Further Pure Mathematics FP1 Mark scheme

Question	Scheme		Marks
1	$\sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$		
	$= \frac{1}{4}n^2(n+1)^2 - 3\left(\frac{1}{2}n(n+1)\right)$	Attempts to expand $r(r^2-3)$ and attempts to substitute at least one correct standard formula into their resulting expression.	M1
		Correct expression (or equivalent)	A1
	$= \frac{1}{4}n(n+1)[n(n+1)-6]$	**dependent on the previous M mark** Attempt to factorise at least $n(n+1)$ having attempted to substitute both the standard formulae	dM1
	$= \frac{1}{4}n(n+1)[n^2+n-6]$	{this step does not have to be written]	
	$= \frac{1}{4}n(n+1)(n+3)(n-2)$	Correct completion with no errors	A1 cso
			(4)
			(4 marks)

Notes:

Applying eg. $n=1, n=2, n=3$ to the printed equation without applying the standard formulae to give $a=1, b=3, c=-2$ or another combination of these numbers is M0A0M0A0.

<u>Alternative Method</u>:

Obtains $\sum_{r=1}^{n} r(r^2-3) \equiv \frac{1}{4}n(n+1)[n(n+1)-6] \equiv \frac{1}{4}n(n+a)(n+b)(n+c)$

So $a=1$. $n=1 \Rightarrow -2 = \frac{1}{4}(1)(2)(1+b)(1+c)$ and $n=2 \Rightarrow 0 = \frac{1}{4}(2)(3)(2+b)(2+c)$

leading to either $b=-2, c=3$ or $b=3, c=-2$

dM1: dependent on the previous M mark.
Substitutes in values of n and solves to find $b=...$ and $c=...$

A1: Finds $a=1, b=3, c=-2$ or another combination of these numbers.

Using **only** a method of "proof by induction" scores 0 marks unless there is use of the standard formulae when the first M1 may be scored.

Allow final dM1A1 for $\frac{1}{4}n^4+\frac{1}{2}n^3-\frac{5}{4}n^2-\frac{3}{2}n$ or $\frac{1}{4}n(n^3+2n^2-5n-6)$

or $\frac{1}{4}(n^4+2n^3-5n^2-6n) \rightarrow \frac{1}{4}n(n+1)(n+3)(n-2)$, from no incorrect working.

Give final A0 for eg. $\frac{1}{4}n(n+1)[n^2+n-6] \rightarrow = \frac{1}{4}n(n+1)(x+3)(x-2)$ unless recovered.

Question	Scheme		Marks
2(a)	$P: y^2 = 28x$ or $P(7t^2, 14t)$		
	$(y^2 = 4ax \Rightarrow a = 7) \Rightarrow S(7,0)$	Accept $(7,0)$ or $x = 7,\ y = 0$ or 7 marked on the x-axis in a sketch	B1
			(1)
(b)	$\{A$ and B have x coordinate$\}$ $\frac{7}{2}$	Divides their x coordinate from (a) by 2 **and** substitutes this into the parabola equation and takes the sqaure root to find $y = ...$ **or applies** $y = \sqrt{\left(2(\text{"}7\text{"}) - \left(\frac{\text{"}7\text{"}}{2}\right)\right)^2 - \left(\frac{\text{"}7\text{"}}{2}\right)^2}$ **or solves** $7t^2 = 3.5$ and finds $y = 2(7)\text{"their } t\text{"}$	M1
	So $y^2 = 28\left(\frac{7}{2}\right) \Rightarrow y^2 = 98 \Rightarrow y = ...$ **or** $y = \sqrt{(2(7) - 3.5)^2 - (3.5)^2}$ $\left\{= \sqrt{(10.5)^2 - (3.5)^2}\right\}$ **or** $7t^2 = 3.5 \Rightarrow t = \sqrt{0.5} \Rightarrow y = 2(7)\sqrt{0.5}$		
	$y = (\pm)7\sqrt{2}$	***At least one*** correct **exact** value of y. Can be un-simplified or simplified.	A1
	A, B have coordinates $\left(\frac{7}{2}, 7\sqrt{2}\right)$ and $\left(\frac{7}{2}, -7\sqrt{2}\right)$		
	Area triangle ABS =		
	• $\frac{1}{2}\left(2(7\sqrt{2})\right)\left(\frac{7}{2}\right)$ • $\frac{1}{2}\begin{vmatrix} 7 & 3.5 & 3.5 & 7 \\ 0 & 7\sqrt{2} & -7\sqrt{2} & 0 \end{vmatrix}$	**dependent on the previous M mark** A **full** method for finding the area of triangle ABS.	dM1
	$= \frac{49}{2}\sqrt{2}$	Correct exact answer.	A1
			(4)
			(5 marks)

Question 2 *continued*
Notes:
(a) You can give B1 for part (a) for correct relevant work seen in either part (a) or part (b).
(b) **1st M1:** Allow a slip when candidates find the x coordinate of their midpoint as long as $0 <$ their midpoint $<$ their a Give 1st M0 if a candidate finds and uses $y = 98$ **1st A1:** Allow any **exact value** of either $7\sqrt{2}, -7\sqrt{2}, \sqrt{98}, -\sqrt{98}, 14\sqrt{0.5}$, awrt 9.9 or awrt -9.9 **2nd dM1:** Either $\frac{1}{2}\left(2\times \text{their "}7\sqrt{2}\text{"}\right)\left(\text{their } x_{\text{midpoint}}\right)$ or $\frac{1}{2}\left(2\times \text{their "}7\sqrt{2}\text{"}\right)\left(\text{their "7"} - x_{\text{midpoint}}\right)$ Condone area triangle $ABS = \left(7\sqrt{2}\right)\left(\frac{7}{2}\right)$, i.e. $\left(\text{their "}7\sqrt{2}\text{"}\right)\left(\frac{\text{their "7"}}{2}\right)$ **2nd A1:** Allow exact answers such as $\frac{49}{2}\sqrt{2}, \frac{49}{\sqrt{2}}, 24.5\sqrt{2}, \frac{\sqrt{4802}}{2}, \sqrt{\frac{4802}{4}}, 3.5\sqrt{2}, 49\sqrt{\frac{1}{2}}$ or $\frac{7}{2}\sqrt{98}$ but do not allow $\frac{1}{2}(3.5)\left(2\sqrt{98}\right)$ seen by itself. Give final A0 for finding 34.64823228… without reference to a correct exact value.

Question	Scheme		Marks
3(a)	$f(x) = x^2 + \frac{3}{x} - 1, \quad x < 0$		
	$f'(x) = 2x - 3x^{-2}$	At one of either $x^2 \to \pm Ax$ or $\frac{3}{x} \to \pm Bx^{-2}$ where A and B are non-zero constants.	M1
		Correct differentiation	A1
	$f(-1.5) = -0.75$, $f'(-1.5) = -\frac{13}{3}$	Either $f(-1.5) = -0.75$ or $f'(-1.5) = -\frac{13}{3}$ or awrt -4.33 **or a correct numerical expression** for either $f(-1.5)$ or $f'(-1.5)$ **Can be implied by later working**	B1
	$\left\{\alpha \simeq -1.5 - \frac{f(-1.5)}{f'(-1.5)}\right\} \Rightarrow \alpha \simeq -1.5 - \frac{-0.75}{-4.333333...}$	**dependent on the previous M mark** Valid attempt at Newton-Raphson using their values of $f(-1.5)$ and $f'(-1.5)$	dM1
	$\left\{\alpha = -1.67307692... \text{ or } -\frac{87}{52}\right\} \Rightarrow \alpha = -1.67$	**dependent on all 4 previous marks** -1.67 on their first iteration (Ignore any subsequent iterations)	A1 cso cao
	Correct differentiation followed by a correct answer scores full marks in (a) **Correct answer with <u>no</u> working scores no marks in (a)**		
			(5)
(b)	**Way 1**		
	$f(-1.675) = 0.01458022...$ $f(-1.665) = -0.0295768...$	Chooses a suitable interval for x, which is within ± 0.005 of their answer to (a) and at least one attempt to evaluate $f(x)$.	M1
	Sign change (positive, negative) (and $f(x)$ is continuous) therefore (a root) $\alpha = -1.67$ (2 dp)	Both values correct awrt (or truncated) 1 sf, sign change and conclusion.	A1 cso
			(2)

Question	Scheme		Marks
3(b) *continued*	**Way 2**		
	Alt 1: Applying Newton-Raphson again Eg. Using $\alpha = -1.67, -1.673$ or $-\frac{87}{52}$		
	• $\alpha \simeq -1.67 - \frac{-0.007507185629...}{-4.415692926...}$ $\{= -1.671700115...\}$ • $\alpha \simeq -1.673 - \frac{0.005743106396...}{-4.41783855...}$ $\{= -1.671700019...\}$ • $\alpha \simeq -\frac{87}{52} - \frac{0.006082942257...}{-4.417893838...}$ $\{= -1.67170036...\}$	Evidence of applying Newton-Raphson for a second time on their answer to part (a)	M1
	So $\alpha = -1.67$ (2 dp)	$\alpha = -1.67$	A1
			(2)
			(7 marks)

Notes:

(a)

Incorrect differentiation followed by their estimate of α with no evidence of applying the NR formula is final dM0A0.

B1: B1 can be given for a correct numerical expression for either $f(-1.5)$ or $f'(-1.5)$

Eg. either $(-1.5)^2 + \frac{3}{(-1.5)} - 1$ or $2(-1.5) - \frac{3}{(-1.5)^2}$ are fine for B1.

Final -This mark can be implied by applying at least one correct value of either $f(-1.5)$ or $f'(-1.5)$

dM1: in $-1.5 - \frac{f(-1.5)}{f'(-1.5)}$. So just $-1.5 - \frac{f(-1.5)}{f'(-1.5)}$ with an incorrect answer and no other evidence scores final dM0A0.

Give final dM0 for applying $1.5 - \frac{f(-1.5)}{f'(-1.5)}$ without first quoting the correct N-R formula.

(b)

A1: **Way 1:** correct solution only

Candidate needs to state both of their values for $f(x)$ to awrt (or truncated) 1 sf along with a reason and conclusion. Reference to change of sign or eg. $f(-1.675) \times f(-1.665) < 0$ or a diagram or < 0 and > 0 or one positive, one negative are sufficient reasons. There must be a (minimal, not incorrect) conclusion, eg. $\alpha = -1.67$, root (or α or part (a)) is correct, QED and a square are all acceptable. Ignore the presence or absence of any reference to continuity.

A minimal acceptable reason and conclusion is "change of sign, hence root".

No explicit reference to 2 decimal places is required.

Stating "root is in between – 1.675 and – 1.665" without some reference to is not sufficient for A1

Accept 0.015 as a correct evaluation of $f(-1.675)$

Question 3 notes *continued*

(b)

A1: **Way 2:** correct solution only

Their conclusion in Way 2 needs to convey that they understand that $\alpha = -1.67$ to 2 decimal places. Eg. "therefore my answer to part (a) [which must be -1.67] is correct" is fine for A1.

$-1.67 - \dfrac{\mathrm{f}(-1.67)}{\mathrm{f}'(1.67)} = -1.67$ (2 dp) is sufficient for M1A1 in part (b).

The root of $\mathrm{f}(x) = 0$ is $-1.67169988...$, so candidates can also choose x_1 which is less than $-1.67169988...$ and choose x_2 which is greater than $-1.67169988...$ with both x_1 and x_2 lying in the interval $[-1.675, -1.665]$ and evaluate $\mathrm{f}(x_1)$ and $\mathrm{f}(x_2)$.

Helpful Table

x	$\mathrm{f}(x)$
-1.675	0.014580224
-1.674	0.010161305
-1.673	0.005743106
-1.672	0.001325627
-1.671	-0.003091136
-1.670	-0.007507186
-1.669	-0.011922523
-1.668	-0.016337151
-1.667	-0.020751072
-1.666	-0.025164288
-1.665	-0.029576802

Question	Scheme		Marks
4(a)	$\mathbf{A} = \begin{pmatrix} k & 3 \\ -1 & k+2 \end{pmatrix}$ where k is a constant and let $g(k) = k^2 + 2k + 3$		
	$\{\det(\mathbf{A}) =\}\ k(k+2)+3$ or k^2+2k+3	Correct $\det(\mathbf{A})$, un-simplified or simplified	B1
	Way 1		
	$= (k+1)^2 - 1 + 3$	Attempts to complete the square [usual rules apply]	M1
	$= (k+1)^2 + 2 > 0$	$(k+1)^2 + 2$ and > 0	A1 cso
			(3)
	Way 2		
	$\{\det(\mathbf{A}) =\}\ k(k+2)+3$ or k^2+2k+3	Correct $\det(\mathbf{A})$, un-simplified or simplified	B1
	$\{b^2 - 4ac =\}\ 2^2 - 4(1)(3)$	Applies "$b^2 - 4ac$" to their $\det(\mathbf{A})$	M1
	All of • $b^2 - 4ac = -8 < 0$ • some reference to k^2+2k+3 being above the x-axis • so $\det(\mathbf{A}) > 0$	Complete solution	A1 cso
			(3)
	Way 3		
	$\{g(k) = \det(\mathbf{A}) =\}\ k(k+2)+3$ or k^2+2k+3	Correct $\det(\mathbf{A})$, un-simplified or simplified	B1
	$g'(k) = 2k+2 = 0 \Rightarrow k = -1$ $g_{\min} = (-1)^2 + 2(-1) + 3$	Finds the value of k for which $g'(k) = 0$ **and** substitutes this value of k into $g(k)$	M1
	$g_{\min} = 2$, so $\det(\mathbf{A}) > 0$	$g_{\min} = 2$ and states $\det(\mathbf{A}) > 0$	A1 cso
			(3)
(b)	$\mathbf{A}^{-1} = \dfrac{1}{k^2+2k+3}\begin{pmatrix} k+2 & -3 \\ 1 & k \end{pmatrix}$	$\dfrac{1}{\text{their } \det(\mathbf{A})}\begin{pmatrix} k+2 & -3 \\ 1 & k \end{pmatrix}$	M1
		Correct answer in terms of k	A1
			(2)
			(5 marks)

Question 4 *continued*
Notes:
(a) **B1:** Also allow $k(k+2)--3$ **Way 2:** Proving $b^2-4ac=-8<0$ by itself could mean that $\det(\mathbf{A})>0$ or $\det(\mathbf{A})<0$. To gain the final A1 mark for Way 2, candidates need to show $b^2-4ac=-8<0$ **and** make some reference to k^2+2k+3 being above the x-axis (eg. states that coefficient of k^2 is positive **or** evaluates $\det(\mathbf{A})$ for any value of k to give a positive result **or** sketches a quadratic curve that is above the x-axis) before then stating that $\det(\mathbf{A})>0$. Attempting to solve $\det(\mathbf{A})=0$ by applying the quadratic formula or finding $-1\pm\sqrt{2}\mathrm{i}$ is enough to score the M1 mark for Way 2. To gain A1 these candidates need to make some reference to k^2+2k+3 being above the x-axis (eg. states that coefficient of k^2 is positive **or** evaluates $\det(\mathbf{A})$ for any value of k to give a positive result **or** sketches a quadratic curve that is above the x-axis) before then stating that $\det(\mathbf{A})>0$.
(b) **A1:** Allow either $\frac{1}{(k+1)^2+2}\begin{pmatrix} k+2 & -3 \\ 1 & k \end{pmatrix}$ or $\begin{pmatrix} \frac{k+2}{k^2+2k+3} & \frac{-3}{k^2+2k+3} \\ \frac{1}{k^2+2k+3} & \frac{k}{k^2+2k+3} \end{pmatrix}$ or equivalent.

Question	Scheme		Marks
5	$2z + z^* = \dfrac{3+4\text{i}}{7+\text{i}}$		
	Way 1		
	$\{2z + z^* =\}\ 2(a+\text{i}b) + (a-\text{i}b)$	Left hand side $= 2(a+\text{i}b) + (a-\text{i}b)$ Can be implied by eg. $3a + \text{i}b$ **Note:** This can be seen anywhere in their solution	B1
	$\text{..........} = \dfrac{(3+4\text{i})}{(7+\text{i})}\dfrac{(7-\text{i})}{(7-\text{i})}$	Multiplies numerator **and** denominator of the right hand side by $7-\text{i}$ or $-7+\text{i}$	M1
	$\text{..........} = \dfrac{25+25\text{i}}{50}$	Applies $\text{i}^2 = -1$ to and collects like terms to give right hand side $= \dfrac{25+25\text{i}}{50}$ or equivalent	A1
	So, $3a + \text{i}b = \dfrac{1}{2} + \dfrac{1}{2}\text{i}$ $\Rightarrow a = \dfrac{1}{6}, b = \dfrac{1}{2}$ **or** $z = \dfrac{1}{6} + \dfrac{1}{2}\text{i}$	**dependent on the previous B and M marks** Equates either real parts or imaginary parts to give at least one of $a = ...$ or $b = ...$	ddM1
		Either $a = \dfrac{1}{6}$ **and** $b = \dfrac{1}{2}$ **or** $z = \dfrac{1}{6} + \dfrac{1}{2}\text{i}$	A1
			(5)
	Way 2		
	$\{2z + z^* =\}\ 2(a+\text{i}b) + (a-\text{i}b)$	Left hand side $= 2(a+\text{i}b) + (a-\text{i}b)$ Can be implied by eg. $3a + \text{i}b$	B1
	$(3a + \text{i}b)(7+\text{i}) = \text{..........}$	Multiplies their $(3a + \text{i}b)$ by $(7+\text{i})$	M1
	$21a + 3a\text{i} + 7b\text{i} - b = \text{..........}$	Applies $\text{i}^2 = -1$ to give left hand side $= 21a + 3a\text{i} + 7b\text{i} - b$	A1
	So, $(21a - b) + (3a + 7b) = 3 + 4\text{i}$ gives $21a - b = 3$, $3a + 7b = 4$ $\Rightarrow a = \dfrac{1}{6}, b = \dfrac{1}{2}$ **or** $z = \dfrac{1}{6} + \dfrac{1}{2}\text{i}$	**dependent on the previous B and M marks** Equates **both** real parts and imaginary parts to give at least one of $a = ...$ or $b = ...$	ddM1
		Either $a = \dfrac{1}{6}$ **and** $b = \dfrac{1}{2}$ **or** $z = \dfrac{1}{6} + \dfrac{1}{2}\text{i}$	A1
			(5)
			(5 marks)

Question 5 *continued*
Notes:
Some candidates may let $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$. So apply the mark scheme with $x \equiv a$ and $y \equiv b$. For the final A1 mark, you can accept exact equivalents for a, b.

Question	Scheme		Marks
6(a)	$H: xy = 25$, $P\left(5t, \frac{5}{t}\right)$ is a general point on H		
	Either $5t\left(\frac{5}{t}\right) = 25$ **or** $y = \frac{25}{x} = \frac{25}{5t} = \frac{5}{t}$ **or** $x = \frac{25}{y} = \frac{25}{\frac{5}{t}} = 5t$ **or** states $c = 5$		B1
			(1)
(b)	$y = \frac{25}{x} = 25x^{-1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -25x^{-2} = -\frac{25}{x^2}$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm k\,x^{-2}$ where k is a numerical value	M1
	$xy = 25 \Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$	Correct use of product rule. The sum of two terms, one of which is correct.	
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{5}{t^2}\left(\frac{1}{5}\right)$	$\frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{1}{\text{their } \frac{\mathrm{d}x}{\mathrm{d}t}}$	
	$\left\{\text{At } A,\ t = \frac{1}{2},\ x = \frac{5}{2},\ y = 10\right\} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -4$	Correct numerical gradient at A, which is found using calculus. **Can be implied by later working**	A1
	So, $m_N = \frac{1}{4}$	Applies $m_N = \frac{-1}{m_T}$, to find a numerical m_N, where m_T is found from using calculus. **Can be implied by later working**	M1
	• $y - 10 = \frac{1}{4}\left(x - \frac{5}{2}\right)$ • $10 = \frac{1}{4}\left(\frac{5}{2}\right) + c \Rightarrow c = \frac{75}{8} \Rightarrow y = \frac{1}{4}x + \frac{75}{8}$	Correct line method for a **normal** where a numerical $m_N\,(\neq m_T)$ is found from using calculus. **Can be implied by later working**	M1
	leading to $8y - 2x - 75 = 0$ (*)	Correct solution only	A1
			(5)

Question	Scheme		Marks
6(c)	$y=\frac{25}{x} \Rightarrow 8\left(\frac{25}{x}\right)-2x-75=0$ **or** $x=\frac{25}{y} \Rightarrow 8y-2\left(\frac{25}{y}\right)-75=0$ **or** $x=5t,\ y=\frac{5}{t} \Rightarrow 8(5t)-2\left(\frac{5}{t}\right)-75=0$ Substitutes $y=\frac{25}{x}$ **or** $x=\frac{25}{y}$ **or** $x=5t$ **and** $y=\frac{5}{t}$ into the printed equation or their normal equation to obtain an equation in either x only, y only or t only		M1
	$2x^2+75x-200=0$ **or** $8y^2-75y-50=0$ **or** $2t^2+15t-8=0$ **or** $10t^2+75t-40=0$		
	$(2x-5)(x+40)=0 \Rightarrow x=...$ **or** $(y-10)(8y+5)=0 \Rightarrow y=...$ **or** $(2t-1)(t+8)=0 \Rightarrow t=...$ **dependent on the previous M mark** Correct attempt of solving a 3TQ to find either $x=...,\ y=...$ or $t=...$		dM1
	Finds at least one of either $x=-40$ or $y=-\frac{5}{8}$		A1
	$B\left(-40,\ -\frac{5}{8}\right)$	Both correct coordinates (If coordinates are not stated they can be paired together as $x=...,\ y=...$)	A1
			(4)
			(10 marks)

Notes:

(a) A conclusion is not required on this occasion in part (a).

B1: Condone reference to $c=5$ (as $xy=c^2$ and $\left(ct,\frac{c}{t}\right)$ are referred in the Formula book.)

(b)

$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=-\frac{5}{t^2}\left(\frac{1}{5}\right)=-\frac{1}{t^2} \Rightarrow m_N=t^2 \Rightarrow y-10=t^2\left(x-\frac{5}{2}\right)$ **scores only the first M1.**

When $t=\frac{1}{2}$ is substituted giving $y-10=\frac{1}{4}\left(x-\frac{5}{2}\right)$ the response then automatically gets A1(implied) M1(implied) M1

Question 6 notes *continued*
(c) You can imply the final three marks (dM1A1A1) for either • $8\left(\frac{25}{x}\right) - 2x - 75 = 0 \rightarrow \left(-40, -\frac{5}{8}\right)$ • $8y - 2\left(\frac{25}{y}\right) - 75 = 0 \rightarrow \left(-40, -\frac{5}{8}\right)$ • $8(5t) - 2\left(\frac{5}{t}\right) - 75 = 0 \rightarrow \left(-40, -\frac{5}{8}\right)$ with no intermediate working. You can also imply the middle dM1A1 marks for either • $8\left(\frac{25}{x}\right) - 2x - 75 = 0 \rightarrow x = -40$ • $8y - 2\left(\frac{25}{y}\right) - 75 = 0 \rightarrow y = -\frac{5}{8}$ • $8(5t) - 2\left(\frac{5}{t}\right) - 75 = 0 \rightarrow x = -40$ or $y = -\frac{5}{8}$ with no intermediate working. **Writing** $x = -40,\ y = -\frac{5}{8}$ followed by $B\left(40, \frac{5}{8}\right)$ or $B\left(-\frac{5}{8}, -40\right)$ is final A0. Ignore stating $B\left(\frac{5}{2}, 10\right)$ in addition to $B\left(-40, -\frac{5}{8}\right)$

Question	Scheme		Marks
7(a)	Rotation	Rotation	B1
	67 degrees (anticlockwise)	Either $\arctan\left(\frac{12}{5}\right)$, $\tan^{-1}\left(\frac{12}{5}\right)$, $\sin^{-1}\left(\frac{12}{13}\right)$, $\cos^{-1}\left(\frac{5}{13}\right)$, awrt 67 degrees, awrt 1.2, truncated 1.1 (anticlockwise), awrt 293 degrees clockwise or awrt 5.1 clockwise	B1 o.e.
	about $(0, 0)$	**The mark is dependent on at least one of the previous B marks being awarded.** About $(0, 0)$ or about O or about the origin	dB1
	Note: Give 2nd B0 for 67 degrees clockwise o.e.		**(3)**
(b)	$\{\mathbf{Q} =\}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Correct matrix	B1
			(1)
(c)	$\{\mathbf{R} = \mathbf{PQ}\}\begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$	Multiplies **P** by their **Q** in the correct order and finds at least one element	M1
		Correct matrix	A1
			(2)
(d)	**Way 1**		
	$\begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}\begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} x \\ kx \end{pmatrix}$	Forming the equation "their matrix $\mathbf{R}$" $\begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} x \\ kx \end{pmatrix}$ **Allow x being replaced by any non-zero number eg. 1.** Can be implied by at least one correct ft equations below.	M1
	$-\frac{12}{13}x + \frac{5kx}{13} = x$ or $\frac{5}{13}x + \frac{12kx}{13} = kx \Rightarrow k = ...$	Uses their matrix equation to form an equation in k and progresses to give $k =$ numerical value	M1
	So $k = 5$	**dependent on only the previous M mark** $k = 5$	A1 **cao**
	Dependent on all previous marks being scored in this part. Either • Solves **both** $-\frac{12}{13}x + \frac{5kx}{13} = x$ **and** $\frac{5}{13}x + \frac{12kx}{13} = kx$ to give $k = 5$ • Finds $k = 5$ and checks that it is true for the other component • Confirms that $\begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}\begin{pmatrix} x \\ 5x \end{pmatrix} = \begin{pmatrix} x \\ 5x \end{pmatrix}$		A1 **cso**
			(4)

Question	Scheme		Marks
7(d) *continued*	**Way 2**		
	Either $\cos 2\theta = -\frac{12}{13}, \sin 2\theta = \frac{5}{13}$ or $\tan 2\theta = -\frac{5}{12}$	Correct follow through equation in 2θ based on their matrix $\mathbf{R}$	M1
		Full method of finding 2θ, then θ and applying $\tan\theta$	M1
	$\{k =\} \tan\left(\frac{1}{2}\arccos\left(-\frac{12}{13}\right)\right)$	$\tan\left(\frac{1}{2}\arccos\left(-\frac{12}{13}\right)\right)$ or $\tan(\text{awrt } 78.7^\circ)$ or $\tan(\text{awrt } 1.37)$. **Can be implied.**	A1
	So $k = 5$	$k = 5$ **by a correct solution only**	A1
			(4)
			(10 marks)

Notes:

(a)

Condone "Turn" for the 1st B1 mark.

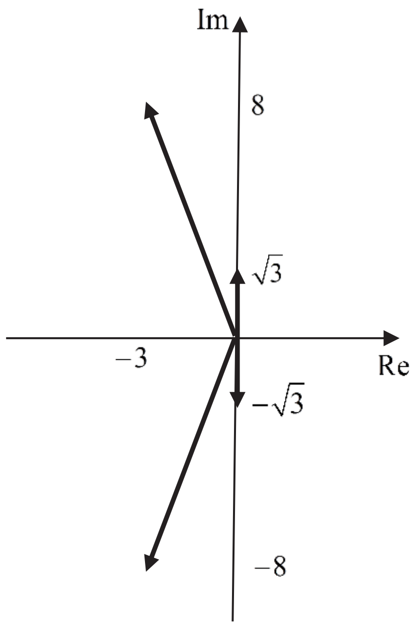
Penalise the first B1 mark for candidates giving a combination of transformations.

(c)

Allow 1st M1 for eg. $\text{"their matrix } \mathbf{R}\text{"}\begin{pmatrix}1\\k\end{pmatrix} = \begin{pmatrix}1\\k\end{pmatrix}$ or $\text{"their matrix } \mathbf{R}\text{"}\begin{pmatrix}k\\k^2\end{pmatrix} = \begin{pmatrix}k\\k^2\end{pmatrix}$

or $\text{"their matrix } \mathbf{R}\text{"}\begin{pmatrix}\frac{1}{k}\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{k}\\1\end{pmatrix}$ or equivalent

$$y = (\tan\theta)x : \begin{pmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{pmatrix} = \begin{pmatrix}-\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13}\end{pmatrix}$$

Question	Scheme		Marks
8(a)	$\mathrm{f}(z) = z^4 + 6z^3 + 76z^2 + az + b$, a, b are real constants. $z_1 = -3 + 8\mathrm{i}$ is given.		
	$-3 - 8\mathrm{i}$	$-3 - 8\mathrm{i}$	B1
			(1)
(b)	$z^2 + 6z + 73$	Attempt to expand $(z - (-3+8\mathrm{i}))(z - (-3-8\mathrm{i}))$ or any valid method ***to establish a quadratic factor*** eg $z = -3 \pm 8\mathrm{i} \Rightarrow z + 3 = \pm 8\mathrm{i} \Rightarrow z^2 + 6z + 9 = -64$ or sum of roots -6, product of roots 73 to give $z^2 \pm (\text{sum})z + \text{product}$	M1
		$z^2 + 6z + 73$	A1
	$\mathrm{f}(z) = (z^2 + 6z + 73)(z^2 + 3)$	Attempts to find the other quadratic factor. eg. using long division to get as far as $z^2 + ...$ or eg. $\mathrm{f}(z) = (z^2 + 6z + 73)(z^2 + ...)$	M1
		$z^2 + 3$	A1
	$\left\{z^2 + 3 = 0 \Rightarrow z = \right\}$ $\pm\sqrt{3}\,\mathrm{i}$	**dependent on only the previous M mark** Correct method of solving the 2nd quadratic factor	dM1
		$\sqrt{3}\,\mathrm{i}$ **and** $-\sqrt{3}\,\mathrm{i}$	A1
			(6)
(c)		<u>**Criteria**</u> • $-3 \pm 8\mathrm{i}$ plotted correctly in quadrants 2 and 3 with some evidence of symmetry • Their other two ***complex roots*** (which are found from solving their 2nd quadratic in (b)) are plotted correctly with some evidence of symmetry about the x-axis	
		Satisfies at least one of the two criteria	B1 ft
		Satisfies both criteria with some indication of scale or coordinates stated. All points (arrows) must be in the correct positions relative to each other.	B1 ft
			(2)
			(9 marks)

Question 8 *continued*
Notes:
(b) Give 3rd M1 for $z^2 + k = 0,\ k > 0 \Rightarrow$ **at least one of either** $z = \sqrt{k}\,\mathrm{i}$ **or** $z = -\sqrt{k}\,\mathrm{i}$ Give 3rd M0 for $z^2 + k = 0,\ k > 0 \Rightarrow z = \pm k\mathrm{i}$ Give 3rd M0 for $z^2 + k = 0,\ k > 0 \Rightarrow z = \pm k$ or $z = \pm\sqrt{k}$ Candidates do not need to find $a = 18,\ b = 219$

Question	Scheme		Marks
9(a)	$2x^2+4x-3=0$ has roots α, β		
	$\alpha+\beta=-\frac{4}{2}$ or -2, $\alpha\beta=-\frac{3}{2}$	**Both** $\alpha+\beta=-\frac{4}{2}$ **and** $\alpha\beta=-\frac{3}{2}$. This may be seen or implied anywhere in this question.	B1
(i)	$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=......$	**Use** of a **correct** identity for $\alpha^2+\beta^2$ (May be implied by their work)	M1
	$=(-2)^2-2\left(-\frac{3}{2}\right)=7$	7 **from correct working**	A1 cso
(ii)	$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=......$ or $=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)=......$	**Use** of an **appropriate** and **correct** identity for $\alpha^3+\beta^3$ (May be implied by their work)	M1
	$=(-2)^3-3\left(-\frac{3}{2}\right)(-2)=-17$ or $=(-2)\left(7--\frac{3}{2}\right)=-17$	-17 **from correct working**	A1 cso
			(5)
(b)	Sum $=\alpha^2+\beta+\beta^2+\alpha$ $=\alpha^2+\beta^2+\alpha+\beta$ $=7+(-2)=5$	Uses at least one of their $\alpha^2+\beta^2$ or $\alpha+\beta$ in an attempt to find a **numerical value** for the sum of $(\alpha^2+\beta)$ and $(\beta^2+\alpha)$	M1
	Product $=(\alpha^2+\beta)(\beta^2+\alpha)$ $=(\alpha\beta)^2+\alpha^3+\beta^3+\alpha\beta$ $=\left(-\frac{3}{2}\right)^2-17-\frac{3}{2}=-\frac{65}{4}$	Expands $(\alpha^2+\beta)(\beta^2+\alpha)$ and uses at least one of their $\alpha\beta$ or $\alpha^3+\beta^3$ in an attempt to find a **numerical value** for the product of $(\alpha^2+\beta)$ and $(\beta^2+\alpha)$	M1
	$x^2-5x-\frac{65}{4}=0$	Applies $x^2-(\text{sum})x+\text{product}$ (Can be implied) (" = 0" not required)	M1
	$4x^2-20x-65=0$	Any integer multiple of $4x^2-20x-65=0$, including the "= 0"	A1
			(4)

Question	Scheme		Marks
9(b) *continued*	**Alternative:** Finding $\alpha^2+\beta$ and $\beta^2+\alpha$ explicitly		
	Eg. Let $\alpha=\frac{-4+\sqrt{40}}{4}$, $\beta=\frac{-4+\sqrt{40}}{4}$ and so $\alpha^2+\beta=\frac{5-3\sqrt{10}}{2}$, $\beta^2+\alpha=\frac{5+3\sqrt{10}}{2}$		
	$\left(x-\left(\frac{5-3\sqrt{10}}{2}\right)\right)\left(x-\left(\frac{5+3\sqrt{10}}{2}\right)\right)$	Uses $\left(x-\left(\alpha^2+\beta\right)\right)\left(x-\left(\beta^2+\alpha\right)\right)$ with exact numerical values. (May expand first)	M1
	$=x^2-\left(\frac{5+3\sqrt{10}}{2}\right)x-\left(\frac{5-3\sqrt{10}}{2}\right)x+\left(\frac{5-3\sqrt{10}}{2}\right)\left(\frac{5+3\sqrt{10}}{2}\right)$	Attempts to expand using exact numerical values for $\alpha^2+\beta$ and $\beta^2+\alpha$	M1
	$\Rightarrow x^2-5x-\frac{65}{4}=0$	Collect terms to give a 3TQ. ("$=0$" not required)	M1
	$4x^2-20x-65=0$	Any integer multiple of $4x^2-20x-65=0$, including the "= 0"	A1
			(4)
			(9 marks)

Notes:

(a)

1st A1: $\alpha+\beta=2,\ \alpha\beta=-\frac{3}{2}\Rightarrow\alpha^2+\beta^2=4-2\left(-\frac{3}{2}\right)=7$ is M1A0 cso

Finding $\alpha+\beta=-2, \alpha\beta=-\frac{3}{2}$ by writing down or applying $\frac{-4+\sqrt{40}}{4}, \frac{-4+\sqrt{40}}{4}$ but then writing $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=4+3=7$ and $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=-8-9=-17$ scores B0M1A0M1A0 in part (a).

Applying $\frac{-4+\sqrt{40}}{4}, \frac{-4+\sqrt{40}}{4}$ explicitly in part (a) will score B0M0A0M0A0

Eg: Give no credit for $\left(\frac{-4+\sqrt{40}}{4}\right)^2+\left(\frac{-4+\sqrt{40}}{4}\right)^2=7$

or for $\left(\frac{-4+\sqrt{40}}{4}\right)^3+\left(\frac{-4+\sqrt{40}}{4}\right)^3=-17$

(b)

Candidates **are allowed** to apply $\frac{-4+\sqrt{40}}{4}, \frac{-4+\sqrt{40}}{4}$ explicitly in part (b).

A correct method leading to a candidate stating $a=4, b=-20, c=-65$ without writing a final answer of $4x^2-20x-65=0$ is **final** M1A0

Question	Scheme		Marks
10	$u_1 = 5,\ u_{n+1} = 3u_n + 2,\ n \geq 1$. Required to prove the result, $u_n = 2\times(3)^n - 1,\ n \in \mathbb{Z}^+$		
(i)	$n=1:\ u_1 = 2(3) - 1 = 5$	$u_1 = 2(3) - 1 = 5$ or $u_1 = 6 - 1 = 5$	B1
	(Assume the result is true for $n = k$)		
	$u_{k+1} = 3\left(2(3)^k - 1\right) + 2$	Substitutes $u_k = 2(3)^k - 1$ into $u_{k+1} = 3u_k + 2$	M1
	$= 2(3)^{k+1} - 1$	**dependent on the previous M mark** Expresses u_{k+1} in term of 3^{k+1}	dM1
		$u_{k+1} = 2(3)^{k+1} - 1$ **by correct solution only**	A1
	If the result is <u>true for $n = k$,</u> then it is <u>true for $n = k+1$.</u> As the result has been shown to be <u>true for $n = 1$,</u> then the result <u>is true for all n</u>		A1 cso
			(5)
	Required to prove the result $\sum_{r=1}^{n} \frac{4r}{3^r} = 3 - \frac{(3+2n)}{3^n}$, $n \in \mathbb{Z}^+$		
(ii)	$n=1: \text{LHS} = \frac{4}{3}$, $\text{RHS} = 3 - \frac{5}{3} = \frac{4}{3}$	Shows or states **both** $\text{LHS} = \frac{4}{3}$ **and** $\text{RHS} = \frac{4}{3}$ **or** states $\text{LHS} = \text{RHS} = \frac{4}{3}$	B1
	(Assume the result is true for $n = k$)		
	$\sum_{r=1}^{k+1} \frac{4r}{3^r} = 3 - \frac{(3+2k)}{3^k} + \frac{4(k+1)}{3^{k+1}}$	Adds the $(k+1)^{\text{th}}$ term to the sum of k terms	M1
	$= 3 - \frac{3(3+2k)}{3^{k+1}} + \frac{4(k+1)}{3^{k+1}}$	**dependent on the previous M mark** Makes 3^{k+1} or $(3)3^k$ a common denominator for their fractions.	dM1
		Correct expression with common denominator 3^{k+1} or $(3)3^k$ for their fractions.	A1
	$= 3 - \left(\frac{3(3+2k) - 4(k+1)}{3^{k+1}}\right)$ $= 3 - \left(\frac{5+2k}{3^{k+1}}\right)$		
	$= 3 - \frac{(3+2(k+1))}{3^{k+1}}$	$3 - \frac{(3+2(k+1))}{3^{k+1}}$ **by correct solution only**	A1
	If the result is <u>true for $n = k$,</u> then it is <u>true for $n = k+1$.</u> As the result has been shown to be <u>true for $n = 1$,</u> then the result <u>is true for all n</u>		A1 cso
			(6)
			(11 marks)

Question 10 *continued*
Notes:
(i) & (ii) **Final A1 for parts (i) and (ii)** is dependent on all previous marks being scored in that part. It is gained by candidates conveying the ideas of **all** four underlined points **either** at the end of their solution **or** as a narrative in their solution.
(i) $u_1 = 5$ by itself is not sufficient for the 1st B1 mark in part (i). $u_1 = 3+2$ without stating $u_1 = 2(3) - 1 = 5$ or $u_1 = 6-1 = 5$ is B0
(ii) $\text{LHS} = \text{RHS}$ by itself is not sufficient for the 1st B1 mark in part (ii).