Homework 1 in EL2620 Nonlinear Control

Yuchao Li
910928-4233
yuchao@kth.se

Anqing Duan
921224-2714
anqingd@kth.se

November 18, 2015

Problem description

The change in population of two interacting species inhabiting a grassy island can be described by the following equations:

$$\dot{x} = x(t)(a + bx(t) + cy(t))$$
$$\dot{y} = y(t)(d + ex(t) + fy(t)) \tag{1}$$

where $a, b, ..., f$ are parameters and x, y are non-negative continuous function with respect to time. The model can be effective only with two preconditions fulfilled. First, the populations here are normalized and therefore can be thought of as continuous numbers. Second, the habitat is isolated area, which means the impact from other creatures can be neglected. While a, b, d and f are very much dependent on food supply and intraspecific competition, c and e reflect biological interactions between those two species. This work seeks to provide more details of this model by analyzing the solutions of (1) and their biological backgrounds when those parameters are assigned with different values. A generalized model for more species is presented in the end.

Problem 1 – Analysis of biological interaction

The relation between x and y can be interpreted in four different ways. It can be predator-prey (x predator, y prey), prey-predator (x prey, y predator), competitive (x and y inhibit each other) or symbiotic (x and y benefit each other). The sign of c and e defines the relation. The following solution elaborates the dependence.

Solution 1 – Analysis of biological interaction

- predator-prey (x predator, y prey)
  c is positive and e is negative. Large population of y benefits x since there are sufficient food, which implies that the population of x shall grow faster when y increases. Therefore c must be positive to show the positive impact y has on $\dot{x}$. However, the more x, the worse situation for y. Increasing of x makes it harder for y to survive. Therefore, e is negative.

- prey-predator (x prey, y predator)
  c is negative and e is positive. The elaboration is the same as previous case except that x and y have inverted roles here.

- competitive (x and y inhibit each other)
  c and e are both negative. Since x and y compete for some common source, increase of one species means higher inhibitive effect on the other, which indicates that c and e shall both be negative.

- symbiotic (x and y benefit each other)
  c and e are both positive. The existence of one species benefits the other, which means the population of one species shall grow faster if the other increases. Therefore both c and e shall be positive to reflect the favourable impact they have on each other.

Problem 2 – System analysis for different cases

The given population model will be analyzed for different signs combination of c and e with phase portraits obtained by pplane and equilibrium point(s) identified analytically. The results should be interpreted under biological context.

Solution 2 – System analysis for different cases

When $a = 3$, $b = f = -1$, and $d = 2$, we can obtain the differential equations as (2a) and (2b).

$$\dot{x} = x(3 - x + cy) \tag{2a}$$
$$\dot{y} = y(2 + ex - y) \tag{2b}$$

The Jacobian matrix is defined as $A = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x})\big|_{\mathbf{x}=\mathbf{x}_0}$, which in those cases is

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) = \begin{bmatrix} 3 - 2x + cy & cx \\ ey & 2 + ex - 2y \end{bmatrix},$$

The eigenvalues of the Jacobian matrix are given by the characteristic equation

$$\det(\lambda I - A) = 0.$$

Note that the phase portrait is only drawn at $x \geq 0$ and $y \geq 0$ for each case, since the physical context is meaningful only when the population is nonnegative.

Case 1: $(c, e) = (-2, -1)$

1. The phase portrait is shown as figure 1.

2. When $(c, e) = (-2, -1)$, the Jacobian matrix can be written as

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}) = \begin{bmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{bmatrix}.$$

There are four equilibrium points $(x^\star, y^\star)$ in total. They are $(0,0), (0,2), (3,0)$ and $(1,1)$ respectively. The Jocabian matrices are

$$A_{(0,0)} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \qquad A_{(0,2)} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix},$$

$$A_{(3,0)} = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}, A_{(1,1)} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}.$$

The characteristic equation of the system linearized around $(0,0)$ is

$$\lambda^2 - 5\lambda + 6 = 0,$$

which gives the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$. The equilibrium $(0,0)$ is an unstable node.

The characteristic equation of the system linearized around $(0,2)$ is

$$\lambda^2 + 3\lambda + 2 = 0,$$

which gives the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. The equilibrium $(0,2)$ is a stable node.

The characteristic equation of the system linearized around $(3,0)$ is

$$\lambda^2 + 4\lambda + 3 = 0,$$

which gives the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3$. The equilibrium $(3,0)$ is a stable node.

The characteristic equation of the system linearized around $(1,1)$ is

$$\lambda^2 + 2\lambda - 1 = 0,$$

which gives the eigenvalues $\lambda_1 = \sqrt{2} - 1 \approx 0.414$ and $\lambda_2 = -1 - \sqrt{2} \approx -2.414$. The equilibrium $(1,1)$ is a saddle point.

3. In this case, the relationship between x and y is competitive. This implies that one species would always tend to become dominate and the other would become extinct based on the initial ratio. There is one point where two species could co-exist but it's not stable since the equilibrium could be broken easily as long as small number disturbance happens.

Case 2: $(c, e) = (-2, 1)$

1. The phase portrait is shown as figure 2.

2. When $(c, e) = (-2, 1)$, there are three equilibrium points $(x^\star, y^\star)$ in total. They are $(0,0), (0,2)$ and $(3,0)$ respectively. The Jocabian matrix can be derived similarly. The corresponding eigenvalues for each equilibrium point are $\lambda_1 = 3$, $\lambda_2 = 2$; $\lambda_1 = -1$, $\lambda_2 = -2$; $\lambda_1 = -3$, $\lambda_2 = 5$. Therefore the equilibrium points are unstable node, stable node and saddle point respectively.

3. In this case, the relationship between the two species is y preying on x. This implies that y would increase faster if the relative population of x is bigger and similarly the amount of x would decrease faster if if the relative population of y is bigger.

Case 3: $(c, e) = (2, -1)$

1. The phase portrait is shown as figure 3.

2. When $(c, e) = (2, -1)$, there are three equilibrium points $(x^\star, y^\star)$ in total. They are $(0,0), (0,2)$ and $(3,0)$ respectively. The Jocabian matrix can be derived similarly. The corresponding eigenvalues for each equilibrium point are $\lambda_1 = 3$, $\lambda_2 = 2$; $\lambda_1 = -2$, $\lambda_2 = 7$; $\lambda_1 = -3$, $\lambda_2 = -1$. Therefore the equilibrium points are unstable node, saddle node and stable point respectively.

3. In this case, the relationship between the two species is x preying on y. This implies that x tends to increase faster if the relative amount of y is bigger and similarly the amount of y would decrease faster if the relative amount of x is bigger.

Case 4: $(c, e) = (2, 1)$

1. The phase portrait is shown as figure 4.

2. When $(c, e) = (2, -1)$, there are three equilibrium points $(x^\star, y^\star)$ in total. They are $(0, 0), (0, 2)$ and $(3, 0)$ respectively. The Jocabian matrix can be derived similarly. The corresponding eigenvalues for each equilibrium point are $\lambda_1 = 3, \lambda_2 = 2$; $\lambda_1 = -2, \lambda_2 = 7$; $\lambda_1 = -3, \lambda_2 = 5$. Therefore the equilibrium points are unstable node, saddle node and saddle node respectively.

3. In this case, the relationship between the two species is symbiotic. This implies that one species would increase faster as the other one increases.

Problem 3 – System analysis for one specific case

Like previous problem, the dynamic behavior of the population model will be analyzed for one specific case. Phase portrait would be obtained by using pplane. The equilibrium point(s) should be identified analytically. The results should be interpreted under the physical context of two species.

Solution 3 – System analysis for one specific case

1. The phase portrait is shown as figure 5.

2. When $a = e = 1$, $b = f = 0$, and $c = d = -1$, we can obtain the differential equations as (3a) and (3b).

$$\dot{x} = x(1 - y) \qquad (3a)$$
$$\dot{y} = y(-1 + x) \qquad (3b)$$

By calculating $(x^\star, y^\star)$, the system has two equilibrium points $(0, 0)$ and $(1, 1)$. The Jocabian matrix can be derived in a similar way. The eigenvalues are $\lambda_{1,2} = \pm 1$ and $\lambda_{1,2} = \pm j$ respectively. It can be derived that the equilibrium $(0, 0)$ is a saddle point. For the equilibrium $(1, 1)$, since it is a non-hyperbolic point, the stability needs to be checked with other methods. From figure 5 we can infer that this equilibrium is a stable point.

3. The physical species model in this case could be y preying on x and x feeding grass which is infinite. When x is comparatively big, y would increase to control the population of x; when y is comparatively big, y would decrease due to the intraspecies competition and lack of food, which would promote the breeding of x. Therefore, the population would stay stable under this natural negative feedback effect.

Problem 4 – Invariant set and periodic orbit

The problem can be solved by finishing the following three tasks:

1. Elaborate that the $x-$ and $y-$axes are invariant sets regardless of the values of parameters.

2. Explain why this feature is necessary for population model.

3. Show that a periodic solution exists for the model described by (3).

Solution 4 – Invariant set and periodic orbit

1. If the initial point is $x(0) = C$, $y(0) = 0$ where C is any constant, then $y \equiv 0$ for $t \geqslant 0$ since $\dot{y} \equiv 0$ according to (3b). This indicates that a trajectory starting on $x-$axis would always stay on $x-$axis. The same analysis can be conducted on solutions with initial points on $y-$axis. Note that the above analysis holds for any values of parameters. Therefore both $x-$ and $y-$axes are always invariant sets.

2. In this biological context, a solution reaching one axis implies extinction of the other species. Once a species is extincted, its population shall be zero permanently. Hence this is a necessary feature of a population model in order to show the irreversibility of species extinction.

3. On any orbit of (3), we have $\dot{y}dx = \dot{x}dy$. Combine the equation with (3), we get the following equation

$$\frac{y-1}{y}dy + \frac{x-1}{x}dx = 0.$$

The trajectory is given by taking integration of the equation, which is

$$x + y - lnx - lny = C,$$

where C is a constant. Define a function V : $D \to R$ as (4)

$$V(x, y) = x + y - lnx - lny - 2 \qquad (4)$$

where $D \subseteq R^2$ is positive section of (x, y). Based on above discussion, we have $\dot{V} \equiv 0$. According to solution 3, $(x^\star, y^\star) = (1, 1)$ is the only equilibrium in positive section. Denote the Jacobian matrix of $V(x, y)$ at the equilibrium as A. Since

$$\frac{\partial V}{\partial x}|_{(x^\star, y^\star)} = 0, \quad \frac{\partial V}{\partial y}|_{(x^\star, y^\star)} = 0,$$

$$\frac{\partial^2 V}{\partial x^2}\Big|_{(x^\star, y^\star)} > 0, \qquad \det(A) > 0,$$

while there is no other stationary point in D, $V(x^\star, y^\star) = 0$ is global minimum in D, which implies that

$$V(x, y) > 0 \text{ in } D - \{(1, 1)\}.$$

According to the continuity of function, for a sufficiently small $\varepsilon > 0$, Ω_ε must be a connected component containing $(x^\star, y^\star)$ where Ω_ε is defined as

$$\Omega_\varepsilon = \{ (x, y) \mid (x, y) \in D, V(x, y) \leqslant \varepsilon \}.$$

Therefore the border of Ω_ε, which does not contain $(1, 1)$ and $(0, 0)$ is a closed and bounded trajectory. The orbit function is shown as (5).

$$x + y - lnx - lny - 2 = \varepsilon \qquad (5)$$

Note that above discussion has created a Lyapunov function which fulfills all the criteria for local stability. Therefore equilibrium $(1, 1)$ is locally stable.

Problem 5 – Generalize the population model

So far the popular model is analyzed when the number of the species is two. A more general model where the species number N is more than two could be built based on the previous regular. Some elaboration would be included.

Solution 5 – Generalize the population model

When the number of species $N > 2$, a general equation to express the population model can be deduced as (6).

$$\dot{x}_i = x_i(\alpha_i + \sum_{j=1}^{N} \beta_{ij} x_j), \ i = 1...N \qquad (6)$$

The coefficients β_{ij} are decided based on the relationship between x_i and x_j. Intuitively, when x_j preys on x_i, then $\beta_{ij} < 0$, $\beta_{ji} > 0$; when x_i preys on x_j, then $\beta_{ij} > 0$, $\beta_{ji} < 0$; when x_j and x_i are competitive, then both β_{ij} and β_{ij} will be negative; when x_j and x_i are symbiotic, then both β_{ij} and β_{ij} will be positive; β_{ij} and β_{ji} might be zero if x_i and x_j are irrelevant.
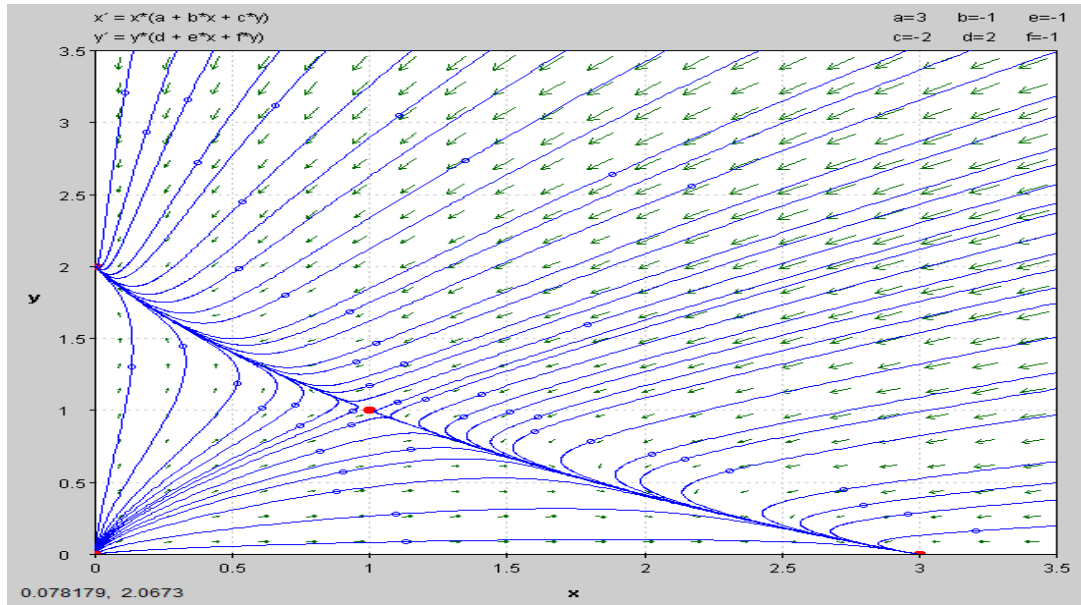
Figure 1: Phase portrait of the system in problem 2 case 1, which has an unstable node at (0,0), two stable points at (0,2) and (3,0) and a saddle point at (1,1). All the equilibriums are marked by large dots and selected trajectories are marked by solid lines. This figure was generated using PPLANE (`http://math.rice.edu/ dfield/dfpp.html`).
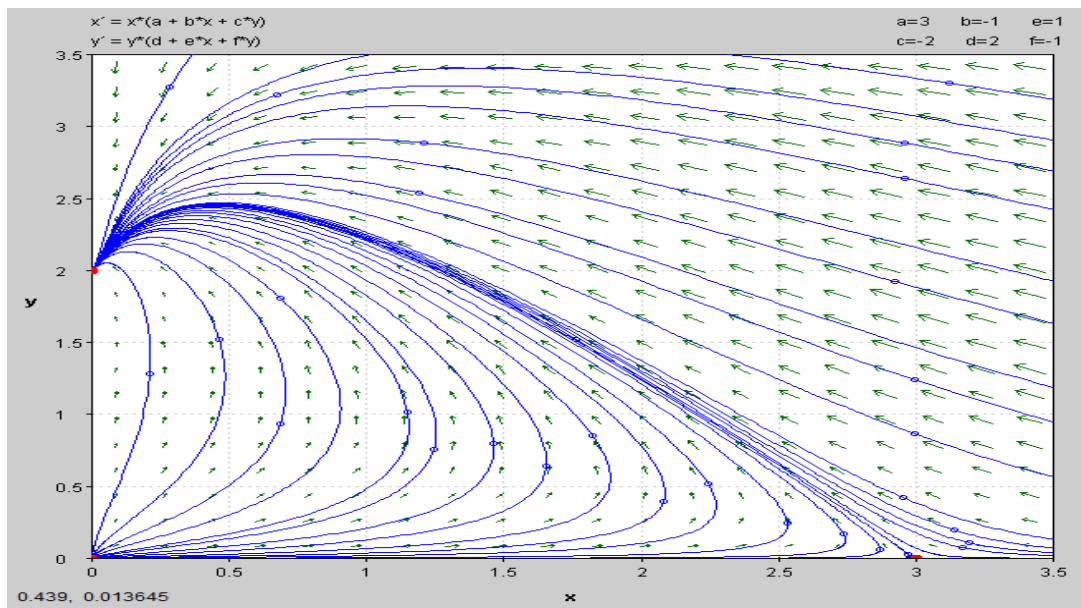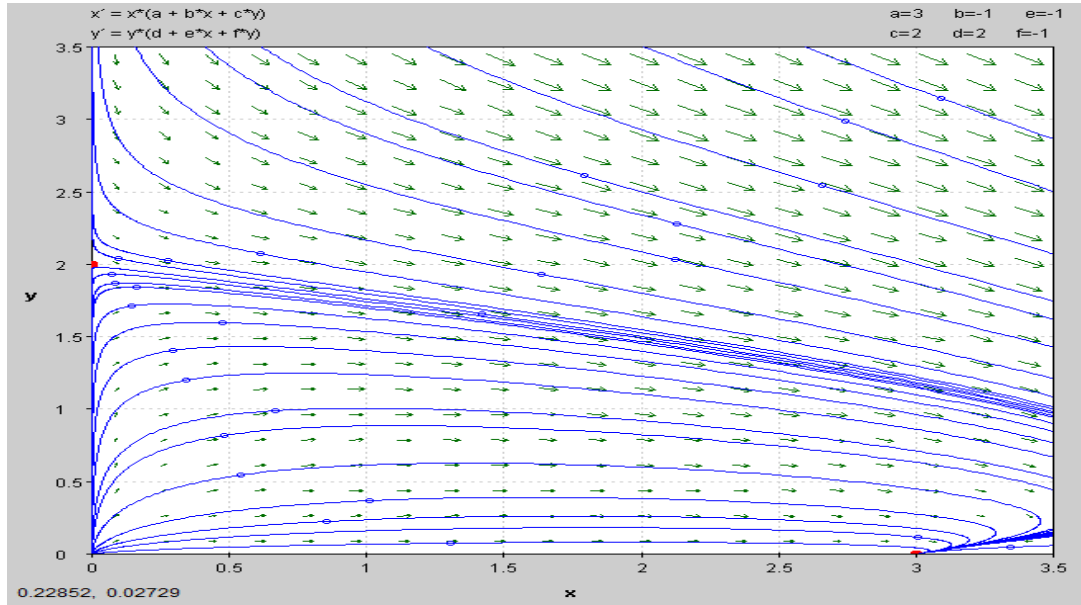

Figure 2: Phase portrait of the system in problem 2 case 2, which has an unstable node at (0,0), a stable point at (0,2) and a saddle at (3,0). All the equilibriums are marked by large dots and selected trajectories are marked by solid lines. This figure was generated using PPLANE (`http://math.rice.edu/ dfield/dfpp.html`).

Figure 3: Phase portrait of the system in problem 2 case 3, which has an unstable node at (0,0), a saddle point at (0,2) and a stable at (3,0). All the equilibriums are marked by large dots and selected trajectories are marked by solid lines. This figure was generated using PPLANE (http://math.rice.edu/ dfield/dfpp.html).
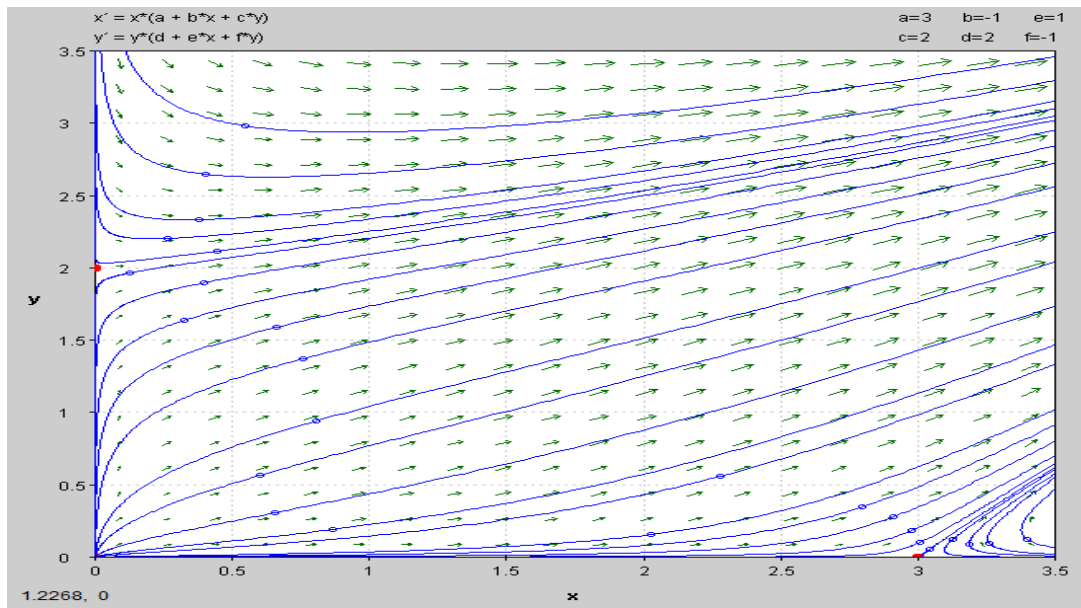

Figure 4: Phase portrait of the system in problem 2 case 4, which has an unstable node at (0,0), two saddle points at (0,2) (3,0). All the equilibriums are marked by large dots and selected trajectories are marked by solid lines. This figure was generated using PPLANE (http://math.rice.edu/ dfield/dfpp.html).
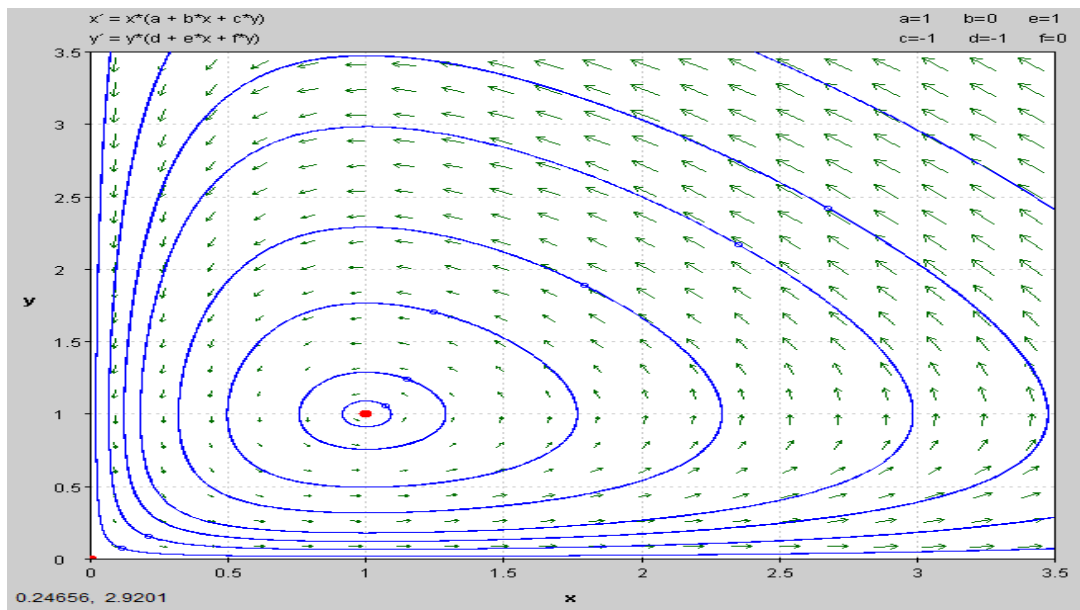
6

Figure 5: Phase portrait of the system in problem 3 and 4, which has a saddle node at (0,0) and a stable point at (1,1). All the equilibriums are marked by large dots and selected trajectories are marked by solid lines. This figure was generated using PPLANE (`http://math.rice.edu/ dfield/dfpp.html`).