Constructing Entire Functions (a summary)

Kirill Lazebnik

SUNY Stony Brook

Kirill.Lazebnik@stonybrook.edu

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Theorem *(Grothendieck)*: *ALL* combinatorial trees occur as $p^{-1}[-1,1]$ for some *Shabat* polynomial $p(z)$.

Theorem *(Bishop)*: Any continua can be ϵ-approximated in the Hausdorff metric by some $p^{-1}[-1,1]$.

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$\mathcal{S}_{2,0}$ - transcendental functions with two critical values ± 1 and no asymptotic values

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critical values: ± 1

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Theorem: *For every choice of parameters* $\lambda, (d_n), (w_n)$ *with* $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$

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(1) $f(\overline{z}) = \overline{f(z)}, f(-z) = f(z)$

$S^+ = \{x+iy : x>0, |y|<\pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1,1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left[\frac{\lambda}{\pi}\cosh(n\pi)\right]\right)$

$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z|<1$ by $z \to (z-z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

Theorem: *For every choice of parameters $\lambda, (d_n), (w_n)$ with $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$, there exists a transcendental f and a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ so that:*

(1) $f(\overline{z}) = \overline{f(z)}, f(-z) = f(z)$
(2)

$$f(z) = \begin{cases} \cosh(\lambda \sinh(\phi(z))) & \text{if } \phi(z) \in S^+ \\ \rho_n((\phi(z) - z_n)^{d_n}) & \text{if } \phi(z) \in D_n \end{cases}$$

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(4) $\phi(0) = 0, \phi(\mathbb{R}) = \mathbb{R}$ and ϕ is conformal in S^+.

Put in somewhere notion of conformally balanced
do a better job with introducting fatou sets/julia sets

References

Chris Bishop (2014)
Constructing Entire Functions By Quasiconformal Folding
Acta Mathematica

Nuria Fagella, Sebastien Godillion, and Xavier Jarque (2014)
Wandering domains for composition of entire functions
Journal of Mathematical Analysis and Applications