Data Structures and Algorithms in Java™

Sixth Edition

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Appendix A

Useful Mathematical Facts

In this appendix we give several useful mathematical facts. We begin with some combinatorial definitions and facts.

Logarithms and Exponents

The logarithm function is defined as

$$\log_b a = c \qquad \text{if} \qquad a = b^c.$$

The following identities hold for logarithms and exponents:

1. $\log_b ac = \log_b a + \log_b c$
2. $\log_b a/c = \log_b a - \log_b c$
3. $\log_b a^c = c \log_b a$
4. $\log_b a = (\log_c a)/\log_c b$
5. $b^{\log_c a} = a^{\log_c b}$
6. $(b^a)^c = b^{ac}$
7. $b^a b^c = b^{a+c}$
8. $b^a/b^c = b^{a-c}$

In addition, we have the following:

Proposition A.1: *If $a > 0$, $b > 0$, and $c > a+b$, then*

$$\log a + \log b < 2\log c - 2.$$

Justification: It is enough to show that $ab < c^2/4$. We can write

$$\begin{aligned} ab &= \frac{a^2 + 2ab + b^2 - a^2 + 2ab - b^2}{4} \\ &= \frac{(a+b)^2 - (a-b)^2}{4} \le \frac{(a+b)^2}{4} < \frac{c^2}{4}. \end{aligned}$$

■

The ***natural logarithm*** function $\ln x = \log_e x$, where $e = 2.71828\ldots$, is the value of the following progression:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

In addition,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots .$$

There are a number of useful inequalities relating to these functions (which derive from these definitions).

Proposition A.2: *If $x > -1$,*

$$\frac{x}{1+x} \le \ln(1+x) \le x.$$

Proposition A.3: *For $0 \le x < 1$,*

$$1 + x \le e^x \le \frac{1}{1-x}.$$

Proposition A.4: *For any two positive real numbers x and n,*

$$\left(1 + \frac{x}{n}\right)^n \le e^x \le \left(1 + \frac{x}{n}\right)^{n+x/2}.$$

Integer Functions and Relations

The “floor” and “ceiling” functions are defined respectively as follows:

1. $\lfloor x \rfloor =$ the largest integer less than or equal to x.
2. $\lceil x \rceil =$ the smallest integer greater than or equal to x.

The ***modulo*** operator is defined for integers $a \ge 0$ and $b > 0$ as

$$a \bmod b = a - \left\lfloor \frac{a}{b} \right\rfloor b.$$

The ***factorial*** function is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n.$$

The binomial coefficient is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

which is equal to the number of different ***combinations*** one can define by choosing k different items from a collection of n items (where the order does not matter). The name “binomial coefficient” derives from the ***binomial expansion***:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

We also have the following relationships.

Proposition A.5: *If $0 \le k \le n$, then*

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \frac{n^k}{k!}.$$

Proposition A.6 (Stirling's Approximation):

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \varepsilon(n)\right),$$

where $\varepsilon(n)$ is $O(1/n^2)$.

The ***Fibonacci progression*** is a numeric progression such that $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$.

Proposition A.7: *If F_n is defined by the Fibonacci progression, then F_n is $\Theta(g^n)$, where $g = (1+\sqrt{5})/2$ is the so-called* **golden ratio**.

Summations

There are a number of useful facts about summations.

Proposition A.8: *Factoring summations:*

$$\sum_{i=1}^{n} af(i) = a\sum_{i=1}^{n} f(i),$$

provided a does not depend upon i.

Proposition A.9: *Reversing the order:*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(i,j) = \sum_{j=1}^{m}\sum_{i=1}^{n} f(i,j).$$

One special form of is a ***telescoping sum***:

$$\sum_{i=1}^{n}(f(i) - f(i-1)) = f(n) - f(0),$$

which arises often in the amortized analysis of a data structure or algorithm.

The following are some other facts about summations that arise often in the analysis of data structures and algorithms.

Proposition A.10: $\sum_{i=1}^{n} i = n(n+1)/2$.

Proposition A.11: $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.

Proposition A.12: *If $k \geq 1$ is an integer constant, then*

$$\sum_{i=1}^{n} i^k \text{ is } \Theta(n^{k+1}).$$

Another common summation is the ***geometric sum***, $\sum_{i=0}^{n} a^i$, for any fixed real number $0 < a \neq 1$.

Proposition A.13:

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1},$$

for any real number $0 < a \neq 1$.

Proposition A.14:

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$$

for any real number $0 < a < 1$.

There is also a combination of the two common forms, called the ***linear exponential*** summation, which has the following expansion:

Proposition A.15: *For* $0 < a \neq 1$*, and* $n \geq 2$*,*

$$\sum_{i=1}^{n} ia^i = \frac{a-(n+1)a^{(n+1)}+na^{(n+2)}}{(1-a)^2}.$$

The n^{th} ***Harmonic number*** H_n is defined as

$$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

Proposition A.16: *If H_n is the n^{th} harmonic number, then H_n is* $\ln n + \Theta(1)$.

Basic Probability

We review some basic facts from probability theory. The most basic is that any statement about a probability is defined upon a ***sample space*** S, which is defined as the set of all possible outcomes from some experiment. We leave the terms "outcomes" and "experiment" undefined in any formal sense.

Example A.17: *Consider an experiment that consists of the outcome from flipping a coin five times. This sample space has 2^5 different outcomes, one for each different ordering of possible flips that can occur.*

Sample spaces can also be infinite, as the following example illustrates.

Example A.18: *Consider an experiment that consists of flipping a coin until it comes up heads. This sample space is infinite, with each outcome being a sequence of i tails followed by a single flip that comes up heads, for $i = 1, 2, 3, \ldots$.*

A ***probability space*** is a sample space S together with a probability function Pr that maps subsets of S to real numbers in the interval $[0, 1]$. It captures mathematically the notion of the probability of certain "events" occurring. Formally, each subset A of S is called an ***event***, and the probability function Pr is assumed to possess the following basic properties with respect to events defined from S:

1. $\Pr(\emptyset) = 0$.
2. $\Pr(S) = 1$.
3. $0 \le \Pr(A) \le 1$, for any $A \subseteq S$.
4. If $A, B \subseteq S$ and $A \cap B = \emptyset$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Two events A and B are ***independent*** if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

A collection of events $\{A_1, A_2, \ldots, A_n\}$ is ***mutually independent*** if

$$\Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \Pr(A_{i_1}) \Pr(A_{i_2}) \cdots \Pr(A_{i_k}).$$

for any subset $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$.

The ***conditional probability*** that an event A occurs, given an event B, is denoted as $\Pr(A|B)$, and is defined as the ratio

$$\frac{\Pr(A \cap B)}{\Pr(B)},$$

assuming that $\Pr(B) > 0$.

An elegant way for dealing with events is in terms of ***random variables***. Intuitively, random variables are variables whose values depend upon the outcome of some experiment. Formally, a ***random variable*** is a function X that maps outcomes from some sample space S to real numbers. An ***indicator random variable*** is a random variable that maps outcomes to the set $\{0, 1\}$. Often in data structure and algorithm analysis we use a random variable X to characterize the running time of a randomized algorithm. In this case, the sample space S is defined by all possible outcomes of the random sources used in the algorithm.

We are most interested in the typical, average, or "expected" value of such a random variable. The ***expected value*** of a random variable X is defined as

$$\mathbf{E}(X) = \sum_x x \Pr(X = x),$$

where the summation is defined over the range of X (which in this case is assumed to be discrete).

Proposition A.19 (The Linearity of Expectation): *Let X and Y be two random variables and let c be a number. Then*

$$E(X+Y) = E(X)+E(Y) \qquad \text{and} \qquad E(cX) = cE(X).$$

Example A.20: *Let X be a random variable that assigns the outcome of the roll of two fair dice to the sum of the number of dots showing. Then $\mathbf{E}(X) = 7$.*

Justification: *To justify this claim, let X_1 and X_2 be random variables corresponding to the number of dots on each die. Thus, $X_1 = X_2$ (i.e., they are two instances of the same function) and $\mathbf{E}(X) = \mathbf{E}(X_1 + X_2) = \mathbf{E}(X_1) + \mathbf{E}(X_2)$. Each outcome of the roll of a fair die occurs with probability $1/6$. Thus,*

$$\mathbf{E}(X_i) = \frac{1}{6}+\frac{2}{6}+\frac{3}{6}+\frac{4}{6}+\frac{5}{6}+\frac{6}{6} = \frac{7}{2},$$

for $i = 1,2$. Therefore, $E(X) = 7$. ■

Two random variables X and Y are ***independent*** if

$$\Pr(X = x|Y = y) = \Pr(X = x),$$

for all real numbers x and y.

Proposition A.21: *If two random variables X and Y are independent, then*

$$\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y).$$

Example A.22: *Let X be a random variable that assigns the outcome of a roll of two fair dice to the product of the number of dots showing. Then $\mathbf{E}(X) = 49/4$.*

Justification: *Let X_1 and X_2 be random variables denoting the number of dots on each die. The variables X_1 and X_2 are clearly independent; hence*

$$\mathbf{E}(X) = \mathbf{E}(X_1X_2) = \mathbf{E}(X_1)\mathbf{E}(X_2) = (7/2)^2 = 49/4.$$ ■

The following bound and corollaries that follow from it are known as ***Chernoff bounds***.

Proposition A.23: *Let X be the sum of a finite number of independent $0/1$ random variables and let $\mu > 0$ be the expected value of X. Then, for $\delta > 0$,*

$$\Pr(X > (1+\delta)\mu) < \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu}.$$

Useful Mathematical Techniques

To compare the growth rates of different functions, it is sometimes helpful to apply the following rule.

Proposition A.24 (L'Hôpital's Rule): *If we have* $\lim_{n\to\infty} f(n) = +\infty$ *and we have* $\lim_{n\to\infty} g(n) = +\infty$, *then* $\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} f'(n)/g'(n)$, *where* $f'(n)$ *and* $g'(n)$ *respectively denote the derivatives of* $f(n)$ *and* $g(n)$.

In deriving an upper or lower bound for a summation, it is often useful to ***split a summation*** as follows:

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{j} f(i) + \sum_{i=j+1}^{n} f(i).$$

Another useful technique is to ***bound a sum by an integral***. If f is a nondecreasing function, then, assuming the following terms are defined,

$$\int_{a-1}^{b} f(x)\,dx \le \sum_{i=a}^{b} f(i) \le \int_{a}^{b+1} f(x)\,dx.$$

There is a general form of recurrence relation that arises in the analysis of divide-and-conquer algorithms:

$$T(n) = aT(n/b) + f(n),$$

for constants $a \ge 1$ and $b > 1$.

Proposition A.25: *Let* $T(n)$ *be defined as above. Then*

1. *If* $f(n)$ *is* $O(n^{\log_b a - \varepsilon})$, *for some constant* $\varepsilon > 0$, *then* $T(n)$ *is* $\Theta(n^{\log_b a})$.
2. *If* $f(n)$ *is* $\Theta(n^{\log_b a} \log^k n)$, *for a fixed nonnegative integer* $k \ge 0$, *then* $T(n)$ *is* $\Theta(n^{\log_b a} \log^{k+1} n)$.
3. *If* $f(n)$ *is* $\Omega(n^{\log_b a + \varepsilon})$, *for some constant* $\varepsilon > 0$, *and if* $af(n/b) \le cf(n)$, *then* $T(n)$ *is* $\Theta(f(n))$.

This proposition is known as the ***master method*** for characterizing divide-and-conquer recurrence relations asymptotically.