SEPARABLE EQUATIONS

The equations $dy/dt = cy$ and $dy/dt = cy + s$ (with constant source s) can be solved by a direct method. ***The idea is to separate y from t***:

$$\frac{dy}{y} = c\,dt \qquad \text{and} \qquad \frac{dy}{y + (s/c)} = c\,dt. \tag{2}$$

All y's are on the left side. All t's are on the right side (and c can be on either side). This separation would not be possible for $dy/dt = y + t$.

Equation (2) contains differentials. They suggest integrals. The t-integrals give ct and the y-integrals give logarithms:

$$\ln y = ct + \text{constant} \qquad \text{and} \qquad \ln\left(y + \frac{s}{c}\right) = ct + \text{constant}. \tag{3}$$

The constant is determined by the initial condition. At $t = 0$ we require $y = y_0$, and the right constant will make that happen:

$$\ln y = ct + \ln y_0 \qquad \text{and} \qquad \ln\left(y + \frac{s}{c}\right) = ct + \ln\left(y_0 + \frac{s}{c}\right). \tag{4}$$

Then the final step isolates y. The goal is a formula for y itself, not its logarithm, so take the exponential of both sides ($e^{\ln y}$ is y):

$$y = y_0 e^{ct} \qquad \text{and} \qquad y + \frac{s}{c} = \left(y_0 + \frac{s}{c}\right)e^{ct}. \tag{5}$$

It is wise to substitute y back into the differential equation, as a check.

This is our fourth method for $y' = cy + s$. Method 1 assumed from the start that $y = Ae^{ct} + B$. Method 2 multiplied all inputs by their growth factors $e^{c(t-T)}$ and added up outputs. Method 3 solved for $y - y_\infty$. Method 4 is ***separation of variables*** (and all methods give the same answer). This separation method is so useful that we repeat its main idea, and then explain it by using it.

To solve $dy/dt = u(y)v(t)$, separate $dy/u(y)$ from $v(t)dt$ and integrate both sides:

$$\int dy/u(y) = \int v(t)dt + C. \tag{6}$$

Then substitute the initial condition to determine C, and solve for $y(t)$.

EXAMPLE 1 $dy/dt = y^2$ separates into $dy/y^2 = dt$. Integrate to reach $-1/y = t + C$. Substitute $t = 0$ and $y = y_0$ to find $C = -1/y_0$. Now solve for y:

$$-\frac{1}{y} = t - \frac{1}{y_0} \qquad \text{and} \qquad y = \frac{y_0}{1 - ty_0}.$$

This solution blows up (Figure 6.15a) when t reaches $1/y_0$. If the bank pays interest on your deposit *squared* ($y' = y^2$), you soon have all the money in the world.

EXAMPLE 2 $dy/dt = ty$ separates into $dy/y = t\,dt$. Then by integration $\ln y = \frac{1}{2}t^2 + C$. Substitute $t = 0$ and $y = y_0$ to find $C = \ln y_0$. The exponential of $\frac{1}{2}t^2 + \ln y_0$ gives $y = y_0 e^{t^2/2}$. When the interest rate is $c = t$, the exponent is $t^2/2$.

EXAMPLE 3 $dy/dt = y + t$ is *not separable*. Method 1 survives by assuming $y =$