Constructing Entire Functions (a summary)

Kirill Lazebnik

SUNY Stony Brook

Kirill.Lazebnik@stonybrook.edu

August 4, 2015

Constructing Entire Functions By Quasiconformal Folding
(a summary)

Kirill Lazebnik

SUNY Stony Brook

Kirill.Lazebnik@stonybrook.edu

August 4, 2015

$p(z) = \frac{z^3}{2} - \frac{3z}{2}$

$p(z) = \frac{z^3}{2} - \frac{3z}{2}$

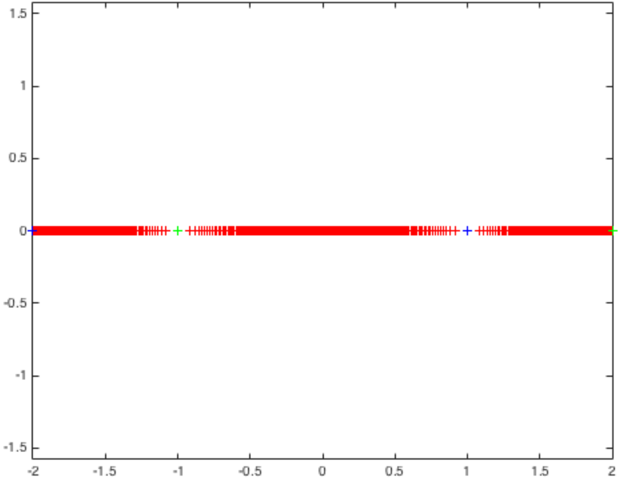
$p'(z) = \frac{3}{2}(z-1)(z+1)$

$p(z) = \frac{z^3}{2} - \frac{3z}{2}$ (two critical values ± 1)

$p'(z) = \frac{3}{2}(z-1)(z+1)$

$p(z) = \frac{z^3}{2} - \frac{3z}{2}$ (two critical values ± 1)

$p'(z) = \frac{3}{2}(z-1)(z+1)$



$$p(z) = \frac{z^4}{4} - \frac{z^3}{3} - \frac{z^2}{2} + z$$

$$p(z) = \frac{z^4}{4} - \frac{z^3}{3} - \frac{z^2}{2} + z$$

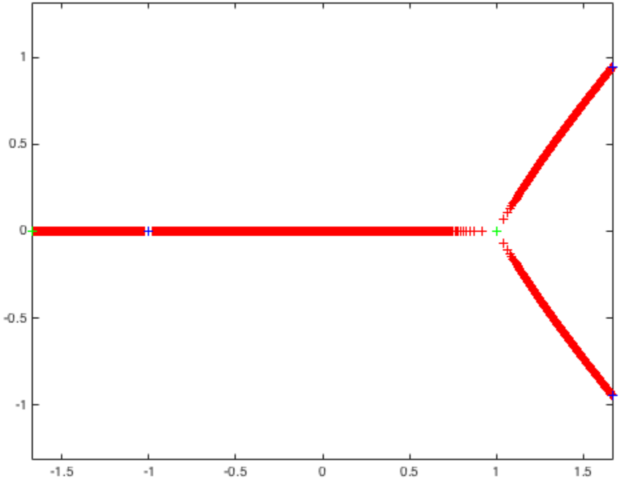
$$p'(z) = (z-1)^2(z+1)$$

$p(z) = \frac{z^4}{4} - \frac{z^3}{3} - \frac{z^2}{2} + z$ (two critical values $5/12, -11/12$)

$p'(z) = (z-1)^2(z+1)$

$p(z) = \frac{z^4}{4} - \frac{z^3}{3} - \frac{z^2}{2} + z$ (two critical values $5/12, -11/12$)

$p'(z) = (z-1)^2(z+1)$



Shabat polynomial -

Shabat polynomial - only has two critical values ± 1

Shabat polynomial - only has two critical values ± 1

Proposition: For any *Shabat* polynomial $p(z)$, it is true that $p^{-1}[-1, 1]$ is a tree.

Shabat polynomial - only has two critical values ± 1

Proposition: For any *Shabat* polynomial $p(z)$, it is true that $p^{-1}[-1, 1]$ is a tree, with $\deg(p)$ edges.

Shabat polynomial - only has two critical values ± 1

Proposition: For any *Shabat* polynomial $p(z)$, it is true that $p^{-1}[-1,1]$ is a tree, with $\deg(p)$ edges.

Theorem *(Grothendieck)***:** *ALL* combinatorial trees occur as $p^{-1}[-1,1]$ for some *Shabat* polynomial $p(z)$.

Shabat polynomial - only has two critical values ± 1

Proposition: For any *Shabat* polynomial $p(z)$, it is true that $p^{-1}[-1,1]$ is a tree, with $\deg(p)$ edges.

Theorem *(Grothendieck)*: *ALL* combinatorial trees occur as $p^{-1}[-1,1]$ for some *Shabat* polynomial $p(z)$.

Theorem *(Bishop)*: Any continua can be ϵ-approximated in the Hausdorff metric by some $p^{-1}[-1,1]$.

trees $\Longleftrightarrow$ *Shabat* polynomials

trees $\iff$ *Shabat* polynomials

infinite trees $\iff$ Transcendental Functions

trees $\iff$ *Shabat* polynomials

infinite trees $\iff$ Subclass of Transcendental Functions

trees $\iff$ *Shabat* polynomials

infinite trees $\iff$ Subclass of Transcendental Functions

$\mathcal{S}_{2,0}$ - transcendental functions with two critical values ± 1 and no asymptotic values

$\cosh(z) := \frac{e^z+e^{-z}}{2}$

$$\cosh(z) := \frac{e^z + e^{-z}}{2}$$
$$\cosh'(z) = \frac{e^z - e^{-z}}{2}$$

$\cosh(z) := \frac{e^z+e^{-z}}{2}$
$\cosh'(z) = \frac{e^z-e^{-z}}{2} = 0 \implies z = \pi i \mathrm{n} : n \in \mathbb{Z}$

$\cosh(z) := \frac{e^z + e^{-z}}{2}$
$\cosh'(z) = \frac{e^z - e^{-z}}{2} = 0 \implies z = \pi i n : n \in \mathbb{Z}$ (critical points)

$\cosh(z) := \frac{e^z + e^{-z}}{2}$
$\cosh'(z) = \frac{e^z - e^{-z}}{2} = 0 \implies z = \pi i \mathrm{n} : n \in \mathbb{Z}$ (critical points)

critical values: ± 1

T - unbounded, locally finite tree

T - unbounded, locally finite tree, with a bipartite labeling of vertices.

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$
$\tau : \cup \Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$
V - the vertices of T.

T - unbounded, locally finite tree, with a bipartite labeling of vertices.

Ω_j - components of $\mathbb{C} - T$.

$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.

V - the vertices of T.

V_j - the image of V under τ restricted to Ω_j.

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e\in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:
(1) The edges of T are C^2 with uniform bounds.

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e\in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:
(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly from zero

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.

(2) The angles between adjacent edges are bounded uniformly from zero

(3) Adjacent edges have uniformly comparable lengths

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.

(2) The angles between adjacent edges are bounded uniformly from zero

(3) Adjacent edges have uniformly comparable lengths

(4) For non-adjacent edges e, f, we have $diam(e)/dist(e, f)$ uniformly bounded.

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly from zero
(3) Adjacent edges have uniformly comparable lengths
(4) For non-adjacent edges e, f, we have $diam(e)/dist(e, f)$ uniformly bounded.

Theorem:

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly from zero
(3) Adjacent edges have uniformly comparable lengths
(4) For non-adjacent edges e, f, we have $diam(e)/dist(e, f)$ uniformly bounded.

Theorem: *Suppose T has bounded geometry*

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly from zero
(3) Adjacent edges have uniformly comparable lengths
(4) For non-adjacent edges e, f, we have $diam(e)/dist(e, f)$ uniformly bounded.

Theorem: *Suppose T has bounded geometry and every edge has τ-size $\geq \pi$.*

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly from zero
(3) Adjacent edges have uniformly comparable lengths
(4) For non-adjacent edges e, f, we have $diam(e)/dist(e, f)$ uniformly bounded.

Theorem: *Suppose T has bounded geometry and every edge has τ-size $\geq \pi$. Then there is an $r_0 > 0$, an entire f, and a K-quasiconformal ϕ*

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly from zero
(3) Adjacent edges have uniformly comparable lengths
(4) For non-adjacent edges e, f, we have $diam(e)/dist(e, f)$ uniformly bounded.

Theorem: *Suppose T has bounded geometry and every edge has τ-size $\geq \pi$. Then there is an $r_0 > 0$, an entire f, and a K-quasiconformal ϕ so that $f \circ \phi = \cosh \circ \tau$ off $T(r_0)$.*

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly from zero
(3) Adjacent edges have uniformly comparable lengths
(4) For non-adjacent edges e, f, we have $diam(e)/dist(e, f)$ uniformly bounded.

Theorem: *Suppose T has bounded geometry and every edge has τ-size $\geq \pi$. Then there is an $r_0 > 0$, an entire f, and a K-quasiconformal ϕ so that $f \circ \phi = \cosh \circ \tau$ off $T(r_0)$. K depends only on the bounded geometry constants of T.*

T - unbounded, locally finite tree, with a bipartite labeling of vertices.
Ω_j - components of $\mathbb{C} - T$.
$\tau : \cup\Omega_j \to \mathbb{C}$ - the map conformal on each Ω_j to $\mathbb{H}_r$.
V - the vertices of T.
V_j - the image of V under τ restricted to Ω_j.
For $r > 0$, define $T(r) = \cup_{e \in T}\{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\}$
The τ-size of edge e is the minimum length of the two images $\tau(e)$

T has uniformly bounded geometry if:

(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly from zero
(3) Adjacent edges have uniformly comparable lengths
(4) For non-adjacent edges e, f, we have $diam(e)/dist(e, f)$ uniformly bounded.

Theorem: *Suppose T has bounded geometry and every edge has τ-size $\geq \pi$. Then there is an $r_0 > 0$, an entire f, and a K-quasiconformal ϕ so that $f \circ \phi = \cosh \circ \tau$ off $T(r_0)$. K depends only on the bounded geometry constants of T. The only critical values of f are ± 1 and f has no asymptotic values.*

$f : \mathbb{C} \to \mathbb{C}$ entire function

$f : \mathbb{C} \to \mathbb{C}$ entire function

$f^{\circ n}$ is normal in an open set U

$f : \mathbb{C} \to \mathbb{C}$ entire function

$f^{\circ n}$ is normal in an open set U if every sequence of $f^{\circ k}$ contains a further subsequence converging locally uniformly to a holomorphic function $g : U \to \overline{\mathbb{C}}$

$f : \mathbb{C} \to \mathbb{C}$ entire function

$f^{\circ n}$ is normal in an open set U if every sequence of $f^{\circ k}$ contains a further subsequence converging locally uniformly to a holomorphic function $g : U \to \overline{\mathbb{C}}$

The Fatou set of f is the set of points $z \in \mathbb{C}$ for which f is normal in some neighborhood of z.

$f : \mathbb{C} \to \mathbb{C}$ entire function

$f^{\circ n}$ is normal in an open set U if every sequence of $f^{\circ k}$ contains a further subsequence converging locally uniformly to a holomorphic function $g : U \to \overline{\mathbb{C}}$

The Fatou set of f is the set of points $z \in \mathbb{C}$ for which f is normal in some neighborhood of z. The components of the Fatou set are called Fatou components.

$f : \mathbb{C} \to \mathbb{C}$ entire function

$f^{\circ n}$ is normal in an open set U if every sequence of $f^{\circ k}$ contains a further subsequence converging locally uniformly to a holomorphic function $g : U \to \overline{\mathbb{C}}$

The Fatou set of f is the set of points $z \in \mathbb{C}$ for which f is normal in some neighborhood of z. The components of the Fatou set are called Fatou components.

A Fatou component U is called *wandering* if $f^n(U) \cap f^m(U) = \emptyset$ for all $n, m \in \mathbb{N}, n \neq m$

Theorem: *(Sullivan) Rational maps don't have wandering domains.*

Theorem: *(Sullivan) Rational maps don't have wandering domains.*

For $f : \mathbb{C} \to \mathbb{C}$, the singular set $S(f)$ consists of the critical values and asymptotic values of f.

Theorem: *(Sullivan) Rational maps don't have wandering domains.*

For $f : \mathbb{C} \to \mathbb{C}$, the singular set $S(f)$ consists of the critical values and asymptotic values of f.

The *Speiser* class $\mathcal{S}$ consists of those transcendental functions for which $S(f)$ is finite.

Theorem: *(Sullivan) Rational maps don't have wandering domains.*

For $f : \mathbb{C} \to \mathbb{C}$, the singular set $S(f)$ consists of the critical values and asymptotic values of f.

The *Speiser* class $\mathcal{S}$ consists of those transcendental functions for which $S(f)$ is finite.

Theorem: *(Golberg and Keen, Eremenko and Lyubich) Functions in the Speiser Class don't have wandering domains.*

Theorem: *(Sullivan) Rational maps don't have wandering domains.*

For $f : \mathbb{C} \to \mathbb{C}$, the singular set $S(f)$ consists of the critical values and asymptotic values of f.

The *Speiser* class $\mathcal{S}$ consists of those transcendental functions for which $S(f)$ is finite.

Theorem: *(Golberg and Keen, Eremenko and Lyubich) Functions in the Speiser Class don't have wandering domains.*

The Eremenko-Lyubich class $\mathcal{B}$ consists of those transcendental functions with bounded singular set.

$$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$.

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by $\cosh$.

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi, D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi, D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$.

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi, D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ of the disc so that:

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi, D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi, D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$

(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor \frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi, D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi, D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor \frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$$

$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

Theorem:

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

Theorem: *For every choice of parameters* $\lambda, (d_n), (w_n)$ *with* $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor \frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

Theorem: *For every choice of parameters $\lambda, (d_n), (w_n)$ with $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$, there exists a transcendental f and a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ so that:*

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor\frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

Theorem: *For every choice of parameters $\lambda, (d_n), (w_n)$ with $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$, there exists a transcendental f and a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ so that:*

(1) $f(\overline{z}) = \overline{f(z)}, f(-z) = f(z)$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor \frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

Theorem: *For every choice of parameters $\lambda, (d_n), (w_n)$ with $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$, there exists a transcendental f and a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ so that:*

(1) $f(\overline{z}) = \overline{f(z)}, f(-z) = f(z)$
(2)

$$f(z) = \begin{cases} \cosh(\lambda \sinh(\phi(z))) & \text{if } \phi(z) \in S^+ \\ \rho_n((\phi(z) - z_n)^{d_n}) & \text{if } \phi(z) \in D_n \end{cases}$$

$S^+ = \{x + iy : x > 0, |y| < \pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1, 1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor \frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z| < 1$ by $z \to (z - z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

Theorem: *For every choice of parameters $\lambda, (d_n), (w_n)$ with $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$, there exists a transcendental f and a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ so that:*

(1) $f(\overline{z}) = \overline{f(z)}, f(-z) = f(z)$
(2)

$$f(z) = \begin{cases} \cosh(\lambda \sinh(\phi(z))) & \text{if } \phi(z) \in S^+ \\ \rho_n((\phi(z) - z_n)^{d_n}) & \text{if } \phi(z) \in D_n \end{cases}$$

(3) f has no asymptotic values and its set of critical values is ± 1 and $\overline{\{w_n : n \geq 1\}}$

$S^+ = \{x+iy : x>0, |y|<\pi/2\}$ is mapped conformally to $\mathbb{H}_r$ by $\lambda \cdot \sinh$. Then holomorphically to $\mathbb{C} - [-1,1]$ by cosh.

$a_n = \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lfloor \frac{\lambda}{\pi}\cosh(n\pi)\right\rfloor\right)$

$z_n = a_n + i\pi$, $D_n = \{z \in \mathbb{C} : |z - z_n| < 1\}$ is mapped holomorphically to $|z|<1$ by $z \to (z-z_n)^{d_n}$. Then by a quasiconformal map ρ_n of the disc so that:

(1) $\rho_n(z) = z$ for $z \in \partial\mathbb{D}$
(2) $\rho_n(0) = w_n$ where w_n is a point near $1/2$.
(3) ρ_n is conformal on $\frac{3}{4}\mathbb{D}$
(4) ρ_n is quasiconformal on $\mathbb{D}$.

Theorem: *For every choice of parameters $\lambda, (d_n), (w_n)$ with $\lambda \in \pi\mathbb{N}, d_n \in 2\mathbb{N}$, there exists a transcendental f and a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ so that:*

(1) $f(\overline{z}) = \overline{f(z)}, f(-z) = f(z)$
(2)

$$f(z) = \begin{cases} \cosh(\lambda \sinh(\phi(z))) & \text{if } \phi(z) \in S^+ \\ \rho_n((\phi(z) - z_n)^{d_n}) & \text{if } \phi(z) \in D_n \end{cases}$$

(3) f has no asymptotic values and its set of critical values is ± 1 and $\overline{\{w_n : n \geq 1\}}$
(4) $\phi(0) = 0, \phi(\mathbb{R}) = \mathbb{R}$ and ϕ is conformal in S^+.

References

Chris Bishop (2014)
Constructing Entire Functions By Quasiconformal Folding
Acta Mathematica

Nuria Fagella, Sebastien Godillion, and Xavier Jarque (2014)
Wandering domains for composition of entire functions
Journal of Mathematical Analysis and Applications