CMPT-413: Computational Linguistics

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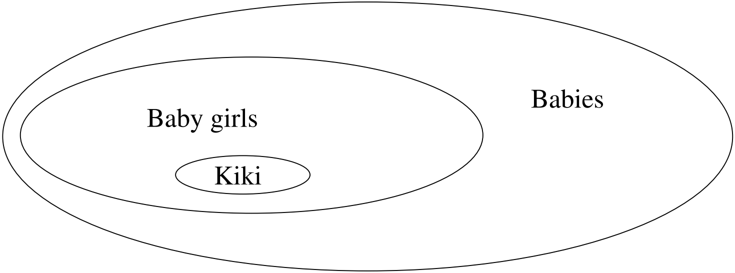
Quick Guide to Probability Theory

Log Probability

Basics of Information Theory

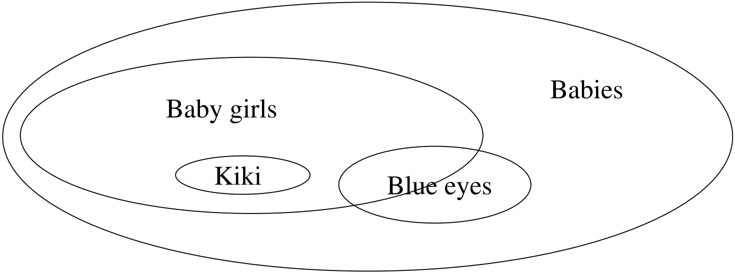
Quick guide to probability theory

- P(X) means probability that X is true
  - P(baby is a girl) $= 0.5$
    percentage of total number of babies that are girls
  - P(baby girl is named Kiki) $= 0.001$
    percentage of total number of babies that are named Kiki



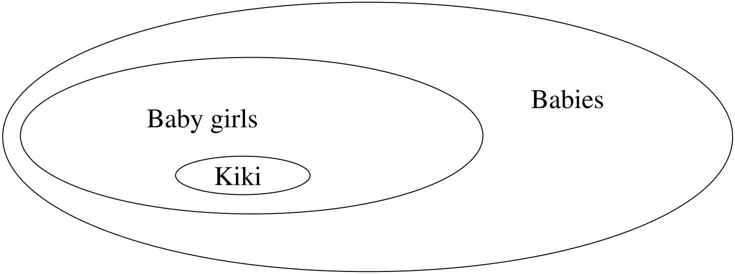
Joint probability

- P(X,Y) means probability that X and Y are both true
  - P(baby girl, blue eyes) percentage of total number of babies that are girls and have blue eyes



Conditional probability

- P(X | Y) means probability that X is true when we already know that Y is true
  - P(baby is named Kiki | baby is a girl) $= 0.002$
  - P(baby is a girl | baby is named Kiki) $= 1$

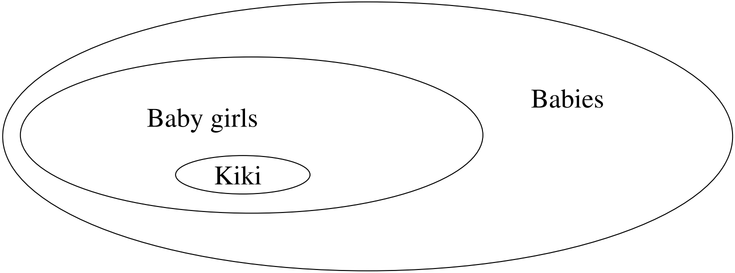


Conditional probability

- Conditional and joint probabilities are related:

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)}$$

  - $P(\text{baby is named Kiki} \mid \text{baby is a girl}) = \frac{P(\text{baby is a girl,baby is named Kiki})}{P(\text{baby is a girl})} = \frac{0.001}{0.5} = 0.002$

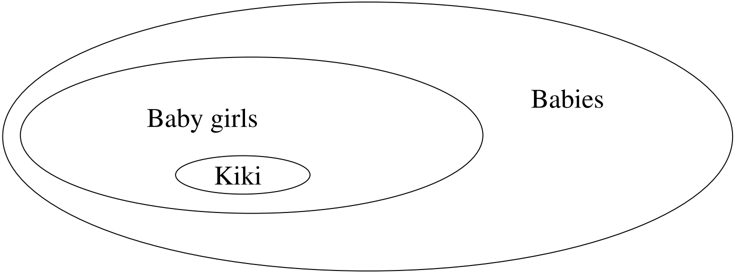


Bayes rule

- Conditional probability re-written as likelihood times prior:

$$P(X \mid Y) = \frac{P(Y \mid X) \times P(X)}{P(Y)}$$

  - $P(\text{named Kiki} \mid \text{girl}) = \frac{P(\text{girl}|\text{named Kiki}) \times P(\text{named Kiki})}{P(\text{girl})} = \frac{1.0 \times 0.001}{0.5} = 0.002$



Bayes Rule

$$\begin{aligned}
P(X \mid Y) &= \frac{P(X, Y)}{P(Y)} && (1)\\
P(Y \mid X) &= \frac{P(Y, X)}{P(X)} && (2)\\
P(X, Y) &= P(Y, X) && (3)\\
P(X \mid Y) \times P(Y) &= P(Y \mid X) \times P(X) && (4)\\
P(X \mid Y) &= \frac{P(Y \mid X) \times P(X)}{P(Y)} && (5)\\
P(X \mid Y) &= P(Y \mid X) \times P(X) && (6)
\end{aligned}$$

Basic Terms

- $P(e)$ – *a priori* probability or just *prior*
- $P(f \mid e)$ – *conditional* probability. The chance of f given e
- $P(e, f)$ – *joint* probability. The chance of e and f both happening.
- If e and f are *independent* then we can write
  $P(e, f) = P(e) \times P(f)$
- If e and f are not *independent* then we can write
  $P(e, f) = P(e) \times P(f \mid e)$
  $P(e, f) = P(f) \times$?

Basic Terms

- Addition of integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n$$

- Product of integers:

$$\prod_{i=1}^{n} i = 1 \times 2 \times 3 \times \ldots \times n$$

- Factoring:

$$\sum_{i=1}^{n} i \times k = k + 2k + 3k + \ldots + nk = k \sum_{i=1}^{n} i$$

- Product with constant:

$$\prod_{i=1}^{n} i \times k = 1k \times 2k \ldots \times nk = k^{n} \times \prod_{i=1}^{n} i$$

Probability: Axioms

- P measures total probability of a set of events
- $P(\emptyset) = 0$
- $P(\text{all events}) = 1$
- $P(X) \leq P(Y)$ for any $X \subseteq Y$
- $P(X) + P(Y) = P(X \cup Y)$ provided that $X \cap Y = \emptyset$
  - $P(\text{Kiki is a girl}) + P(\text{Kiki is fictional}) =$
    $P(\text{Kiki is a fictional girl})$,
    provided there are no real girls called Kiki or persons/objects that are fictional Kiki's.

Probability Axioms

- All events sum to 1:

$$\sum_{e} P(e) = 1$$

- Marginal probability $P(f)$:

$$P(f) = \sum_{e} P(e, f)$$

- Conditional probability:

$$\sum_{e} P(e \mid f) = \sum_{e} \frac{P(e, f)}{P(f)} = \frac{1}{P(f)} \sum_{e} P(e, f) = 1$$

- Computing $P(f)$ from axioms:

$$P(f) = \sum_{e} P(e) \times P(f \mid e)$$

Probability: The Chain Rule

- $P(a, b, c, d \mid e)$
- We cannot simply remove items from the left of $\mid$ (verify that it violates the definitions we have given based on sets)
- In this case we can use the chain rule of probability to rescue us
- $P(a, b, c, d \mid e) =$ $P(d \mid e) \cdot P(c \mid d, e) \cdot P(b \mid c, d, e) \cdot P(a \mid b, c, d, e)$
- To see why this is possible, recall that $P(X \mid Y) = \frac{p(X,Y)}{p(Y)}$
  - $\frac{p(a,b,c,d,e)}{p(e)} = \frac{p(d,e)}{p(e)} \cdot \frac{p(c,d,e)}{p(d,e)} \cdot \frac{p(b,c,d,e)}{p(c,d,e)} \cdot \frac{p(a,b,c,d,e)}{p(b,c,d,e)}$
- Use chain rule and simplify:

$$P(a, b, c, d \mid e) = P(d \mid e) \cdot P(c \mid d, e) \cdot P(b \mid c, e) \cdot P(a \mid b, e)$$

Probability: The Chain Rule

- $P(e_1, e_2, \ldots, e_n) = P(e_1) \times P(e_2 \mid e_1) \times P(e_3 \mid e_1, e_2) \ldots$

$$P(e_1, e_2, \ldots, e_n) = \prod_{i=1}^{n} P(e_i \mid e_{i-1}, e_{i-2}, \ldots, e_1)$$

Probability: Random Variables and Events

- What is y in $P(y)$?
- Shorthand for value assigned to a random variable Y, e.g. $Y = y$
- y is an element of some implicit **event space**: $\mathcal{E}$

Probability: Random Variables and Events

- The *marginal probability* $P(y)$ can be computed from $P(x, y)$ as follows:

$$P(y) = \sum_{x \in \mathcal{E}} P(x, y)$$

- Finding the value that maximizes the probability value:

$$\hat{x} = \underset{x \in \mathcal{E}}{\text{arg max}} \; P(x)$$

Quick Guide to Probability Theory

Log Probability

Basics of Information Theory

Log Probability Arithmetic

- Practical problem with tiny $P(e)$ numbers: underflow
- One solution is to use log probabilities:

$$\begin{aligned}\log(P(e)) &= \log(p_1 \times p_2 \times \ldots \times p_n) \\ &= \log(p_1) + \log(p_2) + \ldots + \log(p_n)\end{aligned}$$

- Note that:

$$x = \exp(\log(x))$$

- Also more efficient: addition instead of multiplication

Log Probability Arithmetic

p	$\log(p)$
0.0	$-\infty$
0.1	-3.32
0.2	-2.32
0.3	-1.74
0.4	-1.32
0.5	-1.00
0.6	-0.74
0.7	-0.51
0.8	-0.32
0.9	-0.15
1.0	0.00

Log Probability Arithmetic

- So: $(0.5 \times 0.5 \times \ldots 0.5) = (0.5)^n$ might get too small but $(-1 - 1 - 1 - 1) = -n$ is manageable
- Another useful fact when writing code ($\log_2$ is *log to the base 2*):

$$\log_2(x) = \frac{\log_{10}(x)}{\log_{10}(2)}$$

Log Probability Arithmetic

- Adding probabilities is expensive to compute: $logadd(x, y) = \log(\exp(x) + \exp(y))$
- A more efficient soln, let *big* be a large constant e.g. 10^{30}:

```
function  $logadd(x, y)$ : $\#$ returns $\log(\exp(x) + \exp(y))$
if  $(y - x) > \log(big)$ return  $y$
elsif  $(x - y) > \log(big)$ return  $x$
else  return
        $min(x, y) + \log(\exp(x - min(x, y)) + \exp(y - min(x, y)))$
endif
```

- There is a more efficient way of computing $\log(\exp(x - min(x, y)) + \exp(y - min(x, y)))$

Log Probability Arithmetic

```
function $logadd(x, y)$ :
    if $(y - x) > \log(big)$ return $y$
    elsif $(x - y) > \log(big)$ return $x$
    elsif $(x \geq y)$ return $x + \log(1 + \exp(y - x))$
        # note that $max(x, y) = x$ and $y - x \leq 0$
    else return $y + \log(\exp(x - y) + 1)$
        # note that $max(x, y) = y$ and $x - y \leq 0$
    endif
```

Also, in ANSI C, `log1p` efficiently computes $\log(1 + x)$

`http://www.ling.ohio-state.edu/~jansche/src/logadd.c`

In Python, numpy.logaddexp2(x1,x2) for base 2

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Log Probability

Basics of Information Theory

Information Theory

- Information theory is the use of probability theory to quantify and measure “information”.
- Consider the task of efficiently sending a message. Sender Alice wants to send several messages to Receiver Bob. Alice wants to do this as efficiently as possible.
- Let’s say that Alice is sending a message where the entire message is just one character a, e.g. $aaaa\ldots$. In this case we can save space by simply sending the length of the message and the single character.

Information Theory

- Now let's say that Alice is sending a completely random signal to Bob. If it is random then we cannot exploit anything in the message to compress it any further.
- The *expected* number of bits it takes to transmit some infinite set of messages is what is called entropy.
- This formulation of entropy by Claude Shannon was adapted from thermodynamics, converting information into a quantity that can be measured.
- Information theory is built around this notion of message compression as a way to evaluate the amount of information.

Expectation

- For a probability distribution p
- **Expectation** with respect to p is a weighted average:

$$\begin{aligned} E_p[x] &= \frac{x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n p_n}{p_1 + p_2 + \ldots + p_n} \\ &= x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n p_n \\ &= \sum_{x \in \mathcal{E}} x \cdot p(x) \end{aligned}$$

- Example: for a six-sided die the expectation is:

$$E_p[x] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = 3.5$$

Entropy

- For a probability distribution p
- **Entropy** of p is:

$$H(p) = -\sum_{x \in \mathcal{E}} p(x) \cdot \log_2 p(x)$$

- Any base can be used for the log, but base 2 means that entropy is measured in bits.
- What is the *expected* number of bits with respect to p:

$$-E_p[\log_2 p(x)] = H(p)$$

- Entropy answers the question: *What is the expected number of bits needed to transmit messages from event space $\mathcal{E}$, where $p(x)$ defines the probability of observing x?*

Entropy

- Alice wants to bet on a horse race. She has to send a message to her bookie Bob to tell him which horse to bet on.
- There are 8 horses. One encoding scheme for the messages is to use a number for each horse. So in bits this would be $001, 010, \ldots$
  (lower bound on message length $=$ 3 bits in this encoding scheme)
- Can we do better?

Entropy

Horse 1	$\frac{1}{2}$	Horse 5	$\frac{1}{64}$
Horse 2	$\frac{1}{4}$	Horse 6	$\frac{1}{64}$
Horse 3	$\frac{1}{8}$	Horse 7	$\frac{1}{64}$
Horse 4	$\frac{1}{16}$	Horse 8	$\frac{1}{64}$

- If we know how likely we are to bet on each horse, say based on the horse's probability of winning, then we can do better.
- Let p be the probability distribution given in the table above. The entropy of p is $H(p)$

Entropy

$$
\begin{aligned}
H(p) = & \\
= & -\sum_{i=1}^{8} p(i) \log_2 \; p(i) \\
= & -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{4}\log_2\frac{1}{4} + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{16}\log_2\frac{1}{16} + 4(\frac{1}{64}\log_2\frac{1}{64})\right) \\
= & -\left(\frac{1}{2}\times -1 + \frac{1}{4}\times -2 + \frac{1}{8}\times -3 + \frac{1}{16}\times -4 + 4(\frac{1}{64}\times -6)\right) \\
= & -\left(-\frac{1}{2} - \frac{1}{2} - \frac{3}{8} - \frac{1}{4} - \frac{3}{8}\right) \\
= & \; 2 \; bits
\end{aligned}
$$

- What is the entropy when the horses are equally likely to win?

$$H(uniform\ distribution) = -8(\frac{1}{8}\times -3) = 3\ bits$$

Entropy

- e.g., most likely horse gets code 0, next most likely gets 10, and then $110, 1110, \ldots$

  many possible coding schemes, this is a simple code to illustrate number of bits needed for a large number of messages ...
- Assume there are 320 messages (one for each race): code 0 occurs 160 times, code 10 occurs 80 times, code 110 occurs 40 times, code 1110 occurs 20 times, code 11110 occurs 5 times.
- Total number of bits for all messages: 160*len(0) + 80*len(10) + 40*len(110) + 20*len(1110) + 5*len(11110)
- Number of bits: 160*1 + 80*2 + 40*3 + 20*4 + 5*5 = 545
- Total number of bits per message (per race): $\frac{545}{320} \approx 1.7$ bits (always less than 2 bits)

Perplexity

- The value $2^{H(p)}$ is called the **perplexity** of a distribution p
- Perplexity is the weighted average number of choices a random variable has to make.
- Choosing between 8 equally likely horses (H=3) is $2^3 = 8$.
- Choosing between the biased horses from before (H=2) is $2^2 = 4$.

Relative Entropy

- In real life, we cannot know for sure the exact winning probability for each horse.
- Let's say q is the estimate and p is the true probability (say we got q by observing previous races with these horses)
- We define the *distance* between q and p as the **relative entropy**: written as $D(q\|p)$

$$D(q\|p) = -\sum_{x \in \mathcal{E}} q(x) \log_2 \frac{p(x)}{q(x)}$$

- Note that

$$D(q\|p) = E_{q(x)} \left[\log_2 \frac{p(x)}{q(x)} \right]$$

- The relative entropy is also called the *Kullback-Leibler divergence*.

Cross Entropy and Relative Entropy

- The **relative entropy** can be written as the sum of two terms:

$$\begin{aligned} D(q\|p) &= -\sum_{x\in\mathcal{E}} q(x)\log_2 \frac{p(x)}{q(x)} \\ &= -\sum_{x} q(x)\log_2 p(x) + \sum_{x} q(x)\log_2 q(x) \end{aligned}$$

- We know that $H(q) = -\sum_x q(x)\log_2 q(x)$
- Similarly define $H_q(p) = -\sum_x q(x)\log_2 p(x)$

$$\begin{array}{lcl} D(q\|p) & = H_q(p) & - H(q) \\ \textbf{relative entropy}(q,p) & = \textbf{cross entropy}(q,p) & - \textbf{entropy}(q) \end{array}$$

- The term $H_q(p)$ is called the **cross entropy**.

Cross Entropy and Relative Entropy

- $H_q(p) \geq H(q)$ always.
- $D(q\|p) \geq 0$ always, and $D(q\|p) = 0$ iff $q = p$
- $D(q\|p)$ is not a true distance:
  - It is asymmetric: $D(q\|p) \neq D(p\|q)$,
  - It does not obey the triangle inequality: $D(p\|r) \nleq D(p\|q) + D(q\|r)$

Conditional Entropy and Mutual Information

- *Entropy* of a random variable X:

$$H(X) = -\sum_{x \in \mathcal{E}} p(x) \log_2 p(x)$$

- *Conditional Entropy* between two random variables X and Y:

$$H(X \mid Y) = -\sum_{x,y \in \mathcal{E}} p(x,y) \log_2 p(x \mid y)$$

- *Mutual Information* between two random variables X and Y:

$$I(X;Y) = D(p(x,y) \| p(x)p(y)) = \sum_{x} \sum_{y} p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)}$$