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3D Point Clouds
Lecture 2
Nearest Neighbors

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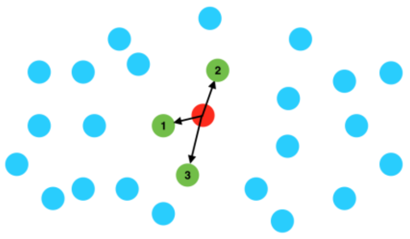
1. Binary Search Tree
2. Kd-tree
3. Octree

Nearest Neighbor (NN) Problem

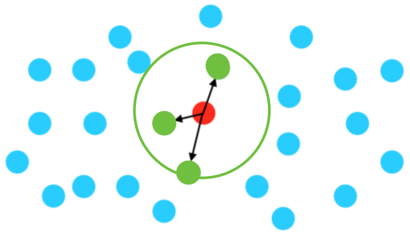
K-NN

- Given a set of points S in a space M, a query point $q \in M$, find the k closest points in S



Fixed Radius-NN

- Given a set of points S in a space M, a query point $q \in M$, find all the points in S, s.t., $\|s - q\| < r$



Why NN problem is important?

- It is almost everywhere
  - What we have covered:
    - Surface normal estimation
    - Noise filtering
    - Sampling
  - What we will cover:
    - Clustering
    - Deep learning
    - Feature detection / description
    - … …

- Why don’t we simply call an open-source library (flann, PCL, etc.)?
  - They are not efficient enough.
    - They are general lib, not optimized for 2D/3D.
    - Most open-source octree implementation is in-efficient, while octree is most effective for 3D.
  - Few GPU based NN library is available

Why NN is difficult for point clouds

- For Images, a neighbor is simply $x + \Delta x, y + \Delta y$

- For point clouds
  - Irregular – no grid based representation
  - Curse of dimensionality
    - Non-trivial to build grids
    - Non-trivial to sort or build spatial partitions
  - Huge data throughput in real-time applications
    - Velodyne HDL-64E – 2.2 million points per second / 110,000 points at 20Hz
    - Brute-force self-NN search is 110,000 x 110,000 x 0.5 = 6×10^9 comparisons @ 20Hz

Lecture Outline

Binary Search Tree (BST)

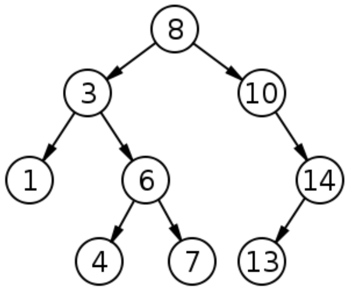
- Basic knowledge about trees
- 1D NN problem
- With Python codes

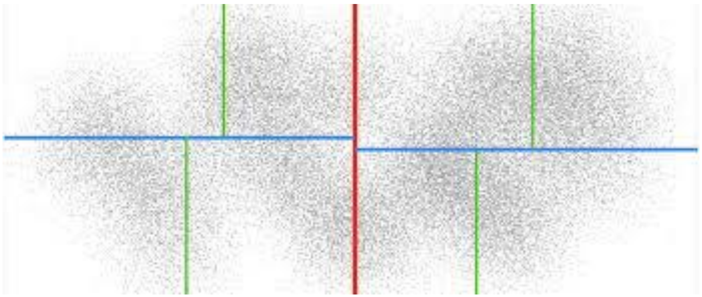
Kd-tree

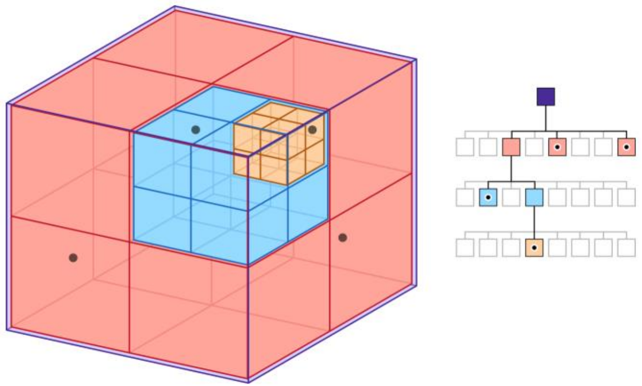
- Works for data of any dimension
- Illustrated in 2D
- With Python codes

Octree

- Specifically designed for 3D data
- Illustrated in 2D/3D
- With Python codes







Core Ideas Shared by BST, kd-tree, octree

NN by space partition

- Split the space into different areas,
- Search some areas only, instead of all the data points

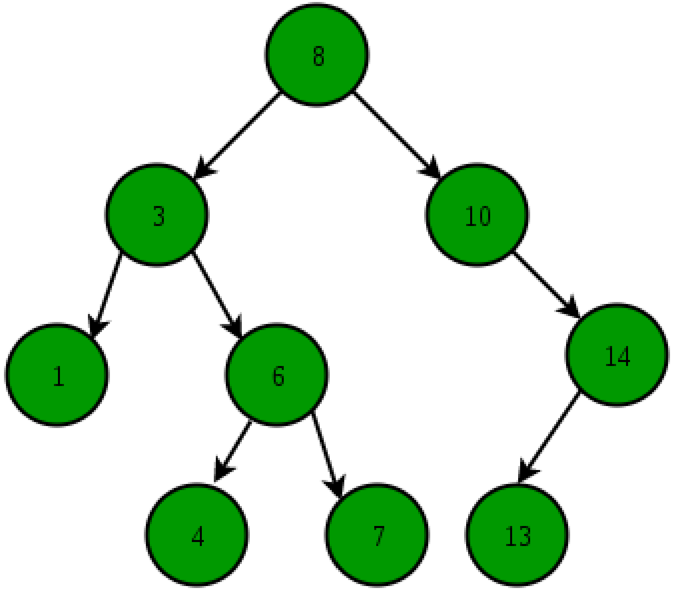
Stopping criteria

- How to skip some partitions?
  - Intersection of the “worst distance” with the partition boundaries
- How to stop the k-NN / Radius-NN search?
  - Search until the root
  - A partition completely contains the “worst distance”

Binary Search Tree (BST)

BST is a node-based tree data structure:

1. A node’ s left subtree contains nodes with keys lesser than its key
2. A node’ s right subtree contains nodes with keys larger than its key
3. The left / right subtree is BST



Binary Search Tree (BST)

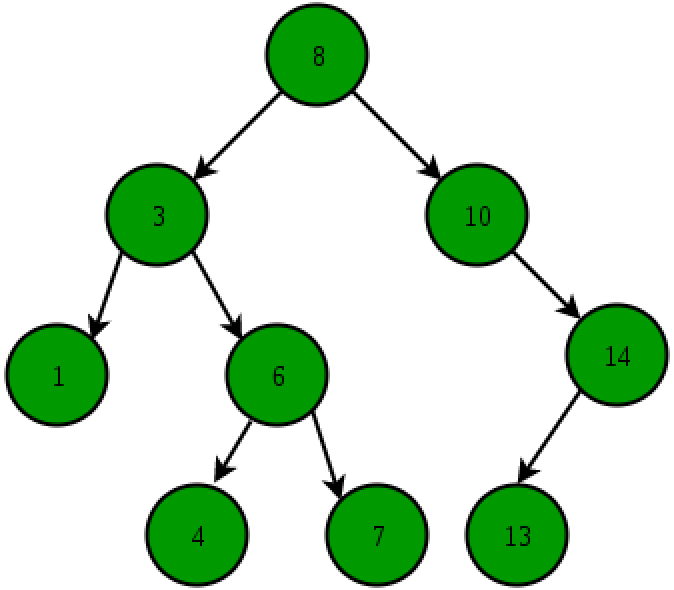
From Wikipedia

Binary search tree

Type	tree
Invented	1960
Invented by	P.F. Windley, A.D. Booth, A.J.T. Colin, and T.N. Hibbard

Time complexity in big O notation

Algorithm	Average	Worst case
Space	$O(n)$	$O(n)$
Search	$O(\log n)$	$O(n)$
Insert	$O(\log n)$	$O(n)$
Delete	$O(\log n)$	$O(n)$

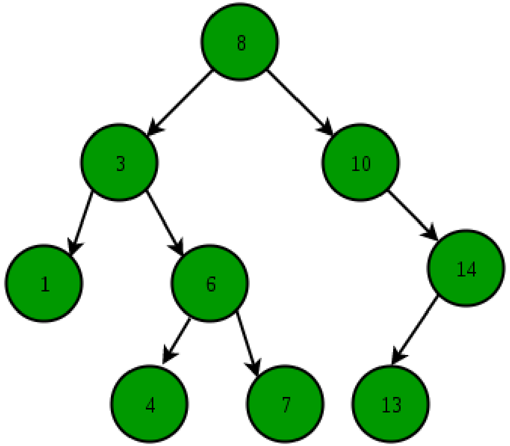


BST – Node definition

- A node contains
    1. Key
    2. Left child
    3. Right child
    4. … …

```python
class Node:
    def __init__(self, key, value=-1):
        self.left = None
        self.right = None
        self.key = key
        self.value = value
```

- The left/right child is a Node as well



BST – Construction / Insertion

- Given N 1D-points (scalar) denoted by an array
  $\{x_1, x_2, \cdots, x_n\},\ x_i \in \mathbb{R}$

- Construct a BST that stores the points and its index in the array, e.g.
  [100, 20, 500, 10, 30, 40]

Data generation

```python
db_size = 10
data = np.random.permutation(db_size).tolist()
```

Recursively insert each an element

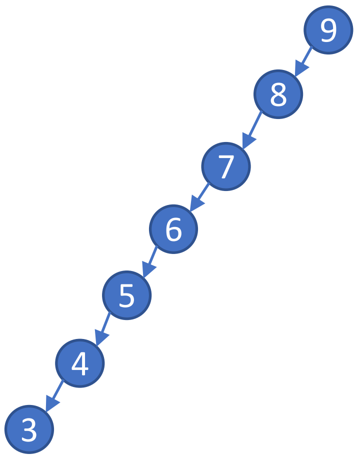
```python
def insert(root, key, value=-1):
    if root is None:
        root = Node(key, value)
    else:
        if key < root.key:
            root.left = insert(root.left, key, value)
        elif key > root.key:
            root.right = insert(root.right, key, value)
        else:  # don't insert if key already exist in the tree
            pass
    return root
```

Insert each element

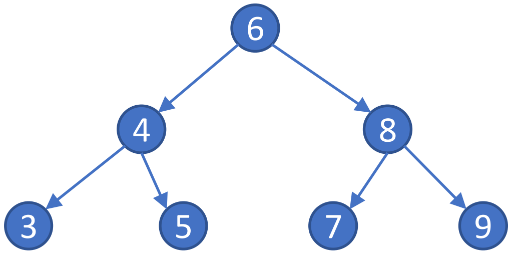
“value” in the Node is the index of a point in the array
Useful in later NN search

```python
root = None
for i, point in enumerate(data):
    root = insert(root, point, i)
```

BST – Insertion Complexity



- The worst case is $O(h)$, where h is the height of the BST
- In the worst case, h is the number of points in BST.
  - Unbalanced tree – a chain in an extreme case
  - E.g., inserting [9, 8, 7, 6, 5, 4, 3] into an empty BST
- Tree balancing is another topic
  - Sort the array and insert as a balanced tree (select median as root)
  - AVL tree
  - Red-Black tree
  - etc.
- Best case $h = \log_2 n$



BST – Search

- Given a BST, and a query (key), determine *which node* equals to that query (key), if not, return *NULL*

- Can be done *recursively* or *iteratively*

BST – Search Recursively

Search till the leaf but not found.

Find a match

For sure just need to look at the left

```
def search_recursive(root, key):
    if root is None or root.key == key:
        return root
    if key < root.key:
        return search_recursive(root.left, key)
    elif key > root.key:
        return search_recursive(root.right, key)
```

For sure just need to look at the right

BST – Search Iteratively

- Use "current_node" to simulate a *Stack*, so that recursion is avoided
- In any case, you can write your own *Stack* to avoid recursion, but that may be complicated sometimes.

```python
def search_iterative(root, key):
    current_node = root
    while current_node is not None:
        if current_node.key == key:
            return current_node
        elif key < current_node.key:
            current_node = current_node.left
        elif key > current_node.key:
            current_node = current_node.right
    return current_node
```

- Search recursively or iteratively complexity same as insertion worst O(h)

BST – Search

Recursion

Pros:

- Easy to understand / implement
- Codes are short

Cons:

- Hard to trace step-by-step
- $O(n)$ storage, n is number of recursion (May be optimized by compiler)

现在很多c++编译器会将递归转化为循环。

Iteration

Pros:

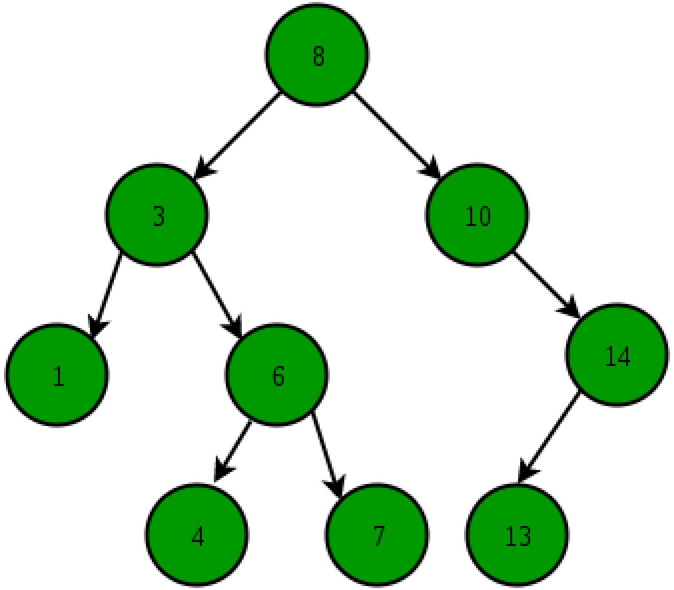
- Avoid stack-overflow, e.g., in embedded system / GPU
- Easier in step-by-step tracing
- $O(1)$ storeage

Cons:

- The logic is complicated

BST – Depth First Traversal

- Inorder – Left, Root, Right
  - E.g., sorting
  - 1, 3, 4, 6, 7, 8, 10, 13, 14
- Preorder – Root, Left, Right
  - E.g., copy a tree
  - 8, 3, 1, 6, 4, 7, 10, 14, 13
- Postorder – Left, Right, Root
  - E.g., delete a node
  - 1, 4, 7, 6, 3, 13, 14, 10, 8



```python
def inorder(root):
    # Inorder (Left, Root, Right)
    if root is not None:
        inorder(root.left)
        print(root)
        inorder(root.right)
```

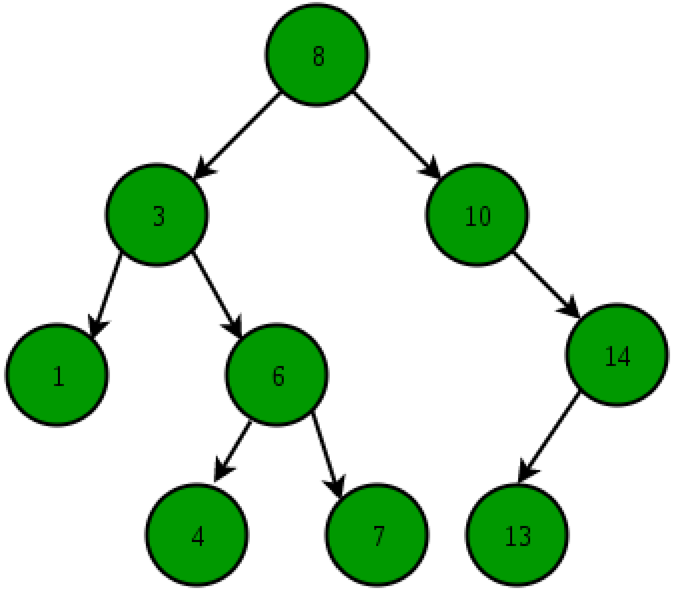
1, 3, 4, 6, 7, 8, 10, 13, 14

```python
def preorder(root):
    # Preorder (Root, Left, Right)
    if root is not None:
        print(root)
        preorder(root.left)
        preorder(root.right)
```

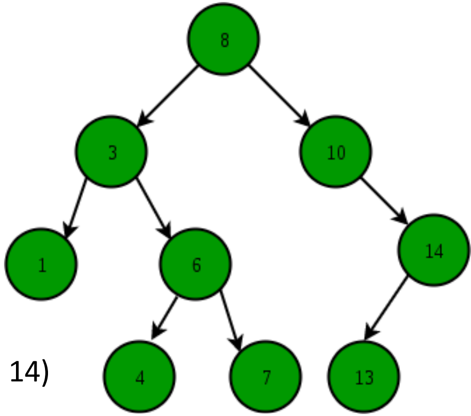
8, 3, 1, 6, 4, 7, 10, 14, 13

```python
def postorder(root):
    # Postorder (Left, Right, Root)
    if root is not None:
        postorder(root.left)
        postorder(root.right)
        print(root)
```

1, 4, 7, 6, 3, 13, 14, 10, 8



BST – 1NN Search



- Query point – 11

1. 8
   a) worst distance = 3 (11-8)
   b) any point in 8's left tree is at least 3 away from query
   c) do I need to go further? Yes! Right subtree is (8, +inf] but worst distance=3 -> need to search (8, 14)

2. 10
   a) worst distance = 1 (11-10)
   b) do I need to go further? Yes! Right subtree is (10, +inf] but worst distance=1 -> need to search (10, 12)

3. 14
   a) worst distance = 1
   b) do I need to further? Yes! Left subtree is (10, 14) but worst distance=1 -> need to search (10, 12)

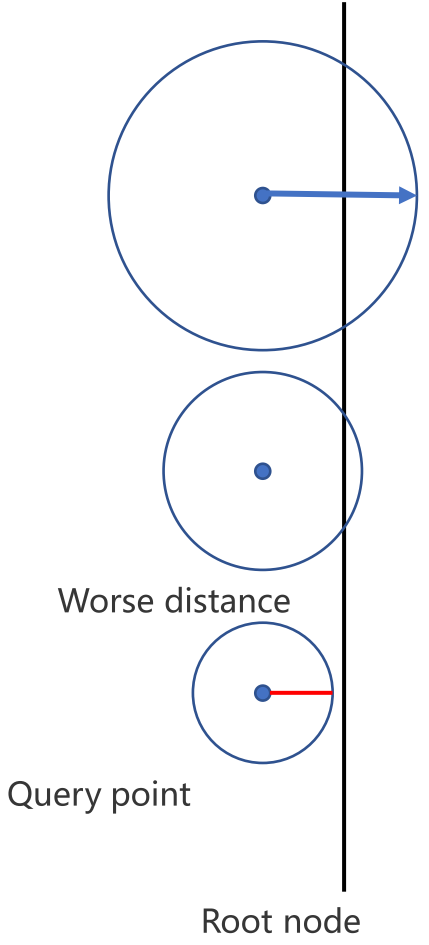
4. Go back to 14
   c) do I need to go right? No! Right subtree is (14, inf] but worst distance=1 -> I just need to search (10, 12)

5. Go back to 10
   c) do I need to go left? No! Left subtree is (8, 10) but worst distance=1 -> I just need to search (10, 12)

6. Go back to 8
   d) do I need to go left? No! Left subtree is [-inf, 8) but worst distance=1 -> just need to search (10, 12)

BST – kNN Search

- Almost same as 1NN search
- Difference is how to compute worst distance
- Worst distance is the largest distance that you should search around the query point
- Areas outside this “worst circle” can be skipped
- In kNN search, the worst distance is dynamic



BST – Worst Distance for kNN

- Build a container to store the kNN results
- k results are sorted
- worst_dist is the last one
- Add a result if

$$dist < worse\_dist$$

- Example:
- Existing container content
- [1, 2, 3, 4, inf, inf]

- **add_point(3.5)**
- *Step 1*. Make space for 3.5
- [1, 2, 3, 4, 4, inf] 将4复制一下，放到后面

- *Step 2*. Put 3.5 in the correct position
- [1, 2, 3, 3.5, 4, inf]

- *Step 3*. Update worst_dist

```python
class KNNResultSet:
    def __init__(self, capacity):
        self.capacity = capacity
        self.count = 0
        self.worst_dist = 1e10
        self.dist_index_list = []
        for i in range(capacity):
            self.dist_index_list.append(DistIndex(self.worst_dist, 0))

        self.comparison_counter = 0

    def size(self):
        return self.count

    def full(self):
        return self.count == self.capacity

    def worstDist(self):
        return self.worst_dist

    def add_point(self, dist, index):
        self.comparison_counter += 1
        if dist > self.worst_dist:
            return

        if self.count < self.capacity:
            self.count += 1

        i = self.count - 1
        while i > 0:
            if self.dist_index_list[i-1].distance > dist:
                self.dist_index_list[i] = copy.deepcopy(self.dist_index_list[i-1])
                i -= 1
            else:
                break

        self.dist_index_list[i].distance = dist
        self.dist_index_list[i].index = index
        self.worst_dist = self.dist_index_list[self.capacity-1].distance
```

Initialized to large value

Container to keep all the k neighbors

If a point is added, put it in a ordered position

```python
class DistIndex:
    def __init__(self, distance, index):
        self.distance = distance
        self.index = index

    def __lt__(self, other):
        return self.distance < other.distance
```

```python
def knn_search(root: Node, result_set: KNNResultSet, key):
    if root is None:
        return False

    # compare the root itself
    result_set.add_point(math.fabs(root.key - key), root.value)
    if result_set.worstDist() == 0:
        return True

    if root.key >= key:
        # iterate left branch first
        if knn_search(root.left, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return knn_search(root.right, result_set, key)
        return False
    else:
        # iterate right branch first
        if knn_search(root.right, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return knn_search(root.left, result_set, key)
        return False
```

A special case – if the worst distance is 0, no need to search anymore

If key != query, need to go through one subtree

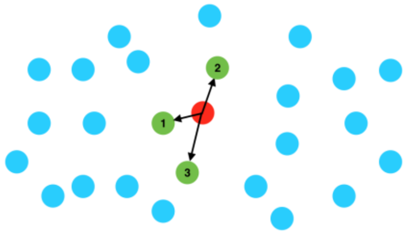
May not need to search for the other subtree, depends on worst distance

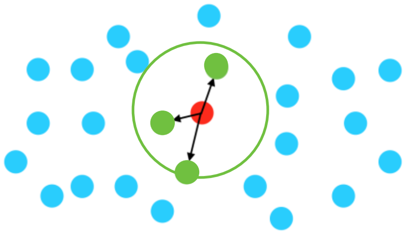
只要有小于worstDist的情况，就要继续找。

Similar to the "if" block above

Radius NN Search

- Same as kNN, in the sense that,
  - Worst distance circle intersects the boundary -> search
  - If not -> skip
- In implementation, we don’t need to change the BST kNN search logics, except that, the worst distance is fixed, instead of dynamic





BST – Radius Result Set Manger

- Simpler than kNN result set manager.
- Worst distance is fixed.
- No need to maintain a sorted result set.

```
def add_point(self, dist, index):
    self.comparison_counter += 1
    if dist > self.radius:
        return

    self.count += 1
    self.dist_index_list.append(DistIndex(dist, index))
```

BST - kNN v.s. Radius NN

```python
def knn_search(root: Node, result_set: KNNResultSet, key):
    if root is None:
        return False

    # compare the root itself
    result_set.add_point(math.fabs(root.key - key), root.value)
    if result_set.worstDist() == 0:
        return True

    if root.key >= key:
        # iterate left branch first
        if knn_search(root.left, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return knn_search(root.right, result_set, key)
        return False
    else:
        # iterate right branch first
        if knn_search(root.right, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return knn_search(root.left, result_set, key)
        return False
```

This part is gone in radius search, because worst_dist = r

```python
def radius_search(root: Node, result_set: RadiusNNResultSet, key):
    if root is None:
        return False

    # compare the root itself
    result_set.add_point(math.fabs(root.key - key), root.value)

    if root.key >= key:
        # iterate left branch first
        if radius_search(root.left, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return radius_search(root.right, result_set, key)
        return False
    else:
        # iterate right branch first
        if radius_search(root.right, result_set, key):
            return True
        elif math.fabs(root.key-key) < result_set.worstDist():
            return radius_search(root.left, result_set, key)
        return False
```

A complete script

- Search in 100 points, takes 7 comparison only
- Complexity is around $O(\log_2(n))$, n is number of database points, if tree is balanced
- Worst O(N)

```python
db_size = 100
k = 5
radius = 2.0

data = np.random.permutation(db_size).tolist()

root = None
for i, point in enumerate(data):
    root = insert(root, point, i)

query_key = 6
result_set = KNNResultSet(capacity=k)
knn_search(root, result_set, query_key)
print('kNN Search:')
print('index - distance')
print(result_set)

result_set = RadiusNNResultSet(radius=radius)
radius_search(root, result_set, query_key)
print('Radius NN Search:')
print('index - distance')
print(result_set)
```

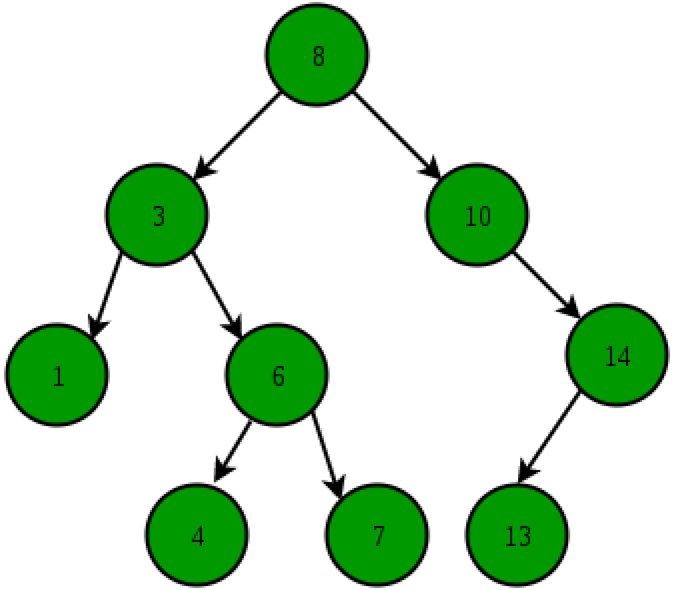
```
kNN Search:
index - distance
73 - 0.00
5 - 1.00
12 - 1.00
1 - 2.00
98 - 2.00
In total 7 comparison operations.

Radius NN Search:
index - distance
73 - 0.00
5 - 1.00
12 - 1.00
1 - 2.00
98 - 2.00
In total 5 neighbors within 2.000000.
There are 7 comparison operations.
```

Binary Search Tree (BST)

BST based 1D kNN/RadiusNN search

- Naïve BST is for 1D data only

Tree based kNN/RadiusNN can be viewed as a Branch-n-Bound algorithm.

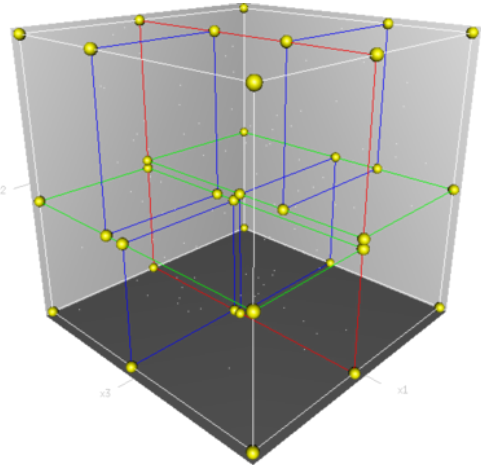


Kd-tree (k-dimensional tree)

- It is an extension of BST into high dimension
  - BST is 1-dimensional, how to extend?
  - BST in each dimension!
- Invented by Jon Louis Bentley in 1975
- The kd-tree is a binary tree where every leaf node is a k-dimensional point

_k_-d tree

Type	Multidimensional BST	
Invented	1975	
Invented by	Jon Louis Bentley	
Time complexity in big O notation		
Algorithm	**Average**	**Worst case**
Space	$O(n)$	$O(n)$
Search	$O(\log n)$	$O(n)$
Insert	$O(\log n)$	$O(n)$
Delete	$O(\log n)$	$O(n)$



A 3-dimensional kd tree:

1. Red
2. Green
3. Blue

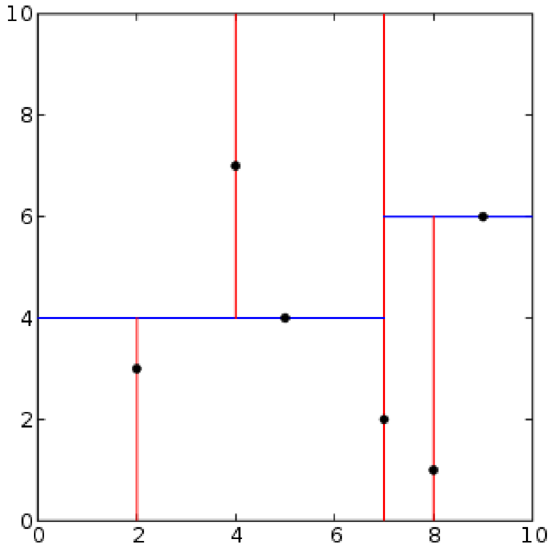
Kd-tree Construction

- If there is only one point, or number of points < leaf_size, build a leaf
- Otherwise, divide the points in half by a hyperplane perpendicular to the selected splitting axis
- Recursively repeat the first two steps.

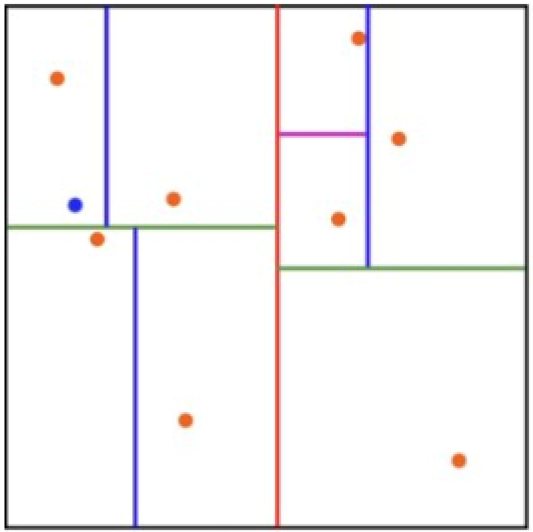
Kd-tree Construction – Two Conventions

Splitting position is one of the points

课件中使用的是这个方法



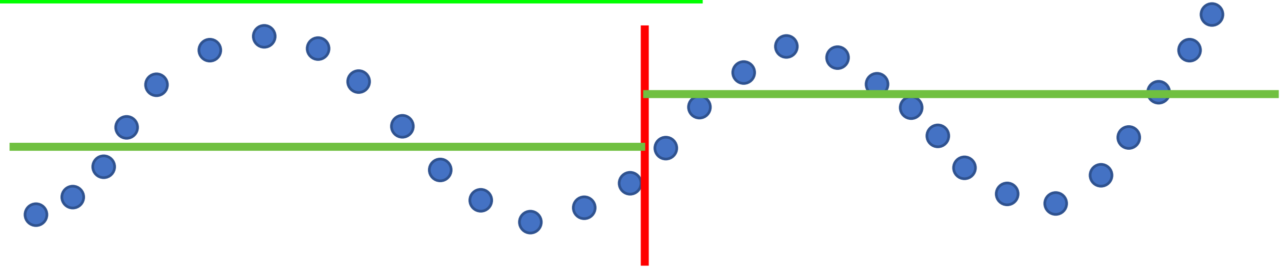
Splitting position is NOT one of the points



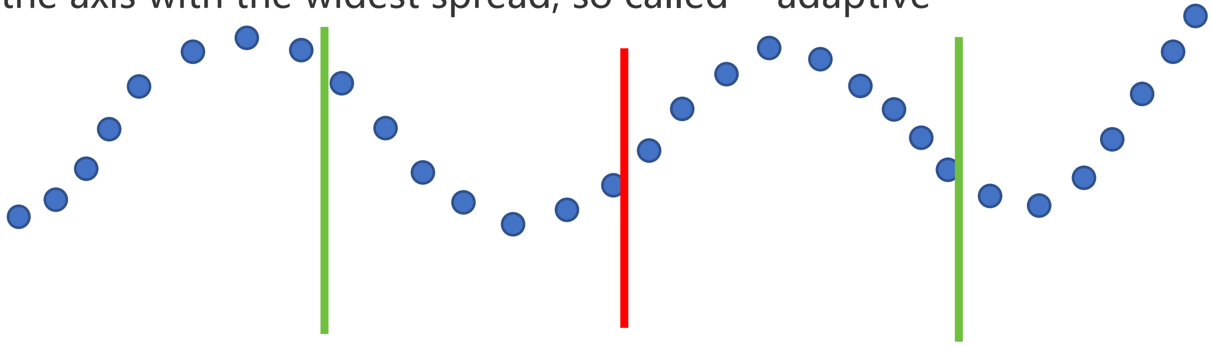
Kd-tree Construction

Division / Splitting Strategy

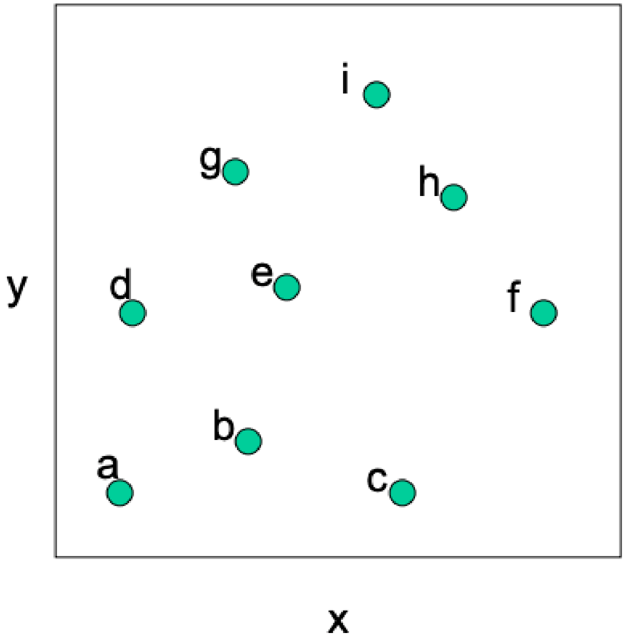
- Dividing axis is round-robin: x-y-z-x-y-z-x-...... 课件中使用的是这个方法



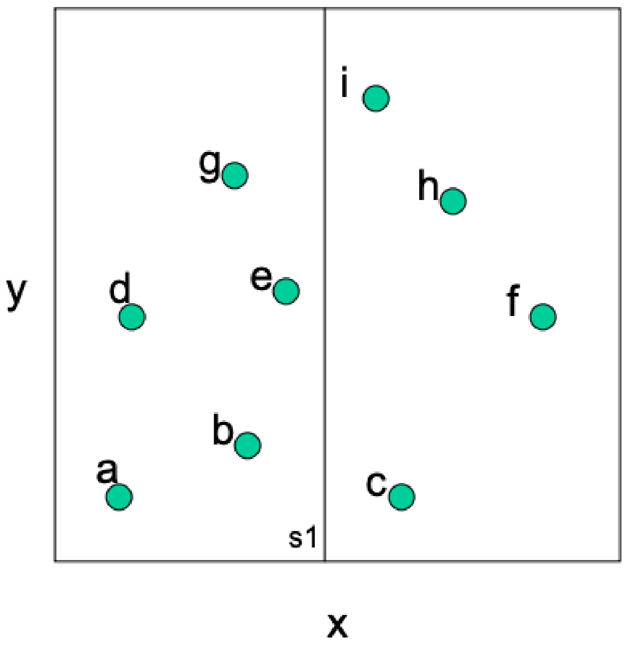
- Select the axis with the widest spread, so called “adaptive”

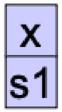


Kd-tree Construction

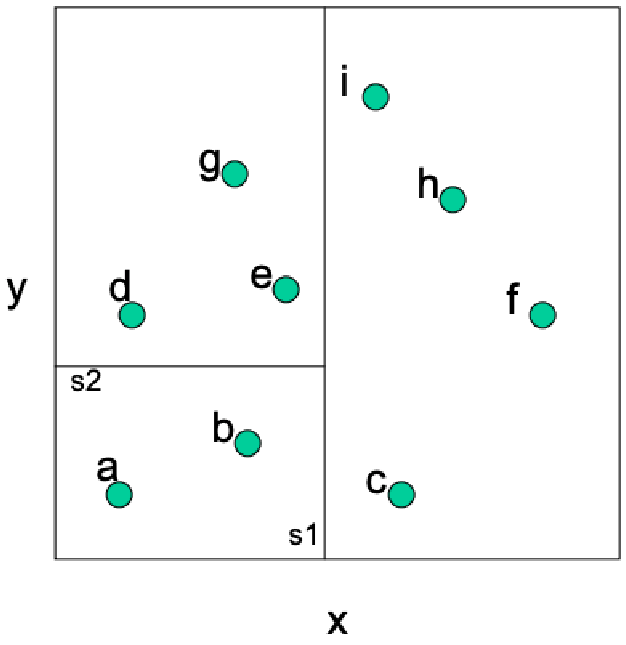


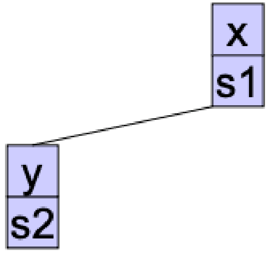
Kd-tree Construction



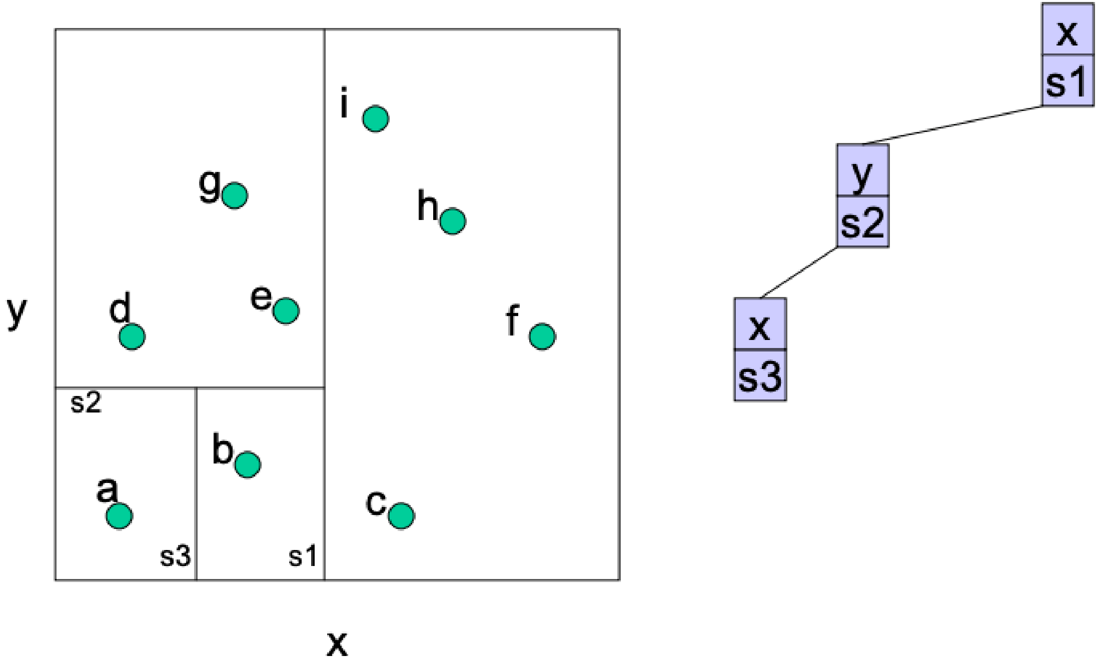


Kd-tree Construction

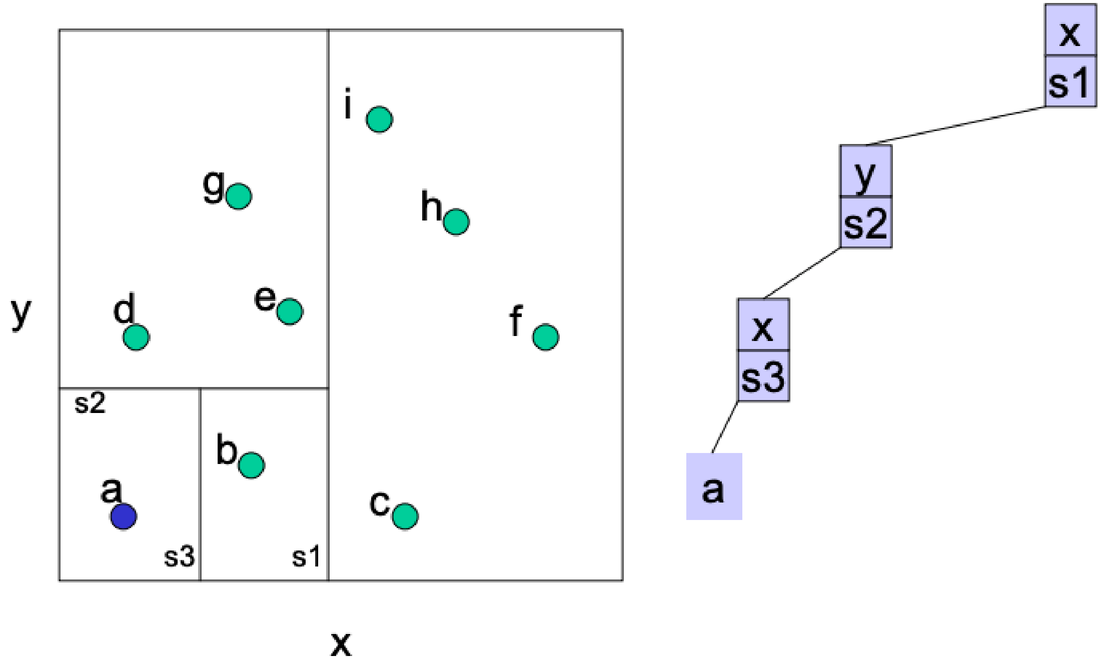




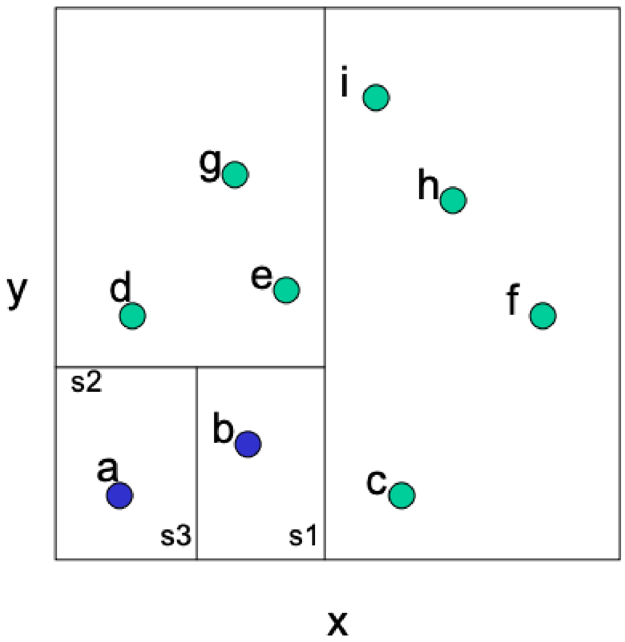
Kd-tree Construction

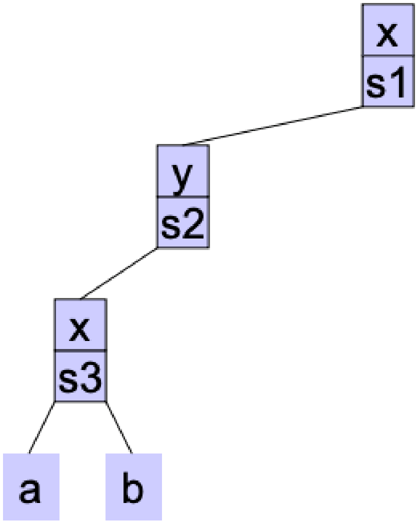


Kd-tree Construction



Kd-tree Construction





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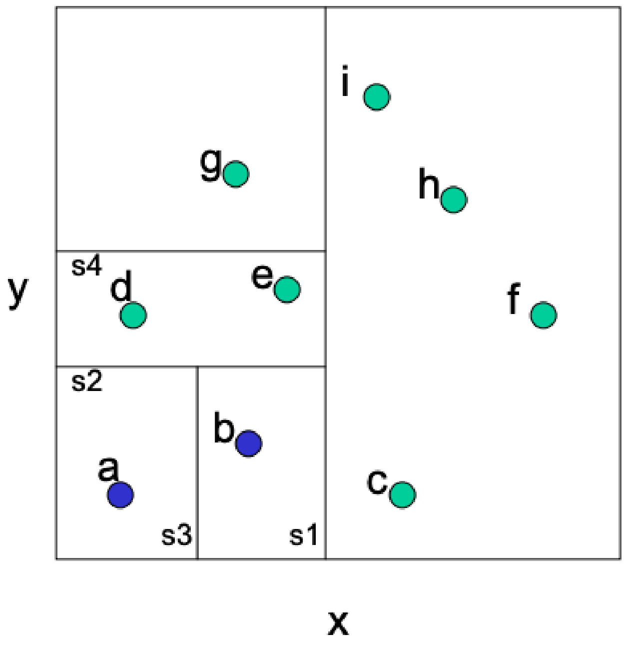

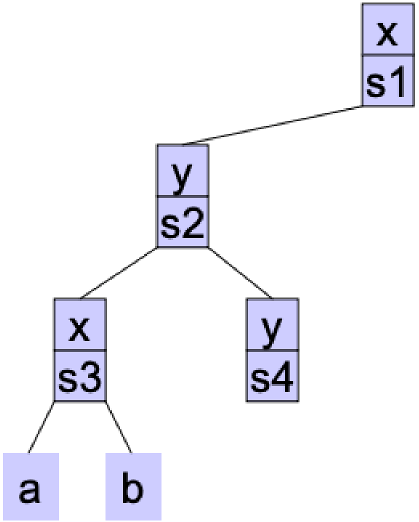

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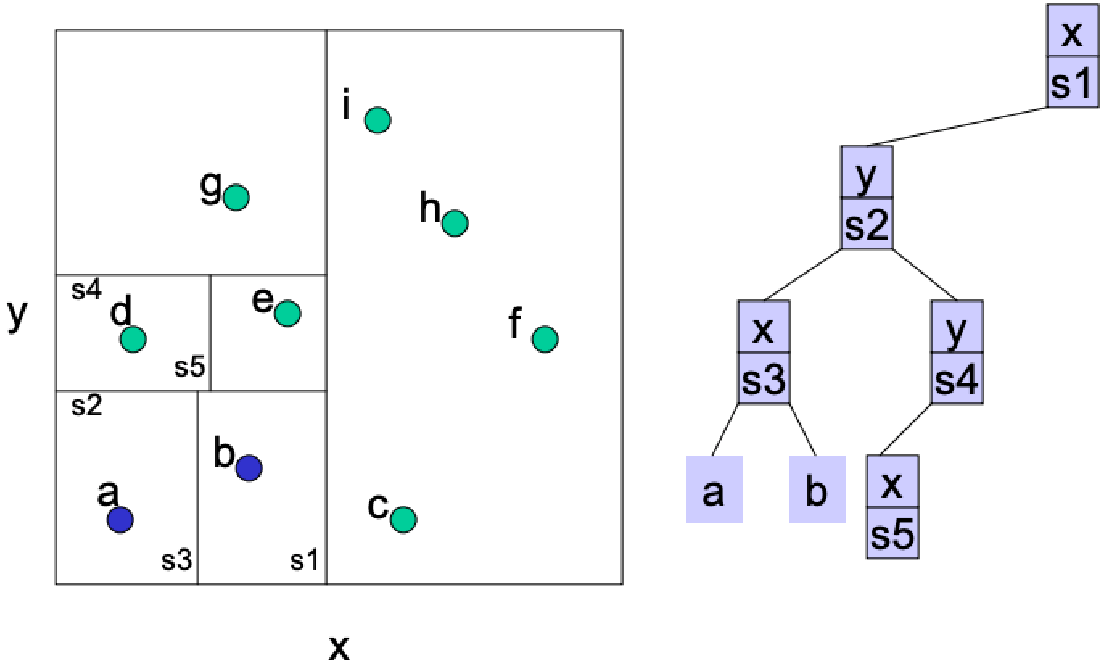

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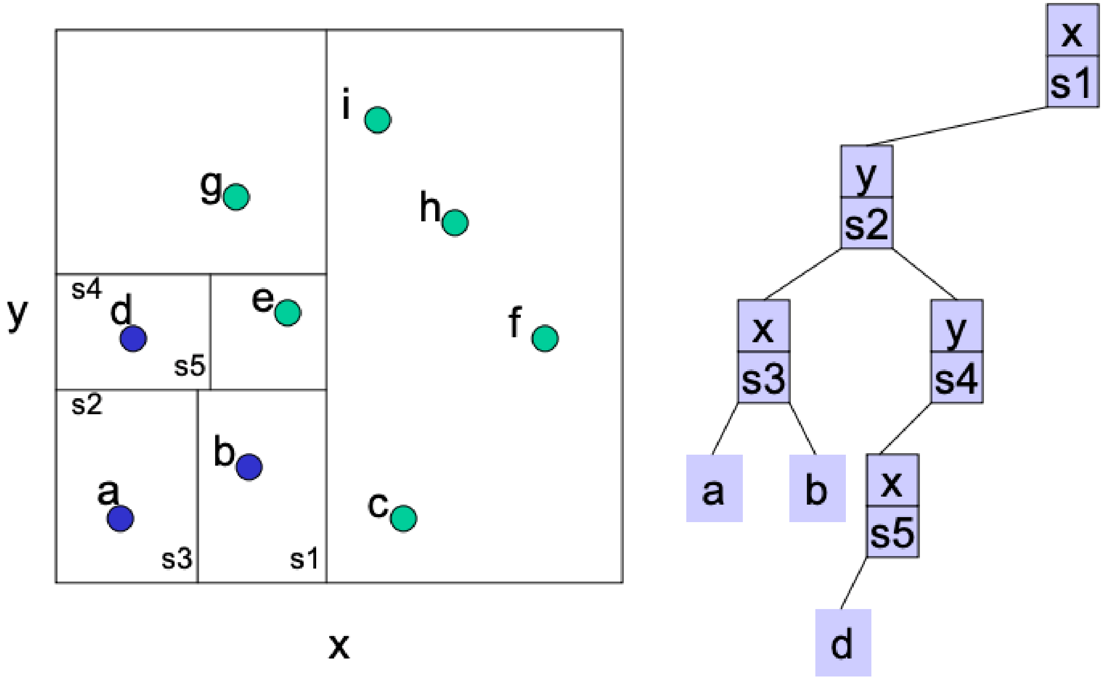

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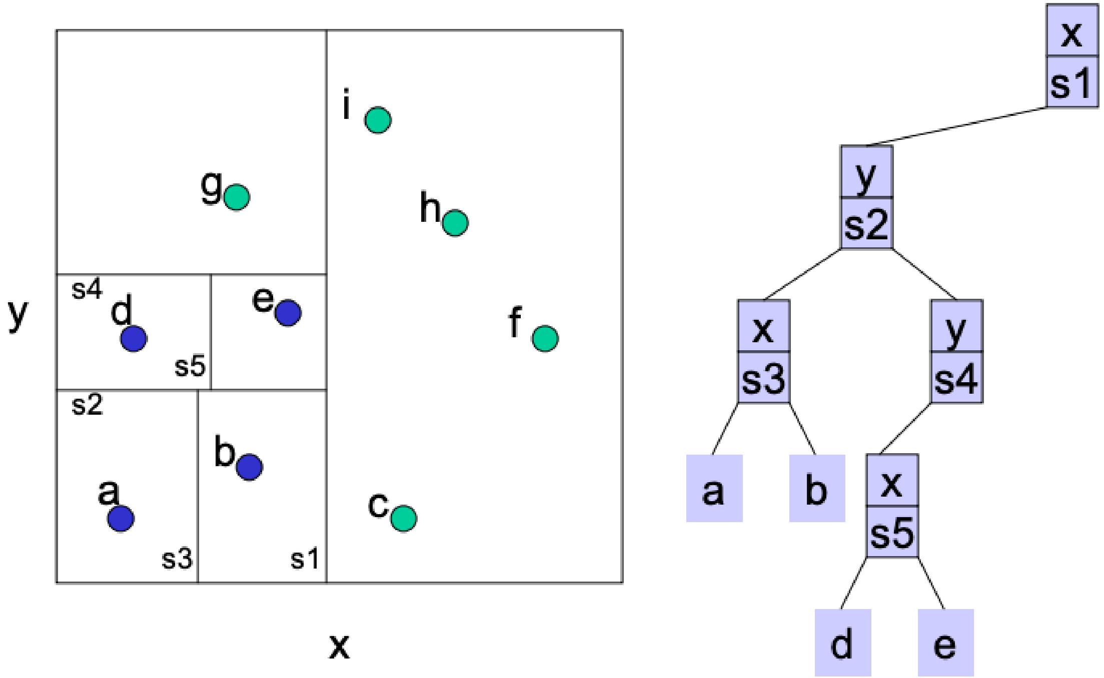

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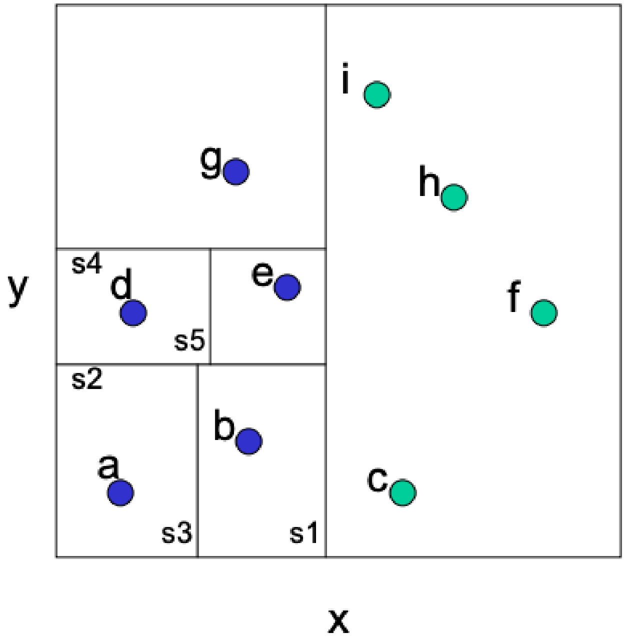

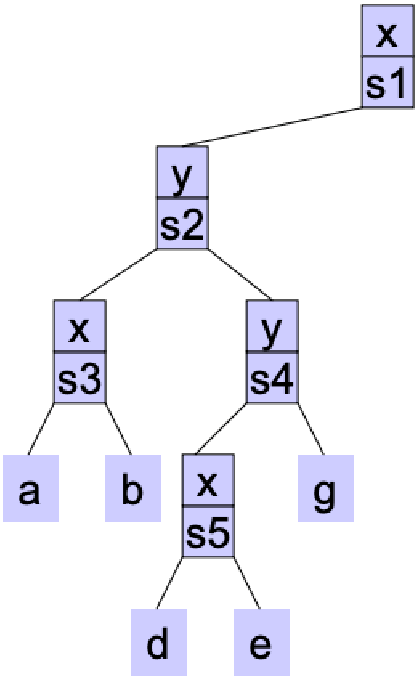

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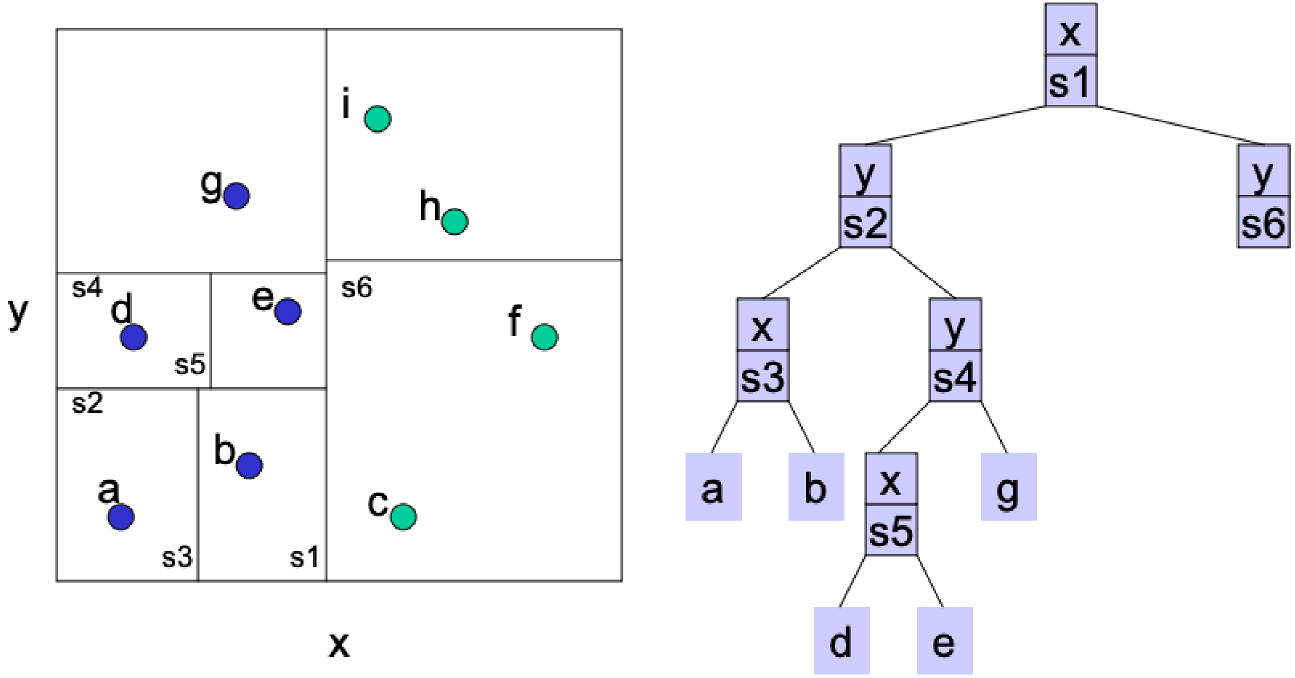

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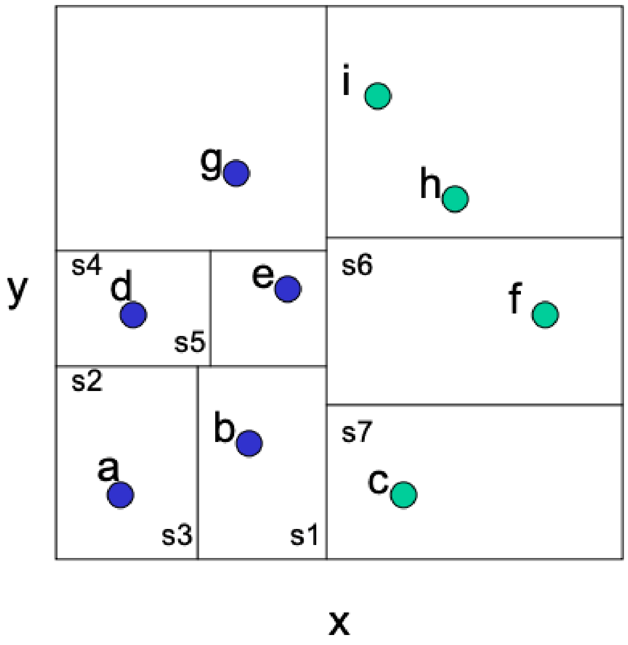
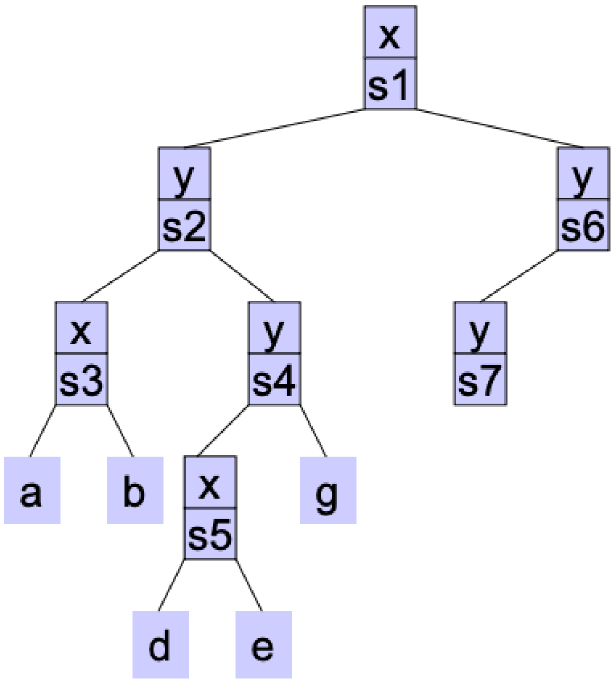
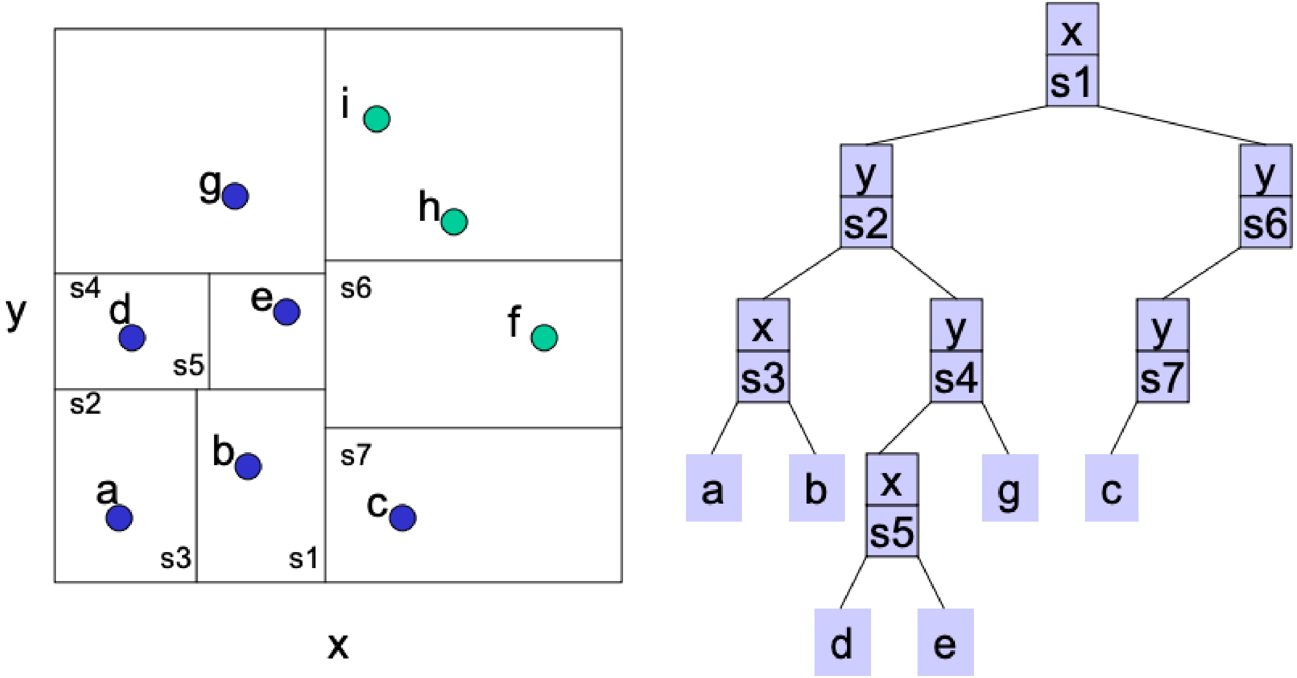




Kd-tree Construction



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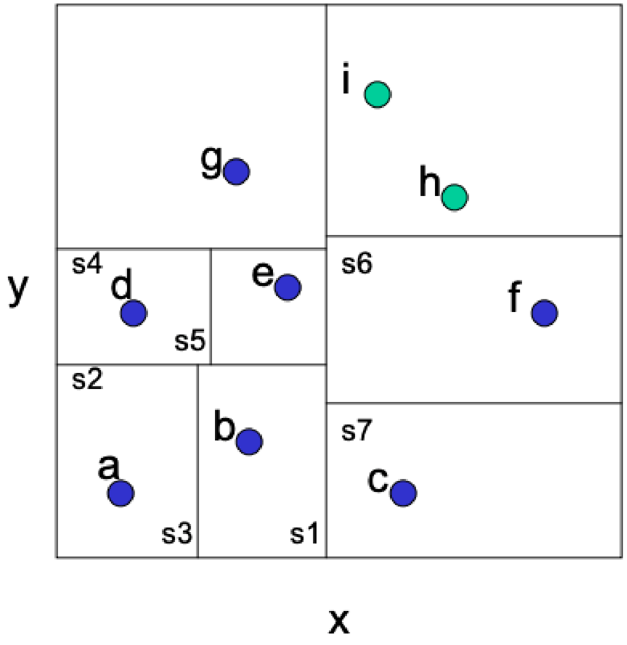

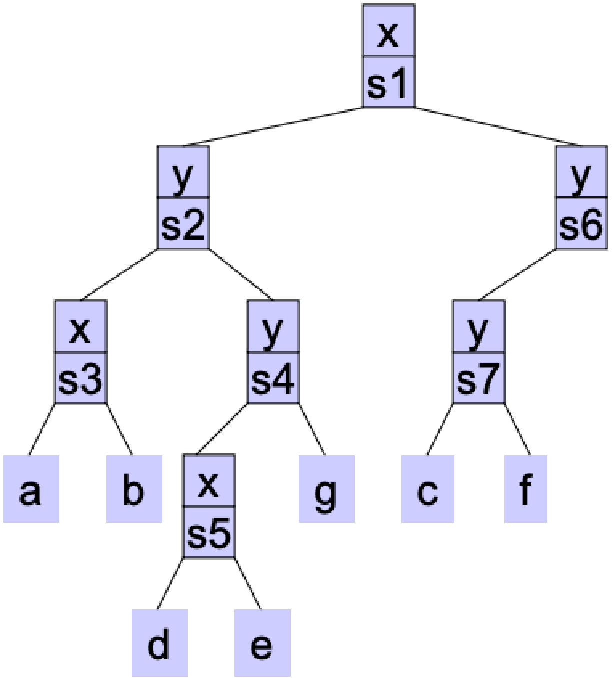

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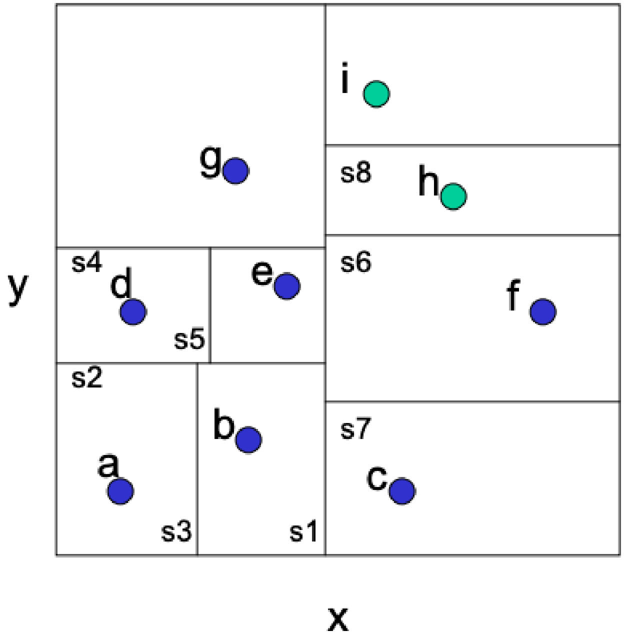

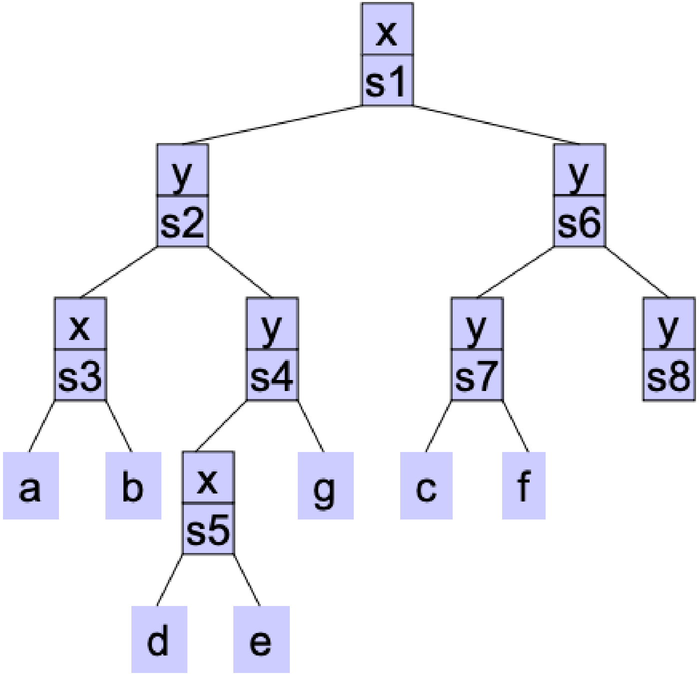

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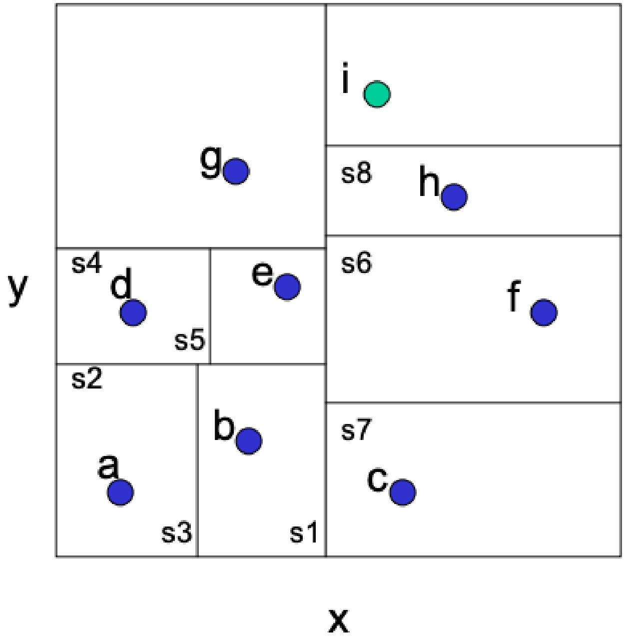
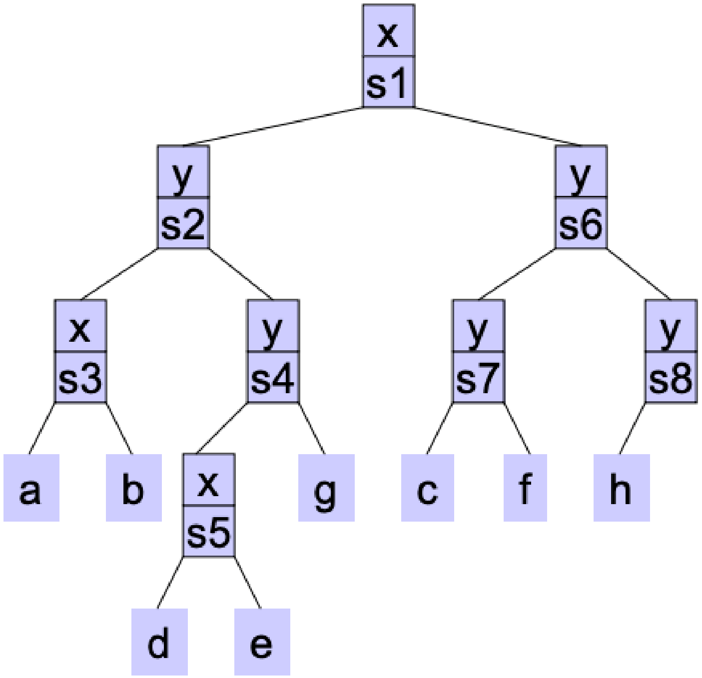
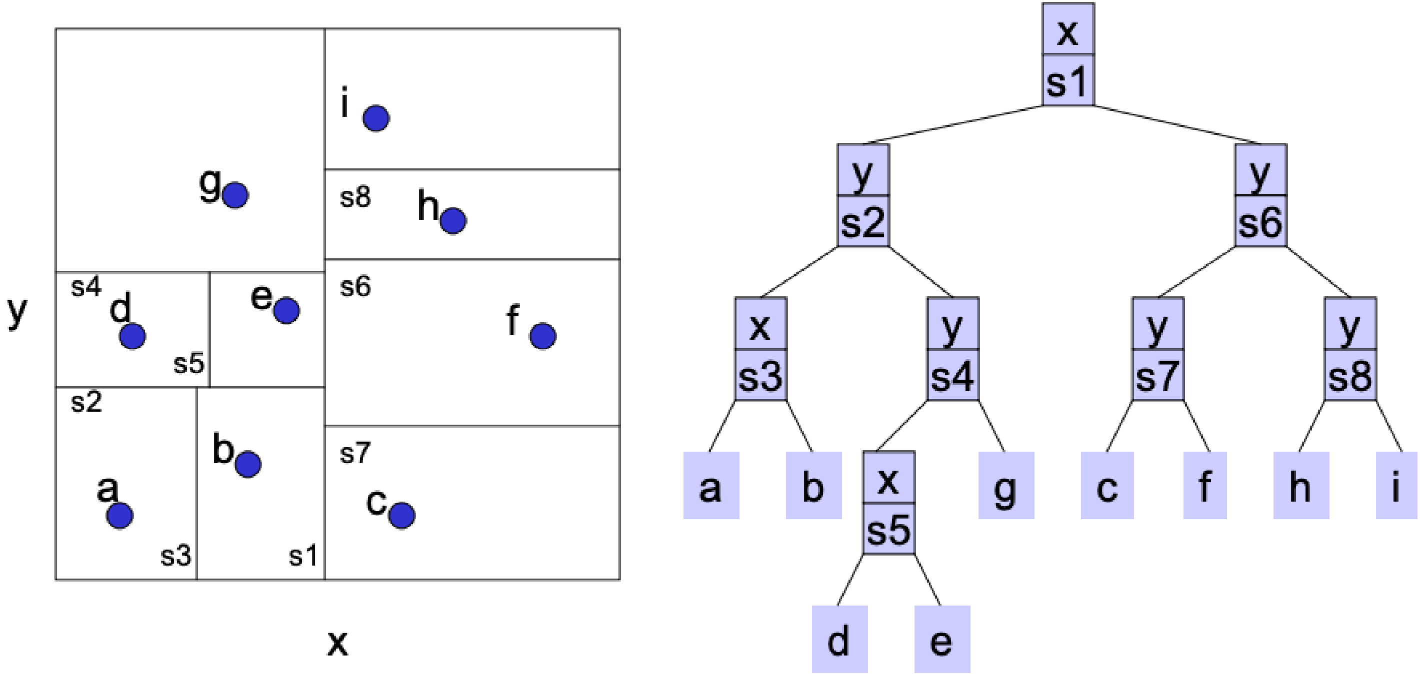




Kd-tree Construction



Kd-tree Construction

Talk is cheap, show me the code.

Linus Torvalds

Kd-tree Node Representation

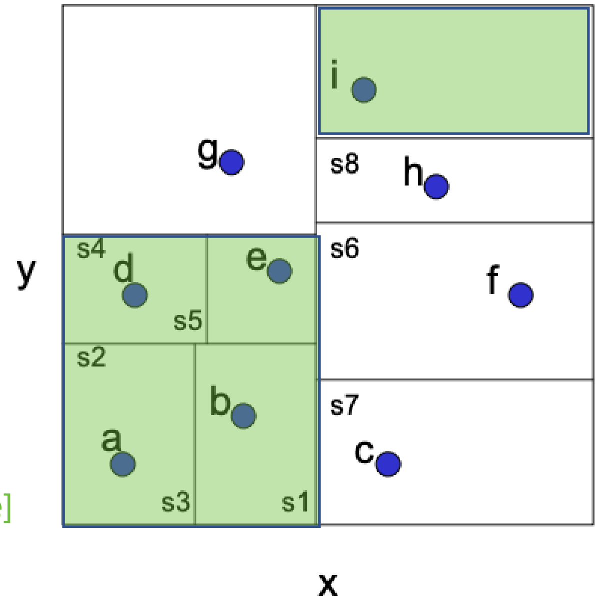
```
class Node:
    def __init__(self, axis, value, left, right, point_indices):
        self.axis = axis
        self.value = value
        self.left = left
        self.right = right
        self.point_indices = point_indices

    def is_leaf(self):
        if self.value is None:
            return True
        else:
            return False
```

Splitting position

Stores points that belongs to this partition

A node with
axis = x
value = ***
points = [a, b, d, e]

A leaf node with
axis = y
value = None
points = i



```python
def kdtree_recursive_build(root, db, point_indices, axis, leaf_size):
    """

    :param root:
    :param db: NxD
    :param db_sorted_idx_inv: NxD
    :param point_idx: M
    :param axis: scalar
    :param leaf_size: scalar
    :return:
    """
    if root is None:
        root = Node(axis, None, None, None, point_indices)

    # determine whether to split into left and right
    if len(point_indices) > leaf_size:
        # --- get the split position ---
        point_indices_sorted, _ = sort_key_by_vale(point_indices, db[point_indices, axis])  # M
        middle_left_idx = math.ceil(point_indices_sorted.shape[0] / 2) - 1
        middle_left_point_idx = point_indices_sorted[middle_left_idx]
        middle_left_point_value = db[middle_left_point_idx, axis]

        middle_right_idx = middle_left_idx + 1
        middle_right_point_idx = point_indices_sorted[middle_right_idx]
        middle_right_point_value = db[middle_right_point_idx, axis]

        root.value = (middle_left_point_value + middle_right_point_value) * 0.5
        # === get the split position ===
        root.left = kdtree_recursive_build(root.left,
                                           db,
                                           point_indices_sorted[0:middle_right_idx],
                                           axis_round_robin(axis, dim=db.shape[1]),
                                           leaf_size)
        root.right = kdtree_recursive_build(root.right,
                                            db,
                                            point_indices_sorted[middle_right_idx:],
                                            axis_round_robin(axis, dim=db.shape[1]),
                                            leaf_size)
    return root
```

A leaf node can contain more than 1 point

Sort the points in this node, get the median position

```python
def axis_round_robin(axis, dim):
    if axis == dim-1:
        return 0
    else:
        return axis + 1
```

Kd-tree Construction Complexity

- The example shown here is not optimal because of sorting at each level of the tree
  - Time complexity of around $O(n \log n \log n)$
  - Space complexity of $O(kn + n \log n)$ → can be easily reduced to $O(kn + n)$
    - Only store points at leaf
- Can we select median instead of sorting?
  - If median finding is $O(n)$
  - Kd-tree is $O(n \log n)$
  - Median finding in $O(n)$ is complicated, but possible!
- $O(kn \log n)$ method
  - Building a Balanced k-d Tree in $O(kn \log n)$ Time

    – Russel A. Brown, Journal of Computer Graphics Techniques, 2015

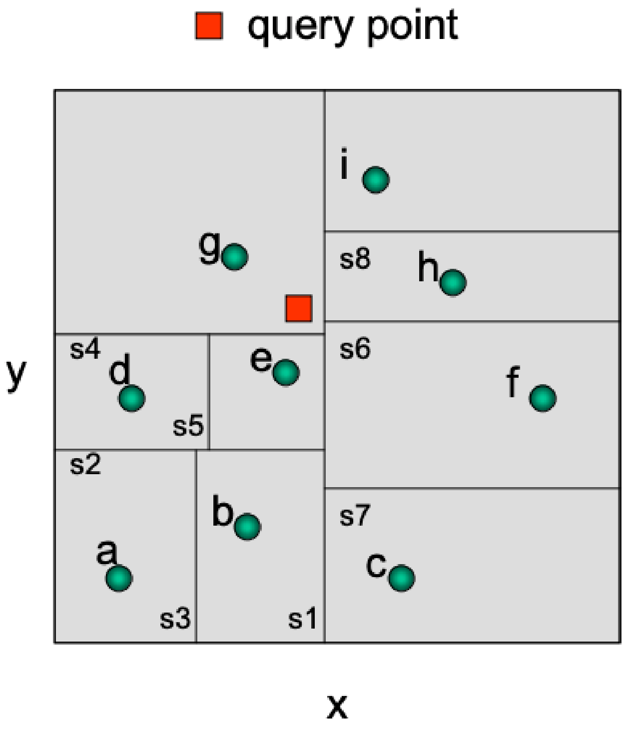
Kd-tree Construction Complexity

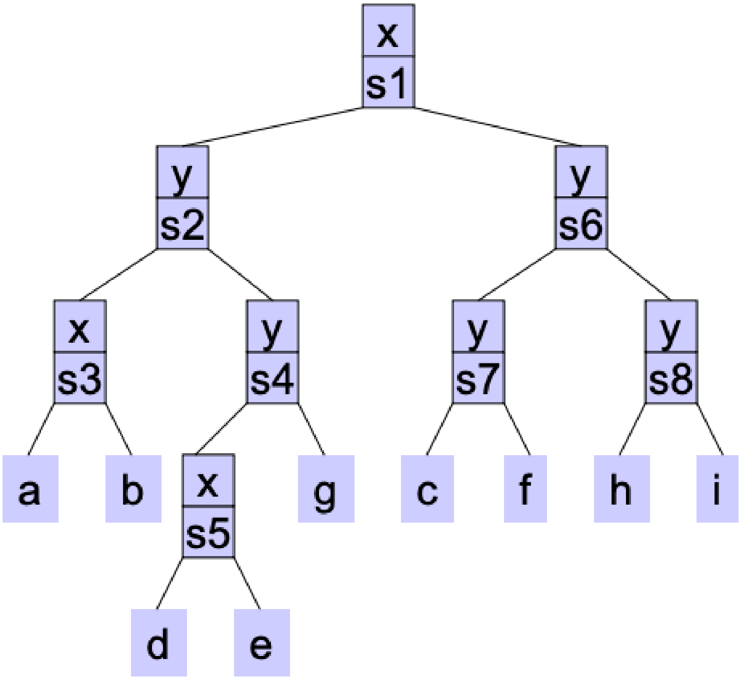
- Simple methods that work well in practice
  - Sample a subset of point in each node for sorting, instead of sorting all points
    - $O(n' \log n)$ or $O(n' \log n' \log n)$
  - Use mean instead of median
    - An easy way to achieve $O(n \log n)$
- They don't guarantee balanced kd-tree
  - Balanced tree - each leaf node is approximately the same distance from the root
  - Similar to the " chain" example in BST.

Kd-tree – kNN Search

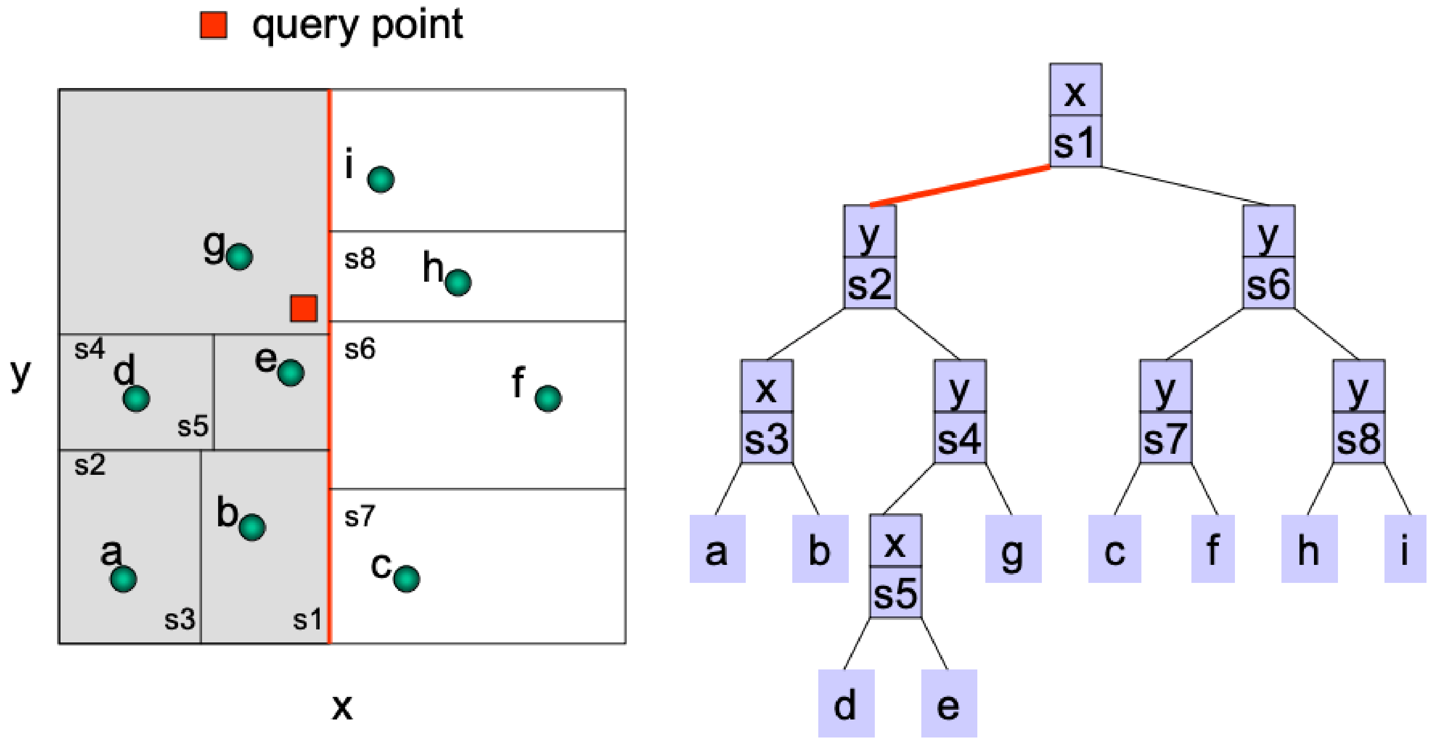
- Start from root
- Reach the leaf node than covers the query point
  - Compare all points in the leaf node
- Go up and traverse the tree

Kd-tree – kNN Search

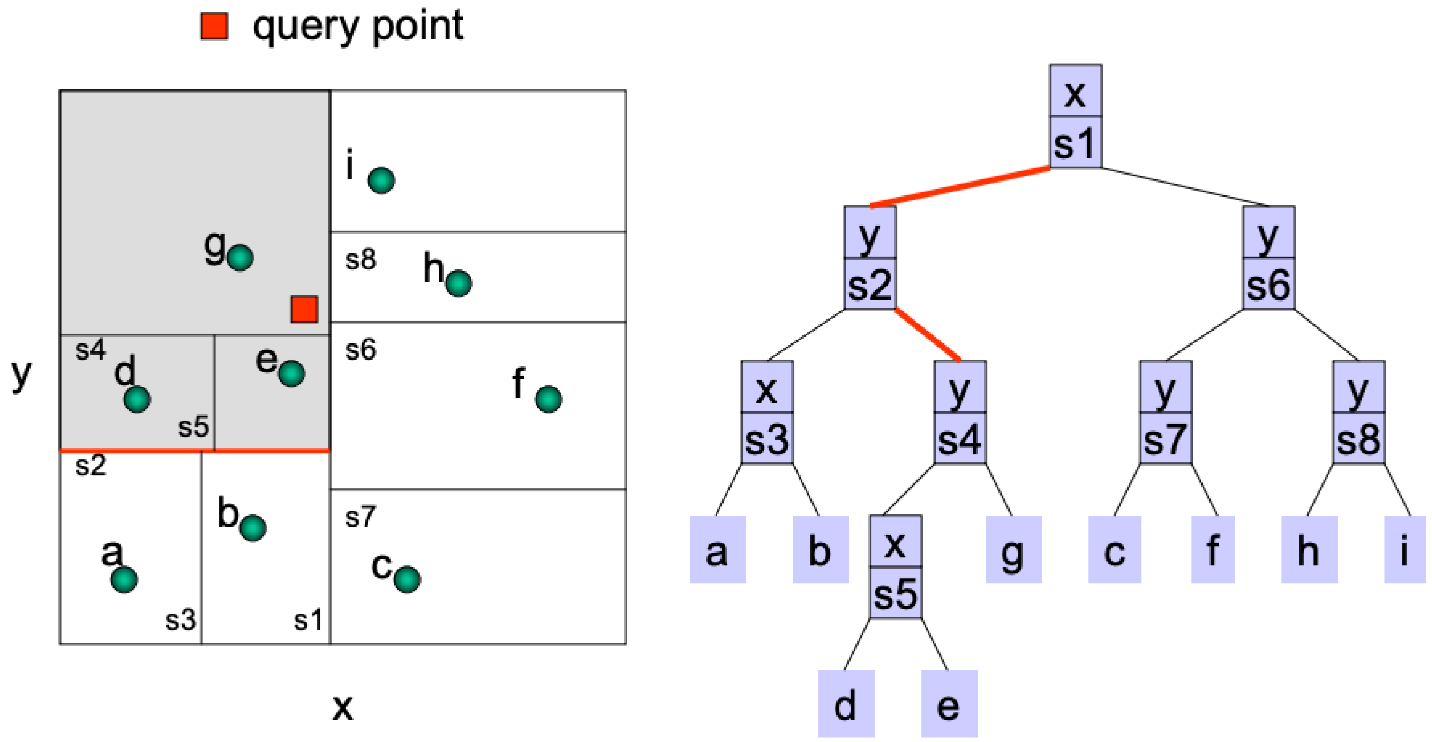




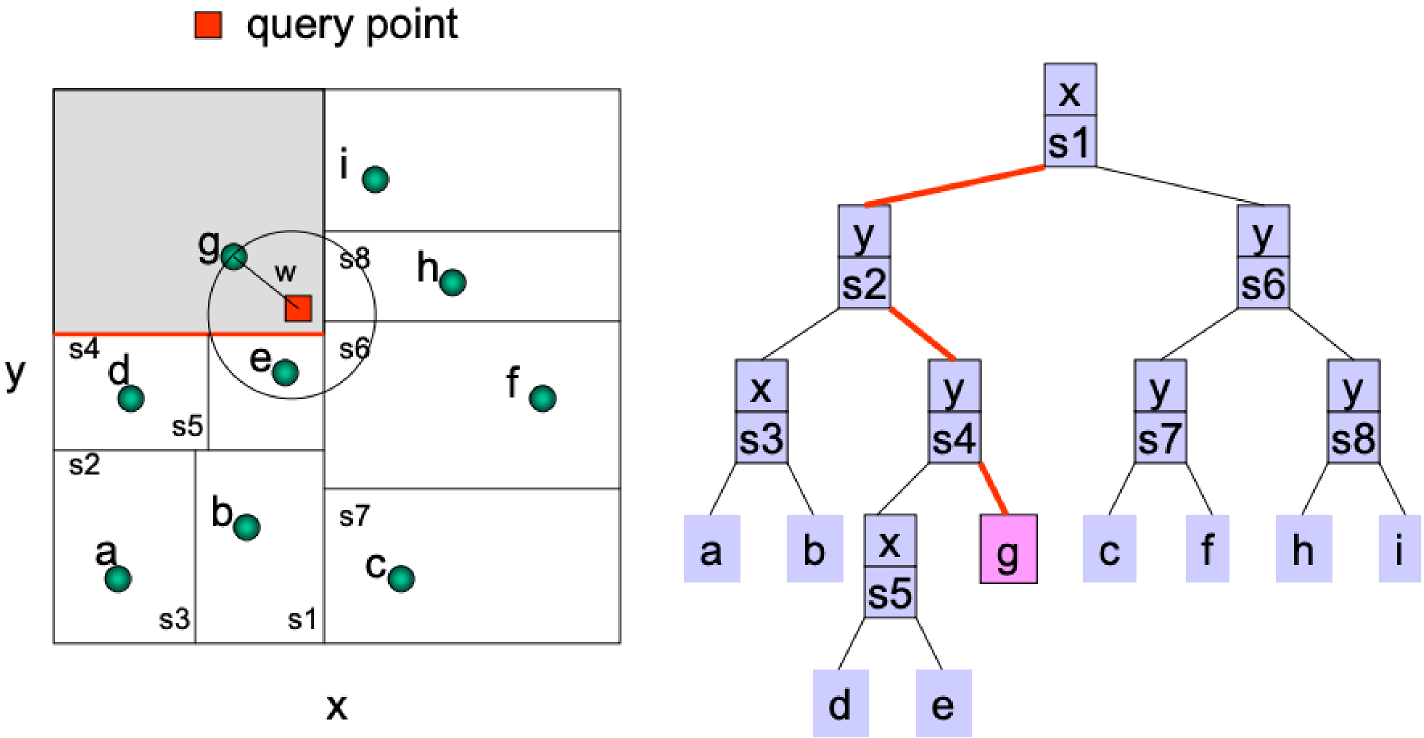
Kd-tree – kNN Search



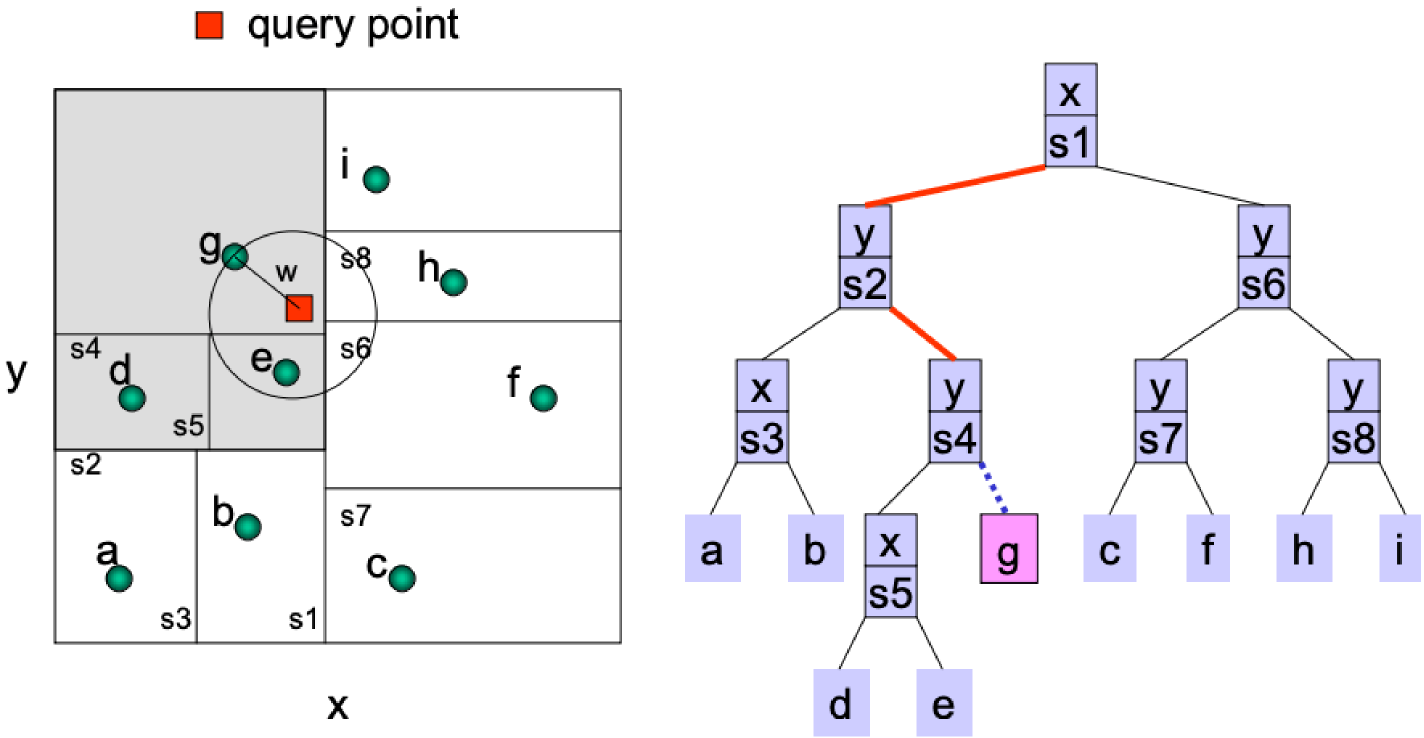
Kd-tree – kNN Search



Kd-tree – kNN Search



Kd-tree – kNN Search



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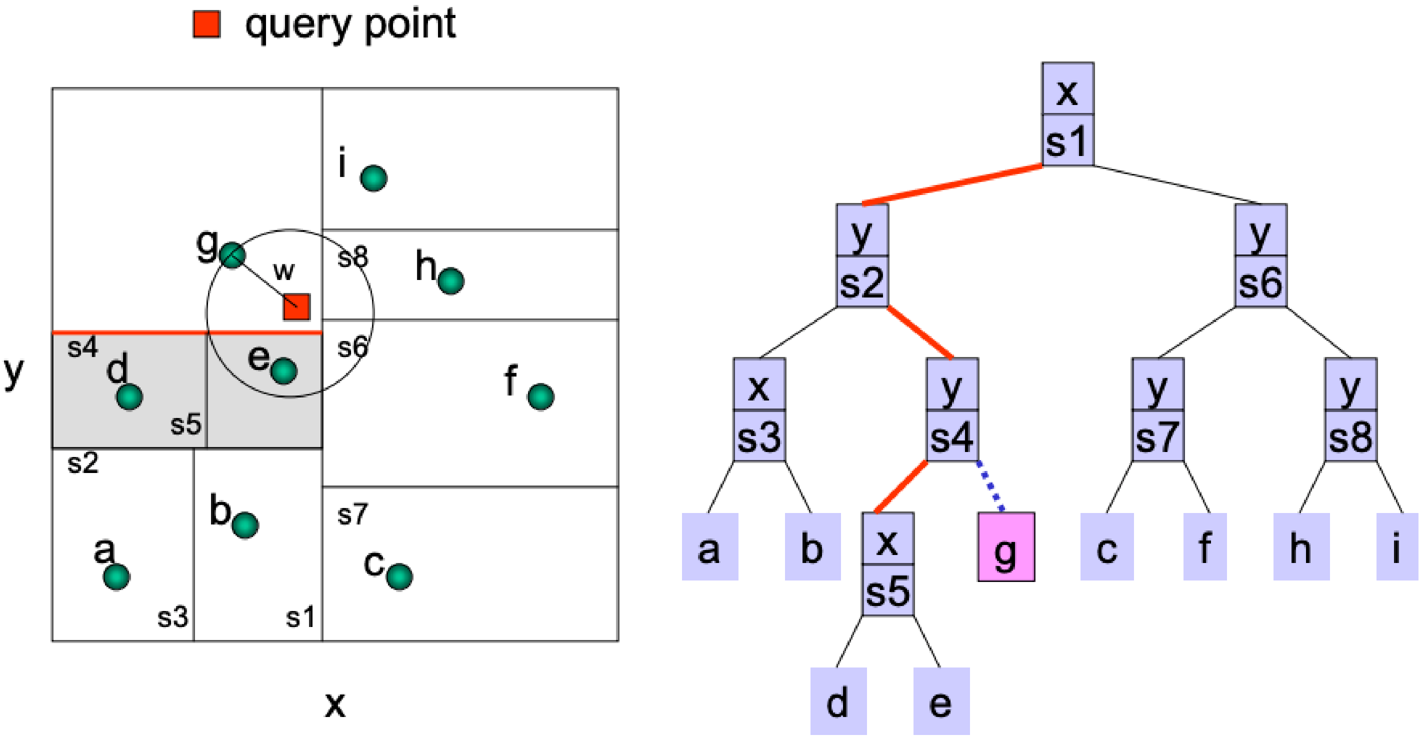

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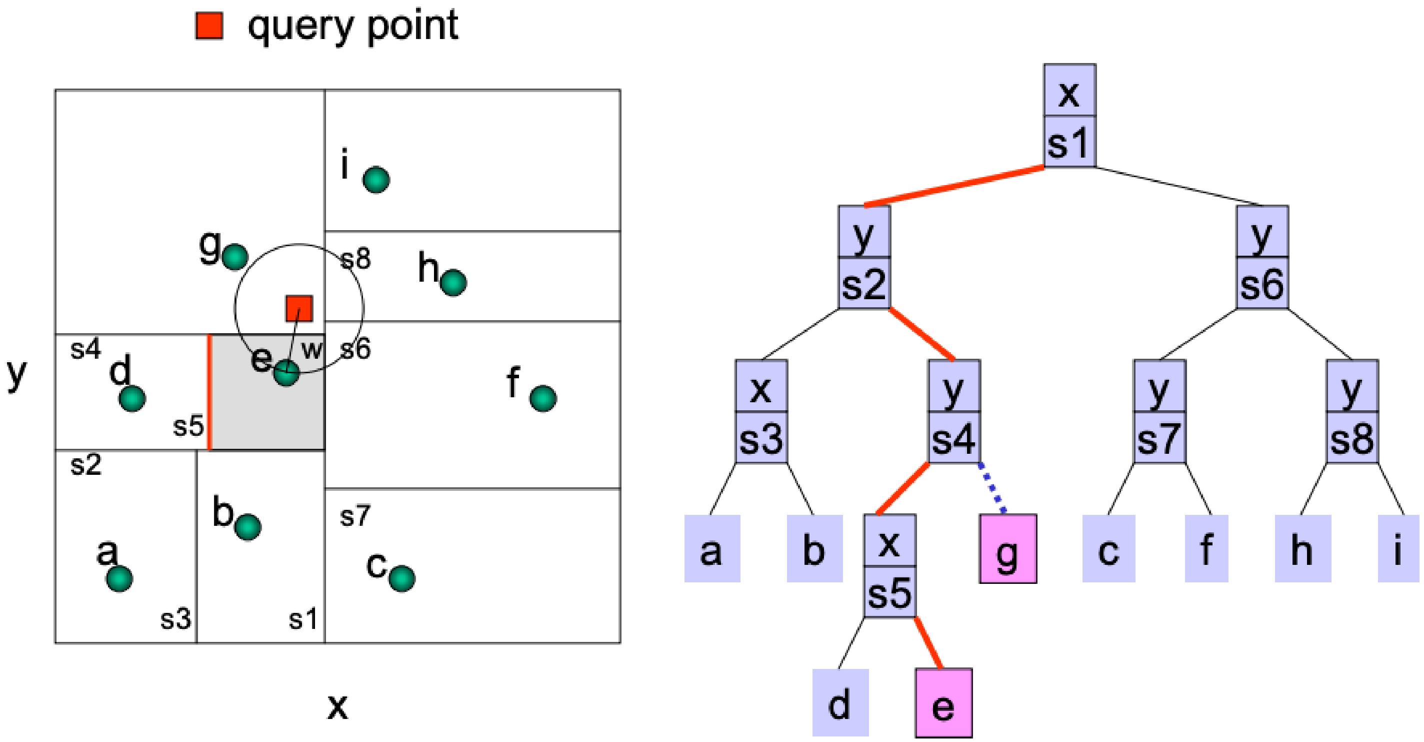

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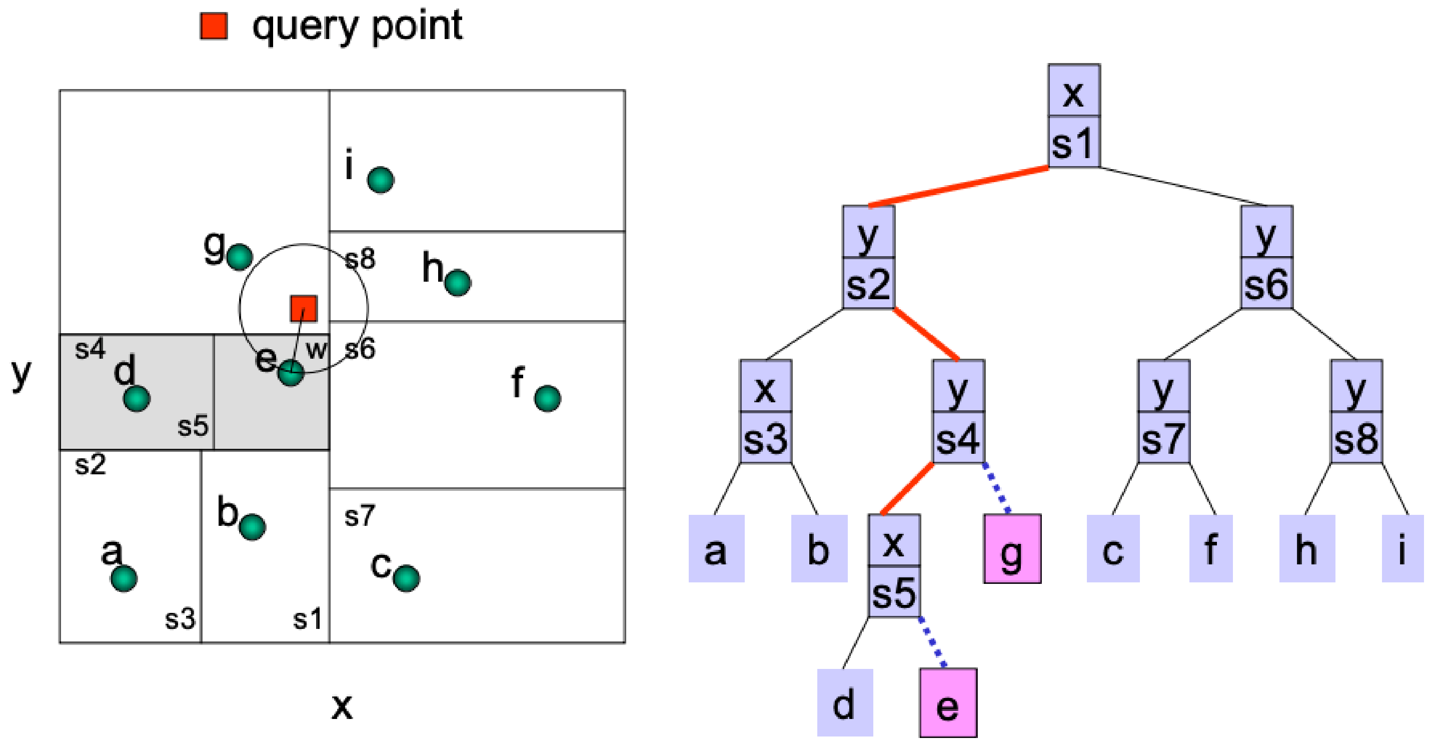

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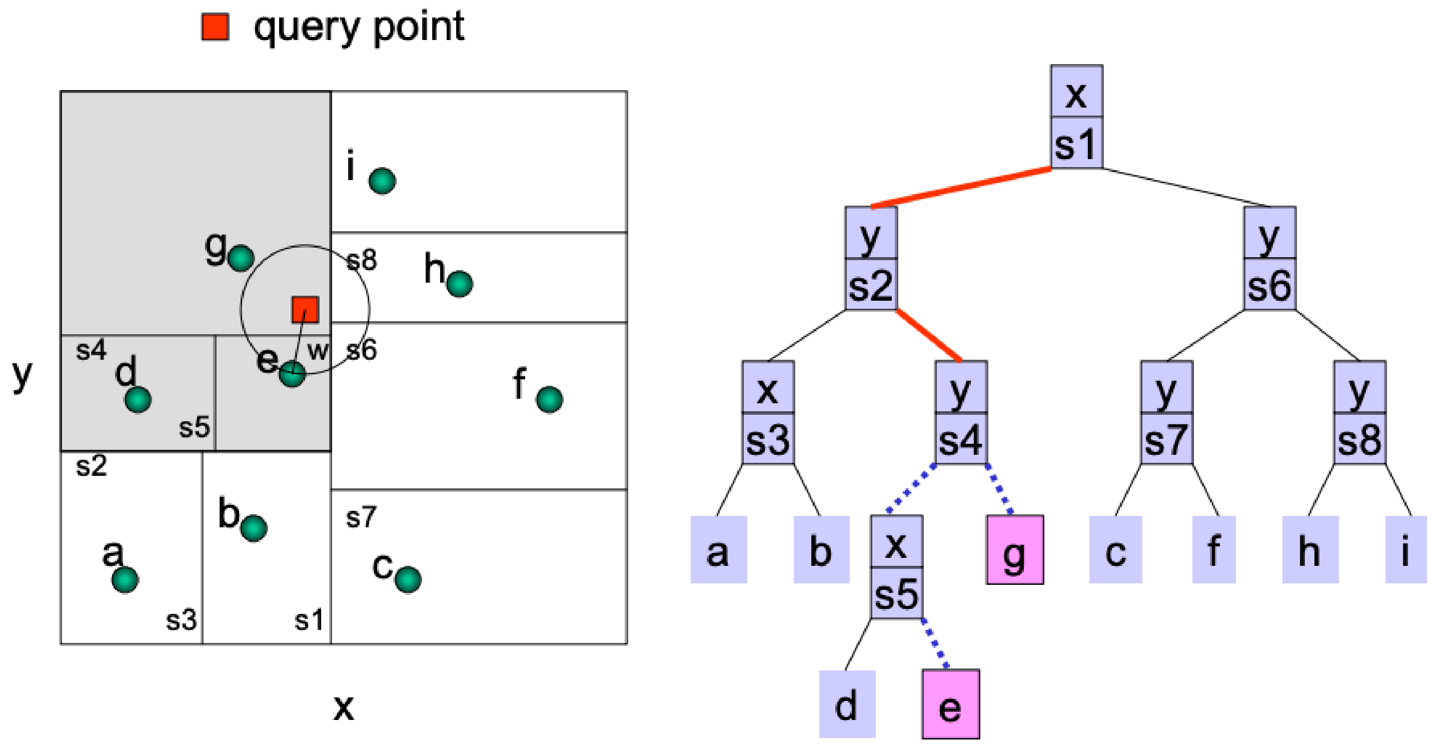

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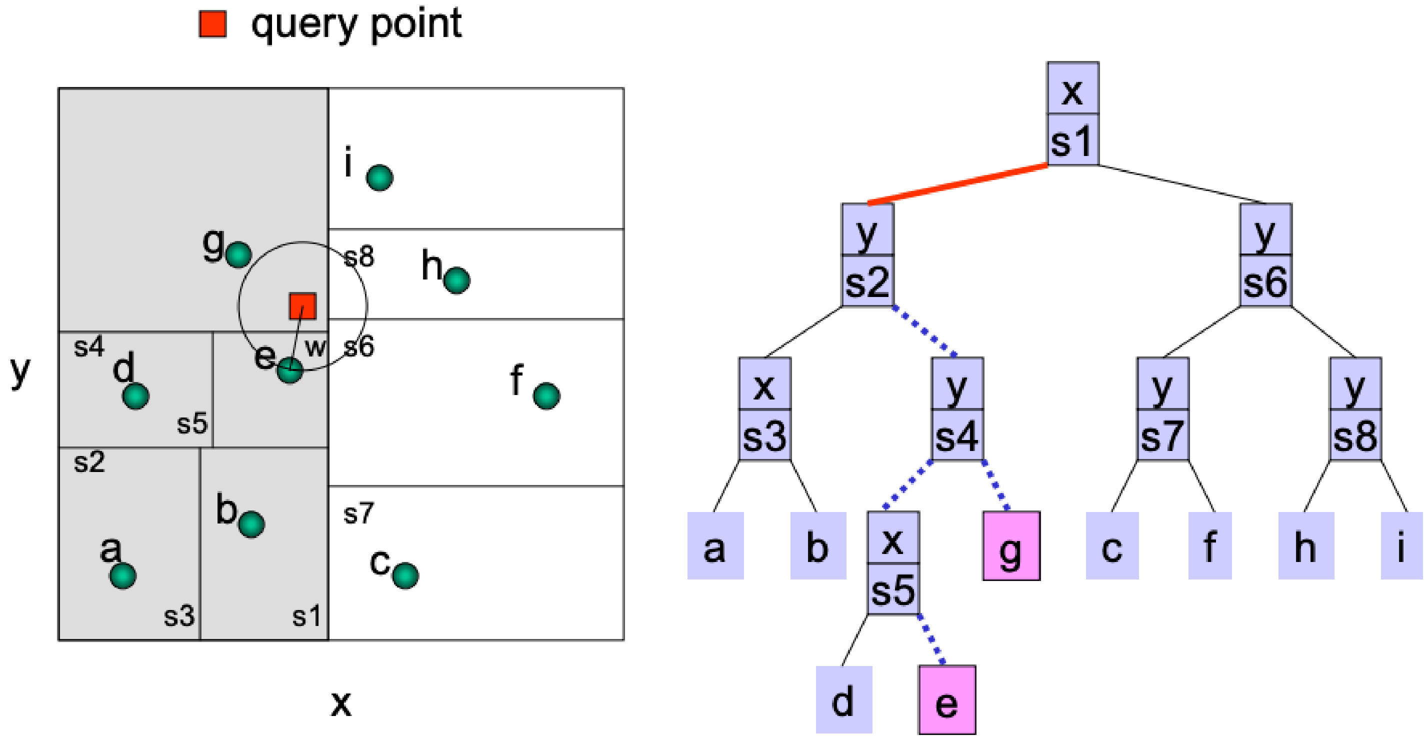

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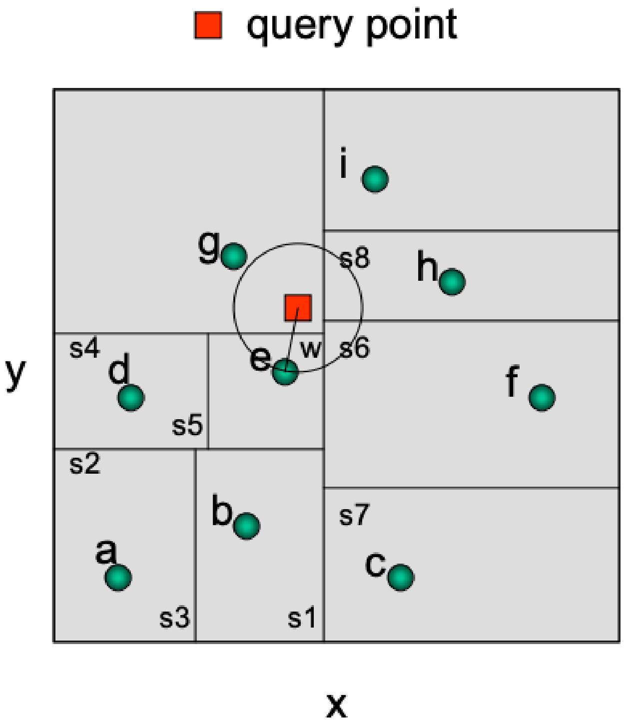

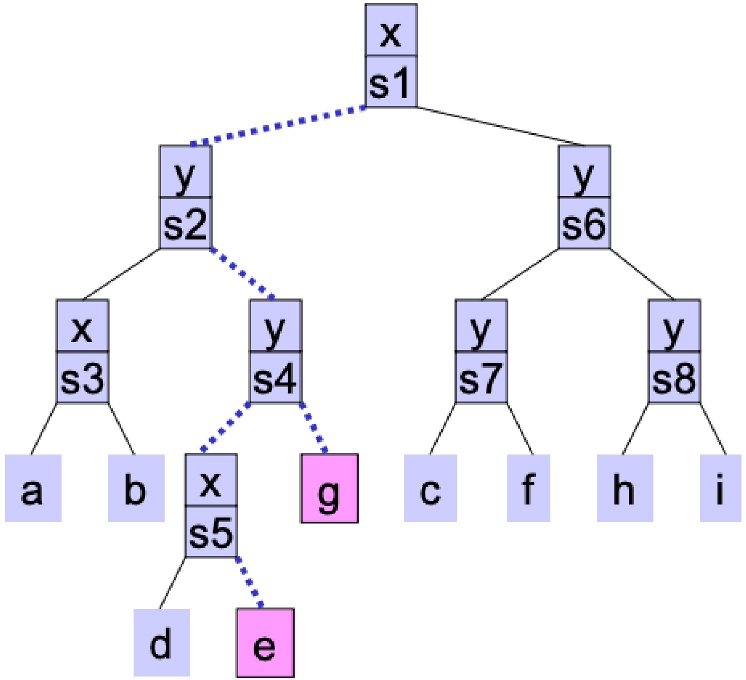

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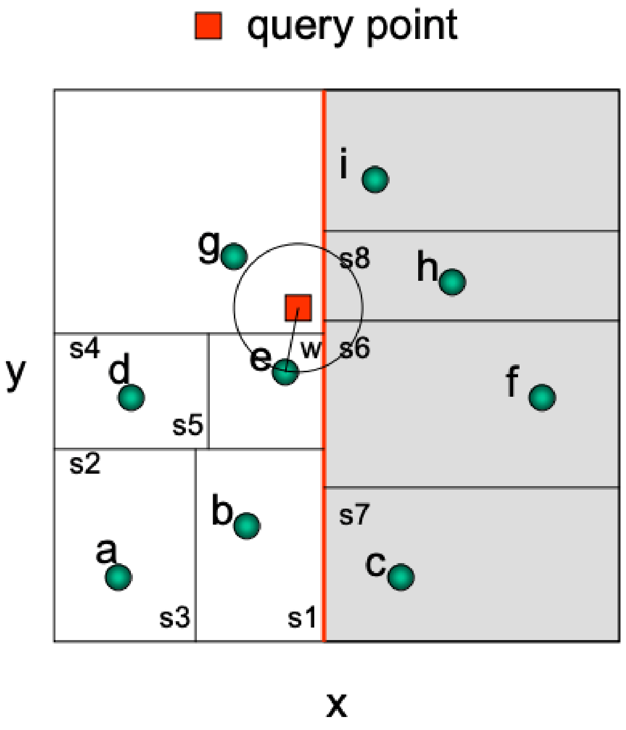

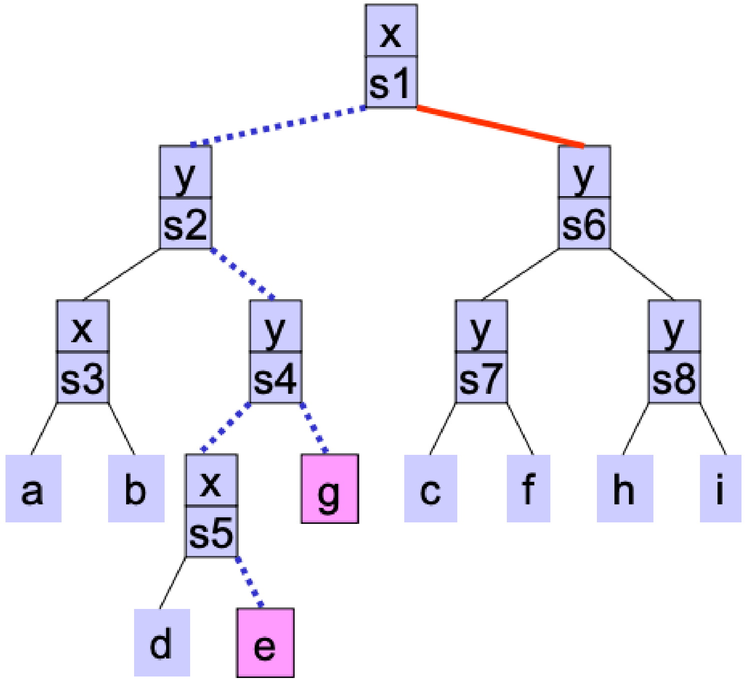

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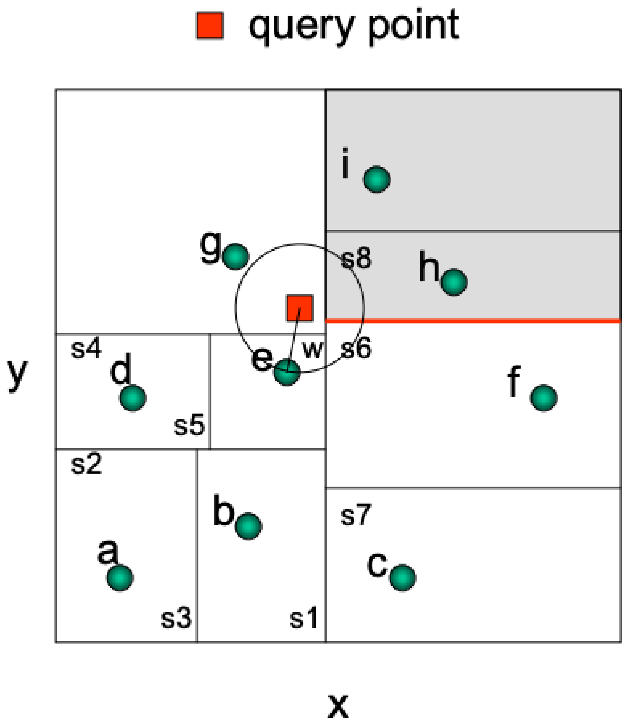

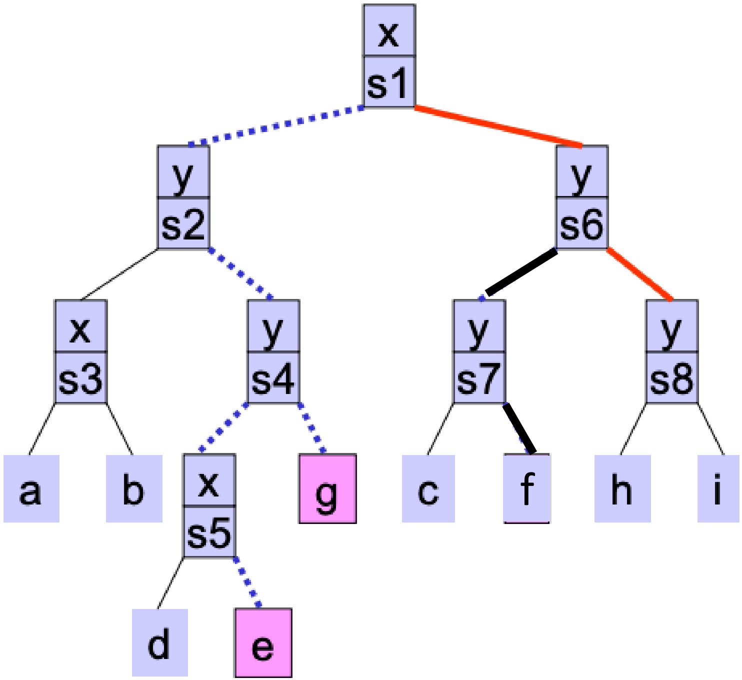

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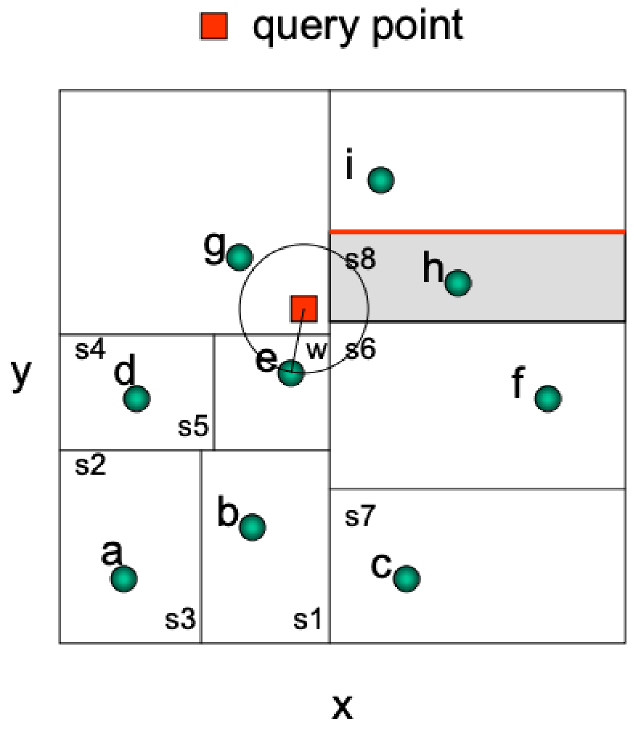

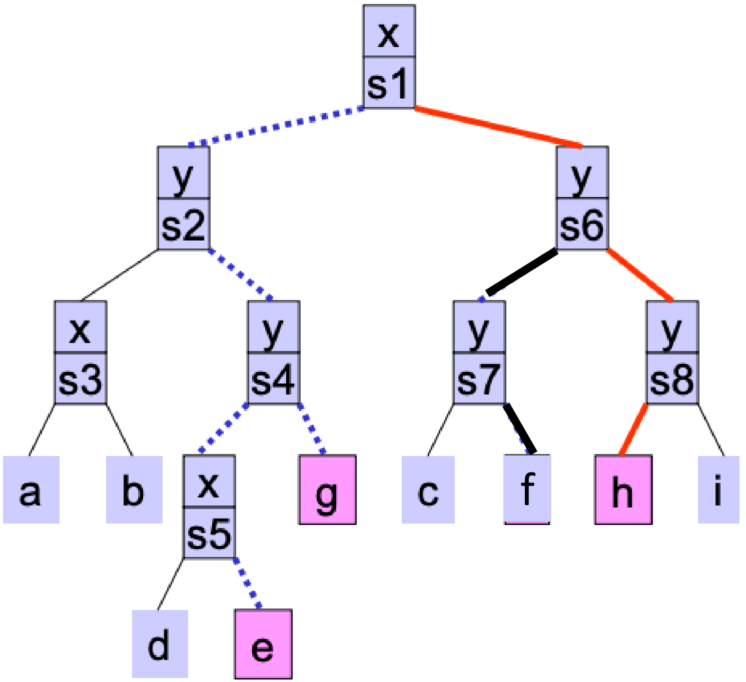

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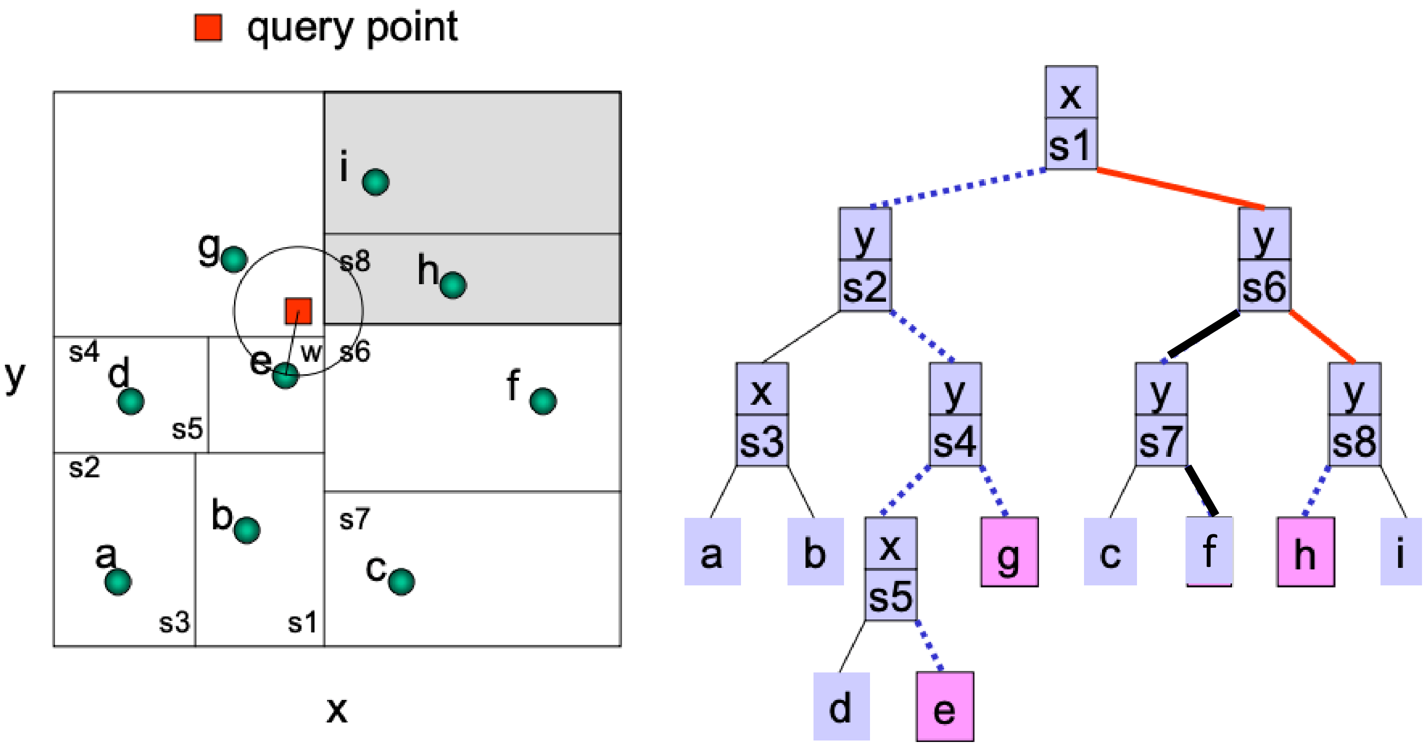

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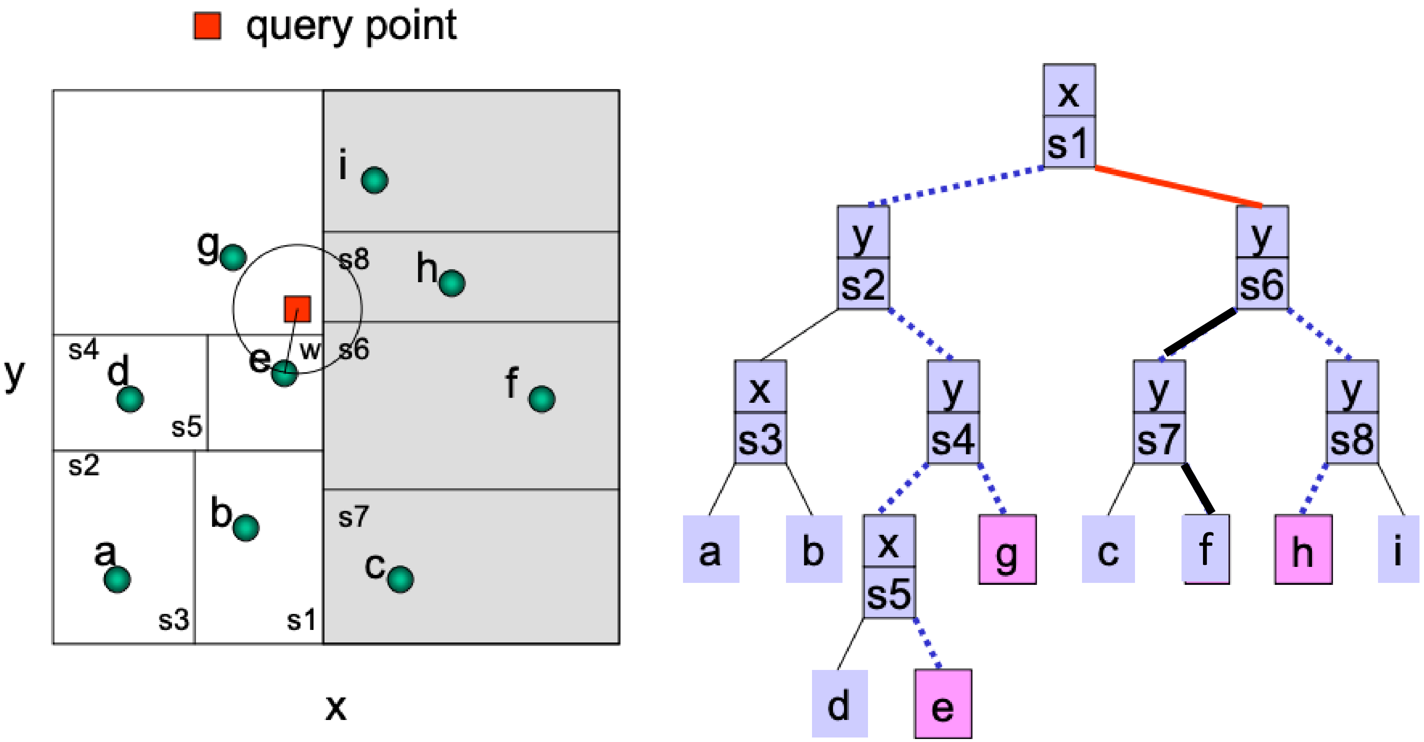

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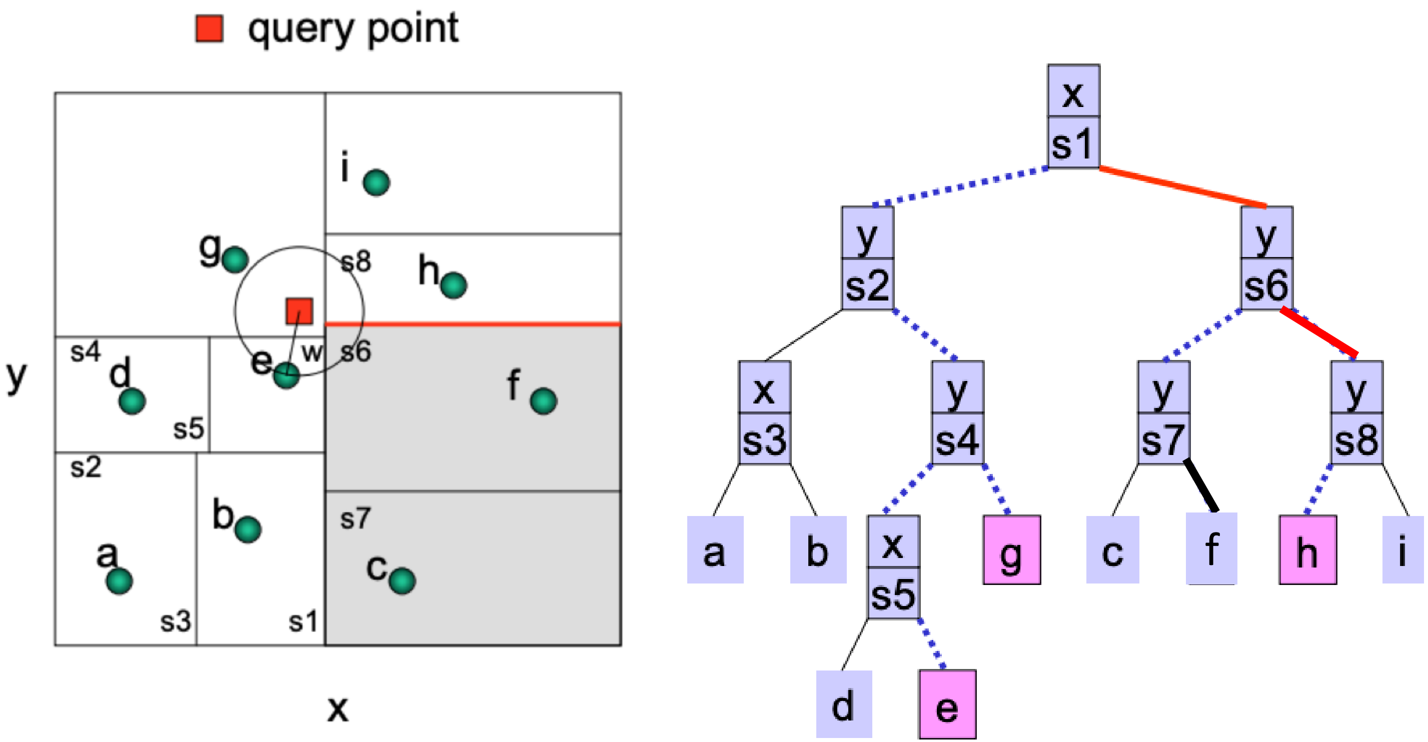

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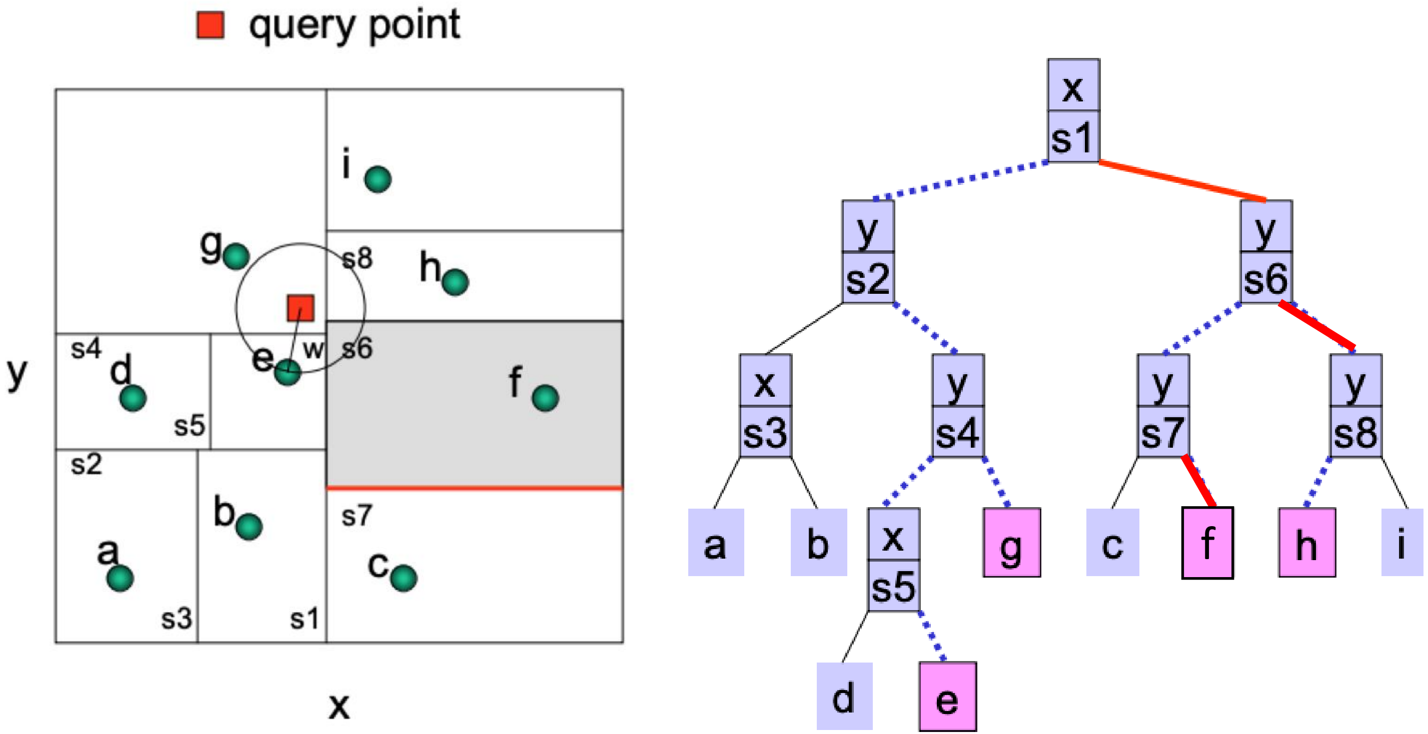

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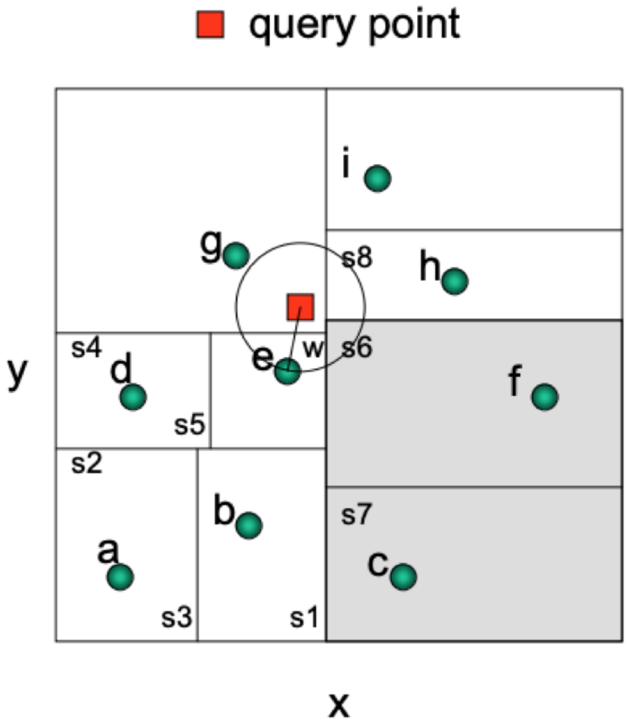

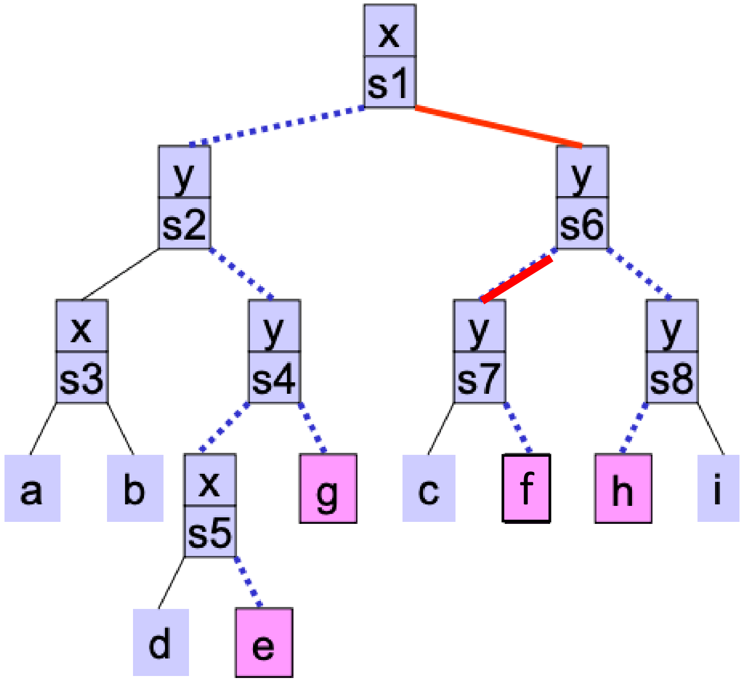

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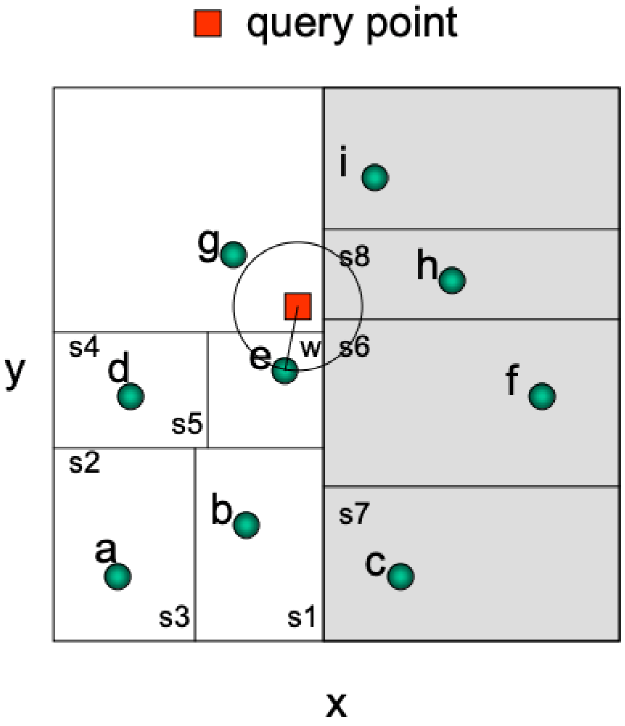

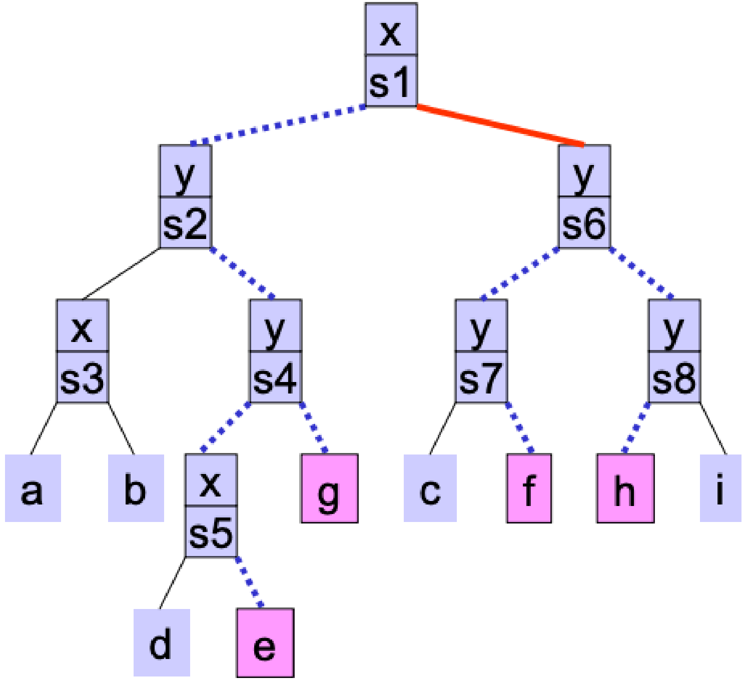

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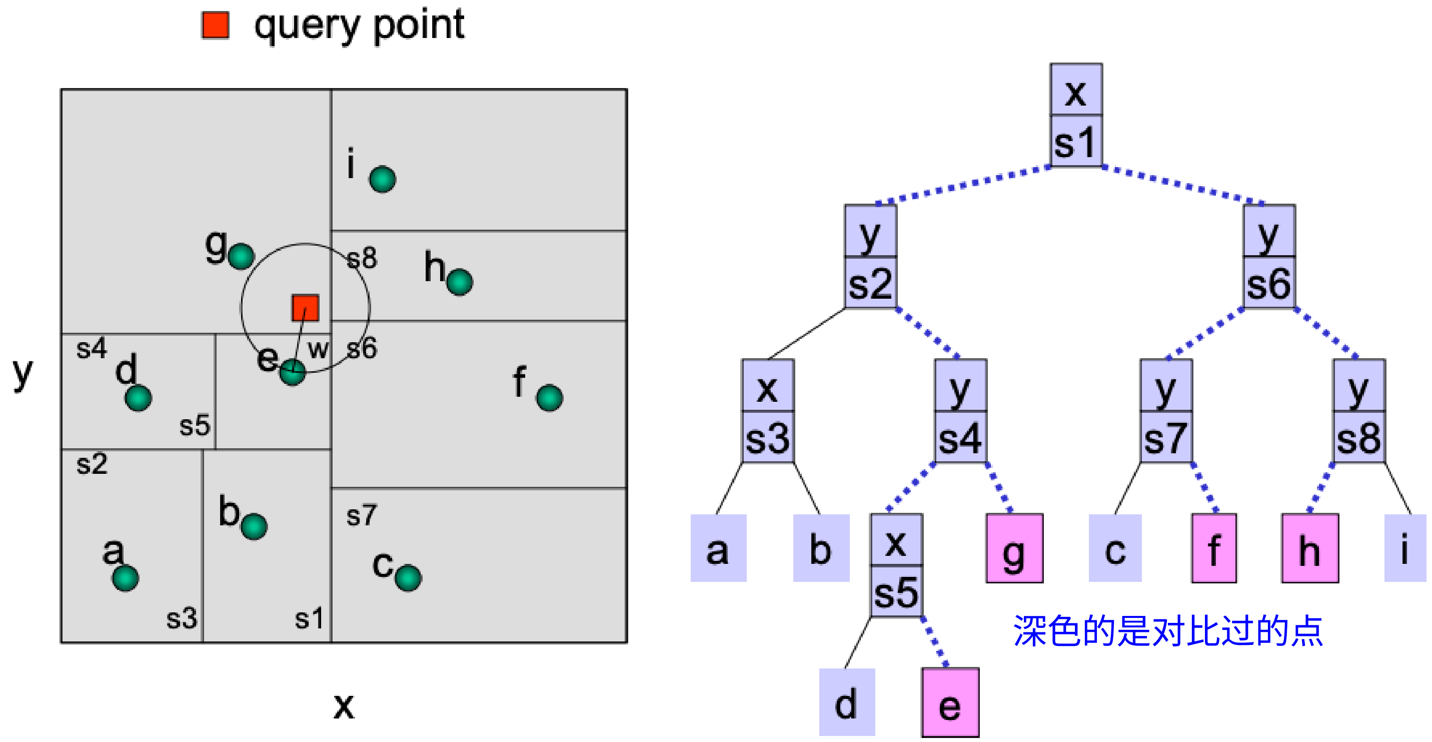

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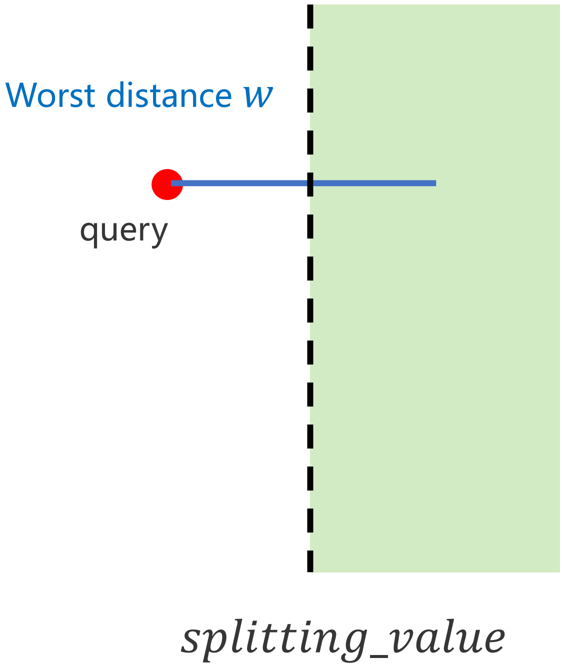


核心思想：给定一个点，我是否要查找该区域。

Criteria of a partition intersects with the worst-distance range:

$q[axis]$ inside the partition

OR

$|q[axis] - splitting\_value| < w$

```python
def knn_search(root: Node, db: np.ndarray, result_set: KNNResultSet, query: np.ndarray):
    if root is None:
        return False

    if root.is_leaf():
        # compare the contents of a leaf
        leaf_points = db[root.point_indices, :]
        diff = np.linalg.norm(np.expand_dims(query, 0) - leaf_points, axis=1)
        for i in range(diff.shape[0]):
            result_set.add_point(diff[i], root.point_indices[i])
        return False

    if query[root.axis] <= root.value:
        knn_search(root.left, db, result_set, query)
        if math.fabs(query[root.axis] - root.value) < result_set.worstDist():
            knn_search(root.right, db, result_set, query)
    else:
        knn_search(root.right, db, result_set, query)
        if math.fabs(query[root.axis] - root.value) < result_set.worstDist():
            knn_search(root.left, db, result_set, query)

    return False
```

Compare query to every point inside the leaf, put into the result set

q[axis] inside the partition

|q[axis] – splitting_value| < w

Kd-tree Radius-NN Search

- Exactly the same as kNN search except:
  - Use *RadiusNNResultSet*, similar to BST search
  - Fixed worst distance, instead of dynamic

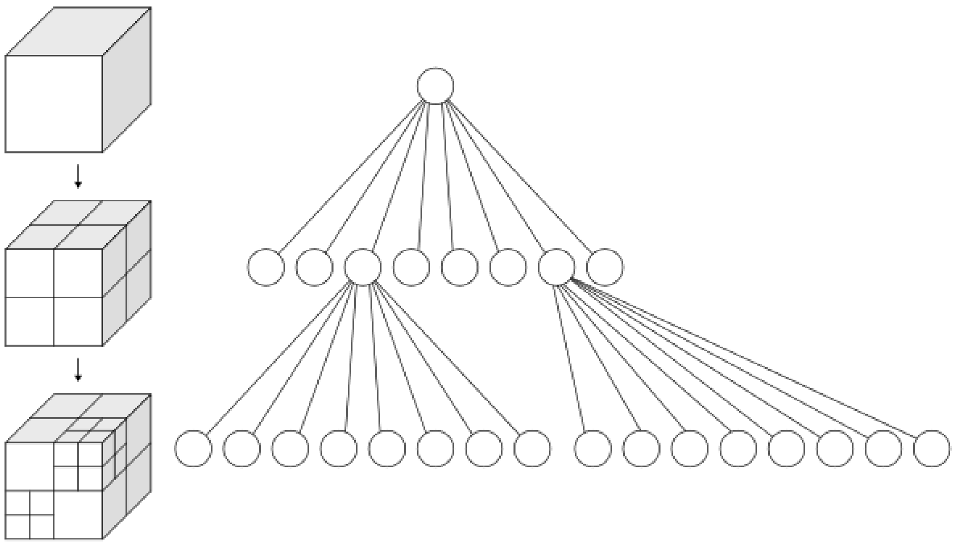
```
if query[root.axis] <= root.value:
    radius_search(root.left, db, result_set, query)
    if math.fabs(query[root.axis] - root.value) < result_set.worstDist():
        radius_search(root.right, db, result_set, query)
else:
    radius_search(root.right, db, result_set, query)
    if math.fabs(query[root.axis] - root.value) < result_set.worstDist():
        radius_search(root.left, db, result_set, query)

return False
```

Kd-tree Search Complexity

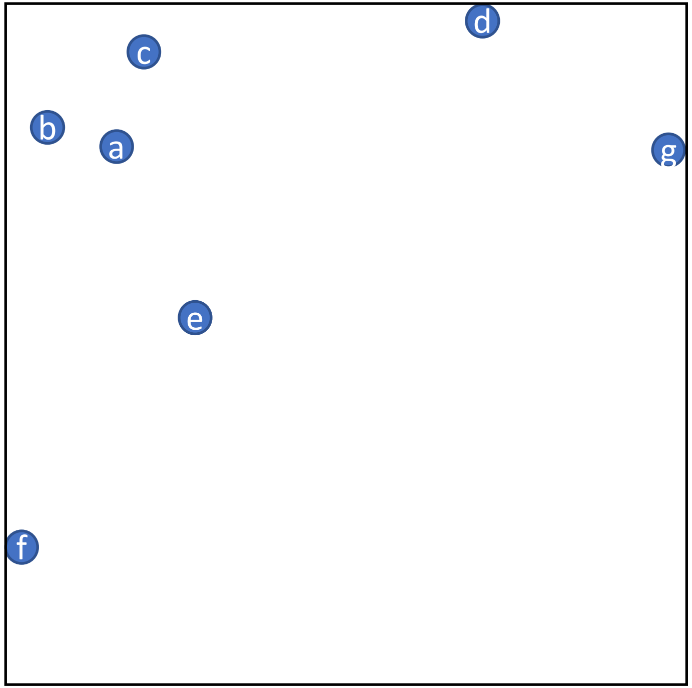
- 1NN search is $O(\log n)$ for a balanced kd-tree
- kNN/radiusNN complexity is hard to analyze
  - Depends on the distribution of points
  - Depends on k or r
  - Varies from $O(\log n)$ to $O(n)$

Octree

- Each node has 8 children
- oct – tree
- Specifically for 3D, $2^3=8$
- In kd-tree, it is non-trivial to determine whether the NN search is done, so we have to go back to root every time
- Octree is more efficient because we can stop without going back to root



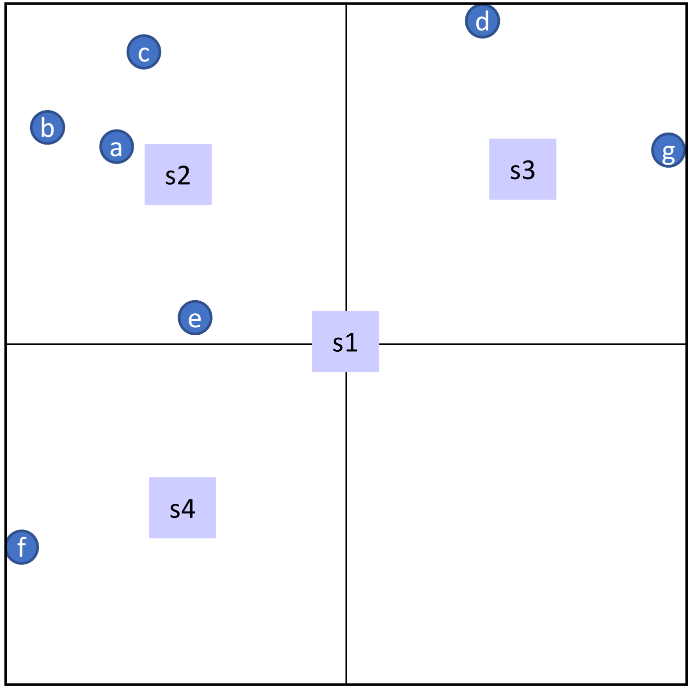
Octree Construction



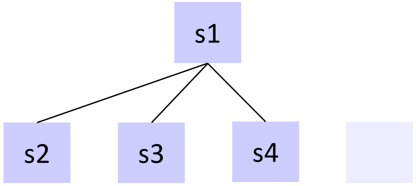
- Determine the extent of the first octant
- Octant is an element in the octree
- Octant is a cube.

s1

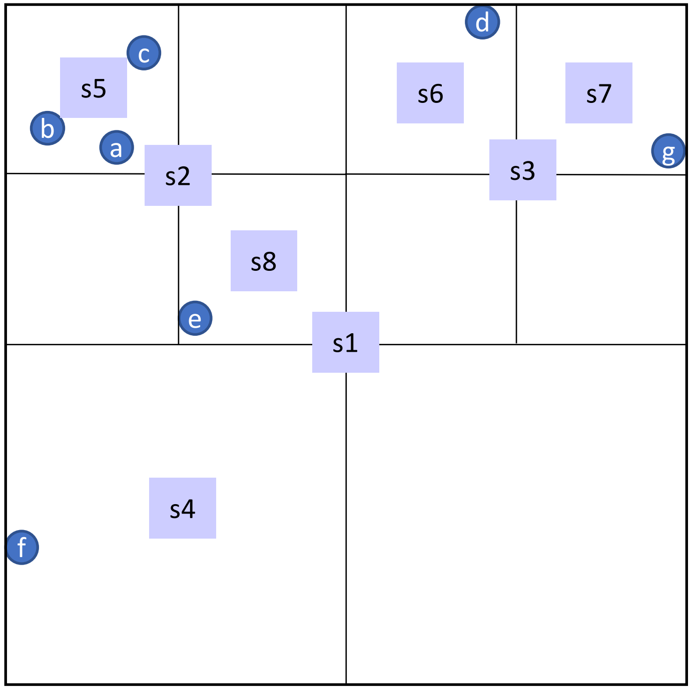
Octree Construction

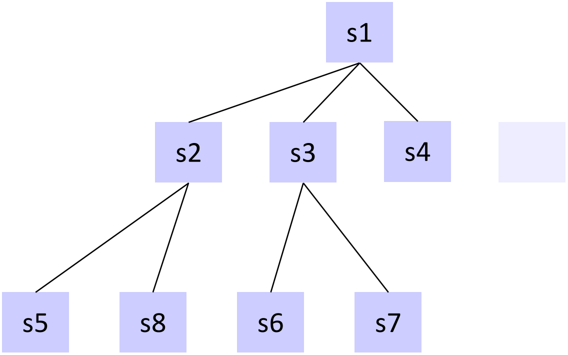


- Determine whether to further split the octant
- leaf_size = 1 here
- min_extent – avoid infinite splitting when there are repeated points

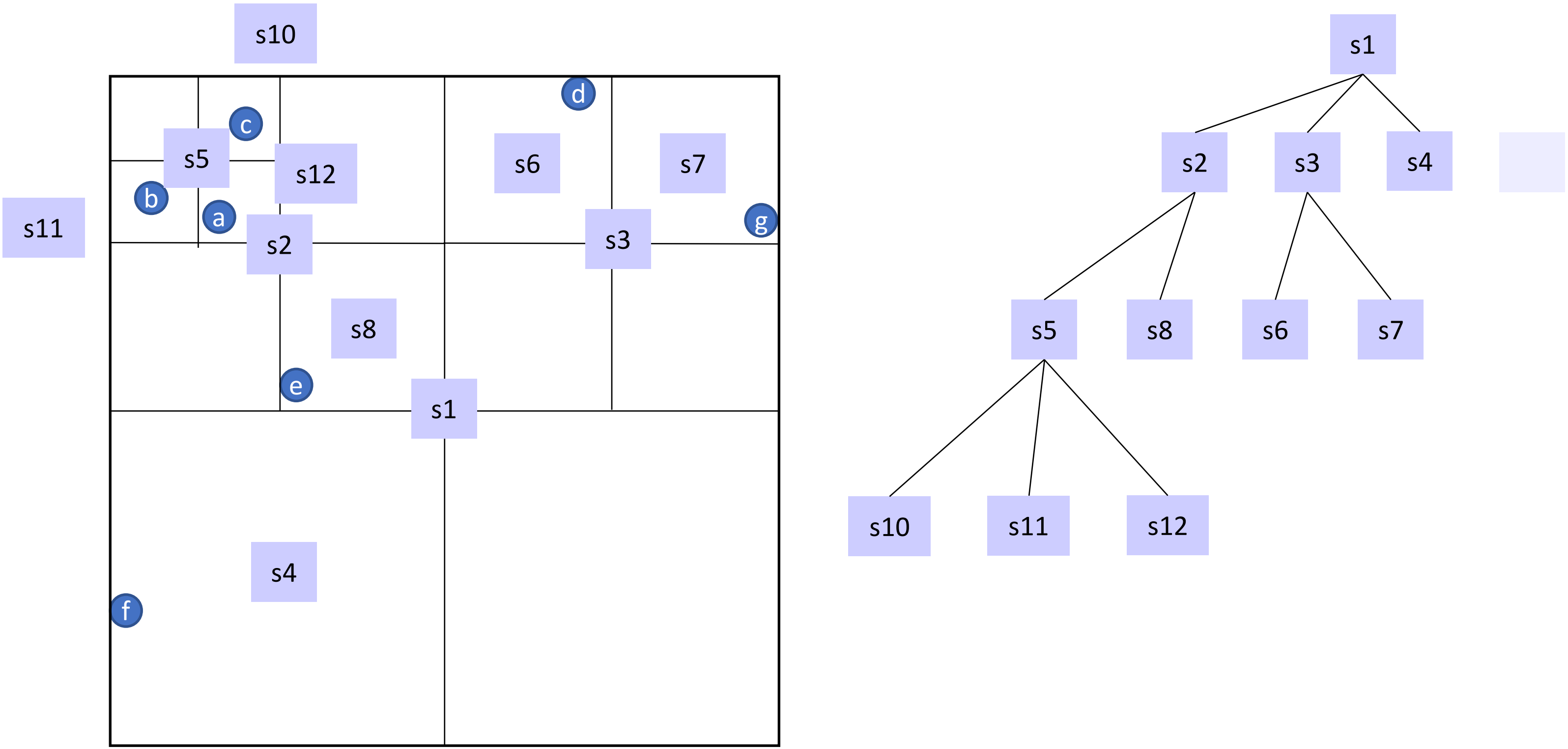


Octree Construction





Octree Construction



Octree Construction

Array of length 8

Center of the cube

Point inside octant

```
class Octant:
    def __init__(self, children, center, extent, point_indices, is_leaf):
        self.children = children
        self.center = center
        self.extent = extent
        self.point_indices = point_indices
        self.is_leaf = is_leaf
```

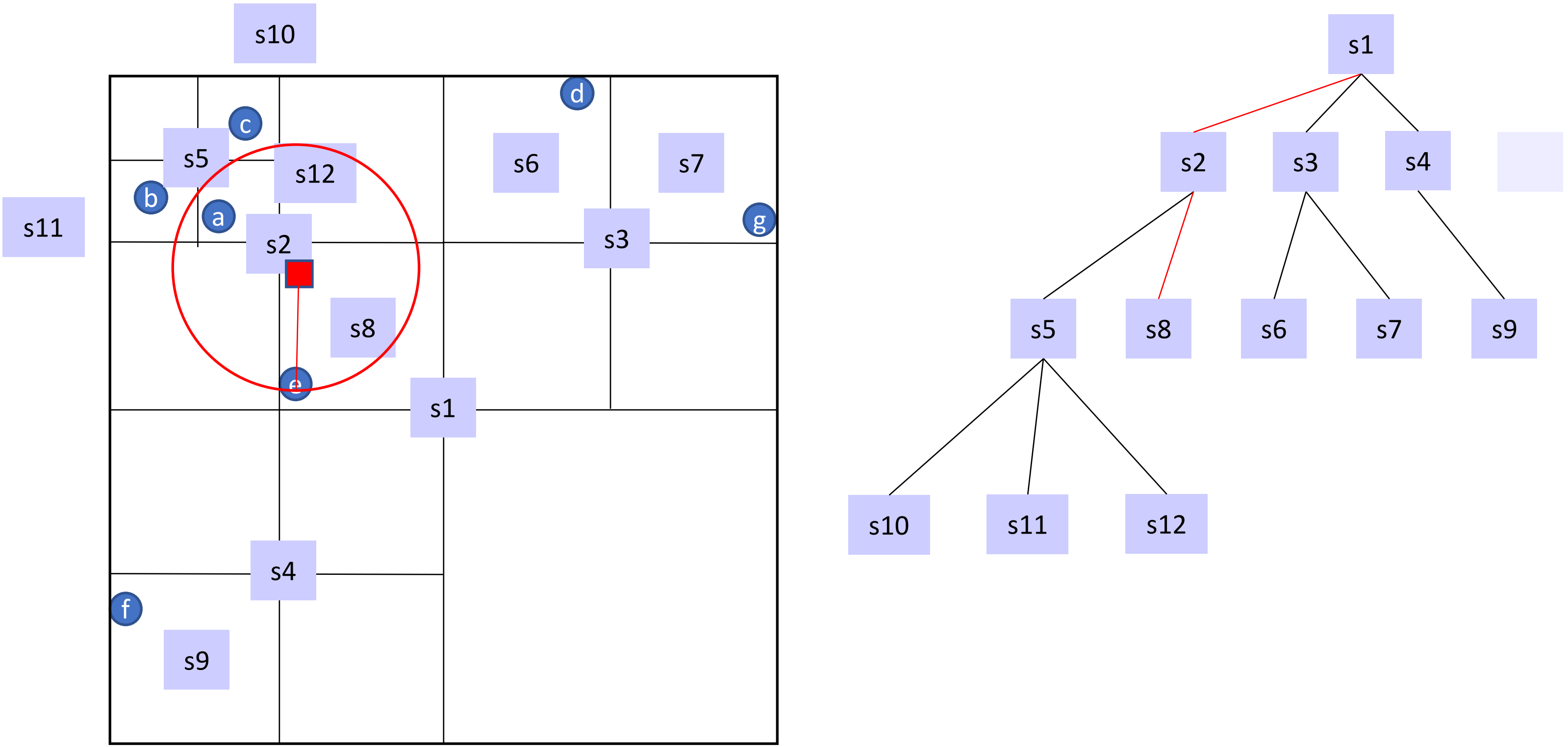
0.5 * length

```python
def octree_recursive_build(root, db, center, extent, point_indices, leaf_size, min_extent):
    if len(point_indices) == 0:
        return None

    if root is None:
        root = Octant([None for i in range(8)], center, extent, point_indices, is_leaf=True)

    # determine whether to split this octant
    if len(point_indices) <= leaf_size or extent <= min_extent:
        root.is_leaf = True
    else:
        root.is_leaf = False
        children_point_indices = [[] for i in range(8)]
        for point_idx in point_indices:
            point_db = db[point_idx]
            morton_code = 0
            if point_db[0] > center[0]:
                morton_code = morton_code | 1
            if point_db[1] > center[1]:
                morton_code = morton_code | 2
            if point_db[2] > center[2]:
                morton_code = morton_code | 4
            children_point_indices[morton_code].append(point_idx)
        # create children
        factor = [-0.5, 0.5]
        for i in range(8):
            child_center_x = center[0] + factor[(i & 1) > 0] * extent
            child_center_y = center[1] + factor[(i & 2) > 0] * extent
            child_center_z = center[2] + factor[(i & 4) > 0] * extent
            child_extent = 0.5 * extent
            child_center = np.asarray([child_center_x, child_center_y, child_center_z])
            root.children[i] = octree_recursive_build(root.children[i],
                                                      db,
                                                      child_center,
                                                      child_extent,
                                                      children_point_indices[i],
                                                      leaf_size,
                                                      min_extent)
    return root
```

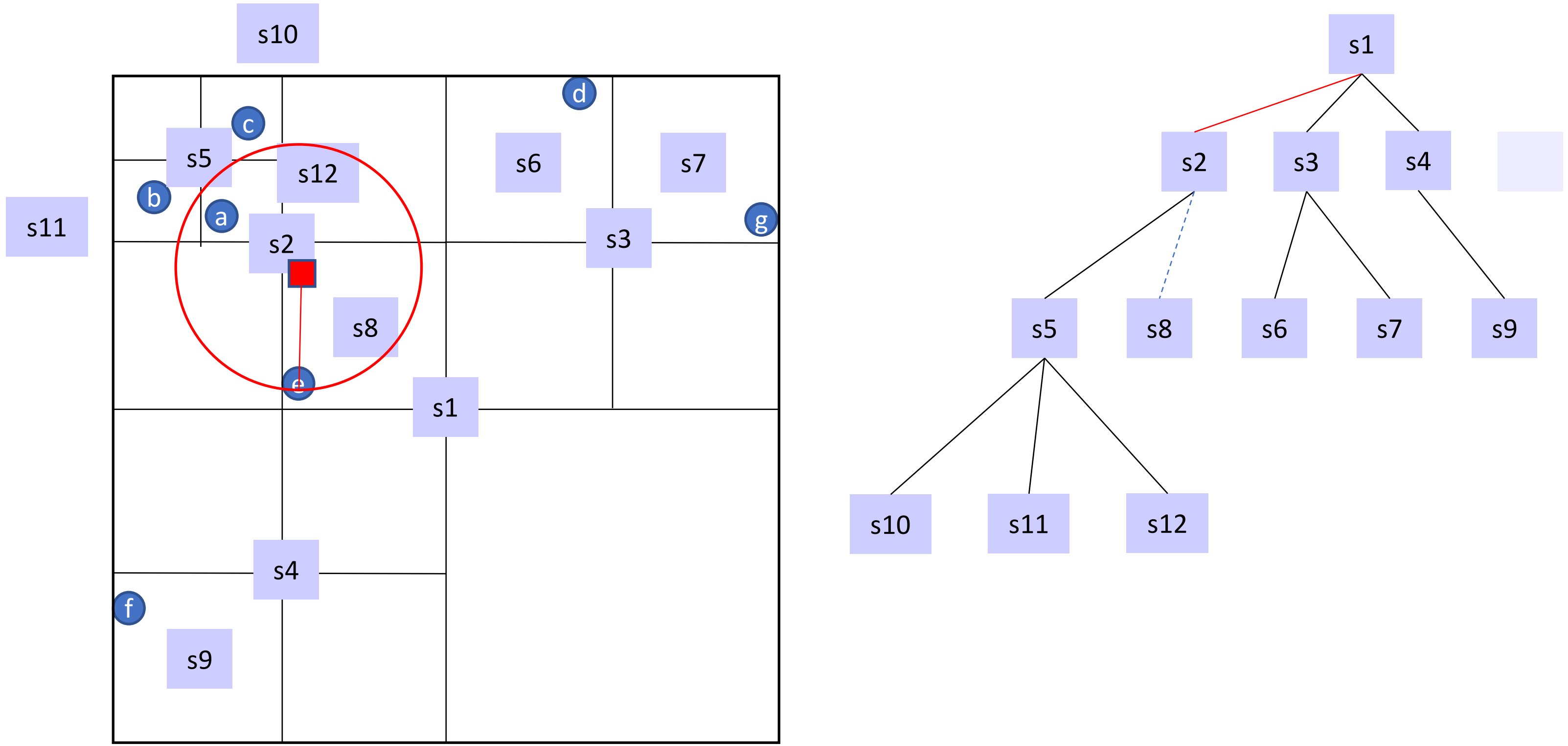
Determine which child a point belongs to

Determine child center & extent

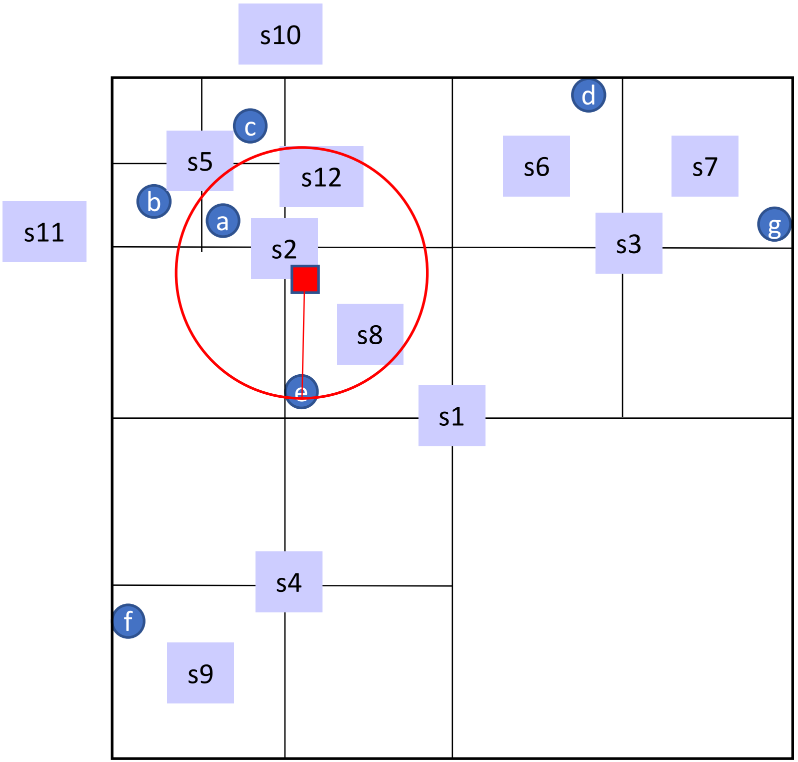
Octree kNN Search

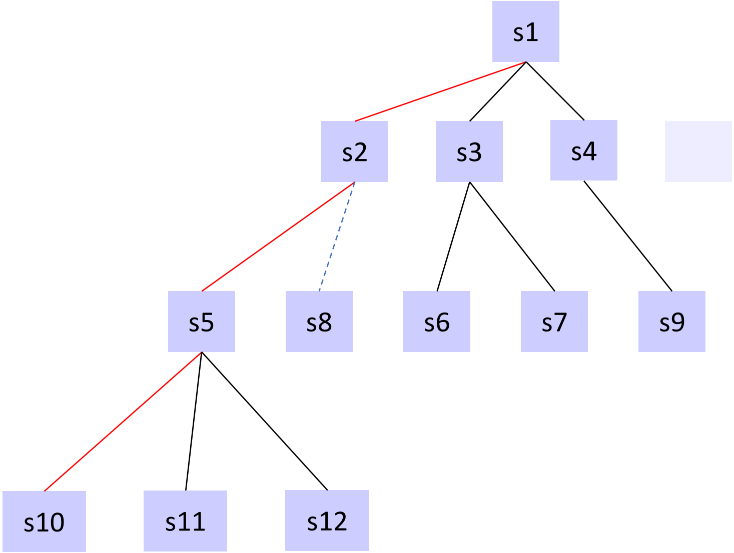


Octree kNN Search

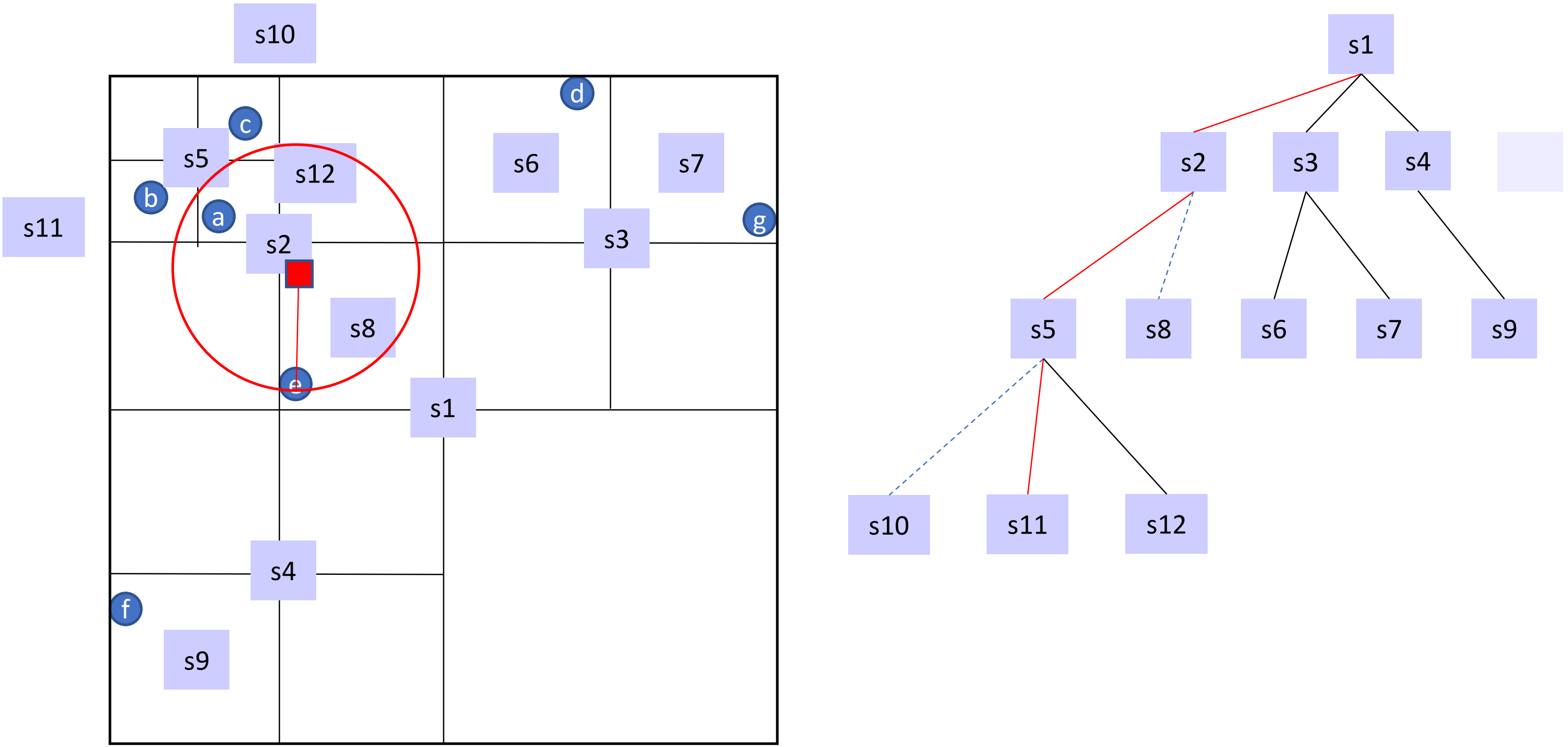


Octree kNN Search

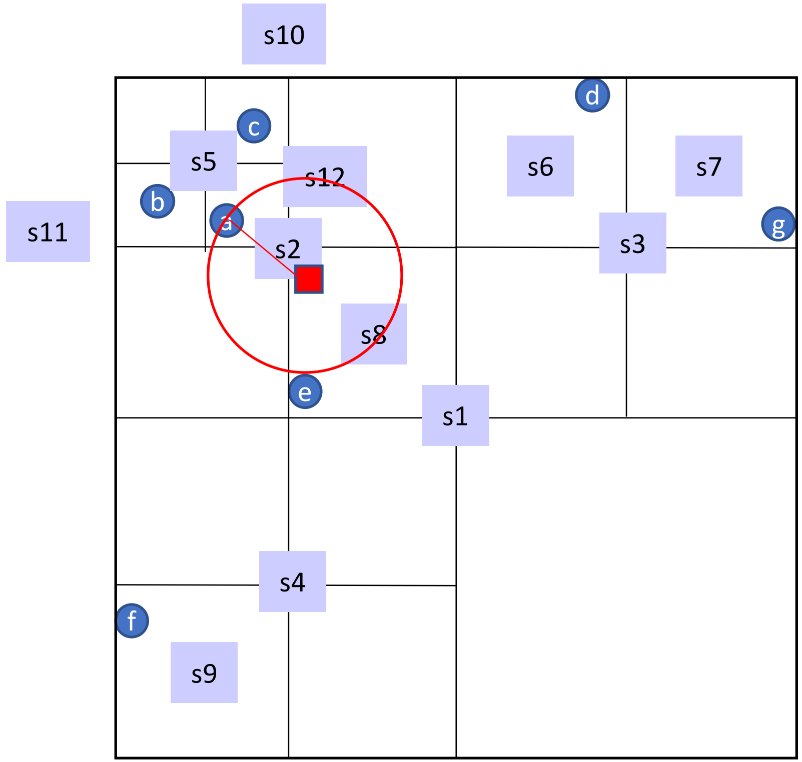


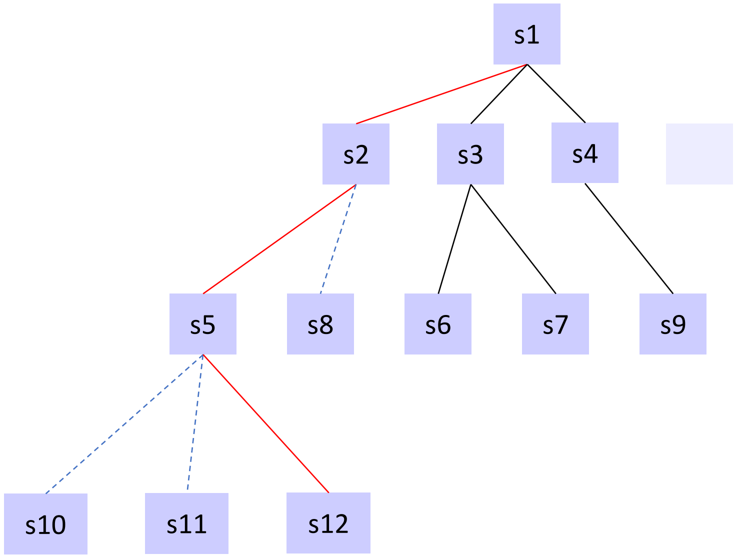


Octree kNN Search

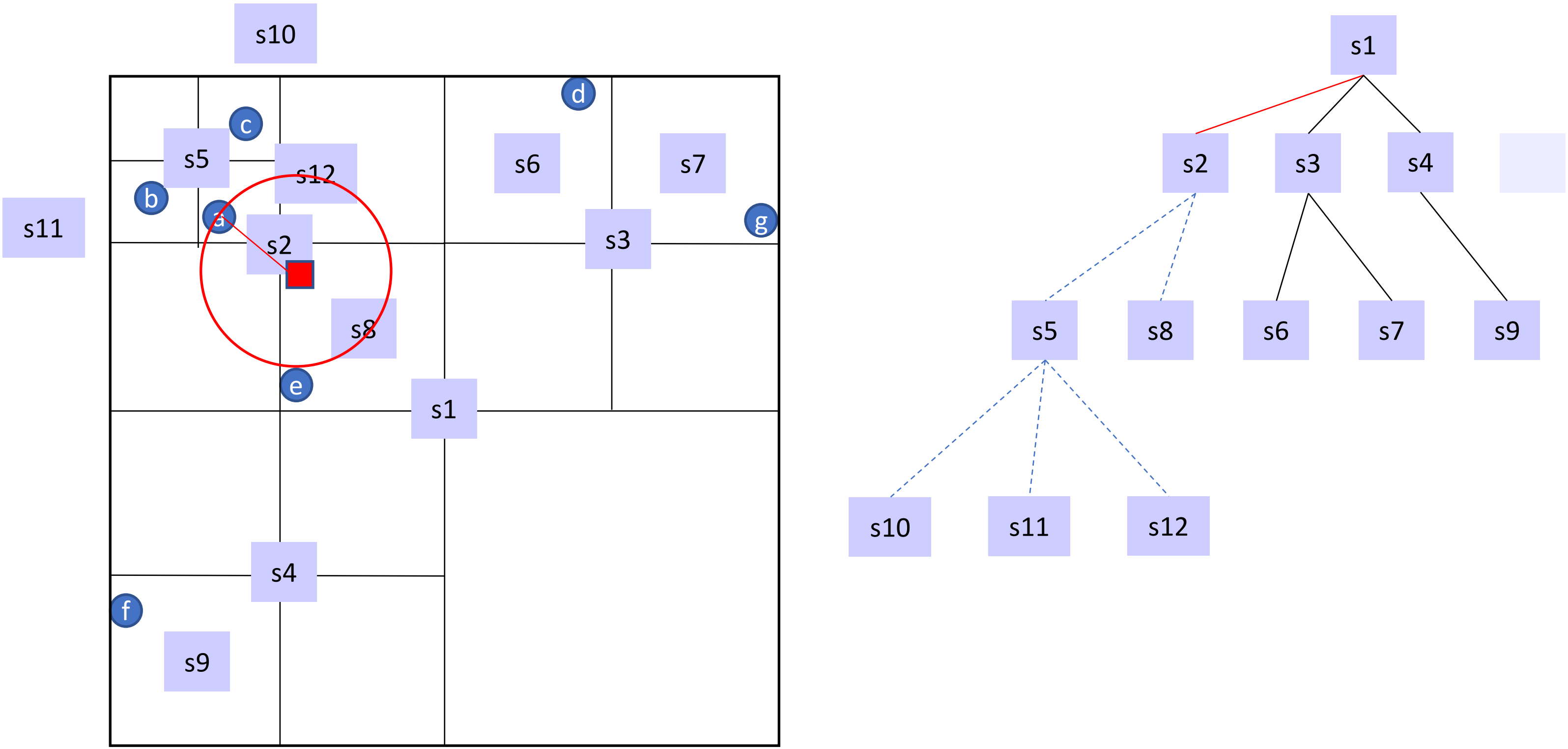


Octree kNN Search

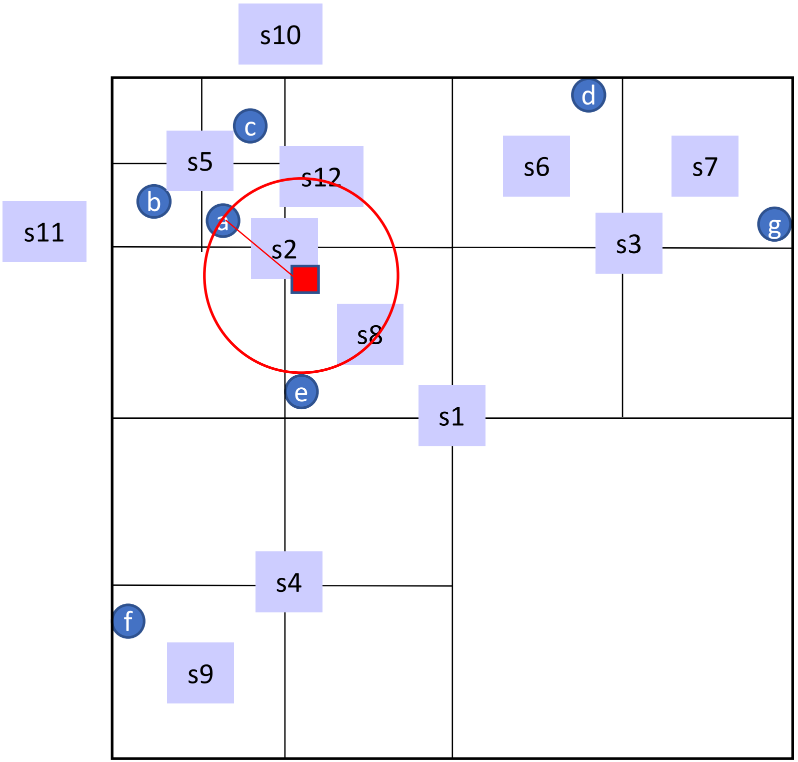




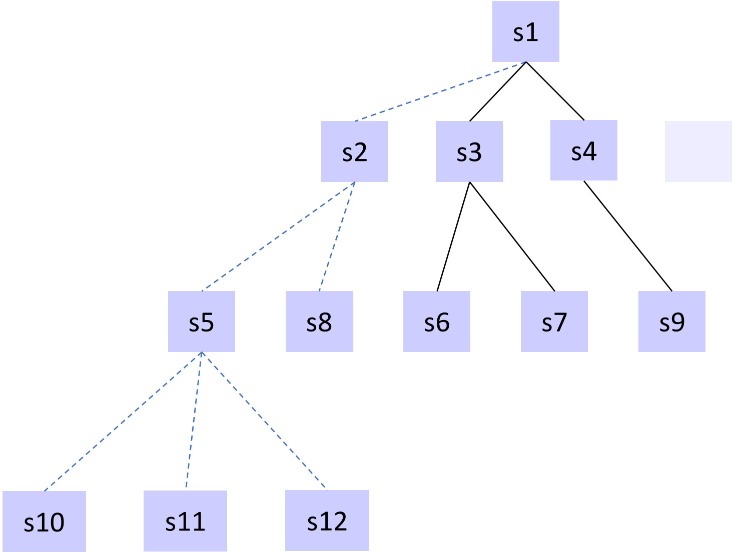
Octree kNN Search



Octree kNN Search



Query ball inside s2, search end!



```python
def octree_knn_search(root: Octant, db: np.ndarray, result_set: KNNResultSet, query: np.ndarray):
    if root is None:
        return False

    if root.is_leaf and len(root.point_indices) > 0:
        # compare the contents of a leaf
        leaf_points = db[root.point_indices, :]
        diff = np.linalg.norm(np.expand_dims(query, 0) - leaf_points, axis=1)
        for i in range(diff.shape[0]):
            result_set.add_point(diff[i], root.point_indices[i])
        # check whether we can stop search now
        return inside(query, result_set.worstDist(), root)

    # go to the relevant child first
    morton_code = 0
    if query[0] > root.center[0]:
        morton_code = morton_code | 1
    if query[1] > root.center[1]:
        morton_code = morton_code | 2
    if query[2] > root.center[2]:
        morton_code = morton_code | 4

    if octree_knn_search(root.children[morton_code], db, result_set, query):
        return True

    # check other children
    for c, child in enumerate(root.children):
        if c == morton_code or child is None:
            continue
        if False == overlaps(query, result_set.worstDist(), child):
            continue
        if octree_knn_search(child, db, result_set, query):
            return True

    # final check of if we can stop search
    return inside(query, result_set.worstDist(), root)
```

Compare all points in a leaf

Determine & search the most relevant child

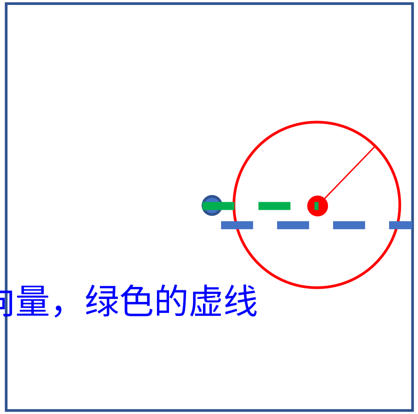
If an octant is not overlapping with query ball, skip

If query ball is inside an octant, stop

Function *inside*

```python
def inside(query: np.ndarray, radius: float, octant:Octant):
    """
    Determines if the query ball is inside the octant
    :param query:
    :param radius:
    :param octant:
    :return:
    """
    query_offset = query - octant.center
    query_offset_abs = np.fabs(query_offset)
    possible_space = query_offset_abs + radius
    return np.all(possible_space < octant.extent)
```

蓝色点指向红色点的向量，绿色的虚线



判断蓝色的线比红色加绿色的线长，那么这个园就在正方体内

Green dash line

Blue dash line

Red line

Function *overlaps*

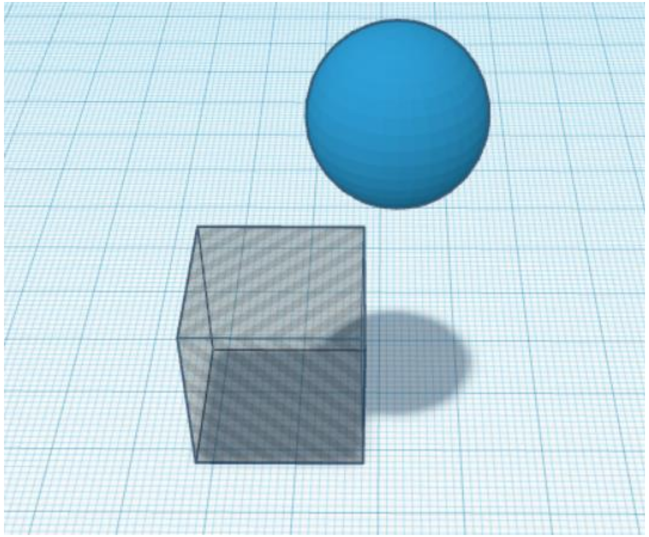
```
def overlaps(query: np.ndarray, radius: float, octant:Octant):
    """
    Determines if the query ball overlaps with the octant
    :param query:
    :param radius:
    :param octant:
    :return:
    """
    query_offset = query - octant.center
    query_offset_abs = np.fabs(query_offset)

    # completely outside, since query is outside the relevant area
    max_dist = radius + octant.extent
    if np.any(query_offset_abs > max_dist):
        return False

    # if pass the above check, consider the case that the ball is contacting the face of the octant
    if np.sum((query_offset_abs < octant.extent).astype(np.int)) >= 2:
        return True

    # conside the case that the ball is contacting the edge or corner of the octant
    # since the case of the ball center (query) inside octant has been considered,
    # we only consider the ball center (query) outside octant
    x_diff = max(query_offset_abs[0] - octant.extent, 0)
    y_diff = max(query_offset_abs[1] - octant.extent, 0)
    z_diff = max(query_offset_abs[2] - octant.extent, 0)

    return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```

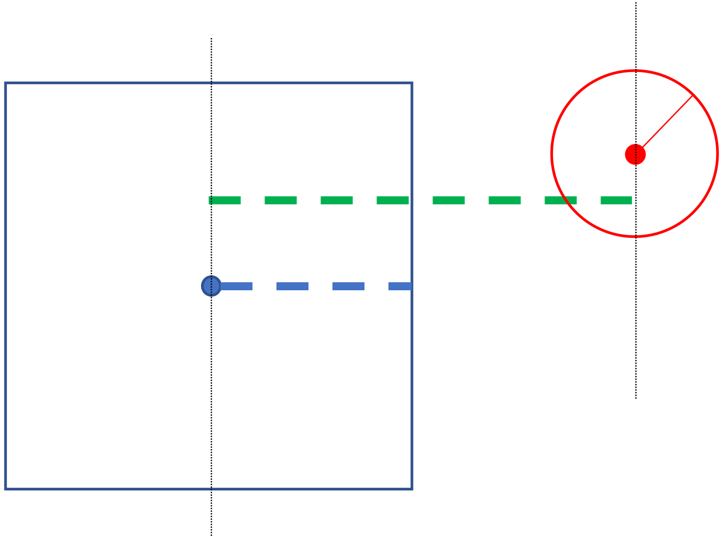
Case 1

Case 2

Case 3

Function *overlaps – Case 1*

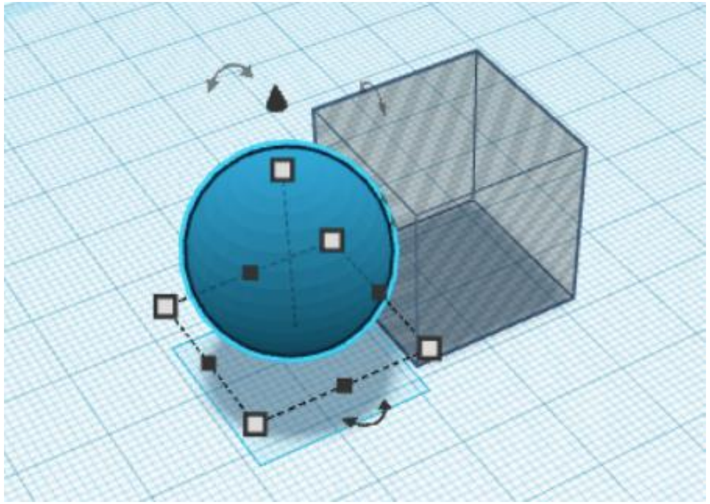


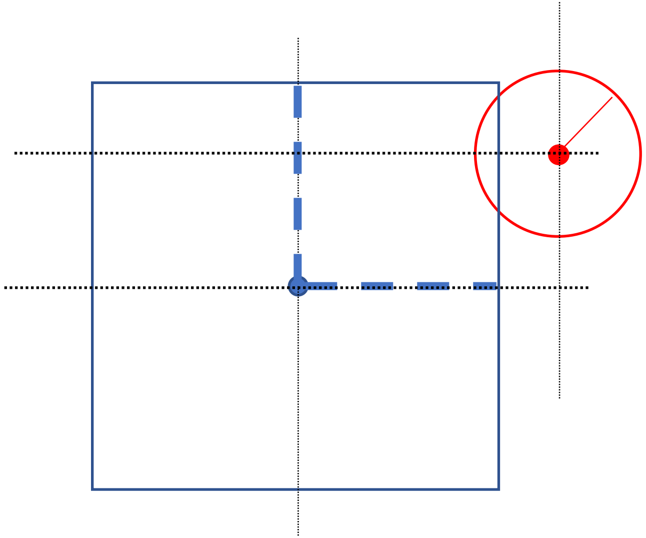


```
# completely outside, since query is outside the relevant area
max_dist = radius + octant.extent
if np.any(query_offset_abs > max_dist):
    return False
```

绿色线

Function *overlaps – Case 2*

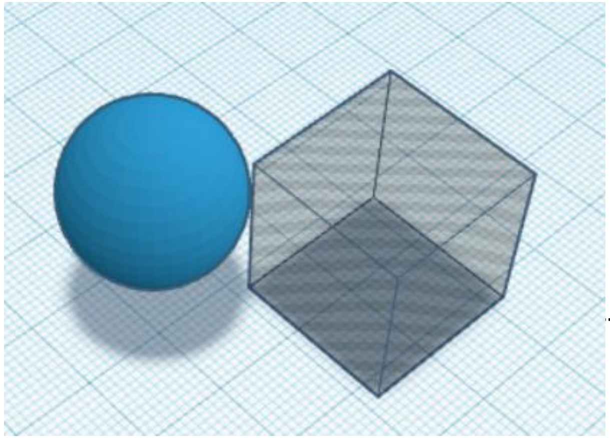


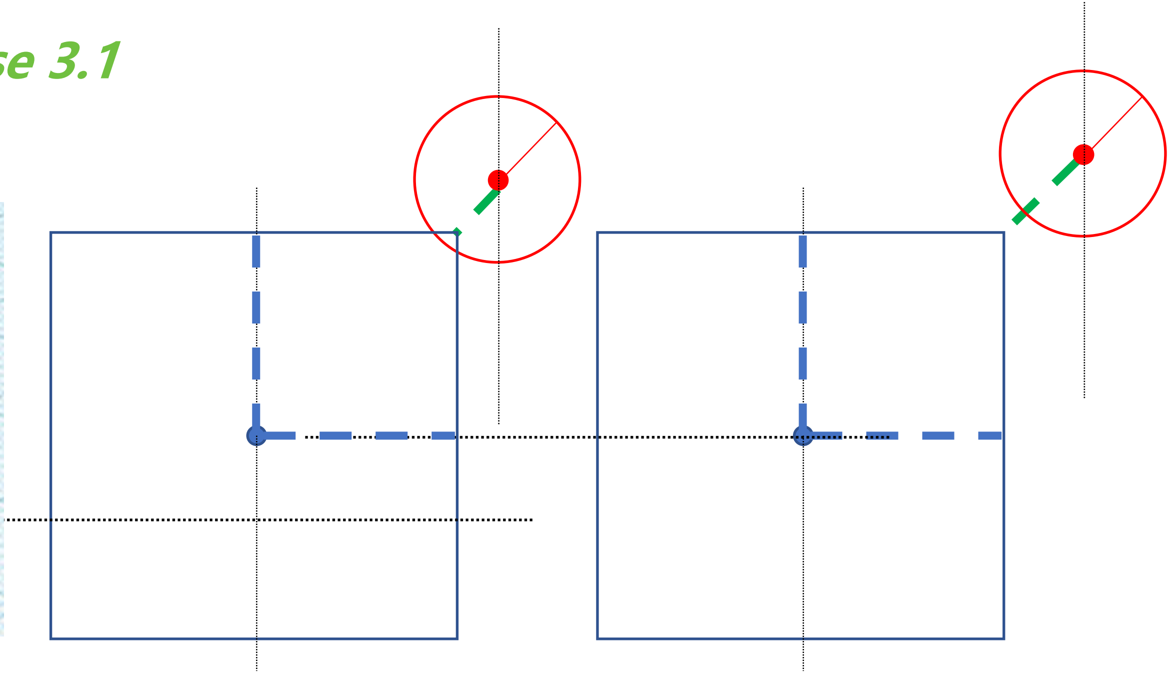


Check if the ball is contacting the face of the octant 在x和y组成的平面上，球和立方体都有接触

```
if np.sum((query_offset_abs < octant.extent).astype(np.int)) >= 2:
    return True
```

Function *overlaps – Case 3.1*



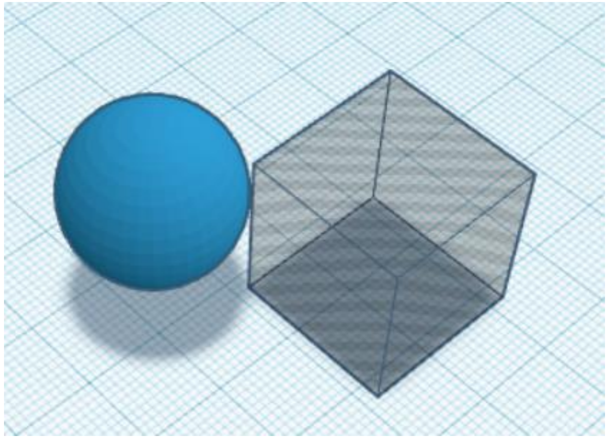


```python
# conside the case that the ball is contacting the edge or corner of the octant
# since the case of the ball center (query) inside octant has been considered,
# we only consider the ball center (query) outside octant
x_diff = max(query_offset_abs[0] - octant.extent, 0)
y_diff = max(query_offset_abs[1] - octant.extent, 0)
z_diff = max(query_offset_abs[2] - octant.extent, 0)

return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```

Function *overlaps – Case 3.2*

In 3D, there is the case that the cube’ s edge cut into the query ball

```
# conside the case that the ball is contacting the edge or corner of the octant
# since the case of the ball center (query) inside octant has been considered,
# we only consider the ball center (query) outside octant
x_diff = max(query_offset_abs[0] - octant.extent, 0)
y_diff = max(query_offset_abs[1] - octant.extent, 0)
z_diff = max(query_offset_abs[2] - octant.extent, 0)

return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```



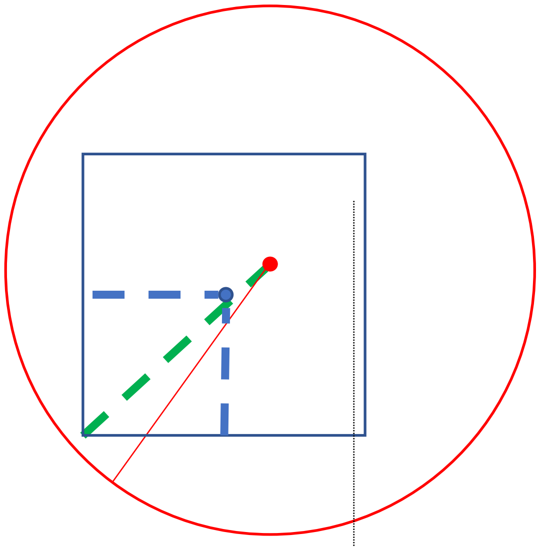
That’s why there is a “*max*” to reduce this case into 3.1

Octree Radius NN Search

- Simple one: replace KNNResultSet with RadiusNNResult Set

- Better one:
  - If the query ball *contains* the octant, just compare the query with all point
  - No need to go into children of that octant

Function *contains*



```
def contains(query: np.ndarray, radius: float, octant:Octant):
    """
    Determine if the query ball contains the octant
    :param query:
    :param radius:
    :param octant:
    :return:
    """
    query_offset = query - octant.center
    query_offset_abs = np.fabs(query_offset)

    query_offset_to_farthest_corner = query_offset_abs + octant.extent
    return np.linalg.norm(query_offset_to_farthest_corner) < radius
```

Green dash line

Red line

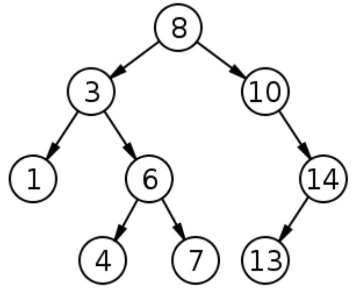
Octree Search Complexity

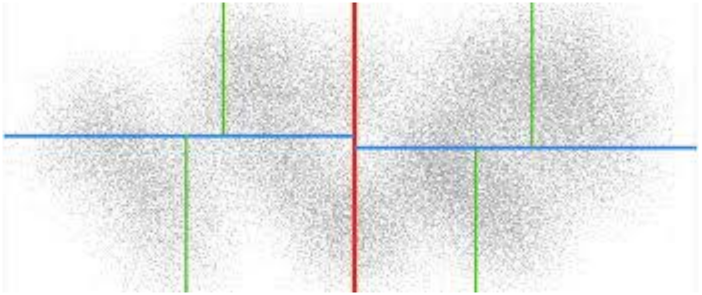
- 1NN search is $O(\log n)$
- kNN/radiusNN complexity is hard to analyze
  - Depends on the distribution of points
  - Depends on k or r
  - Varies from $O(\log n)$ to $O(n)$

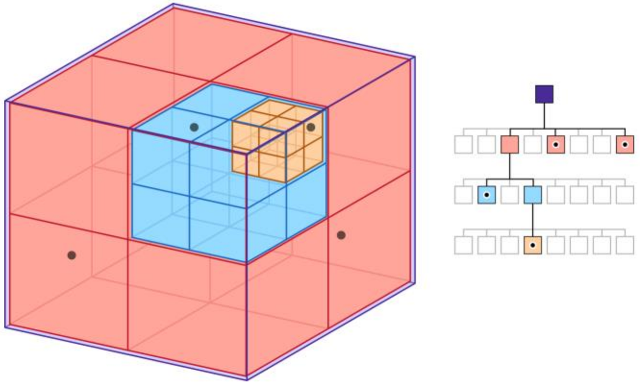
BST / Kd-tree / Octree

Dimension

- BST for one dimension
- Kd-tree works for any dimension
- Octree is optimized for 3D







Idea

- Same – space partition

Summary

- Space partition
- Find a method to skip some partitions
- Pythons codes:
  https://github.com/lijx10/NN-Trees

Homework

- We provide one $N \times 3$ point cloud
- 8-NN search for each point to the point cloud
- Implement 3 NN algorithms
  1. Numpy brute-force search
  2. scipy.spatial.KDTree
  3. Your own kd-tree/octree in python or C++
- Report timing using method 1 as baseline
- This is a competition!
  - Timing of method 3 determine your grade

感谢聆听！

Thanks for Listening