CMPT-413
Computational Linguistics

Anoop Sarkar
http://www.cs.sfu.ca/∼anoop

March 5, 2012

Cross-Entropy and Perplexity

How good is a model

- So far we've seen the probability of a sentence: $P(w_0, \ldots, w_n)$
- What is the probability of a collection of sentences, that is what is the probability of a corpus
- Let $T = s_0, \ldots, s_m$ be a text corpus with sentences s_0 through s_m
- What is $P(T)$?
  Let us assume that we trained $P(\cdot)$ on some *training data*, and T is the *test data*

How good is a model

- $T = s_0, \ldots, s_m$ is the text corpus with sentences s_0 through s_m
- $P(T) = P(s_0, s_1, s_2, \ldots, s_m)$ – but each sentence is independent from the other sentences
- $P(T) = P(s_0) \cdot P(s_1) \cdot P(s_2) \cdot \ldots \cdot P(s_m) = \prod_{i=0}^{m} P(s_i)$
- $P(s_i) = P(w_0^i, \ldots, w_n^i)$
- Let W_T be the length of the text T measured in words
- Then for the unigram model, $P(T) = \prod_{w \in T} P(w)$
- A problem: we want to compare two different models P_1 and P_2 on T
- To do this we use the *per word* perplexity of the model:

$$PP_P(T) = P(T)^{-\frac{1}{W_T}} = \sqrt[W_T]{\frac{1}{P(T)}}$$

How good is a model

- The *per word* perplexity of the model is:

$$PP_P(T) = P(T)^{-\frac{1}{W_T}}$$

- Recall that $PP_P(T) = 2^{H_P(T)}$ where $H_P(T)$ is the cross-entropy of P for text T.
- Therefore, $H_P(T) = \log_2 PP_P(T) = -\frac{1}{W_T}\log_2 P(T)$
- Above we use a unigram model $P(w)$, but the same derivation holds for bigram, trigram, ...

How good is a model

- Lower cross entropy values and perplexity values are better
  Lower values mean that the model is *better*
  Correlation with performance of the language model in various applications
- Performance of a language model is its cross-entropy or perplexity on *test data* (unseen data)
  corresponds to the number bits required to encode that data
- On various real life datasets, typical perplexity values yielded by n-gram models on English text range from about 50 to almost 1000 (corresponding to cross entropies from about 6 to 10 bits/word)

Cross-Entropy and Perplexity

Smoothing *n*-gram Models

- Add-one Smoothing
- Additive Smoothing
- Good-Turing Smoothing
- Backoff Smoothing
- Event Space for *n*-gram Models

Bigram Models

- In practice:

$$
\begin{aligned}
&P(\text{Mork read a book}) = \\
&\quad P(\text{Mork} \mid <\text{start}>) \times P(\text{read} \mid \text{Mork}) \times \\
&\quad P(\text{a} \mid \text{read}) \times P(\text{book} \mid \text{a}) \times \\
&\quad P(<\text{stop}> \mid \text{book})
\end{aligned}
$$

- $P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$
On unseen data, $c(w_{i-1}, w_i)$ or worse $c(w_{i-1})$ could be zero

$$\sum_{w_i} \frac{c(w_{i-1}, w_i)}{c(w_{i-1})} = ?$$

Smoothing

- **Smoothing** deals with events that have been observed zero times
- Smoothing algorithms also tend to improve the accuracy of the model

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- Not just unobserved events: what about events observed once?

Add-one Smoothing

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- Add-one Smoothing:

$$P(w_i \mid w_{i-1}) = \frac{1 + c(w_{i-1}, w_i)}{V + c(w_{i-1})}$$

- Let V be the number of words in our vocabulary
  Assign count of 1 to unseen bigrams

Add-one Smoothing

$$
\begin{aligned}
&P(\text{Mindy read a book}) = \\
&\quad P(\text{Mindy} \mid \; < \text{start} >) \times P(\text{read} \mid \text{Mindy}) \times \\
&\quad P(\text{a} \mid \text{read}) \times P(\text{book} \mid \text{a}) \times \\
&\quad P(< \text{stop} > \; \mid \text{book})
\end{aligned}
$$

- Without smoothing:

$$P(\text{read} \mid \text{Mindy}) = \frac{c(\text{Mindy, read})}{c(\text{Mindy})} = 0$$

- With add-one smoothing (assuming c(Mindy) = 1 but c(Mindy, read) = 0):

$$P(\text{read} \mid \text{Mindy}) = \frac{1}{V+1}$$

Additive Smoothing: (Lidstone 1920, Jeffreys 1948)

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- Add-one smoothing works horribly in practice. Seems like 1 is too large a count for unobserved events.
- Additive Smoothing:

$$P(w_i \mid w_{i-1}) = \frac{\delta + c(w_{i-1}, w_i)}{(\delta \times V) + c(w_{i-1})}$$

- $0 < \delta \leq 1$
  Still works horribly in practice, but better than add-one smoothing.

Good-Turing Smoothing: (Good, 1953)

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- Imagine you're sitting at a sushi bar with a conveyor belt.
- You see going past you 10 plates of tuna, 3 plates of unagi, 2 plates of salmon, 1 plate of shrimp, 1 plate of octopus, and 1 plate of yellowtail
- Chance you will observe a new kind of seafood: $\frac{3}{18}$
- How likely are you to see another plate of salmon: should be $< \frac{2}{18}$

Good-Turing Smoothing

- How many types of seafood (words) were seen once? Use this to predict probabilities for unseen events
  Let n_1 be the number of events that occurred once: $p_0 = \frac{n_1}{N}$
- The Good-Turing estimate states that for any n-gram that occurs r times, we should pretend that it occurs r^* times

$$r^* = (r+1)\frac{n_{r+1}}{n_r}$$

- n_r: number of different objects seen r times

Good-Turing Smoothing

- 10 tuna, 3 unagi, 2 salmon, 1 shrimp, 1 octopus, 1 yellowtail
- How likely is new data? Let n_1 be the number of items occurring once, which is 3 in this case. N is the total, which is 18.

$$p_0 = \frac{n_1}{N} = \frac{3}{18} = 0.166$$

Good-Turing Smoothing

- 10 tuna, 3 unagi, 2 salmon, 1 shrimp, 1 octopus, 1 yellowtail
- How likely is *octopus*? Since $c(\mathit{octopus}) = 1$ The GT estimate is 1^*.

$$r^* = (r+1)\frac{n_{r+1}}{n_r}$$

$$p_{GT} = \frac{r^*}{N}$$

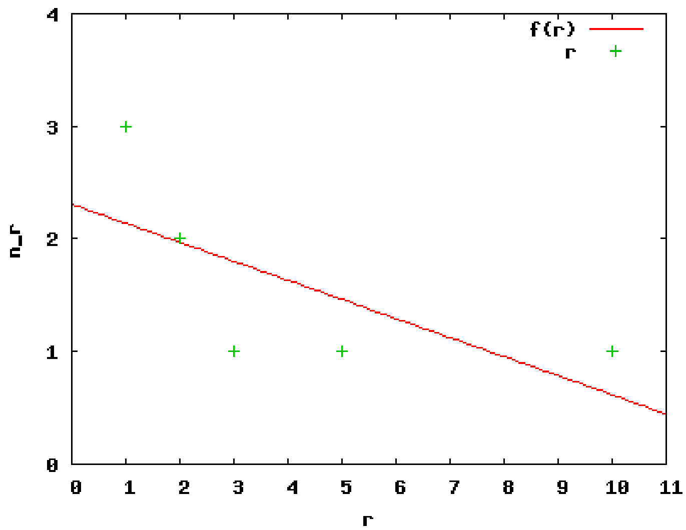
- To compute 1^*, we need $n_1 = 3$ and $n_2 = 1$

$$1^* = 2 \times \frac{1}{3} = \frac{2}{3}$$

$$p_1 = \frac{1^*}{18} = 0.037$$

- What happens when $n_{r+1} = 0$? (smoothing before smoothing)

Simple Good-Turing: linear interpolation for missing n_{r+1}



$$f(r) = a + b * r$$
$$a = 2.3$$
$$b = -0.17$$

r	$n_r = f(r)$
1	2.14
2	1.97
3	1.80
4	1.63
5	1.46
6	1.29
7	1.12
8	0.95
9	0.78
10	0.61
11	0.44

Comparison between Add-one and Good-Turing

freq	num with freq r	NS	Add1	SGT
r	n_r	p_r	p_r	p_r
0	0	0	0.0294	0.12
1	3	0.04	0.0588	0.03079
2	2	0.08	0.0882	0.06719
3	1	0.12	0.1176	0.1045
5	1	0.2	0.1764	0.1797
10	1	0.4	0.3235	0.3691

- $N = (1 * 3) + (2 * 2) + 3 + 5 + 10 = 25$
- $V = 1 + 3 + 2 + 1 + 1 + 1 = 9$
- Important: we added a new word type for unseen words. Let's call it UNK, the unknown word.
- Check that: $1.0 == \sum_r n_r \times p_r$
  $0.12 + (3 * 0.03079) + (2 * 0.06719) + 0.1045 + 0.1797 + 0.3691 = 1.0$

Comparison between Add-one and Good-Turing

freq	num with freq r	NS	Add1	SGT
r	n_r	p_r	p_r	p_r
0	0	0	0.0294	0.12
1	3	0.04	0.0588	0.03079
2	2	0.08	0.0882	0.06719
3	1	0.12	0.1176	0.1045
5	1	0.2	0.1764	0.1797
10	1	0.4	0.3235	0.3691

- NS = No smoothing: $p_r = \frac{r}{N}$
- Add1 = Add-one smoothing: $p_r = \frac{1+r}{V+N}$
- SGT = Simple Good-Turing: $p_0 = \frac{n_1}{N}$, $p_r = \frac{(r+1)\frac{n_{r+1}}{n_r}}{N}$
  with linear interpolation for missing values where $n_{r+1} = 0$
  (Gale and Sampson, 1995) *http://www.grsampson.net/AGtf1.html*

Simple Backoff Smoothing: incorrect version

$$P(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- In add-one or Good-Turing:
  $P(\text{the} \mid \text{string}) = P(\text{Fonz} \mid \text{string})$
- If $c(w_{i-1}, w_i) = 0$, then use $P(w_i)$ (back off)
- Works for trigrams: back off to bigrams and then unigrams
- Works better in practice, but probabilities get mixed up (unseen bigrams, for example will get higher probabilities than seen bigrams)

Backoff Smoothing: Jelinek-Mercer Smoothing

$$P_{ML}(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}$$

- $P_{JM}(w_i \mid w_{i-1}) = \lambda P_{ML}(w_i \mid w_{i-1}) + (1-\lambda) P_{ML}(w_i)$ where, $0 \leq \lambda \leq 1$
- Notice that $P_{JM}(\text{the} \mid \text{string}) > P_{JM}(\text{Fonz} \mid \text{string})$ as we wanted
- Jelinek-Mercer (1980) describe an elegant form of this **interpolation**:

$$P_{JM}(n\text{gram}) = \lambda P_{ML}(n\text{gram}) + (1-\lambda) P_{JM}(n-1\text{gram})$$

- What about $P_{JM}(w_i)$?
For missing unigrams: $P_{JM}(w_i) = \lambda P_{ML}(w_i) + (1-\lambda)\frac{\delta}{V}$

Backoff Smoothing: Many alternatives

$$P_{JM}(n\text{gram}) = \lambda P_{ML}(n\text{gram}) + (1-\lambda) P_{JM}(n-1\text{gram})$$

- Different methods for finding the values for λ correspond to variety of different smoothing methods
- Katz Backoff (include Good-Turing with Backoff Smoothing)

$$P_{katz}(y \mid x) = \begin{cases} \frac{c^*(xy)}{c(x)} & \text{if } c(xy) > 0 \\ \alpha(x) P_{katz}(y) & \text{otherwise} \end{cases}$$

- where $\alpha(x)$ is chosen to make sure that $P_{katz}(y \mid x)$ is a proper probability

$$\alpha(x) = 1 - \sum_y \frac{c^*(xy)}{c(x)}$$

Backoff Smoothing: Many alternatives

$$P_{JM}(n\text{gram}) = \lambda P_{ML}(n\text{gram}) + (1 - \lambda) P_{JM}(n - 1\text{gram})$$

- Deleted Interpolation (Jelinek, Mercer)
  compute λ values to minimize cross-entropy on **held-out** data which is deleted from the initial set of training data
- Improved JM smoothing, a separate λ for each w_{i-1}:

$$P_{JM}(w_i \mid w_{i-1}) = \lambda(w_{i-1}) P_{ML}(w_i \mid w_{i-1}) + (1 - \lambda(w_{i-1})) P_{ML}(w_i)$$

$$\text{where } \sum_i \lambda(w_i) = 1 \text{ because } \sum_{w_i} P(w_i \mid w_{i-1}) = 1$$

Backoff Smoothing: Many alternatives

$$P_{JM}(n\text{gram}) = \lambda P_{ML}(n\text{gram}) + (1 - \lambda) P_{JM}(n - 1\text{gram})$$

- Witten-Bell smoothing
  use the $n - 1$ gram model when the n gram model has too few unique words in the n gram context
- Absolute discounting (Ney, Essen, Kneser)

$$P_{abs}(y \mid x) = \begin{cases} \frac{c(xy) - D}{c(x)} & \text{if } c(xy) > 0 \\ \alpha(x) P_{abs}(y) & \text{otherwise} \end{cases}$$

compute $\alpha(x)$ as was done in Katz smoothing

Backoff Smoothing: Many alternatives

$$P_{JM}(n\text{gram}) = \lambda P_{ML}(n\text{gram}) + (1 - \lambda)P_{JM}(n - 1\text{gram})$$

- Kneser-Ney smoothing
  $P(\text{Francisco} \mid \text{eggplant}) > P(\text{stew} \mid \text{eggplant})$
  - *Francisco* is common, so interpolation gives $P(\text{Francisco} \mid \text{eggplant})$ a high value
  - But *Francisco* occurs in few contexts (only after *San*)
  - *stew* is common, **and** occurs in many contexts
  - Hence weight the interpolation based on number of contexts for the word using discounting

Backoff Smoothing: Many alternatives

$$P_{JM}(n\text{gram}) = \lambda P_{ML}(n\text{gram}) + (1 - \lambda)P_{JM}(n - 1\text{gram})$$

- Modified Kneser-Ney smoothing (Chen and Goodman) multiple discounts for one count, two counts and three or more counts
- Finding λ: use Generalized line search (Powell search) or the Expectation-Maximization algorithm

Trigram Models

- Revisiting the trigram model:
  $P(w_1, w_2, \ldots, w_n) =$
  $P(w_1) \times P(w_2 \mid w_1) \times P(w_3 \mid w_1, w_2) \times P(w_4 \mid w_2, w_3) \times$
  $\ldots P(w_i \mid w_{i-2}, w_{i-1}) \ldots \times P(w_n \mid w_{n-2}, \ldots, w_{n-1})$
- Notice that the length of the sentence n is variable
- What is the event space?

The stop symbol

- Let $\Sigma = \{a, b\}$ and the language be Σ^*
  so $L = \{\epsilon, a, b, aa, bb, ab, ba \ldots\}$
- Consider a unigram model: $P(a) = P(b) = 0.5$
- $P(a) = 0.5,\ P(b) = 0.5,\ P(aa) = 0.5^2 = 0.25,\ P(bb) = 0.25$ and so on.
- But $P(a) + P(b) + P(aa) + P(bb) = 1.5$!!

$$\sum_w P(w) = 1$$

The stop symbol

- What went wrong?
  No probability for $P(\epsilon)$
- Add a special stop symbol:

$$P(a) = P(b) = 0.25$$

$$P(\mathtt{stop}) = 0.5$$

- $P(\mathtt{stop}) = 0.5$,
  $P(a\ \mathtt{stop}) = P(b\ \mathtt{stop}) = 0.25 \times 0.5 = 0.125$,
  $P(aa\ \mathtt{stop}) = 0.25^2 \times 0.5 = 0.03125$ (now the sum is no longer greater than one)

The stop symbol

- With this new stop symbol we can show that $\sum_w P(w) = 1$
  Notice that the probability of any sequence of length n is $0.25^n \times 0.5$
  Also there are 2^n sequences of length n

$$\sum_w P(w) =$$

$$\sum_{n=0}^{\infty} 2^n \times 0.25^n \times 0.5$$

$$\sum_{n=0}^{\infty} 0.5^n \times 0.5 = \sum_{n=0}^{\infty} 0.5^{n+1}$$

$$\sum_{n=1}^{\infty} 0.5^n = 1$$