

Floating Point

15-213/18-213/15-513: Introduction to Computer Systems
4th Lecture, May 22, 2020

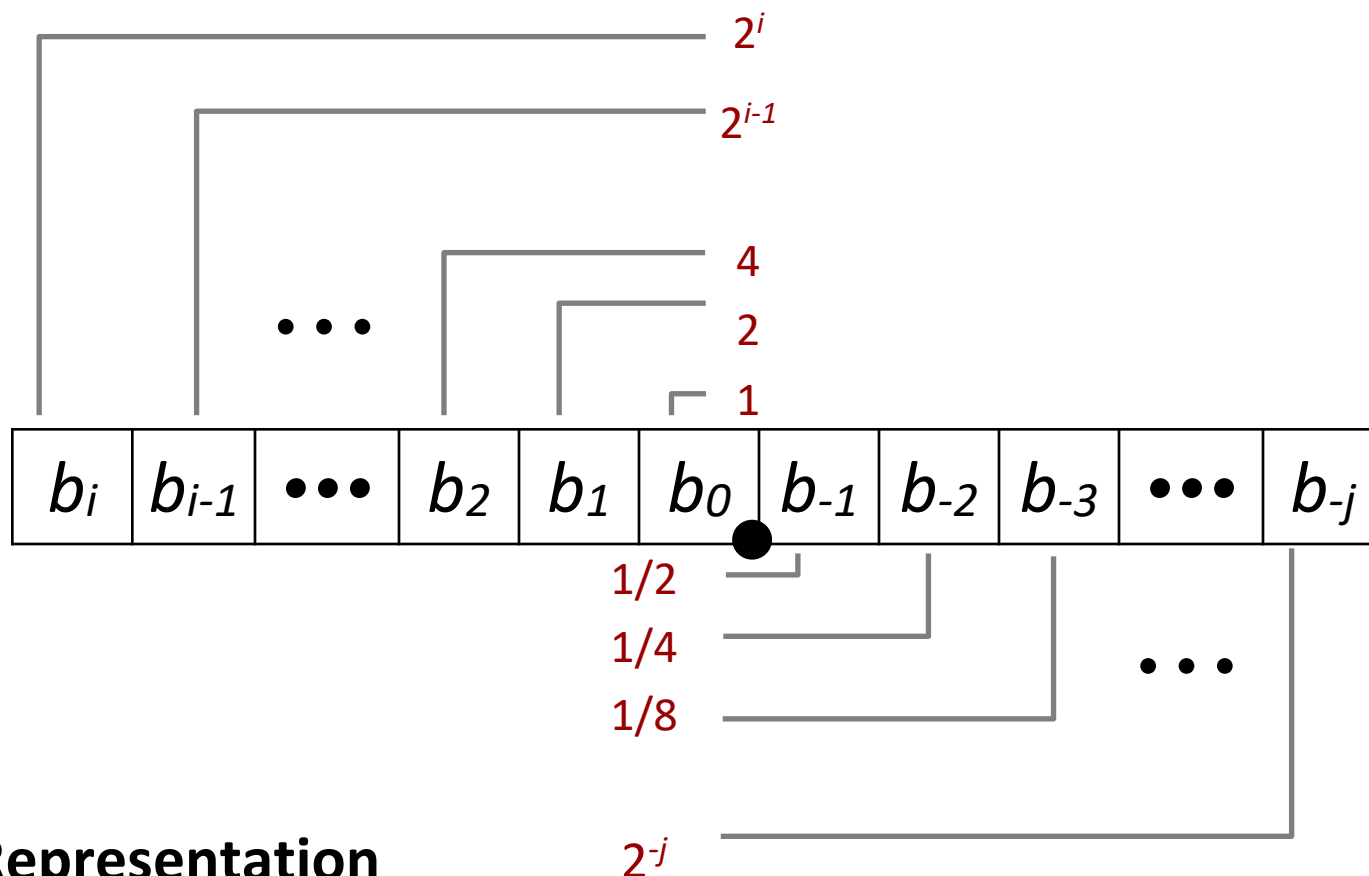
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2 ?

Fractional Binary Numbers



■ Representation

- Bits to right of “binary point” represent fractional powers of 2
- Represents rational number:

$$\sum_{k=-j}^i b_k \times 2^k$$

Fractional Binary Numbers: Examples

Value	Representation	
$5 \frac{3}{4} = 23/4$	101.11_2	$= 4 + 1 + 1/2 + 1/4$
$2 \frac{7}{8} = 23/8$	10.111_2	$= 2 + 1/2 + 1/4 + 1/8$
$1 \frac{7}{16} = 23/16$	1.0111_2	$= 1 + 1/4 + 1/8 + 1/16$

Observations

- Divide by 2 by shifting right (unsigned)
- Multiply by 2 by shifting left
- Numbers of form $0.111111..._2$ are just below 1.0
 - $1/2 + 1/4 + 1/8 + \dots + 1/2^i + \dots \rightarrow 1.0$
 - Use notation $1.0 - \epsilon$

Representable Numbers

■ Limitation #1

- Can only exactly represent numbers of the form $x/2^k$
 - Other rational numbers have repeating bit representations
- Value Representation
 - $1/3$ $0.0101010101 [01] \dots_2$
 - $1/5$ $0.001100110011 [0011] \dots_2$
 - $1/10$ $0.0001100110011 [0011] \dots_2$

■ Limitation #2

- Just one setting of binary point within the w bits
 - Limited range of numbers (very small values? very large?)

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IEEE Floating Point

■ IEEE Standard 754

- Established in 1985 as uniform standard for floating point arithmetic
 - Before that, many idiosyncratic formats
- Supported by all major CPUs
- Some CPUs don't implement IEEE 754 in full
e.g., early GPUs, Cell BE processor

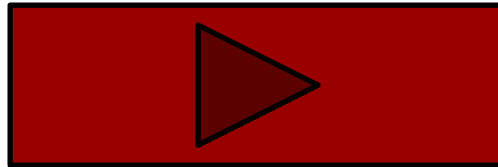
■ Driven by numerical concerns

- Nice standards for rounding, overflow, underflow
- Hard to make fast in hardware
 - **Numerical analysts** predominated over **hardware designers** in defining standard

This is important!

■ Ariane 5 explodes on maiden voyage: \$500 MILLION dollars lost

- 64-bit floating point number assigned to 16-bit integer
- Causes rocket to get incorrect value of horizontal velocity and crash



■ Patriot Missile defense system misses scud – 28 people die

- System tracks time in tenths of second
- Converted from integer to floating point number.
- Accumulated rounding error causes drift. 20% drift over 8 hours.
- Eventually (on 2/25/1991 system was on for 100 hours) causes range mis-estimation sufficiently large to miss incoming missiles.

Floating Point Representation

■ Numerical Form:

Example:

$$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$$

$$(-1)^s M 2^E$$

- **Sign bit** s determines whether number is negative or positive
- **Significand** M normally a fractional value in range $[1.0, 2.0)$.
- **Exponent** E weights value by power of two

■ Encoding

- MSB s is sign bit s
- **exp** field encodes E (but is not equal to E)
- **frac** field encodes M (but is not equal to M)



Precision options

■ Single precision: 32 bits

≈ 7 decimal digits, $10^{\pm 38}$



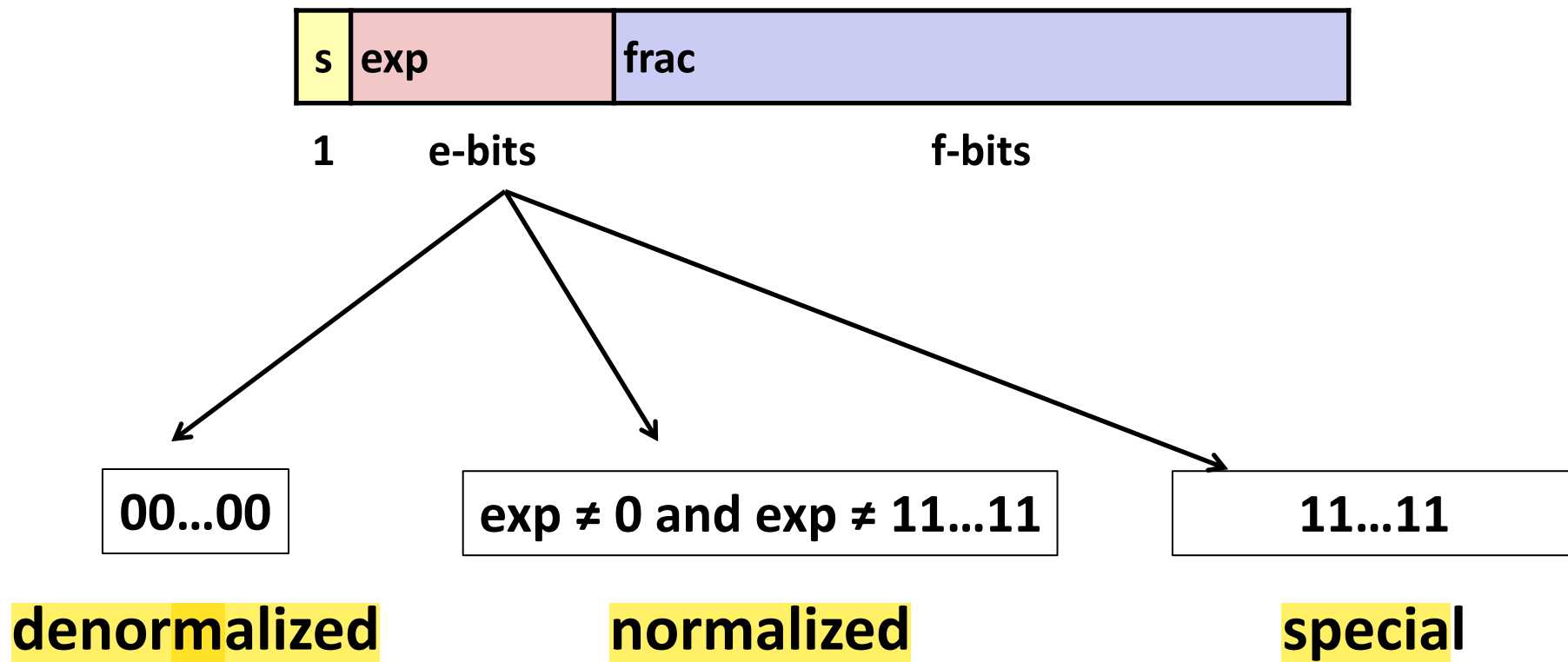
■ Double precision: 64 bits

≈ 16 decimal digits, $10^{\pm 308}$



■ Other formats: half precision, quad precision

Three “kinds” of floating point numbers



“Normalized” Values

$$v = (-1)^s M 2^E$$

- When: **exp** \neq 000...0 and **exp** \neq 111...1

- Exponent coded as a *biased* value: $E = \text{exp} - \text{Bias}$
 - *exp*: unsigned value of exp field
 - **Bias** = $2^{k-1} - 1$, where k is number of exponent bits
 - Single precision: 127 (**exp**: 1...254, E: -126...127)
 - Double precision: 1023 (**exp**: 1...2046, E: -1022...1023)

- Significand coded with implied leading 1: $M = 1.\text{xxx}...\text{x}_2$
 - xxx...x: bits of frac field
 - Minimum when **frac**=000...0 ($M = 1.0$)
 - Maximum when **frac**=111...1 ($M = 2.0 - \epsilon$)
 - Get extra leading bit for “free”

Normalized Encoding Example

$$v = (-1)^s M 2^E$$

$$E = \text{exp} - \text{Bias}$$

■ Value: `float F = 15213.0;`

$$15213_{10} = 11101101101101_2$$

$$= 1.1101101101101_2 \times 2^{13}$$

■ Significand

$$M = 1.\underline{1101101101101}_2$$

$$\text{frac} = \underline{110110110110100000000000}_2$$

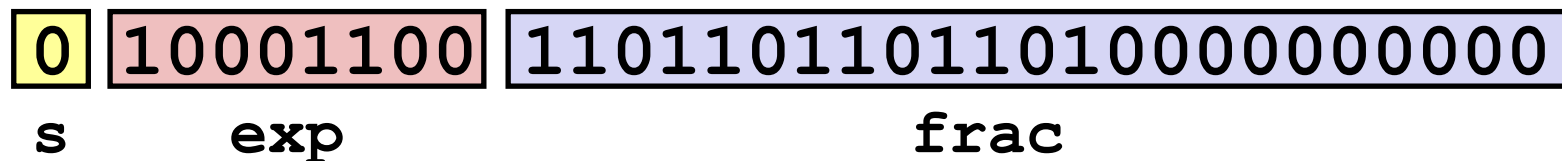
■ Exponent

$$E = 13$$

$$\text{Bias} = 127$$

$$\text{exp} = 140 = 10001100_2$$

■ Result:



Denormalized Values

$$v = (-1)^s M 2^E$$
$$E = 1 - \text{Bias}$$

- **Condition:** $\text{exp} = 000\dots 0$
- **Exponent value:** $E = 1 - \text{Bias}$ (instead of $\text{exp} - \text{Bias}$) (why?)
- **Significand coded with implied leading 0:** $M = 0.\text{xxx}\dots\text{x}_2$
 - $\text{xxx}\dots\text{x}$: bits of `frac`
- **Cases**
 - $\text{exp} = 000\dots 0, \text{frac} = 000\dots 0$
 - Represents zero value
 - Note distinct values: $+0$ and -0 (why?)
 - $\text{exp} = 000\dots 0, \text{frac} \neq 000\dots 0$
 - Numbers closest to 0.0
 - Equispaced

Special Values

- **Condition: $\text{exp} = 111\dots 1$**

- **Case: $\text{exp} = 111\dots 1, \text{frac} = 000\dots 0$**
 - **Represents value ∞ (infinity)**
 - Operation that overflows
 - Both positive and negative
 - E.g., $1.0/0.0 = -1.0/-0.0 = +\infty$, $1.0/-0.0 = -\infty$

- **Case: $\text{exp} = 111\dots 1, \text{frac} \neq 000\dots 0$**
 - **Not-a-Number (NaN)**
 - Represents case when no numeric value can be determined
 - E.g., $\text{sqrt}(-1)$, $\infty - \infty$, $\infty \times 0$

C float Decoding Example

float: 0xC0A00000

binary: _____



1

8-bits

23-bits

E =

S =

M =

$v = (-1)^S M 2^E =$

$$v = (-1)^S M 2^E$$

$$E = \text{exp} - \text{Bias}$$

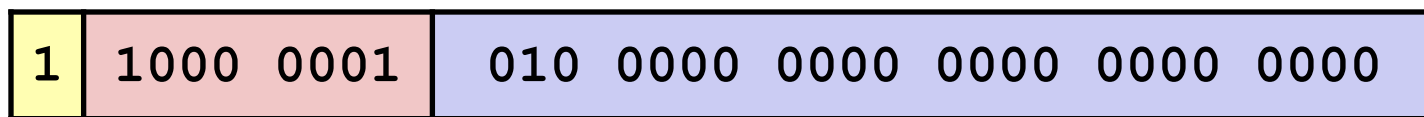
$$\text{Bias} = 2^{k-1} - 1 = 127$$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #1

float: 0xC0A00000

binary: 1100 0000 1010 0000 0000 0000 0000 0000



1

8-bits

23-bits

E =

S =

M = 1.

$v = (-1)^S M 2^E =$

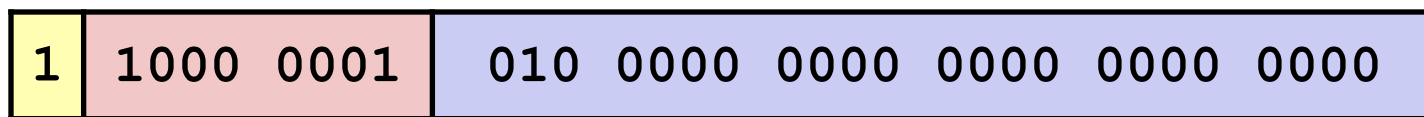
Hex Decimal Binary

0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #1

float: 0xC0A00000

binary: 1100 0000 1010 0000 0000 0000 0000 0000



1

8-bits

23-bits

$$E = \text{exp} - \text{Bias} = 129 - 127 = 2 \text{ (decimal)}$$

$S = 1 \rightarrow$ negative number

$$M = 1.010 \ 0000 \ 0000 \ 0000 \ 0000 \ 0000$$

$$= 1 + 1/4 = 1.25$$

$$v = (-1)^S M 2^E = (-1)^1 * 1.25 * 2^2 = -5$$

$$v = (-1)^S M 2^E$$

$$E = \text{exp} - \text{Bias}$$

$$\text{Bias} = 2^{k-1} - 1 = 127$$

Hex Decimal Binary

0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

12.75

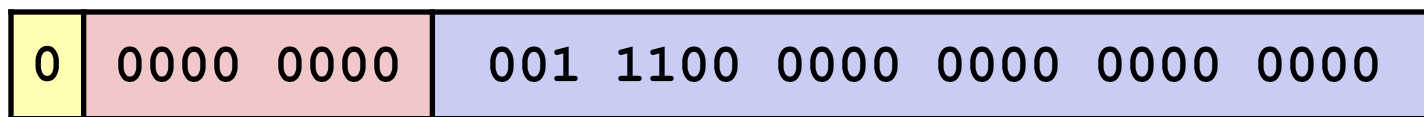
C float Decoding Example #2

$$v = (-1)^s M 2^E$$

$$E = 1 - \text{Bias}$$

float: 0x001C0000

binary: 0000 0000 0001 1100 0000 0000 0000 0000



1

8-bits

23-bits

E =

S =

M = 0.

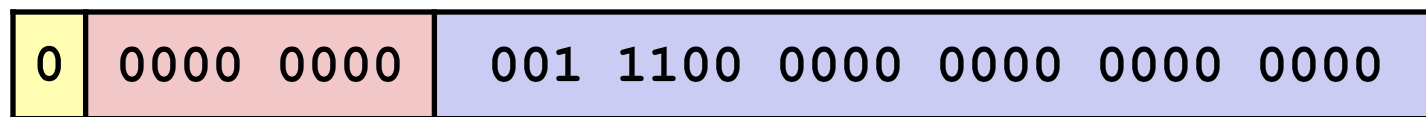
$$v = (-1)^s M 2^E =$$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

C float Decoding Example #2

float: 0x001C0000

binary: 0000 0000 0001 1100 0000 0000 0000 0000



1

8-bits

23-bits

$$E = 1 - \text{Bias} = 1 - 127 = -126 \text{ (decimal)}$$

$S = 0$ -> positive number

$$M = 0.001 \ 1100 \ 0000 \ 0000 \ 0000 \ 0000$$

$$= 1/8 + 1/16 + 1/32 = 7/32 = 7 \cdot 2^{-5}$$

$$v = (-1)^S M 2^E = (-1)^0 * 7 \cdot 2^{-5} * 2^{-126} = 7 \cdot 2^{-131}$$

$$\approx 2.571393892 \times 10^{-39}$$

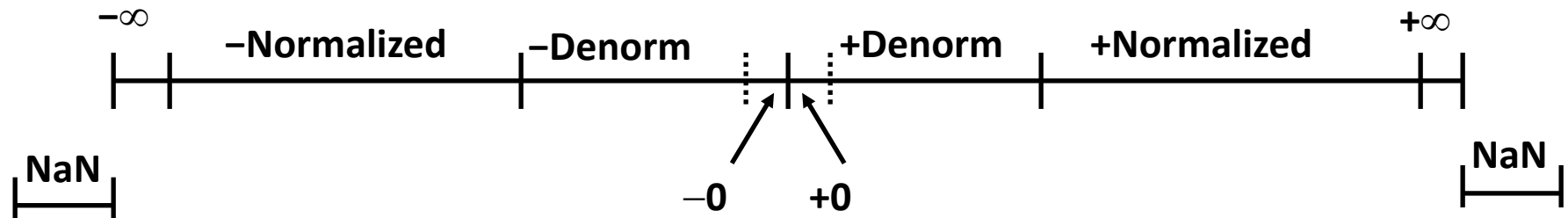
$$v = (-1)^S M 2^E$$

$$E = 1 - \text{Bias}$$

$$\text{Bias} = 2^{k-1} - 1 = 127$$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

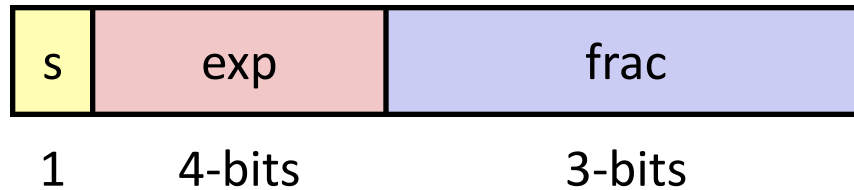
Visualization: Floating Point Encodings



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Tiny Floating Point Example



■ 8-bit Floating Point Representation

- the sign bit is in the most significant bit
- the next four bits are the **exp**, with a bias of 7
- the last three bits are the **frac**

■ Same general form as IEEE Format

- normalized, denormalized
- representation of 0, NaN, infinity

Dynamic Range (s=0 only)

$$v = (-1)^s M 2^E$$

norm: $E = \text{exp} - \text{Bias}$

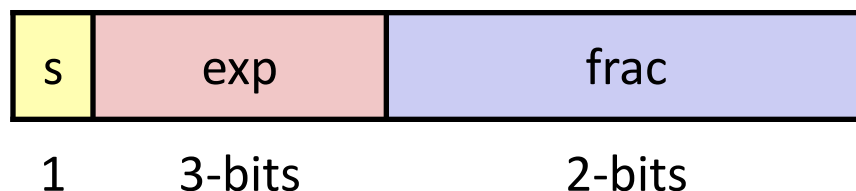
denorm: $E = 1 - \text{Bias}$

	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	$1/8 * 1/64 = 1/512$	closest to zero
	0	0000	010	-6	$2/8 * 1/64 = 2/512$	$(-1)^0 (0 + 1/4) * 2^{-6}$
	...					
	0	0000	110	-6	$6/8 * 1/64 = 6/512$	
	0	0000	111	-6	$7/8 * 1/64 = 7/512$	largest denorm
Normalized numbers	0	0001	000	-6	$8/8 * 1/64 = 8/512$	smallest norm
	0	0001	001	-6	$9/8 * 1/64 = 9/512$	$(-1)^0 (1 + 1/8) * 2^{-6}$
	...					
	0	0110	110	-1	$14/8 * 1/2 = 14/16$	
	0	0110	111	-1	$15/8 * 1/2 = 15/16$	closest to 1 below
	0	0111	000	0	$8/8 * 1 = 1$	
	0	0111	001	0	$9/8 * 1 = 9/8$	closest to 1 above
	0	0111	010	0	$10/8 * 1 = 10/8$	
	...					
	0	1110	110	7	$14/8 * 128 = 224$	
	0	1110	111	7	$15/8 * 128 = 240$	largest norm
	0	1111	000	n/a	inf	

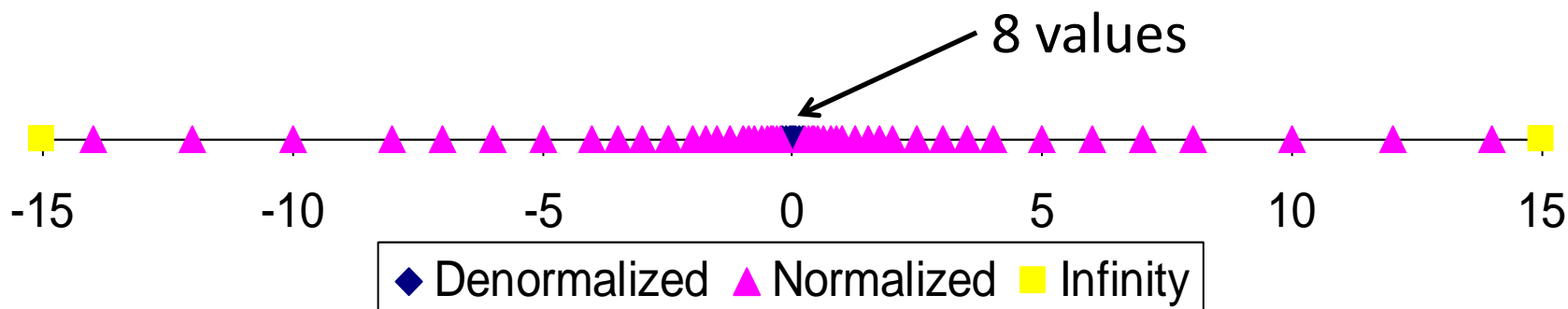
Distribution of Values

■ 6-bit IEEE-like format

- $e = 3$ exponent bits
- $f = 2$ fraction bits
- Bias is $2^{3-1}-1 = 3$



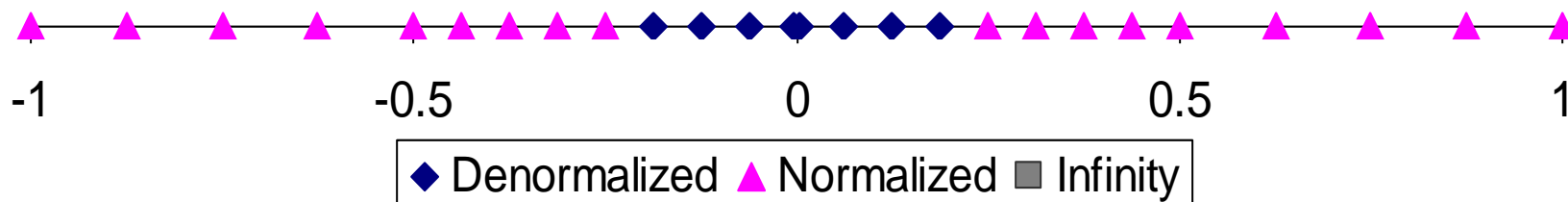
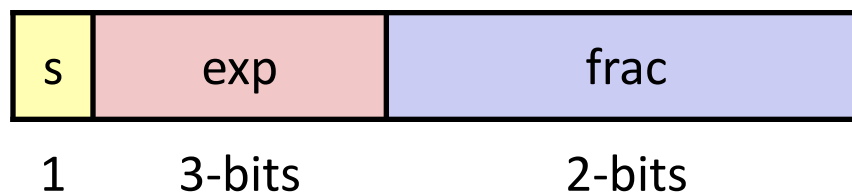
■ Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

■ 6-bit IEEE-like format

- $e = 3$ exponent bits
- $f = 2$ fraction bits
- Bias is 3



Special Properties of the IEEE Encoding

■ FP Zero Same as Integer Zero

- All bits = 0

■ Can (Almost) Use Unsigned Integer Comparison

- Must first compare sign bits
- Must consider $-0 = 0$
- NaNs problematic
 - Will be greater than any other values
 - What should comparison yield? The answer is complicated.
- Otherwise OK
 - Denorm vs. normalized
 - Normalized vs. infinity

Quiz Time!

Check out:

<https://canvas.cmu.edu/courses/16836>

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Floating Point Operations: Basic Idea

$$\blacksquare \mathbf{x} +_{\mathbf{f}} \mathbf{y} = \mathbf{Round}(\mathbf{x} + \mathbf{y})$$

$$\blacksquare \mathbf{x} \times_{\mathbf{f}} \mathbf{y} = \mathbf{Round}(\mathbf{x} \times \mathbf{y})$$

■ Basic idea

- First **compute exact result**
- Make it fit into desired precision
 - Possibly overflow if exponent too large
 - Possibly **round to fit into frac**

Rounding

■ Rounding Modes (illustrate with \$ rounding)

	\$1.40	\$1.60	\$1.50	\$2.50	-\$1.50
■ Towards zero	\$1 ↓	\$1 ↓	\$1 ↓	\$2 ↓	-\$1 ↑
■ Round down ($-\infty$)	\$1 ↓	\$1 ↓	\$1 ↓	\$2 ↓	-\$2 ↓
■ Round up ($+\infty$)	\$2 ↑	\$2 ↑	\$2 ↑	\$3 ↑	-\$1 ↑
■ Nearest Even* (default)	\$1 ↓	\$2 ↑	\$2 ↑	\$2 ↓	-\$2 ↓

*Round to nearest, but if half-way in-between then round to nearest even

Closer Look at Round-To-Even

■ Default Rounding Mode

- Hard to get any other kind without dropping into assembly
 - C99 has support for rounding mode management
- All others are statistically biased
 - Sum of set of positive numbers will consistently be over- or under-estimated

■ Applying to Other Decimal Places / Bit Positions

- When exactly halfway between two possible values
 - Round so that least significant digit is even
- E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

■ Binary Fractional Numbers

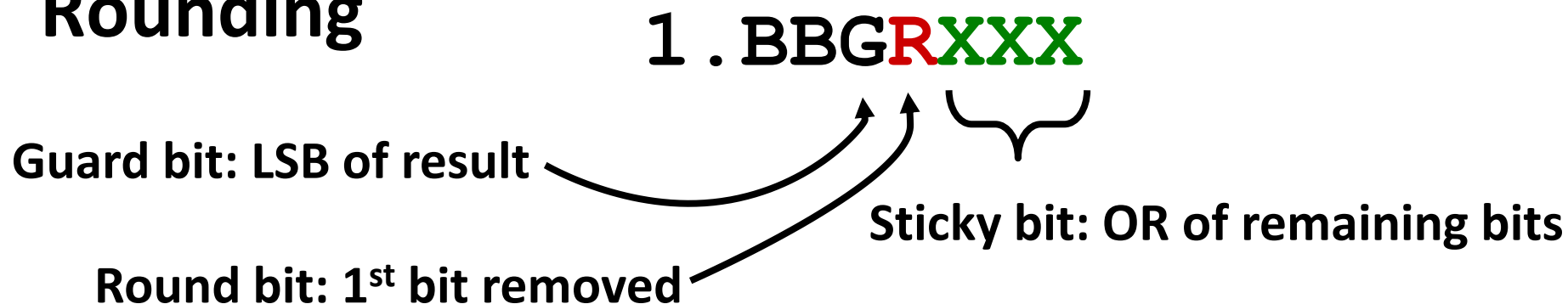
- “Even” when least significant bit is 0
- “Half way” when bits to right of rounding position = $100..._2$

■ Examples

- Round to nearest $1/4$ (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
$2 \frac{3}{32}$	$10.00\textcolor{red}{011}_2$	10.00_2	($<1/2$ —down)	2
$2 \frac{3}{16}$	$10.00\textcolor{red}{110}_2$	10.01_2	($>1/2$ —up)	$2 \frac{1}{4}$
$2 \frac{7}{8}$	$10.11\textcolor{red}{100}_2$	11.00_2	($\textcolor{red}{1/2}$ —up)	3
$2 \frac{5}{8}$	$10.10\textcolor{red}{100}_2$	10.10_2	($\textcolor{red}{1/2}$ —down)	$2 \frac{1}{2}$

Rounding



■ Round up conditions

- Round = 1, Sticky = 1 \rightarrow > 0.5
- Guard = 1, Round = 1, Sticky = 0 \rightarrow Round to even

<i>Fraction</i>	<i>GRS</i>	<i>Incr?</i>	<i>Rounded</i>
1.000 0 000	0 0 0	N	1.000
1.101 0 000	1 0 0	N	1.101
1.000 1 000	0 1 0	N	1.000
1.001 1 000	1 1 0	Y	1.010
1.000 1 010	0 1 1	Y	1.001
1.111 1 100	1 1 1	Y	10.000

FP Multiplication

■ $(-1)^{s1} M1 2^{E1} \times (-1)^{s2} M2 2^{E2}$

■ **Exact Result:** $(-1)^s M 2^E$

- Sign s : $s1 \wedge s2$
- Significand M : $M1 \times M2$
- Exponent E : $E1 + E2$

■ Fixing

- If $M \geq 2$, shift M right, increment E
- If E out of range, overflow
- Round M to fit **frac** precision

■ Implementation

- Biggest chore is multiplying significands

$$\begin{aligned} \text{4 bit significand: } 1.010 \times 2^2 \times 1.110 \times 2^3 &= 1\mathbf{0}.0011 \times 2^5 \\ &= 1.000\mathbf{11} \times 2^6 = 1.00\mathbf{1} \times 2^6 \end{aligned}$$

Floating Point Addition

$$\blacksquare (-1)^{s1} M1 2^{E1} + (-1)^{s2} M2 2^{E2}$$

- Assume $E1 > E2$

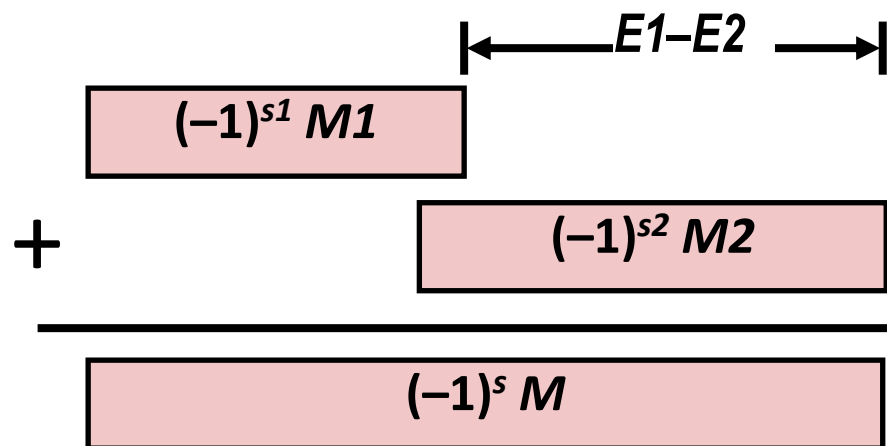
$$\blacksquare \text{Exact Result: } (-1)^s M 2^E$$

- Sign s , significand M :
 - Result of signed align & add
- Exponent E : $E1$

Fixing

- If $M \geq 2$, shift M right, increment E
- if $M < 1$, shift M left k positions, decrement E by k
- Overflow if E out of range
- Round M to fit **frac** precision

Get binary points lined up



$$\begin{aligned}
 1.010 * 2^2 + 1.110 * 2^3 &= (0.1010 + 1.1100) * 2^3 \\
 &= 1\textcolor{red}{0}.0110 * 2^3 = 1.001\textcolor{red}{10} * 2^4 = 1.010 * 2^4
 \end{aligned}$$

Mathematical Properties of FP Add

■ Compare to those of Abelian Group

- Closed under addition? *Yes*
 - But may generate infinity or NaN
- Commutative? *Yes*
- Associative? *No*
 - Overflow and inexactness of rounding
 - $(3.14 + 1e10) - 1e10 = 0$, $3.14 + (1e10 - 1e10) = 3.14$
- 0 is additive identity? *Yes*
- Every element has additive inverse? *Almost*
 - Yes, except for infinities & NaNs

■ Monotonicity

- $a \geq b \Rightarrow a + c \geq b + c$ *Almost*
 - Except for infinities & NaNs

Mathematical Properties of FP Mult

■ Compare to Commutative Ring

- Closed under multiplication? *Yes*
 - But may generate infinity or NaN
- Multiplication Commutative? *Yes*
- Multiplication is Associative? *No*
 - Possibility of overflow, inexactness of rounding
 - Ex: $(1e20 * 1e20) * 1e-20 = \text{inf}$, $1e20 * (1e20 * 1e-20) = 1e20$
- 1 is multiplicative identity? *Yes*
- Multiplication distributes over addition? *No*
 - Possibility of overflow, inexactness of rounding
 - $1e20 * (1e20 - 1e20) = 0.0$, $1e20 * 1e20 - 1e20 * 1e20 = \text{NaN}$

■ Monotonicity

- $a \geq b \ \& \ c \geq 0 \Rightarrow a * c \geq b * c$ *Almost*
 - Except for infinities & NaNs

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Floating Point in C

■ C Guarantees Two Levels

- `float` single precision
- `double` double precision

■ Conversions/Casting

- Casting between `int`, `float`, and `double` changes bit representation
- `double/float` \rightarrow `int`
 - Truncates fractional part
 - Like rounding toward zero
 - Not defined when out of range or NaN: Generally sets to TMin
- `int` \rightarrow `double`
 - Exact conversion, as long as `int` has ≤ 53 bit word size
- `int` \rightarrow `float`
 - Will round according to rounding mode

Floating Point Puzzles

■ For each of the following C expressions, either:

- Argue that it is true for all argument values
- Explain why not true

```
int x = ...;
float f = ...;
double d = ...;
```

Assume neither
d nor **f** is NaN

• `x == (int)(float) x`

✗

• `x == (int)(double) x`

✓

• `f == (float)(double) f`

✓

• `d == (double)(float) d`

✗

• `f == -(-f);`

✓

• `2/3 == 2/3.0`

✗

• `d < 0.0` \Rightarrow `((d*2) < 0.0)`

✓

• `d > f` \Rightarrow `-f > -d`

✓

• `d * d >= 0.0`

✓

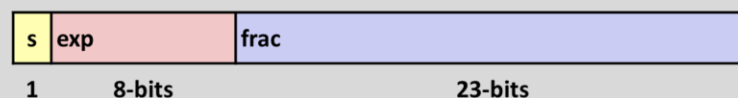
• `(d+f) - d == f`

✗

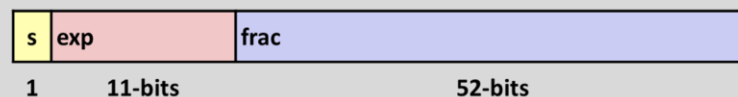
Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form $M \times 2^E$
- One can reason about operations independent of implementation
 - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
 - Violates associativity/distributivity
 - Makes life difficult for compilers & serious numerical applications programmers

Single precision: 32 bits



Double precision: 64 bits

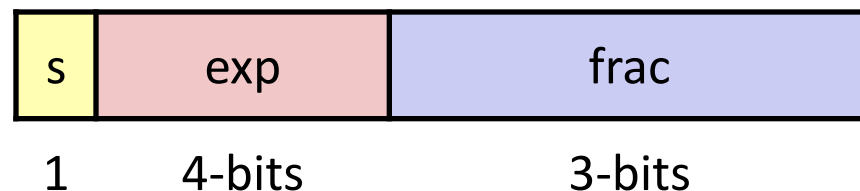


Additional Slides

Creating Floating Point Number

■ Steps

- Normalize to have leading 1
- Round to fit within fraction
- Postnormalize to deal with effects of rounding



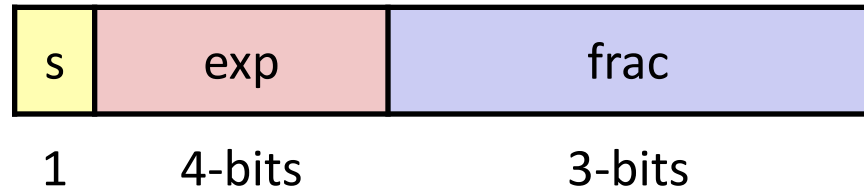
■ Case Study

- Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

128	10000000
15	00001101
33	00010001
35	00010011
138	10001010
63	00111111

Normalize



■ Requirement

- Set binary point so that numbers of form 1.xxxxx
- Adjust all to have leading one
 - Decrement exponent as shift left

<i>Value</i>	<i>Binary</i>	<i>Fraction</i>	<i>Exponent</i>
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Postnormalize

■ Issue

- Rounding may have caused overflow
- Handle by shifting right once & incrementing exponent

<i>Value</i>	<i>Rounded</i>	<i>Exp</i>	<i>Adjusted</i>	<i>Numeric Result</i>
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

Interesting Numbers

{single, double}

<i>Description</i>	<i>exp</i>	<i>frac</i>	<i>Numeric Value</i>
■ Zero	00...00	00...00	0.0
■ Smallest Pos. Denorm.	00...00	00...01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
<ul style="list-style-type: none"> ■ Single $\approx 1.4 \times 10^{-45}$ ■ Double $\approx 4.9 \times 10^{-324}$ 			
■ Largest Denormalized	00...00	11...11	$(1.0 - \epsilon) \times 2^{-\{126,1022\}}$
<ul style="list-style-type: none"> ■ Single $\approx 1.18 \times 10^{-38}$ ■ Double $\approx 2.2 \times 10^{-308}$ 			
■ Smallest Pos. Normalized	00...01	00...00	$1.0 \times 2^{-\{126,1022\}}$
<ul style="list-style-type: none"> ■ Just larger than largest denormalized 			
■ One	01...11	00...00	1.0
■ Largest Normalized	11...10	11...11	$(2.0 - \epsilon) \times 2^{\{127,1023\}}$
<ul style="list-style-type: none"> ■ Single $\approx 3.4 \times 10^{38}$ ■ Double $\approx 1.8 \times 10^{308}$ 			