CMPT-379

Compilers

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Formal Language Theory

- Σ is the alphabet, e.g. $\Sigma = \{a, b\}$

- Σ^* is the set of all strings with alphabet Σ
  A good example of Σ^* is the short story *The Library of Babel* by Jorge Luis Borges

- A (formal) Language is a set of strings

Defining the Set of Regular Languages

- A **regular language** is a set of strings constructed as follows:
  - ϕ is a RL
  - $\forall x \in \Sigma \cup \epsilon, \{x\}$ is a RL
  - If L_1 and L_2 are RLs then the following are RLs,
    1. $L_1 \cdot L_2 = \{xy \mid x \in L_1, y \in L_2\}$
    2. $L_1 \cup L_2$
    3. L_1^*

Programming Languages and Formal Language Theory

- We ask the question: *Does a particular formal language describe some key aspect of a programming language*

- Then we find out if that language **isn't** in a particular language class

Programming Languages and Formal Language Theory

- For example, if we abstract some aspect of the programming language structure to the formal language:
  $\{ww^R \mid \text{ where } w \in \{a,b\}^*, w^R \text{ is the reverse of } w\}$ we can then ask if this language is a regular language

- If this is false, i.e. the language is not regular, then we have to go beyond regular languages

Recursion in Regular Languages

- Consider a regular expression for arithmetic expressions:

  $2 + 3 * 4$

  $8 * 10 + -24$

  $2 + 3 * -2 + 8 + 10$

  ```
  ^\s*-?\s*\d+\s*((\+|\*)\s*-?\s*\d+\s*)*$
  ```

- *Can we compute the* meaning *of these expressions?*

Recursion in Regular Languages

- Construct the finite state automata and associate the meaning with the state sequence

- However, this solution is missing something crucial about arithmetic expressions – *what is it?*

Do Programming Languages belong to Regular Languages

- Consider the following arithmetic expressions
  - $(((2) + (3)) * (4))$
  - $((8) * ((10) + (-24)))$
- Map ($\rightarrow a$ and) $\rightarrow b$. Map everything else to ϵ.
- This results in strings like $aaababbabb$ and $aabaababbb$
- What is a good description of this language? Let's call it L

Pumping Lemma proofs

- Is L a regular language?

- To show something is *not* a regular language, we use the **pumping lemma**

- For any infinite set of strings generated by a finite-state machine if you consider a string that is long enough from this set, there has to be a loop which visits the same state at least twice (from *the pigeonhole principle*)

- Thus, in a regular language L, there are strings x, y, z such that $xy^nz \in L$ for $n \geq 0$ where $y \neq \epsilon$

Pumping Lemma proofs

- Let L' be the intersection of L with the language L_1 defined by the regular expression a^*b^*

- Intersect the set $L = \{\epsilon, ab, abab, aabb, \ldots\}$ with $L_1 = \{\epsilon, a, b, aa, ab, aab, abb, bb, \ldots\}$

- Recall that RLs are closed under intersection, so L' must also be a RL. In fact, we can describe L' as the language $a^n b^n$ for $n \geq 0$

Pumping Lemma proofs

- For any choice of y (consider a^i or a^ib or b^i) if we multiply y^n for $n \geq 0$ we get strings that are not in L'

- For example, for a string $aaabbb$ if we pick $y = ab$ and pick $n = 2$ we get a string $aaababbb$ which is not in L'

- Hence, the pumping lemma leads to the conclusion that L' is **not** regular

- This implies that L is not regular since RLs are closed under intersection

- What lies beyond the set of regular languages?

The Chomsky Hierarchy

- **unrestricted** or **type-0** grammars, generate the *recursively enumerable* languages, automata equals *Turing machines*
- **context-sensitive** or **type-1** grammars, generate the *context-sensitive* languages, automata equals *Linear Bounded Automata*
- **context-free** or **type-2** grammars, generate the *context-free* languages, automata equals *Pushdown Automata*
- **regular** or **type-3** grammars, generate the *regular* languages, automata equals *Finite-State Automata*

The Chomsky Hierarchy
A system of grammars $G = (N, T, P, S)$

- T is a set of symbols called terminal symbols.
  Also called the alphabet Σ

- N is a set of non-terminals, where $N \cap T = \emptyset$
  Some notation: $\alpha, \beta, \gamma \in (N \cup T)^*$
  N is sometimes called the set of variables V

- P is a set of production rules that provide a finite description of an infinite set of strings (a language)

- S is the start non-terminal symbol (similar to the start state in a FSA)

Languages

- Language defined by G: $L(G)$
  - $L(G)$: set of strings $w \in T^*$ derived from S
  - $S \Rightarrow^+ w$ (derives in 1 or more steps using rules in P)
  - w is a sentence of G
  - Sentential form: $S \Rightarrow^+ \alpha$ and α contains a mix of terminals and non-terminals
- Two grammars G_1 and G_2 are equivalent if $L(G_1) = L(G_2)$

The Chomsky Hierarchy: $G = (N, T, P, S)$ where, $\alpha, \beta, \gamma \in (N \cup T)^*$

- **unrestricted** or **type-0** grammars: $\alpha \to \beta$, such that $\alpha \neq \epsilon$
- **context-sensitive** or **type-1** grammars: $\alpha A \beta \to \alpha \gamma \beta$, such that $\gamma \neq \epsilon$
- **context-free** or **type-2** grammars: $A \to \gamma$
- **regular** or **type-3** grammars: $A \to a\ B$ or $A \to a$

Regular grammars: **right-linear CFG**:
$L(G) = \{a^* b^* \mid n \geq 0\}$

$$
\begin{aligned}
A &\rightarrow a\ A && (1)\\
A &\rightarrow \epsilon && (2)\\
A &\rightarrow b\ B && (3)\\
B &\rightarrow b\ B && (4)\\
B &\rightarrow \epsilon && (5)
\end{aligned}
$$

- Input: bb

- Derivation using sentential forms: $A \Rightarrow bB \Rightarrow bbB \Rightarrow bb\epsilon = bb$

Context-free grammars: $L(G) = \{a^n b^n \mid n \geq 0\}$

$$
\begin{array}{rcl}
S & \rightarrow & a\,S\,b \\
S & \rightarrow & \epsilon
\end{array}
$$

- Input: $aabb$

- Derivation using sentential forms:
  $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aa\epsilon bb = aabb$

Context-free grammars: $L(G) = \{a^n \mid n \geq 0\}$

$$\begin{array}{lcl} S & \rightarrow & S\ S \\ S & \rightarrow & a \end{array}$$

- Input: $aaaa$
- Derivation using sentential forms:
  $S \Rightarrow SS \Rightarrow aS \Rightarrow aSS \Rightarrow aaS \Rightarrow aaSS \Rightarrow aaaS \Rightarrow aaaa$
- But what about another derivation:
  $S \Rightarrow SS \Rightarrow SSS \Rightarrow SSSS \Rightarrow aSSS \Rightarrow \ldots \Rightarrow aaaa$
- Key problem with CFGs: **ambiguity**

Context-sensitive grammars: $L(G) = \{a^n b^n \mid n \geq 1\}$

$$
\begin{aligned}
S &\rightarrow S\ B\ C \\
S &\rightarrow a\ C \\
a\ B &\rightarrow a\ a \\
C\ B &\rightarrow B\ C \\
B\ a &\rightarrow a\ a \\
C &\rightarrow b
\end{aligned}
$$

Context-sensitive grammars: $L(G) = \{a^n b^n \mid n \geq 1\}$

$$S_1$$
$$S_2\ B_1\ C_1$$
$$S_3\ B_2\ C_2\ B_1\ C_1$$
$$a_3\ C_3\ B_2\ C_2\ B_1\ C_1$$
$$a_3\ B_2\ C_3\ C_2\ B_1\ C_1$$
$$a_3\ a_2\ C_3\ C_2\ B_1\ C_1$$
$$a_3\ a_2\ C_3\ B_1\ C_2\ C_1$$
$$a_3\ a_2\ B_1\ C_3\ C_2\ C_1$$
$$a_3\ a_2\ a_1\ C_3\ C_2\ C_1$$
$$a_3\ a_2\ a_1\ b_3\ b_2\ b_1$$

Unrestricted grammars: $L(G) = \{a^{2^i} \mid i \geq 1\}$

$$
\begin{array}{rcl}
S & \rightarrow & A\ C\ a\ B \\
C\ a & \rightarrow & a\ a\ C \\
C\ B & \rightarrow & D\ B \\
\mathbf{C\ B} & \rightarrow & \mathbf{E} \\
a\ D & \rightarrow & D\ a \\
A\ D & \rightarrow & A\ C \\
a\ E & \rightarrow & E\ a \\
\mathbf{A\ E} & \rightarrow & \epsilon
\end{array}
$$

Unrestricted grammars: $L(G) = \{a^{2^i} \mid n \geq 1\}$

$$
\begin{array}{r}
S \\
A\ C\ a\ B \\
A\ a\ a\ C\ B \\
A\ a\ a\ E \\
A\ a\ E\ a \\
A\ E\ a\ a \\
a\ a
\end{array}
$$

Unrestricted grammars: $L(G) = \{a^{2i} \mid i \geq 1\}$

- A and B serve as left and right end-markers for sentential forms (derivation of each string)

- C is a marker that moves through the string of a's between A and B, doubling their number using $C\ a \rightarrow a\ a\ C$

- When C hits right end-marker B, it becomes a D or E by $C\ B \rightarrow D\ B$ or $C\ B \rightarrow E$

- If a D is chosen, that D migrates left using $a\ D \rightarrow D\ a$ until left end-marker A is reached

Unrestricted grammars: $L(G) = \{a^{2i} \mid i \geq 1\}$

- At that point D becomes C using $A\ D \rightarrow A\ C$ and the process starts over

- Finally, E migrates left until it hits left end-marker A using $a\ E \rightarrow E\ a$

- Note that $L(G) = \{a^{2i} \mid i \geq 1\}$ can also be written as a context-sensitive grammar

- But consider G', where $L(G') = \{a^{2i} \mid i \geq 0\}$ can only be an unrestricted grammar. Note that $a^0 = \epsilon$

Examples of Languages in the Chomsky Hierarchy

- **context-sensitive** grammars: 0^i, i is not a prime number and $i > 0$
- **indexed** grammars: $0^n 1^n 2^n \ldots m^n$, for any fixed m and $n \geq 0$
- **context-free** grammars: $0^n 1^n$ for $n \geq 0$
- **deterministic context-free** grammars: $S' \to S\ c$, $S \to S\ A \mid A$, $A \to a\ S\ b \mid ab$: the language of ”balanced parentheses”
- **regular** grammars: $(0|1)^* 00 (0|1)^*$

Language	*Automaton*	*Grammar*	*Recognition*	*Dependency*
Recursively Enumerable Languages	Turing Machine	Unrestricted Baa → A	Undecidable	Arbitrary
Context-Sensitive Languages	Linear-Bounded	Context-Sensitive At → aA	NP-Complete	Crossing
Context-Free Languages	Pushdown (stack)	Context-Free S → gSc	Polynomial	Nested
Regular Languages	Finite-State Machine	Regular A → cA	Linear	Strictly Local

Complexity of Parsing Algorithms

- Given grammar G and input x, provide algorithm for: Is $x \in L(G)$?
  - **unrestricted**: undecidable
  - **context-sensitive**: NSPACE[n] – linear non-deterministic space
  - **indexed** grammars: NP-Complete
  - **context-free**: $\mathcal{O}(n^3)$
  - **deterministic context-free**: $\mathcal{O}(n)$
  - **regular** grammars: $\mathcal{O}(n)$

Verifying that $L = L(G)$

- Let's say we have a context-free grammar G and a description of a language L

- How can we say for sure that $L = L(G)$?

- By verifying the statement in two directions:
  $\Rightarrow$ All strings generated by G are in L
  $\Leftarrow$ All strings $w \in L$ can be generated by G

Verifying that $L = L(G)$

- Example: $T = \{a, b\}$. Consider language L to be "all strings with same number of as and bs"

- Consider G to be a CFG: $S \to \epsilon \mid a\ S\ \ b\ S \mid b\ S\ \ a\ S$

- To verify that $L = L(G)$, prove that
  $\Rightarrow$ All strings generated by G are in L
  $\Leftarrow$ All strings $w \in L$ can be generated by G

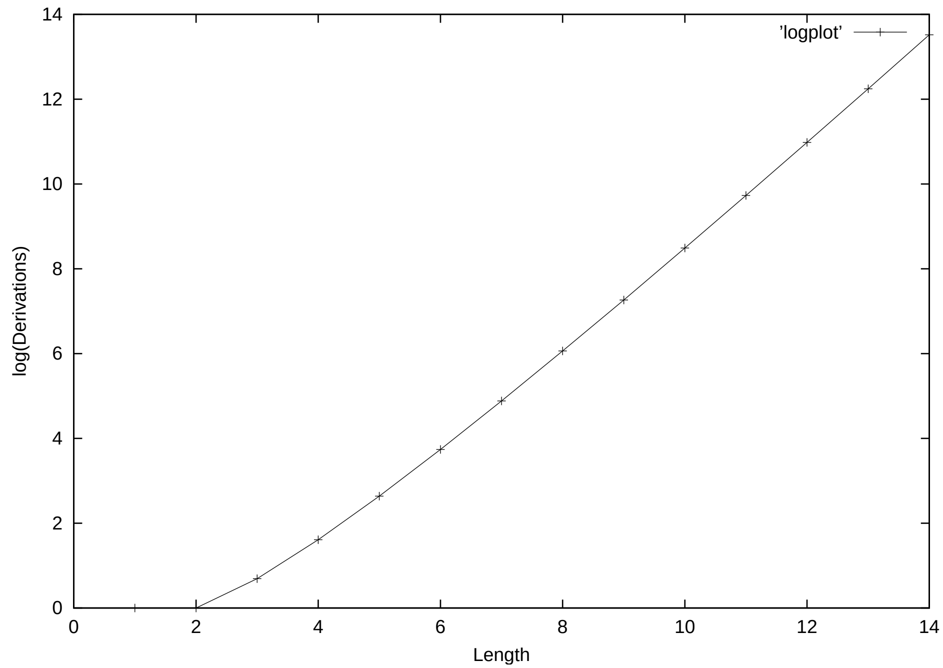
Proof ($\Rightarrow$): All strings generated by G are in L

- Proof by induction:
  - **Base case**: ϵ is in L (trivial)
  - **Inductive hypothesis**: Assume $u \in L$ and $v \in L$. Let w be generated by G with $|u| < |w|$ and $|v| < |w|$
    * Because w is generated by G then either $w \Rightarrow a\ u\ b\ v$ or $w \Rightarrow b\ u\ a\ v$, where u and v are generated by G
    * Since $|u| < |w|$ and $|v| < |w|$ and $u, v \in L$ then since we only added a single matching a, b pair, we can conclude that w is in L

Proof ($\Leftarrow$): All strings $w \in L$ can be generated by G

- Proof by induction (show that $S \Rightarrow^+ w$):
  - **Base case**: $w = \epsilon$ (trivial: $S \rightarrow \epsilon$)
  - **Inductive hypothesis**: For a given $w \in L$, assume that for all $u, v \in L$ where $|u| < |w|$ and $|v| < |w|$ we have $S \Rightarrow^+ u$ and $S \Rightarrow^+ v$
    - ***Case 1* – w starts with a**: Find the first b from the right so that $w = a\ u\ b\ v$ and v has the same number of as and bs
      Because $w \in L$ it has to be true that $u, v \in L$ and by the inductive hypothesis $S \Rightarrow^+ u$ and $S \Rightarrow^+ v$
      Using rule $S \rightarrow a\ S\ b\ S$ and the above step we get $S \Rightarrow^+ w$
    - ***Case 2* – w starts with b**: (analogous to Case 1)

CFG Ambiguity: Number of derivations grows exponentially



$\mathtt{L(G)} = \mathtt{a}+$ using CFG rules $\{\ \mathtt{S} \rightarrow \mathtt{S}\ \mathtt{S}, \mathtt{S} \rightarrow \mathtt{a}\ \}$

CFG Ambiguity

- Algebraic character of parse derivations
- Power Series for grammar for the (simplified) arithmetic expression CFG:
  $\mathtt{E} \rightarrow$ digit $|$ digit $|$ $\mathtt{E\ binop\ E}$
- Write it down as an equation with coefficients equal to number of different analyses possible:

$$\begin{aligned} E &= \text{digit} + \text{digit}\ \mathtt{binop}\ \text{digit} \\ &+ 2(\text{digit}\ \mathtt{binop}\ \text{digit}\ \mathtt{binop}\ \text{digit}) \\ &+ 5(\text{digit}\ \mathtt{binop}\ \text{digit}\ \mathtt{binop}\ \text{digit}\ \mathtt{binop}\ \text{digit}) \\ &+ 14\ldots \end{aligned}$$

CFG Ambiguity

- Coefficients in previous equation equal the number of parses for each string derived from E

- These ambiguity coefficients are Catalan numbers:

$$Cat(n) = \frac{1}{n+1}\binom{2n}{n}$$

- $\binom{a}{b}$ is the *binomial coefficient*

$$\binom{a}{b} = \frac{a!}{(b!(a-b)!)}$$

Catalan numbers

- Why Catalan numbers? Cat(n) is the number of ways to parenthesize an expression of length n with two conditions:

  1. there must be equal numbers of open and close parens

  2. they must be properly nested so that an open precedes a close

Catalan numbers

- For an expression of length n there are a total of $2n$ choose n parenthesis pairs. But $n + 1$ of them have the right parenthesis to the left of its matching left parenthesis () ().

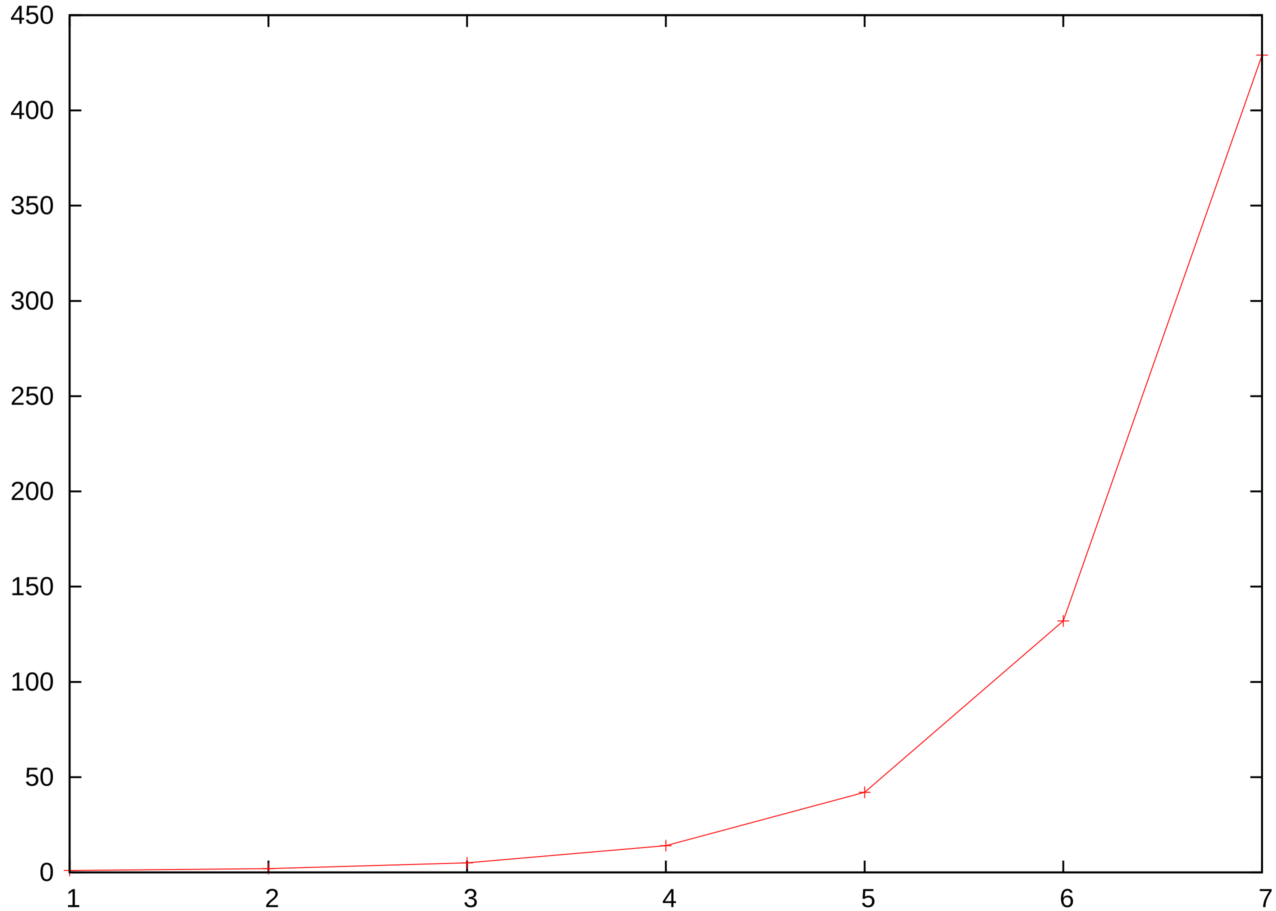
- So we divide $2n$ choose n by $n + 1$:

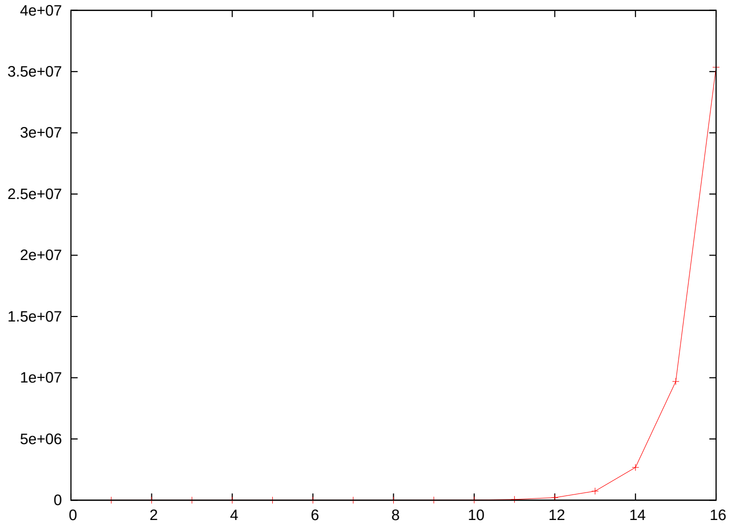
$$Cat(n) = \frac{1}{n+1}\binom{2n}{n}$$

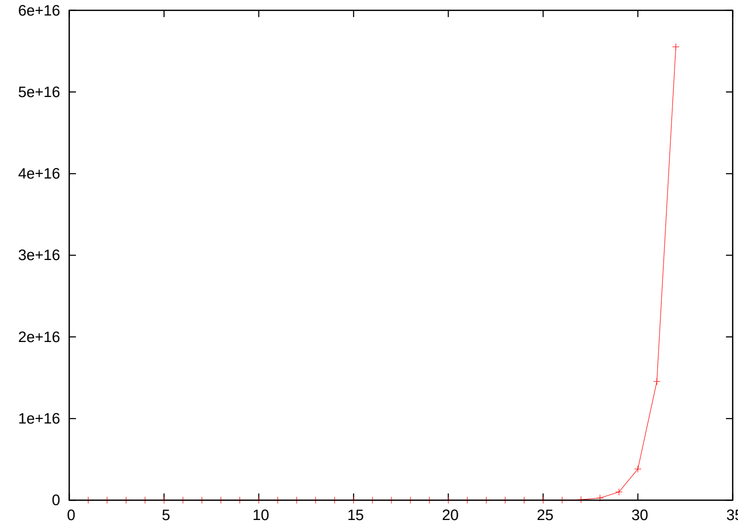
Catalan numbers

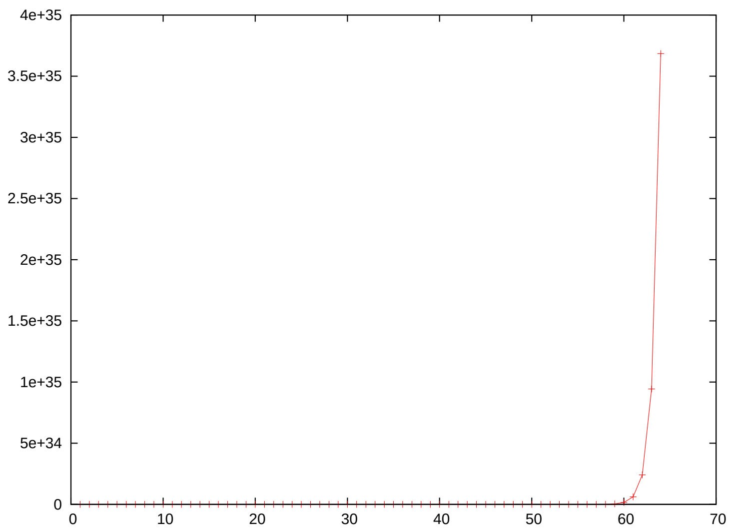
n	catalan(n)
1	1
2	2
3	5
4	14
5	42
6	132
7	429
8	1430
9	4862
10	16796

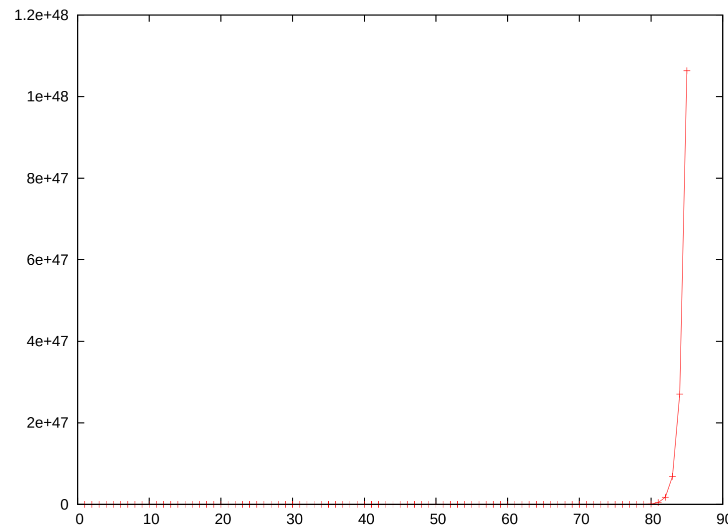
Catalan numbers




4e+07
3.5e+07
3e+07
2.5e+07
2e+07
1.5e+07
1e+07
5e+06
0
0 2 4 6 8 10 12 14 16


6e+16
5e+16
4e+16
3e+16
2e+16
1e+16
0
0 5 10 15 20 25 30 35


4e+35
3.5e+35
3e+35
2.5e+35
2e+35
1.5e+35
1e+35
5e+34
0
0 10 20 30 40 50 60 70


1.2e+48
1e+48
8e+47
6e+47
4e+47
2e+47
0
0 10 20 30 40 50 60 70 80 90

Summary

- Aspects of PL structure cannot be represented by FSAs
- Pumping lemma proofs for proving a language is not regular
- Chomsky hierarchy: from FSAs to Turing machines
- Verifying that a particular language is generated by a grammar G
- Context-free grammars (seems sufficient for PLs) but problems with ambiguity