

Today we will see some concrete one-way function candidates that arise from number theory, and abstract out some of their other special properties that will be useful when we proceed to investigate pseudorandomness.

## 1 Collections of OWFs

Our generic definition of a one-way function is concise, and very useful for complexity-theoretic crypto. However, it tends not to be as appropriate for the kinds of hard functions that we use in “real-life” crypto; below we give a more flexible definition. (In your homework, you will show that the generic OWF definition is equivalent to this one.)

**Definition 1.1.** A *collection of one-way functions* is a family  $F = \{f_s: D_s \rightarrow R_s\}_{s \in S}$  satisfying the following conditions:

1. *Easy to sample a function:* there is a PPT algorithm  $S$  such that  $S()$  outputs some  $s \in S$  (according to some arbitrary distribution).
2. *Easy to sample from domain:* there is a PPT algorithm  $D$  such that  $D(s)$  outputs some  $x \in D_s$  (according to some arbitrary distribution).
3. *Easy to evaluate function:* there is a PPT algorithm  $F$  such that  $F(s, x) = f_s(x)$  for all  $s \in S, x \in D_s$ .
4. *Hard to invert:* for any non-uniform PPT algorithm  $\mathcal{I}$ ,

$$\Pr_{s \leftarrow S(1^n), x \leftarrow D(s)} [\mathcal{I}(s, f_s(x)) \in f_s^{-1}(f_s(x))] = \text{negl}(n).$$

For example, the subset-sum function  $f_{ss}$  is more naturally defined as a collection, as follows. Let  $S_n = (\mathbb{Z}_N)^n$  where  $N = 2^n$ , and let the full index set  $S = \cup_{n=1}^{\infty} S_n$ . Define the domain  $D_{\vec{a}} = \{0, 1\}^n$  and the range  $R_{\vec{a}} = \mathbb{Z}_N$ , for all  $\vec{a} = (a_1, \dots, a_n) \in S_n$ . The corresponding function is defined as

$$f_{\vec{a}}(x) = \sum_{i=1}^n a_i \cdot x_i \bmod N.$$

The algorithms  $S$  (function sampler),  $D$  (domain sampler), and  $F$  (function evaluator) are all straightforward.

In the remainder of the lecture, we will see other examples of OWF collections (some with other special properties) that arise from number theory.

## 2 Number Theory Background

**Definition 2.1.** For positive integers  $a, b \in \mathbb{N}$ , their *greatest common divisor*  $d = \gcd(a, b)$  is the largest integer  $d$  such that  $d \mid a$  and  $d \mid b$ .

As a consequence of Algorithm 1 below, there always exist integers  $x, y \in \mathbb{Z}$  such that  $ax + by = \gcd(a, b)$ . We say that  $a$  and  $b$  are *co-prime* (or *relatively prime*) if  $\gcd(a, b) = 1$ , i.e.,  $ax = 1 \bmod b$ . From this,  $x$  is the multiplicative inverse of  $a$  modulo  $b$ , and likewise  $y$  is the multiplicative inverse of  $b$  modulo  $a$ . The following deterministic algorithm shows that  $\gcd(a, b)$  (and additionally, the integers  $x$  and  $y$ ) can be computed efficiently.

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**Algorithm 1** Algorithm  $\text{ExtendedEuclid}(a, b)$  for computing the greatest common divisor of  $a$  and  $b$ .

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**Input:** Positive integers  $a \geq b > 0$ .

**Output:**  $(x, y) \in \mathbb{Z}^2$  such that  $ax + by = \gcd(a, b)$ .

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1: if  $b \mid a$  then
2:   return  $(0, 1)$ 
3: else
4:   Let  $a = b \cdot q + r$  for  $r \in \{1, \dots, b-1\}$ 
5:    $(x', y') \leftarrow \text{ExtendedEuclid}(b, r)$ 
6:   return  $(y', x' - q \cdot y')$ 
7: end if

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**Theorem 2.2.**  $\text{ExtendedEuclid}$  is correct and runs in polynomial time in the lengths of  $a$  and  $b$ , i.e., in  $\text{poly}(\log a + \log b)$  time.

*Proof.* For correctness, we argue by induction on the second argument  $b$ . Clearly the algorithm is correct when  $b = 1$ . If  $b \mid a$ , then  $\gcd(a, b) = b$ , hence  $\text{ExtendedEuclid}$  correctly returns  $(0, 1)$ . If  $b \nmid a$  then by the inductive hypothesis (using  $b > r$ ), the recursive call correctly returns  $(x', y')$  such that  $bx' + ry' = \gcd(b, r)$ . It can be checked that  $\gcd(a, b) = \gcd(b, r)$ , because any common divisor of  $a$  and  $b$  is also a divisor of  $r$ . Finally, observe that

$$\gcd(b, r) = bx' + ry' = bx' + (a - b \cdot q)y' = ay' + (x' - q \cdot y')b.$$

Hence  $\text{ExtendedEuclid}$  correctly returns  $(y', x' - q \cdot y')$ .

For the running time, observe that all the basic operations (not including the recursive call) can be implemented in polynomial time. The following claim establishes the overall efficiency.

**Claim 2.3.** For  $2^n > a \geq b > 0$ ,  $\text{ExtendedEuclid}$  makes at most  $2n$  recursive calls.

We use induction. The claim is true when  $a < 2^1$ . Suppose the claim is true for all  $a < 2^n$ , and suppose  $a < 2^{n+1}$ . Two cases arise:

- If  $b < 2^n$ , the first recursive call is on  $(b, r)$ . Since  $b < 2^n$ , by the inductive hypothesis we make at most  $2n$  more recursive calls. Hence the total number of recursive calls is at most  $1 + 2n < 2(n+1)$ .
- If  $b \geq 2^n$ , i.e.,  $2^{n+1} > a \geq b \geq 2^n$ , we have  $a = b \cdot 1 + r$  for  $r = a - b < 2^n < b$ . The first recursive call is on  $(b \geq 2^n, r < 2^n)$ . In turn, its recursive call uses  $r < 2^n$  as its first parameter. By the inductive hypothesis, the number of recursive calls following the second one is at most  $2n$ . Hence the total number of recursive calls is at most  $2 + 2n \leq 2(n+1)$ .  $\square$

We frequently work with the ring  $(\mathbb{Z}_N, +, \cdot)$  of integers modulo a positive integer  $N$ .

**Lemma 2.4** (Chinese remainder theorem, special case). *Let  $N = p \cdot q$  for distinct primes  $p, q$ . The ring  $\mathbb{Z}_N$  is isomorphic to the product ring  $\mathbb{Z}_p \times \mathbb{Z}_q$ , via the isomorphism  $h(x) = (x \bmod p, x \bmod q)$ .*

A few remarks about the above lemma:

- In the product ring  $\mathbb{Z}_p \times \mathbb{Z}_q$ , addition and multiplication are coordinate-wise.

- Clearly the isomorphism  $h$  is efficiently computable. Less obvious is that it is also efficiently *invertible*. Suppose we know some elements  $c_p, c_q \in \mathbb{Z}_N$  such that  $h(c_p) = (1, 0)$  and  $h(c_q) = (0, 1)$ ; such a pair is sometimes called a *CRT basis*. Then given  $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_q$ , it is easy to see that  $h^{-1}(x, y) = x \cdot c_p + y \cdot c_q$ . Exercise: show how to compute  $c_p, c_q$  efficiently (hint: use ExtendedEuclid on  $p, q$ ).

**Question 1.** Verify that the isomorphism works in  $\mathbb{Z}_{15}$  by checking that  $h(7 \cdot 9) = h(7) \cdot h(9)$ .

**Question 2.** Show that  $h^{-1}(x, y) = x \cdot c_p + y \cdot c_q$ .

**Definition 2.5.** The multiplicative group  $\mathbb{Z}_N^* := \{x \in \mathbb{Z}_N : x \text{ is invertible mod } N, \text{ i.e., } \gcd(x, N) = 1\}$ .

Here are some useful facts about the multiplicative group  $\mathbb{Z}_N^*$ :

- For a prime  $p$ ,  $\mathbb{Z}_p^* = \{1, \dots, p-1\}$ .
- When  $N = pq$  for distinct primes  $p, q$ , we have  $\mathbb{Z}_N^* \cong \mathbb{Z}_p^* \times \mathbb{Z}_q^*$ .

**Definition 2.6.** For  $N \in \mathbb{Z}^+$ , Euler's *totient function*  $\varphi(n)$  is defined to be  $|\mathbb{Z}_N^*|$ , i.e., the number of positive integers  $a \leq n$  relatively prime to  $n$ .

Here are some useful facts about the totient function:

- For a prime  $p$ , we have  $\varphi(p) = p - 1$ .
- For a prime  $p$  and positive integer  $a$ , we have  $\varphi(p^a) = (p - 1)p^{a-1} = p^a - p^{a-1}$ .
- If  $\gcd(a, b) = 1$ , then  $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ .

**Definition 2.7.** The subgroup of *quadratic residues* is defined as

$$\mathbb{QR}_N^* = \{y \in \mathbb{Z}_N^* : \exists x \in \mathbb{Z}_N^* \text{ s.t. } y = x^2 \bmod N\} \subseteq \mathbb{Z}_N^*.$$

Here are some useful facts about  $\mathbb{QR}_N^*$ :

- For an odd prime  $p$ ,  $|\mathbb{QR}_p^*| = \frac{p-1}{2}$ , because  $x \mapsto x^2$  is 2-to-1 over  $\mathbb{Z}_p^*$ . (Exercise: prove this.)
- When  $N = pq$  for distinct odd primes  $p, q$ , we have  $\mathbb{QR}_N^* \cong \mathbb{QR}_p^* \times \mathbb{QR}_q^*$ , hence  $|\mathbb{QR}_N^*| = \frac{p-1}{2} \cdot \frac{q-1}{2}$ .
- For an odd prime  $p$ , we have  $-1 \in \mathbb{QR}_p^*$  if and only if  $p \equiv 1 \pmod{4}$ .

### 3 Factoring-Related Functions

We can abstract out a modulus generation algorithm  $S$ , which given the security parameter  $1^n$  outputs the product  $N$  of two primes  $p, q$ . For example,  $S$  might choose  $p$  and  $q$  to be uniformly random and independent  $n$ -bit primes.

*Rabin's function*  $f_N: \mathbb{Z}_N^* \rightarrow \mathbb{QR}_N^*$  is defined as follows:

$$f_N(x) = x^2 \bmod N.$$

Precisely defining the collection according to Definition 1.1 is a simple exercise. Note that  $f_N$  is 4-to-1, because each  $y \in \mathbb{QR}_N^*$  has two square roots modulo  $p$ , and two modulo  $q$ .

**Theorem 3.1.** *If factoring is hard (with respect to  $S$ ), then the Rabin collection (with function generator  $S$ ) is one-way.*

*Proof.* First, as already discussed it is easy to generate a function, sample its domain, and evaluate the function. The main fact we use to prove one-wayness is the following.

**Claim 3.2.** *Let  $N = pq$  be the product of distinct odd primes. Given any  $x_1, x_2 \in \mathbb{Z}_N^*$  such that  $x_1^2 = x_2^2 \pmod{N}$  but  $x_1 \not\equiv \pm x_2 \pmod{N}$ , the factors of  $N$  can be computed efficiently.*

*Proof of Claim.* We have  $x_1^2 = x_2^2 \pmod{p}$  and  $x_1^2 = x_2^2 \pmod{q}$ , which implies  $x_1 = \pm x_2 \pmod{p}$  and  $x_1 = \pm x_2 \pmod{q}$ . But we cannot have both  $+$  or both  $-$ , by assumption. Wlog, we have  $x_1 = +x_2 \pmod{p}$  and  $x_1 = -x_2 \pmod{q}$ . Then  $p \mid (x_1 - x_2)$  but  $q \nmid (x_1 - x_2)$ , otherwise we'd have  $q \mid (2x_2) \Rightarrow q \mid x_2 \Rightarrow x_2 \notin \mathbb{Z}_N^*$ . Then  $\gcd(x_1 - x_2, N) = p$ , which we can compute efficiently.  $\square$

Continuing with the proof of Theorem 3.1, we prove one-wayness by contrapositive, via a reduction. Assuming we have an inverter for the Rabin function, the idea is to choose our own  $x_1 \in \mathbb{Z}_N^*$  and invoke the inverter on  $y = f_N(x_1) = x_1^2 \pmod{N}$ . The square root  $x_2$  it returns will be  $\neq \pm x_1$ , with probability  $1/2$ . In such a case, we get the prime factorization of  $N$  by Claim 3.2. We now proceed more formally.

Assume a non-uniform PPT inverter  $\mathcal{I}$  violating the one-wayness of the Rabin collection, i.e.,

$$\Pr_{N \leftarrow S(1^n), x \leftarrow \mathbb{Z}_N^*} [\mathcal{I}(N, y = x^2 \pmod{N}) \in \sqrt{y} \pmod{N}] = \delta(n)$$

is non-negligible.

Our factoring algorithm  $\mathcal{A}(N)$  works as follows: first, generate a uniform  $x_1 \leftarrow \mathbb{Z}_N^*$ . Let  $y = x_1^2 \pmod{N}$  and let  $x_2 \leftarrow \mathcal{I}(N, y)$ . If  $x_2^2 = y \pmod{N}$  but  $x_1 \not\equiv \pm x_2 \pmod{N}$ , then compute the factorization of  $N$  by Claim 3.2.

We now analyze the reduction. First,  $N$  and  $y$  are distributed as expected, so  $\mathcal{I}$  outputs  $x_2$  such that  $x_2^2 = y \pmod{N}$  with probability  $\delta$ . Conditioned on the fixed value of  $y$ , there are four possible values for  $x_1$ , each equally likely by construction. So we have  $x_2^2 = y \pmod{N}$  and  $x_2 \not\equiv \pm x_1 \pmod{N}$  with prob  $\delta/2$ , which is non-negligible by assumption.  $\square$

Suppose  $p, q \equiv 3 \pmod{4}$ . Then  $-1$  is not a square modulo  $p$  (respectively,  $q$ ). So for any  $x \in \mathbb{Z}_p^*$  (resp.,  $\mathbb{Z}_q^*$ ), exactly one of  $\pm x$  is a square modulo  $p$  (resp.,  $q$ ). From this it can be seen that if we restrict the Rabin function to have domain  $\mathbb{QR}_N^*$ , i.e.,  $f_N: \mathbb{QR}_N^* \rightarrow \mathbb{QR}_N^*$ , it becomes a *permutation* (bijection).

**Question:** Our proof that  $f_N$  is one-way used (quite essentially) the fact that  $f_N$  is 4-to-1. Now that we have changed its domain to make  $f_N$  a permutation, is the proof still valid?

**Definition 3.3** (One-Way Permutation). A collection  $F = \{f_s: D_s \rightarrow D_s\}_{s \in S}$  is a collection of *one-way permutations* if it is a collection of one-way functions  $f_s$  under the *uniform* distribution over  $D_s$ , and each  $f_s$  is a *permutation* of  $D_s$  (i.e., a bijection).

## Answers

**Question 1.** Verify that the isomorphism works in  $\mathbb{Z}_{15}$  by checking that  $h(7 \cdot 9) = h(7) \cdot h(9)$ .

**Answer.** Note that 15 is the product of two primes: 3 and 5, so  $\mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ . First, we consider  $h(7 \cdot 9)$ .  $7 \cdot 9 \equiv 3 \pmod{15}$  and  $h(3) = (3 \bmod 3, 3 \bmod 5) = (0, 3)$ . Next,  $h(7) = (7 \bmod 3, 7 \bmod 5) = (1, 2)$  and  $h(9) = (9 \bmod 3, 9 \bmod 5) = (0, 4)$ . Finally we multiply the pairs elementwise, recalling that the first elements of each pair are from  $\mathbb{Z}_3$  and the second elements are from  $\mathbb{Z}_5$ .  $(1, 2) \cdot (0, 4) = (0, 3)$  as expected.

**Question 2.** Show that  $h^{-1}(x, y) = x \cdot c_p + y \cdot c_q$ .

**Answer.** Consider,

$$\begin{aligned}(x, y) &= (x, 0) + (0, y) \\ &= h(x) \cdot (1, 0) + h(y) \cdot (0, 1) \\ &= h(x) \cdot h(c_p) + h(y) \cdot h(c_q) \\ &= h(x \cdot c_p + y \cdot c_q) \\ h^{-1}(x, y) &= x \cdot c_p + y \cdot c_q\end{aligned}$$