

# AA274 (Winter 2017-18): Problem Set 1

Anqi Fu

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## 1 Optimal Control

- (i) Let  $\mathbf{x} = (x, y, \theta)$  denote the robot state and  $\mathbf{u} = (V, \omega)$  be the robot control inputs. Our optimal control problem is

$$\begin{aligned} & \underset{(V, \omega)}{\text{minimize}} && \int_0^{t_f} [\lambda + V(t)^2 + \omega(t)^2] dt \\ & \text{subject to} && \begin{aligned} \dot{x}(t) &= V \cos(\theta(t)), \\ \dot{y}(t) &= V \sin(\theta(t)), \\ \dot{\theta}(t) &= \omega(t), \\ |V(t)| &\leq 0.5, \\ |\omega(t)| &\leq 1.0. \end{aligned} \end{aligned} \tag{1}$$

with initial and final conditions

$$\begin{aligned} x(0) &= 0, & y(0) &= 0, & \theta(0) &= -\pi/2, \\ x(t_f) &= 5, & y(t_f) &= 5, & \theta(t_f) &= -\pi/2. \end{aligned}$$

Here  $\lambda \in \mathbb{R}_+$  is a weighting factor and  $t_f$  is free. The Hamiltonian is

$$H(t) = \lambda + V(t)^2 + \omega(t)^2 + p_1(t)V(t) \cos(\theta(t)) + p_2(t)V(t) \sin(\theta(t)) + p_3(t)\omega(t)$$

where  $\mathbf{p} = (p_1, p_2, p_3)$  are the Lagrange multipliers associated with  $\dot{\mathbf{x}} = (\dot{x}, \dot{y}, \dot{\theta})$ . Our optimality conditions are

$$\begin{aligned} \begin{pmatrix} \dot{x}^*(t) \\ \dot{y}^*(t) \\ \dot{\theta}^*(t) \end{pmatrix} &= \begin{pmatrix} V^*(t) \cos(\theta^*(t)) \\ V^*(t) \sin(\theta^*(t)) \\ \omega^*(t) \end{pmatrix} \\ \begin{pmatrix} \dot{p}_1^*(t) \\ \dot{p}_2^*(t) \\ \dot{p}_3^*(t) \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ p_1^*(t)V^*(t) \sin(\theta^*(t)) - p_2^*(t)V^*(t) \cos(\theta^*(t)) \end{pmatrix} \end{aligned} \tag{2}$$

with the additional constraint (for the control inequality constraints)

$$\begin{aligned} & V^*(t)^2 + \omega^*(t)^2 + p_1^*(t)V^*(t) \cos(\theta^*(t)) + p_2^*(t)V^*(t) \sin(\theta^*(t)) + p_3^*(t)\omega^*(t) \\ & \leq V(t)^2 + \omega(t)^2 + p_1^*(t)V(t) \cos(\theta^*(t)) + p_2^*(t)V(t) \sin(\theta^*(t)) + p_3^*(t)\omega(t) \end{aligned} \tag{3}$$

for all  $(V(t), \omega(t)) \in \mathbb{R}^2$ . Since  $t_f$  is free and  $\mathbf{x}(t_f)$  fixed, the boundary conditions amount to

$$\begin{aligned} x^*(0) &= 0, \quad y^*(0) = 0, \quad \theta^*(0) = -\pi/2, \\ x^*(t_f) &= 5, \quad y^*(t_f) = 5, \quad \theta^*(t_f) = -\pi/2, \\ \lambda + V^*(t_f)^2 + \omega^*(t_f)^2 + p_1^*(t_f)V^*(t_f)\cos(\theta^*(t_f)) + p_2^*(t_f)V^*(t_f)\sin(\theta^*(t_f)) + p_3^*(t_f)\omega^*(t_f) \\ &= \lambda + V^*(t_f)^2 - p_2^*(t_f)V^*(t_f) + \omega^*(t_f)^2 + p_3^*(t_f)\omega^*(t_f) = 0. \end{aligned}$$

We make two modifications in order to solve this problem. First, we ignore the control inequality constraints and manually vary  $\lambda$ . This means that (3) is replaced with

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2V^*(t) + p_1^*(t)\cos(\theta^*(t)) + p_2^*(t)\sin(\theta^*(t)) \\ 2\omega^*(t) + p_3^*(t) \end{pmatrix} \\ \Rightarrow \begin{pmatrix} V^*(t) \\ \omega^*(t) \end{pmatrix} &= -\frac{1}{2} \begin{pmatrix} p_1^*(t)\cos(\theta^*(t)) + p_2^*(t)\sin(\theta^*(t)) \\ p_3^*(t) \end{pmatrix} \end{aligned}$$

Second, we use a change of variables to reformulate the BVP: we rescale the time to  $\tau = \frac{t}{t_f} \in [0, 1]$ , so the derivatives become  $\frac{d}{d\tau} := t_f \frac{d}{dt}$ , then introduce a dummy state variable  $r$  that corresponds to  $t_f$  with dynamic  $\dot{r} = 0$  and replace all instances of  $t_f$  with  $r$ . Let  $\mathbf{z} = (x, y, \theta, p_1, p_2, p_3, r)$  denote the augmented state vector, then  $\frac{d\mathbf{z}}{dt} = \frac{1}{t_f} \frac{d\mathbf{z}}{d\tau} = \frac{1}{r} \frac{d\mathbf{z}}{d\tau}$  and our optimality conditions become

$$\frac{d\mathbf{z}}{d\tau} = z_7 \begin{pmatrix} V \cos(z_3) \\ V \sin(z_3) \\ \omega \\ 0 \\ 0 \\ V(z_4 \sin(z_3) - z_5 \cos(z_3)) \\ 0 \end{pmatrix} \quad (4)$$

with boundary conditions

$$\begin{aligned} z_1(0) &= 0, \quad z_2(0) = 0, \quad z_3(0) = -\pi/2 \\ z_1(1) &= 5, \quad z_2(1) = 5, \quad z_3(1) = -\pi/2 \\ \lambda + V(1)^2 - z_5(1)V(1) + \omega(1)^2 + z_6(1)\omega(1) &= 0 \end{aligned} \quad (5)$$

where we have defined

$$\begin{pmatrix} V \\ \omega \end{pmatrix} := -\frac{1}{2} \begin{pmatrix} z_4 \cos(z_3) + z_5 \sin(z_3) \\ z_6 \end{pmatrix} \quad (6)$$

(ii) See submitted code.

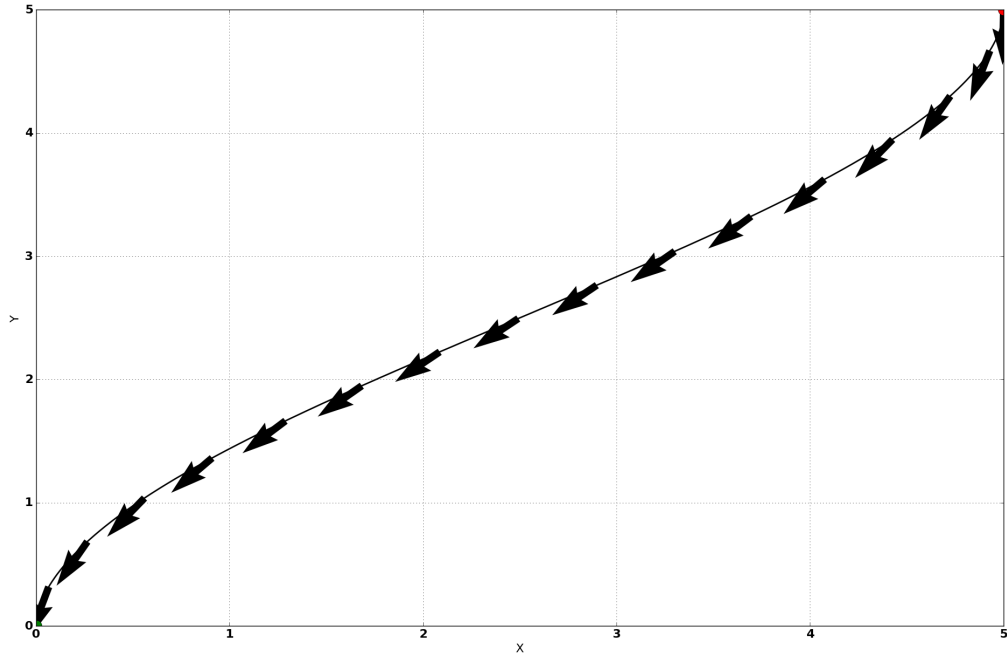
- (iii) By choosing the largest feasible  $\lambda$ , we are solving the control problem with the smallest  $t_f$  feasible, i.e., our optimal controls drive the unicycle to its final waypoint in the shortest possible time. This can be seen by separating the objective into

$$J = \int_0^{t_f} [\lambda + V(t)^2 + \omega(t)^2] dt = \lambda t_f + \int_0^{t_f} [V(t)^2 + \omega(t)^2] dt.$$

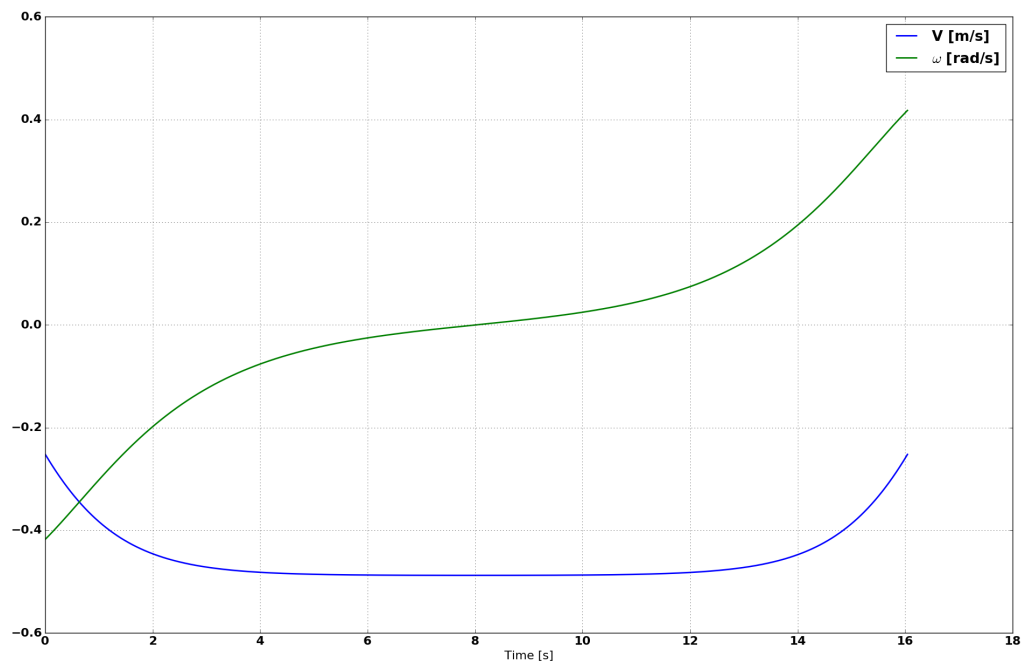
Here  $\lambda \geq 0$  acts like a regularization weight: a larger value of  $\lambda$  upweights  $t_f$  relative to the rest of the objective, which means (since we are minimizing  $J$ ) that we place more importance on reducing  $t_f$  in the optimization. Selecting the largest feasible  $\lambda$  makes sense, since in general, we prefer to travel between the initial and final waypoints as quickly as possible.

- (iv) See submitted code. With  $\lambda = 0.238$  and solution guess  $\mathbf{z}_0 = (0, 0, -\pi/2, 1, 1, 0, 10)$ , we get  $V < 0$  and the robot backs up from  $(0, 0)$  to  $(5, 5)$ . Its trajectory and control histories are shown below.

**Trajectory**  $(x(t), y(t))$

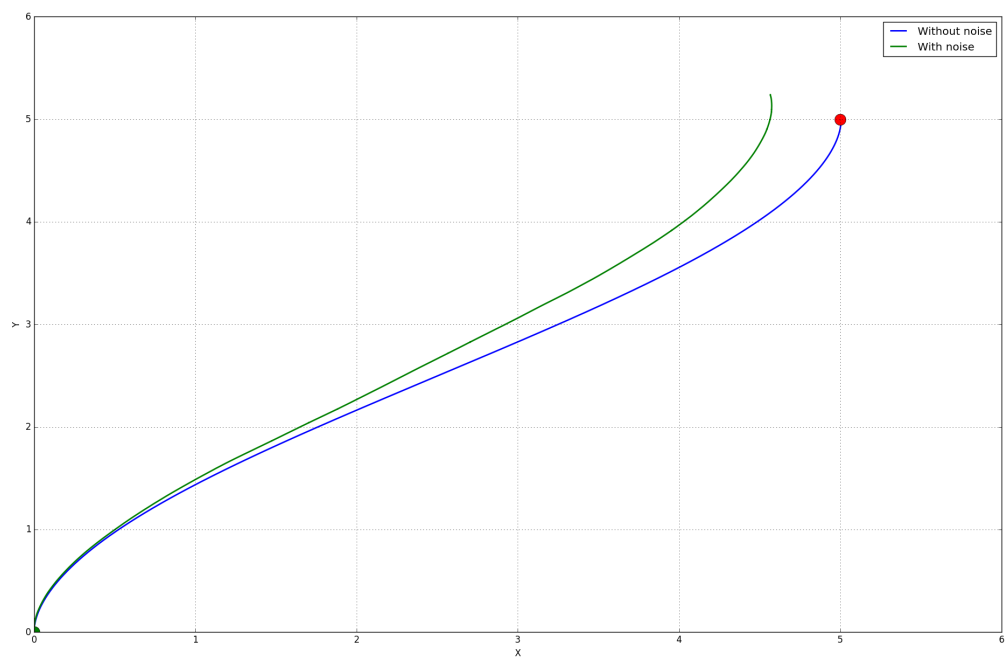


## History of $V$ and $\omega$

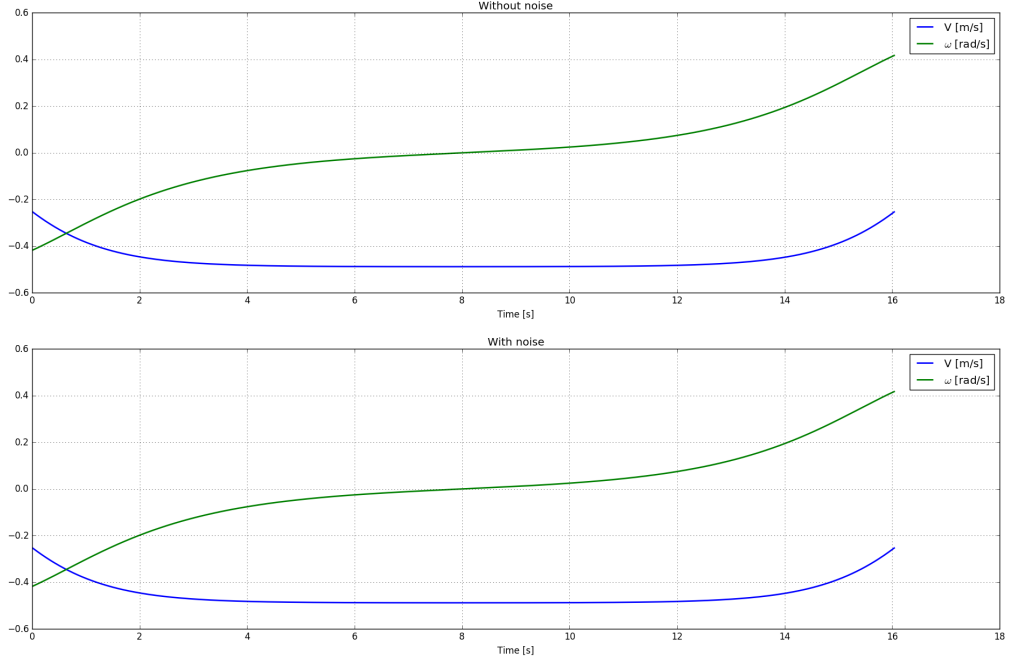


(v) See submitted code.

# Trajectory $(x(t), y(t))$



## History of $V$ and $\omega$



## 2 Differential Flatness

(i) Let  $\psi_1(t) = 1$ ,  $\psi_2(t) = t$ ,  $\psi_3(t) = t^2$ , and  $\psi_4(t) = t^3$ , then  $n = 4$  and

$$x(t) = \sum_{i=1}^n x_i \psi_i(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3$$

$$y(t) = \sum_{i=1}^n y_i \psi_i(t) = y_1 + y_2 t + y_3 t^2 + y_4 t^3$$

$$\dot{x}(t) = \sum_{i=1}^n x_i \dot{\psi}_i(t) = x_2 + 2x_3 t + 3x_4 t^2$$

$$\dot{y}(t) = \sum_{i=1}^n y_i \dot{\psi}_i(t) = y_2 + 2y_3 t + 3y_4 t^2$$

so the initial and final conditions can be expressed as

$$\begin{aligned}
x(0) &= 0 = x_1 \\
y(0) &= 0 = y_1 \\
x(t_f) &= 5 = x_1 + x_2 t_f + x_3 t_f^2 + x_4 t_f^3 \\
y(t_f) &= 5 = y_1 + y_2 t_f + y_3 t_f^2 + y_4 t_f^3 \\
\dot{x}(0) &= V(0) \cos(\theta(0)) = 0.5 \cos(-\pi/2) = 0 = x_2 \\
\dot{y}(0) &= V(0) \sin(\theta(0)) = 0.5 \sin(-\pi/2) = -1 = y_2 \\
\dot{x}(t_f) &= V(t_f) \cos(\theta(t_f)) = 0.5 \cos(-\pi/2) = 0 = x_2 + 2x_3 t_f + 3x_4 t_f^2 \\
\dot{y}(t_f) &= V(t_f) \sin(\theta(t_f)) = 0.5 \sin(-\pi/2) = -1 = y_2 + 2y_3 t_f + 3y_4 t_f^2
\end{aligned}$$

or more succinctly in matrix form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2t_f & 3t_f^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 15 & 225 & 3375 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 30 & 675 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x(0) \\ x(t_f) \\ \dot{x}(0) \\ \dot{x}(t_f) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2t_f & 3t_f^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 15 & 225 & 3375 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 30 & 675 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} y(0) \\ y(t_f) \\ \dot{y}(0) \\ \dot{y}(t_f) \end{pmatrix}$$

We cannot set  $V(t_f) = 0$  because then  $J$  would be singular at time  $t_f$ , and we could not recover the flat outputs from the virtual control inputs.

(ii) After solving for  $x_i, y_i$  for  $i = 1, \dots, n$ , we can recover

$$\begin{pmatrix} x(t) \\ y(t) \\ \dot{x}(t) \\ \dot{y}(t) \\ \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_2 & 2x_3 & 3x_4 & 0 \\ y_2 & 2y_3 & 3y_4 & 0 \\ 2x_3 & 6x_4 & 0 & 0 \\ 2y_3 & 6y_4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

along with the alignment angle

$$\frac{\dot{y}(t)}{\dot{x}(t)} = \frac{V \sin(\theta(t))}{V \cos(\theta(t))} = \tan(\theta(t)) \Rightarrow \theta(t) = \arctan \left( \frac{\dot{y}(t)}{\dot{x}(t)} \right).$$

Since  $V > 0$ , we take the positive root of

$$\dot{x}(t)^2 + \dot{y}(t)^2 = V^2 \cos^2(\theta(t)) + V^2 \sin^2(\theta(t)) = V^2 \Rightarrow V(t) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2},$$

then the matrix  $J$  is invertible, so we can calculate

$$J^{-1} = \begin{pmatrix} \cos(\theta) & -V \sin(\theta) \\ \sin(\theta) & V \cos(\theta) \end{pmatrix}^{-1} = \frac{1}{V} \begin{pmatrix} V \cos(\theta) & V \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\begin{pmatrix} a \\ \omega \end{pmatrix} = J^{-1} \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix} = \begin{pmatrix} \ddot{x}(t) \cos(\theta) + \ddot{y}(t) \sin(\theta) \\ \frac{1}{V}(-\ddot{x}(t) \sin(\theta) + \ddot{y}(t) \cos(\theta)) \end{pmatrix}.$$

Putting this all together, our state-trajectory is

$$\begin{aligned} x(t) &= x_1 + x_2 t + x_3 t^2 + x_4 t^3 \\ y(t) &= y_1 + y_2 t + y_3 t^2 + y_4 t^3 \\ \theta(t) &= \arctan \left( \frac{y_2 + 2y_3 t + 3y_4 t^2}{x_2 + 2x_3 t + 3x_4 t^2} \right) \end{aligned}$$

and our control history is

$$\begin{aligned} V(t) &= \sqrt{(x_2 + 2x_3 t + 3x_4 t^2)^2 + (y_2 + 2y_3 t + 3y_4 t^2)^2} \\ \omega(t) &= \frac{1}{V(t)} (-(2x_3 + 6x_4 t) \sin(\theta(t)) + (2y_3 + 6y_4 t) \cos(\theta(t))) \end{aligned}$$

(iii) Given  $\dot{s}(t) = V(t)$  with  $s(0) = 0$ , we can integrate to get the path parameter

$$s(t) = \int_0^t V(t') dt' - V(0)$$

We wish to find an alternative velocity control  $\tilde{V}(s)$  that satisfies the control saturation constraints. Then, the corresponding angular velocity control and time history are

$$\tilde{\omega}(s) = \omega(s) \frac{\tilde{V}(s)}{V(s)}, \quad \tau(s) = \int_0^s \frac{ds'}{\tilde{V}(s')}.$$

Combining the constraint  $|\tilde{V}(s)| \leq 0.5$  m/s with

$$|\tilde{\omega}(s)| = \left| \omega(s) \frac{\tilde{V}(s)}{V(s)} \right| \leq 1 \text{ rad/s} \Rightarrow |\tilde{V}(s)| \leq \frac{V(s)}{|\omega(s)|} \text{ for } \omega(s) \neq 0$$

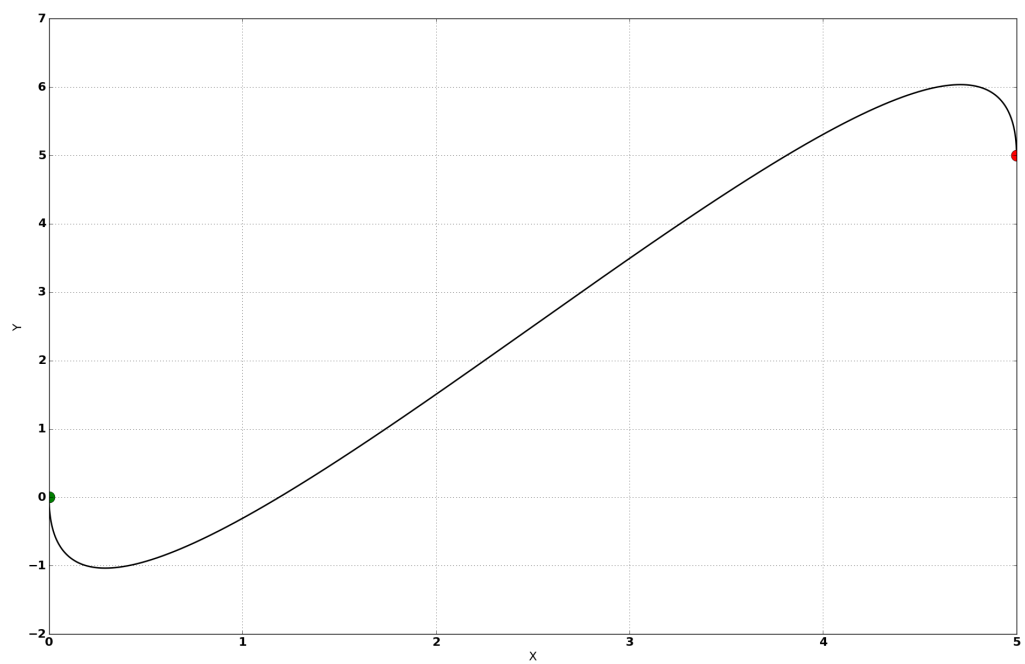
we see that a viable choice is

$$\tilde{V}(s) = \begin{cases} \min(V(s), 0.5) & \text{if } \omega(s) = 0 \\ \min\left(V(s), \frac{V(s)}{|\omega(s)|}, 0.5\right) & \text{if } \omega(s) \neq 0 \end{cases}$$

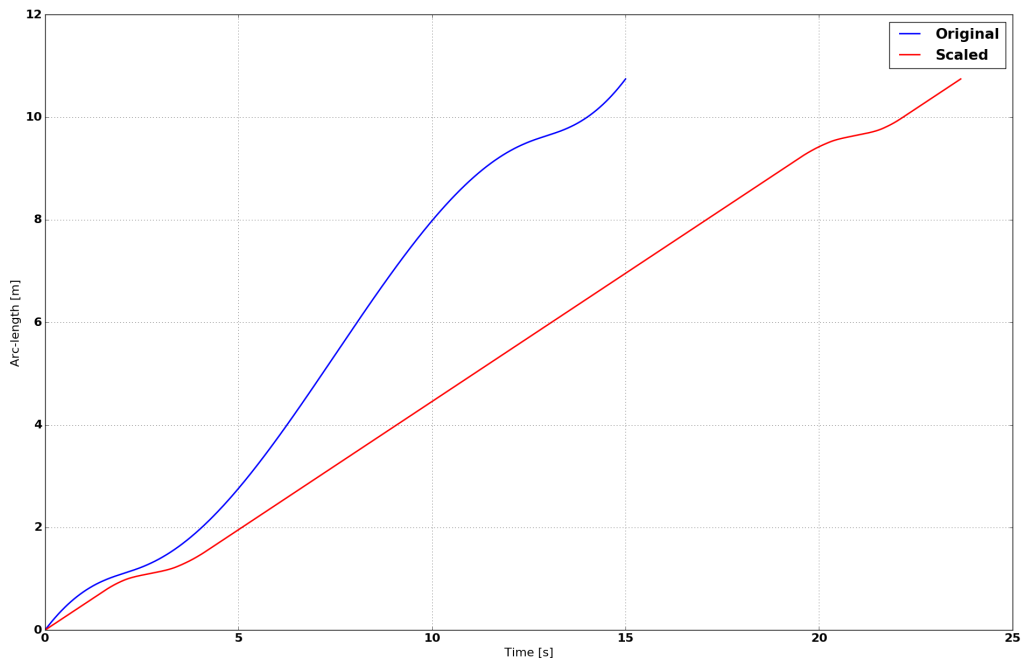
(iv) See submitted code.



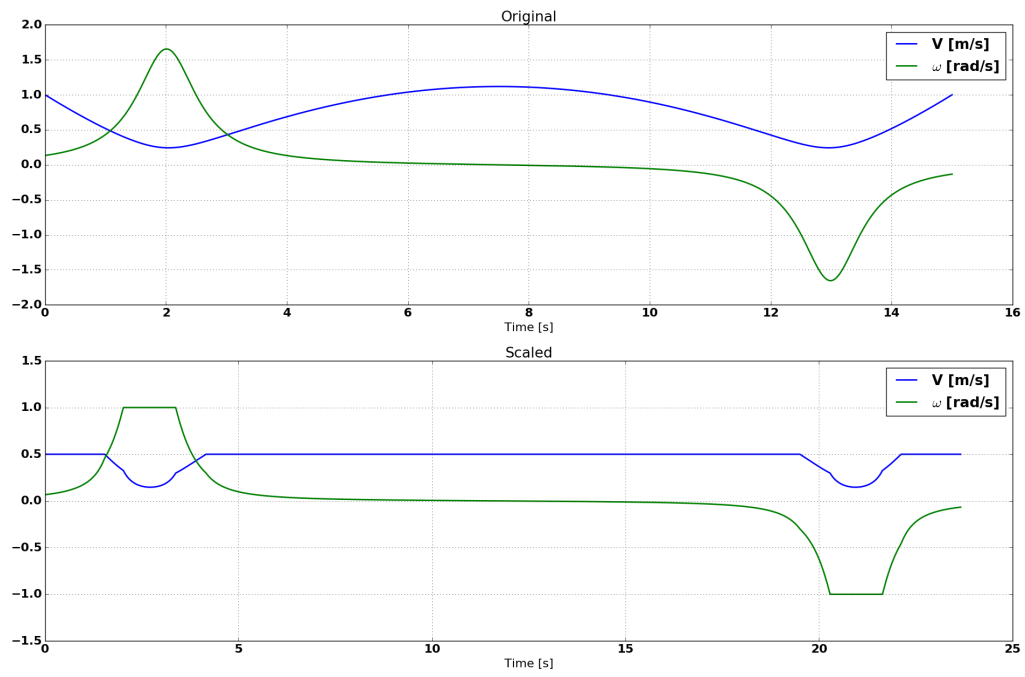
Trajectory  $(x(t), y(t))$



# Arc-length $s(t)$

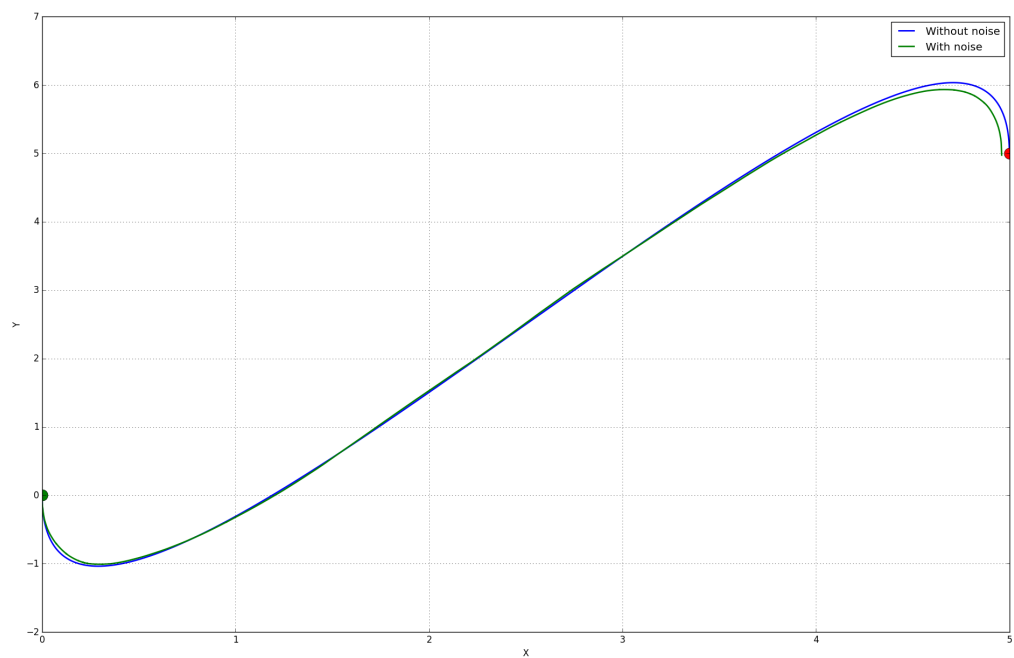


## History of $V$ and $\omega$

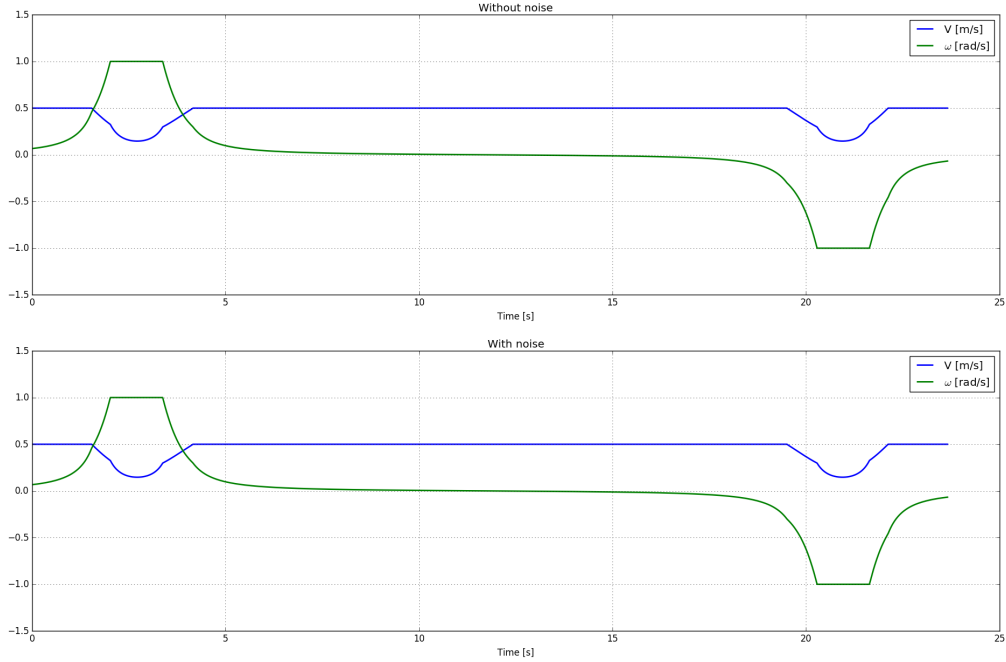


(v) See submitted code.

# Trajectory $(x(t), y(t))$



## History of $V$ and $\omega$



## 3 Closed-loop Control I

(i) We are given the control law

$$V = k_1 \rho \cos(\alpha)$$

$$\omega = k_2 \alpha + k_1 \frac{\sin(\alpha) \cos(\alpha)}{\alpha} (\alpha + k_3 \delta)$$

for the ODE system

$$\dot{\rho}(t) = -V(t) \cos(\alpha(t))$$

$$\dot{\alpha}(t) = V(t) \frac{\sin(\alpha(t))}{\rho(t)} - \omega(t)$$

$$\dot{\delta}(t) = V(t) \frac{\sin(\alpha(t))}{\rho(t)}$$

where  $k_1, k_2, k_3 > 0$  are constants. From Homework 1, Figure 2, we readily see that given a starting position  $(x, y, \theta)$  and final position  $(x_g, y_g, \theta_g)$ , the incremental change

in polar coordinates is

$$\rho = \sqrt{(x_g - x)^2 + (y_g - y)^2}$$

$$\tan(\alpha + \theta) = \frac{y_g - y}{x_g - x} \Rightarrow \alpha = \arctan\left(\frac{y_g - y}{x_g - x}\right) - \theta$$

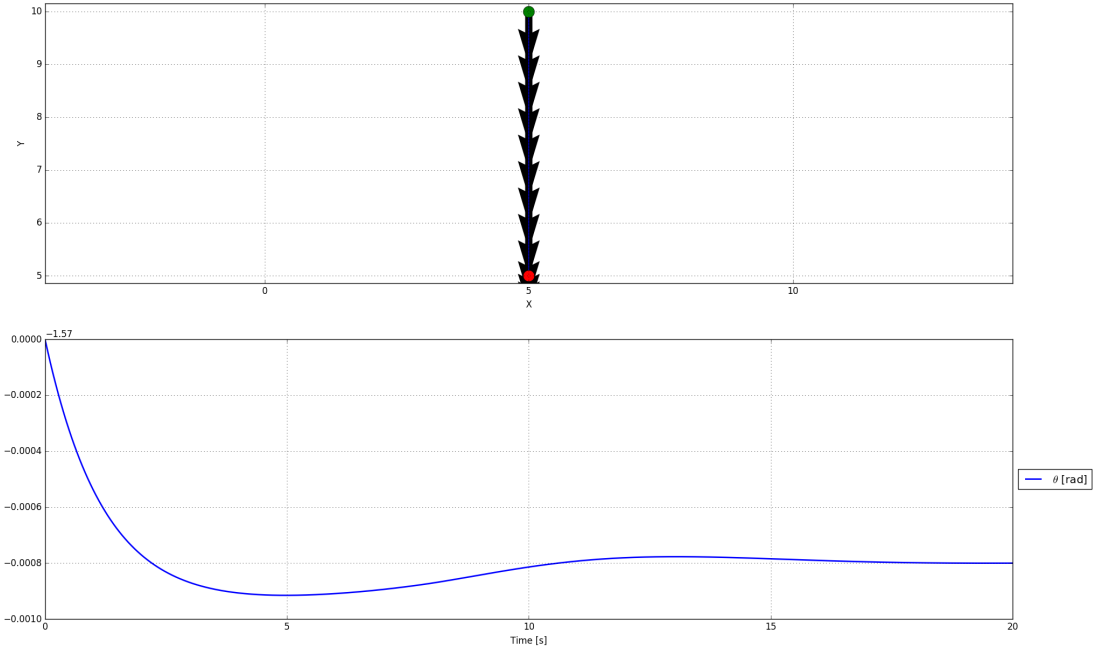
$$\alpha + \theta = \delta + \theta_g \Rightarrow \delta = \alpha + \theta - \theta_g$$

(ii) See submitted code. The parameters used were

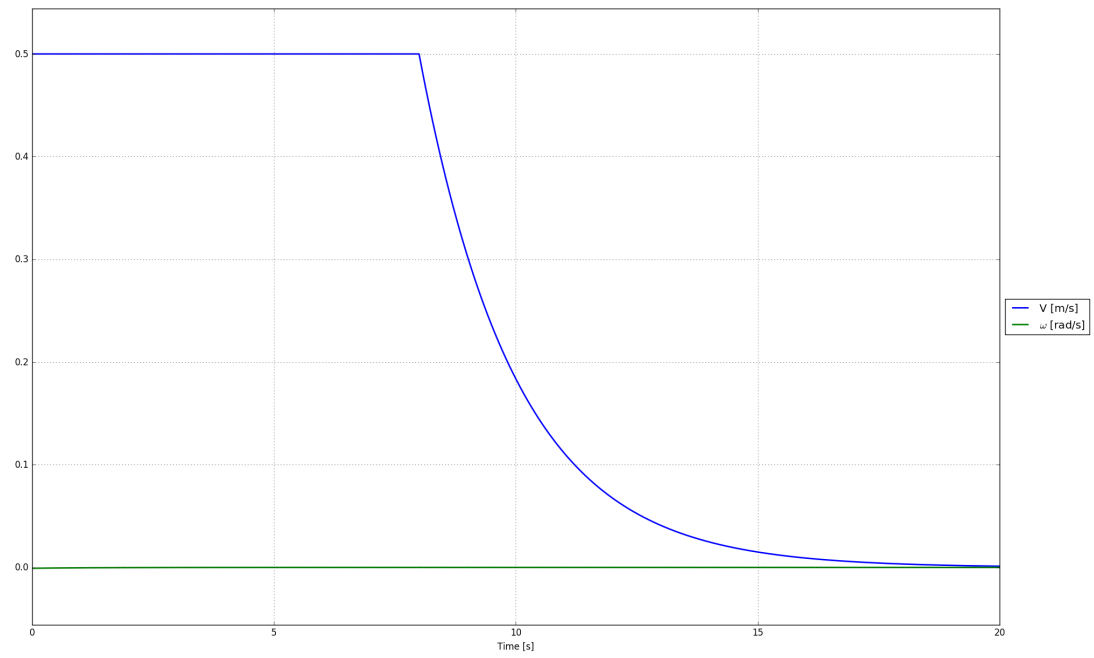
$$(x_0, y_0, \theta_0, t_f) = \begin{cases} (5, 10, -\pi/2, 20) & \text{for forward parking} \\ (5, 0, -\pi/2, 12) & \text{for reverse parking} \\ (0, 5, -\pi/2, 18) & \text{for parallel parking} \end{cases}$$

$$(k_1, k_2, k_3) = \begin{cases} (0.5, 0.5, 1.2) & \text{for forward parking} \\ (1.45, 0.01, 1.45) & \text{for reverse parking} \\ (0.5, 0.5, 1.2) & \text{for parallel parking} \end{cases}$$

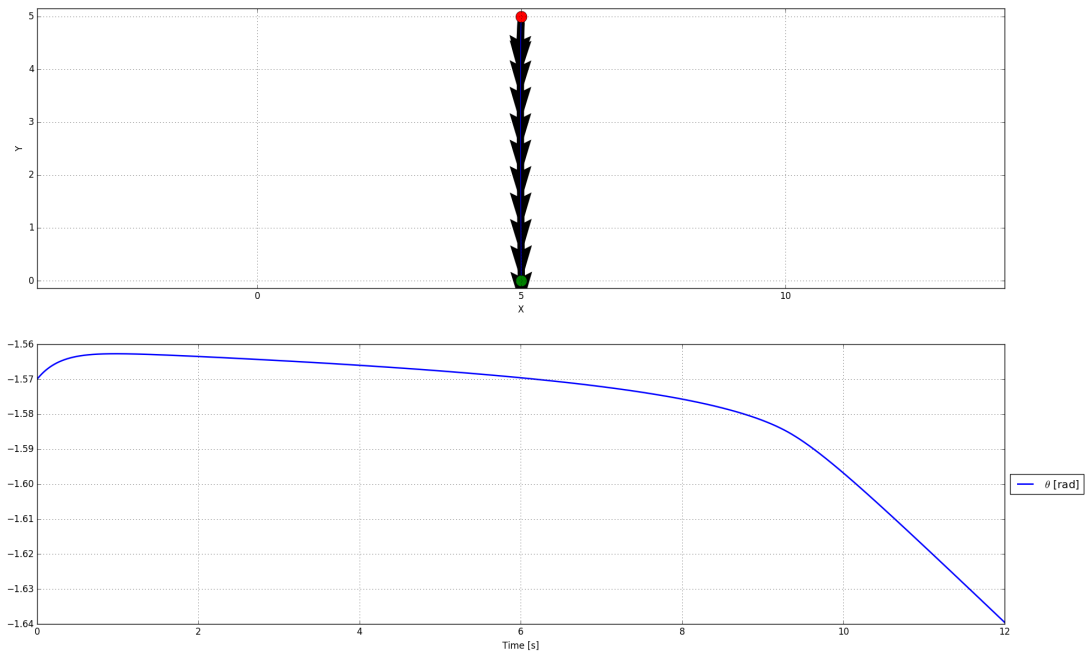
### Forward Trajectory



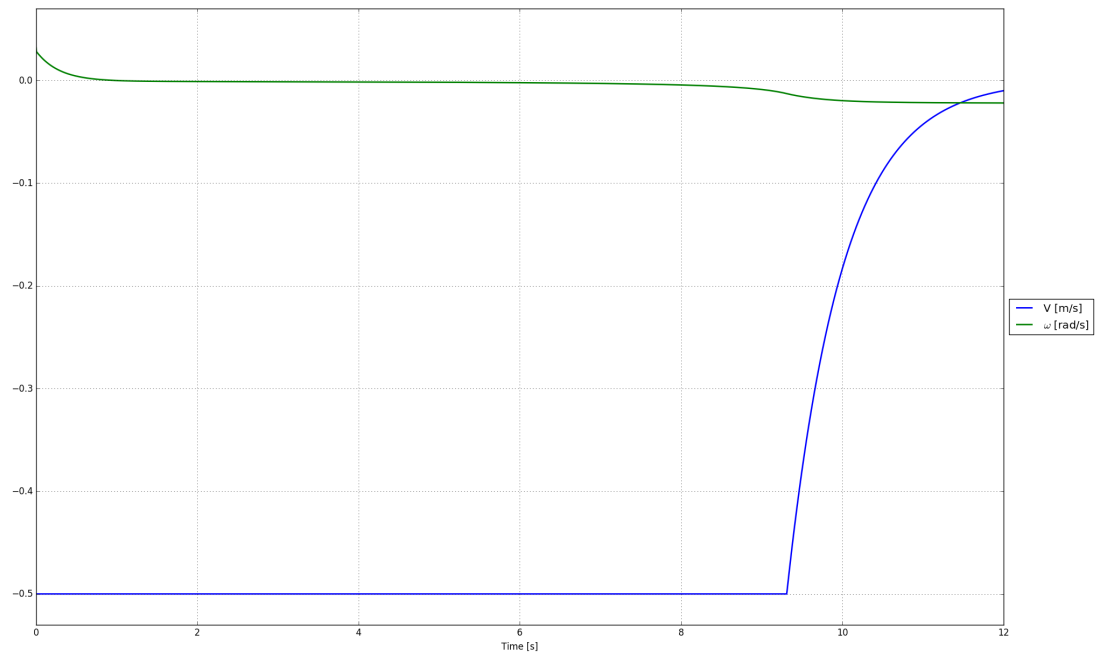
## Forward Control



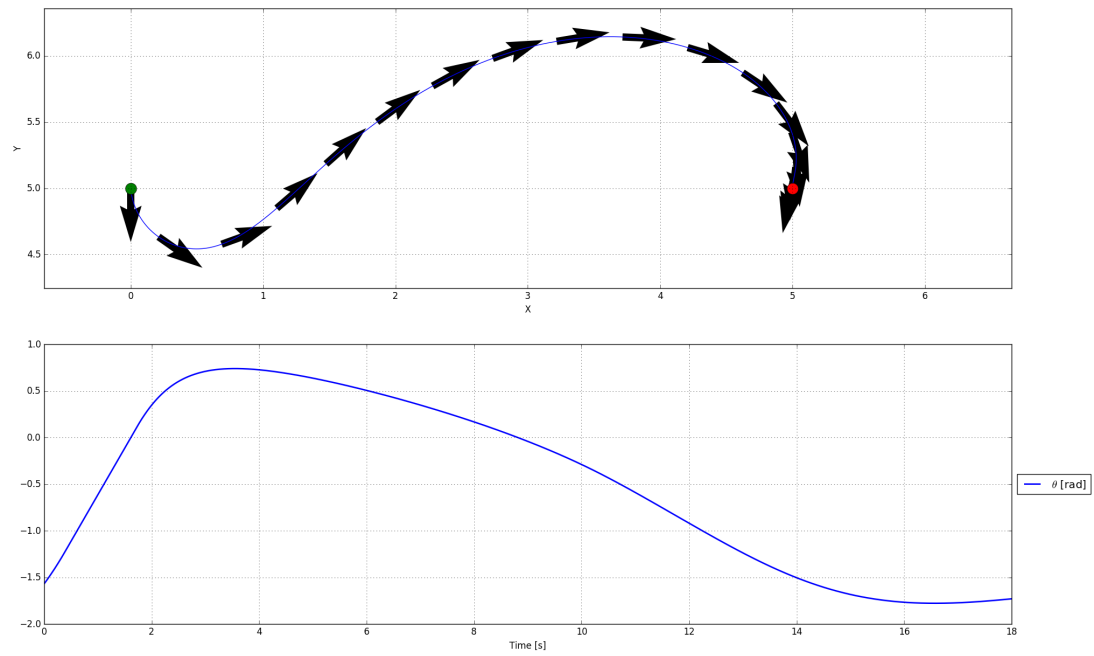
## Reverse Trajectory



## Reverse Control

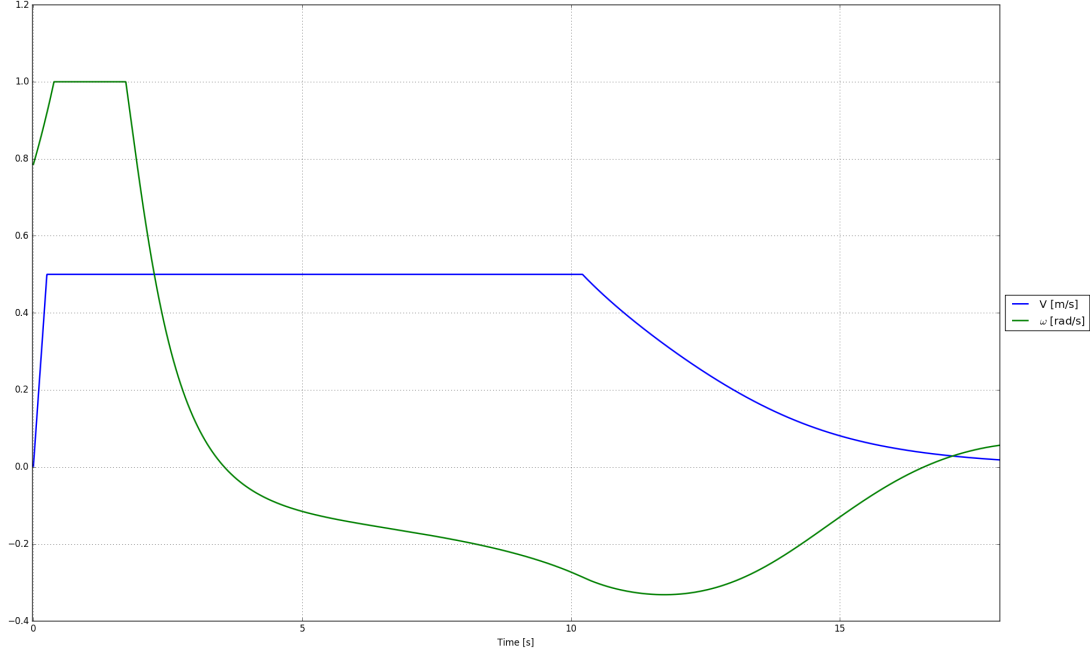


## Parallel Trajectory





## Parallel Control



## 4 Closed-loop Control II

- (i) In Problem 2, part (ii), we showed that for a given state  $\mathbf{x} = (x, y, \theta)$ ,

$$\begin{pmatrix} a \\ \omega \end{pmatrix} = \begin{pmatrix} \dot{V} \\ \omega \end{pmatrix} = \begin{pmatrix} u_1 \cos(\theta) + u_2 \sin(\theta) \\ \frac{1}{V}(-u_1 \sin(\theta) + u_2 \cos(\theta)) \end{pmatrix} \quad (7)$$

where the virtual controls are  $(u_1, u_2) = (\ddot{x}, \ddot{y})$ .

- (ii) See submitted code. We assume the control law is implemented at a high enough frequency that the state variables  $\mathbf{x} = (x, y, \theta)$  can be treated as constant. Furthermore, at each step  $t$ , we consider the current velocity to be that commanded at  $t - 1$ , i.e.,

$$\dot{x}_t = V_{t-1} \cos(\theta), \quad \dot{y}_t = V_{t-1} \sin(\theta),$$

Using this, we can calculate  $(u_1, u_2)$  and plug into (7) to get the current acceleration,  $\dot{V}_t$ . We then apply the Euler method to update

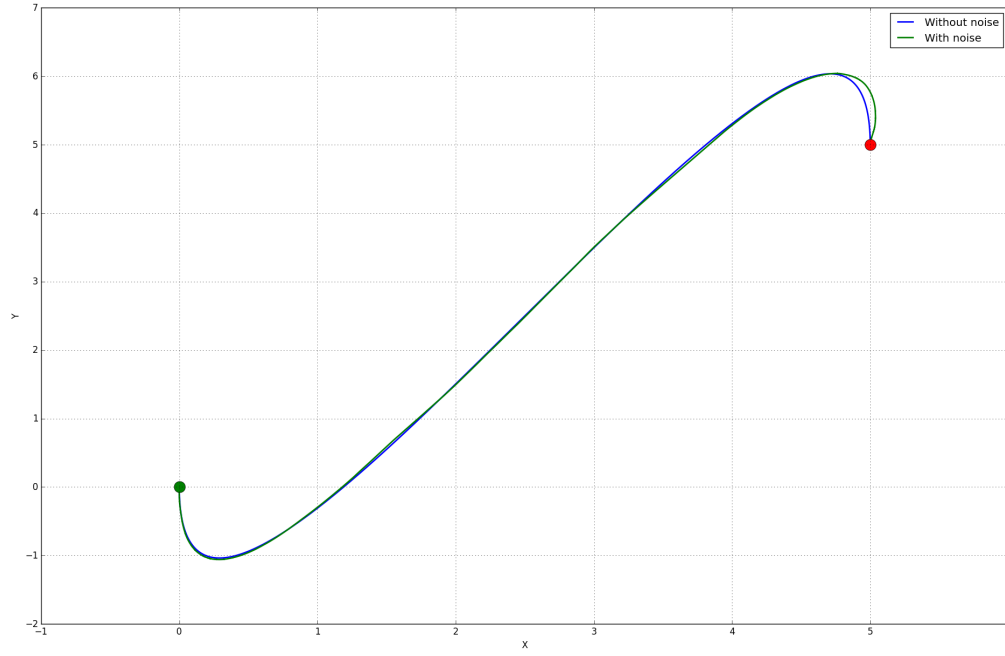
$$V_t = V_{t-1} + \delta \dot{V}_t \quad \text{where } \delta \text{ is the timestep.}$$

If  $V_t \approx 0$ , we reset it to the nominal (desired) velocity  $V_d = \sqrt{\dot{x}_d^2 + \dot{y}_d^2}$  according to the robot kinematic model. Finally, we calculate  $\omega$ .

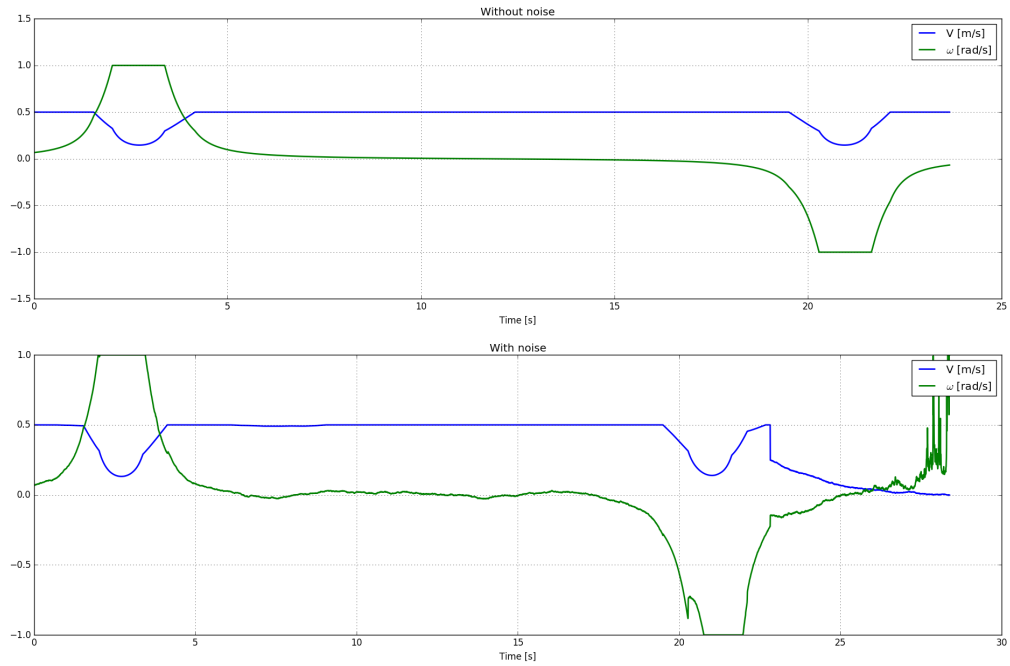
(iii) See submitted code.

(iv) See submitted code. For this simulation, we set  $(k_{px}, k_{py}, k_{dx}, k_{dy}) = (1, 1, 0.5, 0.5)$  and reset to the nominal velocity whenever  $|V_t| \leq \epsilon = 10^{-5}$ .

### Trajectory $(x(t), y(t))$



## History of $V$ and $\omega$



## 5 Robot Operating System

- (i) See submitted code.
- (ii) `roslaunch filename.bag`
- (iii) See submitted code.