Pearson Edexcel

Mark Scheme (Results)

January 2022

Pearson Edexcel International A Level
In Further Pure Mathematics F2 (WFM02)
Paper : WFM02/01

Question	Scheme	Marks
1(a)	$r=\sqrt{(-4)^2+\left(-4\sqrt{3}\right)^2}=...$	M1
	$\tan\theta=\dfrac{-4\sqrt{3}}{-4}\Rightarrow\theta=\tan^{-1}\left(\sqrt{3}\right)\pm\pi$	M1
	$8\left(\cos\left(-\dfrac{2\pi}{3}\right)+\mathrm{i}\sin\left(-\dfrac{2\pi}{3}\right)\right)$	A1
		(3)
(b)	$z=r\mathrm{e}^{\mathrm{i}\theta}\Rightarrow\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^3=-4-4\sqrt{3}\Rightarrow r^3\left(\mathrm{e}^{3\mathrm{i}\theta}\right)=8\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$	
	$\Rightarrow r=\sqrt[3]{8}=2$	M1
	$3\theta=-\dfrac{2\pi}{3}(+2k\pi)\Rightarrow\theta=-\dfrac{2\pi}{9}+\left(\dfrac{2k\pi}{3}\right)$	M1
	So $z=2\mathrm{e}^{-\frac{8\pi}{9}\mathrm{i}},2\mathrm{e}^{-\frac{2\pi}{9}\mathrm{i}},2\mathrm{e}^{\frac{4\pi}{9}\mathrm{i}}$	A1ft A1
		(4)
		(7 marks)

Notes:

(a)

M1: For a correct attempt at the modulus, implied by a correct modulus if no method seen and allow recovery if correct answer follows a minor slip in notation.

M1: For an attempt to find a value of θ in the correct quadrant. Accept $\tan^{-1}\left(\sqrt{3}\right)\pm\pi$ or $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\pm\pi$

May be implied by sight of an of $-\dfrac{2}{3}\pi,\dfrac{4}{3}\pi,-\dfrac{5}{6}\pi,\dfrac{7}{6}\pi$.

A1: cao as in scheme, no other solution.

(b)

M1: Applies De Moivre's Theorem and proceeds to find a value for r ie $\left(\text{their } 8\right)^{\frac{1}{3}}$

M1: Proceeds to find at least one value for θ – ie their argument/3.

A1ft: At least two roots correct for their r and θ. (Must come from correct method, watch for correct roots coming from an incorrect angle due to errors.)

A1: All three correct roots and no others. Accept e.g $2\mathrm{e}^{\mathrm{i}-\frac{8\pi}{9}}$ as a slip in notation, so allow marks.

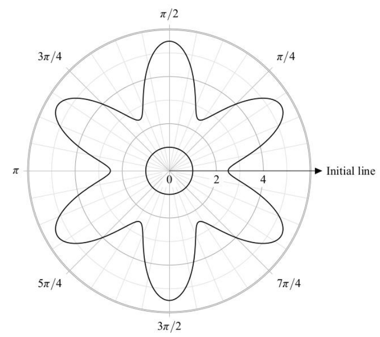
Question	Scheme	Marks
2	$2m^2-5m-3=0\Rightarrow(2m+1)(m-3)=0\Rightarrow m=...$	M1
	So C.F. is $(y_{CF}=)\,A\mathrm{e}^{-\frac{1}{2}x}+B\mathrm{e}^{3x}$	A1
	P.I. is $y_{PI}=ax\mathrm{e}^{3x}$	B1
	$\frac{\mathrm{d}y_{PI}}{\mathrm{d}x}=3ax\mathrm{e}^{3x}+a\mathrm{e}^{3x},\ \frac{\mathrm{d}^2y_{PI}}{\mathrm{d}x^2}=9ax\mathrm{e}^{3x}+3a\mathrm{e}^{3x}+3a\mathrm{e}^{3x}$ $\Rightarrow 2(9ax+6a)\mathrm{e}^{3x}-5(3ax+a)\mathrm{e}^{3x}-3ax\mathrm{e}^{3x}=2\mathrm{e}^{3x}\Rightarrow a=...$	M1
	$a=\frac{2}{7}$	A1
	General solution is $y=A\mathrm{e}^{-\frac{1}{2}x}+B\mathrm{e}^{3x}+\frac{2}{7}x\mathrm{e}^{3x}$	B1ft
		(6)
		(6 marks)

Notes:

M1: Forms and solves the auxiliary equation.
A1: Correct complementary function (no need for $y = \dots$)
B1: Correct form for the particular integral. Accept any PI that includes $ax\mathrm{e}^{3x}$, so e.g. $(ax+b)\mathrm{e}^{3x}$ is fine.
M1: Attempts to differentiate their PI twice and substitutes into the left hand side of the equation. The derivatives must be changed functions. There is no need to reach a value for the unknown(s) but their PI must contain an unknown constant.
A1: Correct value of a (and any other coefficients as zero). Must have had a suitable PI
B1ft: For y = their CF + their PI. Must include the $y =$. The PI must be a function of x that matches their initial choice of PI, with their constants substituted.

Question	Scheme	Marks
3(a)	Meet when $x^2-8x=\frac{4x}{4-x}\Rightarrow\left(x^2-8x\right)\left(4-x\right)=4x\Rightarrow x\left(4x-32-x^2+8x-4\right)=0$	**M1**
	(so $x=0$ or) $x^2-12x+36=0$	**A1**
	$\Rightarrow x\left(x-6\right)^2=0\Rightarrow x=\ldots$	**M1**
	Meet at (6,−12)	**A1**
	e.g. touch at (6,−12) as repeated root.	**B1**
		(5)
Alt	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-8x\right)=2x-8$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{4x}{4-x}\right)=\frac{4(4-x)-4x(-1)}{(4-x)^2}=\frac{16}{(4-x)^2}$	**M1A1**
	$2x-8=\frac{16}{(4-x)^2}\Rightarrow(x-4)^3=8\Rightarrow x=\ldots$	**M1**
	Meet at (6,−12)	**A1**
	e.g. $6^2-6\times9=-12$ and $\frac{4\times6}{4-6}=-12$, so curves meet at tangent at (6,−12)	**B1**
		(5)
(b)	$x^2-8x=\frac{4x}{4+x}\Rightarrow x\left(x-8\right)\left(4+x\right)-4x=0\Rightarrow x\left(x^2-4x-36\right)=0\Rightarrow x=..$	**M1**
	$x=(0),2\pm2\sqrt{10}\Rightarrow$ critical value is (0 and) $2-2\sqrt{10}$	**A1**
	Other C.V.'s are 0, ±4	**B1**
	E.g. extremes are $x<2-2\sqrt{10}$ and $x>6$ or any two suitable ranges.	**M1**
	Solution is $x<2-2\sqrt{10},-4<x<0,4<x<6,x>6$	**A1A1**
		(6)
		(11 marks)

Notes:
(a) **M1:** Attempts to find intersection by setting equations equal and cross multiplies and factorises the x out or cancels. **A1:** Correct quadratic reached. May be implied by solutions of 0,6 seen from the cubic (by calculator) **M1:** Solves the quadratic to find roots. **A1:** Obtains the correct point where the curves meet. **B1:** Correct reason given for why the curves touch. Accepted "repeated root" as reason. As a minimum, accept "$\left(x-6\right)^2=0$ therefore touches". Alternatively, accept discriminant = 0 shown with conclusion, or may find gradient at both points and show equal, with conclusion. Alt: **M1:** Attempts derivatives of both curves **A1:** Both derivatives correct. **M1:** Sets derivatives equal and solves to find x value where gradients agree. **A1:** Obtains the correct point where the curves meet. **B1:** Correct value checked in both curves with conclusion that they meet at a tangent or equivalent working as per main scheme. **(b)** **M1:** Attempts to find the intersection of the other branch of $\dfrac{4x}{4-\lvert x\rvert}$ with x^2-8x. Allow for any attempt at solving $\dfrac{4x}{4+x}=x^2-8x$ that reaches a value for x **A1:** Correct value of $2-2\sqrt{10}$ identified. (No need to see the second root rejected for this mark.) **B1:** Both 0 and ±4 identified as critical values for the ranges needed at some stage in working. **M1:** Forms at least two suitable ranges from their critical values (allow if e.g. $\leqslant$ is used instead of $<$). Likely to be the extreme ranges, so look for $x <$ their $2-2\sqrt{10}$ and $x >$ their 6. However, allow if this latter is included as part of the range $x > 4$ for this mark. **A1:** At least two correct ranges. **A1:** Fully correct answer as in scheme.

Question	Scheme	Marks
4(a)		
	Completes to a closed loop with “petals” containing circle of radius 1 (whether the circle is drawn or not)	**M1**
	Fully correct – 6 petals in roughly the right places, but allow if curvature is not quite smooth.	**A1**
	Circle centre O radius 1.	**B1**
		(3)
(b)	$\left(\frac{1}{2}\right)\int r^2\ \mathrm{d}\theta = \left(\frac{1}{2}\right)\int\left(16 - 12\cos 6\theta + \frac{9}{4}\cos^2 6\theta\right)\ \mathrm{d}\theta$	**M1**
	$= \frac{1}{2}\int_0^{2\pi}\left(16 - 12\cos 6\theta + \frac{9}{8}(1 + \cos 12\theta)\right)\ \mathrm{d}\theta$	**M1**
	$= \frac{1}{2}\left[16\theta - 2\sin 6\theta + \frac{9}{8}\left(\theta + \frac{1}{12}\sin 12\theta\right)\right]_0^{2\pi}$	**M1** **A1**
	$A_{outer} = \frac{1}{2}\int_0^{2\pi} r^2\ \mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi}\left(16 - 12\cos 6\theta + \frac{9}{4}\cos^2 6\theta\right)\ \mathrm{d}\theta$ $= \frac{1}{2}\left(32\pi - 0 + \frac{9}{8}(2\pi + 0) - (0)\right)$	**dM1**
	So Area required is $\frac{1}{2}\left(32\pi + \frac{9\pi}{4}\right) - \pi(1^2) = \ldots$	**B1**
	$= \frac{129}{8}\pi$	**A1**
		(7)
		(10 marks)

Notes:

(a)

M1: Allow for any closed loop that oscillates, though may not have the correct number of “petals” but require at least 4 . Need not have correct places of maximum radius.

A1: Fully correct sketch, 6 “petals” in the right places, with maximum radius between the 5 and 6 radius lines, minimum between the 2 and 3 radius lines.

B1: For a circle of radius 1 and centre O drawn.

(b)

M1: Attempts to square r as part of an integral for the outer curve, achieving a 3 term quadratic in $\cos 6\theta$

M1: Applies the double angle formula to the $\cos^2$ term from their expansion (not dependent on the first M, but must have a $\cos^2$ term). Accept $\cos^2 6\theta \to \frac{1}{2}(\pm 1 \pm \cos 12\theta)$

M1: Attempts to integrate, achieving the form $\alpha\theta + \beta\sin 6\theta + \gamma\sin 12\theta$ where $\alpha, \beta, \gamma \neq 0$

A1: Correct integration – limits and the $\frac{1}{2}$ not needed. Look for $16\theta - 2\sin 6\theta + \frac{9}{8}\left(\theta + \frac{1}{12}\sin 12\theta\right)$ oe.

dM1: Depends on at least two of the previous M's being scored. For a correct overall strategy for the area contained in the outer loop, with an attempt at the r^2 (should be 3 term expansion). Correct appropriate limits and the $\frac{1}{2}$ should be present or implied by working, but note variations on the scheme are possible, e.g. $2\times\frac{1}{2}\int_0^{\pi} r^2 \, d\theta$, in which the $2\times\frac{1}{2}$ may be implied rather than seen.

B1: Subtracts correct area of π for inner circle

A1: cso. Check carefully the integration was correct as the sin terms disappear with the limits.

Question	Scheme	Marks
5(a)	$\dfrac{dy}{dx}=\dfrac{1}{2}(4+\ln x)^{-\frac{1}{2}}\times\dfrac{1}{x}$	M1 A1
	$\dfrac{d^2y}{dx^2}=\dfrac{1}{2}\dfrac{0-\left(\sqrt{4+\ln x}+x\times\frac{1}{2}(4+\ln x)^{-\frac{1}{2}}\times\frac{1}{x}\right)}{x^2(4+\ln x)}$ or $\dfrac{d^2y}{dx^2}=-\dfrac{1}{4x}(4+\ln x)^{-\frac{3}{2}}\times\dfrac{1}{x}-\dfrac{1}{x^2}\times\dfrac{1}{2}(4+\ln x)^{-\frac{1}{2}}$ oe	M1
	$=\dfrac{\ldots}{4x^2(4+\ln x)^{\frac{3}{2}}}=-\dfrac{9+2\ln x}{4x^2(4+\ln x)^{\frac{3}{2}}}$ *	M1 A1*
		(5)
Alt(a)	$y^2=4+\ln x\Rightarrow 2y\dfrac{dy}{dx}=\dfrac{1}{x}$	M1 A1
	$\Rightarrow 2y\dfrac{d^2y}{dx^2}+2\left(\dfrac{dy}{dx}\right)^2=-\dfrac{1}{x^2}$	M1
	$\Rightarrow\dfrac{d^2y}{dx^2}=-\dfrac{1}{2yx^2}-\dfrac{2}{8x^2y^3}=\dfrac{-2y^2-1}{4x^2y^3}$	M1
	$=-\dfrac{9+2\ln x}{4x^2(4+\ln x)^{\frac{3}{2}}}$ *	A1*
		(5)
(b)	$y_{x=1}=2,\ \left.\dfrac{dy}{dx}\right\|_{x=1}=\dfrac{1}{4},\ \left.\dfrac{d^2y}{dx^2}\right\|_{x=1}=-\dfrac{9}{32}$	M1
	So $y=2+\dfrac{1}{4}(x-1)-\dfrac{1}{2!}\times\dfrac{9}{32}(x-1)^2+\ldots$	M1
	$=2+\dfrac{1}{4}(x-1)-\dfrac{9}{64}(x-1)^2+\ldots$	A1
		(3)
		(8 marks)

Notes:

(a)

M1: Attempts the derivative of y using the chain rule, look for $\dfrac{K}{x}(4+\ln x)^{-\frac{1}{2}}$ oe

A1: Correct derivative.

M1: Attempts the second derivative of y using the product or quotient rule and chain rule. Look for the correct form for their $\dfrac{dy}{dx}$ for the answer up to slips in coefficients.

M1: Attempts to simplify to get correct denominator. Must be correct work for their second derivative, but may have been errors in differentiating.

A1*: For a correct unsimplified second derivative, with no errors before reaching the given answer.

Note it is a given answer so needs a suitable intermediate line with at least the formation of the correct common denominator between two fractions before reaching the answer.

Alt:

M1: Squares and uses implicit differentiation to achieve $\alpha y\frac{dy}{dx}=\frac{\beta}{x}$

A1: Correct derivative.

M1: Differentiates again using implicit differentiation and product rule. Look for $\gamma y\frac{d^2y}{dx^2}+\delta\left(\frac{dy}{dx}\right)^2=\frac{\nu}{x^2}$

M1: Makes $\frac{d^2y}{dx^2}$ the subject and forms single fraction with denominator kx^2y^3

A1*: Obtains the correct second derivative, with no errors seen in working.

(b)

M1: Evaluates y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1$, if substitution is not seen, accept stated values for all three following attempts at the first and second derivatives as an attempt to find these.

M1: Applies Taylor's theorem with their values.

A1: Correct expression (don't be concerned if the $y =$ is missing.)

5(b) Alt	$y=\sqrt{4+\ln(1+(x-1))}=\sqrt{4+\left((x-1)-\frac{(x-1)^2}{2}+...\right)}$	**M1**
	$=4^{\frac{1}{2}}+\frac{1}{2}\times4^{-\frac{1}{2}}\times\left((x-1)-\frac{(x-1)^2}{2}\right)+\frac{\frac{1}{2}\times-\frac{1}{2}}{2!}\times4^{-\frac{3}{2}}\times((x-1)-...)^2+...$	**M1**
	$=2+\frac{1}{4}(x-1)-\frac{1}{8}(x-1)^2-\frac{1}{64}(x-1)^2+...=2+\frac{1}{4}(x-1)-\frac{9}{64}(x-1)^2+...$	**A1**
		(3)

Notes:

M1: Writes the x as $1+(x-1)$ and attempts to expand using the Maclaurin series for $\ln(1+x)$ with correct expansion of $\ln(1+(x-1))$.

M1: Attempts a binomial expansion using their ln expansion. Alternatively, may gain this before the first M if they expand using ln's, e.g. $4^{\frac{1}{2}}+\frac{1}{2}4^{-\frac{1}{2}}\ln x+\frac{\frac{1}{2}\times\frac{-1}{2}}{2!}(\ln x)^2$

A1: Fully correct expression (don't be concerned if the $y =$ is missing.)

Question	Scheme	Marks
6(a)	Let $x = \arctan A$ and $y = \arctan B$ then $\tan(x-y) = \dfrac{\tan x - \tan y}{1+\tan x \tan y}$ Or $\tan(\arctan A - \arctan B) = \dfrac{\tan\arctan A - \tan\arctan B}{1+\tan\arctan A \tan\arctan B}$	B1
	$\tan(x-y) = \dfrac{A-B}{1+AB} \Rightarrow x-y = \arctan\left(\dfrac{A-B}{1+AB}\right)$	M1
	So $\arctan A - \arctan B = x - y = \arctan\left(\dfrac{A-B}{1+AB}\right)$ *	A1*
		(3)
(b)	$A = r+2, B = r \Rightarrow \left(\dfrac{A-B}{1+AB} =\right)\dfrac{r+2-r}{1+(r+2)r} = \dfrac{2}{...}$	M1
	$= \dfrac{2}{r^2+2r+1} = \dfrac{2}{(1+r)^2}$ *	A1*
		(2)
(c)	$\sum_{r=1}^{n}\arctan\left(\dfrac{2}{(1+r)^2}\right) = \sum_{r=1}^{n}\left(\arctan(r+2) - \arctan(r)\right) = ...$	M1
	$= \left(\cancel{\arctan 3} - \arctan 1\right) + \left(\cancel{\arctan 4} - \arctan 2\right) + \left(\cancel{\arctan 5} - \cancel{\arctan 3}\right) + ...$ $+\left(\arctan(n+1) - \cancel{\arctan(n-1)}\right) + \left(\arctan(n+2) - \cancel{\arctan n}\right)$	A1
	$= \arctan(n+2) + \arctan(n+1) - \arctan 2 - \arctan 1$	M1
	$= \arctan(n+2) + \arctan(n+1) - \arctan 2 - \dfrac{\pi}{4}$	A1
		(4)
(d)	As $n \to \infty$, $\arctan n \to \dfrac{\pi}{2}$	M1
	So $\sum_{r=1}^{\infty}\arctan\left(\dfrac{2}{(1+r)^2}\right) = \dfrac{\pi}{2} + \dfrac{\pi}{2} - \arctan 2 - \dfrac{\pi}{4} = \dfrac{3\pi}{4} - \arctan 2$	A1
		(2)
		(11 marks)

Notes:
(a) **B1:** For any correct statement or use of the compound angle formula with **consistent variables** of x and y or arctan A and arctan B. Can be either way round (may be working in reverse). **M1:** Attempts to apply tan or arctan on an appropriate identity with either x and y or arctan A and arctan B. Should have $\dfrac{\tan x \pm \tan y}{1 \pm \tan x \tan y}$ (oe with arctans or A's and B's) as part of the identity, and allow if they change between x,y and arctan's during the step. **A1*:** Must have scored the B and M marks. Replaces tan x and tan y by A and B respectively if appropriate with fully correct work leading to the given result and conclusion made and no erroneous statements. **Note: for working in reverse e.g.** Let $x = \arctan A$ and $y = \arctan B$ then $\arctan A - \arctan B = \arctan\left(\dfrac{A-B}{1+AB}\right) \Leftrightarrow x - y = \arctan\left(\dfrac{A-B}{1+AB}\right) \Leftrightarrow \tan(x-y) = \dfrac{A-B}{1+AB}$ Scores M1 $\Leftrightarrow \tan(x-y) = \dfrac{\tan x - \tan y}{1+\tan x \tan y}$ Scores B1 – but enter as the first mark. Which is the correct identity for tan($x - y$) hence the result is true. Score A1 The conclusion here must include reference to the identity being true, e.g. with a tick, or statement, before deducing the final result. **(b)** **M1:** Substitutes in $A = r + 2$ and $B = r$ and simplifies the numerator to 2 (may be implied) **A1*:** Expands the denominator (must be seen) and then factorises to the given result, no errors seen. **(c)** **M1:** Applies the result of (a) to the series – allow if they have a different A and B due to error. **A1:** At least first three and final two brackets of terms correctly written out – must be clear enough to show cancelling. **M1:** Extracts the non-cancelling terms. **A1:** Correct result with no errors seen – must see the arctan 1 before reaching $\dfrac{\pi}{4}$. **Note:** Insufficient terms to gain the first A is not an error, so M1A0M1A1 is possible if e.g. only the first two terms are shown. Condone missing brackets on arctan n + 1 etc. **(d)** **M1:** Identifies the value arctan tends towards as n increase. Need not see limits, as long as the value is identified. **A1:** Correct answer.

Question	Scheme	Marks
7(a)	$z=(0+)\mathrm{i}y \Rightarrow w=\dfrac{(1+\mathrm{i})\mathrm{i}y+2(1-\mathrm{i})}{\mathrm{i}y-\mathrm{i}}=\dfrac{-y+2+\mathrm{i}(y-2)}{\mathrm{i}(y-1)}=\dfrac{y-2+\mathrm{i}(y-2)}{y-1}$	M1
	$\Rightarrow u=v$ or $\operatorname{Im} w=\operatorname{Re} w$	A1
		(2)
(b)	$w=\dfrac{(1+\mathrm{i})z+2(1-\mathrm{i})}{z-\mathrm{i}} \Rightarrow z=\dfrac{2(1-\mathrm{i})+\mathrm{i}w}{w-1-\mathrm{i}}=\dfrac{2-v+\mathrm{i}(u-2)}{u-1+\mathrm{i}(v-1)}$	M1
	$\dfrac{2-v+\mathrm{i}(u-2)}{u-1+\mathrm{i}(v-1)}\times\dfrac{u-1-\mathrm{i}(v-1)}{u-1-\mathrm{i}(v-1)}$ $=\dfrac{(2-v)(u-1)+(u-2)(v-1)+\mathrm{i}\big((u-1)(u-2)-(2-v)(v-1)\big)}{\ldots}$ $\operatorname{Im} z=0 \Rightarrow (u-1)(u-2)-(2-v)(v-1)=0$	M1
	$\Rightarrow (u-1)(u-2)-(2-v)(v-1)=0 \Rightarrow u^2-3u+2+v^2-3v+2=0$	A1
	$\Rightarrow \left(u-\dfrac{3}{2}\right)^2+\left(v-\dfrac{3}{2}\right)^2=\dfrac{1}{2}$	M1
	Centre is $\dfrac{3}{2}+\dfrac{3}{2}\mathrm{i}$ and radius is $\dfrac{\sqrt{2}}{2}$	A1A1
		(6)
		(8 marks)

Notes:

(a)

M1: Correct method to find the equation of the image line – e.g. substitutes in $z=\mathrm{i}y$ and rearranges to Cartesian form. May use $x+\mathrm{i}y$ and later set $x = 0$. Alternatively, may start as in (b) and then set $(2-v)(u-1)+(u-2)(v-1)=0 \Rightarrow 2u-v-uv-2+uv+2-2v-u=0$ etc.

Another alternative is to find the image points of two points on the imaginary axis and to find the line from these.

A1: For $u = v$ oe equation. Accept Im w = Re w, or $x = y$ if they have set $w = x + \mathrm{i}y$.

(b)

Note: Accept work done in part (a) that is relevant to part (b) for credit if appropriate.

M1: Makes z the subject, substitutes $w = u + \mathrm{i}v$ into the equation.

M1: Multiplies the numerator by the complex conjugate of denominator **and** extracts the imaginary part and sets it equal to zero to form an equation in u and v. Do not be concerned about the denominator.

A1: Correct equation in u and v for the circle.

M1: Completes the square on their equation to extract centre and radius. Not dependent, so allow as long as a suitable equation in u and v has been reached.

A1: Correct centre or correct radius. Accept either $\dfrac{3}{2}+\dfrac{3}{2}\mathrm{i}$ or $\left(\dfrac{3}{2},\dfrac{3}{2}\right)$ for the centre.

A1: Correct centre and correct radius. As above. Accept equivalent forms (need not be simplified)

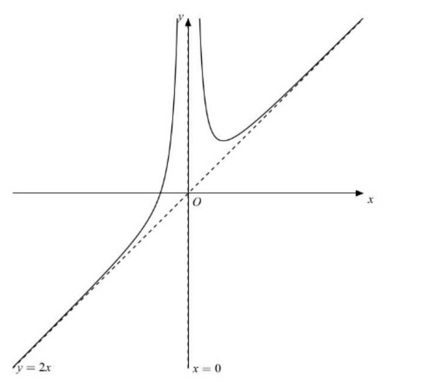
Allow the final two A marks if all that is wrong is an error in the denominator. (M1M0A0M1A1A1 is possible.)

7(b) **Alt1**	Real axis is $z = x\left(+0\text{i}\right)$, so $u+\text{i}v = \dfrac{(1+\text{i})x+2(1-\text{i})}{x-\text{i}} = \dfrac{(1+\text{i})x+2(1-\text{i})}{x-\text{i}} \times \dfrac{x+\text{i}}{x+\text{i}} =$ $\dfrac{(1+\text{i})x^2+2x(1-\text{i})+(\text{i}-1)x+2(\text{i}+1)}{x^2+1} = \dfrac{x^2+x+2+\text{i}\left(x^2-x+2\right)}{x^2+1}$	**M1**
	$u = \dfrac{x^2+x+2}{x^2+1} = 1+\dfrac{x+1}{x^2+1}; v = \dfrac{x^2-x+2}{x^2+1} = 1-\dfrac{x-1}{x^2+1} \Rightarrow u+v = 2+\dfrac{2}{x^2+1}$ $\Rightarrow (u-1)^2+(v-1)^2 = \dfrac{(x+1)^2+(x-1)^2}{\left(x^2+1\right)^2} = \dfrac{2x^2+2}{\left(x^2+1\right)^2} = \dfrac{2}{x^2+1} = u+v-2$	**M1** **A1**
	$\Rightarrow \left(u-\dfrac{3}{2}\right)^2+\left(v-\dfrac{3}{2}\right)^2 = \dfrac{1}{2}$	**M1**
	Centre is $\dfrac{3}{2}+\dfrac{3}{2}\text{i}$ and radius is $\dfrac{\sqrt{2}}{2}$	**A1A1**
		(6)

Notes

M1: Sets $z = x$ in the equation (or uses $x + \text{i}y$ and later sets $y = 0$) and multiplies by complex conjugate.
M1: Eliminates x from the equations (one suitable method is shown, others are possible).
A1: Correct equation in u and v for the circle.
M1: Completes the square on their equation to extract centre and radius
A1: Correct centre or correct radius. Accept either $\dfrac{3}{2}+\dfrac{3}{2}\text{i}$ or $\left(\dfrac{3}{2},\dfrac{3}{2}\right)$ for the centre.
A1: Correct centre and correct radius. As above.

7(b) **Alt 2**	Unlikely to be seen As i and $-\text{i}$ are inverse points of the line, so their images are inverse points of the circle. $\text{i} \to \infty,\ -\text{i} \to \dfrac{-\text{i}+1+2-2\text{i}}{-2\text{i}} = \dfrac{3}{2}+\dfrac{3}{2}\text{i}$ Hence (as ∞ is the other point) the centre is $\dfrac{3}{2}+\dfrac{3}{2}\text{i}$ $0 \to \dfrac{2-2\text{i}}{-\text{i}} = 2+2\text{i}$ So radius is $\left\lvert\dfrac{3}{2}+\dfrac{3}{2}\text{i}-2-2\text{i}\right\rvert = \ldots$ $= \dfrac{\sqrt{2}}{2}$	**M1** **M1** **A1** **M1** **A1** **A1**
(b) Alt 3	M1: Attempt to find images of three different points on the real axis. M1: Correct method to find centre from three points – e.g. intersection of two perpendicular bisectors. A1: Correct equation for the centre. M1: Uses centre and one point to find radius. A1: Correct centre A1: Correct radius	

Question	Scheme	Marks
8(a)	$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} - 2$	**B1**
	$\frac{\mathrm{d}y}{\mathrm{d}x} + 2yx(y-4x) = 2 - 8x^3 \to \frac{\mathrm{d}v}{\mathrm{d}x} + 2 + 2(v+2x)x(v+2x-4x) = 2 - 8x^3$	
	$\to \frac{\mathrm{d}v}{\mathrm{d}x} + 2 + 2x(v^2 - 4x^2) = 2 - 8x^3$	**M1**
	$\to \frac{\mathrm{d}v}{\mathrm{d}x} = -2xv^2 \,*$	**A1***
		(4)
(b)	$\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = -2x \Rightarrow \int v^{-2}\,\mathrm{d}v = -2\int x\,\mathrm{d}x$	**B1**
	$\Rightarrow \frac{v^{-1}}{-1} = -2\frac{x^2}{2}(+c)$	**M1**
	$\Rightarrow \frac{1}{v} = x^2 + c$	**A1**
	$\Rightarrow v = \frac{1}{x^2 + c}$	**A1**
		(4)
(c)	$y = 2x + \frac{1}{x^2 + c}$	**B1ft**
		(1)
(d)	$-1 = 2 \times -1 + \frac{1}{1+c} \Rightarrow c = \ldots$	**M1**
	$y = 2x + \frac{1}{x^2}$	**A1**
	 Attempts the sketch for their equation, with at least one of - One branch correct - Vertical asymptote for their equation - Long term behaviour tends to infinity - Minimum in quadrant 1	**M1**
	Fully correct shape, two branches tending to infinity as x tends to infinity both directions, with minimum in first quadrant No need for oblique asymptote marked.	**A1**
	y-axis a vertical asymptote labelled	**B1ft**
		(5)
		(14 marks)

Notes:

(a)

B1: Correct differentiation of the given transformation. Allow any correct connecting derivative, e.g. $\frac{\mathrm{d}y}{\mathrm{d}v} = 1 + 2\frac{\mathrm{d}x}{\mathrm{d}v}$ or $\frac{\mathrm{d}v}{\mathrm{d}y} = 1 - 2\frac{\mathrm{d}x}{\mathrm{d}y}$

M1: For a complete substitution into the equation (*I*).

M1: Applies difference of squares, or completely expands brackets of the left hand side. Alternatively, may rearrange and factorise to give $8x^2y - 2xy^2 - 8x^3 = -2x\left(y^2 - 4xy + 4x^2\right) = -2x(y-2x)^2$ before substituting.

A1*: Reaches the given answer with no errors seen.

(b)

B1: Correct separation of the variables.

M1: Attempts the integration, usual rule, power increased by 1 on at least one term. No need for + *c* for the method.

A1: Correct integration including the + *c*

A1: Correct expression for *v*.

(c)

B1: Follow through their answer to (b), so $y = 2x +$ their v from (b)

(d)

M1: Uses the point $(-1, -1)$ to find a value for the constant in their equation. Must have had a constant of integration in their equation to score this mark.

A1: Correct equation for *y* following a correct general solution. Withhold this mark for $y = 2x + \frac{1}{x^2} + c$ leading to the correct equation.

Note: the following three marks may be scored from a correct equation that arose from having no constant in (b) or from $y = 2x + \frac{1}{x^2} + c$ (which gives the same equation).

M1: Attempts a sketch for their curve. See scheme. Look for at least one of the key features for their equation shown.

A1: Correct shape, two branches tending to infinity as *x* tends to infinity both directions with a minimum in first quadrant. Not a follow through mark, so must be the correct curve.

B1ft: Correct vertical asymptote at $x = 0$. Need not be labelled if it is clearly the *y*-axis. Follow through their equation as long as there is at least one vertical asymptote (ie for a negative *c* they need a pair of asymptotes symmetric about the *y*-axis).