Theorem 5 (i): Concave in direction (0,1)

In[1]:=

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(* See Mathematica codes at https://bit.ly/48WdIvd *)
(* Define mu *)
mu = -p^4 ((4 - 3 br)^2 + bl^2 (9 - 10 br + 2 br^2) - 2 bl (12 - 16 br + 5 br^2)) -
   pf (br (-2 + pf + pf br - 2 pf^2 br + pf^3 br) +
      bl (-2 + pf + 4 pf br + 6 pf^3 br^2 - 4 pf^2 br (1 + br)) +
      pf bl^2 (1 - 2 pf (1 + 2 br) + pf^2 (1 + 6 br + 2 br^2))) +
   2 p^3 ((-4 + 3 br) (-2 + br + 2 pf br) + bl^2 (3 - 2 br + 2 pf (3 - 6 br + 2 br^2)) -
      2 bl (5 - 5 br + br^2 + 2 pf (2 - 6 br + 3 br^2))) +
   2 p (2 + (-1 - 3 pf + 2 pf^2) br + (pf + pf^2 - 2 pf^3) br^2 +
      bl (-1 + 4 pf^3 (-2 + br) br + pf (-3 + 4 br) + pf^2 (2 + 6 br - 6 br^2)) +
      pf bl^2 (1 + pf - 6 pf br + pf^2 (-2 + 4 br + 4 br^2))) -
   p^2 (8 + (-7 - 16 pf + 8 pf^2) br + (1 + 10 pf - 2 pf^2) br^2 +
      bl (-7 + 4 br + pf^2 (8 - 12 br^2) - 4 pf (4 - 9 br + 3 br^2)) +
      bl^2 (1 - 2 pf (-5 + 6 br) + 2 pf^2 (-1 - 6 br + 6 br^2)))
(* Take second-order derivative w.r.t. b_r *)
SOD = Simplify[D[mu, {br, 2}]]
(* Specify range of parameters *)
conditions = 0 < bl < 1 && 0 < br < 1 && 0 ≤ p < pf ≤ 1/2
(* Verify if it is possible to have SOD ≥ 0;
returns false if SOD < 0 for all parameters within the range *)
Reduce[SOD ≥ 0 && conditions, {bl, br, pf, p}]
```

Out[1]=

$$-\Big(\big((4-3\,br)^2+bl^2\,(9-10\,br+2\,br^2)-2\,bl\,(12-16\,br+5\,br^2)\big)\,p^4\Big)+2\,p^3\,\Big((-4+3\,br)\,(-2+br+2\,br\,pf)+bl^2\,\big(3-2\,br+2\,(3-6\,br+2\,br^2)\,pf\big)-2\,bl\,\big(5-5\,br+br^2+2\,(2-6\,br+3\,br^2)\,pf\big)\Big)-p^2\Big(8+br^2\,(1+10\,pf-2\,pf^2)+br\,(-7-16\,pf+8\,pf^2)+bl\,\big(-7+4\,br-4\,(4-9\,br+3\,br^2)\,pf+(8-12\,br^2)\,pf^2\big)+bl^2\,\big(1-2\,(-5+6\,br)\,pf+2\,(-1-6\,br+6\,br^2)\,pf^2\big)\Big)+2\,p\,\Big(2+br\,(-1-3\,pf+2\,pf^2)+bl^2\,pf\,\big(1+pf-6\,br\,pf+(-2+4\,br+4\,br^2)\,pf^2\big)+br^2\,(pf+pf^2-2\,pf^3)+bl\,\big(-1+(-3+4\,br)\,pf+(2+6\,br-6\,br^2)\,pf^2+4\,(-2+br)\,br\,pf^3\big)\Big)-pf\,\Big(bl^2\,pf\,\big(1-2\,(1+2\,br)\,pf+(1+6\,br+2\,br^2)\,pf^2\big)+br\,(-2+pf+br\,pf-2\,br\,pf^2+br\,pf^3)+bl\,\big(-2+pf+4\,br\,pf-4\,br\,(1+br)\,pf^2+6\,br^2\,pf^3\big)\Big)$$

Out[2]=

$$-2\,(p-pf)^2\,\Big(1+(9-10\,bl+2\,bl^2)\,p^2-2\,(1+2\,bl)\,pf+(1+6\,bl+2\,bl^2)\,pf^2+p\,\big(6\,(-1+pf)-4\,bl^2\,pf+4\,bl\,(1+pf)\big)\Big)$$

Out[3]=

$$0<bl<1\,\&\&\,0<br<1\,\&\&\,0\le p<pf\le\frac{1}{2}$$

Out[4]= False

Theorem 5 (i): Concave in direction (1,1)

In[5]:=
```
(* Define mu(b_l+h, b_r+h) *)
muh = Simplify[mu /. {bl → bl + h, br → br + h}]
(* Take second-order derivative w.r.t. h *)
SOD = Simplify[D[muh, {h, 2}]]
(* Evaluate SOD at h = 0 *)
SOD = Simplify[SOD /. {h → 0}]
(* Specify range of parameters *)
conditions = 0 < bl < 1 && 0 < br < 1 && 0 ≤ p < pf ≤ 1/2
(* Verify if it is possible to have SOD ≥ 0;
returns false if SOD < 0 for all parameters within the range *)
Reduce[SOD ≥ 0 && conditions, {bl, br, pf, p}]
```

Out[5]=

$$\begin{aligned}
&-\Big(\big((4-3(br+h))^2+(bl+h)^2\left(9-10(br+h)+2(br+h)^2\right)-2(bl+h)\left(12-16(br+h)+5(br+h)^2\right)\big)p^4\Big)+\\
&\quad 2p^3\Big((-4+3(br+h))(-2+br+h+2(br+h)pf)+(bl+h)^2\left(3-2(br+h)+2\left(3-6(br+h)+2(br+h)^2\right)pf\right)-\\
&\qquad 2(bl+h)\left(5-5(br+h)+(br+h)^2+2\left(2-6(br+h)+3(br+h)^2\right)pf\right)\Big)-\\
&\quad p^2\Big(8+(br+h)^2\left(1+10\,pf-2\,pf^2\right)+(br+h)\left(-7-16\,pf+8\,pf^2\right)+\\
&\qquad (bl+h)\left(-7+4(br+h)-4\left(4-9(br+h)+3(br+h)^2\right)pf+\left(8-12(br+h)^2\right)pf^2\right)+\\
&\qquad (bl+h)^2\left(1-2(-5+6(br+h))pf+2\left(-1-6(br+h)+6(br+h)^2\right)pf^2\right)\Big)+\\
&\quad 2p\Big(2+(br+h)\left(-1-3\,pf+2\,pf^2\right)+(bl+h)^2pf\left(1+pf-6(br+h)pf+\left(-2+4(br+h)+4(br+h)^2\right)pf^2\right)+\\
&\qquad (br+h)^2\left(pf+pf^2-2\,pf^3\right)+\\
&\qquad (bl+h)\left(-1+(-3+4(br+h))pf+\left(2+6(br+h)-6(br+h)^2\right)pf^2+4(-2+br+h)(br+h)pf^3\right)\Big)-\\
&\quad pf\Big((bl+h)^2pf\left(1-2(1+2(br+h))pf+\left(1+6(br+h)+2(br+h)^2\right)pf^2\right)+\\
&\qquad (br+h)\left(-2+(1+br+h)pf-2(br+h)pf^2+(br+h)pf^3\right)+\\
&\qquad (bl+h)\left(-2+pf+4(br+h)pf-4(br+h)(1+br+h)pf^2+6(br+h)^2pf^3\right)\Big)
\end{aligned}$$

Out[6]=

$$\begin{aligned}
&-4(p-pf)^2\Big(3+\left(25+bl^2+br^2-30h+6h^2+3br(-5+2h)+bl(-15+4br+6h)\right)p^2-\\
&\qquad 2(2+3bl+3br+6h)pf+\left(1+bl^2+br^2+18h+6h^2+br(9+6h)+bl(9+4br+6h)\right)pf^2-\\
&\qquad 2p\left(8-6h-7pf+bl^2pf+br^2pf-6h\,pf+6h^2pf+br(-3-3pf+6h\,pf)+bl(-3+(-3+4br+6h)pf)\right)\Big)
\end{aligned}$$

Out[7]=

$$\begin{aligned}
&-4(p-pf)^2\\
&\quad\Big(3+\left(25+bl^2-15br+br^2+bl(-15+4br)\right)p^2-2(2+3bl+3br)pf+\left(1+bl^2+9br+br^2+bl(9+4br)\right)\\
&\qquad pf^2-2p\left(8-7pf+bl^2pf+br^2pf-3br(1+pf)+bl(-3+(-3+4br)pf)\right)\Big)
\end{aligned}$$

Out[8]= $0 < bl < 1 \;\&\&\; 0 < br < 1 \;\&\&\; 0 \le p < pf \le \frac{1}{2}$

Out[9]= False

Theorem 5 (ii): Convex in direction (1,-1)

In[10]:=

```
(* Define mu(b_l, B - b_l) *)
musigma = Simplify[mu /. {br → B - bl}]
(* Take second-order derivative w.r.t. b_l *)
SOD = Simplify[D[musigma, {bl, 2}]]
(* Specify range of parameters *)
conditions = 0 < bl < 1 && bl < B < bl + 1 && 0 ≤ p < pf ≤ 1/2
(* Verify if it is possible to have SOD ≤ 0;
returns false if SOD > 0 for all parameters within the range *)
Simplify[Reduce[SOD ≤ 0 && conditions, {B, bl, p, pf}]]
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Out[10]=

$$\begin{aligned}&-\left(\left(B^2 \left(9-10\, bl+2\, bl^2\right)-2 B \left(12-7\, bl-5\, bl^2+2\, bl^3\right)+2 \left(8-7\, bl^2+bl^4\right)\right) p^4\right)+\\
&\quad 2 p \left(2+4\, bl^4\, pf^3+2\, bl^2\, pf \left(-1-2\, pf+2\, pf^2\right)+B^2\, pf \left(1+pf-6\, bl\, pf+\left(-2+4\, bl+4\, bl^2\right) pf^2\right)+\right.\\
&\qquad \left. B \left(-1+\left(-3+2\, bl\right) pf+\left(2+4\, bl+6\, bl^2\right) pf^2-4\, bl \left(1+bl+2\, bl^2\right) pf^3\right)\right)-\\
&\quad pf \left(2\, bl^2\, pf \left(-1+\left(1+bl^2\right) pf^2\right)+B^2\, pf \left(1-2 \left(1+2\, bl\right) pf+\left(1+6\, bl+2\, bl^2\right) pf^2\right)+\right.\\
&\qquad \left. B \left(-2+pf+2\, bl\, pf+4\, bl^2\, pf^2-2\, bl \left(1+3\, bl+2\, bl^2\right) pf^3\right)\right)+\\
&\quad 2 p^3 \left(8+4\, bl^4\, pf-4\, bl^2 \left(1+3\, pf\right)+B^2 \left(3+6\, pf+4\, bl^2\, pf-2\, bl \left(1+6\, pf\right)\right)-\right.\\
&\qquad \left. 2 B \left(5+4\, pf+4\, bl^3\, pf-2\, bl \left(1+3\, pf\right)-bl^2 \left(1+6\, pf\right)\right)\right)-\\
&\quad p^2 \left(8+12\, bl^4\, pf^2-2\, bl^2 \left(1+8\, pf+2\, pf^2\right)+B^2 \left(1-2 \left(-5+6\, bl\right) pf+2 \left(-1-6\, bl+6\, bl^2\right) pf^2\right)+\right.\\
&\qquad \left. B \left(-7-16\, pf+8\, pf^2-24\, bl^3\, pf^2+12\, bl^2\, pf \left(1+pf\right)+2\, bl \left(1+8\, pf+2\, pf^2\right)\right)\right)\end{aligned}$$

Out[11]=

$$\begin{aligned}&-4 \left(p-pf\right)^2 \left(-1+\left(-7+B^2+B \left(5-6\, bl\right)+6\, bl^2\right) p^2+pf^2+B^2\, pf^2+\right.\\
&\qquad \left. 6\, bl^2\, pf^2+B\, pf \left(2-3\, pf-6\, bl\, pf\right)-2 p \left(-2+pf+B^2\, pf+6\, bl^2\, pf+B \left(1+pf-6\, bl\, pf\right)\right)\right)\end{aligned}$$

Out[12]=

$$0 < bl < 1\ \&\&\ bl < B < 1 + bl\ \&\&\ 0 \le p < pf \le \frac{1}{2}$$

Out[13]=

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False
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