$ds = |\mathbf{r}'|\,dt = \sqrt{1+4t^2}\,dt$, so the integral is

$$\int_0^2 f(t,t^2)\sqrt{1+4t^2}\,dt = \int_0^2 (t+t^2)\sqrt{1+4t^2}\,dt = \frac{167}{48}\sqrt{17} - \frac{1}{12} - \frac{1}{64}\ln(4+\sqrt{17}).$$

This integral of a function along a curve C is often written in abbreviated form as

$$\int_C f(x,y)\,ds.$$

EXAMPLE 16.2.1 Compute $\displaystyle\int_C ye^x\,ds$ where C is the line segment from $(1,2)$ to $(4,7)$.

We write the line segment as a vector function: $\mathbf{r} = \langle 1,2\rangle + t\langle 3,5\rangle$, $0 \le t \le 1$, or in parametric form $x = 1+3t$, $y = 2+5t$. Then

$$\int_C ye^x\,ds = \int_0^1 (2+5t)e^{1+3t}\sqrt{3^2+5^2}\,dt = \frac{16}{9}\sqrt{34}e^4 - \frac{1}{9}\sqrt{34}\,e.$$

□

All of these ideas extend to three dimensions in the obvious way.

EXAMPLE 16.2.2 Compute $\displaystyle\int_C x^2z\,ds$ where C is the line segment from $(0,6,-1)$ to $(4,1,5)$.

We write the line segment as a vector function: $\mathbf{r} = \langle 0,6,-1\rangle + t\langle 4,-5,6\rangle$, $0 \le t \le 1$, or in parametric form $x = 4t$, $y = 6-5t$, $z = -1+6t$. Then

$$\int_C x^2z\,ds = \int_0^1 (4t)^2(-1+6t)\sqrt{16+25+36}\,dt = 16\sqrt{77}\int_0^1 -t^2+6t^3\,dt = \frac{56}{3}\sqrt{77}.$$

□

Now we turn to a perhaps more interesting example. Recall that in the simplest case, the work done by a force on an object is equal to the magnitude of the force times the distance the object moves; this assumes that the force is constant and in the direction of motion. We have already dealt with examples in which the force is not constant; now we are prepared to examine what happens when the force is not parallel to the direction of motion.