Constructing Entire Functions (a summary)

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November 23, 2015

Constructing Entire Functions By Quasiconformal Folding
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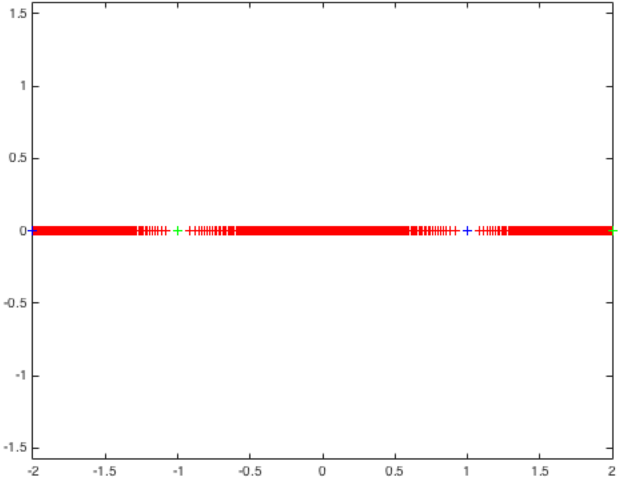

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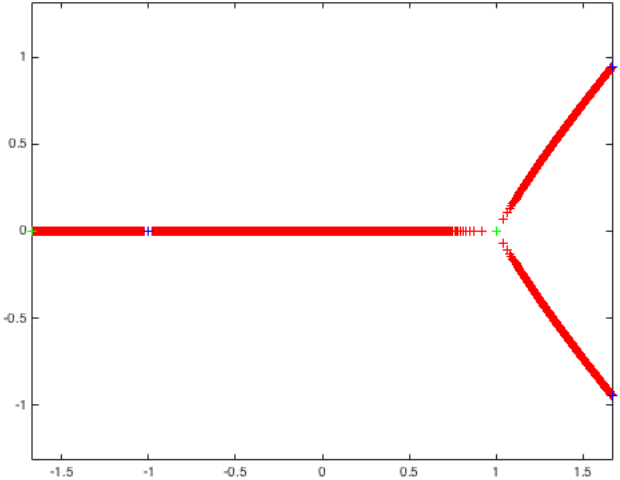

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Theorem *(Bishop)***:** Any continua can be ϵ-approximated in the Hausdorff metric by some $p^{-1}[-1,1]$.

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$\mathcal{S}_{2,0}$ - transcendental functions with two critical values ± 1 and no asymptotic values

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critical values: ± 1

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A Fatou component U is called *wandering* if $f^n(U) \cap f^m(U) = \emptyset$ for all $n, m \in \mathbb{N}, n \neq m$

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Theorem: *(Golberg and Keen, Eremenko and Lyubich) Functions in the Speiser Class don't have wandering domains.*

The Eremenko-Lyubich class $\mathcal{B}$ consists of those transcendental functions with bounded singular set.

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Theorem: *For every choice of parameters* $\lambda, (d_n), (w_n)$ *with* $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$

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References

Chris Bishop (2014)
Constructing Entire Functions By Quasiconformal Folding
Acta Mathematica

Nuria Fagella, Sebastien Godillion, and Xavier Jarque (2014)
Wandering domains for composition of entire functions
Journal of Mathematical Analysis and Applications