Theorem 6: b_l = 1/2

In[1]:=

```
(* See Mathematica codes at https://bit.ly/3vY8Bw1 *)
(* Define mu *)
mu =
 (-8 pf p^3 (-1 + bl) bl (-1 + br) br + 2 p^4 (-1 + bl) bl (-1 + br) br - 2 p (-2 + br + 3 pf br - pf
          br^2 + pf bl^2 (-1 + 4 pf^2 (-1 + br) br) + bl (1 + pf (3 - 4 br) - 4 pf^3 (-1 + br) br)) +
     pf (-br (-2 + pf + pf br) + pf bl^2 (-1 + 2 pf^2 (-1 + br) br) -
        bl (-2 + pf + 4 pf br + 2 pf^3 (-1 + br) br)) + p^2 (-8 + 7 br - br^2 +
        bl^2 (-1 + 12 pf^2 (-1 + br) br) + bl (7 + 4 (-1 + 3 pf^2) br - 12 pf^2 br^2))) /
   (1 + pf^4 (-1 + bl) bl (-1 + br) br - 4 pf p^3 (-1 + bl) bl (-1 + br) br +
     p^4 (-1 + bl) bl (-1 + br) br - pf^2 (bl + br + 2 bl br) -
     2 pf p (bl + br + 2 (-1 + pf^2) bl br + 2 pf^2 bl^2 (-1 + br) br - 2 pf^2 bl br^2) +
     p^2 (-4 + 3 br + 6 pf^2 bl^2 (-1 + br) br + bl (3 - 2 br + 6 pf^2 br - 6 pf^2 br^2)))
(* Take second-order derivative w.r.t. b_r *)
secondDerivative = D[mu, {br, 2}]
(* Take b_l = 1/2 *)
SOD = Simplify[secondDerivative /. {bl → 1/2}]
(* Specify range of parameters *)
conditions = 0 < br < 1 && 0 ≤ p < pf ≤ 1/2
(* Verify if it is possible to have SOD ≥ 0;
returns false if SOD < 0 for all parameters within the range *)
Reduce[SOD ≥ 0 && conditions, {br, bl, pf, p}]
```

Out[1]=

$$\Big(2(-1+bl)\,bl\,(-1+br)\,br\,p^4 - 8(-1+bl)\,bl\,(-1+br)\,br\,p^3\,pf - 2p\left(-2+br+3\,br\,pf-br^2\,pf+bl^2\,pf\left(-1+4(-1+br)\,br\,pf^2\right)+bl\left(1+(3-4\,br)\,pf-4(-1+br)\,br\,pf^3\right)\right) + pf\left(-br\left(-2+pf+br\,pf\right)+bl^2\,pf\left(-1+2(-1+br)\,br\,pf^2\right)-bl\left(-2+pf+4\,br\,pf+2(-1+br)\,br\,pf^3\right)\right) + p^2\left(-8+7\,br-br^2+bl^2\left(-1+12(-1+br)\,br\,pf^2\right)+bl\left(7-12\,br^2\,pf^2+4\,br\left(-1+3\,pf^2\right)\right)\right)\Big) \Big/ \Big(1+(-1+bl)\,bl\,(-1+br)\,br\,p^4-4(-1+bl)\,bl\,(-1+br)\,br\,p^3\,pf-\left(bl+br+2\,bl\,br\right)pf^2 + (-1+bl)\,bl\,(-1+br)\,br\,pf^4-2\,p\,pf\left(bl+br+2\,bl^2(-1+br)\,br\,pf^2-2\,bl\,br^2\,pf^2+2\,bl\,br\left(-1+pf^2\right)\right) + p^2\left(-4+3\,br+6\,bl^2(-1+br)\,br\,pf^2+bl\left(3-2\,br+6\,br\,pf^2-6\,br^2\,pf^2\right)\right)\Big)$$

Out[2]=

$$
\begin{aligned}
&(4 (-1 + bl) bl\, p^4 - 16 (-1 + bl) bl\, p^3 pf + p^2 (-2 - 24 bl\, pf^2 + 24 bl^2 pf^2) + \\
&\quad pf (-2 pf - 4 bl\, pf^3 + 4 bl^2 pf^3) - 2 p (-2 pf - 8 bl\, pf^3 + 8 bl^2 pf^3)) / \\
&\quad (1 + (-1 + bl) bl (-1 + br) br\, p^4 - 4 (-1 + bl) bl (-1 + br) br\, p^3 pf - (bl + br + 2 bl\, br) pf^2 + \\
&\quad (-1 + bl) bl (-1 + br) br\, pf^4 - 2 p\, pf (bl + br + 2 bl^2 (-1 + br) br\, pf^2 - 2 bl\, br^2 pf^2 + 2 bl\, br (-1 + pf^2)) + \\
&\quad p^2 (-4 + 3 br + 6 bl^2 (-1 + br) br\, pf^2 + bl (3 - 2 br + 6 br\, pf^2 - 6 br^2 pf^2))) - \\
&\quad (2 ((-1 + bl) bl (-1 + br) p^4 + (-1 + bl) bl\, br\, p^4 - 4 (-1 + bl) bl (-1 + br) p^3 pf - \\
&\quad\quad 4 (-1 + bl) bl\, br\, p^3 pf - (1 + 2 bl) pf^2 + (-1 + bl) bl (-1 + br) pf^4 + (-1 + bl) bl\, br\, pf^4 - \\
&\quad\quad 2 p\, pf (1 + 2 bl^2 (-1 + br) pf^2 - 4 bl\, br\, pf^2 + 2 bl^2 br\, pf^2 + 2 bl (-1 + pf^2)) + \\
&\quad\quad p^2 (3 + 6 bl^2 (-1 + br) pf^2 + 6 bl^2 br\, pf^2 + bl (-2 + 6 pf^2 - 12 br\, pf^2))) \\
&\quad (2 (-1 + bl) bl (-1 + br) p^4 + 2 (-1 + bl) bl\, br\, p^4 - 8 (-1 + bl) bl (-1 + br) p^3 pf - 8 (-1 + bl) bl\, br\, p^3 pf - \\
&\quad\quad 2 p (1 + 3 pf - 2 br\, pf + bl^2 pf (4 (-1 + br) pf^2 + 4 br\, pf^2) + bl (-4 pf - 4 (-1 + br) pf^3 - 4 br\, pf^3)) + \\
&\quad\quad pf (2 - pf - 2 br\, pf + bl^2 pf (2 (-1 + br) pf^2 + 2 br\, pf^2) - bl (4 pf + 2 (-1 + br) pf^3 + 2 br\, pf^3)) + \\
&\quad\quad p^2 (7 - 2 br + bl^2 (12 (-1 + br) pf^2 + 12 br\, pf^2) + bl (-24 br\, pf^2 + 4 (-1 + 3 pf^2))))) / \\
&\quad (1 + (-1 + bl) bl (-1 + br) br\, p^4 - 4 (-1 + bl) bl (-1 + br) br\, p^3 pf - (bl + br + 2 bl\, br) pf^2 + (-1 + bl) \\
&\quad\quad bl (-1 + br) br\, pf^4 - 2 p\, pf (bl + br + 2 bl^2 (-1 + br) br\, pf^2 - 2 bl\, br^2 pf^2 + 2 bl\, br (-1 + pf^2)) + \\
&\quad\quad p^2 (-4 + 3 br + 6 bl^2 (-1 + br) br\, pf^2 + bl (3 - 2 br + 6 br\, pf^2 - 6 br^2 pf^2)))^2 + \\
&\quad (2 (-1 + bl) bl (-1 + br) br\, p^4 - 8 (-1 + bl) bl (-1 + br) br\, p^3 pf - \\
&\quad\quad 2 p (-2 + br + 3 br\, pf - br^2 pf + bl^2 pf (-1 + 4 (-1 + br) br\, pf^2) + bl (1 + (3 - 4 br) pf - 4 (-1 + br) br\, pf^3)) + \\
&\quad\quad pf (-br (-2 + pf + br\, pf) + bl^2 pf (-1 + 2 (-1 + br) br\, pf^2) - bl (-2 + pf + 4 br\, pf + 2 (-1 + br) br\, pf^3)) + \\
&\quad\quad p^2 (-8 + 7 br - br^2 + bl^2 (-1 + 12 (-1 + br) br\, pf^2) + bl (7 - 12 br^2 pf^2 + 4 br (-1 + 3 pf^2)))) \\
&\quad ((2 ((-1 + bl) bl (-1 + br) p^4 + (-1 + bl) bl\, br\, p^4 - 4 (-1 + bl) bl (-1 + br) p^3 pf - \\
&\quad\quad\quad 4 (-1 + bl) bl\, br\, p^3 pf - (1 + 2 bl) pf^2 + (-1 + bl) bl (-1 + br) pf^4 + (-1 + bl) bl\, br\, pf^4 - \\
&\quad\quad\quad 2 p\, pf (1 + 2 bl^2 (-1 + br) pf^2 - 4 bl\, br\, pf^2 + 2 bl^2 br\, pf^2 + 2 bl (-1 + pf^2)) + \\
&\quad\quad\quad p^2 (3 + 6 bl^2 (-1 + br) pf^2 + 6 bl^2 br\, pf^2 + bl (-2 + 6 pf^2 - 12 br\, pf^2)))^2) / \\
&\quad\quad (1 + (-1 + bl) bl (-1 + br) br\, p^4 - 4 (-1 + bl) bl (-1 + br) br\, p^3 pf - (bl + br + 2 bl\, br) pf^2 + (-1 + bl) \\
&\quad\quad\quad bl (-1 + br) br\, pf^4 - 2 p\, pf (bl + br + 2 bl^2 (-1 + br) br\, pf^2 - 2 bl\, br^2 pf^2 + 2 bl\, br (-1 + pf^2)) + \\
&\quad\quad\quad p^2 (-4 + 3 br + 6 bl^2 (-1 + br) br\, pf^2 + bl (3 - 2 br + 6 br\, pf^2 - 6 br^2 pf^2)))^3 - \\
&\quad\quad (2 (-1 + bl) bl\, p^4 - 8 (-1 + bl) bl\, p^3 pf + 2 (-1 + bl) bl\, pf^4 - 2 p\, pf (-4 bl\, pf^2 + 4 bl^2 pf^2) + \\
&\quad\quad\quad p^2 (-12 bl\, pf^2 + 12 bl^2 pf^2)) / \\
&\quad\quad (1 + (-1 + bl) bl (-1 + br) br\, p^4 - 4 (-1 + bl) bl (-1 + br) br\, p^3 pf - (bl + br + 2 bl\, br) pf^2 + (-1 + bl) \\
&\quad\quad\quad bl (-1 + br) br\, pf^4 - 2 p\, pf (bl + br + 2 bl^2 (-1 + br) br\, pf^2 - 2 bl\, br^2 pf^2 + 2 bl\, br (-1 + pf^2)) + \\
&\quad\quad\quad p^2 (-4 + 3 br + 6 bl^2 (-1 + br) br\, pf^2 + bl (3 - 2 br + 6 br\, pf^2 - 6 br^2 pf^2)))^2)
\end{aligned}
$$

Out[3]=

$$
\begin{aligned}
-\Big(&\big(2\,(p-pf)^2\,\big(-64+(1-3\,br+3\,br^2)\,p^8+256\,pf-352\,pf^2+16\,(7+6\,br)\,pf^3-4\,(-29+36\,br+12\,br^2)\,pf^4+\\
&8\,(-7+6\,br)\,pf^5+2\,(-9+12\,br+16\,br^3)\,pf^6+4\,(1-3\,br+3\,br^2+2\,br^3)\,pf^7+(1-3\,br+3\,br^2)\,pf^8+\\
&2\,p^2\,\big(-176+72\,(5+2\,br)\,pf-36\,(9-4\,br+4\,br^2)\,pf^2-8\,(-25+42\,br)\,pf^3+(-91+204\,br-\\
&\quad 144\,br^2+80\,br^3)\,pf^4+6\,(-1+3\,br-3\,br^2+14\,br^3)\,pf^5+14\,(1-3\,br+3\,br^2)\,pf^6\big)-\\
&4\,p\,\big(-64+208\,pf+36\,(-7+2\,br)\,pf^2-4\,(-25+12\,br+12\,br^2)\,pf^3+(26-36\,br)\,pf^4+\\
&\quad(-25+48\,br-24\,br^2+32\,br^3)\,pf^5+(3-9\,br+9\,br^2+14\,br^3)\,pf^6+(2-6\,br+6\,br^2)\,pf^7\big)-\\
&4\,p^7\,\big(-3+2\,br^3+br\,(9-6\,pf)+2\,pf+br^2\,(-9+6\,pf)\big)+\\
&2\,p^6\,\big(19-34\,pf+14\,pf^2+4\,br^3\,(-4+7\,pf)+6\,br^2\,(8-17\,pf+7\,pf^2)-6\,br\,(10-17\,pf+7\,pf^2)\big)-\\
&4\,p^5\,\big(2+31\,pf-39\,pf^2+14\,pf^3+2\,br^3\,pf\,(-16+21\,pf)+\\
&\quad 3\,br^2\,pf\,(24-39\,pf+14\,pf^2)-3\,br\,(-4+32\,pf-39\,pf^2+14\,pf^3)\big)+\\
&2\,p^4\,\big(-38+20\,pf+49\,pf^2-90\,pf^3+35\,pf^4+20\,br^3\,pf^2\,(-4+7\,pf)+\\
&\quad 3\,br^2\,(-8+32\,pf^2-90\,pf^3+35\,pf^4)-3\,br\,(-40+24\,pf+52\,pf^2-90\,pf^3+35\,pf^4)\big)-\\
&4\,p^3\,\big(-52+4\,pf+68\,pf^2-22\,pf^3-25\,pf^4+70\,br^3\,pf^4+14\,pf^5+\\
&\quad 3\,br\,(8+48\,pf-56\,pf^2+16\,pf^3+25\,pf^4-14\,pf^5)+br^2\,(-48\,pf-48\,pf^3-75\,pf^4+42\,pf^5)\big)\big)\big)\Big/\\
&\big(-4+(-1+br)\,br\,p^4-4\,(-1+br)\,br\,p^3\,pf+(2+8\,br)\,pf^2+(-1+br)\,br\,pf^4+\\
&\quad p\,(4\,pf-4\,(-1+br)\,br\,pf^3)+2\,p^2\,(5+3\,br^2\,pf^2-br\,(4+3\,pf^2))\big)^3\Big)
\end{aligned}
$$

Out[4]= $0 < br < 1 \,\&\&\, 0 \le p < pf \le \frac{1}{2}$

Out[5]= False

Theorem 6: b_l = 0

In[6]:=

```
(* Take second-order derivative w.r.t. b_r *)
secondDerivative = D[mu, {br, 2}]
(* Take b_l = 0 *)
SOD = Simplify[secondDerivative /. {bl → 0}]
(* Specify range of parameters *)
conditions = 0 < br < 1 && 0 ≤ p < pf ≤ 1/2
(* Verify if it is possible to have SOD ≥ 0;
returns false if SOD < 0 for all parameters within the range *)
Reduce[SOD ≥ 0 && conditions, {br, bl, pf, p}]
```

Out[6]=

$$\left(4 (-1 + bl)\, bl\, p^4 - 16 (-1 + bl)\, bl\, p^3\, pf + p^2 \left(-2 - 24\, bl\, pf^2 + 24\, bl^2\, pf^2\right) +\right.$$
$$\left. pf \left(-2\, pf - 4\, bl\, pf^3 + 4\, bl^2\, pf^3\right) - 2\, p \left(-2\, pf - 8\, bl\, pf^3 + 8\, bl^2\, pf^3\right)\right) /$$
$$\left(1 + (-1 + bl)\, bl\, (-1 + br)\, br\, p^4 - 4 (-1 + bl)\, bl\, (-1 + br)\, br\, p^3\, pf - \left(bl + br + 2\, bl\, br\right) pf^2 +\right.$$
$$(-1 + bl)\, bl\, (-1 + br)\, br\, pf^4 - 2\, p\, pf \left(bl + br + 2\, bl^2 (-1 + br)\, br\, pf^2 - 2\, bl\, br^2\, pf^2 + 2\, bl\, br \left(-1 + pf^2\right)\right) +$$
$$\left. p^2 \left(-4 + 3\, br + 6\, bl^2 (-1 + br)\, br\, pf^2 + bl \left(3 - 2\, br + 6\, br\, pf^2 - 6\, br^2\, pf^2\right)\right)\right) -$$
$$\left(2 \left((-1 + bl)\, bl\, (-1 + br)\, p^4 + (-1 + bl)\, bl\, br\, p^4 - 4 (-1 + bl)\, bl\, (-1 + br)\, p^3\, pf -\right.\right.$$
$$4 (-1 + bl)\, bl\, br\, p^3\, pf - \left(1 + 2\, bl\right) pf^2 + (-1 + bl)\, bl\, (-1 + br)\, pf^4 + (-1 + bl)\, bl\, br\, pf^4 -$$
$$2\, p\, pf \left(1 + 2\, bl^2 (-1 + br)\, pf^2 - 4\, bl\, br\, pf^2 + 2\, bl^2\, br\, pf^2 + 2\, bl \left(-1 + pf^2\right)\right) +$$
$$\left.\left. p^2 \left(3 + 6\, bl^2 (-1 + br)\, pf^2 + 6\, bl^2\, br\, pf^2 + bl \left(-2 + 6\, pf^2 - 12\, br\, pf^2\right)\right)\right)\right)$$
$$\left(2 (-1 + bl)\, bl\, (-1 + br)\, p^4 + 2 (-1 + bl)\, bl\, br\, p^4 - 8 (-1 + bl)\, bl\, (-1 + br)\, p^3\, pf - 8 (-1 + bl)\, bl\, br\, p^3\, pf -\right.$$
$$2\, p \left(1 + 3\, pf - 2\, br\, pf + bl^2\, pf \left(4 (-1 + br)\, pf^2 + 4\, br\, pf^2\right) + bl \left(-4\, pf - 4 (-1 + br)\, pf^3 - 4\, br\, pf^3\right)\right) +$$
$$pf \left(2 - pf - 2\, br\, pf + bl^2\, pf \left(2 (-1 + br)\, pf^2 + 2\, br\, pf^2\right) - bl \left(4\, pf + 2 (-1 + br)\, pf^3 + 2\, br\, pf^3\right)\right) +$$
$$\left. p^2 \left(7 - 2\, br + bl^2 \left(12 (-1 + br)\, pf^2 + 12\, br\, pf^2\right) + bl \left(-24\, br\, pf^2 + 4 \left(-1 + 3\, pf^2\right)\right)\right)\right)\Big) /$$
$$\left(1 + (-1 + bl)\, bl\, (-1 + br)\, br\, p^4 - 4 (-1 + bl)\, bl\, (-1 + br)\, br\, p^3\, pf - \left(bl + br + 2\, bl\, br\right) pf^2 + (-1 + bl)\right.$$
$$bl\, (-1 + br)\, br\, pf^4 - 2\, p\, pf \left(bl + br + 2\, bl^2 (-1 + br)\, br\, pf^2 - 2\, bl\, br^2\, pf^2 + 2\, bl\, br \left(-1 + pf^2\right)\right) +$$
$$\left. p^2 \left(-4 + 3\, br + 6\, bl^2 (-1 + br)\, br\, pf^2 + bl \left(3 - 2\, br + 6\, br\, pf^2 - 6\, br^2\, pf^2\right)\right)\right)^2 +$$
$$\left(2 (-1 + bl)\, bl\, (-1 + br)\, br\, p^4 - 8 (-1 + bl)\, bl\, (-1 + br)\, br\, p^3\, pf -\right.$$
$$2\, p \left(-2 + br + 3\, br\, pf - br^2\, pf + bl^2\, pf \left(-1 + 4 (-1 + br)\, br\, pf^2\right) + bl \left(1 + \left(3 - 4\, br\right) pf - 4 (-1 + br)\, br\, pf^3\right)\right) +$$
$$pf \left(-br \left(-2 + pf + br\, pf\right) + bl^2\, pf \left(-1 + 2 (-1 + br)\, br\, pf^2\right) - bl \left(-2 + pf + 4\, br\, pf + 2 (-1 + br)\, br\, pf^3\right)\right) +$$
$$\left. p^2 \left(-8 + 7\, br - br^2 + bl^2 \left(-1 + 12 (-1 + br)\, br\, pf^2\right) + bl \left(7 - 12\, br^2\, pf^2 + 4\, br \left(-1 + 3\, pf^2\right)\right)\right)\right)$$
$$\left(\left(2 \left((-1 + bl)\, bl\, (-1 + br)\, p^4 + (-1 + bl)\, bl\, br\, p^4 - 4 (-1 + bl)\, bl\, (-1 + br)\, p^3\, pf -\right.\right.\right.$$
$$4 (-1 + bl)\, bl\, br\, p^3\, pf - \left(1 + 2\, bl\right) pf^2 + (-1 + bl)\, bl\, (-1 + br)\, pf^4 + (-1 + bl)\, bl\, br\, pf^4 -$$
$$2\, p\, pf \left(1 + 2\, bl^2 (-1 + br)\, pf^2 - 4\, bl\, br\, pf^2 + 2\, bl^2\, br\, pf^2 + 2\, bl \left(-1 + pf^2\right)\right) +$$
$$\left.\left. p^2 \left(3 + 6\, bl^2 (-1 + br)\, pf^2 + 6\, bl^2\, br\, pf^2 + bl \left(-2 + 6\, pf^2 - 12\, br\, pf^2\right)\right)\right)^2\right) /$$
$$\left(1 + (-1 + bl)\, bl\, (-1 + br)\, br\, p^4 - 4 (-1 + bl)\, bl\, (-1 + br)\, br\, p^3\, pf - \left(bl + br + 2\, bl\, br\right) pf^2 + (-1 + bl)\right.$$
$$bl\, (-1 + br)\, br\, pf^4 - 2\, p\, pf \left(bl + br + 2\, bl^2 (-1 + br)\, br\, pf^2 - 2\, bl\, br^2\, pf^2 + 2\, bl\, br \left(-1 + pf^2\right)\right) +$$
$$\left. p^2 \left(-4 + 3\, br + 6\, bl^2 (-1 + br)\, br\, pf^2 + bl \left(3 - 2\, br + 6\, br\, pf^2 - 6\, br^2\, pf^2\right)\right)\right)^3 -$$
$$\left(2 (-1 + bl)\, bl\, p^4 - 8 (-1 + bl)\, bl\, p^3\, pf + 2 (-1 + bl)\, bl\, pf^4 - 2\, p\, pf \left(-4\, bl\, pf^2 + 4\, bl^2\, pf^2\right) +\right.$$
$$\left. p^2 \left(-12\, bl\, pf^2 + 12\, bl^2\, pf^2\right)\right) /$$
$$\left(1 + (-1 + bl)\, bl\, (-1 + br)\, br\, p^4 - 4 (-1 + bl)\, bl\, (-1 + br)\, br\, p^3\, pf - \left(bl + br + 2\, bl\, br\right) pf^2 + (-1 + bl)\right.$$
$$bl\, (-1 + br)\, br\, pf^4 - 2\, p\, pf \left(bl + br + 2\, bl^2 (-1 + br)\, br\, pf^2 - 2\, bl\, br^2\, pf^2 + 2\, bl\, br \left(-1 + pf^2\right)\right) +$$
$$\left. p^2 \left(-4 + 3\, br + 6\, bl^2 (-1 + br)\, br\, pf^2 + bl \left(3 - 2\, br + 6\, br\, pf^2 - 6\, br^2\, pf^2\right)\right)\right)^2\Big)$$

Out[7]=

$$-\frac{2 \left(1 - 2\, p\right)^2 \left(p - p^2 - pf + pf^2\right)^2}{\left(1 + \left(-4 + 3\, br\right) p^2 - 2\, br\, p\, pf - br\, pf^2\right)^3}$$

Out[8]=

$$0 < br < 1 \;\&\&\; 0 \leq p < pf \leq \frac{1}{2}$$

Out[9]= `False`