Theorem 1: One-sided Allocation > Balanced Allocation

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(* See Mathematica codes at https://bit.ly/3HHIETU *)
(* Define mu *)
mu = -p^4 ((4 - 3 br)^2 + bl^2 (9 - 10 br + 2 br^2) - 2 bl (12 - 16 br + 5 br^2)) -
  pf (br (-2 + pf + pf br - 2 pf^2 br + pf^3 br) +
      bl (-2 + pf + 4 pf br + 6 pf^3 br^2 - 4 pf^2 br (1 + br)) +
      pf bl^2 (1 - 2 pf (1 + 2 br) + pf^2 (1 + 6 br + 2 br^2))) +
  2 p^3 ((-4 + 3 br) (-2 + br + 2 pf br) + bl^2 (3 - 2 br + 2 pf (3 - 6 br + 2 br^2)) -
      2 bl (5 - 5 br + br^2 + 2 pf (2 - 6 br + 3 br^2))) +
  2 p (2 + (-1 - 3 pf + 2 pf^2) br + (pf + pf^2 - 2 pf^3) br^2 +
      bl (-1 + 4 pf^3 (-2 + br) br + pf (-3 + 4 br) + pf^2 (2 + 6 br - 6 br^2)) +
      pf bl^2 (1 + pf - 6 pf br + pf^2 (-2 + 4 br + 4 br^2))) -
  p^2 (8 + (-7 - 16 pf + 8 pf^2) br + (1 + 10 pf - 2 pf^2) br^2 +
      bl (-7 + 4 br + pf^2 (8 - 12 br^2) - 4 pf (4 - 9 br + 3 br^2)) +
      bl^2 (1 - 2 pf (-5 + 6 br) + 2 pf^2 (-1 - 6 br + 6 br^2)))
(* Compute mu(B, 0) - mu(B/2, B/2) *)
compare = Simplify[mu /. {bl → B, br → 0}] - Simplify[mu /. {bl → B/2, br → B/2}]
(* Specify range of parameters *)
conditions = 0 < B ≤ 1 && 0 ≤ p < pf ≤ 1/2
(* Verify if it is possible to have mu(B, 0) - mu(B/2, B/2) ≤ 0;
returns False if the difference > 0 for all parameters within the range *)
Reduce[compare ≤ 0 && conditions, {B, pf, p}]
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Out[1]=

$$
\begin{aligned}
&-\Big(\big((4-3\,br)^2+bl^2\,(9-10\,br+2\,br^2)-2\,bl\,(12-16\,br+5\,br^2)\big)\,p^4\Big)+\\
&\quad 2\,p^3\,\big((-4+3\,br)\,(-2+br+2\,br\,pf)+bl^2\,(3-2\,br+2\,(3-6\,br+2\,br^2)\,pf)-\\
&\qquad 2\,bl\,(5-5\,br+br^2+2\,(2-6\,br+3\,br^2)\,pf)\big)-p^2\\
&\quad \big(8+br^2\,(1+10\,pf-2\,pf^2)+br\,(-7-16\,pf+8\,pf^2)+bl\,(-7+4\,br-4\,(4-9\,br+3\,br^2)\,pf+(8-12\,br^2)\,pf^2)+\\
&\qquad bl^2\,(1-2\,(-5+6\,br)\,pf+2\,(-1-6\,br+6\,br^2)\,pf^2)\big)+\\
&\quad 2\,p\,\big(2+br\,(-1-3\,pf+2\,pf^2)+bl^2\,pf\,(1+pf-6\,br\,pf+(-2+4\,br+4\,br^2)\,pf^2)+\\
&\qquad br^2\,(pf+pf^2-2\,pf^3)+bl\,(-1+(-3+4\,br)\,pf+(2+6\,br-6\,br^2)\,pf^2+4\,(-2+br)\,br\,pf^3)\big)-\\
&\quad pf\,\big(bl^2\,pf\,(1-2\,(1+2\,br)\,pf+(1+6\,br+2\,br^2)\,pf^2)+br\,(-2+pf+br\,pf-2\,br\,pf^2+br\,pf^3)+\\
&\qquad bl\,(-2+pf+4\,br\,pf-4\,br\,(1+br)\,pf^2+6\,br^2\,pf^3)\big)
\end{aligned}
$$

Out[2]= $-(4-3B)^2 p^4 + 2(-4+3B) p^3 (-2+B+2Bpf) - Bpf(-2+pf+B(-1+pf)^2 pf) +$
$2p(2+B^2 pf(1+pf-2pf^2)+B(-1-3pf+2pf^2)) - p^2(8+B^2(1+10pf-2pf^2)+B(-7-16pf+8pf^2)) +$
$\frac{1}{8}((-2+B)^2(32-16B+B^2)p^4 - 4(-2+B)p(-4+2B(3-2pf)pf+6B^2(-1+pf)pf^2+B^3 pf^3) +$
$Bpf(-16+4(2+3B)pf-8B(2+B)pf^2+B(4+12B+B^2)pf^3) - 4(-2+B)^2 p^3(8+B^2 pf-2B(1+4pf)) +$
$2(-2+B)p^2(-16+3B^3 pf^2-6B^2 pf(2+pf)+B(6+32pf-16pf^2)))$

Out[3]= $0 < B \le 1 \&\& 0 \le p < pf \le \frac{1}{2}$

Out[4]= False