observe from Equation (5) that the denominator and the numerator are both constantly greater than 0. Therefore, if $0 < m_i < \frac{1+\sqrt{1+4b}}{2}$, u is constantly greater than v, implying that $e < m_{c,i}$. Considering that e is only required to satisfy $e < m_{c,i}$ when $m_i \geq \frac{1+\sqrt{1+4b}}{2}$, we can conclude that if $e < m_{c,i}$ is satisfied, all intermediate results sent from P_i to P_1 satisfy infinite solution security. Since P_1 should interact with all passive parties simultaneously, in order for all intermediate results received by P_1 to satisfy infinite solution security, e is required to be less than the number of continuous features held by any passive party, i.e., $e < m_c^*$, where m_c^* is the minimum value of $\{m_{c,i}\}_{i=2}^{p}$.

Passive party. The intermediate results received by P_i include $\{\sigma^t r^t, \alpha\varphi_i^t K_i^t G_i^t + \mu_i^t, \varphi_i^{t+1}\theta_i^{t+1}\}_{t=1}^{T}$. The random masks contained in these intermediate results ensure that the number of unknown variables is great than the number of equations in the system of equations constructed by each intermediate result. Therefore, these intermediate results satisfy infinite solution security.

In summary, if $e < m_c^*$, we can say that Protocol 1 satisfies infinite solution security.

Lemma 1. *Given a nonlinear system of equations $\mathcal{S}$ consisting of $h_1 = \{\theta_i^t X_i^t\}_{t=1}^{T}$, $h_2 = \{K_i^t G_i^t\}_{t=1}^{T}$, and $h_3 = \left\{K_i^t \theta_i^{t+1}\right\}_{t=1}^{T}$, if the number of features held by P_i satisfies $m_i \geq \frac{1+\sqrt{1+4b}}{2}$ or the number of epochs satisfies $e < \frac{(nm_{c,i}+m_i)b}{(b-m_i^2+m_i)n}$, where n is the number of samples, $m_{c,i}$ is the number of continuous features held by P_i, and b is the batch size, $\mathcal{S}$ has infinitely many solutions.*

Proof. Let n be the number of samples, m_i be the number of features held by P_i, $F_{c,i}$ be the set of continuous features held by P_i, ψ be a function used to count the number of intervals in $F_{c,i}$, $m_{c,i}$ be the size of $F_{c,i}$, $m_{d,i}$ be the number of discrete features held by P_i, b be the batch size, T be the number of iterations, and e be the number of epochs. Since $K_i^t\theta_i^{t+1} = K_i^t\theta_i^t - \alpha K_i^t G_i^t$, we merge h_2 and h_3 into $h_4 = \{K_i^t\theta_i^t\}_{t=1}^{T}$, and $\mathcal{S}$ only consists of h_1 and h_4. To ensure that $\mathcal{S}$ has infinitely many solutions, we should derive a constraint to make the number of variables greater than the number of equations. Specifically, the unknown variables in $\mathcal{S}$ include input features X_i, initial model parameters θ_i^1, random masks $\{K_i^t\}_{t=1}^{T}$, and the artificial variables introduced by considering the value ranges of input features. The total number of unknown variables is $nm_i + m_i + m_i^2 T + n\phi(F_{c,i})$. For the equations in $\mathcal{S}$, they contain not only the equations formed by h_1 and h_4, but also the equations introduced by considering the value ranges of input features. The total number of equations is $ne + m_i T + nm_{d,i} + n\phi(F_{c,i})$. Let the

number of unknown variables be greater than the number of equations, we can get an inequality as follows:

$$\begin{aligned} &nm_i + m_i + m_i^2 T + n\phi(F_{c,i}) - (ne + m_i T + nm_{d,i} + n\phi(F_{c,i})) \\ &= nm_{c,i} + m_i + (m_i^2 - m_i - b)T > 0. \end{aligned} \tag{6}$$

If $m_i^2 - m_i - b \geq 0$, i.e., $m_i \geq \frac{1+\sqrt{1+4b}}{2}$, Equation (6) holds constant. Otherwise, the number of iterations T is required to satisfy $T < \frac{nm_{c,i}+m_i}{b-m_i^2+m_i}$ for guaranteeing that Equation (6) holds. Since $T = \frac{ne}{b}$, it can be further deduced that $e < \frac{(nm_{c,i}+m_i)b}{(b-m_i^2+m_i)n}$. In summary, if $m_i \geq \frac{1+\sqrt{1+4b}}{2}$ or $e < \frac{(nm_{c,i}+m_i)b}{(b-m_i^2+m_i)n}$ is satisfied, $\mathcal{S}$ has infinitely many solutions.

C.2 PS-VLR

In this subsection, we give the efficiency and security analysis of Protocols 2-3, respectively.

Efficiency Let n be the number of samples, m be the number of features, b be the batch size, p be the number of parties, e be the number of epochs, l be the bit length, N_r and N_c are respectively the row and column dimensions of matrices in matrix triples, and N_s is the number of slots in a ciphertext. The efficiency bottleneck of Protocol 2 stems from the frequent communication among involved parties, and the communication complexity of Protocol 2 is $O(((1 + \frac{e}{b})m + (15 - \frac{8}{l})e)np(p-1))$. For Protocol 3, its computational and communication complexities are shown in Table 2.

Comparison with Beaver triples based protocols. For SS-based matrix multiplication in each iteration, the communication complexity of $PS\text{-}VLR$ and Beaver triples based protocols are $O((b+m)p(p-1))$ and $O(4bmp(p-1))$, respectively. It can be seen that $PS\text{-}VLR$ has less communication overhead. Moreover, $PS\text{-}VLR$ has a higher efficiency on the offline triple generation in scenarios that require a large number of triples, such as LR model training and inference. The reason can be explained as follows. $PS\text{-}VLR$ can simultaneously generate k matrix triples, enabling the execution of k matrix multiplications, where $k \leq \frac{N_s}{N_r}$. If using threshold FHE to generate an adequate number of Beaver triples for supporting k matrix multiplications, the computational and communication complexities are shown in Table 2. It can be observed that although the generation of matrix triples requires a slightly higher communication complexity and greater number of encryptions, homomorphic multiplications, and homomorphic additions, it significantly reduces the number of distributed descriptions.

Security We prove that Protocol 2 and Protocol 3 satisfy the provable security by the real/ideal simulation paradigm [26]. This paradigm utilizes a simulator, denoted as $\mathcal{S}$, to generate the adversary's views in the ideal world by executing the ideal functionalities. Then, these views are compared with the adversary's

Table 2: The computational and communication complexities of generating triples

	Matrix triples	Beaver triples
Encryption	$O(2pN_c)$	$O(2p\lceil\frac{N_rN_ck}{N_s}\rceil)$
Addition	$O((2p-1)N_c-1)$	$O(2(p-1)\lceil\frac{N_rN_ck}{N_s}\rceil))$
Multiplication	$O(N_c)$	$O(\lceil\frac{N_rN_ck}{N_s}\rceil)$
Distributed decryption	$O(1)$	$O(\lceil\frac{N_rN_ck}{N_s}\rceil)$
Communication	$O(2(p-1)N_c)$	$O(2(p-1)\lceil\frac{N_rN_ck}{N_s}\rceil))$

Ideal functionality for model training $\mathcal{F}_{Train}$
Parameters: Active party P_1 and passive parties P_2, ..., P_p
Inputs: P_i inputs features X_i for $i \in [1, p]$ and P_1 inputs true labels Y
Outputs: A set of secretly shared model parameters $\{\langle\theta_j\rangle_i^A\}_{j=1}^p$ to P_i for $i \in [1, p]$

views generated by executing these protocols in the real world. If both views are consistent, these protocols satisfy provable security. We identify two ideal functionalities $\mathcal{F}_{Train}$ and $\mathcal{F}_{Triple}$ for Protocol 2 and Protocol 3, respectively, and have the following theorem:

Theorem 3. *Protocol 2 and Protocol 3 securely compute $\mathcal{F}_{Train}$ and $\mathcal{F}_{Triple}$ against a semi-honest adversary who can statically corrupt up to p-1 out of p parties, respectively.*

Proof. An adversary in our threat model can corrupt any subset of parties. Let I be the set of corrupted parties, we construct a simulator $\mathcal{S}$ to generate the view of the adversary in the ideal world. $\mathcal{S}$ submits the corrupted parties' input data to the ideal functionalities $\mathcal{F}_{Train}$ and $\mathcal{F}_{Triple}$, and receives the final outputs. The proof focuses on whether the views generated by $\mathcal{S}$ are indistinguishable from the views generated by the adversary in the real world. During the execution of Protocols 2 and 3, there are three forms of intermediate results available to the adversary, including secretly shared values, homomorphic ciphertexts, and randomly masked values. Since the security of secret sharing and homomorphic encryption has been demonstrated, the secret shared values and homomorphic ciphertexts are consistent in the views of $\mathcal{S}$ and the adversary. For the randomly

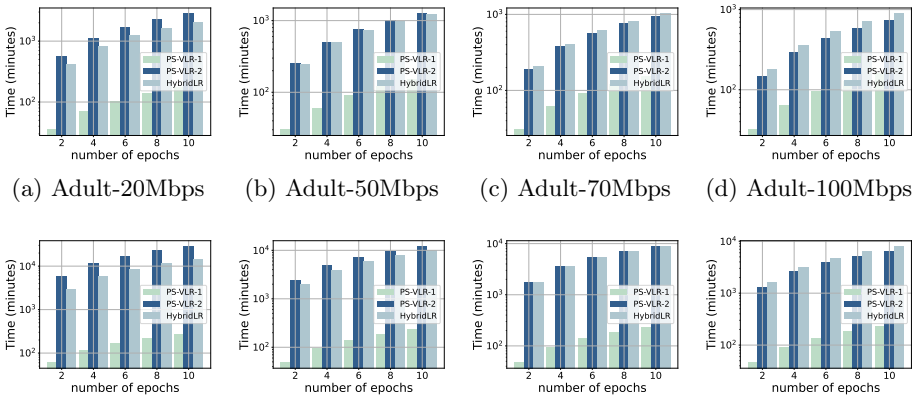
Ideal functionality for matrix triple generation $\mathcal{F}_{Triple}$
Parameters: Active party P_1 and passive parties P_2, ..., P_p
Inputs: P_1 inputs the number of rows N_r, the number of columns N_c, and the number of matrix triples k
Outputs: A set of secretly shared matrix triples $\{\langle U_j\rangle_i^A, \langle V_j\rangle_i^A, \ \langle Z_j\rangle_i^A\}_{j=1}^k$ to P_i for $i \in [1, p]$

masked values, the matrices and vectors used as random masks are obtained by a pseudo-random generator, making the masked values indistinguishable from a random number. Therefore, $\mathcal{S}$ can generate random numbers and put them into its view when executing Steps 3, 6, and 11 of Protocol 2. In summary, benefiting from the security of secret sharing, homomorphic encryption, and the pseudo-random generator, the view generated by S is consistent with the view generated by the adversary.

D Supplementary Experiments

D.1 Comparsion with HybridLR on the overall training time

Taking the running time of sharing input data into consideration, we evaluate the training efficiency of $PS\text{-}VLR$ and HybridLR on Adult and MNIST under different bandwidths and numbers of epochs. Figure 3 shows the comparison results, where $PS\text{-}VLR$-1 reports the running time of $PS\text{-}VLR$ including sharing input data and model training, and $PS\text{-}VLR$-2 reports the running time of $PS\text{-}VLR$ including sharing input data, model training, and offline triple generation. We can observe that if the offline triple generation is not considered, $PS\text{-}VLR$ is more efficient than HybridLR in all settings; otherwise, $PS\text{-}VLR$ is more efficient than HybridLR when the bandwidth is high (e.g., $\geq$ 70 Mbps).



(a) Adult-20Mbps (b) Adult-50Mbps (c) Adult-70Mbps (d) Adult-100Mbps

(e) MNIST-20Mbps (f) MNIST-50Mbps (g) MNIST-70Mbps (h) MNIST-100Mbps

Fig. 3: Comparison of HybridLR and $PS\text{-}VLR$ with input data sharing on the training efficiency.

D.2 Effect of feature sparsity

We compare $PS\text{-}VLR$ with HybridLR in terms of the running time of model training and offline triple generation for one iteration under different sparsities of input features. Specifically, we randomly generate a number of batches with 64 samples, 10,000 features, and different sparsities, and measure the running time of model training and offline triple generation on these batches. For comparing fairness, the running time of the evaluation of the sigmoid function is not recorded. The bandwidth in this experiment is fixed to 70 Mbps. Table 3 illustrates the comparison results. We can observe that the running time of HybridLR decreases as the sparsity grows, while that of $PS\text{-}VLR$ remains constant. This is because HybridLR does not secretly share the input data and can leverage sparse computation techniques to improve computational efficiency.

Table 3: The running time (s) of $PS\text{-}VLR$ and HybridLR under various feature sparsities

Sparsity	0.5	0.9	0.99	0.999
HybridLR	148.781	132.587	129.898	128.591
PS-VLR-training	0.374	0.374	0.374	0.374
PS-VLR-offline	71.998	71.998	71.998	71.998

D.3 Scalability of the number of parties

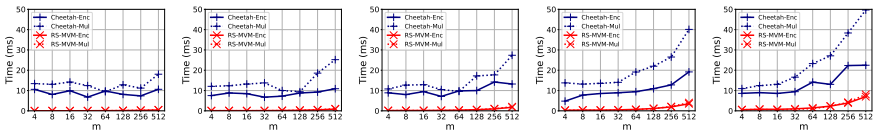
We verify the scalability of $ISS\text{-}VLR$ and $PS\text{-}VLR$ on the number of parties p, respectively. In this experiment, we randomly generate a batch of data for each party, which contains 64 samples with 40 features, and fix the bandwidth to 70 Mbps. Table 4 illustrates the training time of one iteration on this batch of data with different settings of p. We can observe that the training time of $ISS\text{-}VLR$ is consistently 2.8 ms as p grows. This result can be explained by the fact that the interaction between each passive party and the active party can be carried out independently and in parallel. Therefore, $ISS\text{-}VLR$ has favorable scalability as the number of parties increases. We also observe that the training time of $PS\text{-}VLR$ increases quadratically with the growth of p. This is because the communication complexity of $PS\text{-}VLR$ is at least quadratic in p.

Table 4: Effect of the number of parties p

	p=3	p=4	p=5	p=6
$ISS\text{-}VLR$	2.8 ms	2.8 ms	2.8 ms	2.8 ms
$PS\text{-}VLR$	112 ms	127 ms	152 ms	187 ms

D.4 Effectiveness of RS-MVM

We compare the performance of *RS-MVM* against Cheetah [20], the state-of-the-art homomorphic matrix-vector multiplication method, on accelerating matrix triple generation. Cheetah [20] chooses not to employ the SIMD operation to avoid the expensive rotation operation. Instead, it designs a new encoding manner for encrypting the matrix and vector, realizing that the homomorphic matrix-vector multiplication can be performed via a single homomorphic multiplication. Since the open-source code of Cheetah [20] is implemented using the BFV scheme from the SEAL library [30], we adopt the same scheme and library to implement *RS-MVM* for comparing fairness and set the polynomial modulus to 4096. However, SEAL [30] does not support the threshold version of BFV. Therefore, we only measure the running time required for encrypting the matrix and vector, and performing the multiplication of their corresponding ciphertexts during the generation of a matrix triple. Let n and m be the numbers of rows and columns of the matrix, respectively. Figure 4 shows the results on the synthetic data generated under different (n, m) settings. We can observe that *RS-MVM* outperforms Cheetah [20] in terms of the running time required for encryption and multiplication under any (n, m) setting. The reason lies in that *RS-MVM* utilize SIMD operations to gain the ability to generate multiple matrix triples simultaneously through a single matrix-vector multiplication. We also observe that the discrepancy in efficiency between *RS-MVM* and Cheetah [20] widens as n and m increase. This is because Cheetah [20] requires to divide the matrix and the vector into a large number of sub-matrices and sub-vectors when nm is larger than the number of slots in a ciphertext, leading to a superlinear growth in the number of encryptions and multiplications as n and m increase. In contrast, *RS-MVM* solely necessitates $O(m)$ encryptions and multiplications while displaying insensitivity towards variations in n.



(a) $n = 16$ (b) $n = 32$ (c) $n = 64$ (d) $n = 128$ (e) $n = 256$

Fig. 4: Comparison of *RS-MVM* and Cheetah [20] on the running time (ms) of encryptions (Enc) and multiplications (Mul) during matrix triple generation.

E Discussion on Security Constraint

Any vertical LR protocol with infinite solution security should meet our proposed security constraint. When training on a dataset, the effect of this security constraint on model accuracy depends on the number of continuous features in the dataset and the ability of the model to fit the dataset. In other words, when the dataset has a large number of continuous features or can be fitted with a small

number of epochs, the security constraint does not affect the model accuracy. If the training of the model fails to converge under the security constraint, we can serialize the discrete features to increase the number of continuous features. For example, we first encode the discrete feature using one-hot encoding, and then employ techniques such as PCA or autoencoder to obtain its continuous representation.

F Extension to Neural Network

Neural network (NN) is a more complex linear model than LR. Although ISS-VLR and PS-VLR cannot be directly employed to implement vertical federated NN, their key building blocks can be combined with existing studies to address drawbacks in privacy and efficiency:

ISS-VLR. Existing studies [29,35] on vertical federated NN adopt the split learning paradigm, which splits a neural network into a bottom model and a top model. The bottom model is computed collaboratively by multiple parties, while the top model is computed locally on the active party. A key issue is that the bottom model output should be exposed to the active party, who may use this information to invert the input features held by other parties. As with avoiding the privacy risk arising from the exposure of the linear-functional output in ISS-VLR, we can construct a security constraint to ensure that the exposure of the bottom model output satisfies infinite solution security to solve such issue.

PS-VLR. Similar to LR, one of the core operations of NN is matrix multiplication. As a result, the matrix triple and its generation method in PS-VLR can be directly employed to implement efficient vertical federated NN with provable security.