Mark Scheme (Results)

Summer 2021

Pearson Edexcel International A Level
In Further Pure Mathmatics F3
(WFM03/01)

Question Number	Scheme	Notes	Marks
1(a)	$1-\tanh^2 x \equiv \text{sech}^2 x$		
	$1-\tanh^2 x = 1-\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$	Replaces the tanh x on the lhs with a **correct** expression in terms of exponentials.	B1
	$= \dfrac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \dfrac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$ or e.g. $\dfrac{2e^{2x} \times 2e^{-2x}}{(e^x + e^{-x})^2}$ Attempts to find common denominator and expand numerator		M1
	$= \left(\dfrac{4}{(e^x + e^{-x})^2}\right) = \text{sech}^2 x$*	Obtains the rhs with no errors.	A1cso
			(3)
ALT 1	$1-\tanh^2 x = (1-\tanh x)(1+\tanh x)$ $= \left(1-\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)\right)\left(1+\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)\right)$	Uses the difference of 2 squares on the lhs and replaces the tanh x with a **correct** expression in terms of exponentials.	B1
	$= \left(\dfrac{2e^{-x}}{e^x + e^{-x}}\right)\left(\dfrac{2e^x}{e^x + e^{-x}}\right)$	Attempt to find common denominators and simplify numerators.	M1
	$= \left(\dfrac{4}{(e^x + e^{-x})^2}\right) = \text{sech}^2 x$*	Obtains the rhs with no errors.	A1cso
ALT 2	$\text{sech}^2 x = \dfrac{4}{(e^x + e^{-x})^2}$	Replaces the sech x on the rhs with a **correct** expression in terms of exponentials.	B1
	$= \dfrac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = \dfrac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$ Attempts to express the "4" in terms of the denominator.		M1
	$= 1-\left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = 1-\tanh^2 x$*	Obtains the lhs with no errors.	A1cso

(b)	$2\operatorname{sech}^2 x + 3\tanh x = 3 \Rightarrow 2(1-\tanh^2 x) + 3\tanh x = 3$ $\Rightarrow 2\tanh^2 x - 3\tanh x + 1 = 0$ Uses $\operatorname{sech}^2 x = 1 - \tanh^2 x$ and forms a 3 term quadratic in $\tanh x$		M1
	$(2\tanh x - 1)(\tanh x - 1) = 0 \Rightarrow \tanh x = \ldots$	Solves 3TQ by any valid method including calculator.	M1
	$\tanh x = \frac{1}{2} \to x = \ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			(3)
ALT	$2\operatorname{sech}^2 x + 3\tanh x = 3 \Rightarrow 2\left(\frac{4}{(e^x + e^{-x})^2}\right) + 3\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 3$ $\Rightarrow 8 + 3(e^{2x} - e^{-2x}) = 3(e^{2x} + 2 + e^{-2x}) \Rightarrow \ldots$ Substitutes the correct exponential forms, attempts to eliminate fractions and collect terms		M1
	$6e^{-2x} = 2 \Rightarrow e^{-2x} = \frac{1}{3}$	Rearranges to reach $e^{-2x} = \ldots$	M1
	$x = \ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			Total 6

Question Number	Scheme	Notes	Marks
2.	$y=\sqrt{9-x^2},\ 0\le x\le 3$		
(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{\sqrt{9-x^2}}$	Correct derivative in any form.	B1
	Note that the derivative may be obtained implicitly after squaring e.g. $y=\sqrt{9-x^2}\Rightarrow y^2=9-x^2\Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x}=-2x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{\sqrt{9-x^2}}$		
	Length of $C=\int\sqrt{1+\frac{x^2}{9-x^2}}\,\mathrm{d}x$	Uses $\int\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ with their $\frac{\mathrm{d}y}{\mathrm{d}x}$	M1
	Note that the above may be obtained via the implicit route as e.g. $\int\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=\int\sqrt{1+\frac{x^2}{y^2}}\,\mathrm{d}x=\int\sqrt{1+\frac{x^2}{9-x^2}}\,\mathrm{d}x$ In which case the B1 is implied.		
	$=\int\sqrt{\frac{9}{9-x^2}}\,\mathrm{d}x=3\arcsin\frac{x}{3}(+c)\ \left(\text{or }-3\arccos\frac{x}{3}(+c)\right)$ $\int_0^3\sqrt{\frac{9}{9-x^2}}\,\mathrm{d}x=3\arcsin(1)-3\arcsin(0)\ \left(\text{or }-3\arccos(1)+3\arccos(0)\right)$ Finds common denominator, integrates to obtain arcsin… or arccos… and applies the limits 0 and 3.		M1
	$=\frac{3\pi}{2}$ *	Obtains the printed answer with no errors. This mark should be withheld if there is no evidence at all of the limits being applied.	A1
	Special case: If $+\frac{x}{\sqrt{9-x^2}}$ is obtained for $\frac{\mathrm{d}y}{\mathrm{d}x}$ score B0M1M1A1 if otherwise correct but allow full recovery in (b)		
			(4)
(b)	Surface Area $=\int 2\pi\sqrt{9-x^2}\left(\sqrt{\frac{9}{9-x^2}}\right)\mathrm{d}x$	Uses $\int 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ with their $\frac{\mathrm{d}y}{\mathrm{d}x}$	M1
	$=\int_0^3 6\pi\,\mathrm{d}x=6\pi[x]_0^3=\ldots$	Integrates to obtain kx and applies the limits 0 and 3. Condone omission of the lower limit.	M1
	$=18\pi$	18π cao	A1
			(3)
			Total 7

Question Number	Scheme	Notes	Marks
3.	$\mathbf{M}=\begin{pmatrix}3 & 1 & p\\ 1 & 1 & 2\\ -1 & p & 2\end{pmatrix}$		
(a)	$\det\mathbf{M}=\begin{vmatrix}3 & 1 & p\\ 1 & 1 & 2\\ -1 & p & 2\end{vmatrix}$ $=3(2-2p)-1(2+2)+p(p+1)$	Attempts determinant. Requires at least 2 correct "terms". May use other rows/columns or rule of Sarrus.	M1
	$=p^2-5p+2$	Correct simplified determinant.	A1
	$p^2-5p+2=0\Rightarrow p=\ldots$	Solves 3TQ	M1
	$\dfrac{5\pm\sqrt{17}}{2}$	Correct values.	A1
			(4)
(b)	Minors $\begin{pmatrix}2-2p & 4 & p+1\\ (2-p^2) & 6+p & (3p+1)\\ 2-p & (6-p) & 2\end{pmatrix}$	Attempts the matrix of minors. If there is any doubt look for at least 6 correct elements. May be implied by their matrix of cofactors.	M1 **(B1 on EPEN)**
	Cofactors $\begin{pmatrix}2-2p & -4 & p+1\\ -(2-p^2) & 6+p & -(3p+1)\\ 2-p & -(6-p) & 2\end{pmatrix}$	Attempts cofactors.	M1
		Correct matrix	A1
	$\mathbf{M}^{-1}=\dfrac{1}{p^2-5p+2}\begin{pmatrix}2-2p & p^2-2 & 2-p\\ -4 & 6+p & p-6\\ p+1 & -3p-1 & 2\end{pmatrix}$	Transposes matrix of cofactors and divides by determinant.	M1
		Follow though their det **M** from part (a) but the adjoint matrix must be correct.	A1ft
			(5)
			Total 9

Question Number	Scheme	Notes	Marks

4(i)	$\mathrm{f}(x)=x\arccos x,\ -1\le x\le 1,$		
	$\mathrm{f}'(x)=\arccos x-\dfrac{x}{\sqrt{1-x^2}}$ M1: Differentiates using the product rule to obtain an expression of the form: $\arccos x\pm\dfrac{x}{\sqrt{1-x^2}}$ A1: Correct derivative		M1A1
	$\mathrm{f}'(0.5)=\arccos 0.5-\dfrac{0.5}{\sqrt{1-0.5^2}}=\dfrac{\pi-\sqrt{3}}{3}$	$\dfrac{\pi-\sqrt{3}}{3}$ oe e.g. $\dfrac{\pi}{3}-\dfrac{1}{\sqrt{3}}$	A1
			(3)
(ii)	$\mathbf{g(x)=arctan(e^{2x})}$		
	$\mathrm{g}'(x)=\dfrac{2\mathrm{e}^{2x}}{\mathrm{e}^{4x}+1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\dfrac{k\mathrm{e}^{2x}}{\left(\mathrm{e}^{2x}\right)^2+1}$ A1: Correct derivative in any form		M1A1
	$\mathrm{g}'(x)=\dfrac{2}{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}=\operatorname{sech}(2x)$	Introduces sech(2x). Depends on previous M.	**d**M1
	$\mathrm{g}''(x)=-2\operatorname{sech}(2x)\tanh(2x)$	Differentiates $\operatorname{sech}(u)\to\pm\operatorname{sech}u\tanh u$ Depends on both previous M's.	**d**M1
		Correct expression.	A1
			(5)
(ii) **ALT 1**	$\mathrm{g}'(x)=\dfrac{2\mathrm{e}^{2x}}{\mathrm{e}^{4x}+1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\dfrac{k\mathrm{e}^{2x}}{\left(\mathrm{e}^{2x}\right)^2+1}$ A1: Correct derivative in any form		M1A1
	$\mathrm{g}''(x)=\dfrac{4\mathrm{e}^{2x}\left(1+\mathrm{e}^{4x}\right)-4\mathrm{e}^{4x}\times 2\mathrm{e}^{2x}}{(\mathrm{e}^{4x}+1)^2}$	Differentiates using quotient or product rule. Depends on first M.	**d**M1
	$=\dfrac{4\mathrm{e}^{2x}-4\mathrm{e}^{6x}}{(\mathrm{e}^{4x}+1)^2}=\dfrac{-4(\mathrm{e}^{2x}-\mathrm{e}^{-2x})}{(\mathrm{e}^{2x}+\mathrm{e}^{-2x})^2}$	Multiply through by e^{-4x}. Depends on both previous M's.	**d**M1
	$=-2\dfrac{2}{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}\dfrac{\mathrm{e}^{2x}-\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}$ $=-2\operatorname{sech}2x\tanh 2x$	Correct expression.	A1
	Note that the first derivative may be obtained implicitly in either method e.g. $y=\arctan\left(\mathrm{e}^{2x}\right)\Rightarrow\tan y=\mathrm{e}^{2x}\Rightarrow\sec^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^{2x}\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2\mathrm{e}^{2x}}{1+\left(\mathrm{e}^{2x}\right)^2}$		
			Total 8

Question Number	Scheme	Notes	Marks
5.	$I_n=\int\sec^n x\,\mathrm{d}x,\qquad n\ge 0$		

5(a)	$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x dx$	Splits $\sec^n x$ into $\sec^{n-2} x \sec^2 x$	M1
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\,dx$ **Depends on previous M mark** dM1: Uses integration by parts to obtain $\sec^{n-2} x \tan x - k\int \sec^{n-2} x \tan^2 x\,dx$ A1: Correct integration		**d**M1A1
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x(\sec^2 x - 1)\,dx$ Uses $\tan^2 x = \sec^2 x - 1$		B1 **(M1 on EPEN)**
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$ $= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2} \Rightarrow (n-1)I_n = ...$ **Depends on all previous M and B marks** Introduces I_n and I_{n-2} and makes progress to the given result.		**dd**M1
	$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$ *	Fully correct proof.	A1cso
			(6)
ALT	$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x dx$	Splits $\sec^n x$ into $\sec^{n-2} x \sec^2 x$	M1
	$\int \sec^{n-2} x \sec^2 x dx = \int \sec^{n-2} x\left(1+\tan^2 x\right)dx$ $= \int \sec^{n-2} x dx + \int \tan^2 x \sec^{n-2} x dx$	Uses $\sec^2 x = 1 + \tan^2 x$ and splits into 2 integrals.	B1 **(4th mark M1 on EPEN)**
	$\int \tan^2 x \sec^{n-2} x dx = \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}\int \sec^n x\,dx$ Uses integration by parts on $\int \tan^2 x \sec^{n-2} x dx$ to obtain $A\tan x \sec^{n-2} x - B\int \sec^n x\,dx$ **Note this is the 2nd M on EPEN.**		**d**M1
	$\int \sec^n x\,dx = \int \sec^{n-2} x\,dx + \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}\int \sec^n x\,dx$ Fully correct integration		A1
	$\int \sec^n x\,dx = I_{n-2} + \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}I_n \Rightarrow (n-1)I_n = ...$ **Depends on previous M and B marks** Introduces I_n and I_{n-2} and makes progress to the given result.		**dd**M1
	$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$ *	Fully correct proof.	A1cso

5(b)	$I_2 = 1$	Correct value for I_2 seen or implied.	B1
	$I_6 = \frac{1}{5}\tan x \sec^4 x + \frac{4}{5}I_4$ or e.g.	Applies the given reduction formula once.	M1

	$I_6=\frac{1}{5}\tan\frac{\pi}{4}\sec^4\frac{\pi}{4}+\frac{4}{5}I_4$ or e.g. $I_6=\frac{1}{5}(1)\left(\sqrt{2}\right)^4+\frac{4}{5}I_4$		
	$=\frac{1}{5}\tan x\sec^4 x+\frac{4}{5}\left(\frac{1}{3}\tan x\sec^2 x+\frac{2}{3}I_2\right)=\frac{1}{5}(1)\left(\sqrt{2}\right)^4+\frac{4}{15}(1)\left(\sqrt{2}\right)^2+\frac{8}{15}(1)$ Applies the given reduction formula again and uses the limits to reach a numerical expression for I_6		M1
	$=\frac{28}{15}$	Correct value	A1
			(4)
ALT	$I_2=1$	Correct value for I_2 seen or implied.	B1
	$I_4=\frac{1}{3}\tan x\sec^2 x+\frac{2}{3}I_2$ or e.g. $I_4=\frac{1}{3}\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}+\frac{2}{3}I_2$ or e.g. $I_4=\frac{1}{3}(1)\left(\sqrt{2}\right)^2+\frac{2}{3}I_2$	Applies the given reduction formula once.	M1
	$I_6=\frac{1}{5}\tan x\sec^4 x+\frac{4}{5}\left(\frac{1}{3}\tan x\sec^2 x+\frac{2}{3}I_2\right)=\frac{1}{5}(1)\left(\sqrt{2}\right)^4+\frac{4}{15}(1)\left(\sqrt{2}\right)^2+\frac{8}{15}$ Applies the given reduction formula again and uses the limits to reach a numerical expression for I_6		M1
	$=\frac{28}{15}$	Correct value	A1
			Total 10

In part (b), condone confusion with the coefficients provided the intention is clear.

For either method in part (b), all working must be shown and the given reduction formula must be used at least once. So do not allow e.g. I_4 to be evaluated with a calculator but I_4 can be evaluated directly without using the given reduction formula using an alternative method e.g. by parts or by substitution – see below:

<u>Parts:</u>

$$I_4=\int\sec^4 x\,dx=\int\sec^2 x\sec^2 x\,dx=\sec^2 x\tan x-2\int\sec^2 x\tan^2 x\,dx$$

$$=\sec^2 x\tan x-2\int\sec^2 x\left(\sec^2 x-1\right)dx=\sec^2 x\tan x-2\int\sec^4 x\,dx+2\int\sec^2 x\,dx$$

$$=\sec^2 x\tan x-2I_4+2\int\sec^2 x\,dx\Rightarrow 3I_4=\sec^2 x\tan x+2\tan x\Rightarrow I_4=\frac{1}{3}\sec^2 x\tan x+\frac{2}{3}\tan x$$

<u>Substitution:</u>

$$I_4=\int\sec^4 x\,dx=\int\sec^2 x\sec^2 x\,dx=\int\sec^2 x\left(1+\tan^2 x\right)dx$$

$$u=\tan x\Rightarrow\int\sec^2 x\left(1+\tan^2 x\right)dx=\int\sec^2 x\left(1+u^2\right)\frac{du}{\sec^2 x}=\frac{u^3}{3}+u=\frac{\tan^3 x}{3}+\tan x$$

Question Number	Scheme	Notes	Marks

6(a)	Normal to plane given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \ldots$	Attempt cross product of direction vectors. If the method is unclear, look for at least 2 correct components.	M1
	$= 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$	Or any multiple of this vector.	A1
	Substitute appropriate point into $6x + 2y - 2z = d$ e.g. (1, 1, 1) or (2, 1, 4) to find "d"	Use a valid point and use scalar product with normal or substitute into Cartesian equation.	M1
	$6x + 2y - 2z = 6$ $3x + y - z = 3$ *	Given answer. No errors seen	A1* cso
			(4)
6(a) ALT	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ $\Rightarrow x = 1 + \lambda + \mu,\ y = 1 - 2\mu,\ z = 1 + 3\lambda + \mu$ M1: Forms equation of plane using (1, 1, 1) and direction vectors and extracts 3 equations for x, y and z in terms of λ and μ A1: Correct equations		M1A1
	$x = 1 + \frac{1}{2} - \frac{1}{2}y + \frac{1}{3}z - \frac{1}{2} + \frac{1}{6}y$	Eliminates λ and μ and achieves an equation in x, y and z only.	M1
	$3x + y - z = 3$ *	Given answer. No errors seen.	A1
6(b)	$s = -3$	cao	B1
			(1)
6(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$	Attempts cross product of normal vectors. If the method is unclear, look for at least 2 correct components.	M1
	e.g. $x = 0, 2y - 2z = 6, y - 2z = 3$ $\Rightarrow y = 3, z = 0$	Any valid attempt to find a point on the line.	M1
	e.g. (0,3,0)	Any valid point on the line	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "**r** =" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1
			(4)
6(c) ALT 1	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}),\ \mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3$ $\Rightarrow 1 + \lambda + \mu + 1 - 2\mu - 2 - 6\lambda - 2\mu = 3$ Forms equation of first plane using (1, 1, 1) and direction vectors and substitutes into the second plane to form an equation in λ and μ		M1
	$\Rightarrow \mu = \frac{1}{3}(-5\lambda - 3)$	Solves to obtain μ in terms of λ or λ in terms of μ	M1
		Correct equation	A1
	E.g. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \frac{1}{3}(-5\lambda - 3)(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ Correct equation including "**r** ="		A1
6(c) ALT 2	$3x + y - z = 3,\ x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1
	$z = \lambda \Rightarrow x = -\frac{1}{2}\lambda,\ y = 3 + 2z - x = 3 + \frac{5}{2}\lambda$	Introduces parameter and expresses other 2 variables in terms of the parameter	M1
		Correct equations	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "**r** =" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1

6(c) ALT 3	$3x + y - z = 3,\ x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1

	$3x+y-z=3,\ x+y-2z=3 \Rightarrow 5x+y=3$	Uses the Cartesian equations of both planes and eliminates another variable	M1
	$\Rightarrow x=-\dfrac{z}{2},\quad x=\dfrac{3-y}{5}$	Correct equations for one variable in terms of the other 2	A1
	$x=\dfrac{y-3}{-5}=\dfrac{z}{-2}$	Correct equation or equivalent e.g. $x=\dfrac{3-y}{5}=\dfrac{z}{-2}$	A1
6(d)	$(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(\mathbf{i}+\mathbf{j}-2\mathbf{k})=6$	Correct value for scalar product	B1
	$\cos\theta=\dfrac{(3\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{9+1+1}\sqrt{1+1+4}}=\sqrt{\dfrac{6}{11}}$	Full scalar product attempt to reach a value for $\cos\theta$	M1
		For $\cos\theta=\sqrt{\dfrac{6}{11}}$	A1
	$\theta=42.4°$	Correct value. Mark their final answer.	A1
			(4)
6(d) ALT	$\lvert(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j}-2\mathbf{k})\rvert=\sqrt{30}$	Correct value for magnitude of cross product	B1
	$\sin\theta=\dfrac{\lvert(3\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}+\mathbf{j}-2\mathbf{k})\rvert}{\sqrt{9+1+1}\sqrt{1+1+4}}=\dfrac{\sqrt{55}}{11}$	Full attempt to reach a value for $\sin\theta$	M1
		For $\sin\theta=\dfrac{\sqrt{55}}{11}$	A1
	$\theta=42.4°$	Correct value. Mark their final answer.	A1
			Total 13

Question Number	Scheme	Notes	Marks
7(i)	$x^2-4x+5=(x-2)^2+1$	Attempts to complete the square. Allow for $(x-2)^2+c,\ c>0$	M1
	$\int \frac{1}{(x-2)^2+1}\mathrm{d}x=\arctan(x-2)$	Allow for $k\arctan \mathrm{f}(x)$.	M1
	$\left[\arctan(x-2)\right]_1^2=0-\left(-\frac{\pi}{4}\right)=\frac{\pi}{4}$	$\frac{\pi}{4}$ cao	A1
			(3)
7(ii)	$\int\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=-\frac{\sqrt{x^2-3}}{x}+\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x$ Uses integration by parts and obtains $A\frac{\sqrt{x^2-3}}{x}+B\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x$		M1
	$=-\frac{\sqrt{x^2-3}}{x}+\operatorname{arcosh}\frac{x}{\sqrt{3}}$	$B\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x=k\operatorname{arcosh}\mathrm{f}\left(x\right)$	M1
		All correct	A1
	$\int_{\sqrt{3}}^{3}\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=\left[-\frac{\sqrt{x^2-3}}{x}+\operatorname{arcosh}\frac{x}{\sqrt{3}}\right]_{\sqrt{3}}^{3}=\left(-\frac{\sqrt{6}}{3}+\operatorname{arcosh}\sqrt{3}\right)-(0+\operatorname{arcosh}1)$ Applies the limits 3 and $\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\operatorname{arcosh}\sqrt{3}-\frac{1}{3}\sqrt{6}=\ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1
			(5)
7(ii) ALT 1	$\int\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=\int\frac{\sqrt{3\cosh^2u-3}}{3\cosh^2u}\sqrt{3}\sinh u\mathrm{d}u$	A complete substitution using $x=\sqrt{3}\cosh u$	M1
	$=\int\tanh^2u\mathrm{d}u$	Obtains $k\int\tanh^2u\mathrm{d}u$	M1
	$=\int(1-\operatorname{sech}^2u)\mathrm{d}u=u-\tanh u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3}\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=\left[u-\tanh u\right]_0^{\operatorname{arcosh}\sqrt{3}}=\operatorname{arcosh}\sqrt{3}-\tanh\left(\operatorname{arcosh}\sqrt{3}\right)-0$ Applies the limits 0 and arcosh$\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\operatorname{arcosh}\sqrt{3}-\frac{1}{3}\sqrt{6}=\ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1

7(ii) ALT 2	$\int \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \int \frac{\sqrt{3\sec^2 u - 3}}{3\sec^2 u}\sqrt{3}\sec u \tan u \mathrm{d}u$	A complete substitution using $x = \sqrt{3}\sec u$	M1
	$= \int \frac{\tan^2 u}{\sec u}\mathrm{d}u$	Obtains $k\int \frac{\tan^2 u}{\sec u}\mathrm{d}u$	M1
	$= \ln(\sec u + \tan u) - \sin u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \left[\ln(\sec u + \tan u) - \sin u\right]_0^{\mathrm{arcsec}\sqrt{3}}$ $= \ln\left(\sec\left(\mathrm{arcsec}\sqrt{3}\right) + \tan\left(\mathrm{arcsec}\sqrt{3}\right)\right) - \ln\left(\sec(0) + \tan(0)\right) - \sin\left(\mathrm{arcsec}\sqrt{3}\right)$ Applies the limits 0 and arcsec√3 **Depends on both previous M marks**		**d**M1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \ln(\sqrt{2}+\sqrt{3}) - \frac{1}{3}\sqrt{6}$	Correct answer.	A1
			Total 8

Note that there may be other ways to perform the integration in part (ii) e.g. subsequent substitutions. Marks can be awarded if the method leads to something that is integrable and should be awarded as in the main scheme e.g. M1 for a complete method, M2 for simplifying and reaching an expression that itself can be integrated or can be integrated after rearrangement, A1 for correct integration, dM3 for using appropriate limits and A2 as above.

<u>**Alternative approach:**</u>

$$\int \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \int \frac{x^2-3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{1}{\sqrt{x^2-3}}\mathrm{d}x - \int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \mathrm{arcosh}\frac{x}{\sqrt{3}} - \ldots$$

Can score **M0M1A0dM0A0** if there is no creditable attempt at the second integral.

If the second integral is attempted, it must be using a suitable method e.g. with either $x = \sqrt{3}\cosh u$ or $x = \sqrt{3}\sec u$:

$$\int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{3}{3\cosh^2 u\sqrt{3\cosh^2 u - 3}}\sqrt{3}\sinh u \,\mathrm{d}u = \int \mathrm{sech}^2 u \;\mathrm{d}u = \tanh u + c$$

or

$$\int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{3}{3\sec^2 u\sqrt{3\sec^2 u - 3}}\sqrt{3}\sec u \tan u \mathrm{d}u = \int \cos u \;\mathrm{d}u = \sin u + c$$

In these cases the first M can then be awarded and the other marks as defined with the appropriate limits used.

Question Number	Scheme	Notes	Marks
8(a)	Asymptotes are $y = \pm 2x$	$y = \pm 2x$ oe e.g. $x = \pm \frac{y}{2}$	B1
			(1)
8(b)	$4 = e^2 - 1 \Rightarrow e = \sqrt{5}$	Uses the correct eccentricity formula with a = 1 and b = 2 to find a value for e.	M1
	Foci are $(\pm\sqrt{5}, 0)$	Both required.	A1
			(2)
8(c)	$8x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{y} = \frac{4\sec\theta}{2\tan\theta}$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{2\sec^2\theta}{\sec\theta\tan\theta}$ M1: $Ax + By\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(\theta)$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \times \frac{\mathrm{d}\theta}{\mathrm{d}x} = \mathrm{f}(\theta)$ A1: Correct gradient in terms of θ		M1A1
	Explicit differentiation may be seen: $y^2 = 4x^2 - 4 \Rightarrow y = \left(4x^2 - 4\right)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(4x^2 - 4\right)^{-\frac{1}{2}} \times 8x = \frac{4\sec\theta}{\sqrt{4\sec^2\theta - 4}}$ Score M1 for $\frac{\mathrm{d}y}{\mathrm{d}x} = kx\left(4x^2 - 4\right)^{-\frac{1}{2}} = \mathrm{f}(\theta)$ and A1 for correct gradient in terms of θ		
	E.g. $y - 2\tan\theta = \frac{4\sec\theta}{2\tan\theta}(x - \sec\theta)$	Correct straight line method using their gradient in terms of θ and $x = \sec\theta$, $y = 2\tan\theta$	M1
	$y\tan\theta - 2\tan^2\theta = 2x\sec\theta - 2\sec^2\theta$ $\Rightarrow y\tan\theta - 2\tan^2\theta = 2x\sec\theta - 2(1 + \tan^2\theta)$		
	$y\tan\theta = 2x\sec\theta - 2$ *	Obtains the given answer with sufficient working shown as above.	A1cso
			(4)
8(d)	$VP: V(-1,0); P(\sec\theta, 2\tan\theta) \Rightarrow y = \frac{2\tan\theta}{\sec\theta + 1}(x+1)$ or $WQ: W(1,0); Q(\sec\theta. -2\tan\theta) \Rightarrow y = \frac{-2\tan\theta}{\sec\theta - 1}(x-1)$ M1: Correct straight line method for either *VP* or *WQ* A1: One correct equation in any form		M1A1
	$y = \frac{-2\tan\theta}{\sec\theta - 1}(x-1),\ y = \frac{2\tan\theta}{\sec\theta + 1}(x+1)$	Both equations correct in any form.	A1
	$\frac{2\tan\theta}{\sec\theta + 1}(x+1) = \frac{-2\tan\theta}{\sec\theta - 1}(x-1) \Rightarrow x / y = ...$	Attempt to solve and makes progress to achieve either $x = \ldots$ or $y = \ldots$in terms of θ only.	M1
	$x = \cos\theta$ or $y = 2\sin\theta$	One correct coordinate	A1
	$x = \cos\theta$ and $y = 2\sin\theta$	Both correct	A1
	$x^2 + \frac{y^2}{4} = 1$ or $a = 1, b = 2$	Correct equation or correct values for a and b	A1
			(7)
			Total 14