In [1]:
```
using CSV
using DataFrames
using JuMP
using Plots
using Random
using Statistics
using LinearAlgebra
using Distributions
using BipartiteMatching
using Gurobi
using LinearAlgebra
using SymPy
using NLsolve
using LaTeXStrings
```

Proof of Theorem 3

Goal

The notebook aims to verify if the lower bound of $\mu(1,0) - \mu(1/2,1/2) > 0$ within each set [alphaf, alphaf + delta) × [alpha,alpha + delta).

Arguments

- `x_lb` `x1_lb` `x2_lb` `x_ub` `x1_ub` `x2_ub` : The respective lower and upper bounds of x, x_1 and x_2.
- `alphaf` or `alphaf_val` : Value of α^f .
- `alpha` or `alpha_val` : Value of α.
- `delta` or `delta_val` : Small positive increment.

Functions

- `equation_x!(F, x, alphaf_val, alpha_val)` and `equation_x1!(F, x, alphaf_val, alpha_val)` return the numerical solutions of x, x_1 and x_2. `lb_comparison` uses these values to build a lower bound of $\mu(1,0) - \mu(1/2,1/2)$ for a set [alphaf, alphaf + delta) × [alpha,alpha + delta).
- `comp_f_func(alphaf_val, alpha_val, delta_val)` returns whether the lower bound > 0.
- `calculate_lb_comparison_matrix(delta_val)` examines all alphaf and alpha within the claimed region.

Outputs

- `calculate_lb_comparison_matrix(delta_val)` returns a boolean matrix, which we plot in heatmap so that (1) the inequality holds in the red region (2) fails in the blue region, and (3) in the grey region the parameters fall outside the tree-like regime.

In [2]:
```
# Define an expression to compute the lower bound of mu(1,0) - mu(1/2,
1/2) in each set.
@vars delta alphaf alpha x_lb x1_lb x2_lb x_ub x1_ub x2_ub
lb_comparison = -x_ub-exp(-(alphaf+alpha)*x_lb)*(1+(alphaf+alpha+2*del
ta)*x_ub)+0.5*x1_lb+0.5*x2_lb + 0.5*exp(
    -(alphaf+delta)*x1_ub-0.5*(alphaf+alpha+2*delta)*x2_ub)*(1+(alpha
f)*x1_lb+0.5*(alphaf+alpha)*x2_lb)+0.5*exp(
    -0.5*(alphaf+alpha+2*delta)*x1_ub-(alpha+delta)*x2_ub)*(1+0.5*(alp
haf+alpha)*x1_lb+(alpha)*x2_lb)

# Define a Julia function to solve x.
function equation_x!(F, x, alphaf_val, alpha_val)
    F[1] = exp(-(alphaf_val + alpha_val)*x[1]) - x[1]
end

# Define a Julia function to solve x1.
function equation_x1!(F, x, alphaf_val, alpha_val)
    F[1] = exp(-0.5*(alphaf_val + alpha_val)*x[1] + 2*alpha_val*(log(x
[1]) + alphaf_val*x[1])/(alphaf_val + alpha_val)) + 2*(log(x[1]) + alp
haf_val*x[1])/(alphaf_val + alpha_val)
end

# Define a Julia function to verify if the lower bound > 0.
tolerance = 1e-8
function comp_f_func(alphaf_val, alpha_val, delta_val)
    x_sol = nlsolve((F, x) -> equation_x!(F, x, alphaf_val, alpha_va
l), [0.2], autodiff=:forward, ftol=tolerance).zero[1]
    x_sol_ub, x_sol_lb = x_sol+tolerance, x_sol-tolerance
    x1_sol = nlsolve((F, x) -> equation_x1!(F, x, alphaf_val, alpha_va
l), [0.2], autodiff=:forward, ftol=tolerance).zero[1]
    x1_sol_ub, x1_sol_lb = x1_sol+tolerance, x1_sol-tolerance
    x2_sol_ub, x2_sol_lb = -2*(log(x1_sol_lb) + alphaf_val*x1_sol_lb)/
(alphaf_val + alpha_val), -2*(log(x1_sol_ub) + alphaf_val*x1_sol_ub)/
(alphaf_val + alpha_val)
    lb_comparison_val = subs(lb_comparison, (alphaf, alphaf_val), (alp
ha, alpha_val),
        (x_lb, x_sol_lb*(1-delta)), (x_ub, x_sol_ub),
        (x1_lb, x1_sol_lb*(1-2*delta)), (x2_lb, x2_sol_lb*(1-2*delt
a)),
        (x1_ub, x1_sol_ub), (x2_ub, x2_sol_ub),(delta,delta_val))
    return lb_comparison_val > 0
end

# This function iterates over alphaf and alpha, recording 1 if the ine
quality is satisfied, 0 if it fails, and -1 if the parameters fall out
side the tree-like regime.
function calculate_lb_comparison_matrix(delta_val)

    alphaf_range = delta_val:delta_val:exp(1)
    alpha_range = 0.0:delta_val:exp(1)/2
    results_matrix = Matrix{Int}(undef, length(alphaf_range), length(a
lpha_range))

    for (i, alphaf_val) in enumerate(alphaf_range)
        print(alphaf_val, '\n')
        for (j, alpha_val) in enumerate(alpha_range)
            if alpha_val <= min(alphaf_val, exp(1) - alphaf_val)
```

```
                results_matrix[i, j] = comp_f_func(alphaf_val, alpha_v
al, delta_val)
            else
                results_matrix[i, j] = -1
            end
        end
    end

    return results_matrix

end
```

Out[2]: calculate_lb_comparison_matrix (generic function with 1 method)

In [ ]:
```
# delta = 0.01: runtime ~ 15 min
delta_val = 0.01
results_matrix = calculate_lb_comparison_matrix(delta_val)
CSV.write("compare01.csv", DataFrame(results_matrix, :auto), writehead
er=false)
```

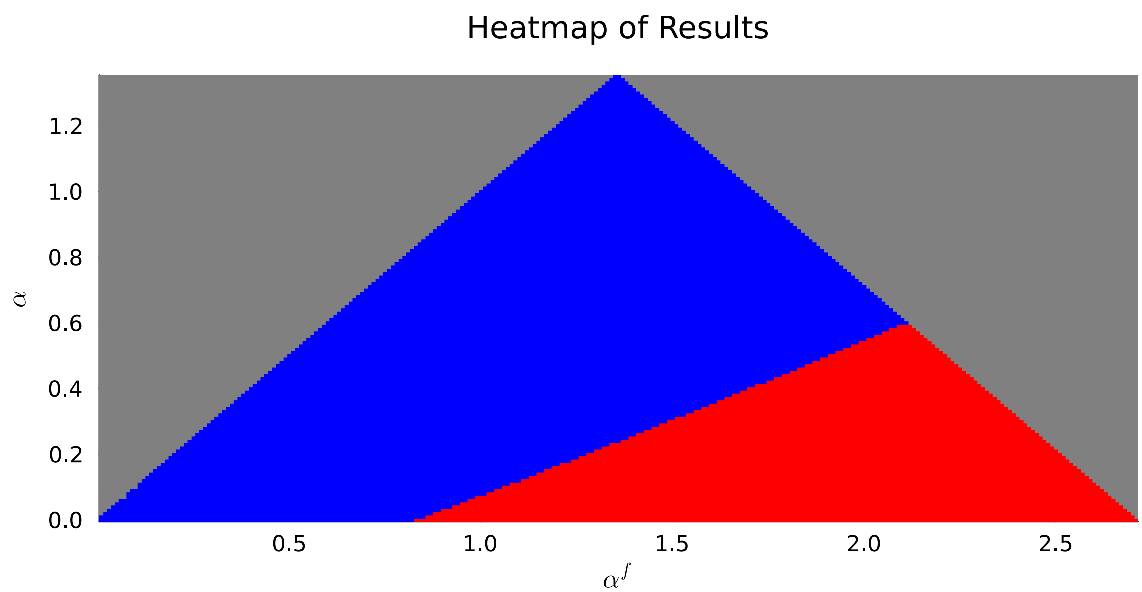
In [3]:
```
# Specify delta and load results
delta_val = 0.01

loaded_dataframe = CSV.File("compare01.csv", header=false) |> DataFram
e
results_matrix = Matrix(loaded_dataframe)

# Create a heatmap to visualize the results
color_palette = cgrad([:gray, :blue, :red], categorical=true)

heatmap(delta_val:delta_val:exp(1), 0.0:delta_val:exp(1)/2, transpose
(results_matrix), color=:auto, colormap=color_palette, colorbar=false,
title="Heatmap of Results",
        xlabel=L"$\alpha^f$", ylabel=L"$\alpha$", tickfont=font(10), g
uidefontsize=12)
plot!(size=(800, 400), margin=5Plots.mm)
```

Out[3]:



In [ ]:
```
# delta_val = 0.005: runtime ~ 60 min
delta_val = 0.005
results_matrix = calculate_lb_comparison_matrix(delta_val)
CSV.write("compare005.csv", DataFrame(results_matrix, :auto), writehea
der=false)
```

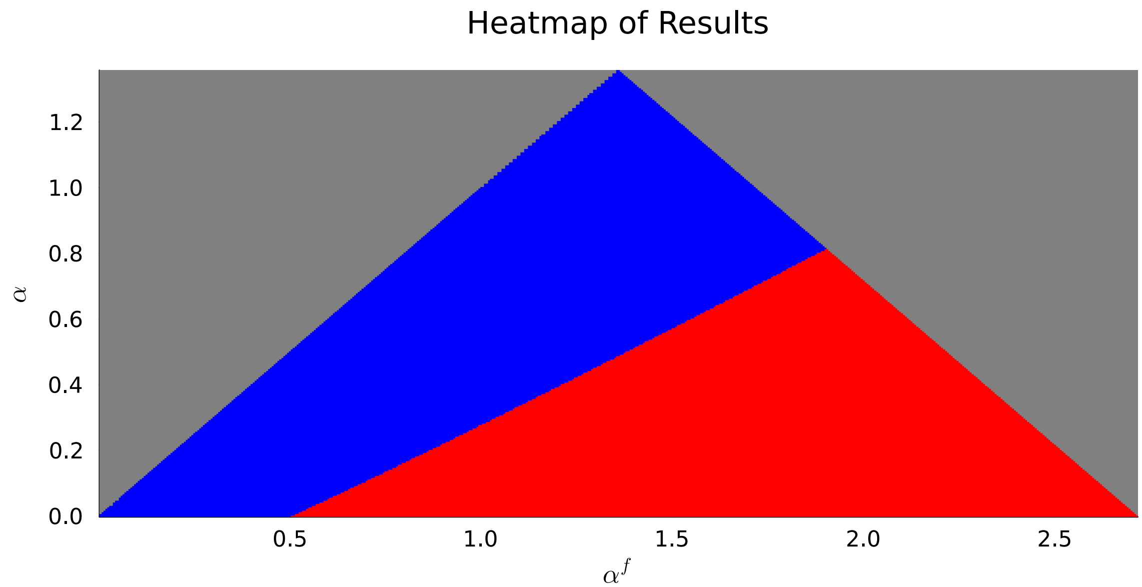
In [4]:
```
# Specify delta and load results
delta_val = 0.005

loaded_dataframe = CSV.File("compare005.csv", header=false) |> DataFra
me
results_matrix = Matrix(loaded_dataframe)

# Create a heatmap to visualize the results
color_palette = cgrad([:gray, :blue, :red], categorical=true)

heatmap(delta_val:delta_val:exp(1), 0.0:delta_val:exp(1)/2, transpose
(results_matrix), color=:auto, colormap=color_palette, colorbar=false,
title="Heatmap of Results",
        xlabel=L"$\alpha^f$", ylabel=L"$\alpha$", tickfont=font(10), g
uidefontsize=12)
plot!(size=(800, 400), margin=5Plots.mm)
```

Out[4]:



In [ ]:
```
# delta_val = 0.0025: runtime ~ 4 hours
delta_val = 0.0025
results_matrix = calculate_lb_comparison_matrix(delta_val)
CSV.write("compare0025.csv", DataFrame(results_matrix, :auto), writehe
ader=false)
```

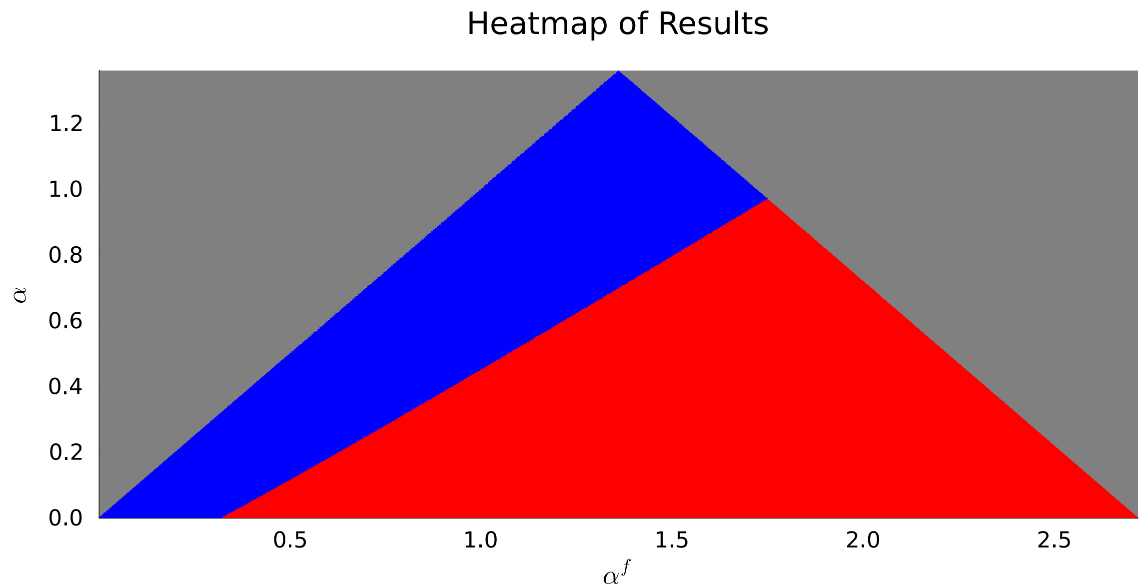
In [5]:
```
# Specify delta and load results
delta_val = 0.0025

loaded_dataframe = CSV.File("compare0025.csv", header=false) |> DataFr
ame
results_matrix = Matrix(loaded_dataframe)

# Create a heatmap to visualize the results
color_palette = cgrad([:gray, :blue, :red], categorical=true)

heatmap(delta_val:delta_val:exp(1), 0.0:delta_val:exp(1)/2, transpose
(results_matrix), color=:auto, colormap=color_palette, colorbar=false,
title="Heatmap of Results",
        xlabel=L"$\alpha^f$", ylabel=L"$\alpha$", tickfont=font(10), g
uidefontsize=12)
plot!(size=(800, 400), margin=5Plots.mm)
```

Out[5]:



In [ ]:
```
# delta_val = 0.001: runtime ~ 20 hours
delta_val = 0.001
results_matrix = calculate_lb_comparison_matrix(delta_val)
CSV.write("compare001.csv", DataFrame(results_matrix, :auto), writehea
der=false)
```

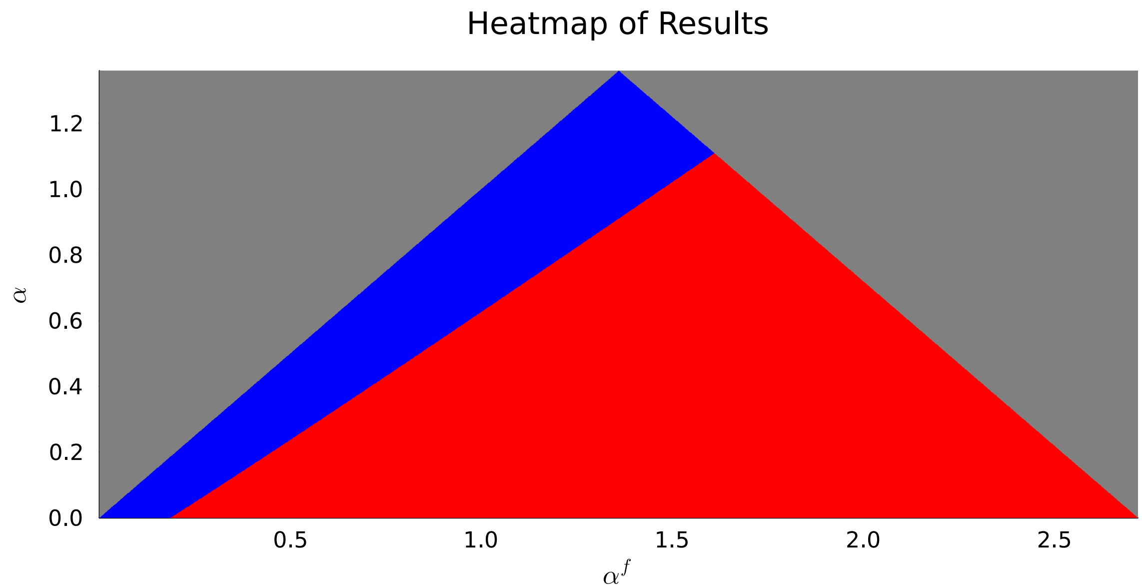
In [6]:
```
# Specify delta and load results
delta_val = 0.001

loaded_dataframe = CSV.File("compare001.csv", header=false) |> DataFrame
results_matrix = Matrix(loaded_dataframe)

# Create a heatmap to visualize the results
color_palette = cgrad([:gray, :blue, :red], categorical=true)

heatmap(delta_val:delta_val:exp(1), 0.0:delta_val:exp(1)/2, transpose(results_matrix), color=:auto, colormap=color_palette, colorbar=false, title="Heatmap of Results",
        xlabel=L"$\alpha^f$", ylabel=L"$\alpha$", tickfont=font(10), guidefontsize=12)
plot!(size=(800, 400), margin=5Plots.mm)
```

Out[6]:



In [ ]: