

An algorithm for making magic cubes

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Magic squares fascinated people throughout centuries. A magic cube is a generalization of a magic square. Probably the first magic cube appeared in a letter of *Pierre de Fermat* from 1640.

A *magic cube* of order n is a cubical array

$$\mathbf{M}_n = [\mathbf{m}_n(i, j, k); 1 \leq i, j, k \leq n], \quad (1)$$

containing natural numbers $1, 2, \dots, n^3$ such that the sums of the numbers along each row (n -tuple of elements having the same coordinates in two places) and also along each of its four great diagonals are the same, e.i. $\frac{n(n^3+1)}{2}$.

Figure 1 shows the magic cube \mathbf{M}_3 which was constructed using the formula (1). The element $\mathbf{m}_3(1, 1, 1) = 8$ is in three rows containing the triples $\{8, 15, 19\}$, $\{8, 24, 10\}$, $\{8, 12, 22\}$. On the four diagonals there are the triples $\{8, 14, 20\}$, $\{19, 14, 9\}$, $\{10, 14, 18\}$ and $\{6, 14, 22\}$.

Here we gives an algorithm for making magic cubes of order $n \neq 2$. The proof of the correctness of our formulas follows from [3] and [4]. We use the following notation:

$x \pmod{n} :=$ the remainder after division of x by n

$\bar{x} := n + 1 - x,$

$x^* := \min\{x, \bar{x}\}$

$\tilde{x} := \begin{cases} 0 & \text{for } 1 \leq x \leq \frac{n}{2} \\ 1 & \text{for } \frac{n}{2} < x \leq n. \end{cases}$

We construct a magic cube $\mathbf{M}_n = [\mathbf{m}_n(i, j, k)]$ of order n using the following three formulas:

1. If $n \equiv 1 \pmod{2}$ then

$$\mathbf{m}_n(i, j, k) = a_{i,j,k} n^2 + b_{i,j,k} n + c_{i,j,k} + 1$$

with $a_{i,j,k} = (i - j + k - 1) \pmod{n}$, $b_{i,j,k} = (i - j - k) \pmod{n}$, $c_{i,j,k} = (i + j + k - 2) \pmod{n}$

2. If $n \equiv 0 \pmod{4}$ then

$$\mathbf{m}_n(i, j, k) = \begin{cases} (i-1)n^2 + (j-1)n + k & \text{if } \mathbb{F}(i, j, k) = 1 \\ (\bar{i}-1)n^2 + (\bar{j}-1)n + \bar{k} & \text{if } \mathbb{F}(i, j, k) = 0 \end{cases}$$

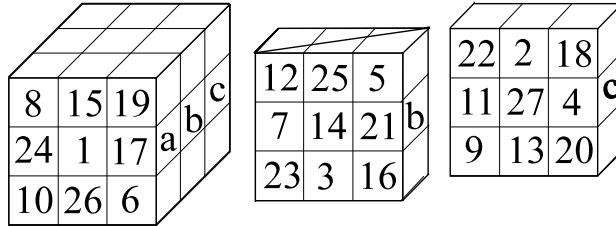


Figure 1:

where

$$\mathbb{F}(i, j, k) = (i + j + k + \widetilde{i} + \widetilde{j} + \widetilde{k}) \pmod{2}$$

3. If $n \equiv 2 \pmod{4}$ (in this case $t = \frac{n}{2}$ is odd) then

$$\mathbf{m}_n(i, j, k) = \mathbf{d}(u, v)t^3 + \mathbf{m}_t(i^*, j^*, k^*)$$

where

$$u = (i^* - j^* + k^*) \pmod{t} + 1,$$

$$v = 4\widetilde{i} + 2\widetilde{j} + \widetilde{k} + 1, \text{ and}$$

$\mathbf{d}(u, v)$ for $1 \leq u \leq t, 1 \leq v \leq 8$ is defined by the table ($x = 1, 2, \dots, \frac{n-6}{4}$)

	$\mathbf{d}(u, 1)$	$\mathbf{d}(u, 2)$	$\mathbf{d}(u, 3)$	$\mathbf{d}(u, 4)$	$\mathbf{d}(u, 5)$	$\mathbf{d}(u, 6)$	$\mathbf{d}(u, 7)$	$\mathbf{d}(u, 8)$
$\mathbf{d}(1, v)$	7	3	6	2	5	1	4	0
$\mathbf{d}(2, v)$	3	7	2	6	1	5	0	4
$\mathbf{d}(3, v)$	0	1	3	2	5	4	6	7
$\mathbf{d}(2x+2, v)$	0	1	2	3	4	5	6	7
$\mathbf{d}(2x+3, v)$	7	6	5	4	3	2	1	0

Using analogous formulas it is easy to make a computer program which constructs a magic square for every $n \neq 2$.

References

- [1] W.S.Andrews, *Magic Squares and Cubes*, Dover, New York, 1960
- [2] W.H.Benson, O.Jacoby, *Magic cubes, New recreations*, Dover, New York, 1981
- [3] M.Trenkler, *A construction of magic cubes*, The Mathematical Gazette, **84**, 36–41, March 2000
- [4] M.Trenkler, *Magic p-dimensional cubes*, Acta Arithmetica **96**, 361–364, 1998

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