Unit 6: Introduction to linear regression

2. Prediction, outliers, and inference for regression

Sta 101 - Fall 2015

Duke University, Department of Statistical Science

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Slides posted at *http://bit.ly/sta101_f15*

1. Housekeeping

2. Main ideas
   1. Predict, but don't extrapolate
   2. Predicted values also have uncertainty around them
   3. R^2 assesses model fit -- higher the better
   4. Inference for regression uses the t-distribution
   5. Conditions for regression
   6. Type of outlier determines how it should be handled

3. Summary

- PA 6 opens today, due Nov 22, Sun
- PS 6 due Nov 20, Fri
- RA 7 (last RA!) on Wednesday
- Project 2 questions?
  - If you want to see sample posters from previous years, stop by office hours
  - Most important advice: Sketch out a meeting / working plan with your team **TODAY**, keeping in mind Thanksgiving break

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Clicker question

Suppose you want to predict annual murder count (per million) for a series of districts that were not included in the dataset. For which of the following districts would you be most comfortable with your prediction?

A district where % in poverty =

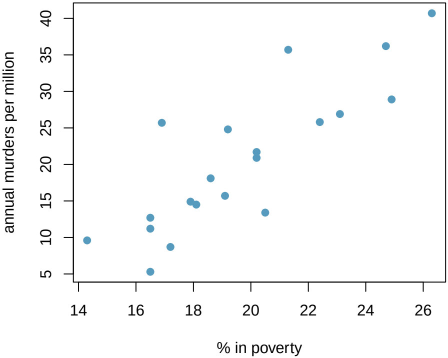
(a) 5%

(b) 15%

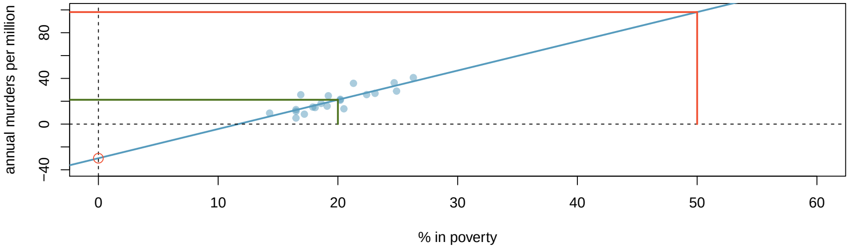
(c) 20%

(d) 26%

(e) 40%



Sometimes the intercept might be an extrapolation: useful for adjusting the height of the line, but meaningless in the context of the data.



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In R:

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murder <- read.csv("https://stat.duke.edu/~mc301/data/murder.csv")
# fit model
m_mur_pov <- lm(annual_murders_per_mil ~ perc_pov, data = murder)
# create new data
newdata <- data.frame(perc_pov = 20)
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- If the model is good, the predictions should be close to the true value of the response variable for this observation, however it may not be exact, i.e. $\hat{y}$ might be different than y.
- With any prediction we can (and should) also report a measure of uncertainty of the prediction.

A *prediction interval* for y for a given $x^\star$ is

$$\hat{y} \pm t^\star_{n-2} s \sqrt{1 + \frac{1}{n} + \frac{(x^\star - \bar{x})^2}{(n-1)s_x^2}}$$

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- Prediction level: If we repeat the study of obtaining a regression data set many times, each time forming a XX% prediction interval at $x^\star$, and wait to see what the future value of y is at $x^\star$, then roughly XX% of the prediction intervals will contain the corresponding actual value of y.

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We are 95% confident that the annual murders per million for a county with 20% poverty rate is between 9.52 and 33.15.

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Response: annual_murders_per_mil
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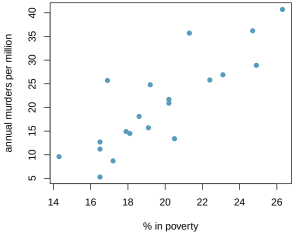
$$R^2 = \frac{explained\ variabilty}{total\ variability} = \frac{SS_{reg}}{SS_{tot}} = \frac{1308.34}{1308.34 + 546.86} = \frac{1308.34}{1855.2} \approx 0.71$$

Clicker question

R^2 for the regression model for predicting annual murders per million based on percentage living in poverty is roughly 71%. Which of the following is the correct interpretation of this value?



(a) 71% of the variability in percentage living in poverty is explained by the model.

(b) 84% of the variability in the murder rates is explained by the model, i.e. percentage living in poverty.

(c) 71% of the variability in the murder rates is explained by the model, i.e. percentage living in poverty.

(d) 71% of the time percentage living in poverty predicts murder rates accurately.

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- Hypothesis testing for a slope: $H_0 : \beta_1 = 0$; $H_A : \beta_1 \neq 0$
  - $T_{n-2} = \frac{b_1 - 0}{SE_{b_1}}$
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- Linearity $\rightarrow$ randomly scattered residuals around 0 in the residuals plot – important regardless of doing inference

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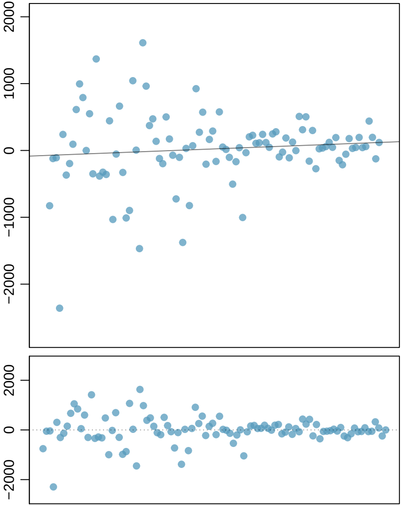
- Constant variability of residuals (*homoscedasticity*) $\rightarrow$ no fan shape in the residuals plot
- Independence of residuals (and hence observations) $\rightarrow$ depends on data collection method, often violated for time-series data

Clicker question

What condition is this linear model obviously and definitely violating?

(a) Linear relationship
(b) Non-normal residuals
(c) Constant variability
(d) Independence of observations



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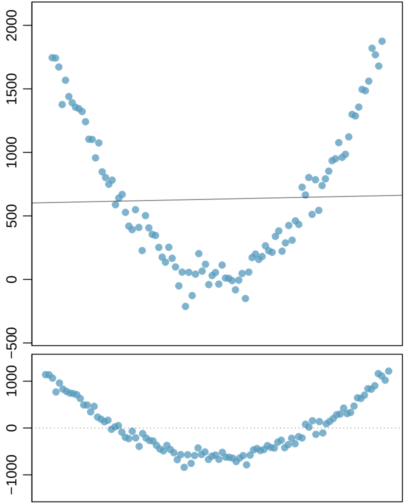

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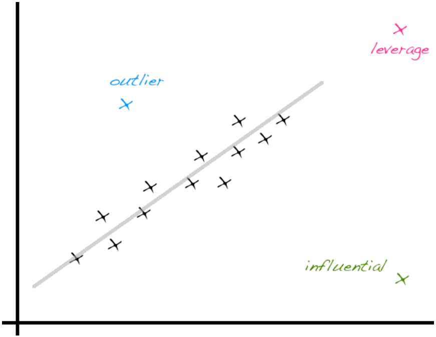
- *Influential* point changes the slope (most likely also has high leverage) – run the regression with and without that point to determine



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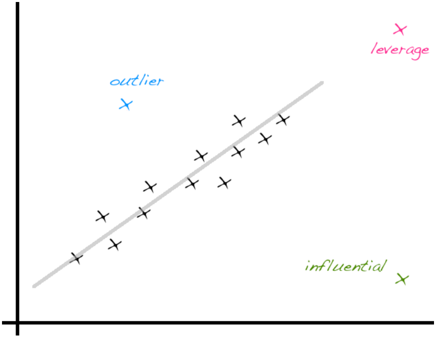

- *Outlier* is an unusual point without these special characteristics (this one likely affects the intercept only)
- If clusters (groups of points) are apparent in the data, it might be worthwhile to model the groups separately.

Application exercise: 6.2 Linear regression

See course website for details

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