

Constructing Entire Functions (a summary)

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August 2, 2015

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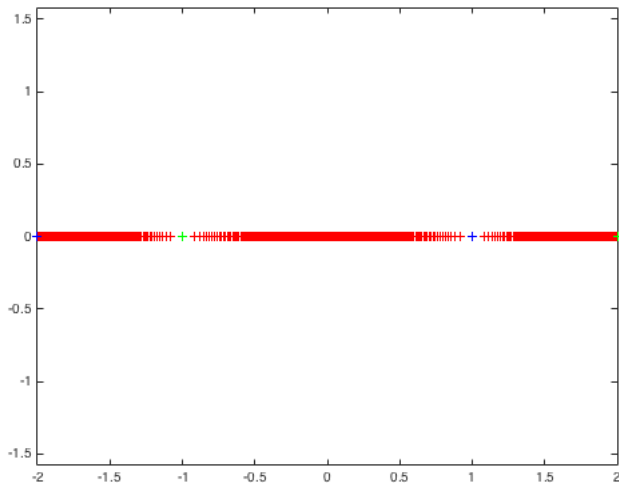
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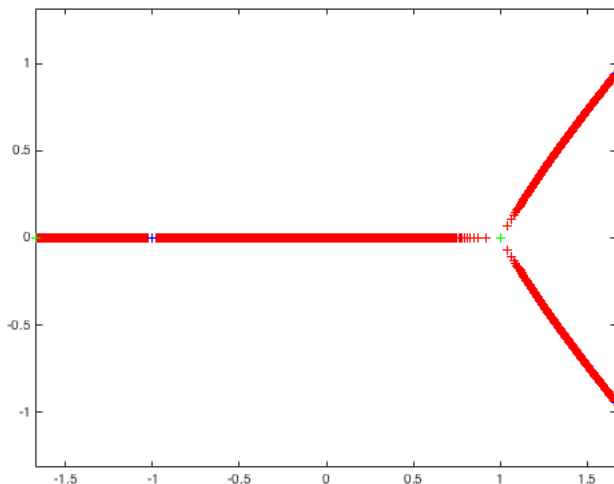
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Theorem (Bishop): Any **continua** can be ϵ -approximated in the **Hausdorff metric** by some $p^{-1}[-1, 1]$.

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$\mathcal{S}_{2,0}$ - transcendental functions with two critical values ± 1 and no asymptotic values

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critical values: ± 1

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Theorem: *Suppose T has bounded geometry and every edge has τ -size $\geq \pi$.*

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A Fatou component U is called *wandering* if $f^n(U) \cap f^m(U) = \emptyset$

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Theorem: *(Golberg and Keen, Eremenko and Lyubich) Functions in the Speiser Class don't have wandering domains.*

The Eremenko-Lyubich class \mathcal{B} consists of those transcendental functions with bounded singular set.

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