



# 3D Point Clouds

## Lecture 2

### Nearest Neighbors

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**1. Binary Search Tree**



**2. Kd-tree**



**3. Octree**

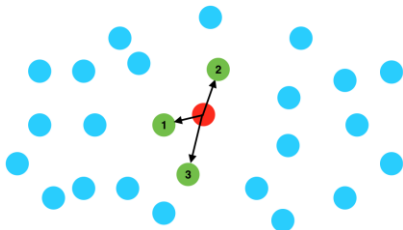


# Nearest Neighbor (NN) Problem



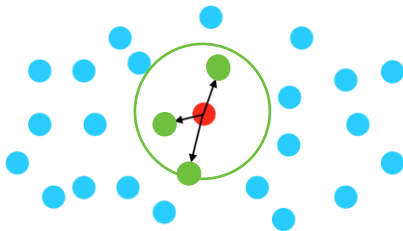
## K-NN

- Given a set of points  $S$  in a space  $M$ , a query point  $q \in M$ , find the  $k$  closest points in  $S$



## Fixed Radius-NN

- Given a set of points  $S$  in a space  $M$ , a query point  $q \in M$ , find all the points in  $S$ , s.t.,  $\|s - q\| < r$





## Why NN problem is important?



It is almost everywhere

- What we have covered:
  - Surface normal estimation
  - Noise filtering
  - Sampling
- What we will cover:
  - Clustering
  - Deep learning
  - Feature detection / description
  - ... ..



Why don't we simply call an open-source library (flann, PCL, etc.)?

- They are not efficient enough.
  - They are general lib, not optimized for 2D/3D.
  - Most open-source octree implementation is in-efficient, while octree is most effective for 3D.
- Few GPU based NN library is available



## Why NN is difficult for point clouds



For Images, a neighbor is simply  $x + \Delta x, y + \Delta y$



For point clouds

- Irregular – no grid based representation
- Curse of dimensionality
  - Non-trivial to build grids
  - Non-trivial to sort or build spatial partitions
- Huge data throughput in real-time applications
  - Velodyne HDL-64E – 2.2 million points per second / 110,000 points at 20Hz
  - Brute-force self-NN search is  $110,000 \times 110,000 \times 0.5 = 6 \times 10^9$  comparisons @ 20Hz



# Lecture Outline



## Binary Search Tree (BST)

- Basic knowledge about trees
- 1D NN problem
- With Python codes



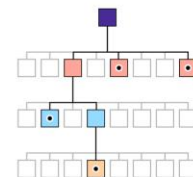
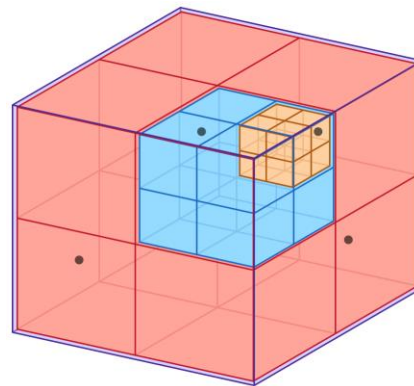
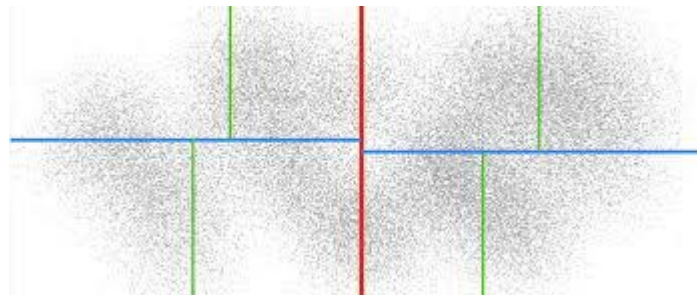
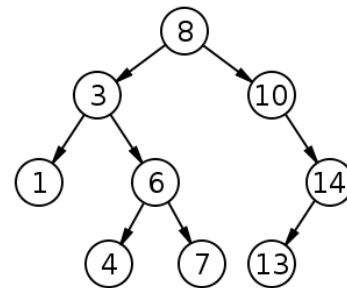
## Kd-tree

- Works for data of any dimension
- Illustrated in 2D
- With Python codes



## Octree

- Specifically designed for 3D data
- Illustrated in 2D/3D
- With Python codes





## Core Ideas Shared by BST, kd-tree, octree



### NN by space partition

- Split the space into different areas,
- Search some areas only, instead of all the data points



### Stopping criteria

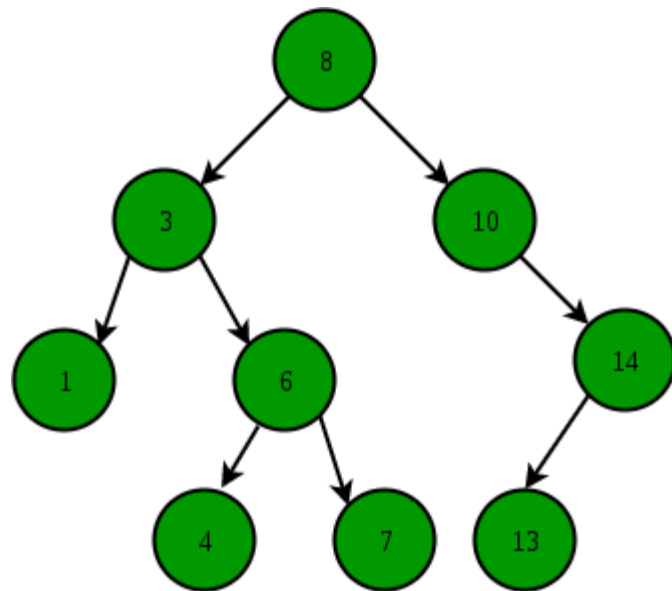
- How to skip some partitions?
  - Intersection of the “worst distance” with the partition boundaries
- How to stop the k-NN / Radius-NN search?
  - Search until the root
  - A partition completely contains the “worst distance”



## Binary Search Tree (BST)

BST is a node-based tree data structure:

1. A node's left subtree contains nodes with keys lesser than its key
2. A node's right subtree contains nodes with keys larger than its key
3. The left / right subtree is BST







# Binary Search Tree (BST)

From Wikipedia

## Binary search tree

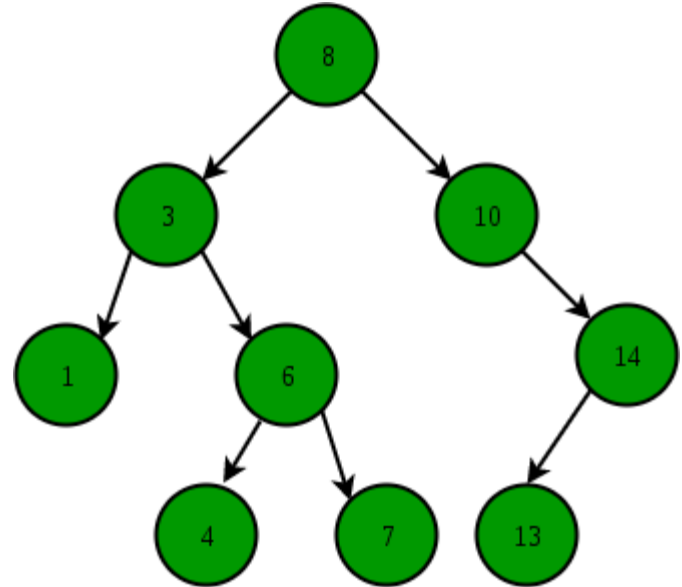
**Type** tree

**Invented** 1960

**Invented by** P.F. Windley, [A.D. Booth](#), [A.J.T. Colin](#), and [T.N. Hibbard](#)

**Time complexity in big O notation**

Algorithm	Average	Worst case
Space	$O(n)$	$O(n)$
Search	$O(\log n)$	$O(n)$
Insert	$O(\log n)$	$O(n)$
Delete	$O(\log n)$	$O(n)$





## BST – Node definition



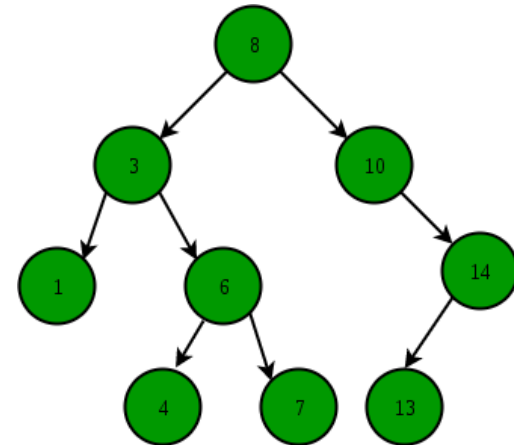
A node contains

1. Key
2. Left child
3. Right child
4. ... ..



The left/right child is a Node as well

```
class Node:
    def __init__(self, key, value=-1):
        self.left = None
        self.right = None
        self.key = key
        self.value = value
```





## BST – Construction / Insertion



Given N 1D-points (scalar) denoted by an array

$$\{x_1, x_2, \dots, x_n\}, x_i \in \mathbb{R}$$



Construct a BST that stores the points and its index in the array, e.g.

[100, 20, 500, 10, 30, 40]

## Data generation

```
db_size = 10  
data = np.random.permutation(db_size).tolist()
```

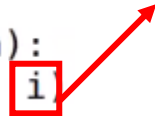
## Recursively insert each an element

```
def insert(root, key, value=-1):  
    if root is None:  
        root = Node(key, value)  
    else:  
        if key < root.key:  
            root.left = insert(root.left, key, value)  
        elif key > root.key:  
            root.right = insert(root.right, key, value)  
        else: # don't insert if key already exist in the tree  
            pass  
    return root
```

## Insert each element

```
root = None  
for i, point in enumerate(data):  
    root = insert(root, point, i)
```

“value” in the Node is the index of a point in the array  
Useful in later NN search





## BST – Insertion Complexity



The worst case is  $O(h)$ , where  $h$  is the height of the BST



In the worst case,  $h$  is the number of points in BST.

- Unbalanced tree – a chain in an extreme case
- E.g., inserting [9, 8, 7, 6, 5, 4, 3] into an empty BST

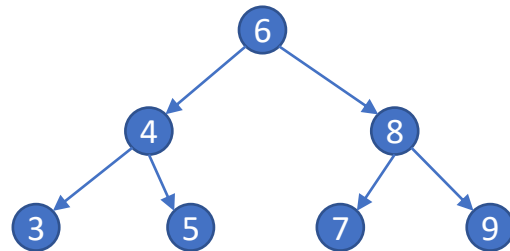
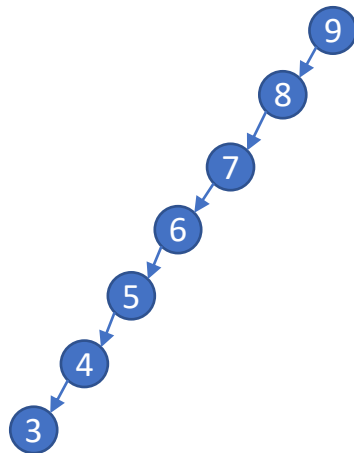


Tree balancing is another topic

- Sort the array and insert as a balanced tree (select median as root)
- AVL tree
- Red-Black tree
- etc.



Best case  $h = \log_2 n$





## BST – Search



Given a BST, and a query (key), determine *which node* equals to that query (key), if not, return *NULL*



Can be done *recursively* or *iteratively*



## BST – Search Recursively

Search till the leaf but not found.

Find a match

For sure just need to look at the left

```
def search_recursive(root, key):  
    if root is None or root.key == key:  
        return root  
    if key < root.key:  
        return search_recursive(root.left, key)  
    elif key > root.key:  
        return search_recursive(root.right, key)
```

For sure just need to look at the right



## BST – Search Iteratively



Use “current\_node” to simulate a *Stack*, so that recursion is avoided



In any case, you can write your own *Stack* to avoid recursion, but that may be complicated sometimes.

```
def search_iterative(root, key):
    current_node = root
    while current_node is not None:
        if current_node.key == key:
            return current_node
        elif key < current_node.key:
            current_node = current_node.left
        elif key > current_node.key:
            current_node = current_node.right
    return current_node
```



Search recursively or iteratively complexity same as insertion  
worst  $O(h)$





## BST – Search

### Recursion

#### Pros:

- Easy to understand / implement
- Codes are short

#### Cons:

- Hard to trace step-by-step
- $O(n)$  storage,  $n$  is number of recursion (May be optimized by compiler)

现在很多C++编译器会将递归转化为循环。

### Iteration

#### Pros:

- Avoid stack-overflow, e.g., in embedded system / GPU
- Easier in step-by-step tracing
- $O(1)$  storage

#### Cons:

- The logic is complicated



# BST – Depth First Traversal



Inorder – Left, Root, Right

- E.g., sorting
- 1, 3, 4, 6, 7, 8, 10, 13, 14



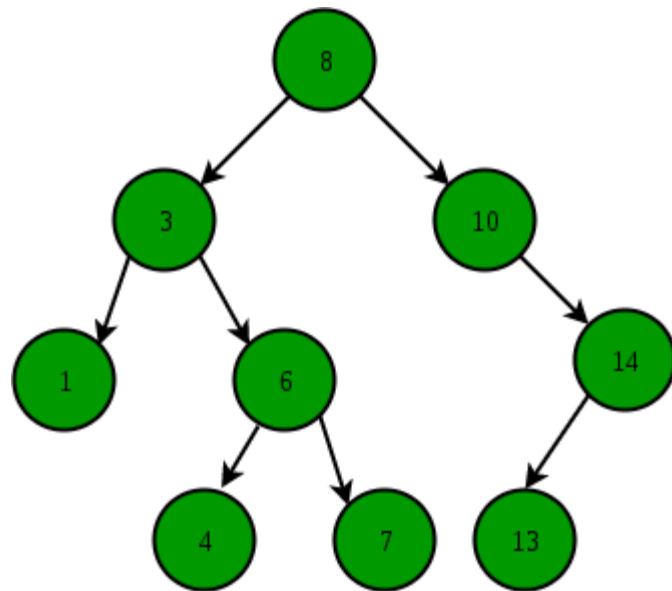
Preorder – Root, Left, Right

- E.g., copy a tree
- 8, 3, 1, 6, 4, 7, 10, 14, 13



Postorder – Left, Right, Root

- E.g., delete a node
- 1, 4, 7, 6, 3, 13, 14, 10, 8



```
def inorder(root):
    # Inorder (Left, Root, Right)
    if root is not None:
        inorder(root.left)
        print(root)
        inorder(root.right)
```

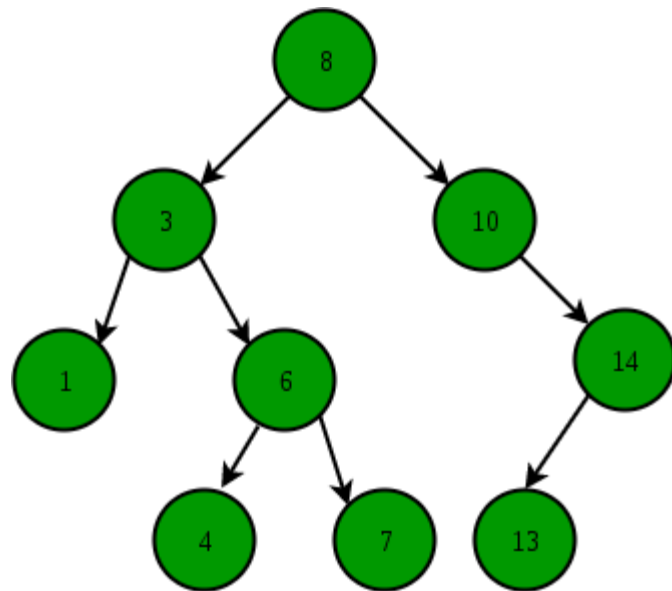
1, 3, 4, 6, 7, 8, 10, 13, 14

```
def preorder(root):
    # Preorder (Root, Left, Right)
    if root is not None:
        print(root)
        preorder(root.left)
        preorder(root.right)
```

8, 3, 1, 6, 4, 7, 10, 14, 13

```
def postorder(root):
    # Postorder (Left, Right, Root)
    if root is not None:
        postorder(root.left)
        postorder(root.right)
        print(root)
```

1, 4, 7, 6, 3, 13, 14, 10, 8

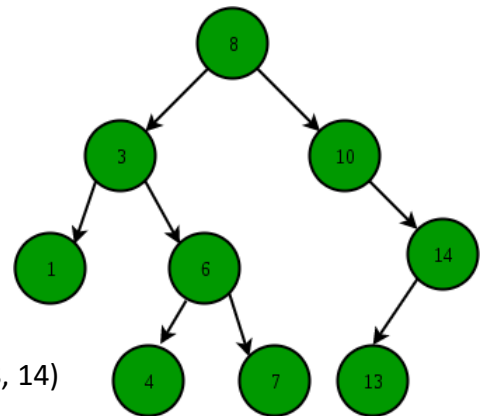




# BST – 1NN Search

- Query point – 11

- 8
  - a) worst distance = 3 (11-8)
  - b) any point in 8's left tree is at least 3 away from query
  - c) do I need to go further? Yes! Right subtree is (8, +inf] but worst distance=3 -> need to search (8, 14)
- 10
  - a) worst distance = 1 (11-10)
  - b) do I need to go further? Yes! Right subtree is (10, +inf] but worst distance=1 -> need to search (10, 12)
- 14
  - a) worst distance = 1
  - b) do I need to go further? Yes! Left subtree is (10, 14) but worst distance=1 -> need to search (10, 12)
- Go back to 14
  - c) do I need to go right? No! Right subtree is (14, inf] but worst distance=1 -> I just need to search (10, 12)
- Go back to 10
  - c) do I need to go left? No! Left subtree is (8, 10) but worst distance=1 -> I just need to search (10, 12)
- Go back to 8
  - d) do I need to go left? No! Left subtree is (-inf, 8) but worst distance=1 -> just need to search (10, 12)





## BST – kNN Search



Almost same as 1NN search



Difference is how to compute **worst distance**



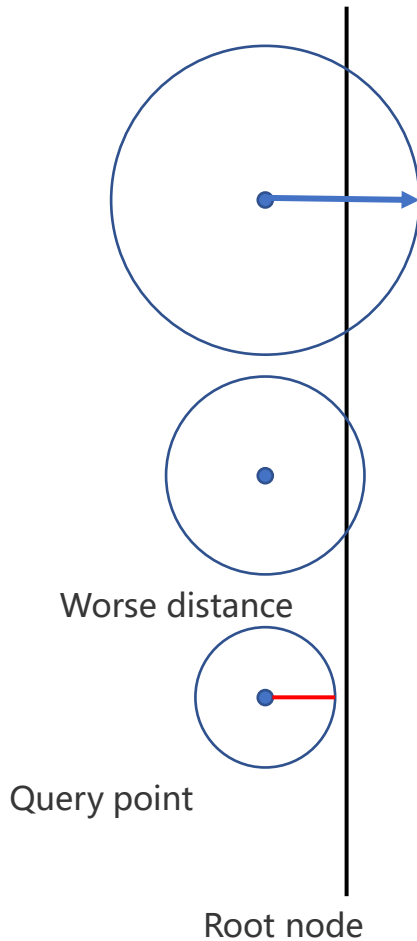
**Worst distance** is the largest distance that you should search around the query point



Areas outside this “worst circle” can be skipped



In kNN search, the worst distance is dynamic





## BST – Worst Distance for kNN



Build a container to store the kNN results



$k$  results are sorted



worst\_dist is the last one



Add a result if

$$dist < worse\_dist$$

- Example:
- Existing container content
- [1, 2, 3, 4, inf, inf]
- **add\_point(3.5)**
- *Step 1.* Make space for 3.5
- [1, 2, 3, 4, 4, inf] 将4复制一下，放到后面
- *Step 2.* Put 3.5 in the correct position
- [1, 2, 3, 3.5, 4, inf]
- *Step 3.* Update worst\_dist

```
class KNNResultSet:
```

```
def __init__(self, capacity):
    self.capacity = capacity
    self.count = 0
    self.worst_dist = 1e10
    self.dist_index_list = []
    for i in range(capacity):
        self.dist_index_list.append(DistIndex(self.worst_dist, 0))
    self.comparison_counter = 0
```

Initialized to large value

Container to keep all the k neighbors

```
def size(self):
    return self.count

def full(self):
    return self.count == self.capacity
```

```
def worstDist(self):
    return self.worst_dist
```

If a point is added, put it in a ordered position

```
def add_point(self, dist, index):
    self.comparison_counter += 1
    if dist > self.worst_dist:
        return

    if self.count < self.capacity:
        self.count += 1

    i = self.count - 1
    while i > 0:
        if self.dist_index_list[i-1].distance > dist:
            self.dist_index_list[i] = copy.deepcopy(self.dist_index_list[i-1])
            i -= 1
        else:
            break

    self.dist_index_list[i].distance = dist
    self.dist_index_list[i].index = index
    self.worst_dist = self.dist_index_list[self.capacity-1].distance
```

```
class DistIndex:
```

```
def __init__(self, distance, index):
    self.distance = distance
    self.index = index

def __lt__(self, other):
    return self.distance < other.distance
```

```

def knn_search(root: Node, result_set: KNNResultSet, key):
    if root is None:
        return False

    # compare the root itself
    result_set.add_point(math.fabs(root.key - key), root.value)
    if result_set.worstDist() == 0:
        return True

    if root.key >= key:
        # iterate left branch first
        if knn_search(root.left, result_set, key):
            return True
        elif math.fabs(root.key - key) < result_set.worstDist():
            return knn_search(root.right, result_set, key)
        return False
    else:
        # iterate right branch first
        if knn_search(root.right, result_set, key):
            return True
        elif math.fabs(root.key - key) < result_set.worstDist():
            return knn_search(root.left, result_set, key)
        return False

```

A special case – if the worst distance is 0, no need to search anymore

If key != query, need to go through one subtree

May not need to search for the other subtree, depends on worst distance

只要有小于worstDist的情况，就要继续找。

Similar to the “if” block above





# Radius NN Search

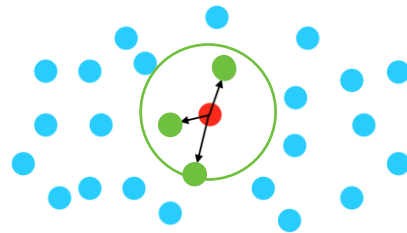
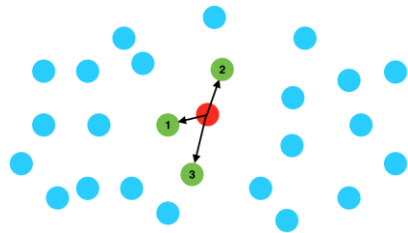


Same as kNN, in the sense that,

- Worst distance circle intersects the boundary -> search
- If not -> skip



In implementation, we don't need to change the BST kNN search logics, except that, the worst distance is **fixed**, instead of dynamic





## BST – Radius Result Set Manger



Simpler than kNN result set manager.



Worst distance is fixed.



No need to maintain a sorted result set.

```
def add_point(self, dist, index):  
    self.comparison_counter += 1  
    if dist > self.radius:  
        return  
  
    self.count += 1  
    self.dist_index_list.append(DistIndex(dist, index))
```



## BST - kNN v.s. Radius NN

```
def knn_search(root: Node, result_set: KNNResultSet, key):  
    if root is None:  
        return False  
  
    # compare the root itself  
    result_set.add_point(math.fabs(root.key - key), root.value)  
    if result_set.worstDist() == 0:  
        return True  
    if root.key >= key:  
        # iterate left branch first  
        if knn_search(root.left, result_set, key):  
            return True  
        elif math.fabs(root.key-key) < result_set.worstDist():  
            return knn_search(root.right, result_set, key)  
        return False  
    else:  
        # iterate right branch first  
        if knn_search(root.right, result_set, key):  
            return True  
        elif math.fabs(root.key-key) < result_set.worstDist():  
            return knn_search(root.left, result_set, key)  
        return False
```

This part is gone in  
radius search, because  
worst\_dist = r

```
def radius_search(root: Node, result_set: RadiusNNResultSet, key):  
    if root is None:  
        return False  
  
    # compare the root itself  
    result_set.add_point(math.fabs(root.key - key), root.value)  
  
    if root.key >= key:  
        # iterate left branch first  
        if radius_search(root.left, result_set, key):  
            return True  
        elif math.fabs(root.key-key) < result_set.worstDist():  
            return radius_search(root.right, result_set, key)  
        return False  
    else:  
        # iterate right branch first  
        if radius_search(root.right, result_set, key):  
            return True  
        elif math.fabs(root.key-key) < result_set.worstDist():  
            return radius_search(root.left, result_set, key)  
        return False
```



## A complete script

```
db_size = 100
k = 5
radius = 2.0

data = np.random.permutation(db_size).tolist()

root = None
for i, point in enumerate(data):
    root = insert(root, point, i)

query_key = 6
result_set = KNNResultSet(capacity=k)
knn_search(root, result_set, query_key)
print('kNN Search:')
print('index - distance')
print(result_set)

result_set = RadiusNNResultSet(radius=radius)
radius_search(root, result_set, query_key)
print('Radius NN Search:')
print('index - distance')
print(result_set)
```

- Search in 100 points, takes 7 comparison only
- Complexity is around  $O(\log_2(n))$ ,  $n$  is number of database points, if tree is balanced
- Worst  $O(N)$

kNN Search:

index - distance

73 - 0.00

5 - 1.00

12 - 1.00

1 - 2.00

98 - 2.00

In total 7 comparison operations.

Radius NN Search:

index - distance

73 - 0.00

5 - 1.00

12 - 1.00

1 - 2.00

98 - 2.00

In total 5 neighbors within 2.000000.

There are 7 comparison operations.

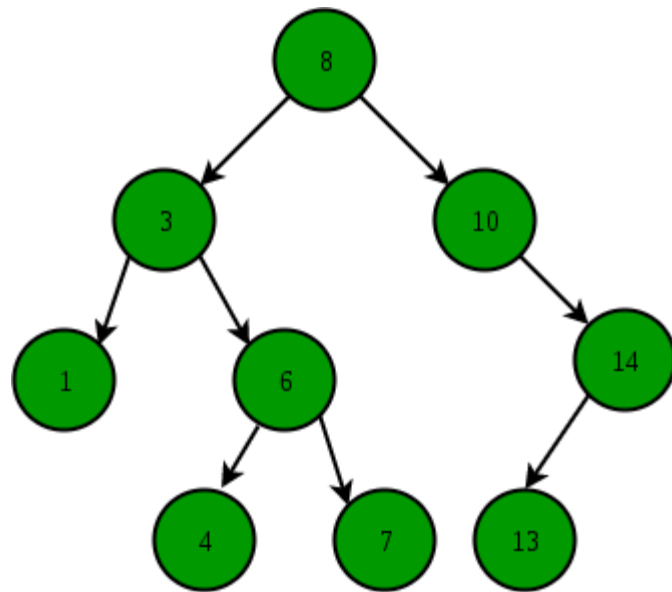


## Binary Search Tree (BST)

BST based 1D kNN/RadiusNN search

- Naïve BST is for 1D data only

Tree based kNN/RadiusNN can be viewed as a Branch-n-Bound algorithm.





# Kd-tree (**k-dimensional** tree)



It is an extension of BST into high dimension

- BST is 1-dimensional, how to extend?
- BST in each dimension!

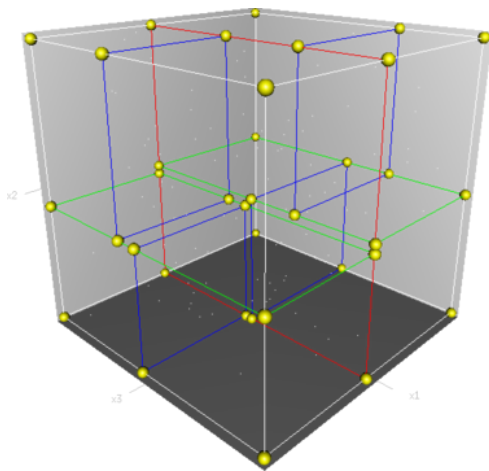


Invented by Jon Louis Bentley in 1975



The kd-tree is a binary tree where every leaf node is a **k-d**imensional point

<b>k-d tree</b>		
Type	Multidimensional <b>BST</b>	
Invented	1975	
Invented by	Jon Louis Bentley	
<b>Time complexity in big O notation</b>		
Algorithm	Average	Worst case
Space	$O(n)$	$O(n)$
Search	$O(\log n)$	$O(n)$
Insert	$O(\log n)$	$O(n)$
Delete	$O(\log n)$	$O(n)$



A 3-dimensional kd tree:

1. Red
2. Green
3. Blue



## Kd-tree Construction



If there is only one point, or number of points  $<$  leaf\_size, build a leaf



Otherwise, divide the points in half by a hyperplane perpendicular to the selected splitting axis



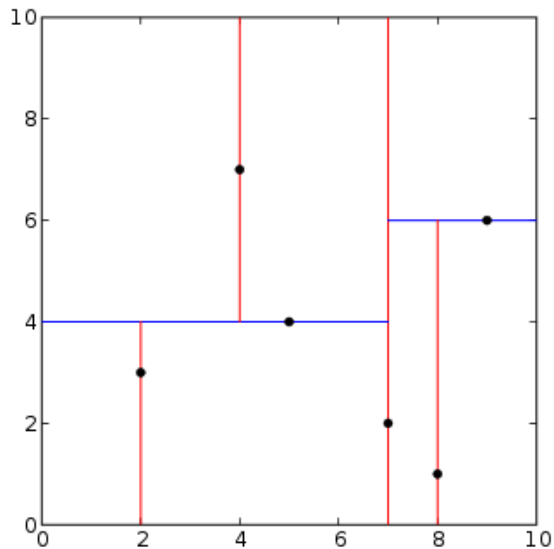
Recursively repeat the first two steps.



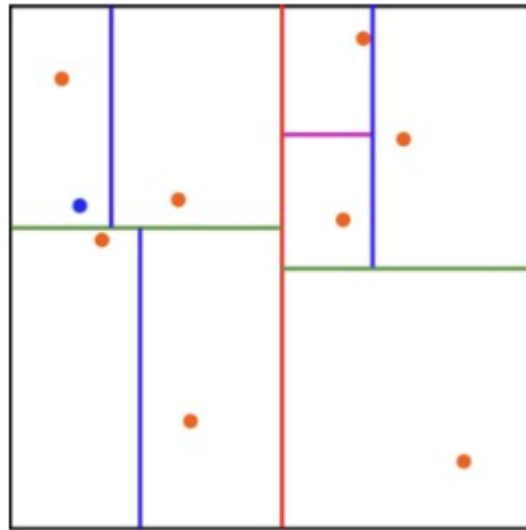
## Kd-tree Construction – Two Conventions

Splitting position is one of the points

课件中使用的是这个方法



Splitting position is NOT one of the points



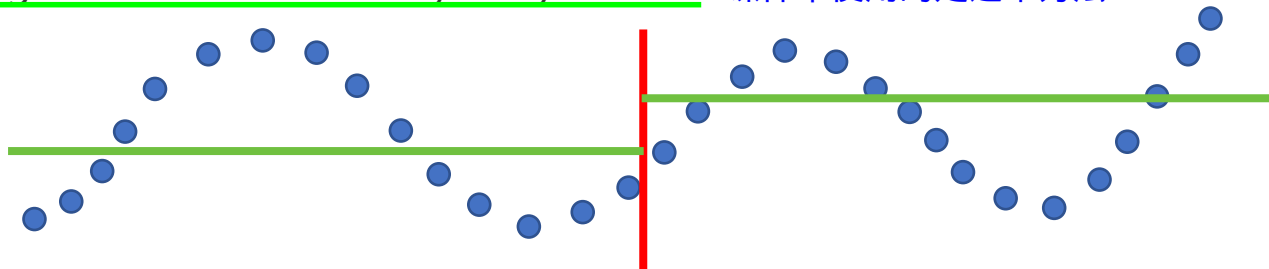




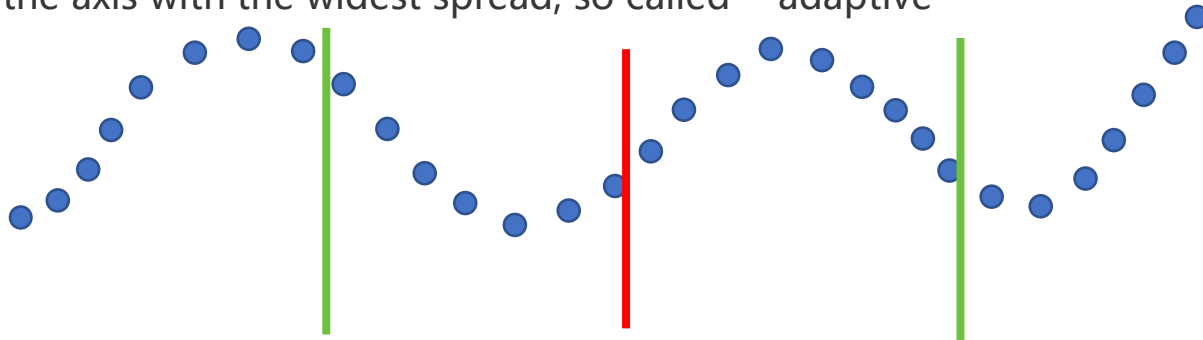
# Kd-tree Construction

## Division / Splitting Strategy

- Dividing axis is round-robin: x-y-z-x-y-z-x-..... 课件中使用的是这个方法

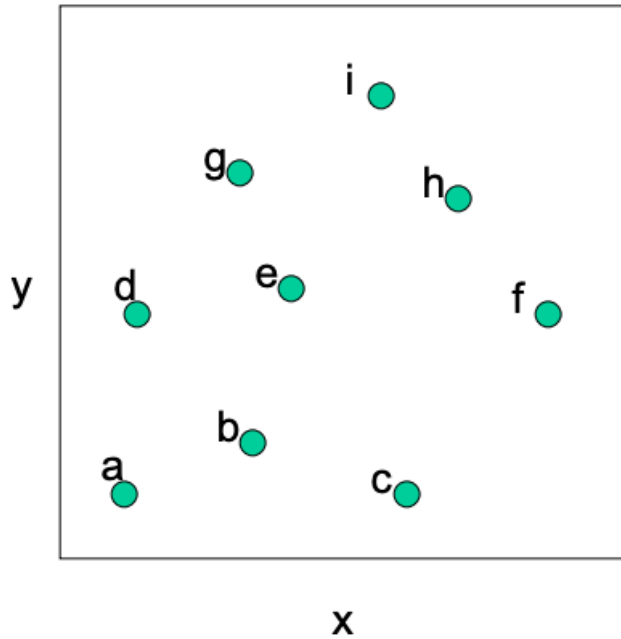


- Select the axis with the widest spread, so called "adaptive"



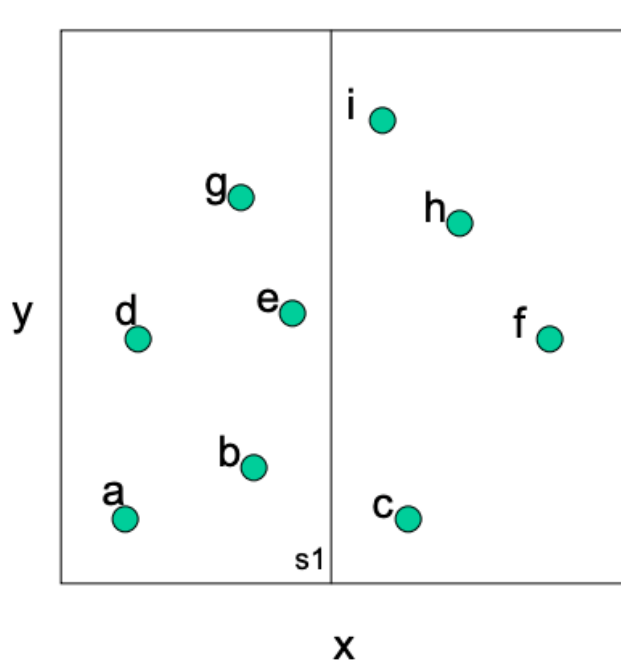


# Kd-tree Construction





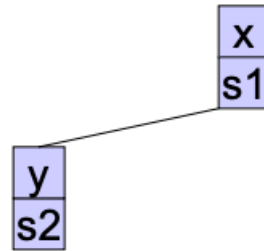
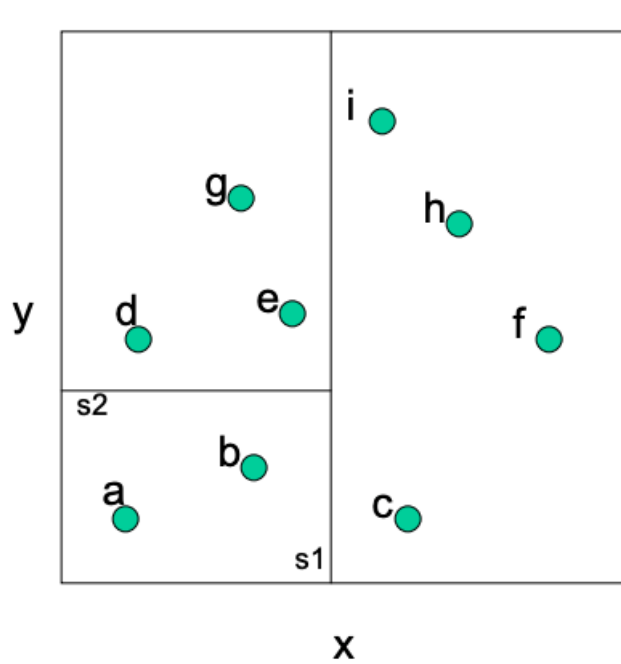
# Kd-tree Construction



x
s1

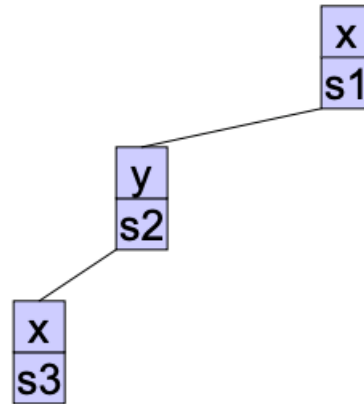
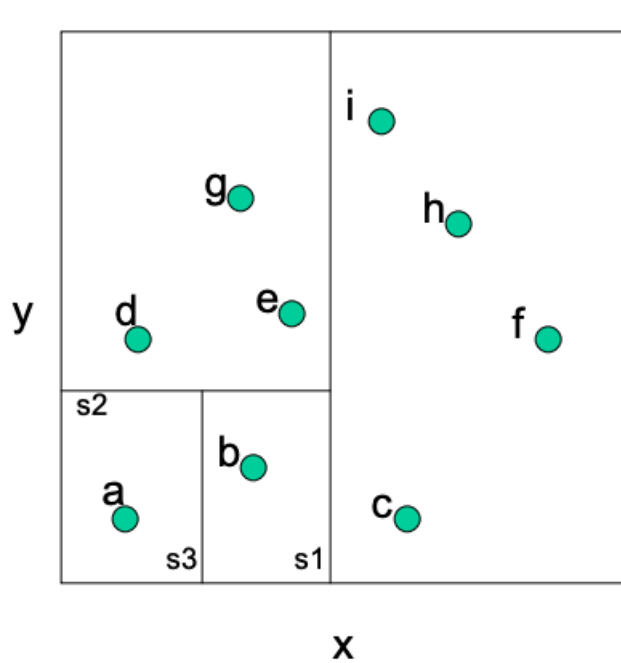


# Kd-tree Construction



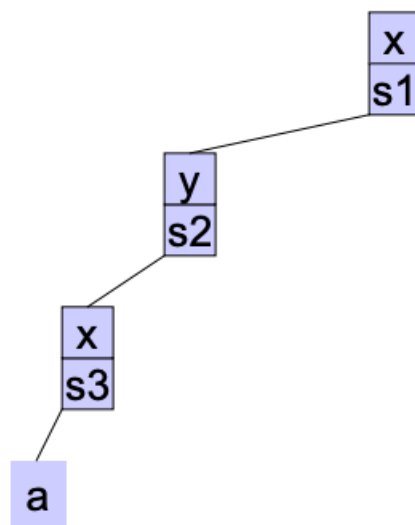
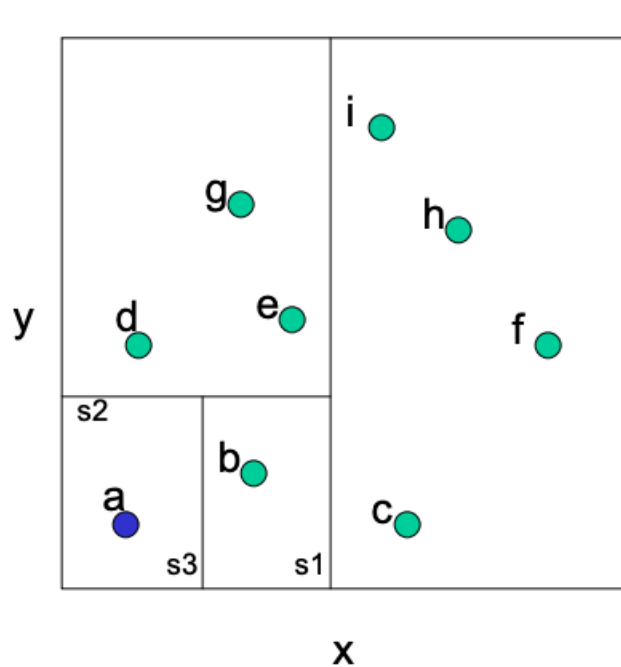


# Kd-tree Construction



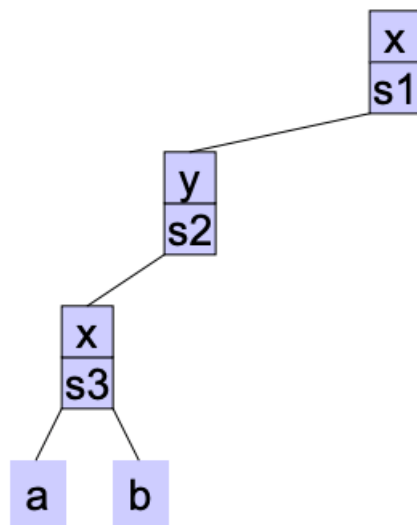
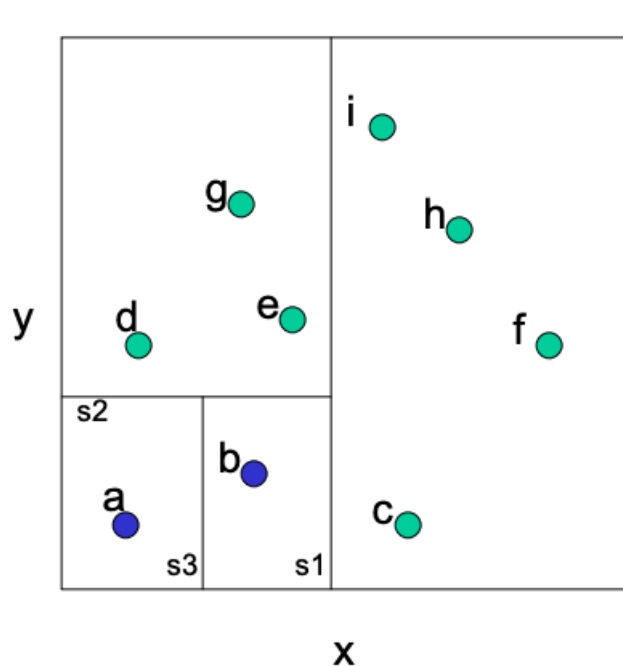


# Kd-tree Construction



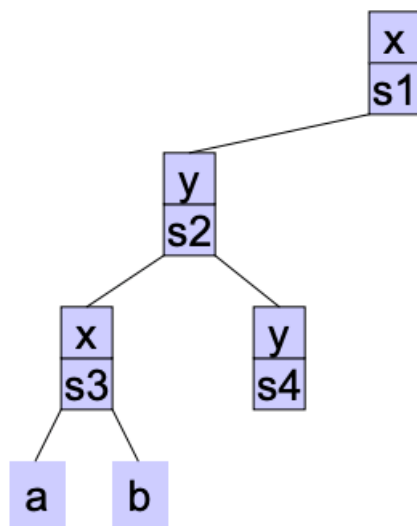
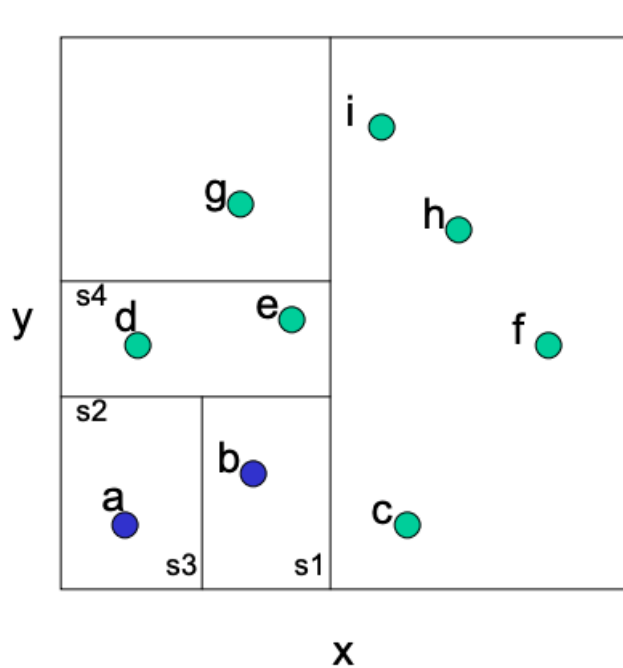


# Kd-tree Construction





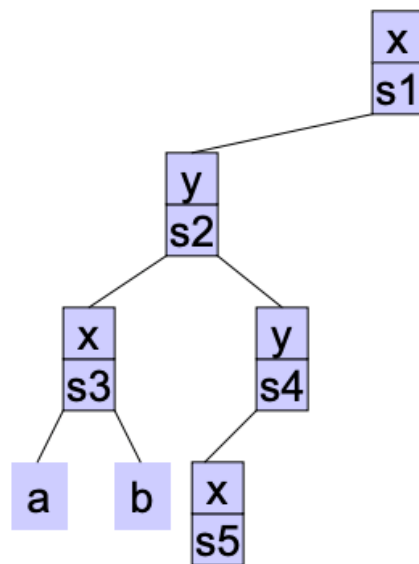
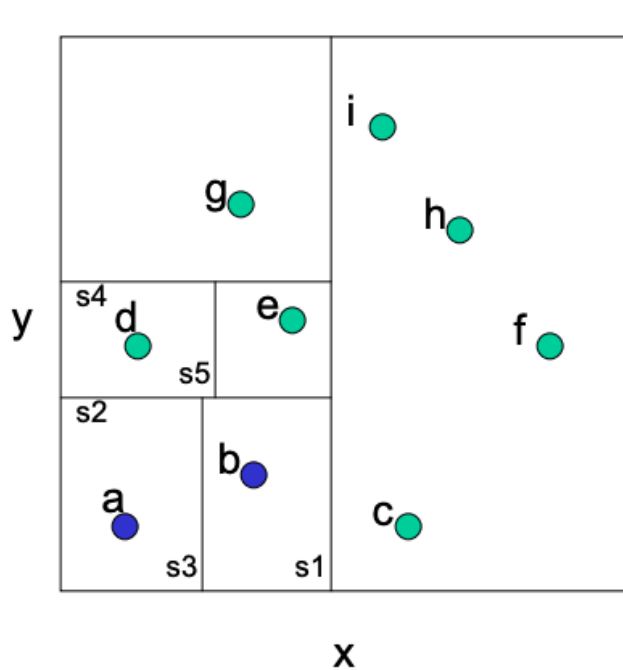
# Kd-tree Construction





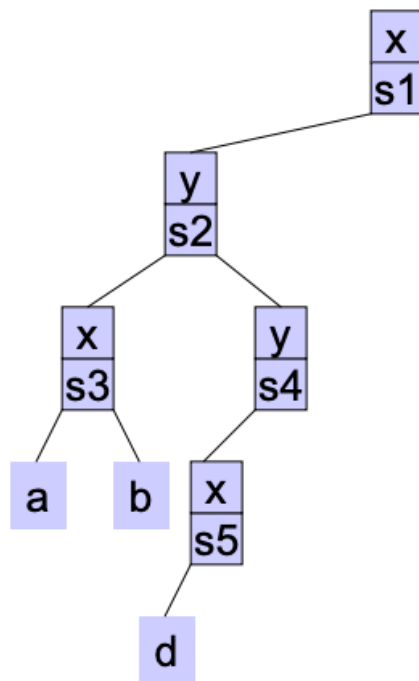
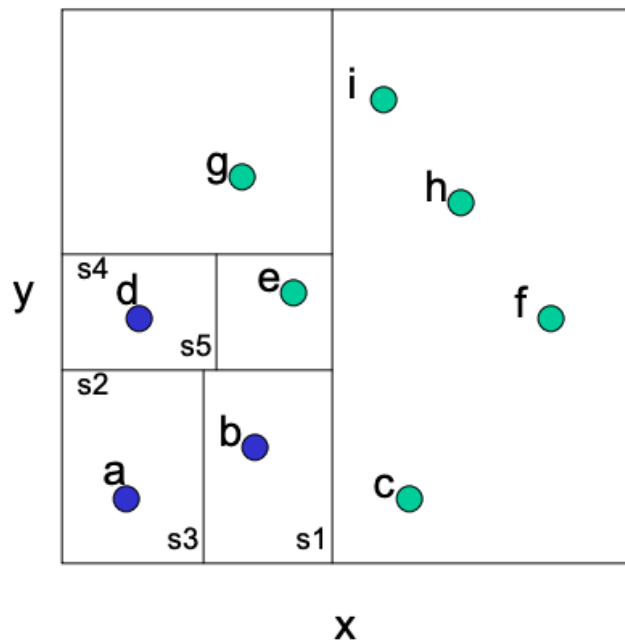


# Kd-tree Construction



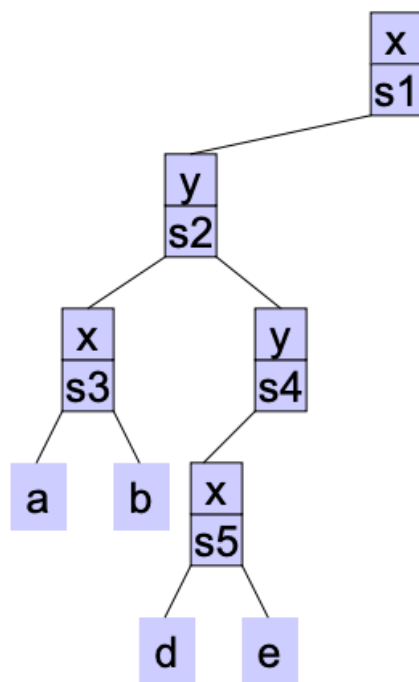
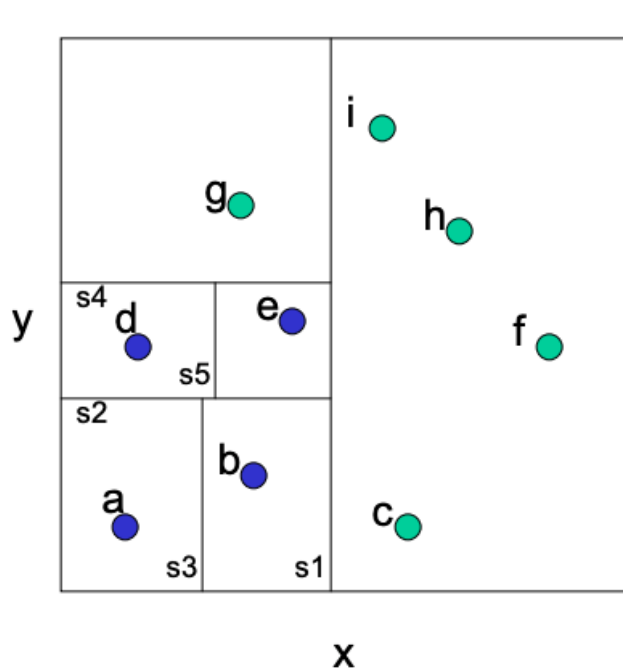


# Kd-tree Construction



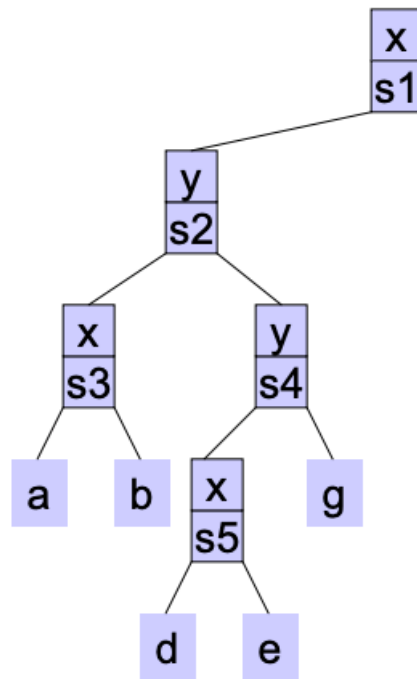
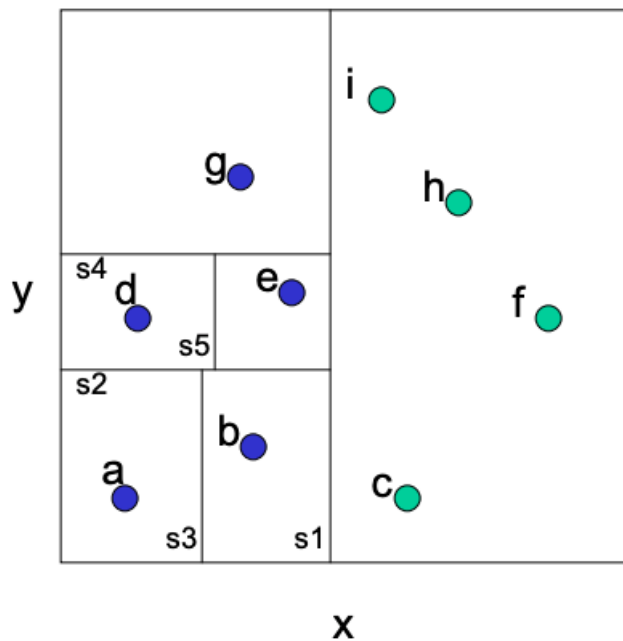


# Kd-tree Construction



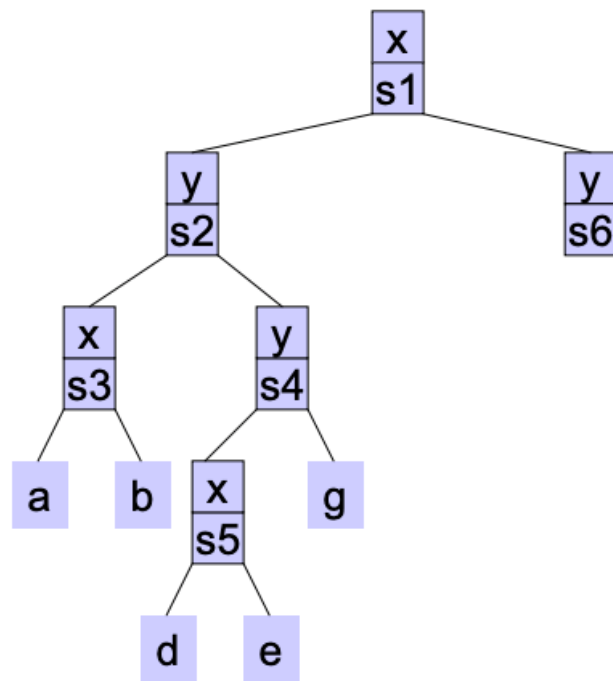
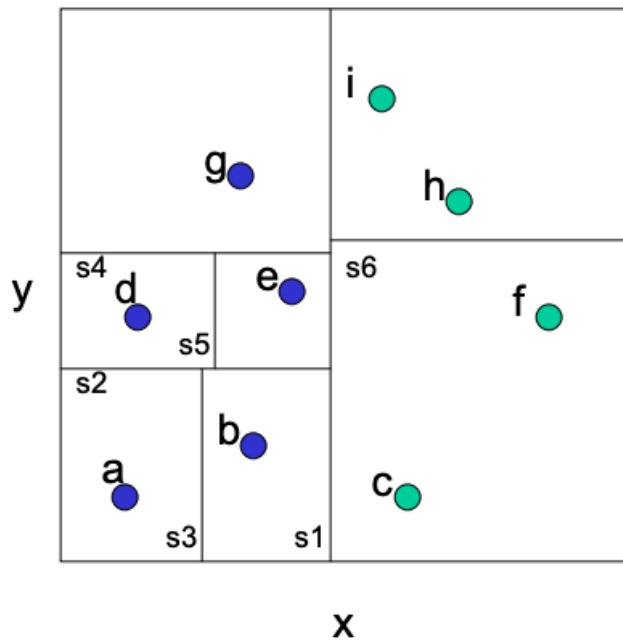


# Kd-tree Construction



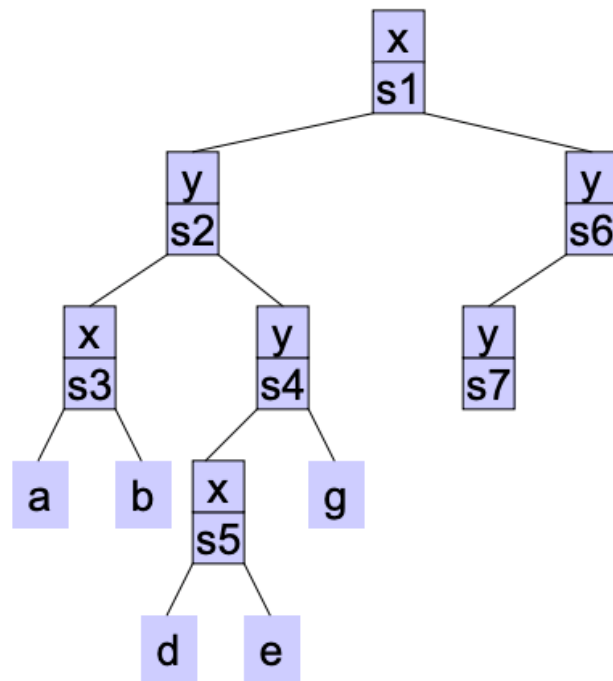
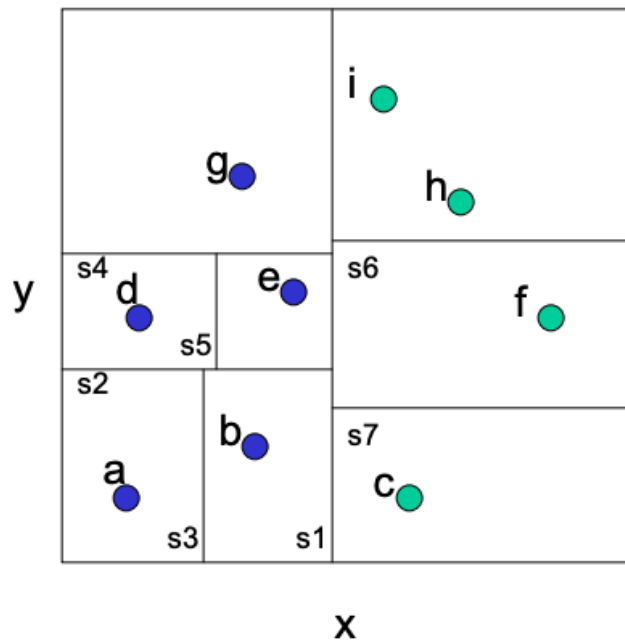


## Kd-tree Construction



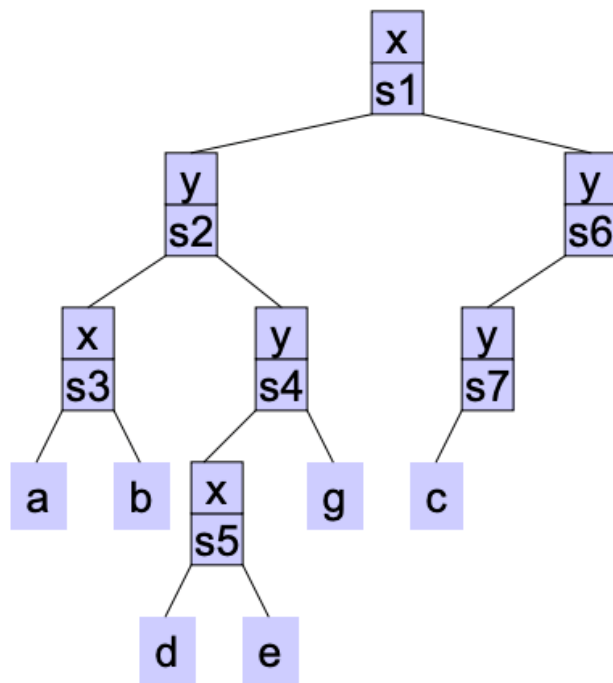
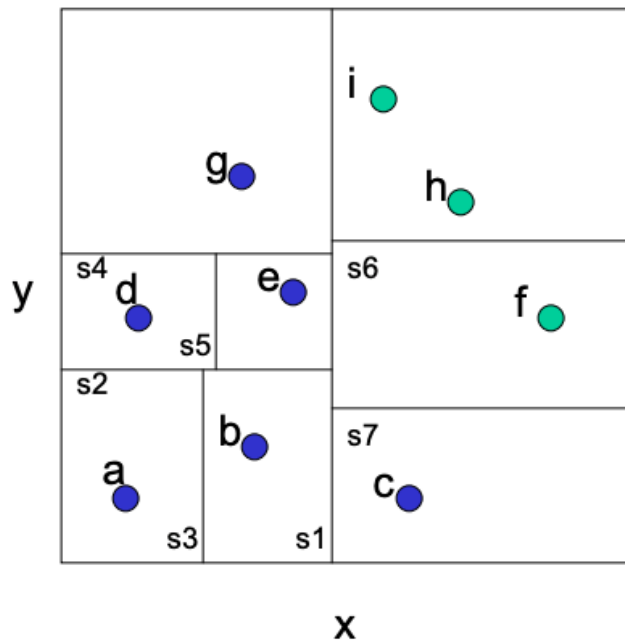


## Kd-tree Construction



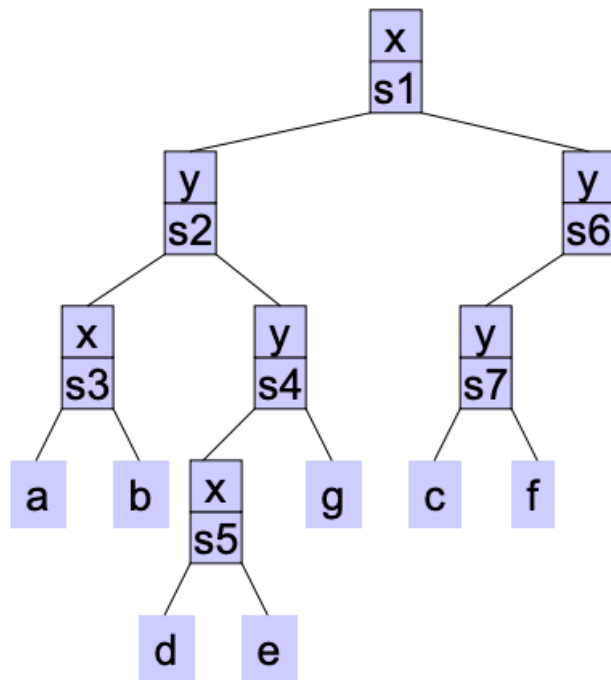
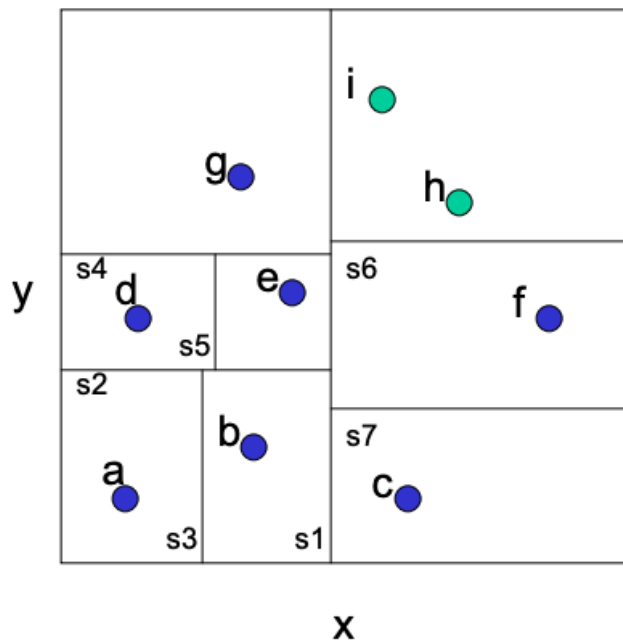


## Kd-tree Construction





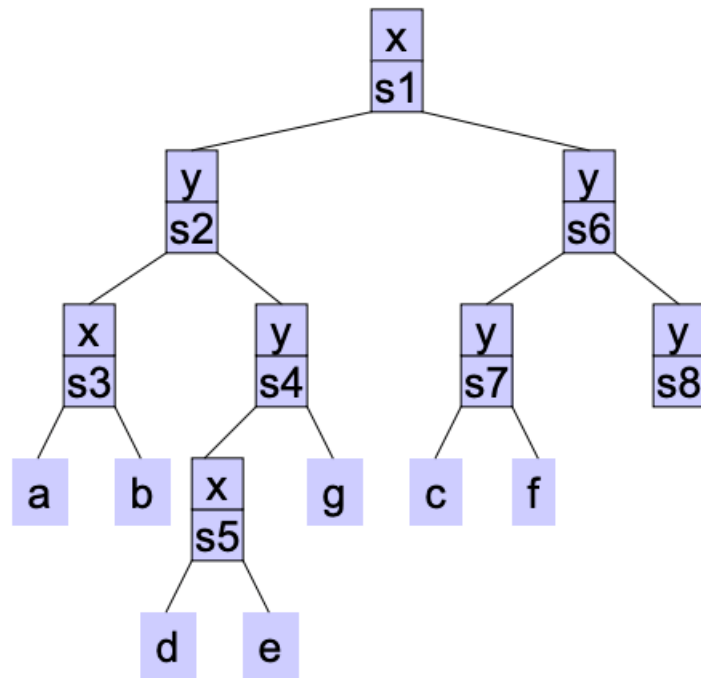
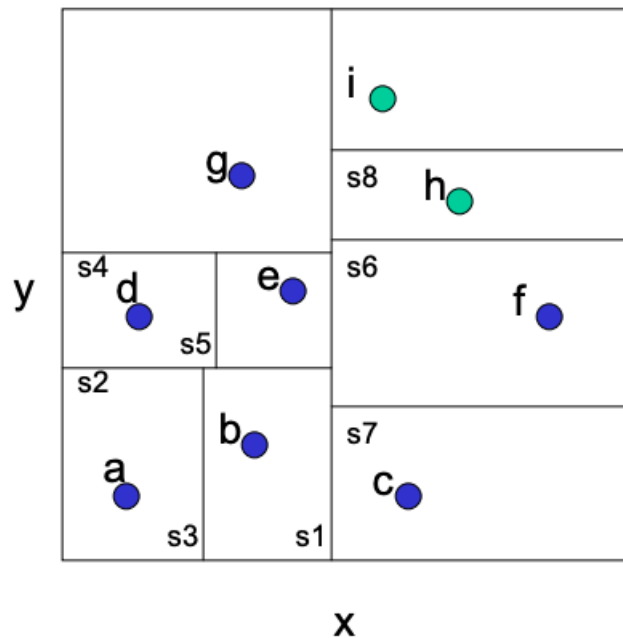
## Kd-tree Construction





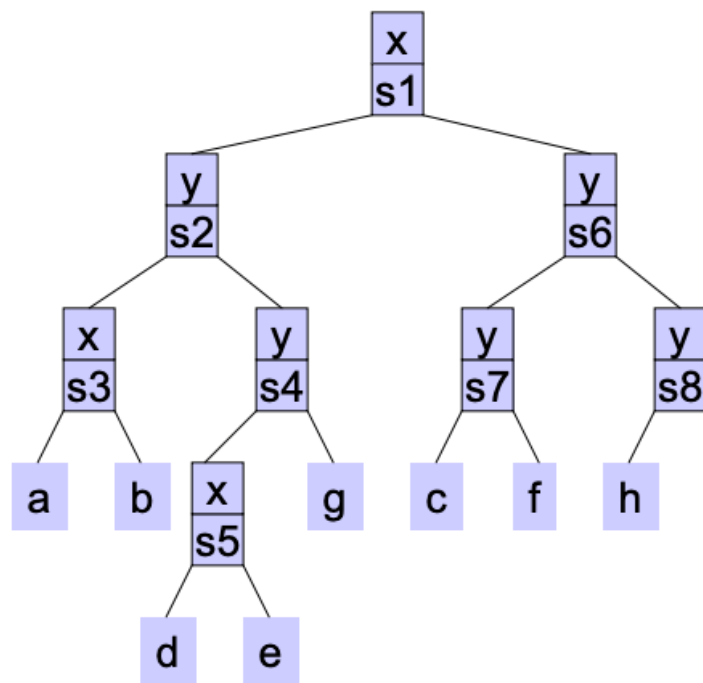
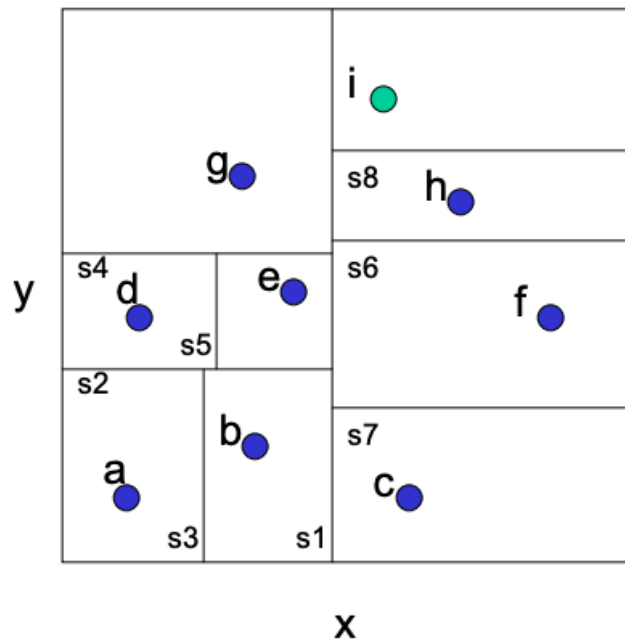


## Kd-tree Construction



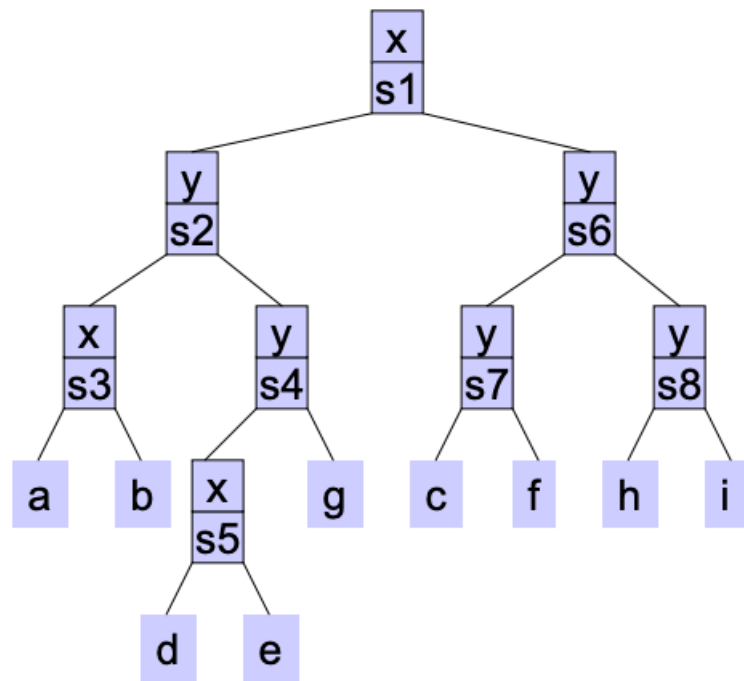
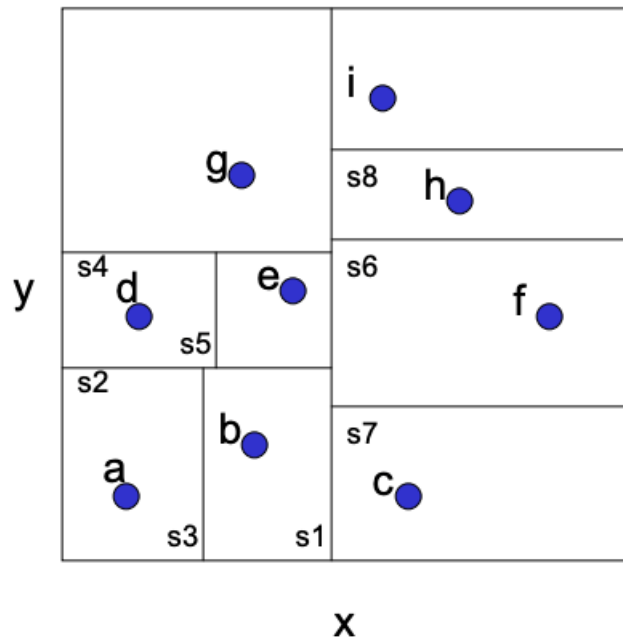


## Kd-tree Construction





## Kd-tree Construction





## Kd-tree Construction



Talk is cheap, show me the code.

**Linus Torvalds**



# Kd-tree Node Representation

```
class Node:
    def __init__(self, axis, value, left, right, point_indices):
        self.axis = axis
        self.value = value
        self.left = left
        self.right = right
        self.point_indices = point_indices

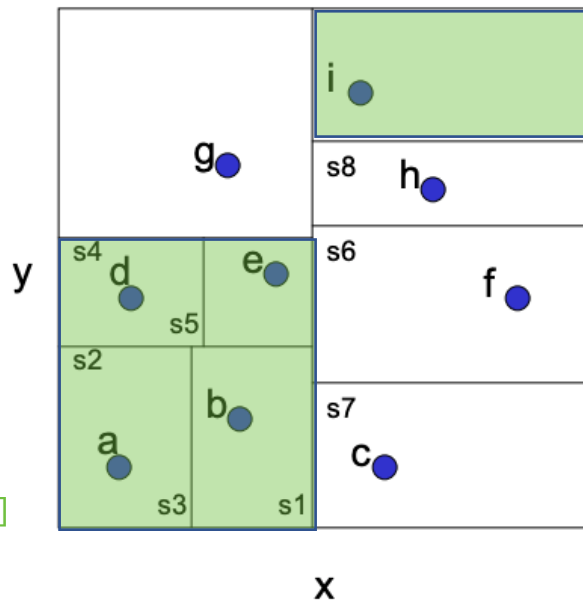
    def is_leaf(self):
        if self.value is None:
            return True
        else:
            return False
```

Splitting position

Stores points that belongs to this partition

A node with  
axis = x  
value = \*\*\*  
points = [a, b, d, e]

A leaf node with  
axis = y  
value = None  
points = i



```
def kdtree_recursive_build(root, db, point_indices, axis, leaf_size):
```

A leaf node can contain more than 1 point

```
    :param root:
    :param db: NxM
    :param db_sorted_idx_inv: NxM
    :param point_idx: M
    :param axis: scalar
    :param leaf_size: scalar
    :return:
```

```
    """
    if root is None:
        root = Node(axis, None, None, None, point_indices)
```

```
    # determine whether to split into left and right
```

```
    if len(point_indices) > leaf_size:
```

```
        # --- get the split position ---
```

```
        point_indices_sorted, _ = sort_key_by_val(point_indices, db[point_indices, axis]) # M
```

```
        middle_left_idx = math.ceil(point_indices_sorted.shape[0] / 2) - 1
```

```
        middle_left_point_idx = point_indices_sorted[middle_left_idx]
```

```
        middle_left_point_value = db[middle_left_point_idx, axis]
```

```
        middle_right_idx = middle_left_idx + 1
```

```
        middle_right_point_idx = point_indices_sorted[middle_right_idx]
```

```
        middle_right_point_value = db[middle_right_point_idx, axis]
```

```
        root.value = (middle_left_point_value + middle_right_point_value) * 0.5
```

```
        # === get the split position ===
```

```
        root.left = kdtree_recursive_build(root.left,
```

```
            db,
```

```
            point_indices_sorted[0:middle_right_idx],
```

```
            axis_round_robin(axis, dim=db.shape[1]),
```

```
            leaf_size)
```

```
        root.right = kdtree_recursive_build(root.right,
```

```
            db,
```

```
            point_indices_sorted[middle_right_idx:],
```

```
            axis_round_robin(axis, dim=db.shape[1]),
```

```
            leaf_size)
```

```
    return root
```

Sort the points in this node, get the median position

```
def axis_round_robin(axis, dim):
    if axis == dim-1:
        return 0
    else:
        return axis + 1
```



## Kd-tree Construction Complexity



The example shown here is not optimal because of sorting at each level of the tree

- Time complexity of around  $O(n \log n \log n)$
- Space complexity of  $O(kn + n \log n) \rightarrow$  can be easily reduced to  $O(kn + n)$ 
  - Only store points at leaf



Can we select **median** instead of sorting?

- If median finding is  $O(n)$
- Kd-tree is  $O(n \log n)$
- Median finding in  $O(n)$  is complicated, but **possible**!



$O(kn \log n)$  method

- Building a Balanced k-d Tree in  $O(kn \log n)$  Time
  - Russel A. Brown, Journal of Computer Graphics Techniques, 2015



# Kd-tree Construction Complexity



## Simple methods that work well in practice

- Sample a **subset** of point in each node for sorting, instead of sorting all points
  - $O(n' \log n)$  or  $O(n' \log n' \log n)$
- Use **mean** instead of median
  - An easy way to achieve  $O(n \log n)$



## They don't guarantee balanced kd-tree

- Balanced tree - each leaf node is approximately the same distance from the root
- Similar to the "chain" example in BST.





## Kd-tree – kNN Search



Start from root



Reach the leaf node than covers the query point

- Compare all points in the leaf node

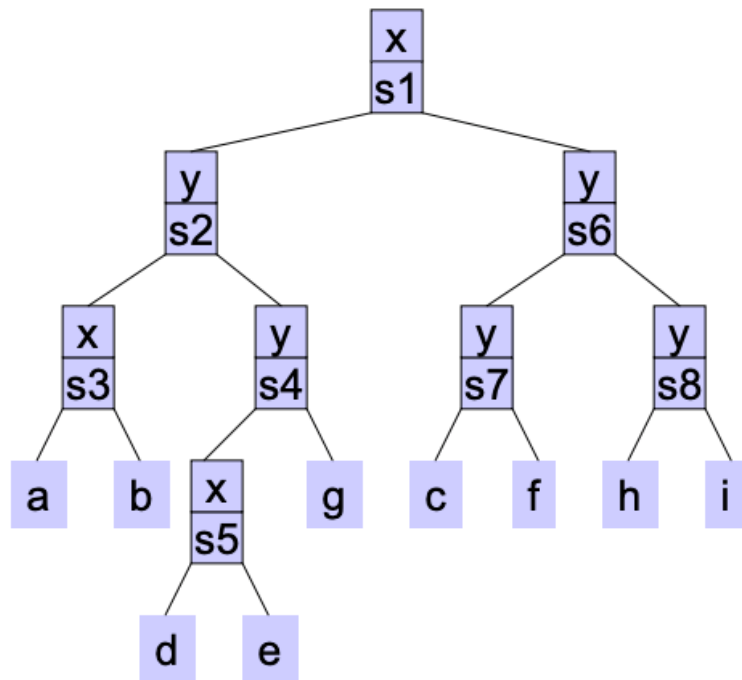
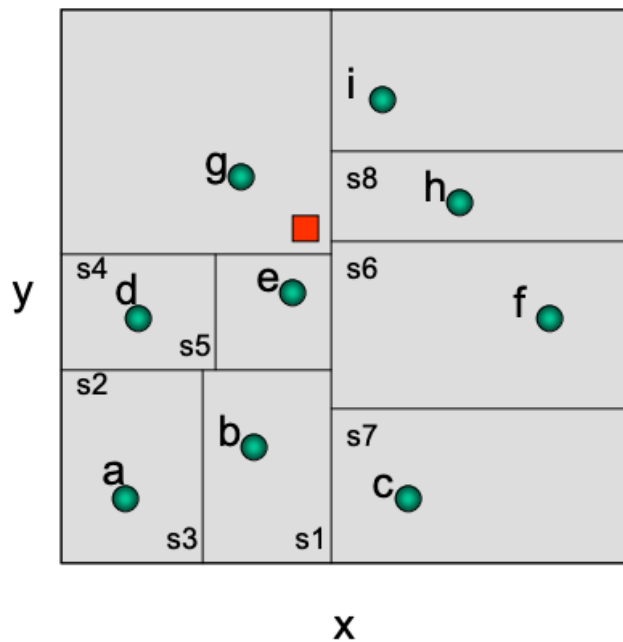


Go up and traverse the tree



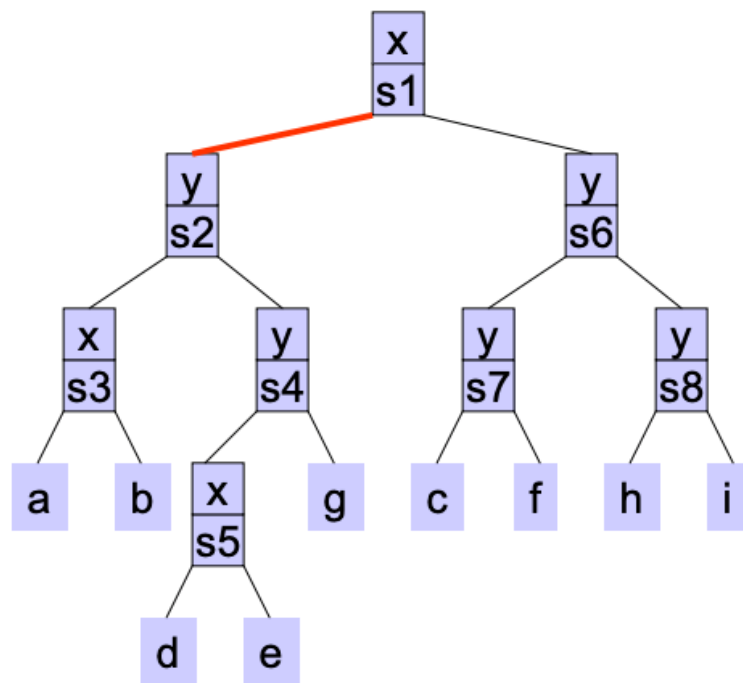
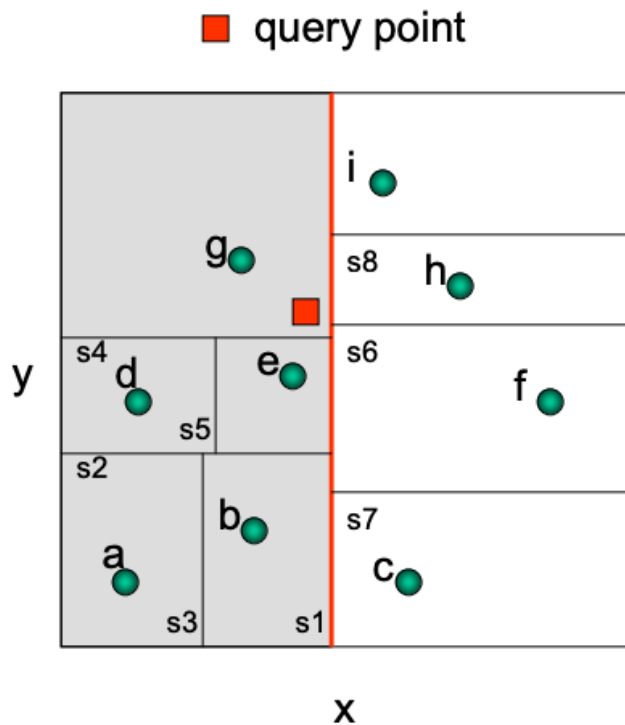
## Kd-tree – kNN Search

■ query point



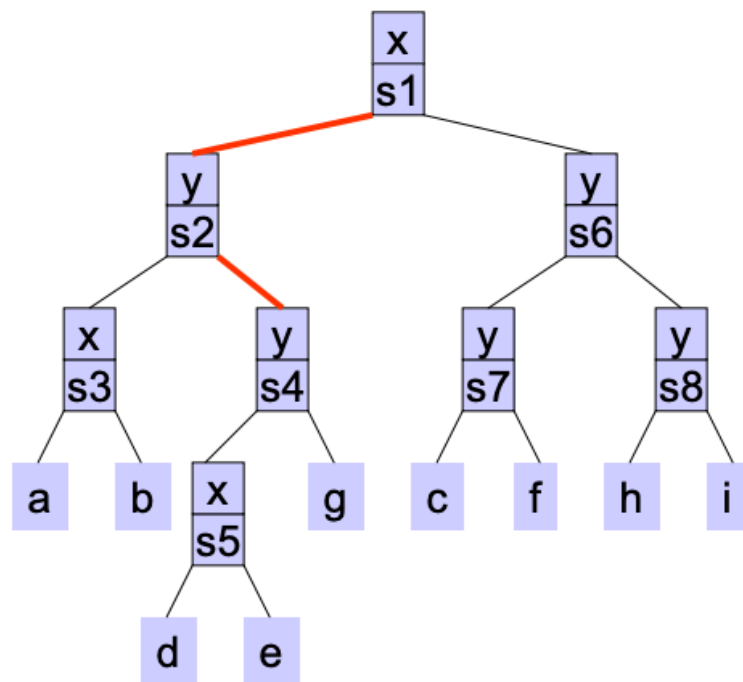
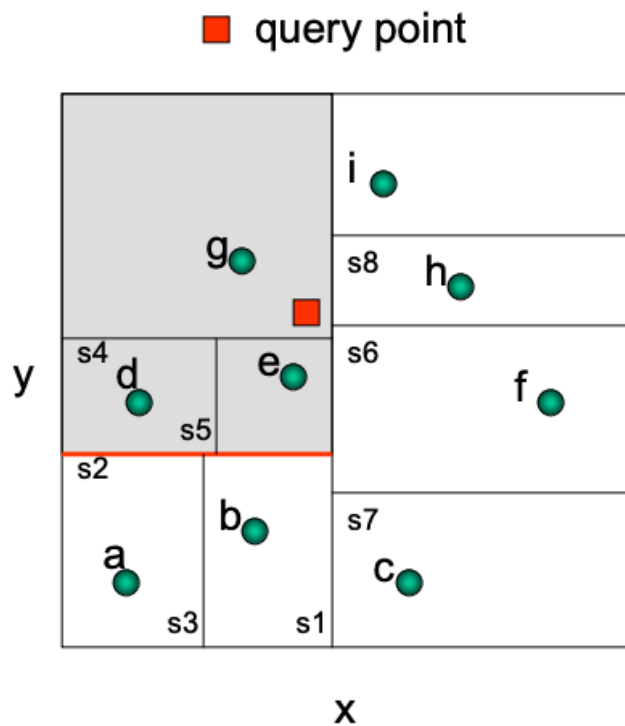


## Kd-tree – kNN Search



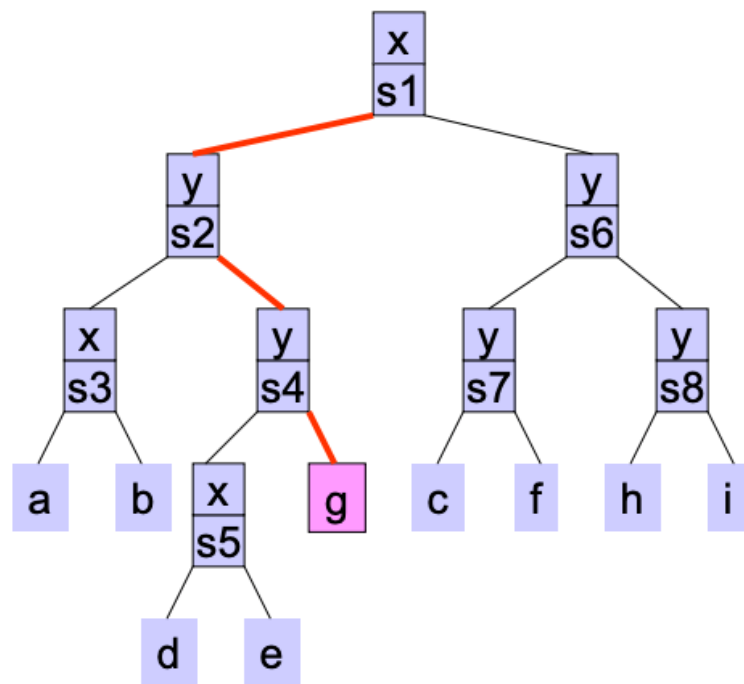
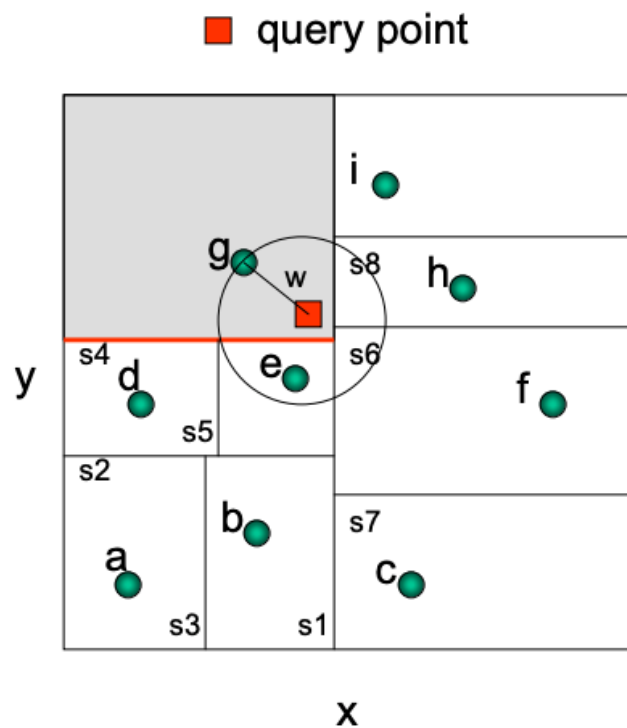


## Kd-tree – kNN Search



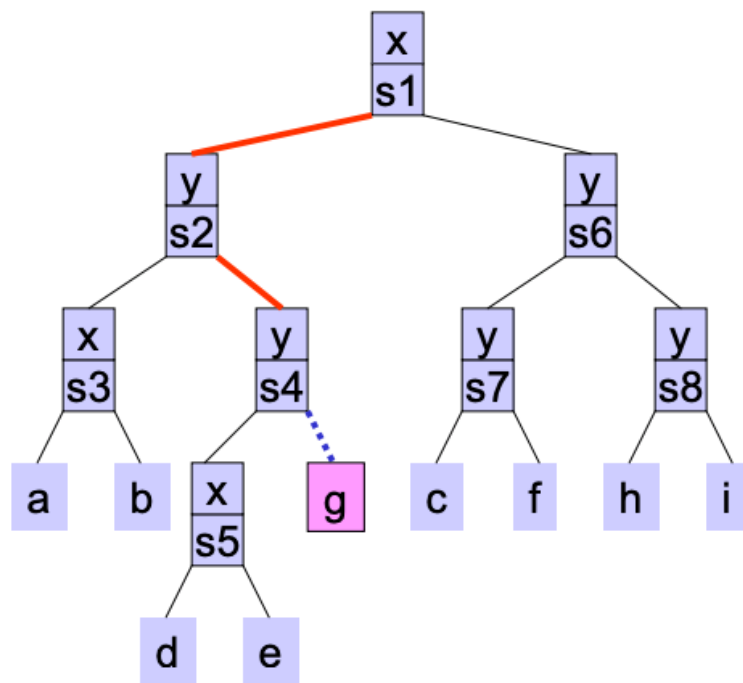
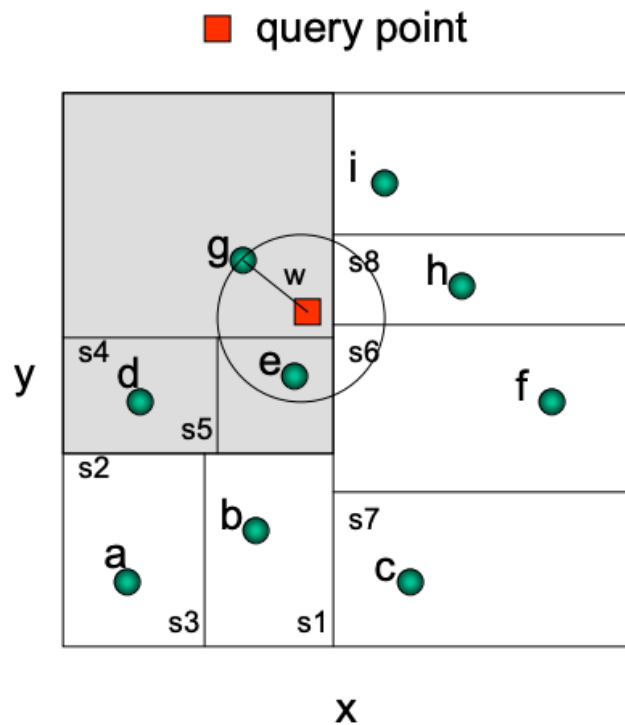


## Kd-tree – kNN Search





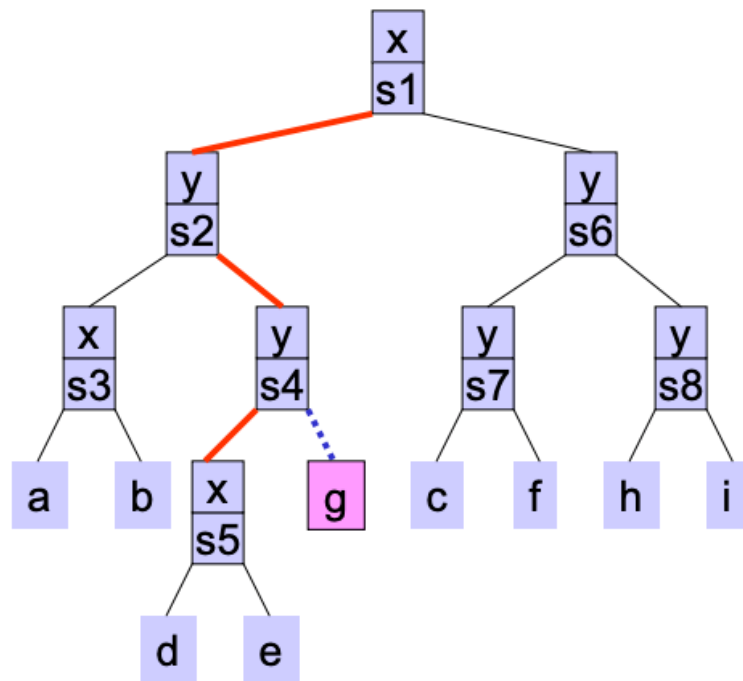
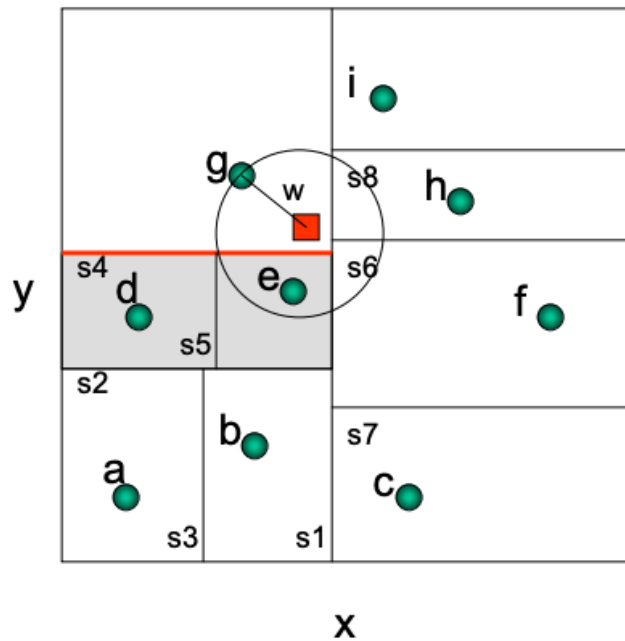
## Kd-tree – kNN Search





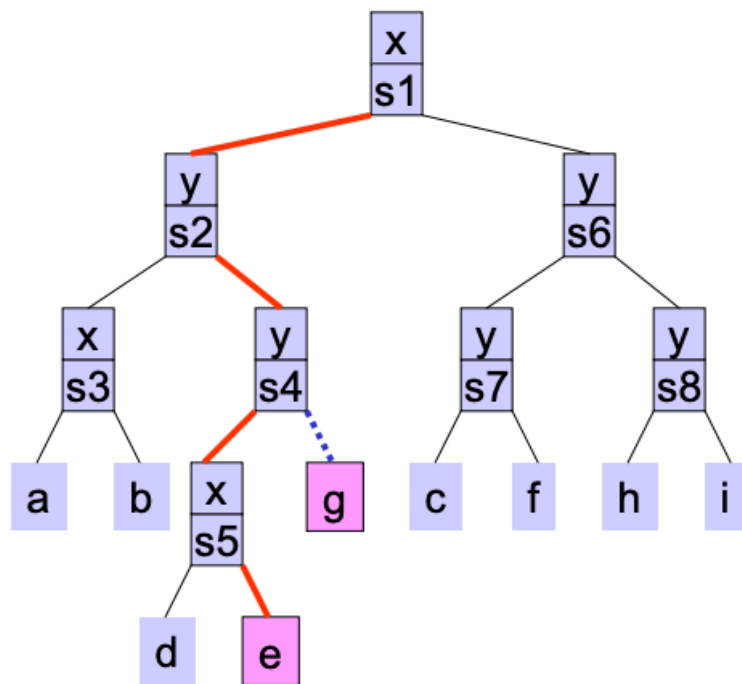
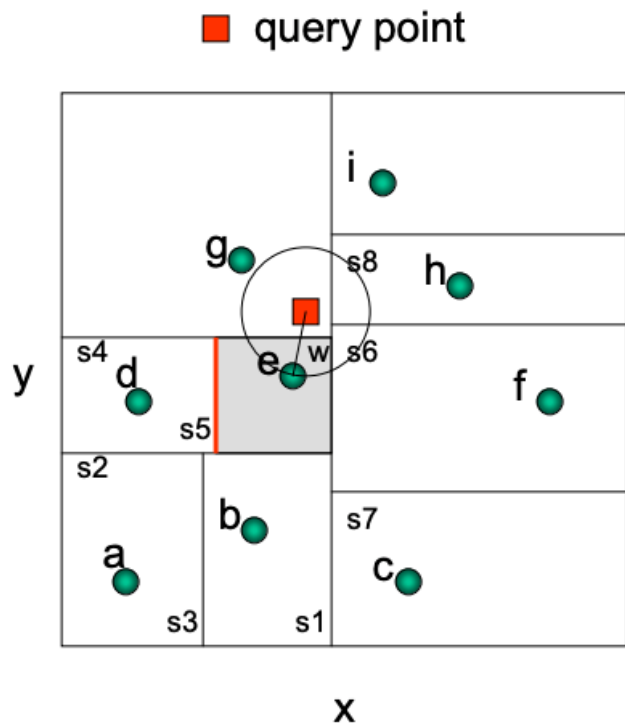
## Kd-tree – kNN Search

■ query point





## Kd-tree – kNN Search

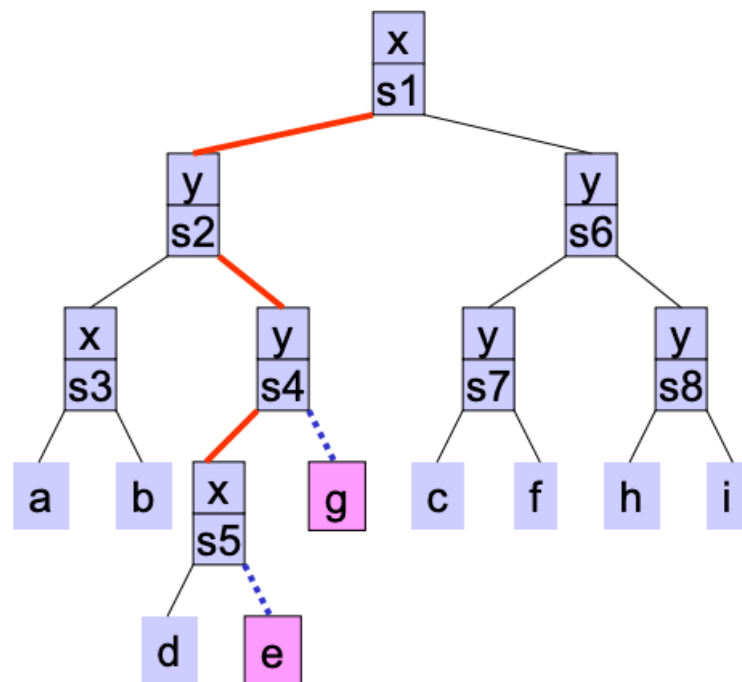
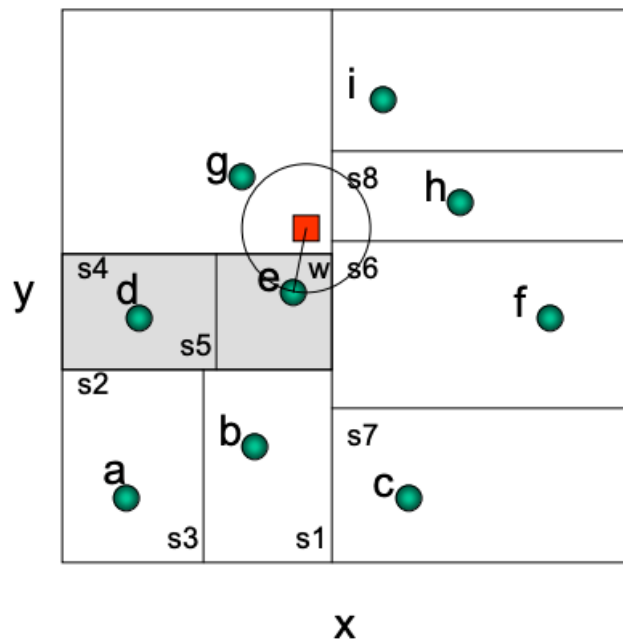






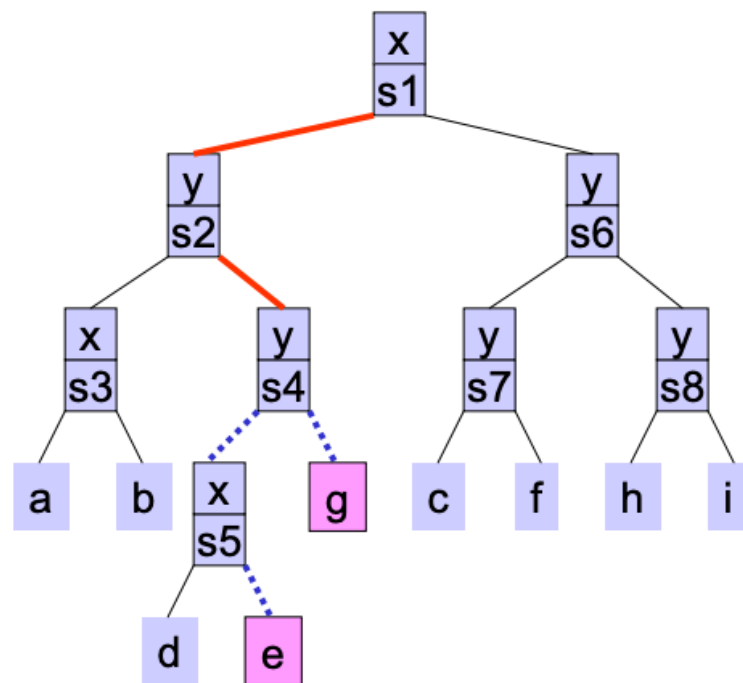
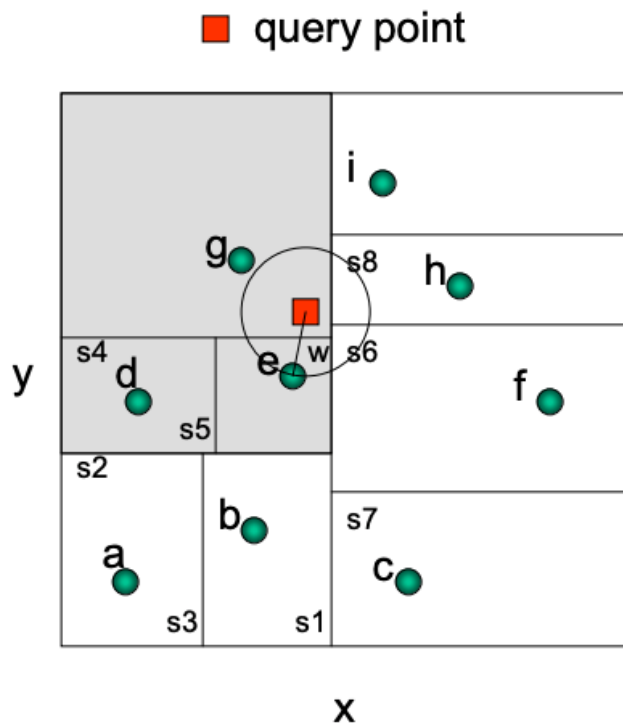
## Kd-tree – kNN Search

■ query point



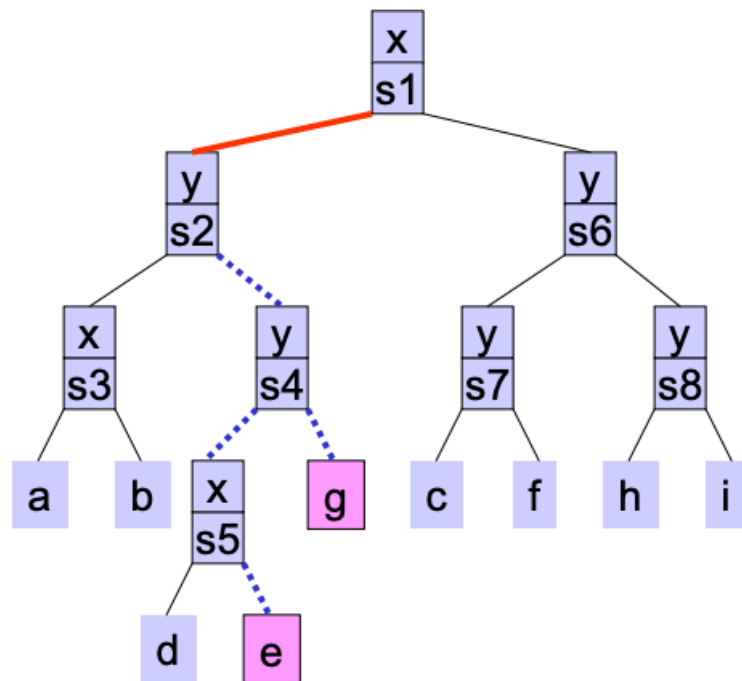
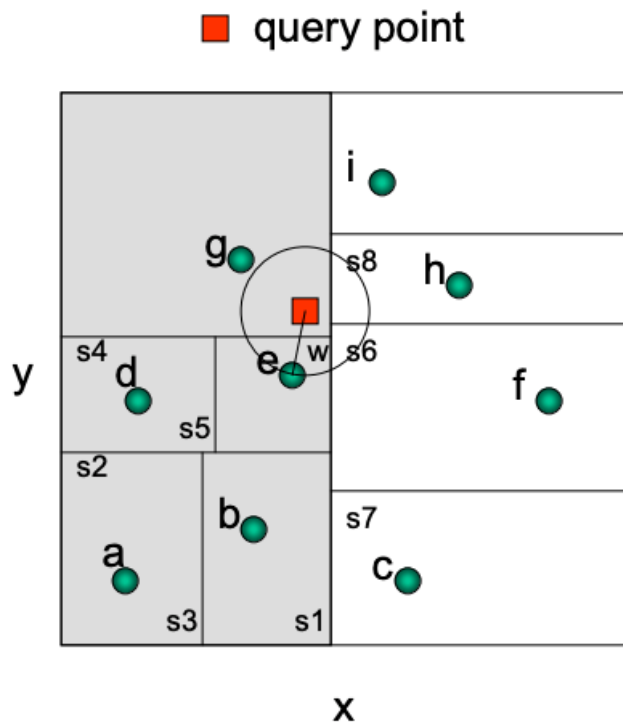


## Kd-tree – kNN Search



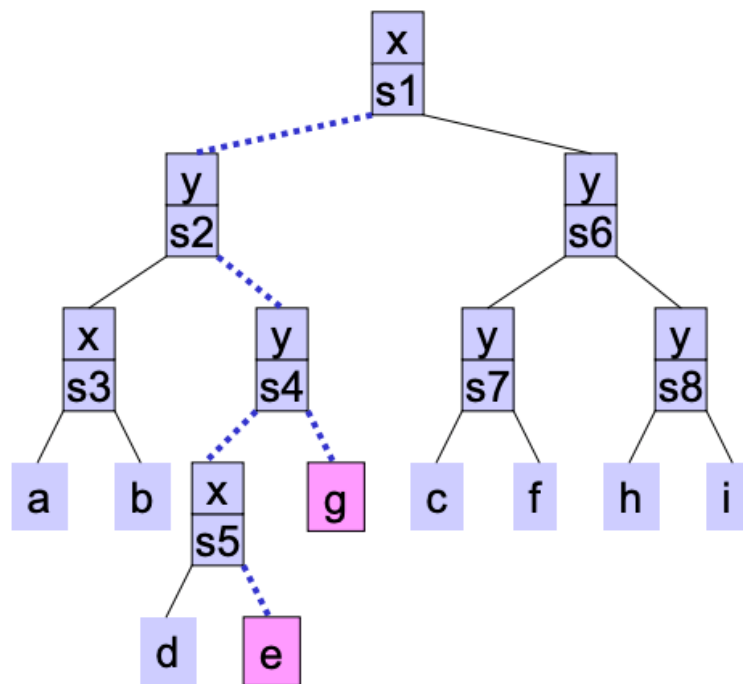
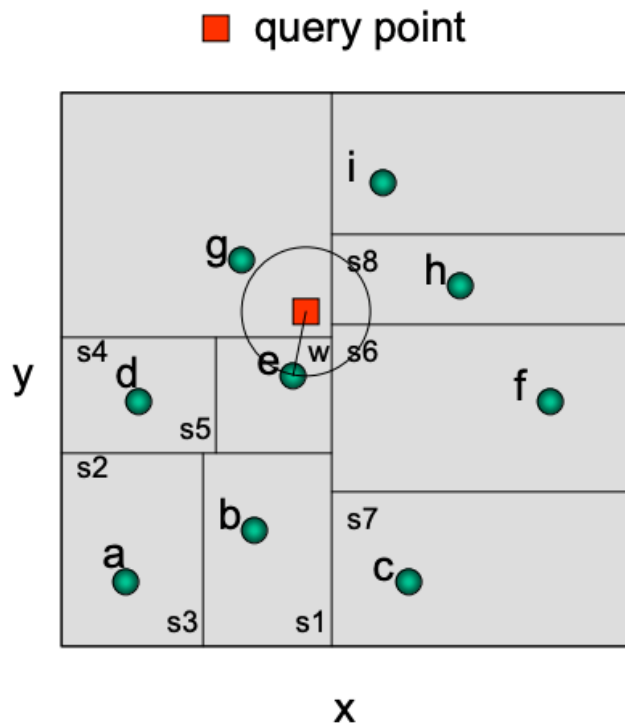


## Kd-tree – kNN Search



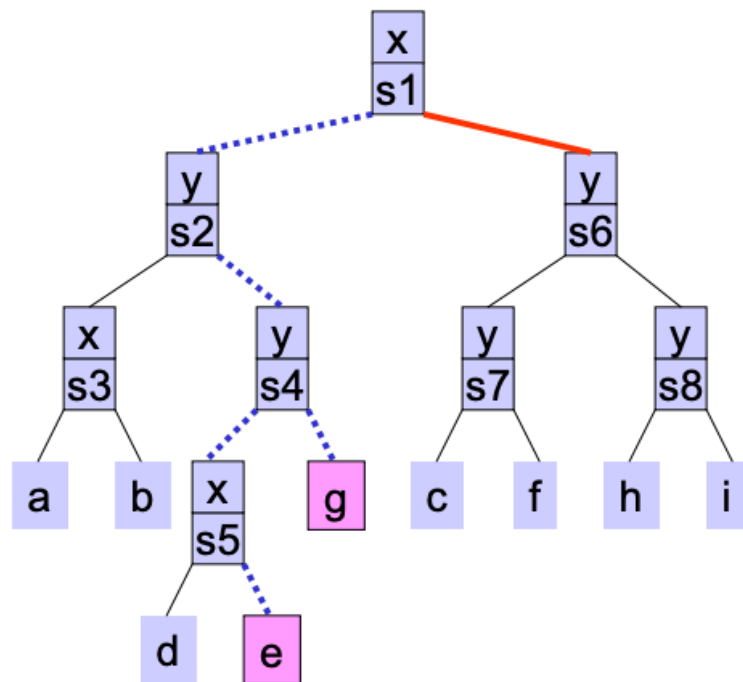
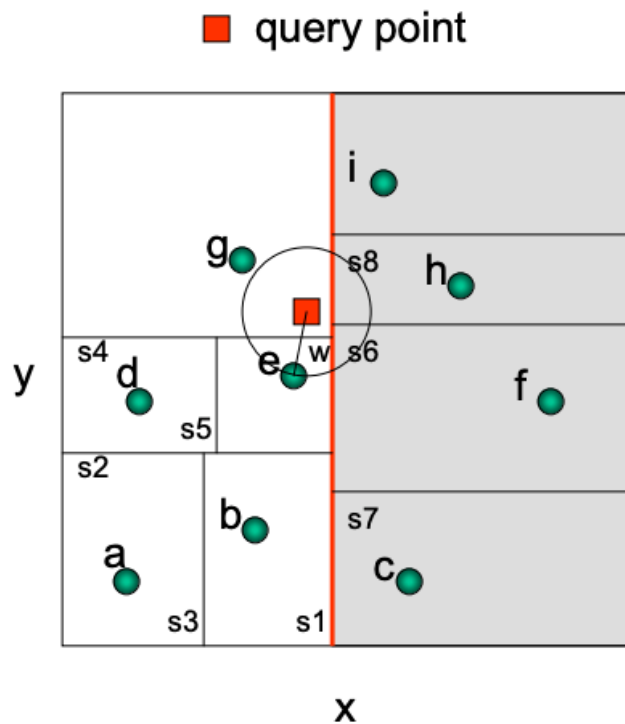


## Kd-tree – kNN Search



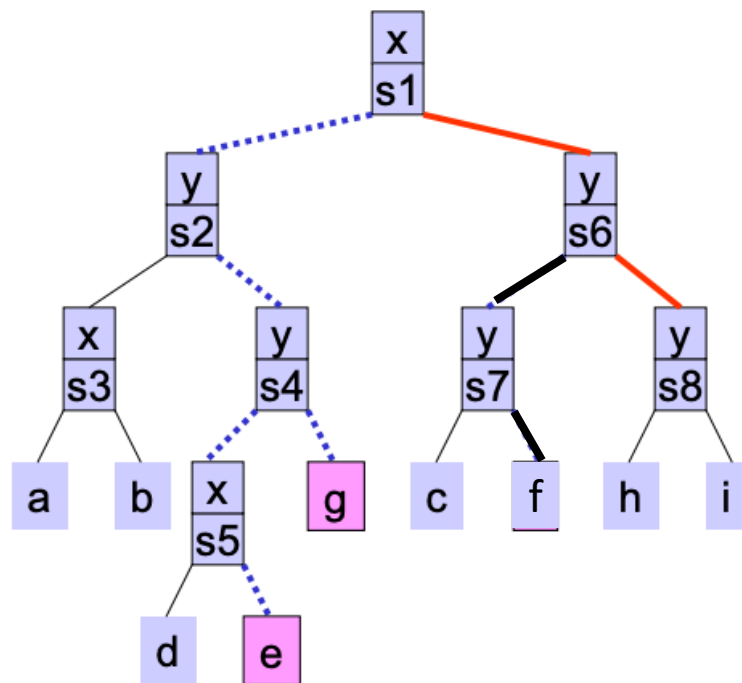
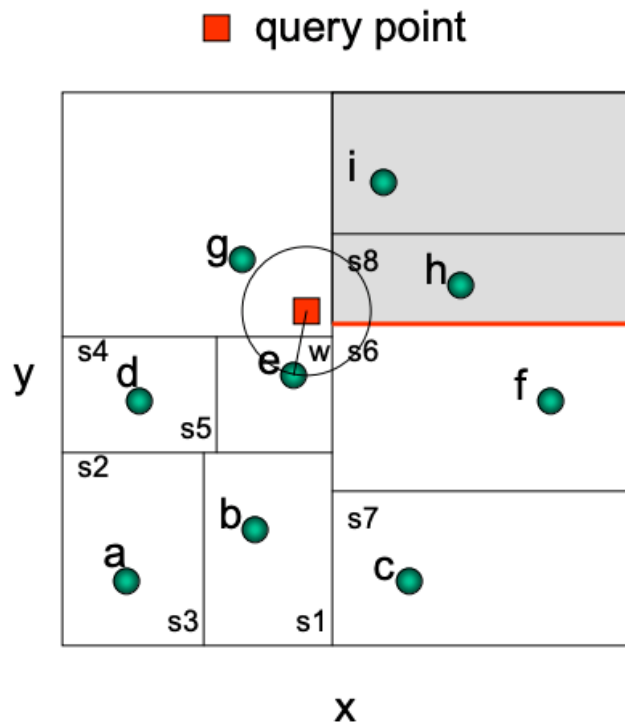


## Kd-tree – kNN Search



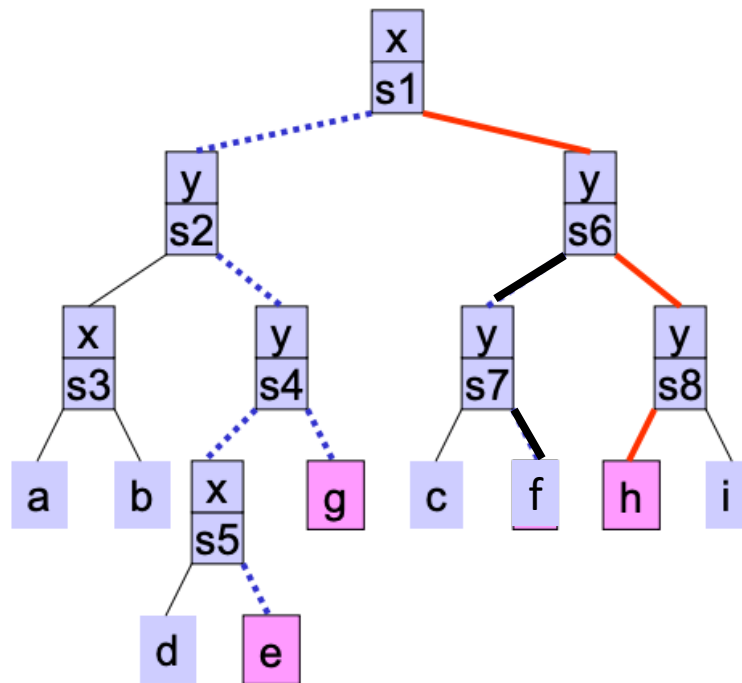
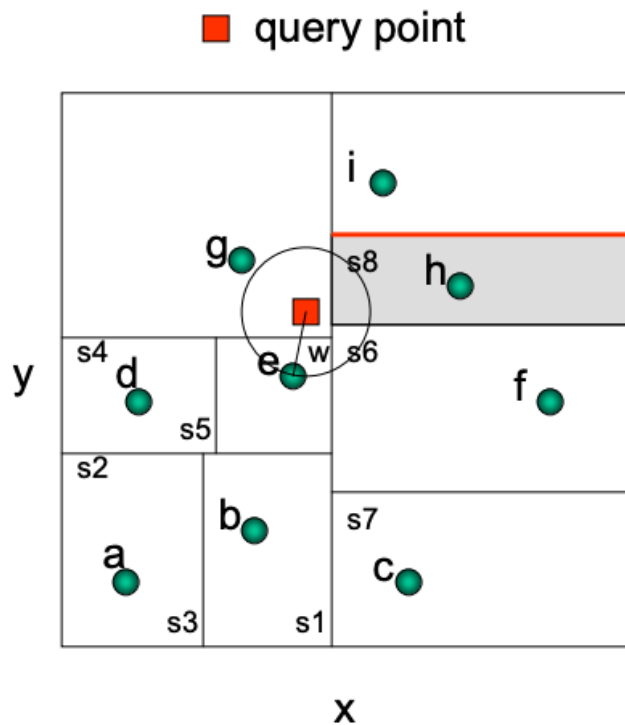


## Kd-tree – kNN Search



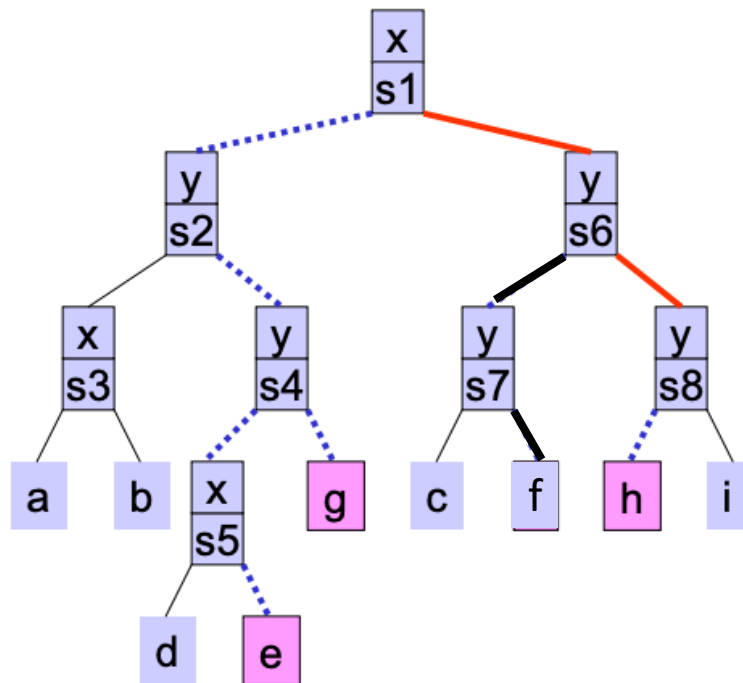
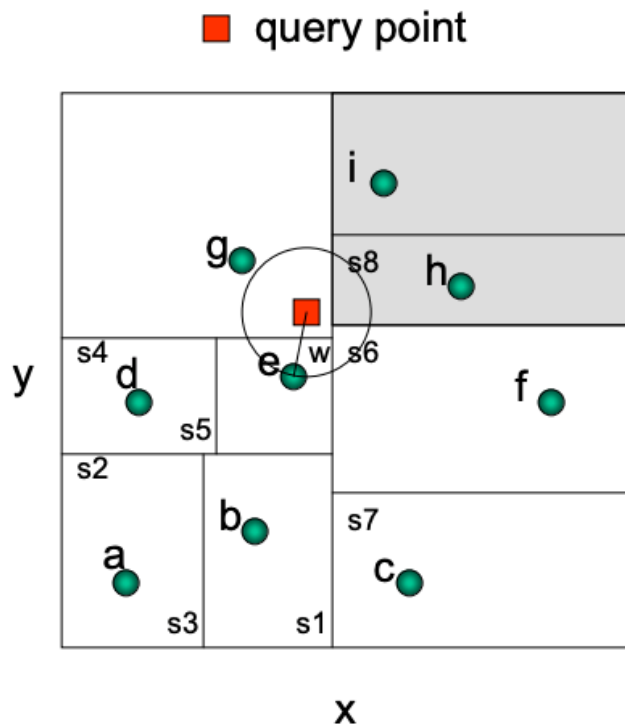


## Kd-tree – kNN Search





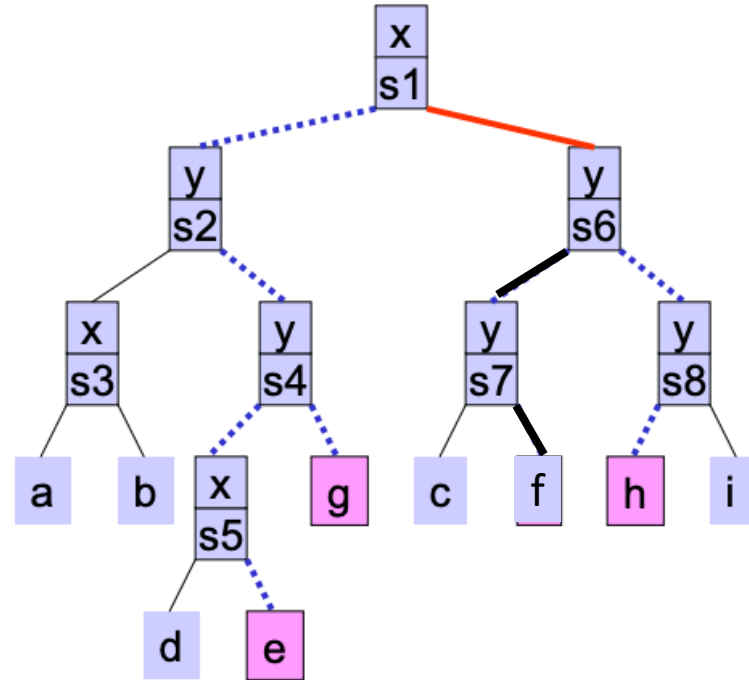
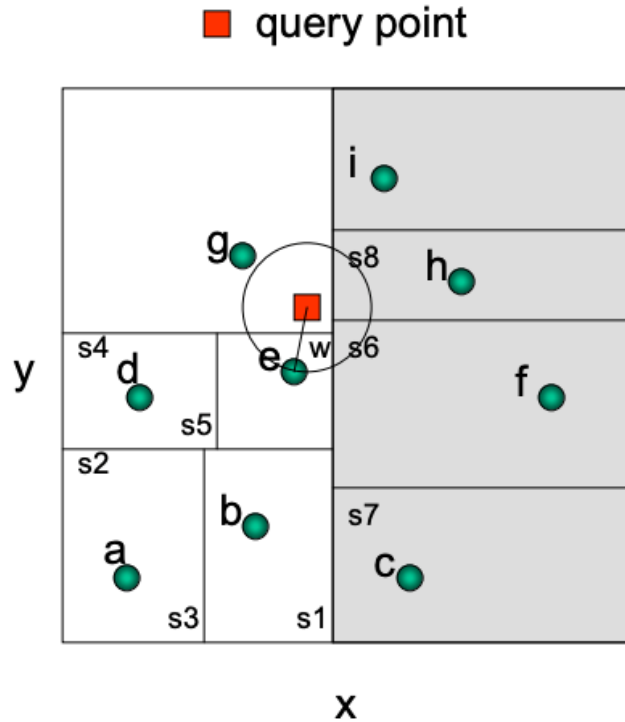
## Kd-tree – kNN Search





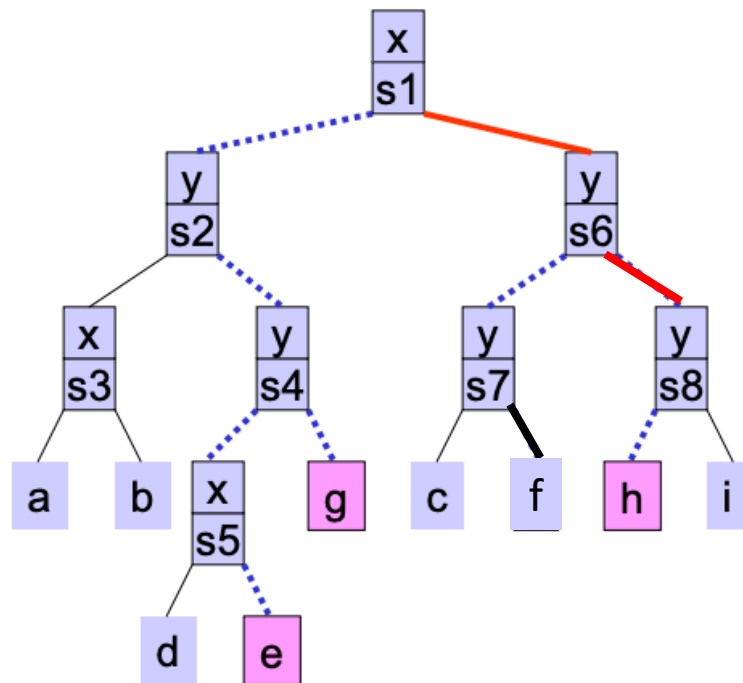
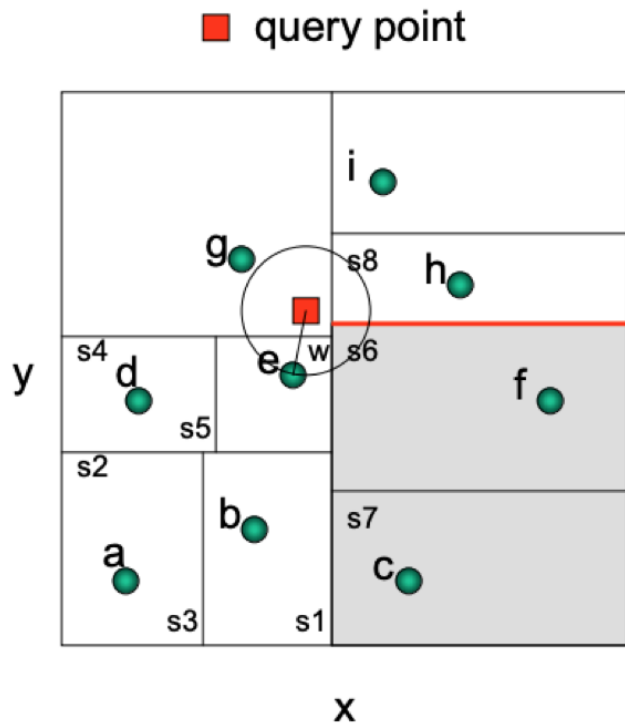


## Kd-tree – kNN Search



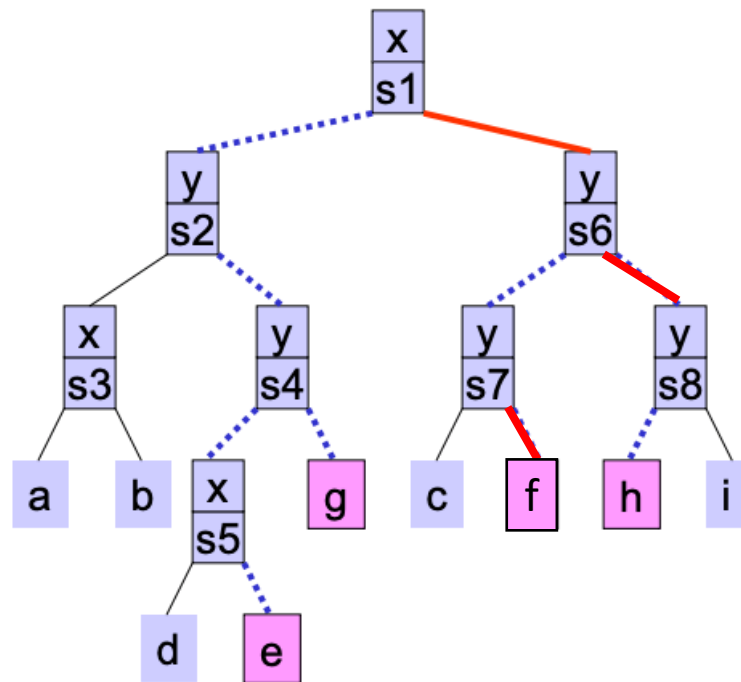
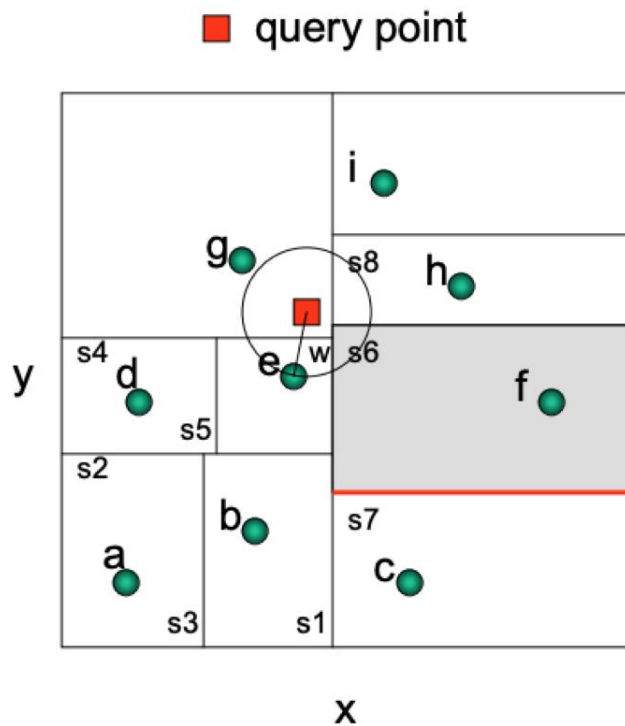


## Kd-tree – kNN Search



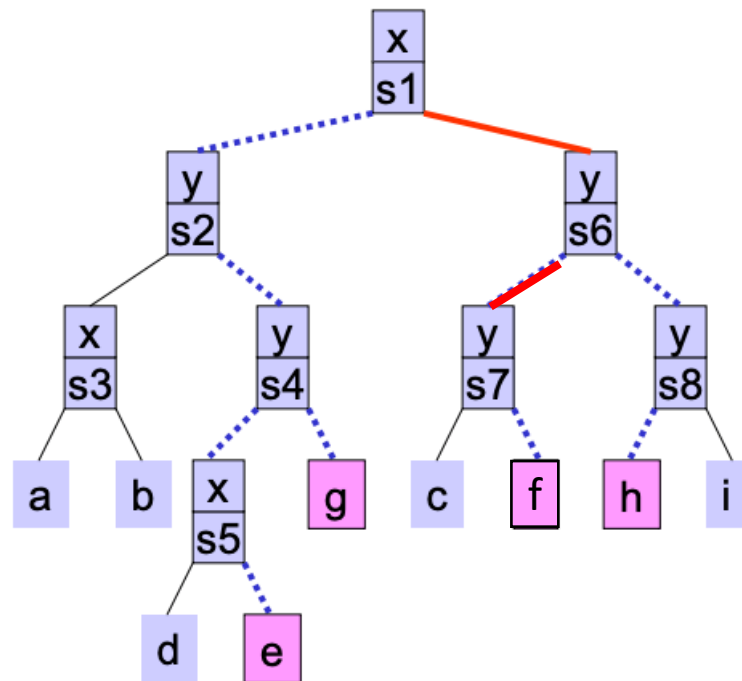
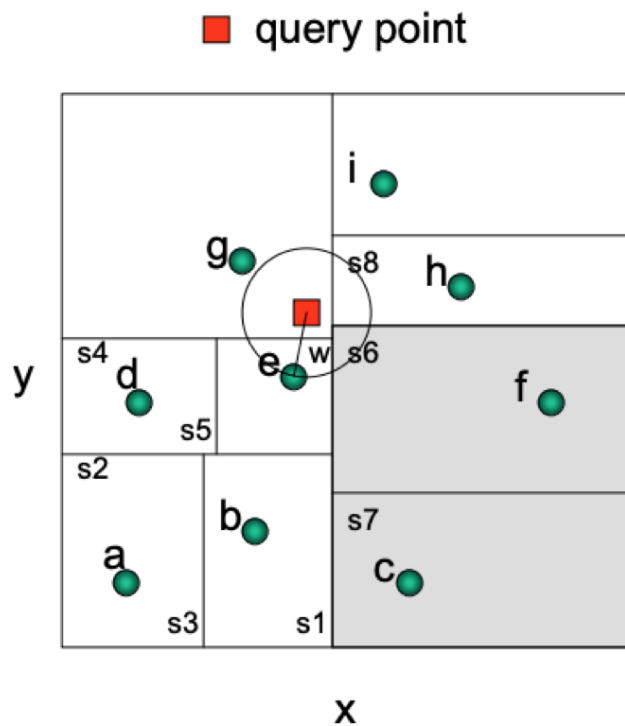


## Kd-tree – kNN Search



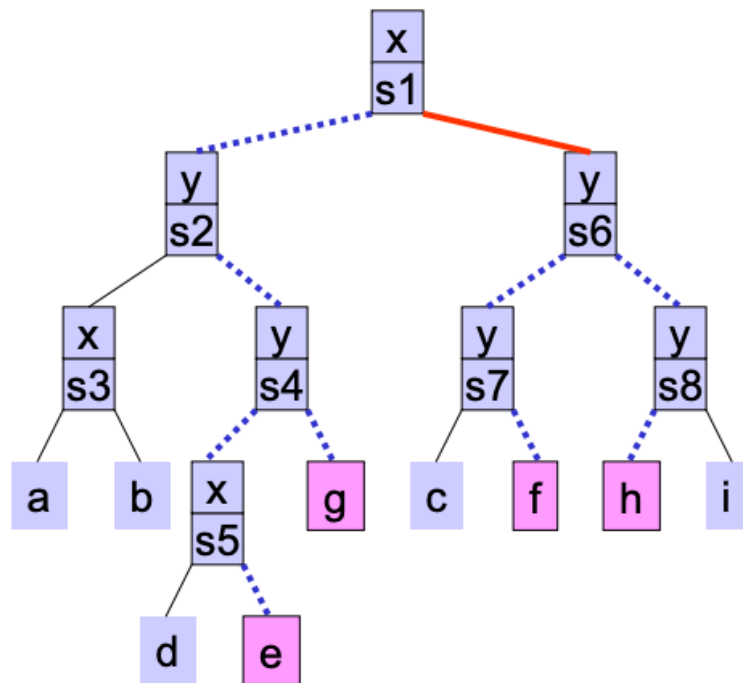
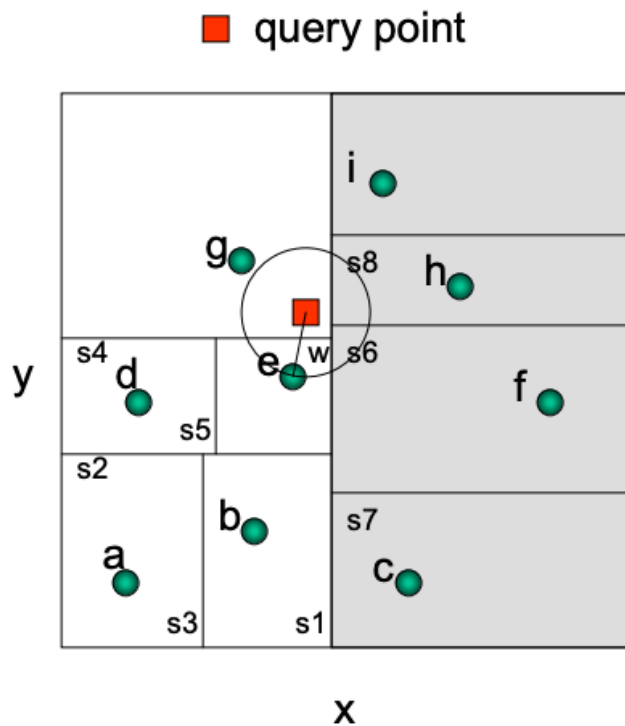


## Kd-tree – kNN Search



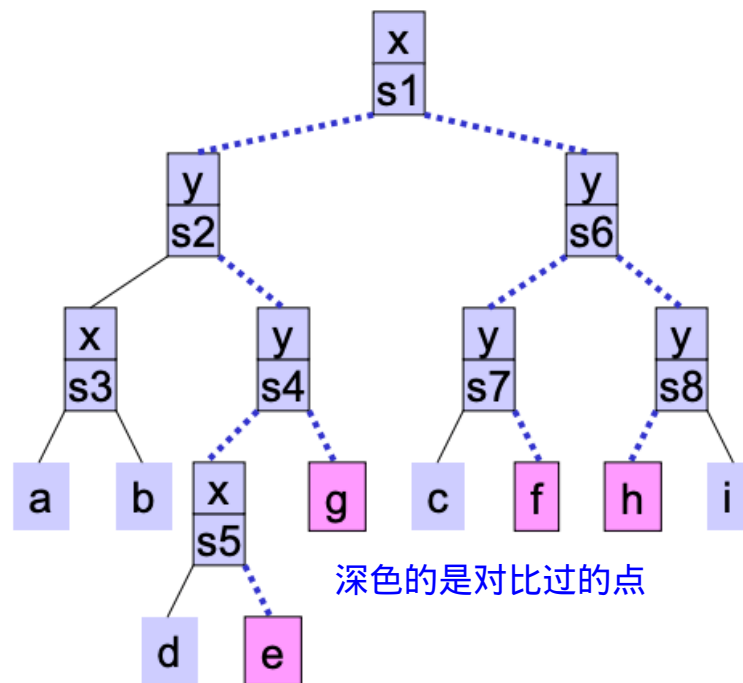
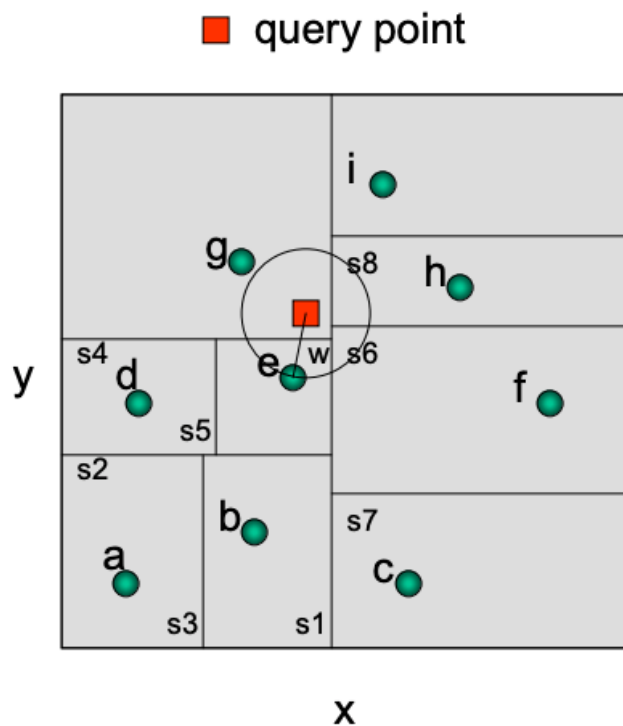


## Kd-tree – kNN Search





## Kd-tree – kNN Search

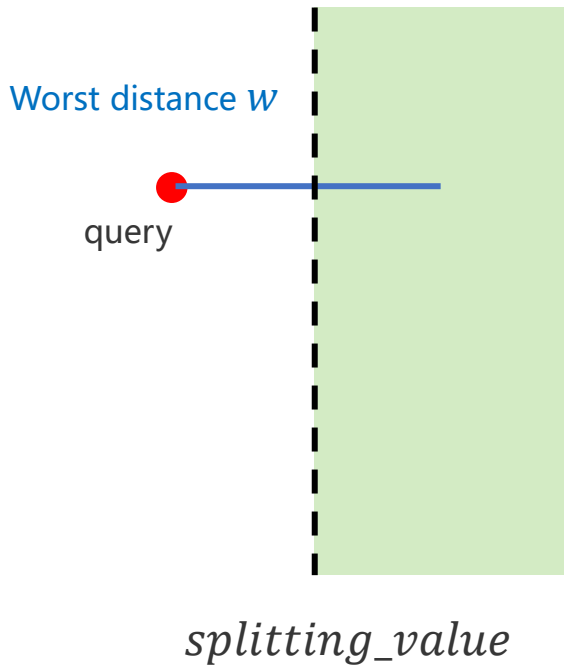




## Kd-tree – kNN Search

核心思想：给定一个点，我是否要查找该区域。

Criteria of a partition intersects with the worst-distance range:



$q[axis]$  inside the partition

OR

$$|q[axis] - splitting\_value| < w$$

```
def knn_search(root: Node, db: np.ndarray, result_set: KNNResultSet, query: np.ndarray):  
    if root is None:  
        return False
```

```
    if root.is_leaf():
```

Compare query to every point inside the leaf, put into the result set

```
        # compare the contents of a leaf  
        leaf_points = db[root.point_indices, :]  
        diff = np.linalg.norm(np.expand_dims(query, 0) - leaf_points, axis=1)  
        for i in range(diff.shape[0]):  
            result_set.add_point(diff[i], root.point_indices[i])  
        return False
```

```
    if query[root.axis] <= root.value:
```

q[axis] inside the partition

```
        knn_search(root.left, db, result_set, query)
```

```
        if math.fabs(query[root.axis] - root.value) < result_set.worstDist():  
            knn_search(root.right, db, result_set, query)
```

```
    else:
```

|q[axis] - splitting value| < w

```
        knn_search(root.right, db, result_set, query)  
        if math.fabs(query[root.axis] - root.value) < result_set.worstDist():  
            knn_search(root.left, db, result_set, query)
```

```
    return False
```





## Kd-tree Radius-NN Search



Exactly the same as kNN search except:

- Use *RadiusNNResultSet*, similar to BST search
- Fixed worst distance, instead of dynamic

```
if query[root.axis] <= root.value:
    radius_search(root.left, db, result_set, query)
    if math.fabs(query[root.axis] - root.value) < result_set.worstDist():
        radius_search(root.right, db, result_set, query)
else:
    radius_search(root.right, db, result_set, query)
    if math.fabs(query[root.axis] - root.value) < result_set.worstDist():
        radius_search(root.left, db, result_set, query)

return False
```



## Kd-tree Search Complexity



1NN search is  $O(\log n)$  for a balanced kd-tree



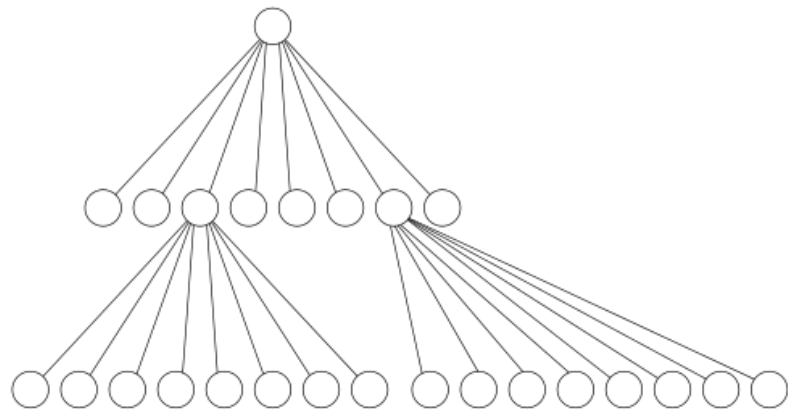
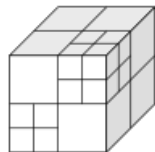
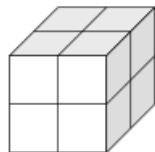
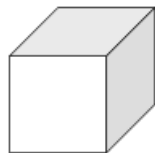
kNN/radiusNN complexity is hard to analyze

- Depends on the distribution of points
- Depends on  $k$  or  $r$
- Varies from  $O(\log n)$  to  $O(n)$



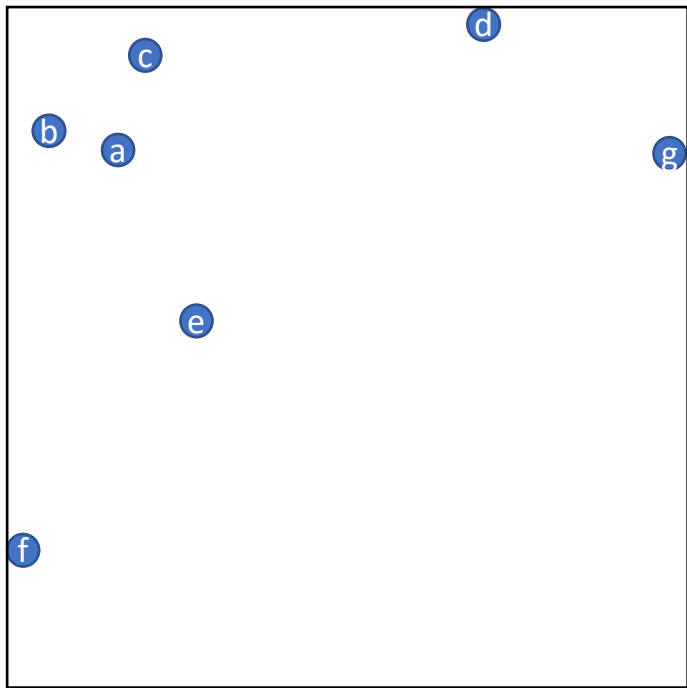
# Octree

- Each node has 8 children
- oct – tree
- Specifically for 3D,  $2^3=8$
- In kd-tree, it is non-trivial to determine whether the NN search is done, so we have to go back to root every time
- Octree is more efficient because we can stop without going back to root





## Octree Construction

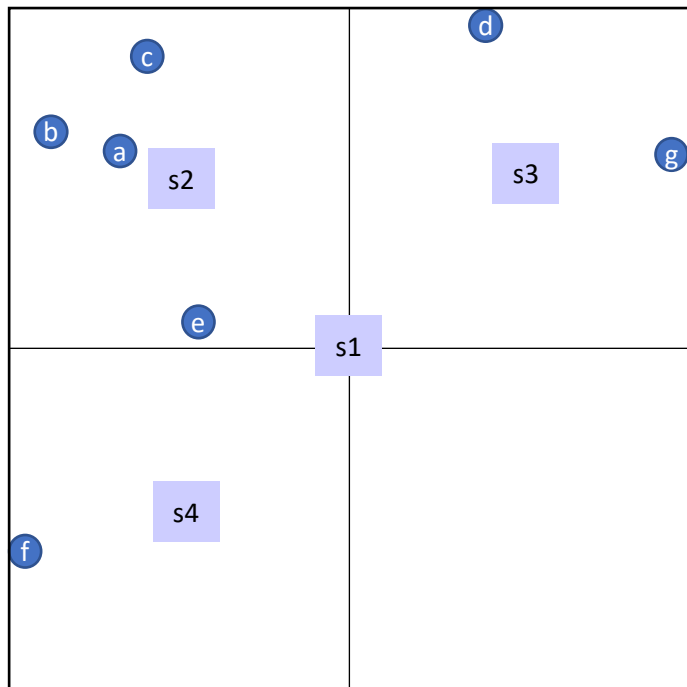


- Determine the extent of the first octant
- Octant is an element in the octree
- Octant is a cube.

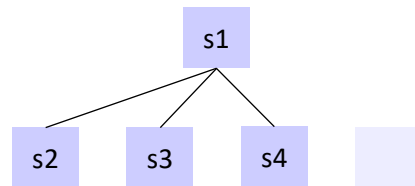
s1



# Octree Construction

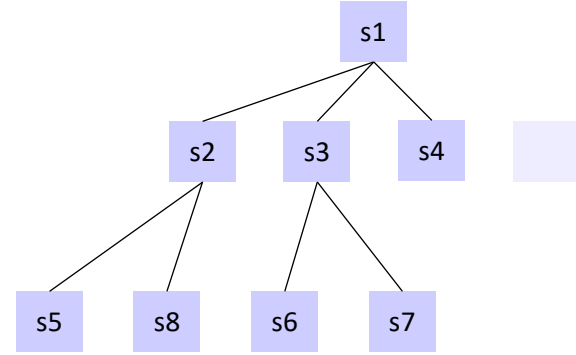
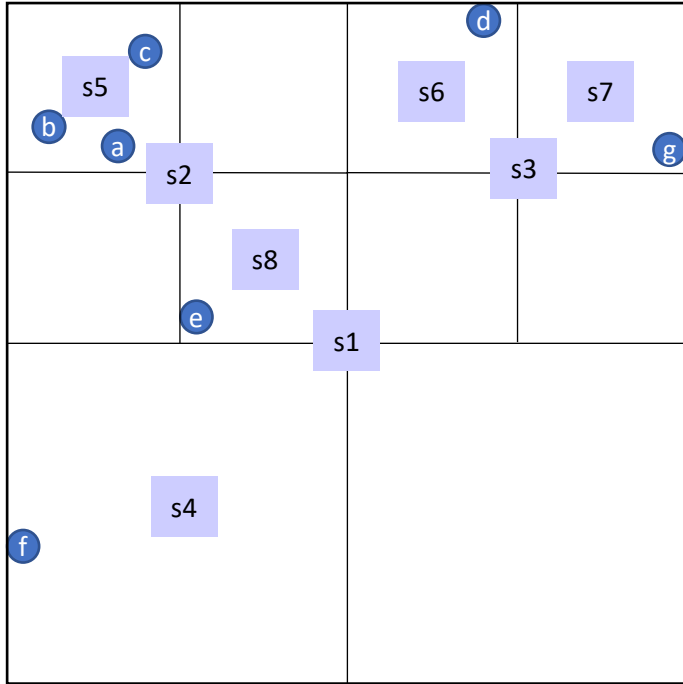


- Determine whether to further split the octant
- **leaf\_size** = 1 here
- **min\_extent** – avoid infinite splitting when there are repeated points



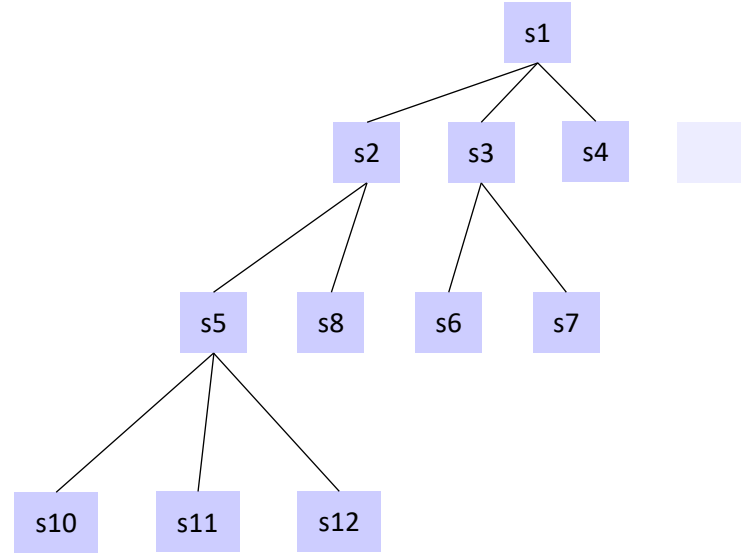
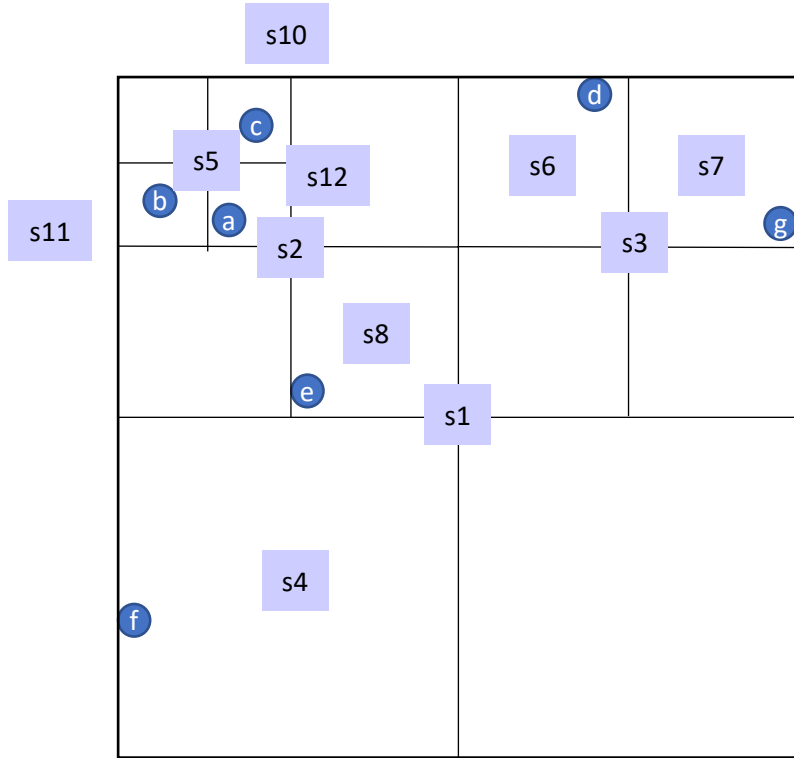


# Octree Construction





# Octree Construction





## Octree Construction

```
class Octant:
    def __init__(self, children, center, extent, point_indices, is_leaf):
        self.children = children
        self.center = center
        self.extent = extent
        self.point_indices = point_indices
        self.is_leaf = is_leaf
```

Array of length 8

Center of the cube

Point inside octant

$0.5 * \text{length}$



```

def octree_recursive_build(root, db, center, extent, point_indices, leaf_size, min_extent):
    if len(point_indices) == 0:
        return None

    if root is None:
        root = Octant([None for i in range(8)], center, extent, point_indices, is_leaf=True)

    # determine whether to split this octant
    if len(point_indices) <= leaf_size or extent <= min_extent:
        root.is_leaf = True
    else:
        root.is_leaf = False
        children_point_indices = [[] for i in range(8)]
        for point_idx in point_indices:
            point_db = db[point_idx]
            morton_code = 0
            if point_db[0] > center[0]:
                morton_code = morton_code | 1
            if point_db[1] > center[1]:
                morton_code = morton_code | 2
            if point_db[2] > center[2]:
                morton_code = morton_code | 4
            children_point_indices[morton_code].append(point_idx)

        # create children
        factor = [-0.5, 0.5]
        for i in range(8):
            child_center_x = center[0] + factor[(i & 1) > 0] * extent
            child_center_y = center[1] + factor[(i & 2) > 0] * extent
            child_center_z = center[2] + factor[(i & 4) > 0] * extent
            child_extent = 0.5 * extent
            child_center = np.asarray([child_center_x, child_center_y, child_center_z])
            root.children[i] = octree_recursive_build(root.children[i],
                                                        db,
                                                        child_center,
                                                        child_extent,
                                                        children_point_indices[i],
                                                        leaf_size,
                                                        min_extent)

    return root

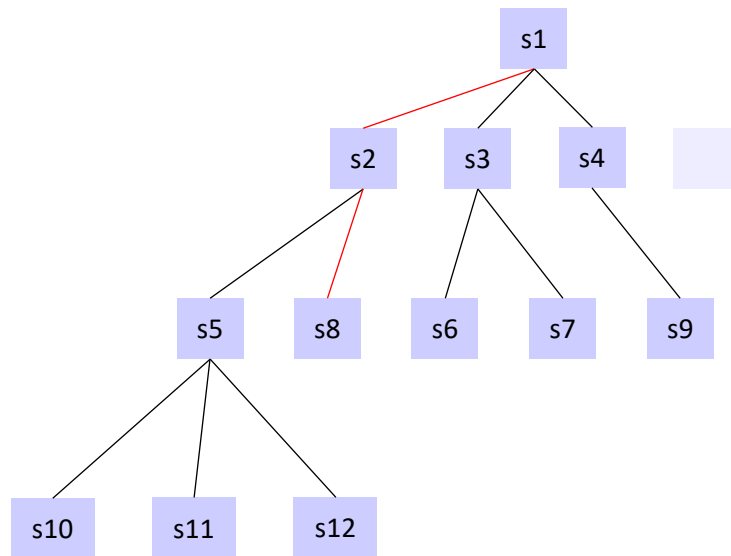
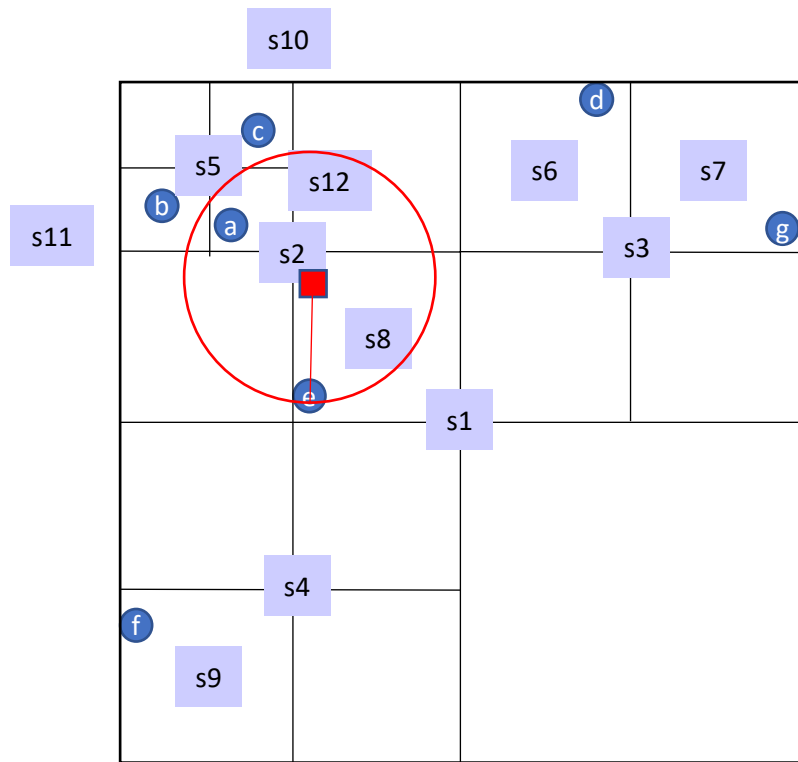
```

Determine which child a point belongs to

Determine child center & extent

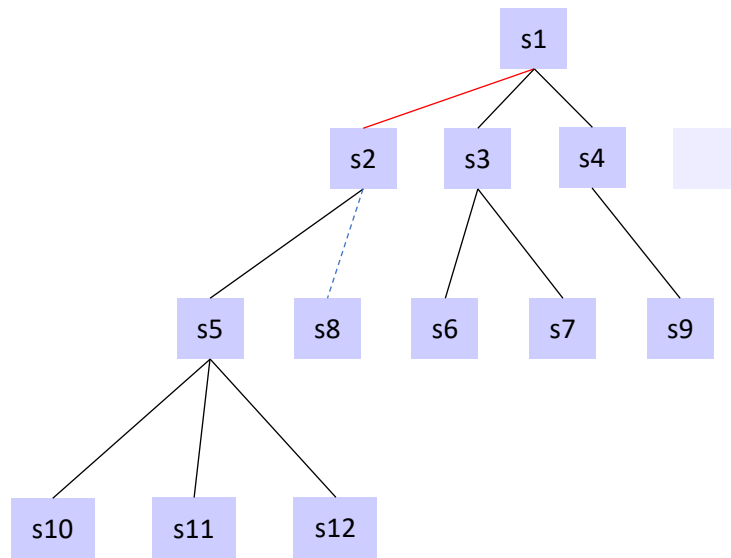
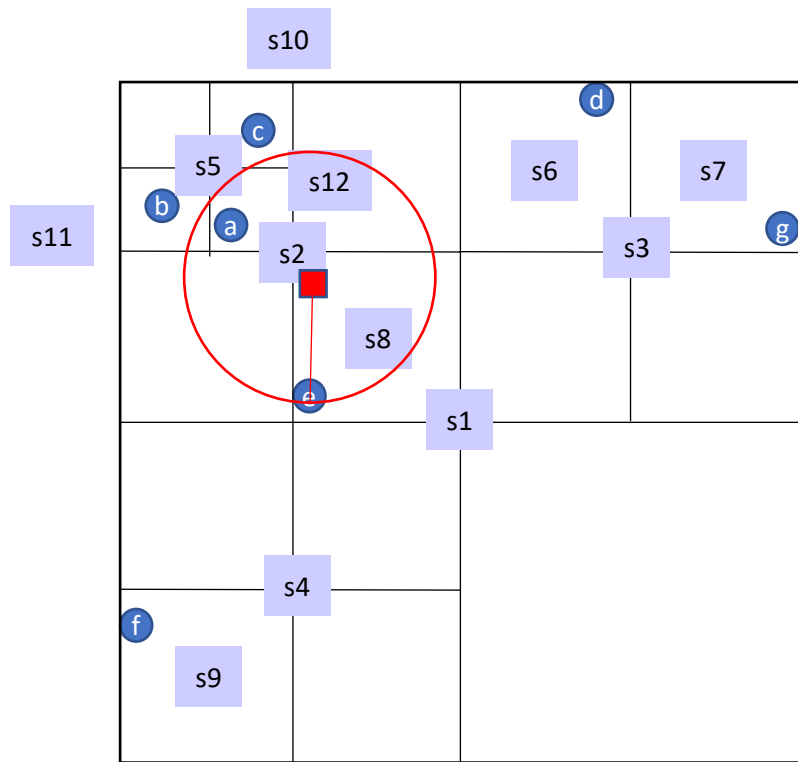


# Octree kNN Search



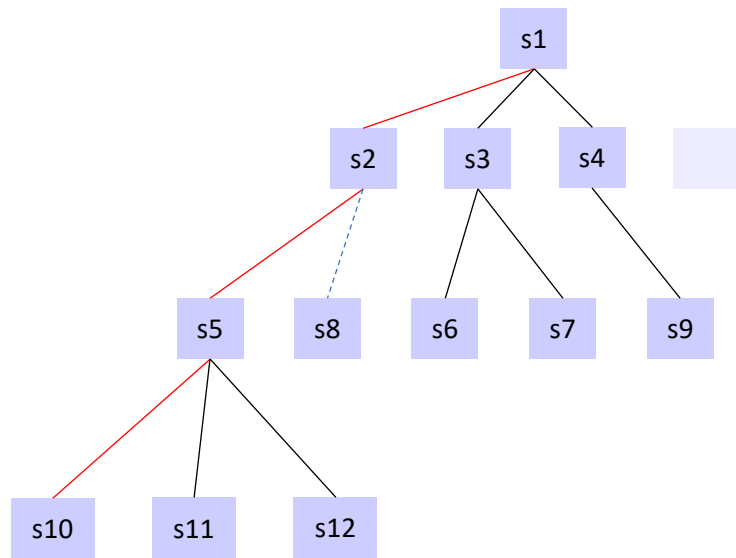
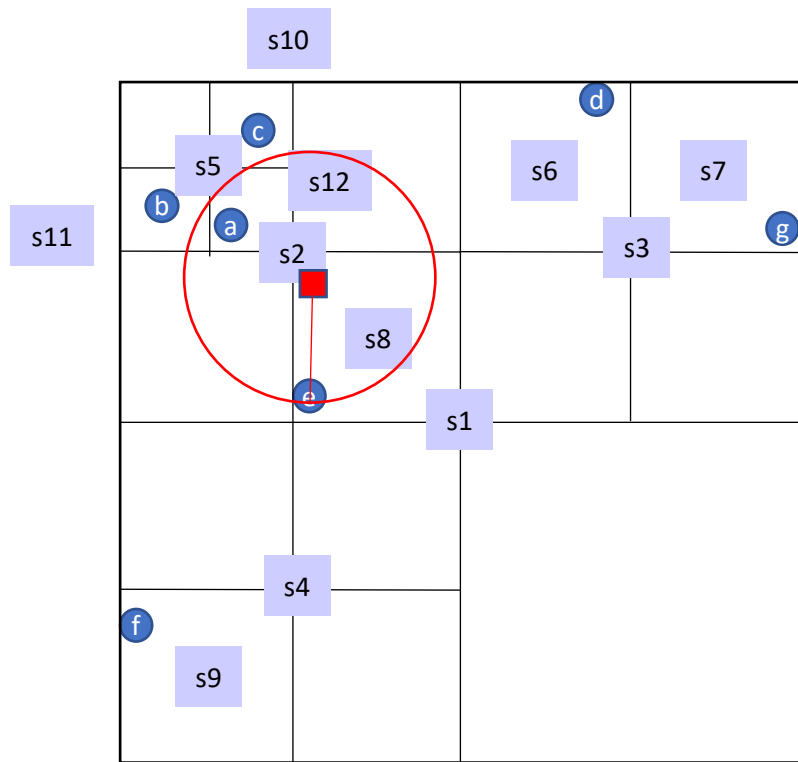


# Octree kNN Search



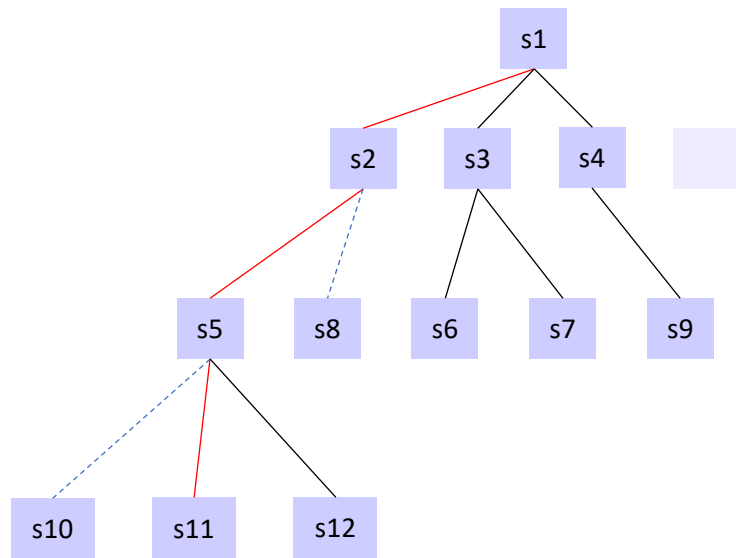
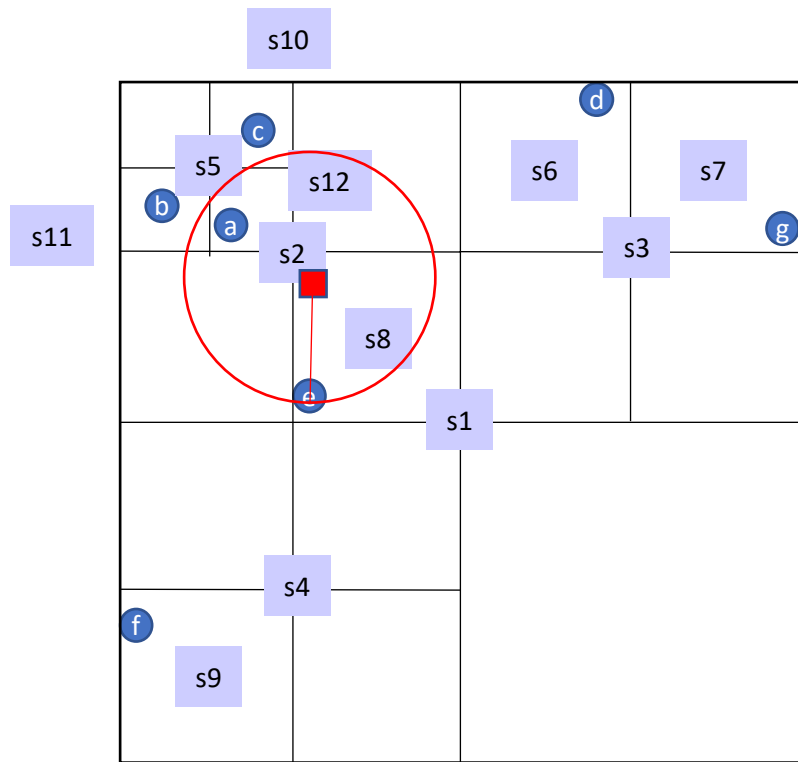


# Octree kNN Search



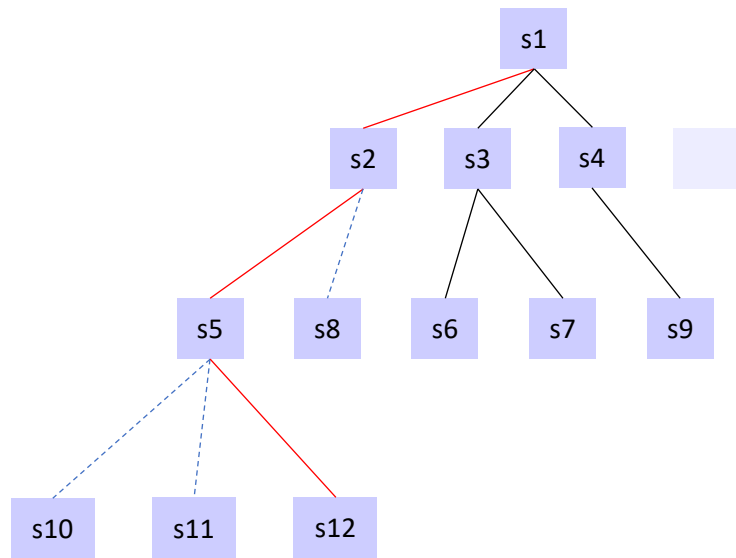
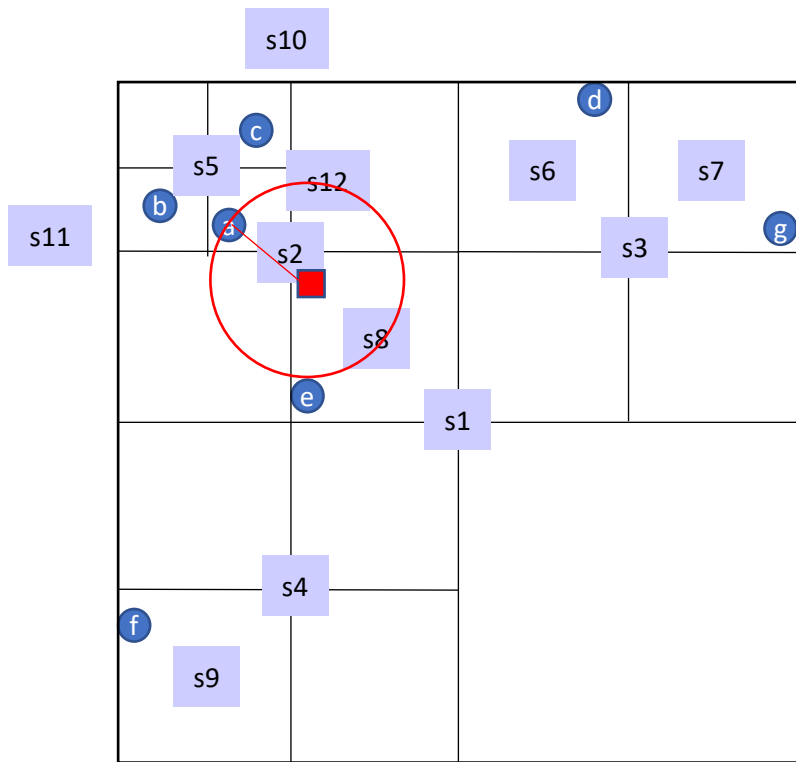


# Octree kNN Search



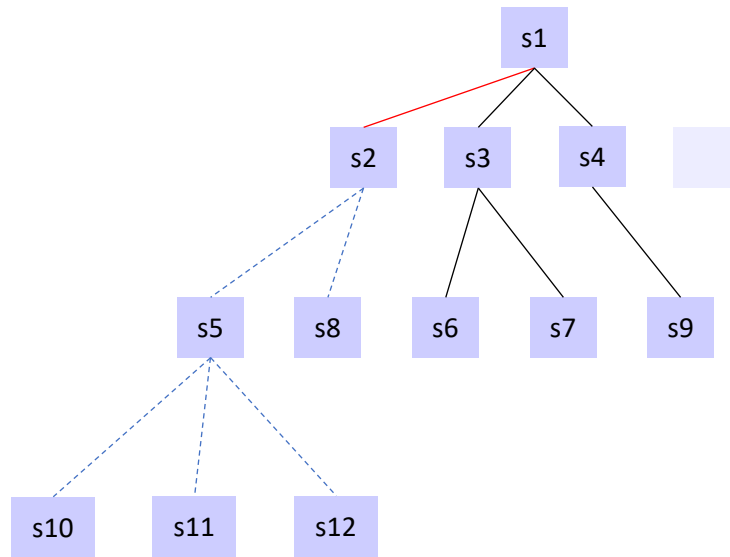
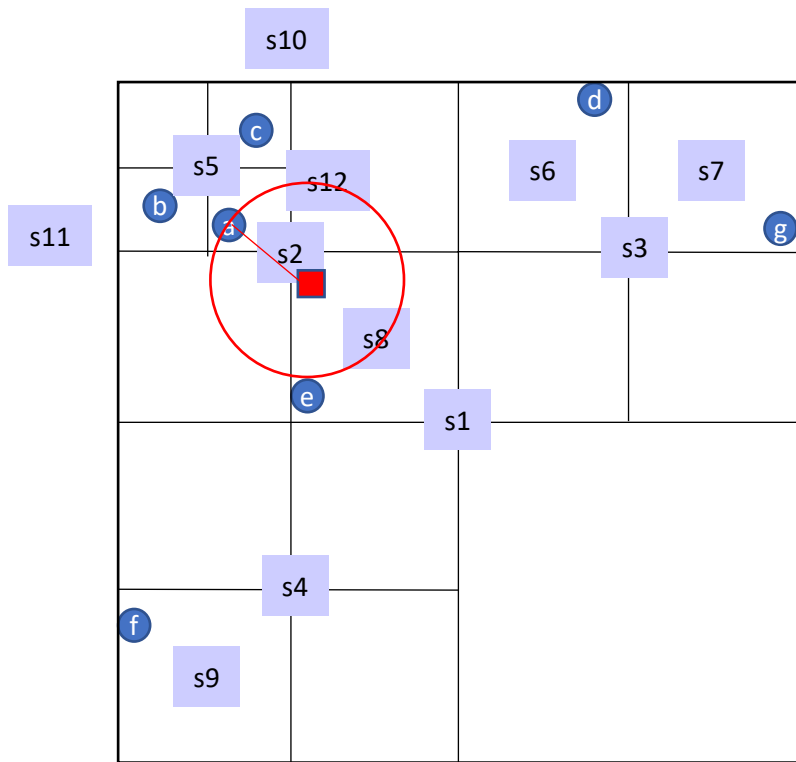


# Octree kNN Search



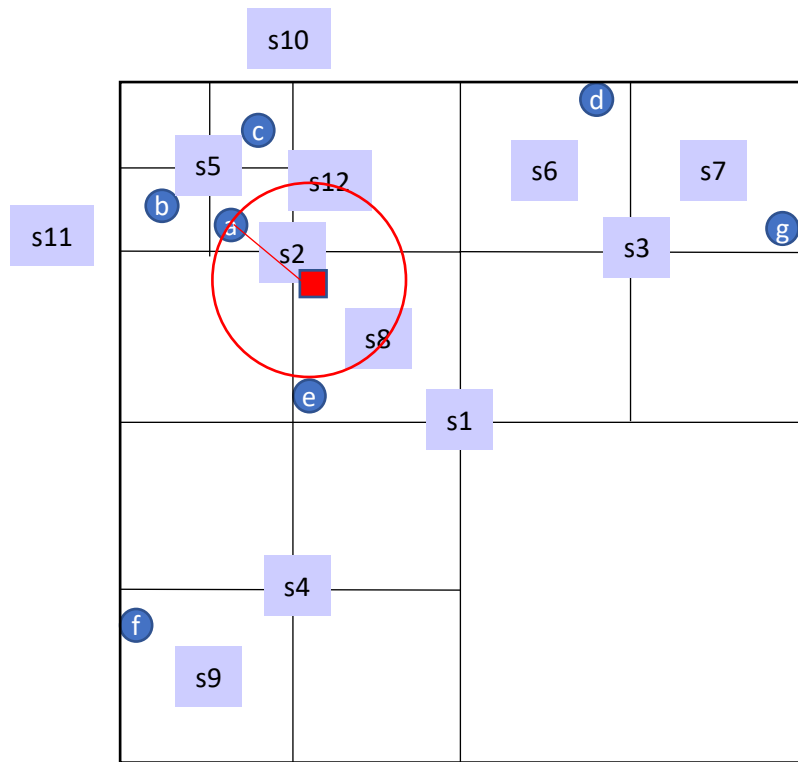


# Octree kNN Search

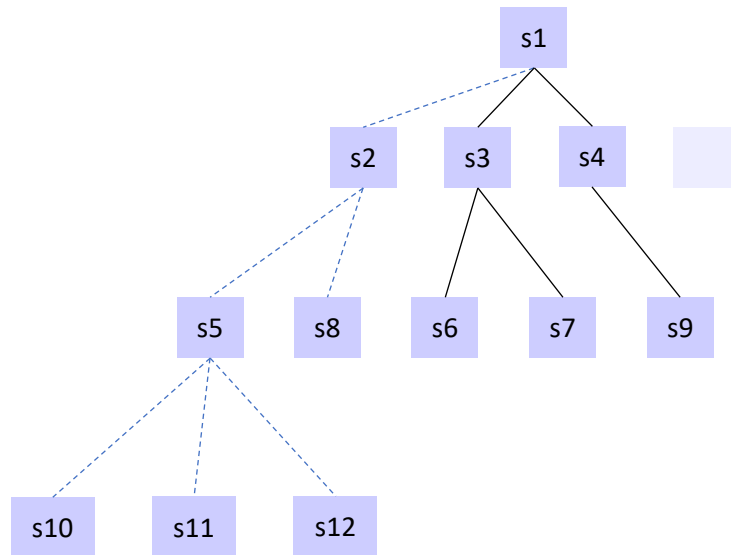




# Octree kNN Search



Query ball inside s2, search end!





```
def octree_knn_search(root: Octant, db: np.ndarray, result_set: KNNResultSet, query: np.ndarray):
```

```
    if root is None:
        return False
```

```
    if root.is_leaf and len(root.point_indices) > 0:
```

```
        # compare the contents of a leaf
        leaf_points = db[root.point_indices, :]
        diff = np.linalg.norm(np.expand_dims(query, 0) - leaf_points, axis=1)
        for i in range(diff.shape[0]):
            result_set.add_point(diff[i], root.point_indices[i])
        # check whether we can stop search now
        return inside(query, result_set.worstDist(), root)
```

Compare all points in a leaf

```
    # go to the relevant child first
```

```
    morton_code = 0
```

```
    if query[0] > root.center[0]:
```

```
        morton_code = morton_code | 1
```

```
    if query[1] > root.center[1]:
```

```
        morton_code = morton_code | 2
```

```
    if query[2] > root.center[2]:
```

```
        morton_code = morton_code | 4
```

Determine & search the most relevant child

```
    if octree_knn_search(root.children[morton_code], db, result_set, query):
        return True
```

```
    # check other children
```

```
    for c, child in enumerate(root.children):
```

```
        if c == morton_code or child is None:
```

```
            continue
```

```
        if False == overlaps(query, result_set.worstDist(), child):
```

```
            continue
```

```
        if octree_knn_search(child, db, result_set, query):
```

```
            return True
```

If an octant is not overlapping with query ball, skip

If query ball is inside an octant, stop

```
    # final check of if we can stop search
```

```
    return inside(query, result_set.worstDist(), root)
```



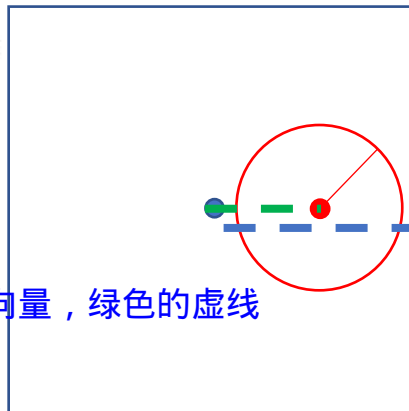
## Function *inside*

```
def inside(query: np.ndarray, radius: float, octant: Octant):  
    """  
    Determines if the query ball is inside the octant  
    :param query:  
    :param radius:  
    :param octant:  
    :return:  
    """  
    query_offset = query - octant.center  
    query_offset_abs = np.fabs(query_offset)  
    possible_space = query_offset_abs + radius  
    return np.all(possible_space < octant.extent)
```

Green dash line

Blue dash line

Red line



判断蓝色的线比  
红色加绿色的线  
长，那么这个圆  
就在正方体内

蓝色点指向红色点的向量，绿色的虚线



## Function *overlaps*

```
def overlaps(query: np.ndarray, radius: float, octant: Octant):
```

```
    """
```

```
    Determines if the query ball overlaps with the octant
```

```
    :param query:
```

```
    :param radius:
```

```
    :param octant:
```

```
    :return:
```

```
    """
```

```
    query_offset = query - octant.center
```

```
    query_offset_abs = np.fabs(query_offset)
```

```
    # completely outside, since query is outside the relevant area
```

```
    max_dist = radius + octant.extent
```

```
    if np.any(query_offset_abs > max_dist):
```

```
        return False
```

Case 1

```
    # if pass the above check, consider the case that the ball is contacting the face of the octant
```

```
    if np.sum((query_offset_abs < octant.extent).astype(np.int)) >= 2:
```

```
        return True
```

Case 2

```
    # consider the case that the ball is contacting the edge or corner of the octant
```

```
    # since the case of the ball center (query) inside octant has been considered,
```

```
    # we only consider the ball center (query) outside octant
```

```
    x_diff = max(query_offset_abs[0] - octant.extent, 0)
```

```
    y_diff = max(query_offset_abs[1] - octant.extent, 0)
```

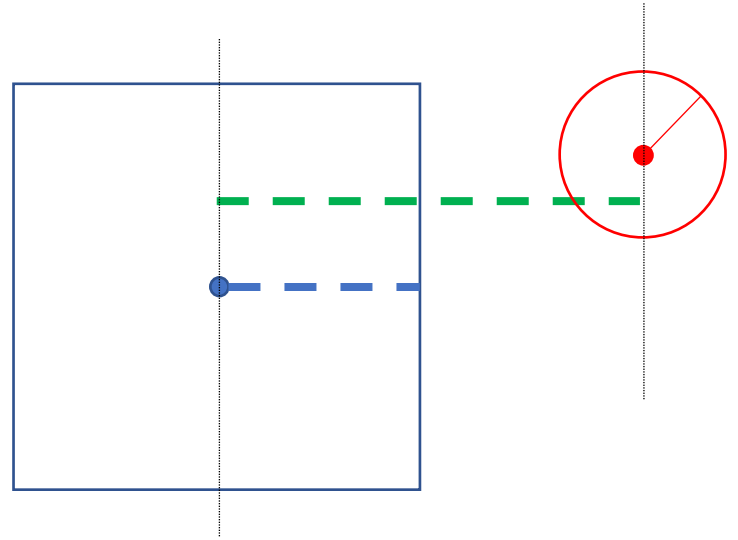
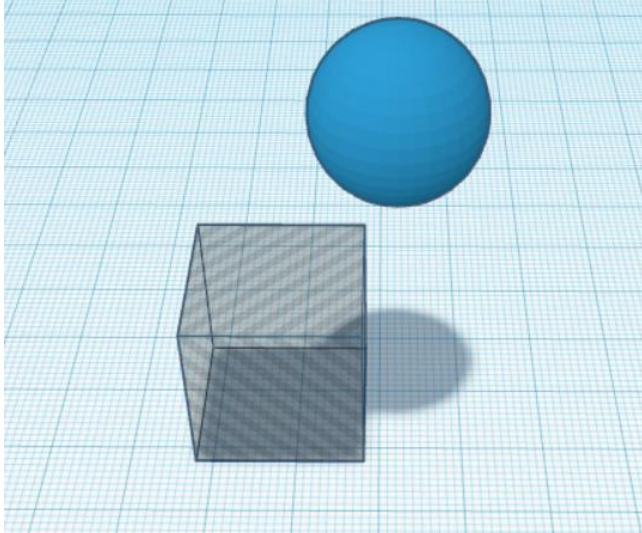
```
    z_diff = max(query_offset_abs[2] - octant.extent, 0)
```

Case 3

```
    return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```



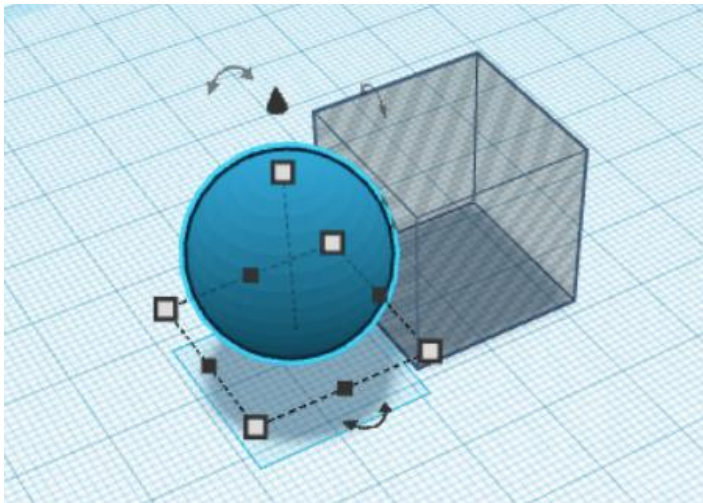
## Function *overlaps* – Case 1



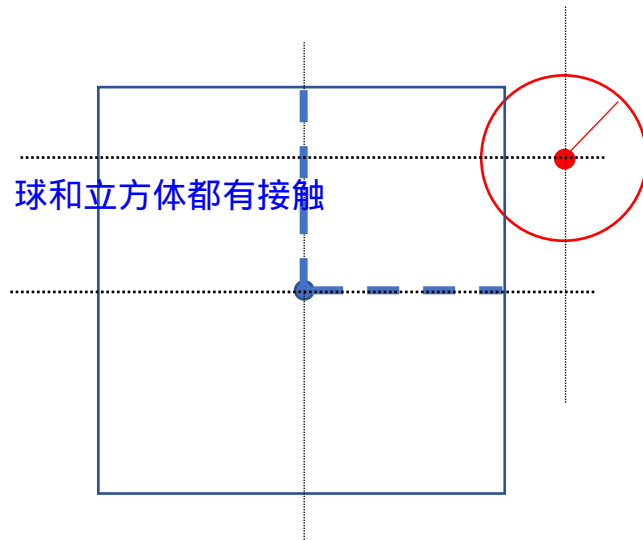
```
# completely outside, since query is outside the relevant area  
max_dist = radius + octant.extent  
if np.any(query_offset_abs > max_dist):  
    return False
```



## Function *overlaps* – Case 2



在x和y方向上，球和立方体都有接触

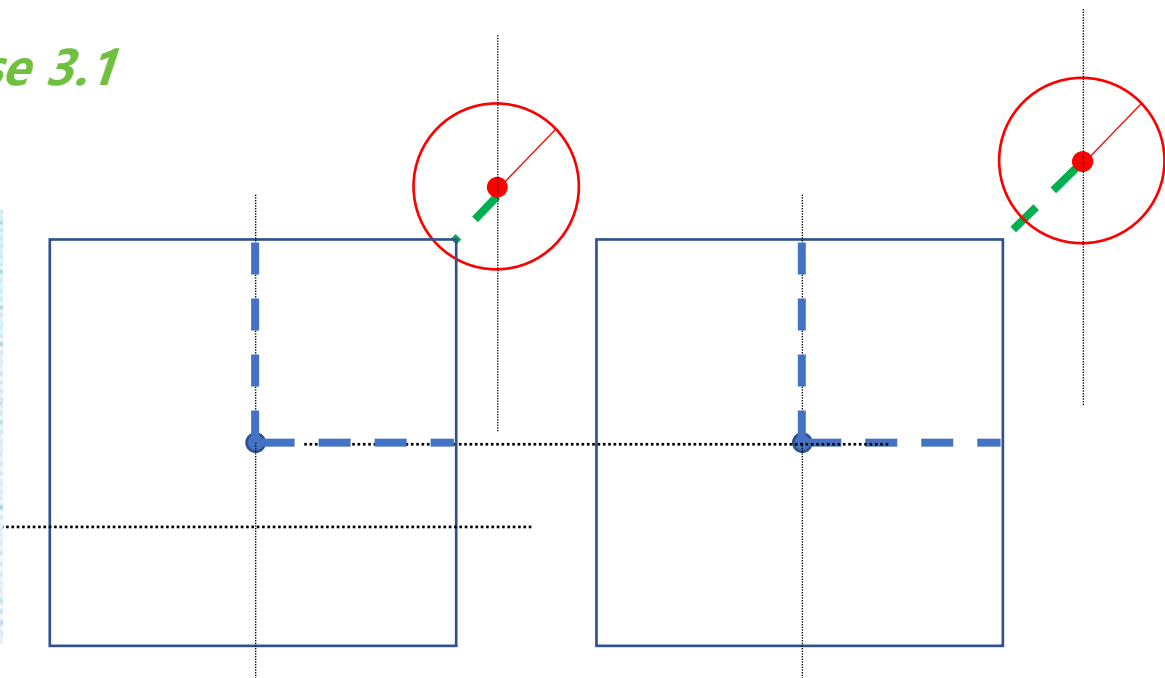
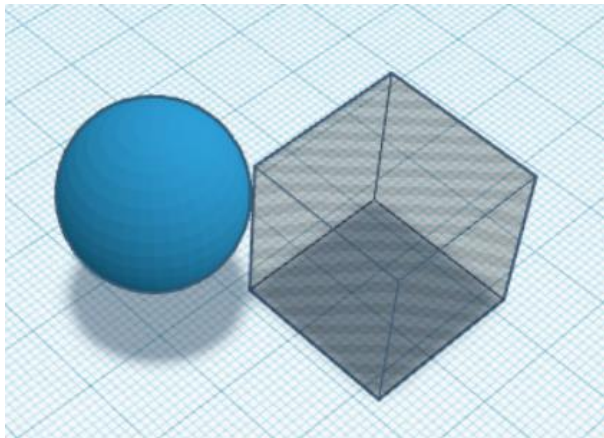


Check if the ball is contacting the face of the octant

```
if np.sum((query_offset_abs < octant.extent).astype(np.int)) >= 2:  
    return True
```



## Function *overlaps* – Case 3.1



```
# consider the case that the ball is contacting the edge or corner of the octant
# since the case of the ball center (query) inside octant has been considered,
# we only consider the ball center (query) outside octant
x_diff = max(query_offset_abs[0] - octant.extent, 0)
y_diff = max(query_offset_abs[1] - octant.extent, 0)
z_diff = max(query_offset_abs[2] - octant.extent, 0)

return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```

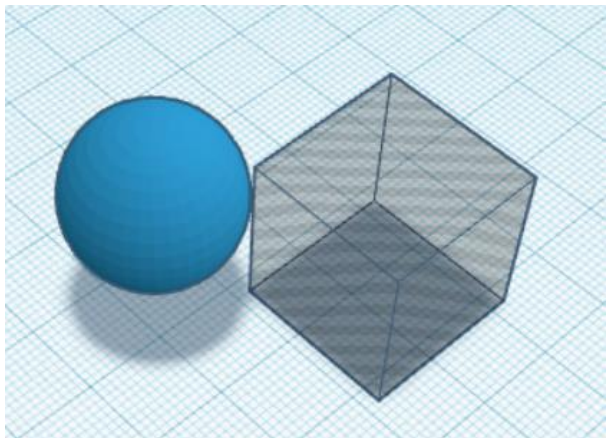




## Function *overlaps* – Case 3.2

In 3D, there is the case that the cube's **edge** cut into the query ball

```
# consider the case that the ball is contacting the edge or corner of the octant  
# since the case of the ball center (query) inside octant has been considered,  
# we only consider the ball center (query) outside octant  
x_diff = max(query_offset_abs[0] - octant.extent, 0)  
y_diff = max(query_offset_abs[1] - octant.extent, 0)  
z_diff = max(query_offset_abs[2] - octant.extent, 0)  
  
return x_diff * x_diff + y_diff * y_diff + z_diff * z_diff < radius * radius
```



That's why there is a “*max*” to reduce this case into 3.1



## Octree Radius NN Search



Simple one: replace KNNResultSet with RadiusNNResult Set



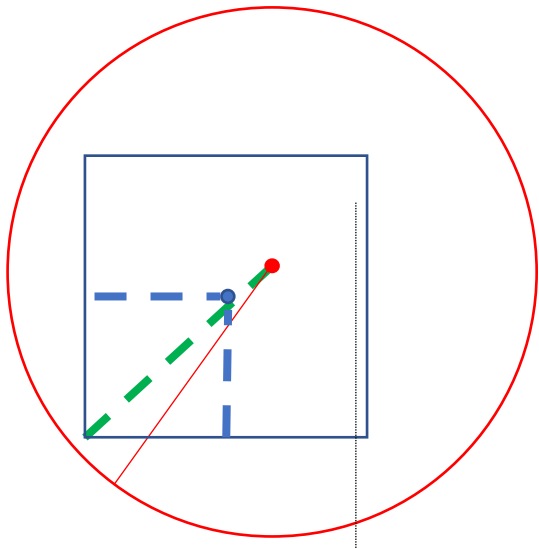
Better one:

- If the query ball *contains* the octant, just compare the query with all point
- No need to go into children of that octant





## Function *contains*



```
def contains(query: np.ndarray, radius: float, octant: Octant):  
    """  
    Determine if the query ball contains the octant  
    :param query:  
    :param radius:  
    :param octant:  
    :return:  
    """  
  
    query_offset = query - octant.center  
    query_offset_abs = np.fabs(query_offset)  
  
    query_offset_to_farthest_corner = query_offset_abs + octant.extent  
    return np.linalg.norm(query_offset_to_farthest_corner) < radius
```

Green dash line

Red line



## Octree Search Complexity



1NN search is  $O(\log n)$



kNN/radiusNN complexity is hard to analyze

- Depends on the distribution of points
- Depends on  $k$  or  $r$
- Varies from  $O(\log n)$  to  $O(n)$

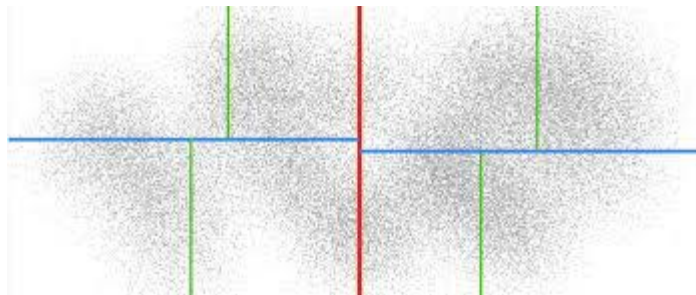
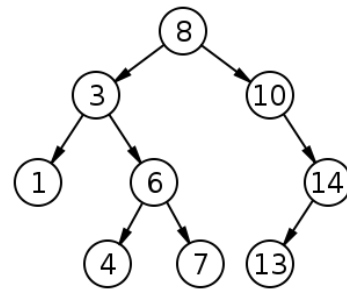


# BST / Kd-tree / Octree



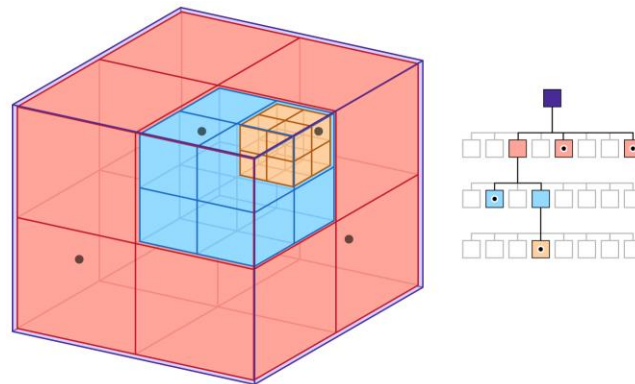
## Dimension

- BST for one dimension
- Kd-tree works for any dimension
- Octree is optimized for 3D



## Idea

- Same – space partition





## Summary



Space partition



Find a method to skip some partitions



Pythons codes:

<https://github.com/lijx10/NN-Trees>



## Homework

- We provide one  $N \times 3$  point cloud
- 8-NN search for each point to the point cloud
- Implement 3 NN algorithms
  1. Numpy brute-force search
  2. `scipy.spatial.KDTree`
  3. Your own kd-tree/octree in python or C++
- Report timing using method 1 as baseline
- This is a competition!
  - Timing of method 3 determine your grade

**感谢聆听 !**

**Thanks for Listening**