SEMIPARAMETRIC BAYESIAN ESTIMATION OF DYNAMIC DISCRETE CHOICE MODELS

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We propose a tractable semiparametric estimation method for dynamic discrete choice models. The distribution of additive utility shocks is modeled by location-scale mixtures of extreme value distributions with varying numbers of mixture components. Our approach exploits the analytical tractability of extreme value distributions and the flexibility of the location-scale mixtures. We implement the Bayesian approach to inference using Hamiltonian Monte Carlo and an approximately optimal reversible jump algorithm from Norets (2021). For binary dynamic choice model, our approach delivers estimation results that are consistent with the previous literature. We also apply the proposed method to multinomial choice models, for which previous literature does not provide tractable estimation methods in general settings without distributional assumptions on the utility shocks. We develop theoretical results on approximations by location-scale mixtures in an appropriate distance and posterior concentration of the set identified utility parameters and the distribution of shocks in the model.

KEYWORDS: Dynamic Discrete choice, Bayesian nonparametrics, set identification, location-scale mixtures, MCMC, Hamiltonian Monte Carlo, reversible jump.

1. INTRODUCTION

A dynamic discrete choice model is a dynamic program with discrete controls. These models have been used widely in various fields of economics, including labour economics, health economics, and industrial organization. See, for example, Rust (1994) and Aguirregabiria and Mira (2010) for literature surveys. In such models, a forward-looking decision-maker chooses an action from a finite set in each time period. The actions affect decision-maker's

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per-period payoff and the evolution of state variables. The decision-maker maximizes the expected sum of current and discounted future per-period payoffs.

Some state variables in these models are usually assumed to be unobserved by the econometrician. The introduction of the unobserved state variable is motivated by the fact that the decision maker usually has more information than the econometrician. If the econometrician observes all state variables, then economic theory implies that the pair of the control and the state variables should obey some deterministic relationship which is never the case with real data (see, for example, page 1008 of Rust (1987) for further discussion).

Most of the previous work on estimation of dynamic discrete choice models imposes specific parametric assumptions on the distribution of the unobserved states or utility shocks. The most commonly used parametric assumption is that the unobserved states are extreme value independently identically distributed (i.i.d.). This assumption alleviates the computational burden of solving the dynamic program and computing the likelihood function. At the same time, it is well known in the literature that imposing parametric distributional assumptions can cause problems, see, for example, Manski (1999). The researcher would not know a priori if such an assumption indeed is problematic and therefore it is better to relax it if possible.

There are several previous papers that treat the unobserved state distribution nonparametrically for the binary choice case. Aguirregabiria (2010) shows the nonparametric identification of the shock distribution under particular assumptions on the per-period payoffs. Norets and Tang (2013) show that under unknown distribution of the unobserved state and discrete observed states, the utility parameters and the unobserved state distribution are only set-identified. They also show how to compute the identified sets. Buchholz et al. (2020) provide idenitifcation results for the per-period payoffs when the observed state is continuous.

For the mutinomial choice case, Chen (2017) uses exclusion restrictions (a subset of the state variables affects only current utility, but not state transition probabilities) to obtain identification and estimation results. In settings without exclusion restrictions, Norets (2011) shows that it is in principle possible to extend the method from Norets and Tang (2013) to compute the identified set in multinomial case, but it is computationally infeasible.

In this paper, we propose a tractable semiparametric estimation method applicable to the multinomial choice case. It is based on modeling the unknown distribution of shocks by a finite mixture of extreme value distributions with a varying number of mixture components. Our approach exploits the analytical tractability of extreme value distributions and the flexibility of the location-scale mixtures. The unobserved utility shocks can be integrated out analytically in the likelihood function and the expected value functions, similarly to the case with extreme value distributed shocks. At the same time, we show that the location-scale mixtures can approximate densities from a large nonparametric class in an appropriate distance and that for any given distribution of utility shocks, a finite mixture of extreme value distributions can deliver exactly the same conditional choice probabilities. Posterior concentration on the identified sets of utility parameters and the distribution of shocks is an implication of these results. We implement the Bayesian approach to inference for the model using Hamiltonian Monte Carlo and an approximately optimal reversible jump algorithm from Norets (2021). We apply our framework to binary and multinomial choice models. For the binary dynamic choice model from Rust (1987), our approach delivers estimation results that are consistent with the previous literature on semiparametric estimation (Norets and Tang (2013)). For the multinomial choice model of medical care use and work absence from Gilleskie (1998), we demonstrate how uncertainty about model parameters and counterfactuals increases when the distributional assumptions on the shocks are relaxed.

The rest of the paper is organized as follows. Section 2 describes the general model setup. In Section 3, we introduce our semiparametric framework. In Section 4, we describe the Bayesian estimation method. Section 5 presents theoretical results. Sections 6 and 7 contain the applications. Derivations, proofs, and implementation details are given in appendices.

2. GENERAL MODEL SETUP

In the infinite-horizon version of the model, the decision maker maximizes the expected discounted sum of the per-period payoffs

$$(1) \qquad \max_{d_t, d_{t+1}, \ldots} E_t \left(\sum_{j=0}^{\infty} \beta_j u\left(x_{t+j}, d_{t+j}, \epsilon_{t+j}\right) \right),$$

where $d_t \in \{0, 1, ..., J\} = D$ is the control variable, $x_t \in X$ is the state variable observed by the econometrician, $\epsilon_t = (\epsilon_{t0}, \epsilon_{t1}, ..., \epsilon_{tJ})^T \in R^{J+1}$ is the state variable unobserved by the econometrician, β is the time discount factor, and $u(x_t, d_t, \epsilon_t)$ is the per-period payoff. The decision-maker observes both x_t and ϵ_t at time t before making the decision.

Following Rust (1987) and the subsequent literature, we assume that (i) the per-period payoffs are additively separable in ϵ_t, $u(x_t, d_t, \epsilon_t) = u(x_t, d_t) + \epsilon_{td_t}$; (ii) ϵ_t's are independent of other variables and independently identically distributed (i.i.d.) over time according to a distribution F with zero mean; (iii) the observed states evolve according to a controlled Markov chain $G_x^j = Pr(x_{t+1}|x_t = x, d_t = j)$. In most applications, the utility functions are assumed to depend on a vector of unknown parameters, $\theta \in \mathbb{R}^{d_\theta}$, that are estimated. Below, we often omit θ in $u(x_t, d_t; \theta)$ and related objects such as value functions for notation brevity.

Under mild regularity conditions (Bhattacharya and Majumdar (1989)), the decision problem in (1) admits the following Bellman representation

$$Q(x) = \int \max_{j=0,1,...,J} \left[u(x, j) + \beta G_x^j Q + \epsilon_j \right] dF(\epsilon), \tag{2}$$

where Q is called the *Emax* function and $G_x^j Q$ denotes $E(Q(x_{t+1})|x_t = x, d_t = j)$. The conditional choice probability (CCP) can be expressed as

$$p(d|x) = \int 1\left\{ u(x, d) + \beta G_x^d Q + \epsilon_d \geq u(x, j) + \beta G_x^j Q + \epsilon_j, \forall j \right\} dF(\epsilon). \tag{3}$$

For a panel of observations, $\{x_{it}, d_{it},\ i = 1, \ldots, n,\ t = 1, \ldots, T\}$, of n decision makers over T time periods, the partial likelihood function (with the fixed G_x^j pre-estimated from the observed transitions as is commonly done in practice) can be expressed as

$$\log L(x_{it}, d_{it},\ i = 1, \ldots, n,\ t = 1, \ldots, T) = \sum_{i,t} \log p(d_{it}|x_{it}). \tag{4}$$

Rust (1987) proposed to solve the dynamic optimization problem by first iterating on the Bellman equation (2) to get close to the fixed point Q and then using a Newton method that quickly converges to the fixed point from a close starting point. With Q at hand, one can compute the CCPs in (3) and evaluate the likelihood function at a given $(u;\beta;G_x^j,\ j\in D,\ x\in X)$. Alternatively, Su and Judd (2008) proposed to use constrained optimization to maximize the likelihood function subject to (2). In either scenario, assuming that ϵ_{tj}'s are i.i.d. Gumbel (or extreme value type I) delivers analytical expressions for the integrals in (2) and (3),

$$p(d|x)=\frac{e^{u(x,d)+\beta G_x^d Q}}{\sum_{j=0}^J e^{u(x,j)+\beta G_x^j Q}},\ Q(x)=\log\sum_{d=0}^J e^{u(x,d)+\beta G_x^d Q}. \tag{5}$$

In the resulting dynamic logit specification, the computational burden of the model solution and estimation is considerably alleviated. Hence, the dynamic logit is predominantly used in applications of the estimable dynamic discrete choice models. At the same time, the econometrics literature suggests that the distributional assumptions could be problematic in general, see, for example, Manski (1999). In the present context, in the special case of binary choice and discrete observed states, Norets and Tang (2013) show that the utility parameters and the distribution of shocks are only set-identified. In the following section, we specify a non-parametric model for the distribution of shocks for the general multinomial choice case that provides analytical simplifications comparable to those of the dynamic logit.

3. SEMIPARAMETRIC MODEL

Rather than making a particular parametric assumption, we model the distribution of unobserved states using a flexible mixture specification. In order to reduce the number of parameters, we use an innocuous normalization $\epsilon_{t0}=0$ (the agent's decisions and value functions do not change if ϵ_{t0} is subtracted from the per-period payoff $u(x_t,d_t,\epsilon_t)$ for all $d_t\in\{0,1,\dots,J\}$).

For $\mu\in\mathbb{R}^J$ and $\sigma>0$, let us define a multivariate Gumbel density by

$$\phi(z;\mu,\sigma)=\prod_{j=1}^J\frac{1}{\sigma}\phi\left(\frac{z_j-\mu_j}{\sigma}\right),\ \text{where}\ \phi(z_j)=e^{-z_j-\gamma-e^{-z_j-\gamma}}, \tag{6}$$

is the univariate Gumbel density and γ is the Euler-Mascheroni constant. Some relevant properties of the Gumbel distribution are outlined in Appendix D.2.

For $\psi = \{\mu_k \in \mathbb{R}^J,\ \sigma_k \in \mathbb{R}_+, \omega_k \in (0,1),\ k = 1, 2, \ldots\}$, we model the unknown density by a location-scale mixture of Gumbel densities

$$\epsilon_t \sim p(\epsilon_t|\psi, m) = \sum_{k=1}^{m} \omega_k \phi(\epsilon_t; \mu_k, \sigma_k), \tag{7}$$

with a variable number of mixture components m for which a prior distribution on the set of positive integers is specified. It is well known that location-scale mixtures with a variable or infinite number of components can approximate any continuous or smooth density arbitrarily well. For example, Bayesian models based on normal mixtures deliver optimal up to a log factor posterior contraction rates in adaptive estimation of smooth densities (Rousseau (2010) and Shen et al. (2013)). To develop intuition for this type of results note that the standard nonparametric density estimator based on kernel ϕ is a special case of (7), or, alternatively and more in line with the actual proofs, the expectation of the standard kernel density estimator is a continuous mixture that can be discretized into a special case of (7). Thus, it is reasonable to expect that the specification (7) is very flexible. Indeed, in Section 5, we show that it can approximate smooth multivariate densities arbitrarily well in an appropriate distance so that the conditional choice probabilities and the *Emax* function implied by the model with (7) approximate those from the model with an arbitrary smooth density for ϵ_t.

The model specification with (7) also possesses attractive analytical properties. If a normalization $\epsilon_{t0} = 0$ is not imposed and $(J + 1)$-dimensional version of (7) is used, then Q and p could be expressed as mixtures of the appropriately recentered and rescaled expressions from the logit model (5). However, even if the normalization $\epsilon_{t0} = 0$ is imposed, which is preferred as it reduces the dimension of the distribution we model nonparametrically, closed form expressions for Q and p are still available. They are presented in the following lemma.

Lemma 1 *Suppose $\epsilon \sim p(\epsilon|\psi, m)$. Then,*

$$p(d|x) = \begin{cases} \sum_{k=1}^{m} \omega_k \exp\left[-e^{-a_{kx}}\right], & \text{if } \ d = 0 \\ \sum_{k=1}^{m} \omega_k \exp\left[\frac{u(x,d)+\beta G_x^d Q+\mu_{dk}}{\sigma_k} - A_{kx}\right]\left\{1 - \exp\left[-e^{-a_{kx}}\right]\right\}, & \text{if } \ d = 1, ..., J; \end{cases} \tag{8}$$

$$Q(x) = \sum_{k=1}^{m} \omega_k \sigma_k \left[A_{kx} + E1(e^{-a_{kx}}) \right], \ \textit{where } E1(z) = \int_z^\infty e^{-t}/t dt, \tag{9}$$

$$A_{kx} = \log \sum_{j=1}^{J} \exp\left(\frac{u(x,j) + \beta G_x^j Q + \mu_{jk}}{\sigma_k} \right) \ \textit{and } a_{kx} = \frac{u(x,0) + \beta G_x^0 Q}{\sigma_k} + \gamma - A_{kx}.$$

The derivations of (8) and (9) can be found in Appendix A. The derivatives of (8) and (9) that are useful for the model solution and estimation are given in Appendix D.4. Similarly to Rust (1987), we obtain the solution of the Belman equation (9) by a Newton-Kantorovich method described in Appendix D.1.

4. BAYESIAN INFERENCE

In estimation of models based on location-scale mixtures with a variable number of components, the econometrician faces several problems. First, the scale parameters need to be bounded away from zero; otherwise, the likelihood function is unbounded. Second, the likelihood function is a rather complex function of parameters with multiple modes. Third, the number of mixture components needs to be selected in the estimation procedure. Finally, there is usually considerable uncertainty about the estimated parameter values and it should be taken into account in model predictions and counterfactual analysis.

The Bayesian approach to inference and the associated simulation methods are well suited for solving these problems. Prior distributions can provide soft constraints for the scale parameters and an appropriate penalization for the number of mixture components or model complexity. MCMC methods can successfully explore very complex posterior or likelihood surfaces. Posterior predictive distributions for objects of interest automatically incorporate the uncertainty about parameter values including the number of mixture components.

Our MCMC algorithm for simulating from the model posterior distribution combines Hamiltonian Monte Carlo for simulating parameters conditional on the number of mixture components and an approximately optimal reversible jump algorithm from Norets (2021) for simulating the number of mixture components. HMC is a very popular and efficient way of constructing proposal distributions for the Metropolis-Hastings algorithm. See, for example, Neal (2012) for an introduction. It requires only evaluation of the likelihood and the prior and their derivatives. The proposals are obtained following the Hamiltonian dynamics on the parameter space that describe the movement of a puck on a frictionless surface with some initial random momentum. For implementing the HMC step of the algorithm we utilize the HMC sampler from the Matlab Statistics and Machine Learning toolbox.

The reversible jump algorithm from Norets (2021) also uses the values of the likelihood, the prior and their derivatives with respect to a part of the parameter vector. As noted in Section 3, for our specification, the derivatives are available in closed form given a solution to the simplified Bellman equation (9). A detailed description of the MCMC algorithm is given in Appendix B.

In addition to the normalization $\epsilon_{t0} = 0$, the location and the scale of ϵ_t can be innocuously fixed. Instead we impose a location and scale normalization on the per-period payoffs and keep the location and scale of (7) unrestricted, which simplifies the MCMC algorithm.

Let us introduce the prior distributions for the parameters of the mixture in (7). We use the following prior distributions on the number of mixture components and the mixing weights,

$$\Pi(m) \propto e^{-\underline{\mathrm{A}}_m m (\log m)^\tau}, \tag{10}$$

$$\Pi(\omega_1, ..., \omega_m | m) = \text{Dirichlet}(\underline{\mathrm{a}}/m, ..., \underline{\mathrm{a}}/m), \tag{11}$$

where the hyperparameters $\underline{\mathrm{a}}$, $\underline{\mathrm{A}}_m$, and τ are specified in the applications below. For the theoretical results obtained in the present paper, we only need $\Pi(m) > 0, \forall m$ and full support on the simplex for $\Pi(\omega_1, ..., \omega_m | m)$. Nevertheless, the functional forms in (10) and (11) perform well in applications and deliver optimal posterior contraction rates in nonparametric multivariate density estimation by mixtures of normal distributions, see, for example, Shen et al. (2013) and Norets and Pelenis (2021). We allow the scale parameter σ_k to have a multiplicative part σ that is common across the mixture components: $\sigma_k = \tilde{\sigma}_k \cdot \sigma$. This multiplicative specification performs well in a variety of applications of location-scale mixture models (see, for example, Geweke (2005)) and is also important for the aforementioned optimal posterior concentration results for mixtures of normals. In the applications, we use finite mixtures of normals as flexible priors for $\log \sigma$, $\log \tilde{\sigma}_k$ and the location parameters μ_{kj}.

5. APPROXIMATION RESULTS AND ASYMPTOTICS

In this section, we show that location-scale mixtures of Gumbel densities can arbitrarily well approximate densities from a large nonparametric class. These approximation results combined with the Schwartz (1965)'s theorem imply a posterior consistency result for the set identified model parameters. We also show that a model with a finite mixture of Gumbels can exactly match the CCPs from a model with an arbitrary distribution of shocks.

5.1. *Approximation Results*

Let us first define a distance for distributions of utility shocks: for F_i with density f_i, $i = 1, 2$,

$$\rho(F_1, F_2) = \int (1 + \sum_{j=0}^{J} |\epsilon_j|) |f_1(\epsilon) - f_2(\epsilon)| d\epsilon.$$

This distance is appropriate for our purposes as the *Emax* function and the conditional choice probabilities are continuous in that distance as shown in the following lemma.

LEMMA 2 *Suppose (i) $u(x, j) \leq \bar{u} < \infty$ for all $x \in X$ and $j = 0, 1, ..., J$; (ii) under F, the density for $\epsilon_j - \epsilon_d$ is bounded for all $j \neq d$; (iii) under F, $E(|\epsilon_j|)$ is finite for all j. Then, the Emax function and the conditional choice probabilities are locally Lipschitz continuous in F,*

$$\sup_x |Q(x; F) - Q(x; \tilde{F})| \leq C \cdot \rho(F, \tilde{F}),$$

$$\sup_{d,x} |p(d|x; F) - p(d|x; \tilde{F})| \leq C' \cdot \rho(F, \tilde{F}),$$

where constants C and C' depend on β, $\bar{u}$ and the bounds on the densities and moments in conditions (ii)-(iii).

The lemma holds irrespective of whether the innocuous normalization $\epsilon_{t0} = 0$ is imposed. Its proof is given in Appendix C.

The following lemma shows that densities satisfying smoothness and finite moment conditions can be approximated by mixtures of Gumbels in distance ρ.

LEMMA 3 *Let f be a density on $\mathbb{R}^I$ satisfying a moment existence condition*

$$\int ||\mu||_2 f(\mu) d\mu < \infty,$$

and a smoothness condition

$$|f(z+h) - f(z)| \leq ||h||_2 L_f(z) e^{\tau ||h||_2}, \tag{12}$$

for some $\tau > 0$ and an envelope function $L_f(\cdot)$ such that

$$\int (1 + |z_i|) L_f(z) dz < \infty, i = 1, ..., I. \tag{13}$$

Then, for any $\delta > 0$, there exist (m, ω, μ, σ) where $m \in \mathbb{Z}^+$, $\omega_j \in [0,1]$ with $\sum_{j=1}^m \omega_j = 1$, $\mu_j \in \mathbb{R}^I$, $j = 1, ..., m$, and $\sigma > 0$ such that

$$\rho\left(f(\cdot), \sum_{j=1}^m \omega_j \phi(\,\cdot\,; \mu_j, \sigma)\right) < \delta.$$

We conjecture that the smoothness and tail conditions on f in the lemma can be weakened at the expense of the proof simplicity. The lemma is proved in Appendix C. The proof uses only smoothness and tail conditions on ϕ that are shown to hold for Gumbel densities in Lemmas 6 and 7 in Appendix C. Thus, Lemma 3 holds for more general location-scale mixtures. These generalizations do not seem essential and we do not elaborate on them here for brevity.

The final intermediate result that we need for establishing posterior consistency is the continuity of finite Gumbel mixtures in parameters in distance ρ, which we present in the following lemma.

LEMMA 4 *Let F^1 and F^2 denote two mixtures of Gumbel densities on $\mathbb{R}^J$ with densities $f^i(\epsilon) = \sum_{k=1}^m \omega_k^i \phi(\epsilon; \mu_k^i, \sigma^i)$. Then, for a given $\delta > 0$ and F_1, there exists $\tilde{\delta} > 0$ such that* *for any F_2 with parameters satisfying: $|\sigma^1 - \sigma^2| < \tilde{\delta}$, $|\omega_k^1 - \omega_k^2| < \tilde{\delta}$, and $|\mu_k^1 - \mu_k^2| < \tilde{\delta}$, $k = 1, \ldots, m$, we have $\rho(F^1, F^2) < \delta$.*

5.2. *Posterior Consistency*

Let us denote the short panel dataset by $D^n = \{d_{it}, x_{it}, t = 1, \ldots, T, i = 1, \ldots, n\}$; the observations are assumed to be independently identically distributed over i, with a small T and a large n. The utility function is parameterized by a vector $\theta \in \mathbb{R}^{d_\theta}$, $u(x, d; \theta)$. Let $P(\theta, F) = \{p(d|x; \theta, F),\ x \in X,\ d = 0, \ldots, J\}$ denote the collection of the CCPs for the distribution of shocks F and parameters θ.

Theorem 1 *Suppose (i) The observed state space is finite, $X = \{1, \ldots, K\}$; (ii) G^d, $d = 0, \ldots, J$ and the distribution of the initial observed state x_{i1} are known and fixed; (iii) $\forall x \in X$, $\exists t \in \{1, \ldots, T\}$, such that $Pr(x_{it} = x) > 0$; (iv) $u(x, d; \theta)$ is continuous in θ; (v) (θ_0, F_0) are the data generating values of parameters, F_0 satisfies the conditions of Lemma 3; (vi) For any $\delta > 0$, $\Pi(B_\delta(\theta_0)) > 0$, where $B_\delta(\theta_0)$ is a ball with radius δ and center θ_0. (vii) For any $\delta > 0$, positive integer m, $\mu_k \in \mathbb{R}^J$, $\sigma_k > 0$, $w_k \geq 0$, $k = 1, \ldots, m$, $\sum_{k=1}^m w_k = 1$, $\Pi(B_\delta(\mu_1, \sigma_1, \ldots, \mu_m, \sigma_m, w_1, \ldots, w_{k-1})|m) > 0$. Then, for any $\delta > 0$,*

$$\Pi\big(\theta, F : ||P(\theta_0, F_0) - P(\theta, F)|| > \delta \big| D^n\big) \to 0 \text{ almost surely.}$$

The theorem shows that the posterior concentrates on the set of parameters and distributions of shocks (θ, F) such that their implied CCPs $P(\theta, F)$ are arbitrarily close to the data generating CCPs $P(\theta_0, F_0)$. To prove this result we use Schwartz (1965) posterior consistency theorem: if the prior puts positive mass on any Kullback-Leibler neighborhood of the data generating distribution then the posterior puts probability converging to 1 on any weak neighborhood of the data generating distribution. Since X is finite, the convergence in weak topology and Kullback-Leibler divergence for distributions on $\{d_{it}, x_{it}, t = 1, \ldots, T\}$ are equivalent to convergence for vectors $\{p(d|x),\ x \in X,\ d = 0, \ldots, J\}$ in a euclidean metric when G^d and the distribution of the initial x_{i1} are fixed and satisfy our theorem condition (iii). Thus, to obtain the conclusion of the theorem we only need to establish that the prior puts positive probability on any euclidean neighborhood of $P(\theta_0, F_0)$. First, note that when $u(x, d; \theta)$ is continuous in θ, $P(\theta, F)$ is also continuous in θ in our settings, see, for example Norets (2010); and, thus, Lipschitz continuity of $P(\theta, F)$ in F from Lemma 2 delivers continuity of $P(\theta, F)$ in (θ, F). The finite mixture approximation result in Lemma 3, the continuity of $P(\theta, F)$ in (θ, F), the continuity of finite mixtures in parameters in Lemma 4, and the theorem conditions (vi) and (vii) on the priors, imply a positive prior probability for any neighborhood of $P(\theta_0, F_0)$, and thus, the theorem conclusion.

The finiteness of the observed state space is important for our posterior consistency argument even though our approximation and continuity results in lemmas above do not require it. The finiteness assumption is not limiting in practice as in most applications even continuous observed states are discretized. The assumption of known G^d can be easily relaxed at the expense of a more involved notation.

Theorem 1 characterizes the support of the posterior in the limit but not its shape, which can also be of interest. Note that the data depend on (θ, F) only through CCPs $P(\theta, F)$ and the posterior for CCPs concentrates at $P(\theta_0, F_0)$. Therefore, the posterior for (θ, F) converges to the conditional prior $\Pi(\theta, F|P)$ at $P = P(\theta_0, F_0)$ under continuity conditions on $\Pi(\theta, F|P)$, see, for example, Plagborg-Møller (2019). As the distribution of shocks is an infinite dimensional object and the solution to the dynamic program does not have a simple explicit form, it appears difficult to characterize the conditional prior $\Pi(\theta, F|P)$, which is implied by the map $P(\theta, F)$ and the prior on (θ, F) . Nevertheless, we can deduce from our approximation and continuity results that under the conditions of Theorem 1, for $\delta > 0$ there exists $\tilde{\delta} > 0$ such that $\theta \in B_{\tilde{\delta}}(\theta_0)$ and $F \in B_{\tilde{\delta}}(F_0)$ imply $P(\theta, F) \in B_{\delta}(P(\theta_0, F_0))$ and

$$\begin{aligned}
&\Pi\left(\theta \in B_{\tilde{\delta}}(\theta_0), F \in B_{\tilde{\delta}}(F_0) \middle| P \in B_{\delta}(P(\theta_0, F_0))\right) = \frac{\Pi(\theta \in B_{\tilde{\delta}}(\theta_0), F \in B_{\tilde{\delta}}(F_0))}{\Pi\big(P \in B_{\delta}(P(\theta_0, F_0))\big)} \\
&\geq \Pi\big(\theta \in B_{\tilde{\delta}}(\theta_0), F \in B_{\tilde{\delta}}(F_0)\big) > 0,
\end{aligned}$$

which suggests that the conditional prior would not rule out the data generating parameter values.

5.3. *Exact Matching of CCPs*

In this subsection, we show that for a finite observed state space, our model formulation based on finite mixtures can exactly match the CCPs from a model with an arbitrary distribution of shocks.

Lemma 5 *Suppose (i) The observed state space is finite, $X = \{1, \ldots, K\}$; (ii) (θ_0, F_0) are the data generating values of parameters; (iii) F_0 has a density that is positive on $\mathbb{R}^J$. Then there exists a finite mixture of Gumbels F such that $P(\theta_0, F_0) = P(\theta_0, F)$.*

The result in Lemma 5 holds not only for mixtures of Gumbels but more generally for location-scale mixtures of distributions with finite first moments, which is evident from the proof presented in Appendix C. Lemma 5 can be used to weaken the smoothness assumptions on F_0 in the posterior consistency results of Theorem 1 (the approximation results in Lemma 3 can be replaced by exact CCPs matching in Lemma 5). Nevertheless, the approximation results in Lemma 3 have independent value. First, they hold for infinite X. Furthermore, they imply that the prior on the distribution of shocks is flexible in a sense that it puts positive probability on any metric ρ neighborhood in a large nonparametric class of distributions, which suggests that the conditional prior for the distribution of shocks and parameters given CCPs, $\Pi(\theta, F|P)$, is also flexible as discussed at the end of Section 5.2.

6. APPLICATION I: RUST'S BINARY CHOICE MODEL

Norets and Tang (2013) propose a method for computing identified sets for parameters in dynamic binary choice models and apply their method to the Rust (1987)'s model. In this section, we show that our semiparametric model can also recover the identified set for that model.

6.1. *Rust (1987)'s Optimal Bus Engine Replacement Problem*

In each time period t, the agent decides whether to replace the bus engine ($d_t = 1$) or not ($d_t = 0$) given the current mileage x_t of the bus. Replacing an engine costs θ_0. If $d_t = 0$, then the owner conducts a regular maintenance which costs $-\theta_1 x$. The utility function of the owner is $u(x,0) = \theta_0 + \theta_1 x$ and $u(x,1) = \epsilon$. Rust (1987) assumes that ϵ follows the logistic distribution; hence, it is the dynamic logit model. The mileage $x_t \in \{1,\dots,K=90\}$ evolves over time following the transition probabilities: $Pr(x_{t+1}|x_t, d_t = 0; \theta) = \theta_2$ for $x_{t+1} - x_t = 0$; $Pr(x_{t+1}|x_t, d_t = 0; \theta) = \theta_3$ for $x_{t+1} - x_t = 1$; $Pr(x_{t+1}|x_t, d_t = 0; \theta) = 1 - \theta_2 - \theta_3$ for $x_{t+1} - x_t = 2$; and $Pr(x_{t+1}|x_t, d_t = 0; \theta) = 0$ otherwise. When the engine is replaced ($d_t = 1$), the mileage restarts at $x_t = 1$.

As in Norets and Tang (2013), we use the following data generating process: logistic distribution for ϵ, $\theta_0 = 5.0727, \theta_1 = -0.002293, \theta_2 = 0.3919, \theta_3 = 0.5953$ and the discount factor $\beta = 0.999$.

At the data generating parameters, we solve the dynamic program to obtain the vector of CCPs, $(p(d|1),\dots,p(d|K))$ for $d = 0,1$. Rather than using simulated observations in this exercise, we use the true CCPs as the sample frequencies and report the results for different sample sizes. Specifically, for a given $N \in Z^+$, we set n_{dx}, the number of times d was chosen at each state x as follows, $n_{0x} = p(0|x) \times N$ and $n_{1x} = p(1|x) \times N$ for $x = 1,...,K$. In this way, we can check if our MCMC algorithm for the semiparametric model specification can recover the identified set computed by the algorithm from Norets and Tang (2013) for a given fixed vector of CCPs. Estimation results for simulated data are presented for the multinational choice application in Section 7.

Since we do not impose a location and scale normalization on the mixture specification for the distribution of ϵ and Rust's model has only two utility parameters that are defined by the location and the scale, the values of (θ_0, θ_1) corresponding to the location and scale of the logistic distribution (used in Rust (1987) and Norets and Tang (2013)) can be computed from the values of $(\sigma, \omega_k, \mu_k, \sigma_k,\ k = 1,\dots,m)$ as described in Appendix E. We use the following flexible prior distributions that were tuned to spread out the probability over a large region for (θ_0, θ_1) that includes the identified set.

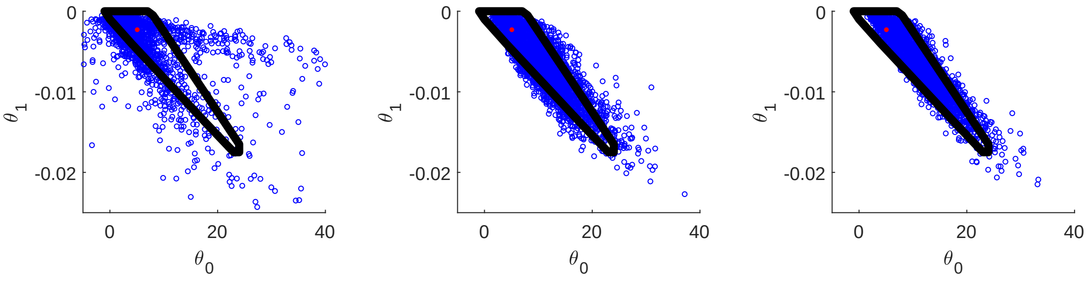
$$\pi(m=k) \propto e^{-\underline{\mathrm{A}}_m k (\log k)^\tau},\ \underline{\mathrm{A}}_m = 0.05,\ \tau = 5,$$

$$(\omega_1,...,\omega_m) \sim \text{Dirichlet}(\underline{\mathrm{a}}/m,...,\underline{\mathrm{a}}/m),\ \underline{\mathrm{a}} = 10,$$

$$\mu_k \sim 0.5N(2.5, 1^2) + 0.5N(-3, 7^2),$$

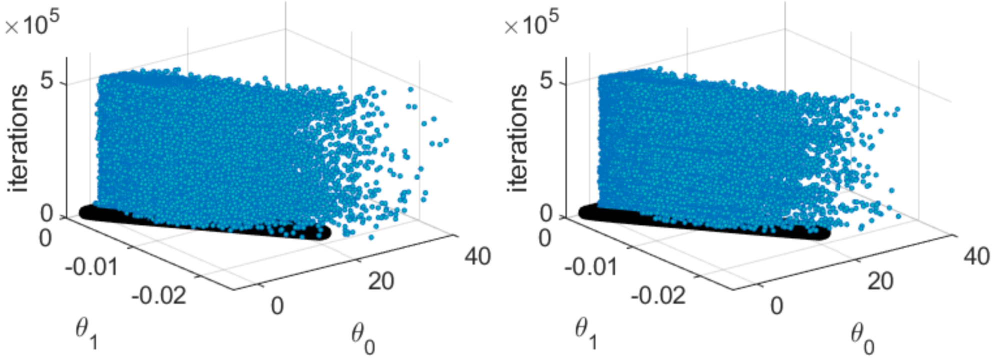
$$\log \sigma_k \sim 0.4N(0, 1^2) + 0.6N(-6, 1^2),\ \log \sigma \sim N(0, 0.01^2).$$

Below, we report the draws of (θ_0, θ_1) obtained from the draws of $(\sigma, \omega_k, \mu_k, \sigma_k,\ k = 1, \ldots, m)$ for the prior and the posterior for $N \in \{3, 10\}$ and compare them to the true identified set. Panel (a) of Figure 1 shows the prior draws of the utility parameters. First note that the prior draws are not uniformly distributed on the utility parameter space. In practice, it is difficult to come up with a prior for the distribution parameters that implies a uniform prior in the θ space. Second, we can see that many prior-draws are outside of the identified set. The other two panels in Figure 1 show posterior draws of utility parameters with different number of observations $N \in \{3, 10\}$. Compared to the prior-predicted draws, we see that the posterior mass concentrates more on the identified set as N increases. To assess the convergence of the MCMC algorithm, consider Figure 2 showing the draws of utility parameters for each MCMC iteration starting from 0 and ending at 500,000. We see that the chain sweeps through the identified set repeatedly during the MCMC run. Thus, we conclude that our approach can be used to recover the identified sets of utility parameters.



(a) prior (b) $N = 3$ (c) $N = 10$

Figure 1: Utility parameter draws. Prior (left). Posterior draws for $N = 3$ (center) and $N = 10$ (right). 500,000 MCMC iterations. The first 100,000 draws discarded as burn-in. Every 10th draw is shown. The true identified set is shown by solid black lines. The red dot corresponds to the point-identified utility parameter values under the logistic distribution for ϵ.



(a) $N = 3$ (b) $N = 10$

Figure 2: Utility parameter draws. The z-axis represents the MCMC iterations. The true identified set is shown in black.

Figure 5 in Appendix E shows additional evidence of MCMC convergence including trace plots of m, $\sum_{k=1} \omega_k \mu_k$, and other parameters.

7. APPLICATION II: GILLESKIE'S MULTINOMIAL MODEL OF MEDICAL CARE USE AND WORK ABSENCE

In this section, we illustrate our methodology using Gilleskie (1998) model of medical care use and work absence.

7.1. *The model*

In the model, individuals occupy one of $K + 1$ distinct health states; well, $k = 0$, or sick with an illness type $k \in \{1, \ldots, K\}$. An individual receives the utility associated with being well until contracting an illness of a specific type. We make the simplifying assumption that there is only one illness type, ($K = 1$), although Gilleskie (1998) works with two illness types, $K = 2$. An illness episode lasts for T periods and enumerated by $t = 1, \ldots, T$. $t = 0$ corresponds to the state of being well, $k = 0$.

7.1.1. *Alternatives*

An individual who became sick makes decisions about doctor visits and and work absences. In each period t of an illness, alternatives available to an employed individual who is sick are: $d_t = 0$ - work and don't visit a doctor, $d_t = 1$ - work and visit a doctor, $d_t = 2$ - don't work and don't visit a doctor, and $d_t = 3$ - don't work and visit a doctor. The utility of the agent depends on the elapsed length of the current illness t, the accumulated number of physician visits v_t, and the accumulated number of work absences a_t. The state variables observed by the econometrician and the agent at t are $x_t = (t, v_t, a_t)$. Note that $k = 1$ if and only if $t > 0$, so k does not appear in the definition of x_t.

7.1.2. *State variable transitions*

The state variables evolve in the following way. An individual always starts with the state of being well, $x_0 = (0, 0, 0)$. The individual contracts an illness and moves to the state $x = (1, 0, 0)$ with probability $\pi^S(H)$, where H is a vector of exogenous indicators for a health status and being between 45-64 years of age.

The accumulated number of physician visits v_t and the accumulated number of illness-related absence from work a_t both take values in $\{0, 1, \ldots, T-1\}$. They start from $v_1 = a_1 = 0$ and evolve in the following way: $v_{t+1} = v_t + 1(d_t = 1 \text{ or } 3)$ and $a_{t+1} = a_t + 1(d_t = 2 \text{ or } 3)$.

In each illness period $t \in \{1, \ldots, T\}$, the individual recovers and returns to the state of being well with probability $\pi^W(x_t, d_t)$. Gilleskie parametrizes and estimates $\pi^W(x_t, d_t)$ and $\pi^S(H)$ prior to estimating the preference parameters. We use those estimate in our application.

7.1.3. *Utility*

The per-period consumption is defined as $C(x_t, d_t) = Y - \big[PC1(d_t = 1 \text{ or } 3) + Y\big(1 - L\Phi(x_t, d_t)\big)1(d_t = 2 \text{ or } 3)\big]1(t > 0)$, where Y is the per-period labor income, PC is the cost of a medical visit, and $\Phi(x_t, d_t)$ is the portion of income that sick leave coverage replaces.

After rearrangements presented in Appendix F.2, the per-period utilities can be expressed in the following form.

$$
\begin{aligned}
u(d_t = 1, x_t, \epsilon_t) &= \theta_1 + \theta_4 1(t = 0) + \theta_6 C(x_t, 1)1(t > 0) + \epsilon_{t1},\\
u(d_t = 2, x_t, \epsilon_t) &= \theta_2 + \theta_4 1(t = 0) + \theta_6 C(x_t, 2)1(t > 0) + \epsilon_{t2},\\
u(d_t = 3, x_t, \epsilon_t) &= \theta_3 + \theta_4 1(t = 0) + \theta_6 C(x_t, 3)1(t > 0) + \epsilon_{t3},
\end{aligned}
$$

$$u(d_t = 0, x_t, \epsilon_t) = \qquad \theta_5 1(t = 0) + \theta_6 C(x_t, 0)1(t > 0) + \epsilon_{t0},$$

$\theta_4 = -\infty$ so that when $t = 0$ the decision $d_t = 0$ is always chosen. Since we do not restrict the location and scale of ϵ_t, the values θ_p, $p = 1, 2, 3$ can be set to arbitrary values and θ_5 can be set to an arbitrary positive value in our semiparametric estimation procedure.

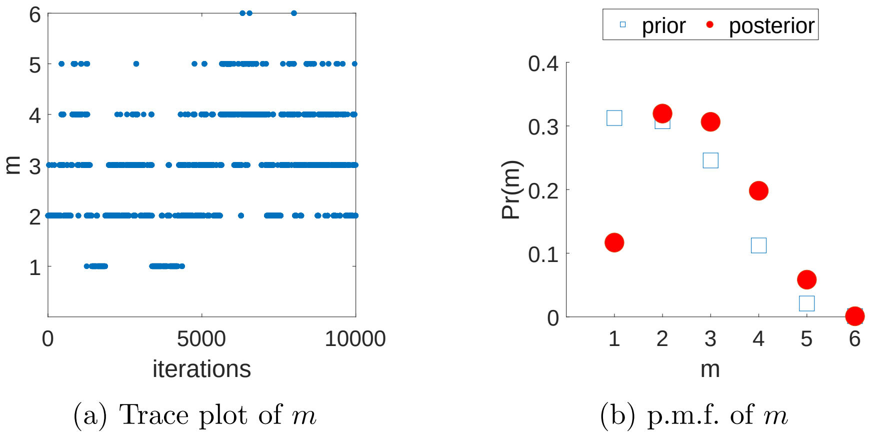
7.2. *Estimation*

For data generation we use parameter values based on estimates in Gilleskie (1998) for Type 2 illness with some adjustments so that the expected number of doctor visits and work absences roughly match with Gilleskie's sample as described in Appendix F.3. The panel data $\{x_{it}, d_{it},\ i = 1, \ldots, n,\ t = 1, \ldots, T\}$ is sequentially simulated for $n = 100$ individuals and $T = 8$ periods.

The priors are specified as follows, $\underline{a} = 10$, $A_m = 0.05$, and $\tau = 5$, $\mu_{jk} \sim N(0, 2^2)$, $\log \sigma_k \sim N(0, 1)$, $\log \sigma \sim N(0, 0.01^2)$, and $\theta_6 \sim N(0, 4^2)$. This gives normal prior on μ_{jk}'s and θ_6 with large variances. The log-normal prior on the component specific scale parameters also implies sufficiently large prior probabilities for large values of σ_k's. Prior sensitivity checks presented in Appendix F.6 show that the obtained estimation results are not substantively affected by moderate changes in the prior.

7.2.1. *MCMC results*

We use 10,000 MCMC iterations to explore the posterior distribution. Figure 3 shows a trace plot and a p.m.f. of the number of mixture components m. The posterior has its peak at $m = 2$ and the posterior probability of $m = 1$ is small.



(a) Trace plot of m

(b) p.m.f. of m

Figure 3: Trace plot and p.m.f. of m

Figure 4 shows the posterior densities of the utility function parameters in the location and scale normalization corresponding to the original model in Gilleskie (1998) described in Appendix F.4.

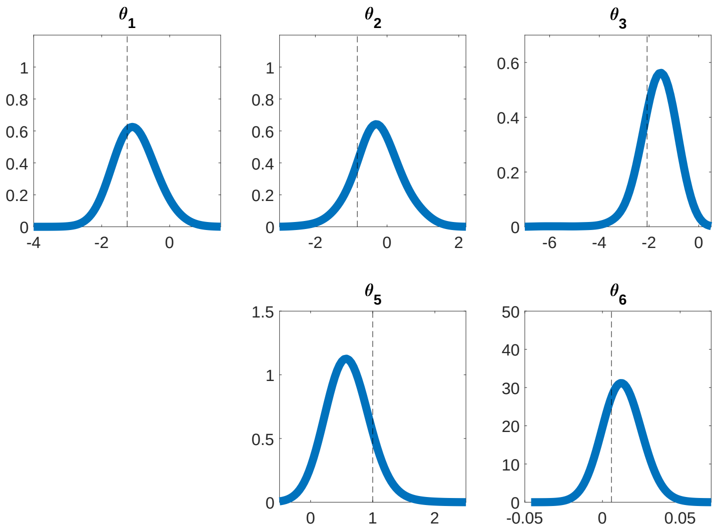


Figure 4: Posterior densities of utility parameters (solid) and the data generating values (dashed).

The traditional approach that Gilleskie takes is to assume that the shocks are extreme value i.i.d. and to estimate the model by the maximum likelihood method. In Table I, we

	MLE			Bayes		
	Estimate	95%CI	IL	Estimate	95%CI	IL
$E(v)$	1.506	(1.392, 1.620)	0.228	1.529	(1.314, 1.749)	0.435
$E(a)$	1.918	(1.797, 2.040)	0.242	1.919	(1.747, 2.089)	0.342

Table I: Estimation results: the MLE, its 95% asymptotic confidence interval, the Bayesian posterior mean and HPD 95% credible interval. IL=length of 95% CI.

compare our semiparametric estimation results with the traditional MLE approach. For the comparison we use the mean values of doctor visits $E(v)$ and work absences $E(a)$ implied by the model. They are functions of the model parameters and they are objects of interest in the application. As can be seen from the table, the Bayesian credible intervals are up to 2 times wider than the corresponding confidence intervals. This additional uncertainty stems from relaxing the extreme value distributional assumption on the shocks. Figure 8 in Appendix presents the posterior density for $E(v)$ and $E(a)$, along with the MLE and the 95% confidence intervals.

7.2.2. *Counterfactual analysis*

One of the main advantages of structural estimation is that it provides an attractive framework for counterfactual experiments. We consider the counterfactual experiment presented in Section 6.2 of Gilleskie's paper (Experiment 1). In this experiment, we are interested in the behavior of individuals when the coinsurance rate paid out of pocket is set to zero. Thus, we examine the counterfactual model solution when PC is set to 0 while the transition probabilities and (θ, F) are unchanged. Table II displays the estimated expected number of doctor visits and work absences in the counterfactual environment for the parametric MLE and our semiparametric Bayesian method. The predicted increase in doctor visits is about 14% for both approaches. The predicted change for work absences is relatively small for both methods. These observations (the increased doctor visits and the very small change in work absences) roughly match Gilleskie's findings (see her Table XII). At the same time, the 95% Bayesian credible intervals for the counterfactual $E(v)$ and $E(a)$ in the semiparametric

	MLE			Bayes		
	Estimate	95%CI	IL	Estimate	95%CI	IL
$E(v)$	1.716	(1.592, 1.839)	0.246	1.743	(1.340, 2.162)	0.821
$E(a)$	1.921	(1.796, 2.045)	0.249	1.926	(1.750, 2.103)	0.353

Table II: Counterfactual analysis: the MLE, its 95% confidence interval computed via Delta method, the Bayesian posterior mean and HPD 95% credible interval. IL=length of 95% CI.

model are up to 3 times wider than the 95% confidence intervals for the MLE in the parametric settings. Figure 8 in Appendix presents the posterior densities for the counterfactual $E(v)$ and $E(a)$, along with the MLEs and the 95% confidence intervals. The higher precision of the MLE results is misleading as it relies on unjustified distributional assumptions, while our semiparametric approach provides a more credible description of uncertainty.

8. CONCLUSION

In this paper, we propose and implement a semiparametric Bayesian estimation method for dynamic discrete choice models that uses flexible mixture specifications for modeling the distribution of unobserved state variables. We establish approximation and posterior consistency results that provide frequentist asymptotic guarantees for the method. Our approach is shown to perform well in practice for binary and mutlinomial choice models. The proposed framework is a robust and computationally tractable semiparametric alternative to the standard dynamic logit model.

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APPENDIX A: CLOSED FORM EXPRESSIONS FOR CCPS AND EMAX

PROOF OF LEMMA 1: Note that for $E1(z) = \int_z^\infty e^{-t}/tdt$, $-E1(z) = Ei(-z)$ for positive $z > 0$, where $Ei(z) = -\int_{-z}^\infty e^{-t}/tdt$ denotes the exponential integral function.

The mixture distribution in (7) can be represented as $\epsilon | Z = k \sim \phi(\epsilon; \mu_k, \sigma_k)$, where the random variable Z indicates a mixture component, $Pr(Z = k) = \omega_k$. Then,

$$p(d) = \sum_{k=1}^{m} \omega_k p(d|Z = k) \text{ and } E\left[\max_{j=0,1,...,J} v_j + \epsilon_j\right] = \sum_{k=1}^{m} \omega_k E\left[\max_{j=0,1,...,J} v_j + \epsilon_j | Z = k\right],$$

where the dependence on the observed state x is suppressed to simplify the notation.

For each $k = 1, ..., m$, $Pr(d = 0|Z = k)$ equals to

$$Pr(\epsilon_j \leq v_0 - v_j \forall j = 1, ..., J|Z = k) = \prod_{j=1}^{J} \exp\left[-e^{-\left(\frac{v_0 - v_j - \mu_{jk}}{\sigma_k}\right)} - e^{-\gamma}\right]$$
$$= \exp\left[-e^{-\gamma} e^{-\frac{v_0}{\sigma_k}} \sum_{j=1}^{J} e^{\frac{v_j + \mu_{jk}}{\sigma_k}}\right] = \exp\left[-e^{-\gamma} e^{-\frac{v_0}{\sigma_k}} e^{A_k}\right] = \exp[-e^{-a_k}].$$

For $d \neq 0$, for each $k = 1, ..., m$, note that $P(d|Z = k) = \int p(d|Z = k, \epsilon_d) f(\epsilon_d|Z = k) d\epsilon_d$ and

$$p(d|Z = k, \epsilon_d) = Pr\big(v_d + \epsilon_d \geq v_j + \epsilon_j \forall j \neq d | Z = k, \epsilon_d\big)$$
$$= Pr\big(0 \leq \epsilon_d + v_d - v_0 \text{ and } \epsilon_j \leq \epsilon_d + v_d - v_j \forall j \neq 0, d | Z = k, \epsilon_d\big)$$
$$= Pr\big(\epsilon_j \leq \epsilon_d + v_d - v_j \forall j \neq 0, d | Z = k, \epsilon_d\big) 1\left(0 \leq \epsilon_d + v_d - v_0\right)$$
$$= \prod_{j=1, j \neq d}^{J} \exp\left[-e^{-\left(\frac{\epsilon_d + v_d - v_j - \mu_{jk}}{\sigma_k}\right)} e^{-\gamma}\right] 1\left(0 \leq \epsilon_d + v_d - v_0\right)$$
$$= \prod_{j=1}^{J} \exp\left[-e^{-\left(\frac{\epsilon_d + v_d - v_j - \mu_{jk}}{\sigma_k}\right)} e^{-\gamma}\right] \exp\left[e^{-\left(\frac{\epsilon_d - \mu_{dk}}{\sigma_k}\right)} e^{-\gamma}\right] 1\left(0 \leq \epsilon_d + v_d - v_0\right).$$

Hence, $p(d|Z = k)$ equals to

$$= \int \prod_{j=1}^{J} \exp\left[-e^{-\left(\frac{s + v_d - v_j - \mu_{jk}}{\sigma_k}\right)} e^{-\gamma}\right] \exp\left[e^{-\left(\frac{s - \mu_{dk}}{\sigma_k}\right)} e^{-\gamma}\right] 1\left(0 \leq s + v_d - v_0\right)$$

$$\times \frac{1}{\sigma_k} e^{-\left(\frac{s-\mu_{dk}}{\sigma_k}\right)} e^{-\gamma} \exp\left[-e^{-\left(\frac{s-\mu_{dk}}{\sigma_k}\right)} e^{-\gamma}\right] ds$$

$$= \int \prod_{j=1}^{J} \exp\left[-e^{-\left(\frac{s+v_d-v_j-\mu_{jk}}{\sigma_k}\right)} e^{-\gamma}\right] 1\left(0 \leq s + v_d - v_0\right) \times \frac{1}{\sigma_k} e^{-\left(\frac{s-\mu_{dk}}{\sigma_k}\right)} e^{-\gamma} ds$$

$$= \int_{s=-\infty}^{s=\infty} \prod_{j=1}^{J} \exp\left[-e^{-\left(\frac{s-\mu_{dk}}{\sigma_k}\right)} e^{-\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)} e^{\left(\frac{u_j+\mu_{jk}}{\sigma_k}\right)} e^{-\gamma}\right] 1\left(0 \leq s + v_d - v_0\right)$$

$$\times \frac{1}{\sigma_k} e^{-\left(\frac{s-\mu_{dk}}{\sigma_k}\right)} e^{-\gamma} ds.$$

Let $t = e^{\frac{s-\mu_{dk}}{\sigma_k}}$. Then $ds = -\sigma_k \frac{1}{t} dt$. Note that $0 \leq s + v_d - v_0 \iff \frac{v_0 - v_d - \mu_{dk}}{\sigma_k} \leq \frac{s-\mu_{dk}}{\sigma_k} \iff e^{-\left(\frac{v_0-v_d-\mu_{dk}}{\sigma_k}\right)} \geq t$ and that $s = \infty \iff t = 0, s = -\infty \iff t = \infty$. Now

$$p(d|Z=k) = \int_{t=\infty}^{t=0} \prod_{j=1}^{J} \exp\left[-te^{-\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)} e^{-\gamma}\right] 1\left(t \leq e^{-\left(\frac{v_0-v_d-\mu_{dk}}{\sigma_k}\right)}\right)$$

$$\times \frac{1}{\sigma_k} t e^{-\gamma} \left(-\sigma_k \frac{1}{t} dt\right)$$

$$= e^{-\gamma} \int_{t=0}^{t=e^{-\left(\frac{v_0-v_d-\mu_{dk}}{\sigma_k}\right)}} \exp\left[-te^{-\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)} e^{-\gamma} \sum_{j=1}^{J} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)}\right] dt$$

$$= e^{-\gamma} \frac{\exp\left[-te^{-\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)} e^{-\gamma} \sum_{j=1}^{J} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)}\right]}{-e^{-\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)} e^{-\gamma} \sum_{j=1}^{J} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)}} \Bigg|_{t=0}^{t=e^{-\left(\frac{v_0-v_d-\mu_{dk}}{\sigma_k}\right)}}$$

$$= \frac{-e^{\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)}}{\sum_{j=1}^{J} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)}} \left\{ \exp\left[-e^{-\left(\frac{v_0-v_d-\mu_{dk}}{\sigma_k}\right)} e^{-\left(\frac{v_d+\mu_{jk}}{\sigma_k}\right)} e^{-\gamma} \sum_{j=1}^{J} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)}\right] - 1 \right\}$$

$$= \frac{e^{\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)}}{\sum_{j=1}^{J} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)}} \left\{ 1 - \exp\left[-e^{-\gamma} e^{-\left(\frac{v_0}{\sigma_k}\right)} e^{-A_k}\right] \right\} = \frac{e^{\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)}}{\sum_{j=1}^{J} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)}} \left[1 - \exp[-e^{-a_k}]\right].$$

We see that the probabilities sum to 1,

$$\sum_{d=0}^{J} P(d|Z=k) = [1 - \exp[-e^{-a_k}] + \sum_{d=1}^{J} \frac{e^{\left(\frac{v_d+\mu_{dk}}{\sigma_k}\right)}}{\sum_{j=1}^{J} e^{\left(\frac{v_j+\mu_{jk}}{\sigma_k}\right)}} \left[1 - \exp[-e^{-a_k}]\right] = 1.$$

This proves (8). To prove (9), we have for each $k = 1, ..., m$,

$$\begin{aligned}
&E\left[\max_{j=0,1,\ldots,J} v_j+\epsilon_j | Z=k\right] = \\
&E\left[\max_{j=0,1,\ldots,J} v_j+\epsilon_j | \max_{j=1,\ldots,J} v_j+\epsilon_j \leq v_0, Z=k\right] Pr\left[\max_{j=1,\ldots,J} v_j+\epsilon_j \leq v_0 | Z=k\right] \\
+&E\left[\max_{j=0,1,\ldots,J} v_j+\epsilon_j | \max_{j=1,\ldots,J} v_j+\epsilon_j > v_0, Z=k\right] Pr\left[\max_{j=1,\ldots,J} v_j+\epsilon_j > v_0 | Z=k\right].
\end{aligned}$$

First, note that $Pr\left[\max_{j=1,\ldots,J} v_j+\epsilon_j \leq v_0 | Z=k\right] = Pr\left[v_j+\epsilon_j \leq v_0, \forall j=1,...,J | Z=k\right]$

$$= \prod_{j=1}^{J} \exp\left[-e^{-\left(\frac{v_0-v_j-\mu_{jk}}{\sigma_k}\right)} e^{-\gamma}\right] = \exp\left[-e^{-\gamma} \sum_{j=1}^{J} e^{-\left(\frac{v_0-v_j-\mu_{jk}}{\sigma_k}\right)}\right].$$

Note that $\sum_{j=1}^J e^{-\left(\frac{v_0-v_j-\mu_{jk}}{\sigma_k}\right)} = e^{-\frac{v_0}{\sigma_k}} \sum_{j=1}^J e^{-\frac{v_j+\mu_{jk}}{\sigma_k}} = e^{-\frac{v_0}{\sigma_k}} e^{A_k} = \exp\left[-\frac{v_0}{\sigma_k} + A_k\right]$ $= \exp\left[-\frac{v_0-A_k\sigma_k}{\sigma_k}\right]$. Hence, $Pr\left[\max_{j=1,\ldots,J} v_j+\epsilon_j \leq v_0 | Z=k\right] = \exp\left[-e^{\left(-\frac{v_0-A_k\sigma_k}{\sigma_k}\right)} e^{-\gamma}\right] =$ $\exp[-e^{-a_k}]$. This means, $\max_{j=1,\ldots,J} v_j+\epsilon_j \leq v_0 | Z=k \sim \phi\left(\cdot; A_k\sigma_k, \sigma_k\right)$. Next, we have

$$\begin{aligned}
&E\left[\max_{j=0,1,\ldots,J} v_j+\epsilon_j | \max_{j=1,\ldots,J} v_j+\epsilon_j > v_0, k\right] \\
&= E\left[\max_{j=1,\ldots,J} v_j+\epsilon_j | \max_{j=1,\ldots,J} v_j+\epsilon_j > v_0, k\right] = \int_{v_0}^{\infty} \frac{y f_{\max_{j=1,\ldots,J} v_j+\epsilon_j}(y|Z=k)}{Pr\left[\max_{j=1,\ldots,J} v_j+\epsilon_j > v_0\right]} dy.
\end{aligned}$$

Note that $\int_{v_0}^\infty y f_{\max_{j=1,\ldots,J} v_j+\epsilon_j}(y|Z=k) dy = \int_{v_0}^\infty y \frac{1}{\sigma_k} e^{\left(-\frac{v_0-A_k\sigma_k}{\sigma_k}\right)} e^{-\gamma}\left[-e^{\left(-\frac{v_0-A_k\sigma_k}{\sigma_k}\right)} e^{-\gamma}\right] dy$. Let $z = \gamma + \frac{y}{\sigma_k} - A_k$. Then $dz = \frac{1}{\sigma_k} dy$. Note that $y = v_0 \to z = \gamma + \frac{v_0}{\sigma_k} - A_k = a_k$ and $y = \infty \to z = \infty$. Note that $y = \sigma_k(z - \gamma + A_k)$. We have

$$\begin{aligned}
&\int_{v_0}^{\infty} y f_{\max_{j=1,\ldots,J} v_j+\epsilon_j}(y|Z=k) dy = \int_{a_k}^{\infty} \sigma_k(z-\gamma+A_k)\frac{1}{\sigma_k} e^{-z} \exp\left[-e^{-z}\right](\sigma_k dz) \\
&= \sigma_k \int_{a_k}^{\infty} z e^{-z} \exp\left[-e^{-z}\right] dz + \sigma_k(A_k-\gamma)\int_{a_k}^{\infty} e^{-z}\exp\left[-e^{-z}\right] dz \\
&= \sigma_k\left[\gamma - a_k \exp\left(-e^{-a_k}\right) + E1(e^{-a_k})\right] + \sigma_k(A_k-\gamma)\left[1-\exp\left(-e^{-a_k}\right)\right].
\end{aligned}$$

Finally, $E\left[\max_{j=0,1,\ldots,J} v_j + \epsilon_j | Z = k\right] = v_0 \exp\left(-e^{-a_k}\right) + \int_{v_0}^{\infty} y f_{\max_{j=1,\ldots,J} v_j+\epsilon_j}(y|Z=k)dy$

$$\begin{aligned}
&= v_0 \exp\left(-e^{-a_k}\right) + \\
&\sigma_k \left[\gamma - a_k \exp\left(-e^{-a_k}\right) + E1(e^{-a_k}) + A_k - \gamma - A_k \exp\left(-e^{-a_k}\right) + \gamma \exp\left(-e^{-a_k}\right)\right] \\
&= \sigma_k \left[\exp\left(-e^{-a_k}\right)\left(\frac{v_0}{\sigma_k} + \gamma - A_k - a_k\right) + E1(e^{-a_k}) + A_k\right] \\
&= \sigma_k \left[A_k + E1(e^{-a_k})\right].
\end{aligned}$$

This proves (9). *Q.E.D.*

APPENDIX B: MCMC ALGORITHM

For the application of a reversible jump algorithm, we need to transform the mixing weights into unnormalized weights γ_k, $k = 1, \ldots$, so that conditional on m, $\omega_k = \gamma_k / \sum_{l=1}^{m} \gamma_l$ and the Dirichlet prior on $(\omega_1, \ldots, \omega_m)$ corresponds to a gamma prior for the unnormalized weights: $\gamma_k | m \sim Gamma(\underline{a}/m, 1)$, $k = 1, \ldots, m$. Let $\psi_k = (\mu_k, \tilde{\sigma}_k, \gamma_k)$ and $\psi_{1m} = (\theta, \sigma, \psi_1, \ldots, \psi_m)$, where θ includes model parameters such as coefficients in the utility functions. With this notation, the likelihood function is denoted by $p(D_n | m, \psi_{1m})$. Our MCMC algorithm alternates the following two blocks: (i) HMC for $\Pi(\psi_{1m} | m, D_n)$ and (ii) an optimal reversible jump for m. The following subsections provide details for each block.

B.1. *Optimal Reversible Jump from Norets (2021)*

The following short description of the reversible jump algorithm is adapted from Norets and Pelenis (2021), see Norets (2021) for more details. Denote a proposal distribution for the parameter of a new mixture component $m+1$ by $\tilde{\pi}_{m+1}(\psi_{m+1} | D_n, \psi_{1m})$. The algorithm works as follows. Simulate proposal m^* from $Pr(m^* = m+1 | m) = Pr(m^* = m-1 | m) = 1/2$. If $m^* = m+1$, then also simulate $\psi_{m+1} \sim \tilde{\pi}_{m+1}(\psi_{m+1} | D_n, \psi_{1m})$. Accept the proposal with probability $\min\{1, \alpha(m^*, m)\}$, where

$$\alpha(m^*, m) = \frac{p(D_n | m^*, \psi_{1m^*}) \Pi(\psi_{1m^*} | m^*) \Pi(m^*)}{p(D_n | m, \psi_{1m}) \Pi(\psi_{1m} | m) \Pi(m)}$$

$$\cdot\left(\frac{1\{m^* = m+1\}}{\tilde{\pi}_m(\psi_{m+1}|\psi_{1m}, Y)} + 1\{m^* = m-1\}\tilde{\pi}_{m-1}(\psi_m|\psi_{1m-1}, D_n)\right). \tag{14}$$

Norets (2021) shows that an optimal choice of proposal $\tilde{\pi}_m$ is the conditional posterior $p(\psi_{m+1}|D_n, m+1, \psi_{1m})$. The conditional posterior can be evaluated up to a normalization constant; however, it seems hard to directly simulate from it and compute the required normalization constant. Hence, we use a Gaussian approximation to $p(\psi_{m+1}|D_n, m+1, \psi_{1m})$ as the proposal (with the mean equal to the conditional posterior mode, obtained by a Newton method, and the variance equal to the inverse of the negative of the Hessian evaluated at the mode).

B.2. *HMC and Transformation of Parameters*

We utilize a built-in HMC package in Matlab. The package requires the parameters to be unbounded. Recall that the vector of parameters is $\psi_{1m} = \big(\theta, \sigma, \omega_k, \mu_k, ...\mu_{Jk}, \tilde{\sigma}_k,\ k = 1, ...\big)$, where the weights and the scales are not unbounded. For a given fixed m, we denote the transformed parameters by $\chi = (\theta, \log\sigma, \alpha_k, \mu_k, \log\tilde{\sigma}_k,\ k = 1, ..., m)$, where

$$\omega_k = \frac{e^{\alpha_k}}{1 + \sum_{\ell=1}^{m-1} e^{\alpha_\ell}}, k = 1, ..., m-1, \text{ and } \alpha_m = 0$$

so that all the components of χ are unbounded.

We only need to supply the Matlab HMC algorithm with a function that evaluates the log of the likelihood times prior and its gradient. The package can choose HMC's parameters, such as a step size, automatically, and we perform this automatic initialization once for each value of m that we encounter in the MCMC run. The form of the prior for the transformed mixing weights and the derivatives of the likelihood used in the algorithm are reported in Appendix D.

APPENDIX C: PROOFS AND INTERMEDIATE RESULTS

PROOF OF LEMMA 2: We first show Lipschitz continuity of the Emax function. Suppose there are two distributions F_1 and F_2. Define the corresponding Emax functions $Q_i(x) =$

$Q(x; F_i)$.

$$\begin{aligned} Q_2(x) - Q_1(x) =& \Big(Q_2(x) - \int \max_d \Big[u(x,d) + \beta G_x^d(Q_1) + \epsilon_d \Big] dF_2(\epsilon) \Big) \\ &+ \Big(\int \max_d \Big[u(x,d) + \beta G_x^d(Q_1) + \epsilon_d \Big] dF_2(\epsilon) - Q_1(x) \Big). \end{aligned}$$

The term in the first parentheses of the right hand side is bounded by

$$\begin{aligned} &\int \max_d \Big[\big(u(x,d) + \beta G_x^d(Q_2) + \epsilon_d \big) - \big(u(x,d) + \beta G_x^d(Q_1) + \epsilon_d \big) \Big] dF_2(\epsilon) \\ &= \beta \max_d \Big[G_x^d(Q_2) - G_x^d(Q_1) \Big] \\ &\leq \beta \max_d \sum_{x' \in X} |Q_2(x') - Q_1(x')| G_x^d(x') \\ &\leq \beta ||Q_2 - Q_1||, \end{aligned}$$

where $||Q_2 - Q_1|| = \sup_{x \in X} |Q_1(x) - Q_1(x)|$. The term in the second parentheses is

$$\begin{aligned} &\int \sum_d \Big[u(x,d) + \beta G_x^d(Q_1) + \epsilon_d \Big] d(F_2 - F_1)(\epsilon) \leq \int \Big[\bar{u} + \sup_x Q_1(x) + \sum_d |\epsilon_d| \Big] d(F_2 - F_1)(\epsilon) \\ &\leq c \int \Big[1 + \sum_d |\epsilon_d| \Big] d(F_2 - F_1)(\epsilon) = c\rho(F_1, F_2) \end{aligned}$$

for $c = \max\{1, \bar{u} + \bar{u}/(1-\beta) + \max_d \int |\epsilon_d| dF_1(\epsilon)\}$. Thus,

$$Q_2(x) - Q_1(x) \leq \beta ||Q_2 - Q_1|| + c\rho(F_1, F_2),$$

for each $x \in X$. Finally, we have

$$||Q_2 - Q_1|| \leq \frac{c}{1-\beta} \rho(F_1, F_2) = C\rho(F_1, F_2).$$

This proves the local Lipschitz continuity of the Emax function.

Given some $d^* \in \{0, 1, ..., J\}$ and x, define the set

$$S_1 = \left\{ \epsilon : u(x, d^*) + \beta G_x^{d^*}(Q_1) + \epsilon_{d^*} \geq u(x, d) + \beta G_x^d(Q_1) + \epsilon_d, \forall d \right\},$$

on which d^* is optimal under F_1. Similarly, define S_2. With this notation,

$$p(d^*|x; F_i) = \int 1(S_i) dF_i(\epsilon),$$

for $i = 1, 2$, where $1(\cdot)$ is an indicator function. Now,

$$p(d^*|x; F_1) - p(d^*|x; F_2) = \int [1(S_1) - 1(S_2)] dF_1 + \int 1(S_2) d(F_1 - F_2).$$

The second integral is bounded by $\rho(F_1, F_2)$. Note that $1(S_1) - 1(S_2)$ is bounded above by

$$\begin{aligned} &\sum_d 1(\{\ d^* \text{ is optimal under } F_1 \text{ but } d \text{ is optimal under } F_2\}) \\ &\quad +1(\{\ d \text{ is optimal under } F_1 \text{ but } d^* \text{ is optimal under } F_2\}). \end{aligned}$$

If d^* is optimal under F_1 and d is optimal under F_2, then

$$u(x,d) - u(x,d^*) + \beta G_x^d(Q_1) - \beta G_x^{d^*}(Q_1) \leq \epsilon_{d^*} - \epsilon_d \leq u(x,d) - u(x,d^*) + \beta G_x^d(Q_2) - \beta G_x^{d^*}(Q_2).$$

Denote this interval by A_1. Similarly, if d is optimal under F_1 and d^* is optimal under F_2, then

$$u(x,d^*) - u(x,d) + \beta G^{d^*}(Q_1) - \beta G_x^d(Q_1) \leq \epsilon_{d^*} - \epsilon_d \leq u(x,d^*) - u(x,d) + \beta G^{d^*}(Q_2) - \beta G_x^d(Q_2).$$

Denote this interval by A_2. Note that the length of the intervals A_1 and A_2 is bounded by $2\beta||Q_2 - Q_1||$. Also, by condition (ii) of the lemma, the density of the difference $\epsilon_{d^*} - \epsilon_d$

implied by F_1 is bounded by some positive constant $\bar{f} > 0$. Thus,

$$\begin{aligned}\int [1(S_1) - 1(S_2)] dF_1 &\leq \sum_d \int 1(\epsilon_{d^*} - \epsilon_d \in A_1) dF_1(\epsilon) + \sum_d \int 1(\epsilon_{d^*} - \epsilon_d \in A_2) dF_1(\epsilon) \\ &\leq 4(J+1)\beta ||Q_2 - Q_1|| \bar{f} \leq 4(J+1)\beta C \rho(F_1, F_2),\end{aligned}$$

where the last inequality follows by the proven Lipschitz continuity of the Emax function. In summary, we have, for each $d^* = 0, 1, ..., J$,

$$|p(d^*|x; F_1) - p(d^*|x; F_2)| \leq 4(J+1)\beta C \rho(F_1, F_2) + \rho(F_1, F_2) = C' \rho(F_1, F_2),$$

for all $x \in X$. *Q.E.D.*

Proof of Lemma 3:

$$\begin{aligned}\rho\left(f(\cdot),\ \sum_{j=1}^m \omega_j \phi\left(\cdot; q_j, \sigma\right)\right) &= \int \left(1 + \sum_{i=1}^I |z_i|\right) \left| f(z) - \sum_{j=1}^m \omega_j \phi\left(z; q_j, \sigma\right) \right| dz \\ &\leq \sum_{i=1}^I \int (1 + |z_i|) \left| f(z) - \sum_{j=1}^m \omega_j \phi\left(z; q_j, \sigma\right) \pm \int \phi\left(z; \mu, \sigma\right) f(\mu) d\mu \right| dz \\ &\leq \sum_{i=1}^I \int (1 + |z_i|) \left| f(z) - \int \phi\left(z; \mu, \sigma\right) f(\mu) d\mu \right| dz \qquad (15) \\ &+ \int (1 + |z_i|) \left| \int \phi\left(z; \mu, \sigma\right) f(\mu) d\mu - \sum_{j=1}^m \omega_j \phi\left(z; q_j, \sigma\right) \right| dz. \qquad (16)\end{aligned}$$

With the change of variable of μ_i to $\theta_i = \frac{z_i - \mu_i}{\sigma}, i = 1, ..., I$, (15) is bounded by

$$\begin{aligned}&\sum_{i=1}^I \int \int (1 + |z_i|) \left| f(z) - f(z - \sigma\theta) \right| \phi(\theta) d\theta dz \\ &\leq \sum_{i=1}^I \int \int (1 + |z_i|) L_f(z) e^{\tau\sigma||\theta||_2} \sigma ||\theta||_2 \phi(\theta) d\theta dz \\ &= \sigma \left[\int ||\theta||_2 e^{\tau\sigma||\theta||_2} \phi(\theta) d\theta \right] \sum_{i=1}^d \left[\int (1 + |z_i|) L_f(z) dz \right],\end{aligned}$$

where the inequality follows from the smoothness assumption (12). Lemma 6 shows that the term in the first square brackets is bounded for sufficiently small σ. The last term is finite by assumption (13). Hence, we can choose σ small enough to make the term above arbitrarily small.

To bound (16), let $A_j, j = 0, 1, ..., m$, be a partition of $\mathbb{R}^I$ consisting of adjacent hypercubes $A_1, ..., A_m$ with sides $h_m^{1/I}$ so that they are collectively centered at zero and the rest of the space is A_0. As m increases, the fine part of the partition becomes finer, $h_m \to 0$ and $m \to \infty$. Also, it covers larger and larger parts of $\mathbb{R}^I$; that is, $m \cdot h_m \to \infty$ as $m \to \infty$. Define $\omega_j = \int_{A_j} f(\mu)d\mu$. This implies,

$$\begin{aligned}
&\int \phi\left(z; \mu, \sigma\right) f(\mu)d\mu - \sum_{j=1}^{m} \omega_j \phi\left(z; q_j, \sigma\right)\\
&= \sum_{j=0}^{m} \int_{A_j} \phi\left(z; \mu, \sigma\right) f(\mu)d\mu - \sum_{j=1}^{m} \int_{A_j} \phi\left(z; q_j, \sigma\right) f(\mu)d\mu\\
&= \sum_{j=1}^{m} \int_{A_j} \Big[\phi\left(z; \mu, \sigma\right) - \phi\left(z; q_j, \sigma\right)\Big] + \int_{A_0} \phi\left(z; \mu, \sigma\right) f(\mu)d\mu.
\end{aligned}$$

The expression in (16) can be bounded as follows,

$$\begin{aligned}
&\sum_{i=1}^{I} \int (1 + |z_i|) \bigg| \int \phi\left(z; \mu, \sigma\right) f(\mu)d\mu - \sum_{j=1}^{m} \omega_j \phi\left(z; q_j, \sigma\right) \bigg| dz\\
&\leq \sum_{i=1}^{I} \sum_{j=1}^{m} \int \int_{A_j} (1 + |z_i|) \bigg| \phi\left(z; \mu, \sigma\right) - \phi\left(z; q_j, \sigma\right) \bigg| f(\mu)d\mu dz\\
&+ \sum_{i=1}^{I} \int \int_{A_0} (1 + |z_i|) \phi\left(z; \mu, \sigma\right) f(\mu)d\mu dz.
\end{aligned}$$

Consider the change of variable of z to $y = \frac{z-\mu}{\sigma}$ and define $\delta_j = \frac{q_j-\mu}{\sigma}$, $j = 1, ..., m$ so that $\frac{z-q_j}{\sigma} = y - \delta_j$ and the above equals to

$$\sum_{i=1}^{I} \sum_{j=1}^{m} \int \int_{A_j} (1+|\mu_i+\sigma y_i|) \bigg| \phi(y) - \phi(y-\delta_j) \bigg| f(\mu)d\mu dy + \int \int_{A_0} (1+|\mu_i+\sigma y_i|) \phi(y) f(\mu)d\mu dy.$$

As $\mu, q_j \in A_j$, we have $||\delta_j||_\infty \leq \left\| \frac{q_j - \mu}{\sigma} \right\|_1 \leq \frac{I \cdot h_m^{1/d}}{I \cdot \sigma} \equiv \bar{\delta}$, which will be made small. By Lemma 7, $\phi(\cdot)$ is Lipschitz continuous and $|\phi(y) - \phi(y - \delta_j)| \leq \bar{L}_\phi(y)||\delta_j||_\infty$, which implies that $|\phi(y) - \phi(y - \delta_j)| \leq \bar{L}_\phi(y)\bar{\delta}$.

The first term above is bounded by $\bar{\delta}$ times

$$\sum_{i=1}^{I} \left(\sum_{j=1}^{m} c_{1j} \int_{A_j} (1 + |\mu_i|) f(\mu) d\mu + c_{2j}\sigma \right),$$

where $c_{1j} = \int \bar{L}_\phi(y)dy$ and c_{2j} is such that $\int |y_i| \bar{L}_\phi(y)dy < c_{2j}$ for $i = 1, ..., I$. The bound approaches to zero as $m \to \infty$. Since $\phi(\cdot)$ has bounded first moment, the second integral is bounded by

$$\sum_{i=1}^{I} \int_{A_0} (1 + |\mu_i| + \sigma c) f(\mu) d\mu,$$

for some constant $c > 0$. As $mh_m \to \infty$, this term goes to zero.

Q.E.D.

Proof of Lemma 4: We have

$$\begin{aligned}
\rho\left(F^1, F^2\right) &= \int (1 + \sum_j |\epsilon_j|) \left| f^1(\epsilon) - f^2(\epsilon) \right| d\epsilon \\
&= \int (1 + \sum_j |\epsilon_j|) \left| \sum_{k=1}^{m} \omega_k^1 \phi\left(\epsilon; \mu_k^1, \sigma_k^1\right) - \omega_k^2 \phi\left(\epsilon; \mu_k^2, \sigma_k^2\right) \pm \omega_k^1 \phi\left(\epsilon; \mu_k^2, \sigma_k^2\right) \right| d\epsilon \\
&\leq \sum_{k=1}^{m} \omega_k^1 \int (1 + \sum_j |\epsilon_j|) \left| \phi\left(\epsilon; \mu_k^1, \sigma_k^1\right) - \phi\left(\epsilon; \mu_k^2, \sigma_k^2\right) \right| d\epsilon \qquad (17) \\
&+ \sum_{k=1}^{m} |\omega_k^1 - \omega_k^2| \int (1 + \sum_j |\epsilon_j|) \phi\left(\epsilon; \mu_k^2, \sigma_k^2\right) d\epsilon. \qquad (18)
\end{aligned}$$

Expression 18 is bounded by a positive constant times $\sum_{k=1}^{m} |\omega_k^1 - \omega_k^2|$.

To bound 17 note that

$$\begin{aligned}
&\int (1+\sum_j |\epsilon_j|)\Big|\phi\left(\epsilon;\mu_k^1,\sigma_k^1\right)-\phi\left(\epsilon;\mu_k^2,\sigma_k^2\right)\Big|d\epsilon \\
&=\int (1+\sum_j |\epsilon_j|)\Big|\phi\left(\epsilon;\mu_k^1,\sigma_k^1\right)-\phi\left(\epsilon;\mu_k^2,\sigma_k^2\right)\pm\phi\left(\epsilon;\mu_k^2,\sigma_k^1\right)\Big|d\epsilon \\
&\leq\int (1+\sum_j |\epsilon_j|)\Big|\phi\left(\epsilon;\mu_k^1,\sigma_k^1\right)-\phi\left(\epsilon;\mu_k^2,\sigma_k^1\right)\Big|d\epsilon \\
&+\int (1+\sum_j |\epsilon_j|)\Big|\phi\left(\epsilon;\mu_k^2,\sigma_k^1\right)-\phi\left(\epsilon;\mu_k^2,\sigma_k^2\right)\Big|d\epsilon.
\end{aligned}$$

With the change of variable $z=\frac{\epsilon-\mu_k^1}{\sigma_k^1}$ and by Lemma 7, the first part can be bounded as follows

$$\begin{aligned}
&\int \left(1+\sum_j |\sigma_k^1 z_j+\mu_{jk}^1|\right)\Big|\phi(z;0_{J+1},1)-\phi\left(z-\frac{\mu_k^2-\mu_k^1}{\sigma_k^1};0_{J+1},1\right)\Big|dz \\
&\leq\left\|\frac{\mu_k^2-\mu_k^1}{\sigma_k^1}\right\|_\infty\int\left(1+\sum_j\left|\sigma_k^1 z_j+\mu_{jk}^1\right|\right)\bar{L}_\phi(z)dz,
\end{aligned}$$

where the integral is bounded. By Lemma 8, the second part can be bounded as follows

$$\int (1+\sum_j |\epsilon_j|)\Big|\phi\left(\epsilon;0_{J+1},\sigma_k^1\right)-\phi\left(\epsilon;0_{J+1},\sigma_k^2\right)\Big|d\epsilon\leq\left|\frac{1}{\sigma_k^1}-\frac{1}{\sigma_k^2}\right|\int(1+\sum_j|\epsilon_j|)\bar{M}_\phi(\epsilon)d\epsilon,$$

where the integral is bounded. To summarize, we have

$$\rho\left(F^1,F^2\right)\leq\sum_{k=1}^m\omega_k^1\left[c_1\left\|\frac{\mu_k^2-\mu_k^1}{\sigma_k^1}\right\|_\infty+c_2\left|\frac{1}{\sigma_k^1}-\frac{1}{\sigma_k^2}\right|\right]+c_3\sum_{k=1}^m|\omega_k^1-\omega_k^2|,$$

for some positive finite constants c_i's, which implies the claimed result. *Q.E.D.*

Proof of Lemma 5: Let us denote the set of shock values at which alternative j is optimal at the observed state k by

$$E_{jk} = \{(\epsilon_1,\dots,\epsilon_J):\ u(k,j;\theta_0)+\beta G_k^j Q+\epsilon_j \geq u(k,d;\theta_0)+\beta G_k^d Q+\epsilon_d, \forall d=0,1,\dots,J\},$$

where the normalization $\epsilon_0 = 0$ is used. Sets $\{E_{jk}, j = 0,1,\dots,J\}$ define a partition of $\mathbb{R}^J$ (up to the overlapping boundaries) for each $k \in \{1,\dots,K\}$. Consider a refinement of these K partitions,

$$\Big\{A:\ A = \bigcap_{k=1}^{K} E_{j_k k},\ j_k \in \{0,\dots,J\},\ \lambda(A) > 0\Big\} = \{A_1,\dots,A_L\},$$

where λ is the Lebesgue measure. Let us define

$$q_l = \int_{A_l} dF_0(\epsilon_{1J}) \in (0,1),$$

$$r_l = \int_{A_l} \epsilon_{1J} dF_0(\epsilon_{1J}) \in \mathbb{R}^J,$$

where $\epsilon_{1J} = (\epsilon_1,\dots,\epsilon_J)'$. It follows from a characterization of the identified sets under unknown distribution of shocks in Norets (2011) that any distribution that implies the same $\{q_l, r_l, l = 1,\dots,L\}$ delivers the same CCPs. The key to this result is that the shocks enter the utility additively and the integrals over the shocks in the Bellman equations can be replaced with expressions depending on the distribution of shocks only through $\{q_l, r_l, l = 1,\dots,L\}$. In what follows, we construct a finite mixture distribution that delivers the same $\{q_l, r_l, l = 1,\dots,L\}$ as F_0. Specifically, consider a mixture with $L\cdot(J+1)$ components

$$\sum_{l=1}^{L}\sum_{j=0}^{J} \omega_{lj}\phi(\cdot;\mu_{lj},\sigma),$$

where we fix the locations as follows. Since F_0 is assumed to have a positive density, by Lemma 2 in Norets (2011), $r_l/q_l \in int(A_l)$ for all l. This is an implication of the supporting hyperplane theorem and the closedness and convexity of E_{jk}'s and thus A_l, $l = 1,\dots,L$

(r_l/q_l is the expectation of ϵ_{1J} conditional on $\epsilon_{1J} \in A_l$). Therefore, $\exists \Delta > 0$ such that

$$\mu_{lj} = r_l/q_l + e_j \cdot \Delta \in int(A_l),\ j = 1, ..., J,$$
$$\mu_{l0} = r_l/q_l - (1, \ldots, 1)' \cdot \Delta \in int(A_l),\ l = 1, ..., L,$$

where e_j is a column vector of length J with 1 in the jth coordinate and 0's in the others. The mixing weights ω_{lj} are chosen as a solution to the following linear system of equations that matches $\{q_l, r_l, l = 1, \ldots, L\}$,

$$q_l = \sum_{\tilde{l}=1}^{L}\sum_{j=0}^{J} \omega_{\tilde{l}j} \int_{A_l} \phi(\epsilon_{1J}; \mu_{\tilde{l}j}, \sigma) d\epsilon_{1J},$$

$$r_l = \sum_{\tilde{l}=1}^{L}\sum_{j=0}^{J} \omega_{\tilde{l}j} \int_{A_l} \epsilon_{1J}\phi(\epsilon_{1J}; \mu_{\tilde{l}j}, \sigma) d\epsilon_{1J}, \qquad l \in \{1, ..., L\}. \tag{19}$$

Let us show that for sufficiently small σ, this linear system has a unique solution that belongs to the $(L(J+1)-1)$-simplex. First, note that since $\mu_{lj} \in int(A_l)$,

$$\lim_{\sigma\to 0} \int_{A_l} \phi(\epsilon_{1J}; \mu_{\tilde{l}j}, \sigma) d\epsilon_{1J} = 1\{l = \tilde{l}\} \text{ and } \lim_{\sigma\to 0} \int_{A_l} \epsilon_{1J}\phi(\epsilon_{1J}; \mu_{\tilde{l}j}, \sigma) d\epsilon_{1J} = \mu_{lj} 1\{l = \tilde{l}\}.$$

Thus, the limiting system corresponding to $\sigma = 0$ in (19) is

$$q_l = \sum_{j=0}^{J} \omega_{lj}, \quad r_l = \sum_{j=0}^{J} \omega_{lj}\mu_{lj}, \qquad l \in \{1, ..., L\}. \tag{20}$$

Plugging the definition of μ_{lj} into (20), we obtain $\omega_{lj} = \omega_{l0}$, $j = 1, \ldots, J$ and, thus, the limiting system (20) has the unique solution

$$\omega^*_{lj} = q_l/(J+1),\ l = 1, \ldots, L,\ j = 0, \ldots, J.$$

It follows that the matrix of the linear coefficients in the limiting system is invertible. Since matrix inversion is continuous and $\omega^*_{lj} > 0$, the system (19) has a unique strictly positive solution $\{\omega^{**}_{lj},\ l = 1, \ldots, L,\ j = 0, \ldots, J\}$ for all sufficiently small σ. Since $\sum_{l=1}^{L} q_l = 1$,

$\sum_{l,j} \omega_{lj}^{**} = 1$ as well.

Q.E.D.

Lemma 6 *The term*

$$\int ||z||_2 e^{\tau\sigma \cdot ||z||_2} \phi(z; 0_d, 1) dz$$

is bounded for any $\tau > 0$ *and sufficiently small* $\sigma > 0$ *where* 0_d *is a column vector of zeros with length* d*.*

Proof: By Cauchy-Schwartz inequality, the term is bounded by

$$\left(\int ||z||_2^2 \phi(z; 0_d, 1) dz\right)^{1/2} \left(\int e^{2\tau\sigma \cdot ||z||_2} \phi(z; 0_d, 1) dz\right)^{1/2}.$$

The first term in the product is bounded. As 2-norms are bounded by 1-norms, the term in the second parentheses is bounded by

$$\int e^{2\tau\sigma \cdot ||z||_1} \phi(z; 0_d, 1) dz = \prod_{i=1}^{d} \int_{-\infty}^{\infty} e^{2\tau\sigma|z_i|} \phi(z_i) dz_i$$
$$\leq \prod_{i=1}^{d} \int_{-\infty}^{\infty} e^{2\tau\sigma|z_i|} e^{-z_i} dz_i = 2^d \prod_{i=1}^{d} \int_{0}^{\infty} e^{2\tau\sigma z_i - z_i} dz_i,$$

which is bounded for small enough σ.

Q.E.D.

Lemma 7 *The density* $\phi(\,\cdot\,; \mu, \sigma)$ *is locally Lipschitz continuous in the location parameter with envelope* $\bar{L}_\phi$ *in a sense that, for an arbitrary* $\bar{\delta}$*, if* $\delta \in \mathbb{R}^d$ *is bounded by* $\bar{\delta}$*, then*

$$|\phi(z; 0_d, 1) - \phi(z - \delta; 0_d, 1)| \leq \bar{L}_\phi(z) ||\delta||_\infty,$$

where 0_d is a column vector of zeros with length d, $||\delta||_\infty = \max_{i=1,\dots,d}|\delta_i|$, and

$$\bar{L}_\phi(z) = L_\phi(z_1)\prod_{i=2}^{d}\phi(z_i) + L_\phi(z_2)\phi(z_1-\delta_1)\prod_{i=3}^{d}\phi(z_i) + \cdots + L_\phi(z_d)\prod_{i=1}^{d-1}\phi(z_i-\delta_i),$$

$$L_\phi(z) = \begin{cases} K_1 e^{-(z+\gamma)}e^{-e^{-K_2(z+\gamma)}}, & \textit{if } z > -\gamma \\ K_3 e^{-2(z+\gamma)}e^{-e^{-K_4(z+\gamma)}}, & \textit{otherwise}, \end{cases}$$

for some positive K_i's that could depend on $\bar{\delta}$.

Proof: First, we establish local Lipschitz continuity for the univariate case. We have for $z, \delta \in \mathbb{R}$

$$|\phi(z) - \phi(z-\delta)| = |\phi'(z-\tilde{\delta})| \cdot |\delta|,$$

for some $\tilde{\delta}$ between 0 and $|\delta|$ where $\phi'(z) = \phi(z)(e^{-z-\gamma} - 1)$. Note that if $z > -\gamma$, then $|\phi'(z)| \le c_1\phi(z)$ and if $z < -\gamma$, then $|\phi'(z)| \le c_2 e^{-2z-2\gamma-c_3e^{-z-\gamma}}$ for some constants $c_1, c_2, c_3 > 0$. From this, it follows that for $\tilde{\delta}$ which is bounded, $|\phi'(z-\tilde{\delta})| \le L_\phi(z)$ where

$$L_\phi(z) = \begin{cases} K_1 e^{-(z+\gamma)}e^{-e^{-K_2(z+\gamma)}}, & \text{if } z > -\gamma \\ K_3 e^{-2(z+\gamma)}e^{-e^{-K_4(z+\gamma)}}, & \text{otherwise}, \end{cases}$$

for some positive K_i's that could depend on $\bar{\delta}$. Now, to show the result for the multivariate case, let $z, \delta \in \mathbb{R}^d$. With $||\delta||_\infty = \max_{i=1,\dots,d}|\delta_i|$, we have

$$\begin{aligned}\phi(z;0_d,1) - \phi(z-\delta;0_d,1) = \prod_{i=1}^{d}\phi(z_i) - \prod_{i=1}^{d}\phi(z_i-\delta_i) \pm \phi(z_1-\delta_1)\prod_{i=2}^{d}\phi(z_i) \\ \pm \phi(z_1-\delta_1)\phi(z_2-\delta_2)\prod_{i=3}^{d}\phi(z_i)\cdots \pm \phi(z_1-\delta_1)\phi(z_2-\delta_2)\cdots\phi(z_d-\delta_d)\end{aligned}$$

$$= [\phi(z_1) - \phi(z_1 - \delta_1)] \prod_{i=2}^{d} \phi(z_i) + \phi(z_1 - \delta_1)[\phi(z_2) - \phi(z_2 - \delta_2)] \prod_{i=3}^{d} \phi(z_i)$$
$$+ \cdots + \prod_{i=1}^{d-1} \phi(z_i - \delta_i)[\phi(z_d) - \phi(z_d - \delta_d)].$$

By the proven locally Lipschitz continuity for the univariate case, this is bounded by

$$L_\phi(z_1)|\delta_1| \prod_{i=2}^{d} \phi(z_i) + L_\phi(z_2)|\delta_2|\phi(z_1 - \delta_1) \prod_{i=3}^{d} \phi(z_i) + \cdots + L_\phi(z_d)|\delta_d| \prod_{i=1}^{d-1} \phi(z_i - \delta_i)$$
$$\leq ||\delta||_\infty \Big[L_\phi(z_1) \prod_{i=2}^{d} \phi(z_i) + L_\phi(z_2)\phi(z_1 - \delta_1) \prod_{i=3}^{d} \phi(z_i) + \cdots + L_\phi(z_d) \prod_{i=1}^{d-1} \phi(z_i - \delta_i) \Big].$$

Q.E.D.

Lemma 8 *The density $\phi(z;\mu,\sigma)$ is locally Lipschitz continuous in the inverse of the scale parameter with envelope $\bar{M}_\phi$ in a sense that given two scale parameters $\sigma^1, \sigma^2 > 0$,*

$$\left| \phi\left(z; 0_d, \sigma^1\right) - \phi\left(z; 0_d, \sigma^2\right) \right| \leq \left| \frac{1}{\sigma^1} - \frac{1}{\sigma^2} \right| \bar{M}_\phi(z), \text{ where}$$

$$\bar{M}_\phi(z) = M_\phi(z_1) \prod_{i=2}^{d} \phi\left(z_i; 0, \sigma^1\right) + M_\phi(z_2)\phi\left(z_1; 0, \sigma^2\right) \prod_{i=3}^{d} \phi\left(z_i; 0, \sigma^1\right) + \cdots + M_\phi(z_d) \prod_{i=1}^{d-1} \phi\left(z_i; 0, \sigma^2\right)$$

and $M_\phi(z_i) = \frac{1}{\sigma^1} L_\phi\left(\frac{z_i}{\sigma^1}\right)|z_i| + \phi\left(\frac{z_i}{\sigma^2}\right)$, $i = 1, ..., d$ for $L_\phi(\cdot)$ defined in Lemma 7.

Proof: We first show the result for the univariate case. Let $\sigma^1, \sigma^2 > 0$ and $z \in \mathbb{R}$.

$$\left| \phi\left(z; 0, \sigma^1\right) - \phi\left(z; 0, \sigma^2\right) \right| = \left| \frac{1}{\sigma^1}\phi\left(\frac{z}{\sigma^1}\right) - \frac{1}{\sigma^2}\phi\left(\frac{z}{\sigma^2}\right) \pm \frac{1}{\sigma^1}\phi\left(\frac{z}{\sigma^2}\right) \right|$$
$$\leq \left| \frac{1}{\sigma^1}\phi\left(\frac{z}{\sigma^1}\right) - \frac{1}{\sigma^1}\phi\left(\frac{z}{\sigma^2}\right) \right| + \left| \frac{1}{\sigma^1}\phi\left(\frac{z}{\sigma^2}\right) - \frac{1}{\sigma^2}\phi\left(\frac{z}{\sigma^2}\right) \right|$$
$$= \frac{1}{\sigma^1}\left| \phi\left(\frac{z}{\sigma^1}\right) - \phi\left(\frac{z}{\sigma^1} - \left(\frac{z}{\sigma^1} - \frac{z}{\sigma^2}\right)\right) \right| + \phi\left(\frac{z}{\sigma^2}\right)\left| \frac{1}{\sigma^1} - \frac{1}{\sigma^2} \right|$$
$$\leq \frac{1}{\sigma^1} L_\phi\left(\frac{z}{\sigma^1}\right)\left| \frac{1}{\sigma^1} - \frac{1}{\sigma^2} \right||z| + \phi\left(\frac{z}{\sigma^2}\right)\left| \frac{1}{\sigma^1} - \frac{1}{\sigma^2} \right| = M_\phi(z)\left| \frac{1}{\sigma^1} - \frac{1}{\sigma^2} \right|,$$

for $L_\phi(\cdot)$ defined in Lemma 7. In the multivariate case,

$$\begin{aligned}
&\phi\left(z;0_d,\sigma^1\right)-\phi\left(z;0_d,\sigma^2\right)=\prod_{i=1}^{d}\phi\left(z_i;0,\sigma^1\right)-\prod_{i=1}^{d}\phi\left(z_i;0,\sigma^2\right)\\
&\pm\phi\left(z_1;0,\sigma^2\right)\prod_{i=2}^{d}\phi\left(z_i;0,\sigma^1\right)\pm\phi\left(z_1;0,\sigma^2\right)\phi\left(z_2;0,\sigma^2\right)\prod_{i=3}^{d}\phi\left(z_i;0,\sigma^1\right)\\
&\pm\cdots\pm\prod_{i=1}^{d}\phi\left(z_i;0,\sigma^2\right)\\
&=\left[\phi\left(z_1;0,\sigma^1\right)-\phi\left(z_1;0,\sigma^2\right)\right]\prod_{i=2}^{d}\phi\left(z_i;0,\sigma^1\right)\\
&+\phi\left(z_1;0,\sigma^2\right)\left[\phi\left(z_2;0,\sigma^1\right)-\phi\left(z_2;0,\sigma^2\right)\right]\prod_{i=3}^{d}\phi\left(z_i;0,\sigma^1\right)\\
&+\cdots+\prod_{i=1}^{d-1}\phi\left(z_i;0,\sigma^2\right)\left[\phi\left(z_d;0,\sigma^1\right)-\phi\left(z_d;0,\sigma^2\right)\right].
\end{aligned}$$

With the result for the univariate case, the last expression is bounded by

$$\begin{aligned}
&\left|\frac{1}{\sigma^1}-\frac{1}{\sigma^2}\right|\left\{M_\phi(z_1)\prod_{i=2}^{d}\phi\left(z_i;0,\sigma^1\right)+M_\phi(z_2)\phi\left(z_1;0,\sigma^2\right)\prod_{i=3}^{d}\phi\left(z_i;0,\sigma^1\right)\right.\\
&\left.+\cdots+M_\phi(z_d)\prod_{i=1}^{d-1}\phi\left(z_i;0,\sigma^2\right)\right\}.
\end{aligned}$$

Q.E.D.

APPENDIX D: AUXILIARY RESULTS AND DETAILS

D.1. *Newton-Kantorovich algorithm for DP solution*

For a given value of the model parameter ψ, the Emax function Q_ψ is the fixed point of the operator $T_\psi(\cdot)$ defined by

$$T_\psi(\mathbf{Q})[x]=\sum_{k=1}^{m}\omega_k\sigma_k\left[A_{kx}+E1(e^{-a_{kx}})\right]$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} \omega_k \sigma_k \Bigg\{ \\
&\log \sum_{j=1}^{J} \exp\left(\frac{u(x,j,;\psi) + \beta \sum_{y=1}^{K} G_{xy}^{j} \mathbf{Q}(y) + \mu_{jk}}{\sigma_k} \right) + \\
&E1\Bigg(\exp\Bigg[-\frac{u(0,j,;\psi) + \beta \sum_{y=1}^{K} G_{xy}^{j} \mathbf{Q}(y)}{\sigma_k} - \gamma \\
&\qquad - \log \sum_{j=1}^{J} \exp\left\{ \frac{u(x,j,;\psi) + \beta \sum_{y=1}^{K} G_{xy}^{j} \mathbf{Q}(y) + \mu_{jk}}{\sigma_k} \right\} \Bigg] \Bigg) \Bigg\}.
\end{aligned}
$$

To find the fixed point of T_ψ, following Rust (1987), we use the Newton-Kantorovich algorithm, which is essentially a Newton method for solving the nonlinear system of equations, because it is more efficient than the iterations on T_ψ.

Lemma 9 *The Newton-Kantorovich algorithm has the update rule*

$$
Q^{(n+1)} = Q^{(n)} - \Big[I - T'_\psi(Q^{(n)}) \Big]^{-1} \Big[I - T_\psi \Big](Q^n), \tag{21}
$$

where

$$
T'_\psi(Q) = \beta \begin{pmatrix} \sum_{d=0}^{J} G_{11}^{d} P(d|x=1) & \dots & \sum_{d=0}^{J} G_{1K}^{d} P(d|x=1) \\ \vdots & & \vdots \\ \sum_{d=0}^{J} G_{K1}^{d} P(d|x=K) & \dots & \sum_{d=0}^{J} G_{KK}^{d} P(d|x=K) \end{pmatrix}.
$$

proof of Lemma 9: Note that

$$
T'_\psi(Q) = \begin{pmatrix} \frac{\partial}{\partial Q(1)} T_\psi(Q)(1) & \dots & \frac{\partial}{\partial Q(K)} T_\psi(Q)(1) \\ \vdots & & \vdots \\ \frac{\partial}{\partial Q(1)} T_\psi(Q)(K) & \dots & \frac{\partial}{\partial Q(K)} T_\psi(Q)(K) \end{pmatrix}.
$$

Recall that $T_\psi(Q)(x) = \sum_{k=1}^m \omega_k \sigma_k \left[A_{kx} + E1(e^{-a_{kx}})\right]$. Denoting $f(a_{kx}) = E1(e^{-a_{kx}})$, we have

$$\frac{\partial}{\partial Q(1)} T_\psi(Q)(x) = \sum_{k=1}^m \omega_k \sigma_k \left[\frac{\partial}{\partial Q(1)} A_{kx} + \frac{\partial}{\partial Q(1)} f(a_{kx})\right], \text{ for } x = 1, \ldots, K.$$

Recall that

$$A_{kx} = \log \sum_{j=1}^J \exp\left[\frac{v(x,j) + \mu_{jk}}{\sigma_k}\right] = \log \sum_{j=1}^J \exp\left[\frac{u(x,j) + \beta \sum_{y=1}^K G_{xy}^j Q(y) + \mu_{jk}}{\sigma_k}\right],$$

$$a_{kx} = \frac{v_0}{\sigma_k} + \gamma - A_k = \frac{1}{\sigma_k}\left[u(x,0) + \beta \sum_{y=1}^K G_{xy}^j Q(y)\right] + \gamma - A_{kx}.$$

Hence, for example,

$$\begin{aligned}
\frac{\partial A_{k1}}{\partial Q(1)} &= \frac{1}{\sum_{j=1}^J \exp\left[\frac{v(1,j)+\mu_{jk}}{\sigma_k}\right]} \sum_{d=1}^J \frac{\partial \exp\left[\frac{v(1,d)+\mu_{dk}}{\sigma_k}\right]}{\partial \left[\frac{v(1,d)+\mu_{dk}}{\sigma_k}\right]} \frac{\partial \left[\frac{v(1,d)+\mu_{dk}}{\sigma_k}\right]}{\partial Q(1)} \\
&= \frac{1}{\sum_{j=1}^J \exp\left[\frac{v(1,j)+\mu_{jk}}{\sigma_k}\right]} \sum_{d=1}^J \exp\left[\frac{v(1,d)+\mu_{dk}}{\sigma_k}\right] \frac{\beta}{\sigma_k} G_{11}^d \\
&= \frac{\beta}{\sigma_k} \sum_{d=1}^J \frac{\exp\left[\frac{v(1,d)+\mu_{dk}}{\sigma_k}\right]}{\sum_{j=1}^J \exp\left[\frac{v(1,j)+\mu_{jk}}{\sigma_k}\right]} G_{11}^d \\
&= \frac{\beta}{\sigma_k} \sum_{d=1}^J \exp\left[\frac{v(1,d)+\mu_{dk}}{\sigma_k} - A_{k1}\right] G_{11}^d.
\end{aligned}$$

Also,

$$\frac{\partial f(a_{k1})}{\partial Q(1)} = \frac{\partial g(e^{-a_{k1}})}{\partial e^{-a_{k1}}} \frac{\partial e^{-a_{k1}}}{\partial(-a_{k1})} \frac{\partial(-a_{k1})}{\partial a_{k1}} \frac{\partial a_{k1}}{\partial Q(1)} = g'(e^{-a_{k1}}) e^{-a_{k1}} (-1) \frac{\partial a_{k1}}{\partial Q(1)},$$

where we let $g(y) = E1(y)$. We know that $g'(y) = -\frac{e^{-y}}{y}$. Note that $\frac{\partial a_{k1}}{\partial Q(1)} = \frac{\beta}{\sigma_k} G_{11}^1 - \frac{\partial A_{k1}}{\partial Q(1)}$. Hence

$$\frac{\partial f(a_{k1})}{\partial Q(1)} = -\exp\left[-e^{-a_{k1}}\right] e^{a_{k1}} e^{-a_{k1}} (-1) \frac{\partial a_{k1}}{\partial Q(1)} = \exp\left[-e^{-a_{k1}}\right] \left[\frac{\beta}{\sigma_k} G_{11}^1 - \frac{\partial A_{k1}}{\partial Q(1)}\right].$$

Now,

$$\begin{aligned}\frac{\partial}{\partial Q(1)}T_{\psi}(Q)(1) &= \sum_{k=1}^{m}\omega_k\sigma_k\left[\frac{\partial A_{k1}}{\partial Q(1)} + \exp\left[-e^{-a_{k1}}\right]\left[\frac{\beta}{\sigma_k}G_{11}^{1} - \frac{\partial A_{k1}}{\partial Q(1)}\right]\right] \\ &= \sum_{k=1}^{m}\omega_k\sigma_k\left[\frac{\partial A_{k1}}{\partial Q(1)}\Big(1-\exp\left[-e^{-a_{k1}}\right]\Big) + \exp\left[-e^{-a_{k1}}\right]\frac{\beta}{\sigma_k}G_{11}^{1}\right].\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{k=1}^{m}\omega_k\sigma_k\frac{\partial A_{k1}}{\partial Q(1)}\Big(1-\exp\left[-e^{-a_{k1}}\right]\Big) \\ &= \sum_{k=1}^{m}\omega_k\sigma_k\underbrace{\frac{\beta}{\sigma_k}\sum_{d=1}^{J}\exp\left[\frac{v(1,d)+\mu_{dk}}{\sigma_k} - A_{k1}\right]G_{11}^{d}}_{\frac{\partial A_{k1}}{\partial Q(1)}}\Big(1-\exp\left[-e^{-a_{k1}}\right]\Big) \\ &= \beta\sum_{d=1}^{J}G_{11}^{d}\sum_{k=1}^{m}\omega_k\exp\left[\frac{v(1,d)+\mu_{dk}}{\sigma_k} - A_{k1}\right]\Big(1-\exp\left[-e^{-a_{k1}}\right]\Big) = \beta\sum_{d=1}^{J}G_{11}^{d}P(d|x=1) \\ &\sum_{k=1}^{m}\omega_k\sigma_k\exp\left[-e^{-a_{k1}}\right]\frac{\beta}{\sigma_k}G_{11}^{1} = \beta\sum_{k=1}^{m}\omega_k\exp\left[-e^{-a_{k1}}\right]G_{11}^{1} = \beta G_{11}^{1}P(0|x=1).\end{aligned}$$

Hence,

$$\frac{\partial}{\partial Q(1)}T_{\psi}(Q)(1) = \beta\sum_{d=0}^{J}G_{11}^{d}P(d|x=1).$$

We can similarly compute other elements. *Q.E.D.*

D.2. *Properties of univariate extreme value distributions*

Let $Z \sim \phi(\cdot)$. Then it has zero mean and variance $\frac{\pi^2}{6}$. Its density is $e^{-z-\gamma-e^{-z-\gamma}}$ and its cdf is $e^{-e^{-z-\gamma}}$, where γ is the Euler-Mascheroni constant.

Lemma 10 and Lemma 11 show how to compute median and truncated integrals of a random variable that follows a mixture of extreme value distributions, which is helpful for imposing the location and scale normalizations after a MCMC run.

Lemma 10 *Median*

1. *Let $X \sim \phi(\cdot;\mu,\sigma)$. Then its median is $\mu - \sigma\log\log 2 - \sigma\gamma$.*
2. *Let $X \sim \sum_{k=1}^{m}\omega_k\phi(\cdot;\mu_k,\sigma_k)$. Then there is no closed form for its median. It has to be solved for via a root-finding algorithm.*

Proof: Let $X \sim \phi(\cdot;\mu,\sigma)$. By definition of median, we want to find M such that $0.5 = Pr(X < M)$. Letting $t(x) = e^{-\left(\frac{x-\mu}{\sigma}\right)}e^{-\gamma}$, this is equivalent to

$$\begin{aligned} 0.5 = \exp\left[-t(M)\right] &\iff t(M) = \log 2 \iff e^{-\left(\frac{M-\mu}{\sigma}\right)}e^{-\gamma} = \log 2 \\ \iff -\left(\frac{M-\mu}{\sigma}\right) - \gamma = \log\log 2 &\iff M = \mu - \sigma\log\log 2 - \sigma\gamma \end{aligned}$$

Q.E.D.

Lemma 11 *Truncated Integral*

1. *Let $X \sim \phi(\cdot;\mu,\sigma)$. Then*

$$\int_M^\infty x\phi(x;\mu,\sigma)dx = \mu - M\exp\left[-e^{-b}\right] + \sigma E1(e^{-b})$$

where $b = \frac{M-\mu}{\sigma} + \gamma$

2. *Let $X \sim p(\cdot|\psi,m) = \sum_{k=1}^{m}\omega_k\phi(\cdot;\mu_k,\sigma_k)$. Then*

$$\int_M^\infty xp(x|\psi,m)dx = \sum_{k=1}^{m}\omega_k\left\{\mu_k - M\exp\left[-e^{-b_k}\right] + \sigma_k E1(e^{-b_k})\right\}$$

where $b_k = \frac{M-\mu_k}{\sigma_k} + \gamma$

Proof: Let $X \sim \phi(\cdot;\mu,\sigma)$.

$$\begin{aligned} \int_M^\infty x\phi(x;\mu,\sigma)dx &= \int_M^\infty x\frac{1}{\sigma}e^{-\left(\frac{x-\mu}{\sigma}\right)}e^{-\gamma}\exp\left[-e^{-\left(\frac{x-\mu}{\sigma}\right)}e^{-\gamma}\right]dx \\ &= \int_b^\infty \left[\sigma z + (\mu - \sigma\gamma)\right]\frac{1}{\sigma}e^{-z}\exp\left[-e^{-z}\right]\sigma dz \end{aligned}$$

$$= \sigma \underbrace{\int_b^\infty ze^{-z}\exp\left[-e^{-z}\right]dz}_{\mathrm{I}} + (\mu-\sigma\gamma)\underbrace{\int_b^\infty e^{-z}\exp\left[-e^{-z}\right]dz}_{\mathrm{II}}$$

where we let $z = \left(\frac{x-\mu}{\sigma}\right) + \gamma$ which means that $x = \sigma z + (\mu - \sigma\gamma), dx = \sigma dz$. $x = \infty \implies z = \infty, x = M \implies z = \frac{M-\mu}{\sigma} + \gamma = b$. For computing the first integral, let $y = e^{-z}$ which means that $dz = -\frac{1}{y}dy, z = -\log y$. We have

$$I = \int_{e^{-b}}^0 (-\log y)\, ye^{-y}(-\frac{1}{y}dy) = -\int_0^{e^{-b}} \log ye^{-y}dy$$

Recall that $\gamma = -\int_0^\infty \log ye^{-y}dy = -\left(\int_0^{e^{-b}} \log ye^{-y}dy + \int_{e^{-b}}^\infty \log ye^{-y}dy\right)$. Hence, $I = \gamma + \int_{e^{-b}}^0 \log ye^{-y}dy$. By integration by parts, letting $u = \log y, dv = e^{-y}dy \implies du\frac{1}{y}dy, v = -e^{-y}$,

$$\begin{aligned} I &= \gamma + \int_{e^{-b}}^0 \log ye^{-y}dy = \gamma + \int udv - \int vdu \\ &= \gamma - \left[\log ye^{-y}\right]_{e^{-b}}^\infty + \int_{e^{-b}}^\infty \frac{1}{y}e^{-y}dy = \gamma - \left[0 - (-b)\exp\left(-e^{-b}\right)\right] + E1\left(e^{-b}\right) \\ &= \gamma - b\exp\left(-e^{-b}\right) + E1\left(e^{-b}\right) \end{aligned}$$

where by definition $E1(z) = \int_z^\infty \frac{1}{t}e^{-t}dt$. For the second integral, letting $y = e^{-z}$ which means that $dz = -\frac{1}{y}dy$, we have

$$\begin{aligned} II &= \int_b^\infty e^{-z}\exp\left[-e^{-z}\right]dz \\ &= \int_{e^{-b}}^0 ye^{-y}(-\frac{1}{y}dy) = \int_0^{e^{-b}} e^{-y}dy = -\left(\exp\left(-e^{-b}\right) - 1\right) \\ &= 1 - \exp\left(-e^{-b}\right) \end{aligned}$$

Finally,

$$\int_M^\infty xdF(x) = \sigma I + (\mu - \sigma\gamma)II = \mu - M\exp\left[-e^{-b}\right] + \sigma E1(e^{-b})$$

Q.E.D.

D.3. *The Prior for the Transformed Mixing Weights*

We want to define a prior on $\big(\alpha_1, \ldots, \alpha_{m-1}\big)$ which implies the prior $\big(\omega_1, \ldots, \omega_m\big) \sim Dir(a_1, \ldots, a_m)$ or equivalently $\gamma_\ell \sim Gamma(a_\ell, 1)$ for $\ell = 1, \ldots, m$. Note that the inverse map $g(\alpha) = \omega$ is defined by $\omega_\ell = \frac{e^{\alpha_\ell}}{1+\sum_{s=1}^{m-1} e^{\alpha_s}}$, for $\ell = 1, \ldots, m-1$. Denote the Jacobian $A(\alpha) = \frac{dg(\alpha)}{d\alpha}$. By the change of variable formula, we have $f_\alpha(\alpha) = f_\omega(g(\alpha)) det \frac{dg(\alpha)}{d\alpha} = f_\omega(g(\alpha)) det A(\alpha)$.

In order to have $f_\omega(\omega) = Dir(\omega_1, \ldots, \omega_m; \bar{\omega}_1, \ldots, \bar{\omega}_m)$, we can use the above expression for the density $f_\alpha(\alpha)$ to determine the desired prior for $\alpha_1, \ldots, \alpha_m$.

In addition, for using HMC, we need the first order derivative $\frac{\partial f_\alpha(\alpha)}{\partial \alpha}$. Note that in general, $\frac{\partial}{\partial \alpha} det A(\alpha) = det A(\alpha) tr\Bigg(A(\alpha)^{-1} \frac{\partial}{\partial \alpha} A(\alpha)\Bigg)$. Hence, $\frac{\partial f_\alpha(\alpha)}{\partial \alpha} = \Bigg[\frac{\partial}{\partial \alpha} f_\omega(g(\alpha))\Bigg] det A(\alpha) + f_\omega(g(\alpha)) \Bigg[det A(\alpha) tr\Bigg(A(\alpha)^{-1} \frac{\partial}{\partial \alpha} A(\alpha)\Bigg)\Bigg]$.

D.4. *Derivatives of the Log-likelihood*

Let $L(\chi) = \log p(D_n|\chi, m) = \sum_{d,x} n_{dx} \times \log p(d|x; \chi, m)$ denote the log-likelihood as a function of χ, where n_{dx} is the number of decision makers in the data set that chose d at the observed state x. Below, we present the derivatives of $L(\chi)$ with respect to χ for $J = 1$. The computation for $J > 1$ can be done similarly.

Lemma 12 *Derivatives of $L(\chi)$ with respect to χ for $J = 1$.*

(1) For $k = 1, ..., m-1$,

$$\frac{\partial L(\chi)}{\partial \alpha_k} = \sum_{k=1}^{m-1} \frac{\partial L(\chi)}{\partial \omega_\ell} \frac{\partial \omega_\ell}{\partial \alpha_k}, \textit{ where}$$

$$\begin{aligned}\frac{\partial L(\chi)}{\partial \omega_\ell} =& \sum_x n_{0x} \frac{p(0|x,\ell) - p(0|x,m) + \sum_k \omega_k \frac{\partial p(0|x,k)}{\partial \omega_\ell}}{\sum_{k'} \omega_{k'} p(0|x,k')} \\ &+ \sum_x n_{1x} \frac{p(1|x,\ell) - p(1|x,m) + \sum_k \omega_k \frac{\partial p(1|x,k)}{\partial \omega_\ell}}{\sum_{k'} \omega_{k'} p(1|x,k')},\end{aligned}$$

$$\frac{\partial p(0|x,k)}{\partial \omega_\ell} = \exp\big[-e^{-a_{xk}} - a_{xk}\big]\frac{\beta}{\sigma_k}\sum_{y=1}^{K}(G^0_{xy} - G^1_{xy})\frac{\partial Q(y)}{\partial \omega_\ell},$$

$$\frac{\partial \omega_\ell}{\partial \alpha_k} = \frac{e^{\alpha_\ell}}{(1+\sum_{s=1}^{m-1} e^{\alpha_s})^2}\big[(1+\sum_{s\neq k}^{m-1} e^{\alpha_s})1(k=\ell) - e^{\alpha_k}1(k\neq\ell)\big].$$

(2) For $k = 1, ..., m$,

$$\frac{\partial L(\chi)}{\partial \mu_k} = \sum_x n_{0x}\frac{\sum_\ell \omega_\ell \frac{\partial p(0|x,\ell)}{\partial \mu_k}}{\sum_{k'}\omega_{k'}p(0|x,k')} + \sum_x n_{1x}\frac{\sum_\ell \omega_\ell \frac{\partial p(1|x,\ell)}{\partial \mu_k}}{\sum_{k'}\omega_{k'}p(1|x,k')},\ \textit{where}$$

$$\frac{\partial p(0|x,\ell)}{\partial \mu_k} = \exp\big[-e^{-a_{x\ell}} - a_{x\ell}\big]\frac{\partial a_{x\ell}}{\partial \mu_k},$$

$$\frac{\partial a_{x\ell}}{\partial \mu_k} = \frac{\beta}{\sigma_\ell}\sum_{y=1}^{K}(G^0_{xy} - G^1_{xy})\frac{\partial Q(y)}{\partial \mu_k} - \frac{1}{\sigma_\ell}1(k=\ell).$$

(3) For $s_k = \log\tilde{\sigma}_k$, $k = 1, ..., m$,

$$\frac{\partial L(\chi)}{\partial s_k} = \frac{\partial L(\chi)}{\partial \sigma_k}\sigma_k,\ \textit{where}$$

$$\frac{\partial L(\chi)}{\partial \sigma_k} = \sum_x n_{0x}\frac{\sum_\ell \omega_\ell \frac{\partial p(0|x,\ell)}{\partial \sigma_k}}{\sum_{k'}\omega_{k'}p(0|x,k')} + \sum_x n_{1x}\frac{\sum_\ell \omega_\ell \frac{\partial p(1|x,\ell)}{\partial \mu_k}}{\sum_{k'}\omega_{k'}p(1|x,k')},$$

$$\frac{\partial p(0|x,\ell)}{\partial \sigma_k} = \exp\big[-e^{-a_{x\ell}} - a_{x\ell}\big]\frac{\partial a_{x\ell}}{\partial \sigma_k},$$

$$\frac{\partial a_{x\ell}}{\partial \sigma_k} = \frac{\beta}{\sigma_\ell}\sum_{y=1}^{K}(G^0_{xy} - G^1_{xy})\frac{\partial Q(y)}{\partial \mu_k} + \frac{1}{\sigma_\ell}[\gamma - a_{xk}]1(k=\ell).$$

(4) With respect to $s = \log(\sigma)$

$$\frac{\partial L(\chi)}{\partial s} = \sum_{\ell=1}^{m}\frac{\partial L(\chi)}{\partial s_\ell}.$$

(5) The derivatives of the Emax function can be computed by the implicit function theorem:

$$\frac{\partial Q(x)}{\partial \omega_k} = -\frac{\frac{\partial f_x}{\partial \omega_k}}{\frac{\partial f_x}{\partial Q(x)}}, \quad \frac{\partial Q(x)}{\partial \mu_k} = -\frac{\frac{\partial f_x}{\partial \mu_k}}{\frac{\partial f_x}{\partial Q(x)}}, \quad \frac{\partial Q(x)}{\partial \sigma_k} = -\frac{\frac{\partial f_x}{\partial \sigma_k}}{\frac{\partial f_x}{\partial Q(x)}},$$

where $f_x = Q(x) - \sum_{k=1}^{m} \omega_k \sigma_k \big[A_{xk} + E1(e^{-a_{xk}})\big]$ *and*

$$\frac{\partial f_x}{\partial Q(x)} = 1 - \beta G^1_{xx} - \beta(G^0_{xx} - G^1_{xx}) \sum_{k=1}^{m} \omega_k \exp\big[-e^{-a_{xk}}\big],$$

$$\frac{\partial f_x}{\partial \omega_k} = -\sigma_k\big[A_{xk} + E1(e^{-a_{xk}})\big] + \sigma_m\big[A_{xm} + E1(e^{-a_{xm}})\big],$$

$$\frac{\partial f_x}{\partial \mu_k} = -\omega_k\big[1 - \exp(-e^{-a_x k})\big],$$

$$\frac{\partial f_x}{\partial \sigma_k} = -\omega_k\big[E1(e^{-a_{xk}}) + \exp(-e^{-a_x k})(\gamma - a_{xk})\big].$$

PROOF: For (1),

$$\frac{\partial L(\chi)}{\partial \alpha_k} = \sum_{k=1}^{m-1} \frac{\partial L(\chi)}{\partial \omega_\ell}\frac{\partial \omega_\ell}{\partial \alpha_k},$$

$$\frac{\partial L(\chi)}{\partial \omega_\ell} = \sum_x n_{0x} \frac{\frac{\partial}{\partial \omega_\ell}\sum_k \omega_k p(0|x,k)}{\sum_{k'} \omega_{k'} p(0|x,k')} + \sum_x n_{1x} \frac{\frac{\partial}{\partial \omega_\ell}\sum_k \omega_k p(1|x,k)}{\sum_{k'} \omega_{k'} p(1|x,k')},$$

$$\sum_k^m \omega_k p(d|x,k) = \sum_k^{m-1} \omega_k p(d|x,k) + (1 - \sum_k^{m-1} \omega_k) p(d|x,m),$$

$$\frac{\partial}{\partial \omega_\ell}\sum_k^m \omega_k p(d|x,k) = p(d|x,\ell) + \sum_k^{m-1} \omega_k \frac{\partial p(d|x,k)}{\partial \omega_\ell} - p(d|x,m) + (1 - \sum_k^{m-1}\omega_k)\frac{\partial p(d|x,m)}{\partial \omega_\ell}$$

$$= p(d|x,\ell) - p(d|x,m) + \sum_k^m \omega_k \frac{\partial p(d|x,k)}{\partial \omega_\ell}$$

$$\Longrightarrow \frac{\partial L(\chi)}{\partial \omega_\ell} = \sum_x n_{0x} \frac{p(0|x,\ell) - p(0|x,m) + \sum_k \omega_k \frac{\partial p(0|x,k)}{\partial \omega_\ell}}{\sum_{k'} \omega_{k'} p(0|x,k')}$$

$$+ \sum_x n_{1x} \frac{p(1|x,\ell) - p(1|x,m) + \sum_k \omega_k \frac{\partial p(1|x,k)}{\partial \omega_\ell}}{\sum_{k'} \omega_{k'} p(1|x,k')},$$

$$\frac{\partial p(0|x,k)}{\partial \omega_\ell} = \exp\big[-e^{-a_{xk}} - a_{xk}\big]\frac{\partial a_{xk}}{\partial \omega_\ell}.$$

For $J = 1$, we have,

$$\begin{aligned} a_{x\ell} &= \frac{1}{\sigma_\ell}\big[v(x,0) - v(x,1) - \mu_\ell\big] + \gamma \\ &= \frac{1}{\sigma_\ell}\left[u(x,0) - u(x,1) + \beta\sum_{y=1}^{K}(G^0_{xy} - G^1_{xy})Q(y) - \mu_\ell\right] + \gamma. \end{aligned}$$

Hence,

$$\frac{\partial p(0|x,k)}{\partial \omega_\ell} = \exp\big[-e^{-a_{xk}} - a_{xk}\big]\frac{\beta}{\sigma_k}\sum_{y=1}^{K}(G^0_{xy} - G^1_{xy})\frac{\partial Q(y)}{\partial \omega_\ell}.$$

Note that

$$\frac{\partial p(1|x,k)}{\partial \omega_\ell} = 1 - \frac{\partial p(0|x,k)}{\partial \omega_\ell}.$$

Moreover, recall that $\omega_\ell = \frac{e^{\alpha_\ell}}{1+\sum_{s=1}^{m-1}e^{\alpha_s}}$.
If $k = \ell$, then $\frac{\partial \omega_\ell}{\partial \alpha_k} = \frac{e^{\alpha_\ell}(1+\sum_{s=1}^{m-1}e^{\alpha_s}) - e^{\alpha_\ell}e^{\alpha_\ell}}{(1+\sum_{s=1}^{m-1}e^{\alpha_s})^2} = \frac{e^{\alpha_\ell}(1+\sum_{s\neq\ell}^{m-1}e^{\alpha_s})}{(1+\sum_{s=1}^{m-1}e^{\alpha_s})^2}$.
If $k \neq \ell$, then $\frac{\partial \omega_\ell}{\partial \alpha_k} = \frac{-e^{\alpha_\ell}e^{\alpha_k}}{(1+\sum_{s=1}^{m-1}e^{\alpha_s})^2}$.

For (2),

$$\begin{aligned} \frac{\partial L(\chi)}{\partial \mu_k} &= \sum_x n_{0x}\frac{\frac{\partial}{\partial \mu_k}\sum_k \omega_k p(0|x,k)}{\sum_{k'}\omega_{k'}p(0|x,k')} + \sum_x n_{1x}\frac{\frac{\partial}{\partial \mu_k}\sum_k \omega_k p(1|x,k)}{\sum_{k'}\omega_{k'}p(1|x,k')} \\ &= \sum_x n_{0x}\frac{\sum_\ell \omega_\ell \frac{\partial p(0|x,\ell)}{\partial \mu_k}}{\sum_{k'}\omega_{k'}p(0|x,k')} + \sum_x n_{1x}\frac{\sum_\ell \omega_\ell \frac{\partial p(1|x,\ell)}{\partial \mu_k}}{\sum_{k'}\omega_{k'}p(1|x,k')}, \\ \frac{\partial p(0|x,\ell)}{\partial \mu_k} &= \exp\big[-e^{-a_{x\ell}} - a_{x\ell}\big]\frac{\partial a_{x\ell}}{\partial \mu_k}, \\ \frac{\partial a_{x\ell}}{\partial \mu_k} &= \begin{cases} \frac{\beta}{\sigma_\ell}\sum_{y=1}^{K}(G^0_{xy} - G^1_{xy})\frac{\partial Q(y)}{\partial \mu_k} - \frac{1}{\sigma_\ell}, & \text{if } k = \ell \\ \frac{\beta}{\sigma_\ell}\sum_{y=1}^{K}(G^0_{xy} - G^1_{xy})\frac{\partial Q(y)}{\partial \mu_k}, & \text{otherwise.} \end{cases} \end{aligned}$$

(3) Recall that $\sigma_k = \sigma\tilde{\sigma}_k$. $\tilde{\sigma}_k = e^{s_k}$, so $\frac{\partial\tilde{\sigma}_k}{\partial s_k} = \tilde{\sigma}_k$. We have

$$\begin{aligned}
\frac{\partial L(\chi)}{\partial s_k} &= \frac{\partial L(\chi)}{\partial \sigma_k}\frac{\partial \sigma\tilde{\sigma}_k}{\partial \tilde{\sigma}_k}\frac{\partial \tilde{\sigma}_k}{\partial s_k} = \frac{\partial L(\chi)}{\partial \sigma_k}\sigma\tilde{\sigma}_k = \frac{\partial L(\chi)}{\partial \sigma_k}\sigma_k, \\
\frac{\partial L(\chi)}{\partial \sigma_k} &= \sum_x n_{0x}\frac{\frac{\partial}{\partial\sigma_k}\sum_k \omega_k p(0|x,k)}{\sum_{k'}\omega_{k'}p(0|x,k')} + \sum_x n_{1x}\frac{\frac{\partial}{\partial\sigma_k}\sum_k \omega_k p(1|x,k)}{\sum_{k'}\omega_{k'}p(1|x,k')} \\
&= \sum_x n_{0x}\frac{\sum_\ell \omega_\ell \frac{\partial p(0|x,\ell)}{\partial\sigma_k}}{\sum_{k'}\omega_{k'}p(0|x,k')} + \sum_x n_{1x}\frac{\sum_\ell \omega_\ell \frac{\partial p(1|x,\ell)}{\partial\sigma_k}}{\sum_{k'}\omega_{k'}p(1|x,k')}, \\
\frac{\partial p(0|x,\ell)}{\partial \sigma_k} &= \exp\big[-e^{-a_{x\ell}} - a_{x\ell}\big]\frac{\partial a_{x\ell}}{\partial\sigma_k}, \\
\frac{\partial a_{x\ell}}{\partial\sigma_k} &= \begin{cases} \frac{1}{\sigma_\ell}\left[\gamma - a_{xk} + \beta\sum_{y=1}^K (G^0_{xy} - G^1_{xy})\frac{\partial Q(y)}{\partial\sigma_k}\right], & \text{if } k = \ell \\ \frac{\beta}{\sigma_\ell}\sum_{y=1}^K (G^0_{xy} - G^1_{xy})\frac{\partial Q(y)}{\partial\sigma_k}, & \text{otherwise.}\end{cases}
\end{aligned}$$

(4) Note that $\sigma = e^s$, so $\frac{\partial\sigma}{\partial s} = \sigma$

$$\frac{\partial L(\chi)}{\partial s} = \sum_{k=1}^m \frac{\partial L(\chi)}{\partial\sigma_k}\frac{\partial\sigma\tilde{\sigma}_k}{\partial\sigma}\frac{\partial\sigma}{\partial s} = \sum_{k=1}^m \frac{\partial L(\chi)}{\partial\sigma_k}\sigma_k = \sum_{k=1}^m \frac{\partial L(\chi)}{\partial s_k}.$$

(5) Recall that

$$Q(x) = \sum_{k=1}^m \omega_k\sigma_k\big[A_{xk} + E1(e^{-a_{xk}})\big],$$

$$f_x = Q(x) - \sum_{k=1}^m \omega_k\sigma_k\big[A_{xk} + E1(e^{-a_{xk}})\big],$$

$$a_{xk} = \frac{1}{\sigma_k}\left[u(x,0) - u(x,1) + \beta\sum_{y=1}^K (G^0_{xy} - G^1_{xy})Q(y) - \mu_k\right] + \gamma,$$

$$A_{xk} = \frac{1}{\sigma_k}\left[u(x,1) + \beta\sum_{y=1}^K G^1_{xy}Q(y) + \mu_k\right].$$

Hence, we have

$$\begin{aligned}
\frac{\partial f_x}{\partial Q(x)} &= 1 - \sum_{k=1}^{m} \omega_k \sigma_k \left[\frac{\partial A_{xk}}{\partial Q(x)} + \frac{\partial E1(e^{-a_{xk}})}{\partial Q(x)}\right], \\
\frac{\partial A_{xk}}{\partial Q(x)} &= \frac{\beta}{\sigma_k} G^1_{xx}, \\
\frac{\partial E1(e^{-a_{xk}})}{\partial Q(x)} &= \exp\big[-e^{-a_{xk}}\big] \frac{\partial a_{xk}}{\partial Q(x)} = \exp\big[-e^{-a_{xk}}\big] \frac{\beta}{\sigma_k}(G^0_{xx} - G^1_{xx}). \\
\Longrightarrow \quad \frac{\partial f_x}{\partial Q(x)} &= 1 - \beta G^2_{xx} - \beta(G^1_{xx} - G^2_{xx}) \sum_{k=1}^{m} \omega_k \exp\big[-e^{-a_{xk}}\big].
\end{aligned}$$

It is easy to show that $\frac{\partial f_x}{\partial \omega_k} = -\sigma_k\big[A_{xk} + E1(e^{-a_{xk}})\big] + \sigma_m\big[A_{xm} + E1(e^{-a_{xm}})\big]$ and $\frac{\partial f_x}{\partial \mu_k} = -\omega_k\sigma_k\left[\frac{\partial A_{xk}}{\partial \mu_k} + \frac{\partial E1(e^{-a_{xk}})}{\partial \mu_k}\right] = -\omega_k\big[1 - \exp(-e^{-a_x k})\big]$.

Finally,

$$\begin{aligned}
\frac{\partial f_x}{\partial \sigma_k} &= -\omega_k\big[A_{xk} + E1(e^{-a_{xk}})\big] - \omega_k\sigma_k\left[\frac{\partial A_{xk}}{\partial \sigma_k} + \frac{\partial E1(e^{-a_{xk}})}{\partial \sigma_k}\right], \\
\frac{\partial A_{xk}}{\partial \sigma_k} &= \frac{1}{\sigma_k^2}\big[v(x,1) + \mu_k\big] = \frac{1}{\sigma_k} A_{xk}, \\
\frac{\partial E1(e^{-a_{xk}})}{\partial Q(x)} &= \exp\big[-e^{-a_{xk}}\big] \frac{\partial a_{xk}}{\partial \sigma_k} = \exp\left[-e^{-a_{xk}}\right] \left[-\frac{1}{\sigma_k^2}(v(x,0) - v(x,1) - \mu_k)\right] \\
&= \exp\big[-e^{-a_{xk}}\big] \frac{1}{\sigma_k}\big[\gamma - a_{xk}\big], \\
\Longrightarrow \quad \frac{\partial f_x}{\partial \sigma_k} &= -\omega_k\big[E1(e^{-a_{xk}}) + \exp(-e^{-a_x k})(\gamma - a_{xk})\big].
\end{aligned}$$

Q.E.D.

APPENDIX E: IMPLEMENTATION DETAILS FOR RUSTS'S MODEL

E.1. *Norets and Tang (2013)*

Norets and Tang (2013) showed that if we assume that the distribution is unknown, then the preference-parameter $\theta = (\theta_0, \theta_1)$ is only set-identified and proposed an algorithm to compute the identified set. In Section 5.1 of their paper, they applied the method to the

Rust's model. In their paper, the following setting of the utility function is used:

$$\begin{aligned} u(x,0) &= \theta_0 + \theta_1 x + \Delta\epsilon, \\ u(x,1) &= 0, \end{aligned}$$

where θ_0 and θ_1 are the data generating values of preference parameters. The term $\Delta\epsilon = \epsilon_1 - \epsilon_2$ follows some unknown distribution F, which is assumed to have the same location-scale normalization as the logistic distribution. Specifically, $\int z dF(z) = 0$ and $\int_{M_F}^{\infty} z dF(z) = \log 2$, where M_F is the median of F or $F(M_F) = 0.5$. Figures 1 and 2 show the identified set of preference-parameters computed by their algorithm. For each θ inside of the black lines, there is a corresponding unknown distribution F such that the pair (θ, F) implies the true vector of CCPs.

E.2. *Semiparametric estimation*

We now apply our semiparametric estimation method to the same example.

E.2.1. *Normalization*

It is convenient not to impose the location and scale normalization on the distribution of $\Delta\epsilon$ during the MCMC estimation. We impose the location and scale normalization on utility parameters instead:

$$\begin{aligned} u(x,0) &= \hat{\theta}_0 + \hat{\theta}_1 x, \\ u(x,1) &= \epsilon \sim \sum_{k=1}^{m} \omega_k \phi(\epsilon; \mu_k, \sigma_k), \end{aligned}$$

where $\hat{\theta}_0$ and $\hat{\theta}_1$ are some fixed reference values of parameters (we are assuming here that we know the sign of the coefficient on x in $u(x,0)$). After we obtain posterior draws of $\{\omega_k, \mu_k, \sigma_k, k = 1, \dots, m\}$, we can renormalize the distribution as in Norets and Tang (2013) and obtain the corresponding draws of (θ_0, θ_1).

Note that the model does not change when we (i) add a constant to all utilities, (ii) multiply all utilities by a positive constant, and (iii) add a random variable to all utilities. Therefore, we can equivalently write our model as

$$\begin{aligned} u(x,0) &= \theta_0' + \theta_1' x + \epsilon', \\ u(x,1) &= 0, \end{aligned}$$

where

$$\begin{aligned} &\epsilon' = s(\mu - \epsilon) \sim F', \quad \theta_0' = s(\hat{\theta}_0 - \mu), \quad \theta_1' = s\hat{\theta}_1, \\ &s = \frac{\log 2}{E[X1(X \geq M_X)]}, \quad X = \mu - \epsilon, \quad \mu = \sum_{k=1}^{m} \omega_k \mu_k, \end{aligned}$$

and M_X is the median of X so that F' has the same location-scale normalization as F, the distribution in Norets and Tang (2013).

To see this, note that we first subtracted ϵ from both utilities and defined $X = \mu - \epsilon$ by demeaning $-\epsilon$, so X has the desired location-normalization of a zero-mean. Second, we let $\epsilon' = sX$ for some $s \geq 0$ and get the desired scale-normalization by choosing s, that is, $\log 2 = E[\epsilon' 1(\epsilon' \geq M_{\epsilon'})]$. Note that the specific choice of s above guarantees that s is nonnegative since the truncated expectation of a zero mean random variable truncated at the median is nonnegative. Since $s \geq 0$, we have

$$0.5 = Pr(X \leq M_X) = Pr(sX \leq sM_X) = Pr(\epsilon' \leq sM_X) \implies M_{\epsilon'} = sM_X.$$

Hence $E[\epsilon' 1(\epsilon' \geq M_{\epsilon'})]$ equals to

$$E[sX1(sX \geq sM_X)] = sE[X1(X \geq M_X)].$$

Therefore, our choice of s gives the desired scale-normalization.

For each MCMC draw of $\psi^{(\tau)}$ $\tau \geq 1$, we can compute the values of $\left(\theta_0^{'(\tau)}, \theta_1^{'(\tau)}\right)$ as follows

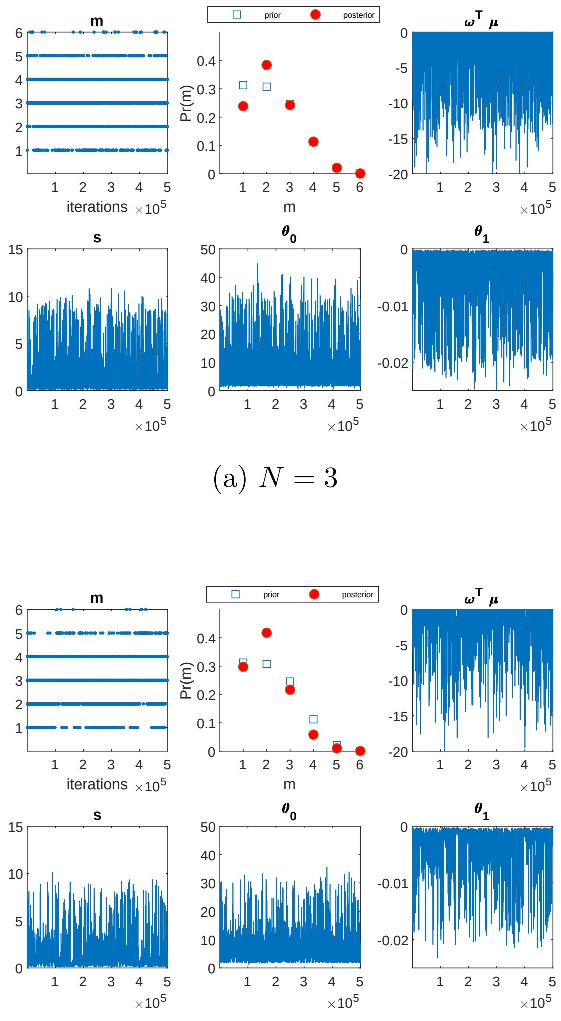
1. Transform $\chi^{(\tau)}$ back to $\psi^{(\tau)}$.

2. Compute $\mu^{(\tau)} = \sum_{k=1}^{m} \omega_k^{(\tau)} \mu_k^{(\tau)}$ and $s^{(\tau)}$.
3. Compute $\theta_0^{'(\tau)} = s^{(\tau)} \left(\hat{\theta}_0 - \mu^{(\tau)} \right)$ and $\theta_1^{'(\tau)} = s^{(\tau)} \hat{\theta}_1$.

E.3. *MCMC convergence*

Let N be the number of samples per state. We generate the data by multiplying N by the true CCP. We run MCMC with variable m with $N \in \{3, 10\}$. After each jump proposal block, we run HMC 10 iterations. We obtained 500,000 draws.

When checking for convergence of the chain, we have to be careful because we are using a mixture model. It is well-known that if the likelihood of a mixture model with m components has one mode for a fixed labeling of the components, then it can have $m!$ modes because the likelihood is invariant to a re-labeling of the components and there are $m!$ ways to label components. Although it is not possible to empirically detect label-switching(s) of our chain, if there was label-switching(s), it would be misleading, for example, to focus our attention on μ_1 when checking convergence of the chain. Geweke (2007) points out that this is not a problem as long as both the object of interest and the prior distribution are permutation-invariant. Note that we are using exchangeable priors and the object of our interest, the density estimate, is also permutation-invariant. We conduct a convergence diagnosis on the mean of ϵ or $\sum_{k=1}^{m} \omega_k \mu_k$ which is a permutation-invariant object. To check the convergence we perform the mean equality test for the first 10% and the last 50% of the samples. We conclude that the HMC samples for both $N = 3$ and $N = 10$ come from stationary distributions after a burn-in period of 10,000 draws.



(a) $N = 3$

(b) $N = 10$

Figure 5: Posterior draws from Rust example. Trace plot of m (upper-left), p.m.f. of m (upper-middle), trace plots of $\sum_{k=1}^{m} \omega_k \mu_k$ (upper-right), s (bottom-left), θ_0 (bottom-middle), and θ_1 (bottom-right).

APPENDIX F: IMPLEMENTATION DETAILS FOR GILLESKIE'S MODEL

F.1. *More on State Transitions*

The individual contracts an illness and moves to the state $x = (1, 0, 0)$ with probability $\pi^S(H) = 1/[1 + \exp(\delta_0 + \delta_1 H)]$.

In each illness period $t \in \{1, \dots, T\}$, the individual recovers and returns to the state of

time	value	# possible values
t=0	$(0,0,0)$	1
t=1	$(1,0,0)$	1
t=2	$(2,v,a)$	$2^2 = 4\ (v=0,1, a=0,1)$
t=3	$(3,v,a)$	$3^2 = 9\ (v=0,1,2, a=0,1,2)$
$\vdots$	$\vdots$	$\vdots$
t	(t,v,a)	$t^2\ (v=0,1,...t-1, a=0,1,...t-1)$
$\vdots$	$\vdots$	$\vdots$
T	(T,v,a)	$T^2\ (v=0,1,...T-1, a=0,1,...T-1)$
Total		$K = 1+1+2^2+3^2+\cdots T^2 = 1+\sum_{j=1}^{T} j^2 = 1+\frac{T(T+1)(2T+1)}{6}$

Table III: State transitions

being well with probability $\pi^W(x_t,d_t) = \exp(\eta^T E(x_t,d_t))/[1+\exp(\eta^T E(x_t,d_t))]$, where $\eta^T E(x_t,d_t) = \eta_0+\eta_1 v_{t+1}+\eta_2(v_{t+1})^2+\eta_3 a_{t+1}+\eta_4(a_{t+1})^2+\eta_5 v_{t+1}a_{t+1}+\eta_6 t+\eta_7 t^2+\eta_8 t^3+\eta_9 H$. As shown in Table III, at the beginning of time t, there are t^2 possible values of the current state.

We summarize the state transitions below.

1. The initial state is $(0,0,0)$.
2. If the state is currently at $(0,0,0)$,
   - wp π^S, $(0,0,0) \to (1,0,0)$,
   - wp $1-\pi^S$, $(0,0,0) \to (0,0,0)$.
3. In general, for $t = 1,...,T-1$, if currently at (t,v_t,a_t),
   - wp $1-\pi^W(x_t,d_t)$,

$$(t,v_t,a_t) \to \begin{cases} (t+1,v_t,a_t), & \text{if } \ d_t = 0 \\ (t+1,v_t+1,a_t), & \text{if } \ d_t = 1 \\ (t+1,v_t,a_t+1), & \text{if } \ d_t = 2 \\ (t+1,v_t+1,a_t+1), & \text{if } \ d_t = 3 \end{cases}$$

   - wp $\pi^W(x_t,d_t)$, $(t,v_t,a_t) \to (0,0,0)$.

$\boldsymbol{d = 0}$

	000	100	200	201	210	211
000	$1-\pi^S$	π^S	0	0	0	0
100	$\pi^W(100,0)$	0	$1-\pi^W(100,0)$	0	0	0
200	1	0	0	0	0	0
201	1	0	0	0	0	0
210	1	0	0	0	0	0
211	1	0	0	0	0	0

$\boldsymbol{d = 1}$

	000	100	200	201	210	211
000	$1-\pi^S$	π^S	0	0	0	0
100	$\pi^W(100,1)$	0	0	0	$1-\pi^W(100,1)$	0
200	1	0	0	0	0	0
201	1	0	0	0	0	0
210	1	0	0	0	0	0
211	1	0	0	0	0	0

$\boldsymbol{d = 2}$

	000	100	200	201	210	211
000	$1-\pi^S$	π^S	0	0	0	0
100	$\pi^W(100,2)$	0	0	$1-\pi^W(100,2)$	0	0
200	1	0	0	0	0	0
201	1	0	0	0	0	0
210	1	0	0	0	0	0
211	1	0	0	0	0	0

$\boldsymbol{d = 3}$

	000	100	200	201	210	211
000	$1-\pi^S$	π^S	0	0	0	0
100	$\pi^W(100,3)$	0	0	0	0	$1-\pi^W(100,3)$
200	1	0	0	0	0	0
201	1	0	0	0	0	0
210	1	0	0	0	0	0
211	1	0	0	0	0	0

Figure 6: Example of transition matrices with $T = 2$.

4. The end of an episode

$$
\begin{aligned}
(T, v, a) &\to (0,0,0) \quad \forall v, a,\\
(t, T-1, a) &\to (0,0,0) \quad \forall t, a,\\
(t, v, T-1) &\to (0,0,0) \quad \forall t, v.
\end{aligned}
$$

Figure 6 shows the state transition matrices when $T = 2$. In this case, the possible states are $x = (t, v, a) \in \{000, 100, 200, 201, 210, 211\}$. The number of states is $K = 6$.

F.2. *More on Utility Function*

First, consider the per-period utilities when the individual is ill, as defined in Gilleskie (1998) page 17:

$$
\begin{aligned}
u(d_t = 1, x_t, \epsilon_t, t > 0) &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 C(x_t, 1) + \epsilon_{t1},\\
u(d_t = 2, x_t, \epsilon_t, t > 0) &= \alpha_0 + \qquad\qquad + \alpha_3 C(x_t, 2) + \epsilon_{t2},\\
u(d_t = 3, x_t, \epsilon_t, t > 0) &= \alpha_0 + \alpha_1 \qquad\quad + \alpha_3 C(x_t, 3) + \epsilon_{t3},\\
u(d_t = 0, x_t, \epsilon_t, t > 0) &= \alpha_0 \qquad\quad + \alpha_2 + \alpha_3 C(x_t, 0) + \epsilon_{t0},
\end{aligned}
$$

where α_0 is the disutility of illness α_1 is the direct utility of doctor visit, α_2 is the direct utility of attending work when ill, and α_3 is the marginal utility of consumption when ill. Note that the marginal utility of consumption when well is set to 1 and hence the per-period utility equals to Y. The marginal utility of consumption α_3 when the individual is ill is lower than 1.

The per-period consumption is defined as $C(x_t, d_t) = Y - \big[PC1(d_t = 1 \text{ or } 3) + Y\big(1 - L\Phi(x_t, d_t)\big)1(d_t = 2 \text{ or } 3)\big]1(t > 0)$, where Y is the per-period labor income, PC is the cost of a medical visit. $\Phi(x_t, d_t) = \frac{exp(\phi_1+\phi_2 a'(x_t,d_t))}{1+exp(\phi_1+\phi_2 a'(x_t,d_t))}$ is the portion of income that sick leave coverage replaces with $a'(x_t, d_t)$ denoting a_{t+1} given x_t, d_t. For the data-generating value of ϕ's, $\Phi(x_t, d_t)$ is decreasing in $a'(x_t, d_t)$. $L \in (0, 1)$ is the sick leave coverage rate.

In order apply our estimation scheme, we augment the utility so that $d = 0$ is always chosen when $t = 0$ (and the individual receives per-period utility Y):

$$\begin{aligned}
u(d_t = 1, x_t, \epsilon_t, t = 0) &= -\infty + \epsilon_{t1},\\
u(d_t = 2, x_t, \epsilon_t, t = 0) &= -\infty + \epsilon_{t2},\\
u(d_t = 3, x_t, \epsilon_t, t = 0) &= -\infty + \epsilon_{t3},\\
u(d_t = 0, x_t, \epsilon_t, t = 0) &= Y \quad\; + \epsilon_{t0}.
\end{aligned}$$

Combining the above we have,

$$\begin{aligned}
u(d_t = 1, x_t, \epsilon_t) &= \Big[\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 C(x_t, 1)\Big]1(t > 0) - \infty 1(t = 0) + \epsilon_{t1},\\
u(d_t = 2, x_t, \epsilon_t) &= \Big[\alpha_0 + \qquad\quad + \alpha_3 C(x_t, 2)\Big]1(t > 0) - \infty 1(t = 0) + \epsilon_{t2},\\
u(d_t = 3, x_t, \epsilon_t) &= \Big[\alpha_0 + \alpha_1 \qquad + \alpha_3 C(x_t, 3)\Big]1(t > 0) - \infty 1(t = 0) + \epsilon_{t3},\\
u(d_t = 0, x_t, \epsilon_t) &= \Big[\alpha_0 \qquad + \alpha_2 + \alpha_3 C(x_t, 0)\Big]1(t > 0) + Y1(t = 0) \; + \epsilon_{t0}.
\end{aligned} \tag{22}$$

We can re-write Eqs (22) as:

$$u(d_t = 1, x_t, \epsilon_t) = \Big[\psi_1 + \psi_6 C(x_t, 1)\Big]1(t > 0) - \infty 1(t = 0) + \epsilon_{t1},$$

$$
\begin{aligned}
(23)\qquad u(d_t=2,x_t,\epsilon_t) &= \Big[\psi_2{+}\psi_6 C(x_t,2)\Big]1(t>0)-\infty 1(t=0)+\epsilon_{t2},\\
u(d_t=3,x_t,\epsilon_t) &= \Big[\psi_3{+}\psi_6 C(x_t,3)\Big]1(t>0)-\infty 1(t=0)+\epsilon_{t3},\\
u(d_t=0,x_t,\epsilon_t) &= \Big[\qquad \psi_6 C(x_t,0)\Big]1(t>0)+Y1(t=0),
\end{aligned}
$$

where we normalized $\epsilon_0 = 0$ and normalized the intercept for $d = 0$ to be zero. We can fix the values of $\psi_j = \hat{\psi}_j\ j = 1, 2, 3$ at some (researcher-specified) arbitrary values and fix (ψ_4, ψ_5) at $(\hat{\psi}_4, \hat{\psi}_5) = (-\infty, Y)$. So we equivalently have,

$$
\begin{aligned}
u(d_t=1,x_t,\epsilon_t) &= \hat{\psi}_1 1(t>0){+}\hat{\psi}_4 1(t=0)+\psi_6 C(x_t,1)1(t>0)\ +\epsilon_{t1},\\
u(d_t=2,x_t,\epsilon_t) &= \hat{\psi}_2 1(t>0){+}\hat{\psi}_4 1(t=0)+\psi_6 C(x_t,2)1(t>0)\ +\epsilon_{t2},\\
u(d_t=3,x_t,\epsilon_t) &= \hat{\psi}_3 1(t>0){+}\hat{\psi}_4 1(t=0)+\psi_6 C(x_t,3)1(t>0)\ +\epsilon_{t3},\\
u(d_t=0,x_t,\epsilon_t) &= \qquad\qquad\quad \hat{\psi}_5 1(t=0)+\psi_6 C(x_t,0)1(t>0).
\end{aligned}
$$

As $1 = 1(t > 0) + 1(t = 0)$, $\hat{\psi}_1 1(t > 0) + \hat{\psi}_4 1(t = 0) = \hat{\psi}_1\big(1 - 1(t = 0)\big) + \hat{\psi}_4 1(t = 0) = \hat{\psi}_1 + \big(\hat{\psi}_4 - \hat{\psi}_1\big)1(t = 0) = \hat{\psi}_1 + \hat{\psi}_4 1(t = 0)$ where the last equality is due to the fact that $\hat{\psi}_4 = -\infty$. The same applies to $d = 2, 3$. Hence, the system is equivalent to:

$$
\begin{aligned}
u(d_t=1,x_t,\epsilon_t) &= \hat{\psi}_1{+}\hat{\psi}_4 1(t=0)+\psi_6 C(x_t,1)1(t>0)\ +\epsilon_{t1},\\
u(d_t=2,x_t,\epsilon_t) &= \hat{\psi}_2{+}\hat{\psi}_4 1(t=0)+\psi_6 C(x_t,2)1(t>0)\ +\epsilon_{t2},\\
u(d_t=3,x_t,\epsilon_t) &= \hat{\psi}_3{+}\hat{\psi}_4 1(t=0)+\psi_6 C(x_t,3)1(t>0)\ +\epsilon_{t3},\\
u(d_t=0,x_t,\epsilon_t) &= \qquad \hat{\psi}_5 1(t=0)+\psi_6 C(x_t,0)1(t>0).
\end{aligned}
$$

Gilleskie (1998) assumes that $\epsilon_j \sim \phi(\epsilon_j; \mu, \rho)$ iid $j = 0, 1, 2, 3$ (see page 38). In this case, it can be shown that

$$
p(d|x) = \frac{e^{\frac{u(x,d)+\beta G_x^d Q}{\rho}}}{\sum_{j=0}^{J} e^{\frac{u(x,j)+\beta G_x^j Q}{\rho}}} \text{ and } Q(x) = \mu + \rho \log\left[\sum_{j=0}^{J} e^{\frac{u(x,j)+\beta G_x^j Q}{\rho}}\right].
$$

In Gilleskie (1998), it appears that μ is set to zero so that $E(\epsilon_j) = 0$. Unfortunately, Gilleskie does not show which value of ρ was used. We assume that $\rho = 100$. We can scale the utility coefficients by ρ so that the error terms follow $\phi(\cdot; 0, 1)$ in the DGP. Defining $\theta = \psi/\rho$'s and re-defining ϵ's, we have

$$\begin{aligned}
u(d_t = 1, x_t, \epsilon_t) &= \hat{\theta}_1 + \hat{\theta}_4 1(t=0) + \theta_6 C(x_t, 1) 1(t > 0) \ + \epsilon_{t1}, \\
u(d_t = 2, x_t, \epsilon_t) &= \hat{\theta}_2 + \hat{\theta}_4 1(t=0) + \theta_6 C(x_t, 2) 1(t > 0) \ + \epsilon_{t2}, \\
u(d_t = 3, x_t, \epsilon_t) &= \hat{\theta}_3 + \hat{\theta}_4 1(t=0) + \theta_6 C(x_t, 3) 1(t > 0) \ + \epsilon_{t3}, \\
u(d_t = 0, x_t, \epsilon_t) &= \qquad \hat{\theta}_5 1(t=0) + \theta_6 C(x_t, 0) 1(t > 0).
\end{aligned}$$

Define $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)'$. Then

$$\begin{aligned}
u(d = 1, x, \epsilon) &= \Big(1,0,0,1(t=0),0, \qquad C(x,1)1(t > 0)\Big)\boldsymbol{\theta} + \epsilon_1 = \boldsymbol{Z_1}(x)\boldsymbol{\theta} + \epsilon_1, \\
u(d = 2, x, \epsilon) &= \Big(0,1,0,1(t=0),0, \qquad C(x,2)1(t > 0)\Big)\boldsymbol{\theta} + \epsilon_2 = \boldsymbol{Z_2}(x)\boldsymbol{\theta} + \epsilon_2, \\
u(d = 3, x, \epsilon) &= \Big(0,0,1,1(t=0),0, \qquad C(x,3)1(t > 0)\Big)\boldsymbol{\theta} + \epsilon_3 = \boldsymbol{Z_3}(x)\boldsymbol{\theta} + \epsilon_3, \\
u(d = 0, x, \epsilon) &= \Big(0,0,0,0, \qquad 1(t=0), C(x,0)1(t > 0)\Big)\boldsymbol{\theta} \qquad = \boldsymbol{Z_0}(x)\boldsymbol{\theta},
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{Z_1}(x) &= \big(Z_{11}(x), Z_{12}(x), Z_{13}(x), Z_{14}(x), Z_{15}(x), Z_{16}(x)\big) = \Big(1, 0, 0, 1(t=0), 0, C(x,1)1(t > 0)\Big), \\
\boldsymbol{Z_2}(x) &= \big(Z_{21}(x), Z_{22}(x), Z_{23}(x), Z_{24}(x), Z_{25}(x), Z_{26}(x)\big) = \Big(0, 1, 0, 1(t=0), 0, C(x,2)1(t > 0), \Big), \\
\boldsymbol{Z_3}(x) &= \big(Z_{31}(x), Z_{32}(x), Z_{33}(x), Z_{34}(x), Z_{35}(x), Z_{36}(x)\big) = \Big(0, 0, 1, 1(t=0), 0, C(x,3)1(t > 0)\Big), \\
\boldsymbol{Z_0}(x) &= \big(Z_{01}(x), Z_{02}(x), Z_{03}(x), Z_{04}(x), Z_{05}(x), Z_{06}(x)\big) = \Big(0, 0, 0, 0, 1(t=0), c(x,0)1(t > 0)\Big).
\end{aligned}$$

Then we have

$$\begin{aligned}
u(1,x,\epsilon) &= \boldsymbol{Z}_{1,1:5}(x)\hat{\boldsymbol{\theta}}_{1:5} + \theta_6 Z_{16}(x) + \epsilon_1,\\
u(2,x,\epsilon) &= \boldsymbol{Z}_{2,1:5}(x)\hat{\boldsymbol{\theta}}_{1:5} + \theta_6 Z_{26}(x) + \epsilon_2,\\
u(3,x,\epsilon) &= \boldsymbol{Z}_{3,1:5}(x)\hat{\boldsymbol{\theta}}_{1:5} + \theta_6 Z_{36}(x) + \epsilon_3,\\
u(0,x,\epsilon) &= \boldsymbol{Z}_{0,1:5}(x)\hat{\boldsymbol{\theta}}_{1:5} + \theta_6 Z_{06}(x).
\end{aligned}$$

Here θ_6 is treated as a parameter.

F.3. *Data Generating Parameters*

We compute the data generating CCPs based on the following values, mostly based on estimates in Gilleskie (1998) for Type 2 illness with some adjustments so that the expected number of doctor visits and work absences roughly match with Gilleskie's sample.

- $T = 8$
- $\beta = 0.9$
- $\hat{\theta}_1 = \alpha_1/\rho, \hat{\theta}_2 = \alpha_2/\rho, \hat{\theta}_3 = (\alpha_1 - \alpha_2)/\rho.$

  Comparing Eqs (22) and (23), we have that

  $$\begin{aligned}
  \psi_1 &= (\alpha_0 + \alpha_1 + \alpha_2) - (\alpha_0 + \alpha_2) = \alpha_1,\\
  \psi_2 &= \alpha_0 - (\alpha_0 + \alpha_2) = -\alpha_2,\\
  \psi_3 &= (\alpha_0 + \alpha_1) - (\alpha_0 + \alpha_2) = \alpha_1 - \alpha_2.
  \end{aligned}$$

  We define $\alpha_1 = -125$ and $\alpha_2 = 83$ [1].
- $\hat{\theta}_6 = 0.58/\rho$. The marginal utility of consumption is estimated $\hat{\alpha}_3 = 0.58$ (page 23).
- $L = 0.7$.
- $Y = 100 \implies \psi_5 = 100 \implies \theta_5 = 100/\rho$. The middle class daily income ranges between \$70 and \$125 (page 8).

[1] In Gilleskie's paper, the estimates for Type 2 illness are approximately $\alpha_1 = -67$ and $\alpha_2 = 153$. We use the quantities above to roughly match the expected numbers of visits and absences.

- $PC = 15$. P (the cost of a physician visit, page 22) and C (the median out-of-pocket payment rate, page 9).
- $\phi_1 = 5.6, \phi_2 = -1.75$. These are Gilleskie's estimates (page 23).
- $\hat{\eta}_k$ $k = 0, ..., 11$ (estimates from page 23).
- $\hat{\delta}_k$ $k = 0, ..., 3$ (estimates from page 23).

Assuming that Gilleskie (1998) uses $\rho = 100$ as the scale parameter for the extreme value distribution, we can compute the data generating CCPs based on the parameter values mentioned above. The expected numbers of doctor visits and work absences, computed based on the data generating CCPs and the state transition probabilities, roughly match with the corresponding values in Gilleskie's sample (see page 11, Table V). We then used the CCPs and the state-transition probabilities to sequentially generate 100 illness episodes.

F.4. *Normalization after MCMC*

The system of utilities can be written as (see Appendix F.2 for detail):

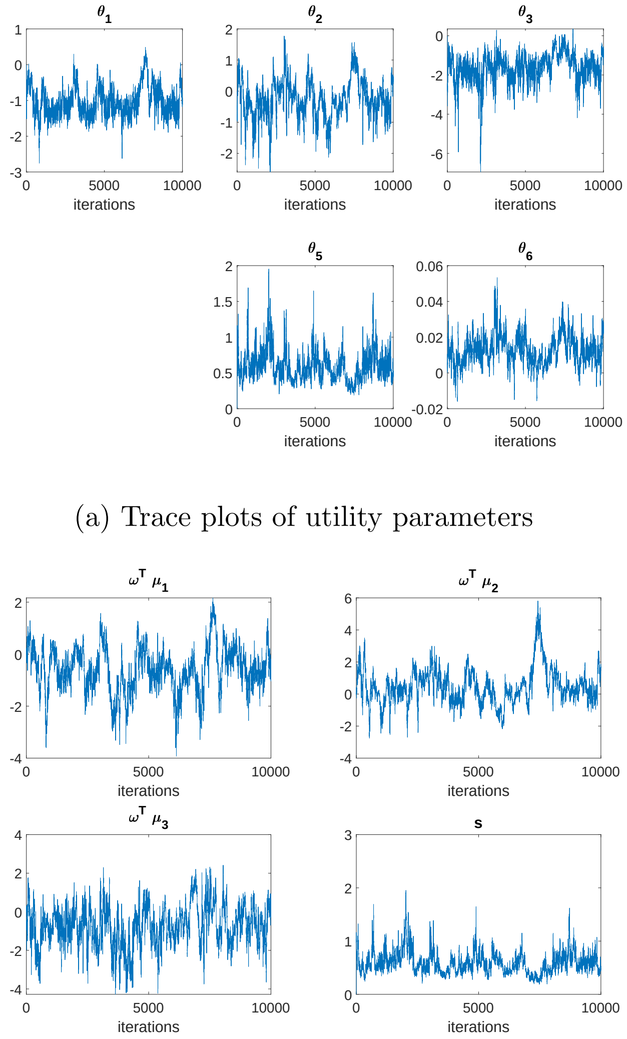
$$\begin{aligned}
u(d=1,x,\epsilon) &= s\big(\hat{\theta}_1+\mu_1\big)+s\hat{\theta}_4 1(t=0)+s\theta_6 C(x,1)1(t>0) \ +s\big(\epsilon_1-\mu_1\big),\\
u(d=2,x,\epsilon) &= s\big(\hat{\theta}_2+\mu_2\big)+s\hat{\theta}_4 1(t=0)+s\theta_6 C(x,2)1(t>0) \ +s\big(\epsilon_2-\mu_1\big),\\
u(d=3,x,\epsilon) &= s\big(\hat{\theta}_3+\mu_3\big)+s\hat{\theta}_4 1(t=0)+s\theta_6 C(x,3)1(t>0) \ +s\big(\epsilon_3-\mu_1\big),\\
u(d=0,x,\epsilon) &= \qquad\qquad\qquad\quad s\hat{\theta}_5 1(t=0)+s\theta_6 C(x,0)1(t>0),
\end{aligned}$$

where

$$s = \frac{\log 2}{E[X_1 1(X_1 \geq M_{X_1})]}, \quad X_1 = \epsilon_1 - \mu_1, \quad \mu_j = \sum_{k=1}^{m} \omega_k \mu_{jk}, \quad \epsilon'_j = s\Big(\epsilon_j - \mu_j\Big).$$

We have (1) $E\epsilon'_j = 0$ for $j = 1, \ldots, 3$ and (2) the scale $E\epsilon'_1 1(\epsilon'_1 > M_{\epsilon'_1}) = \log 2$. Note that we have the logistic location/scale normalization for the first error term. We plot the posterior draws of $\hat{\theta}'_j$ $j = 1, 2, 3, 5$ and θ'_6 to study the identified region (note that $\hat{\theta}_4$ is set to ∞).

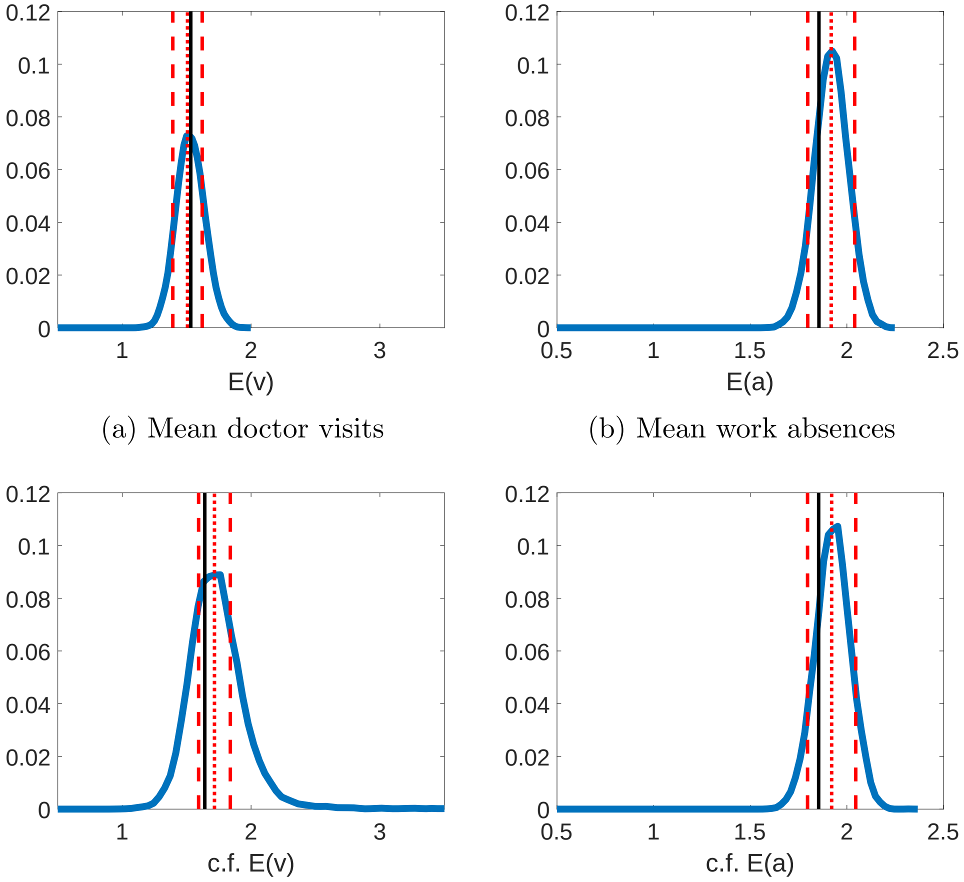
F.5. *MCMC Plots from Gilleskie's Application*



(a) Trace plots of utility parameters

(b) Trace plots of $\sum_{k=1}^{m} \omega_k \mu_{1k}$ (upper-left), $\sum_{k=1}^{m} \omega_k \mu_{2k}$ (upper-right), $\sum_{k=1}^{m} \omega_k \mu_{3k}$ (bottom-left), s (bottom-right), which is defined in Appendix F.4.

Figure 7: Trace plots in Gilleskie model



(a) Mean doctor visits

(b) Mean work absences

(c) Mean c.f. doctor visits

(d) Mean c.f. work absences

Figure 8: Actual and counterfactual (c.f.) expected number of doctor visits and work absences. Posterior density (blue-solid), MLE (red-dotted), 95% confidence interval (red-dashed), and data generating value (black-solid).

F.6. *Prior sensitivity check for Gilleskie's model*

We consider two additional sets of priors for estimating Gilleskie's model in Section 7 and present results of estimation and counterfactual analysis. Despite slight differences, the overall findings remain the same. That is, the credible intervals from our semiparametric approach for objects of interest are wider than the confidence intervals computed via MLE assuming the dynamic logit.

F.6.1. *Prior sensitivity check 1*

First, we consider increasing the prior variances for the component specific scale parameters σ_k's and the preference parameter θ_6. The priors are now specified as follows, $\underline{a} = 10$, $A_m = 0.05$, and $\tau = 5$, $\mu_{jk} \sim N(0, 3^2)$, $\log \sigma_k \sim N(0, 1)$, $\log \sigma \sim N(0, 0.01^2)$, and $\theta_6 \sim N(0, 5^2)$. The results are presented below.

	MLE			Bayes		
	Estimate	95%CI	IL	Estimate	95%CI	IL
$E(v)$	1.506	(1.392, 1.620)	0.228	1.534	(1.319, 1.760)	0.441
$E(a)$	1.918	(1.797, 2.040)	0.242	1.925	(1.747, 2.099)	0.351

Table IV: Estimation results for prior sensitivity check 1: the MLE, its 95% asymptotic confidence interval, the Bayesian posterior mean and HPD 95% credible interval. IL=length of 95% CI.

	MLE			Bayes		
	Estimate	95%CI	IL	Estimate	95%CI	IL
$E(v)$	1.716	(1.592, 1.839)	0.246	1.760	(1.383, 2.145)	0.761
$E(a)$	1.921	(1.796, 2.045)	0.249	1.944	(1.763, 2.119)	0.355

Table V: Counterfactual analysis for prior sensitivity check 1: the MLE, its 95% confidence interval computed via Delta method, the Bayesian posterior mean and HPD 95% credible interval. IL=length of 95% CI.

F.6.2. *Prior sensitivity check 2*

Second, we consider using a set of priors similar to the one used to estimate Rust model in Section 6. The priors are now specified as follows, $\underline{a} = 10$, $A_m = 0.05$, and $\tau = 5$, $\mu_{jk} \sim 0.5N(2, 2^2) + 0.5N(-3, 2^2)$, $\log \sigma_k \sim 0.4N(0, 1) + 0.6N(-3, 1)$, $\log \sigma \sim N(0, 0.01^2)$, and $\theta_6 \sim N(0, 3^2)$. The results are shown below.

	MLE			Bayes		
	Estimate	95%CI	IL	Estimate	95%CI	IL
$E(v)$	1.506	(1.392, 1.620)	0.228	1.533	(1.317, 1.755)	0.438
$E(a)$	1.918	(1.797, 2.040)	0.242	1.903	(1.724, 2.090)	0.365

Table VI: Estimation results for prior sensitivity check 2: the MLE, its 95% asymptotic confidence interval, the Bayesian posterior mean and HPD 95% credible interval. IL=length of 95% CI.

	MLE			Bayes		
	Estimate	95%CI	IL	Estimate	95%CI	IL
$E(v)$	1.716	(1.592, 1.839)	0.246	1.689	(1.383, 2.022)	0.639
$E(a)$	1.921	(1.796, 2.045)	0.249	1.914	(1.734, 2.101)	0.367

Table VII: Counterfactual analysis for prior sensitivity check 2: the MLE, its 95% confidence interval computed via Delta method, the Bayesian posterior mean and HPD 95% credible interval. IL=length of 95% CI.