ARMA Processes

Econ 211C – Unit 1, Section 5

Eric M. Aldrich

UC Santa Cruz

$ARMA(p, q)$ Process

Given white noise $\{\varepsilon_t\}$, consider the process

$$Y_t = c + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots \theta_q \varepsilon_{t-q},$$

where c, $\{\phi_i\}_{i=1}^{p}$ and $\{\theta_i\}_{i=1}^{q}$ are constants.

- This is an $ARMA(p, q)$ process.
- We can rewrite in terms of lag operators:

$$\phi(L) Y_t = c + \theta(L) \varepsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p$$
$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q.$$

$ARMA(p, q)$ as $MA(\infty)$

Recall

- $\phi(L) = (1 - \lambda_1 L)(1 - \lambda_2 L) \cdots (1 - \lambda_p L)$.
- If the roots, $\frac{1}{|\lambda_i|} > 1$, $\forall i$ then $|\lambda_i| < 1$, $\forall i$ and

$$\begin{aligned}
\phi(L)^{-1} &= (1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1} \cdots (1 - \lambda_p L)^{-1} \\
&= \left(\sum_{j=0}^{\infty} \lambda_1^j L^j\right) \left(\sum_{j=0}^{\infty} \lambda_2^j L^j\right) \cdots \left(\sum_{j=0}^{\infty} \lambda_p^j L^j\right).
\end{aligned}$$

$ARMA(p, q)$ as $MA(\infty)$

Thus, if the roots of $\phi(L)$ all lie outside the unit circle,

$$Y_t = \mu + \psi(L)\varepsilon_t,$$

where $\mu = \phi(L)^{-1}c$ and $\psi(L) = \phi(L)^{-1}\theta(L)$.

- This restriction on the roots of $\phi(L)$ results in

$$\sum_{i=1}^{\infty} |\psi_i| < \infty.$$

- Hence, Y_t is an $MA(\infty)$ process and is weakly stationary.
- The stationarity of an $ARMA(p, q)$ depends only on $\{\phi_i\}_{i=1}^{p}$ and not on $\{\theta_i\}_{i=1}^{q}$.

Expectation of $ARMA(p, q)$

Assume Y_t is weakly stationary: the roots of $\phi(L)$ lie outside the unit circle.

$$
\begin{aligned}
\mathrm{E}\left[Y_t\right] &= c + \phi_1 \mathrm{E}\left[Y_{t-1}\right] + \ldots + \phi_p \mathrm{E}\left[Y_{t-p}\right] \\
&\qquad + \mathrm{E}\left[\varepsilon_t\right] + \theta_1 \mathrm{E}\left[\varepsilon_{t-1}\right] + \ldots + \theta_q \mathrm{E}\left[\varepsilon_{t-q}\right] \\
&= c + \phi_1 \mathrm{E}\left[Y_t\right] + \ldots + \phi_p \mathrm{E}\left[Y_t\right] \\
\Rightarrow \mathrm{E}\left[Y_t\right] &= \frac{c}{1 - \phi_1 - \ldots - \phi_p} = \mu.
\end{aligned}
$$

- This is the same mean as an $AR(p)$ process with parameters c and $\{\phi_i\}_{i=1}^{p}$.

Autocovariances of $ARMA(p, q)$

Given that $\mu = c/(1 - \phi_1 - \ldots - \phi_p)$ for weakly stationary Y_t:

$$
\begin{aligned}
Y_t &= \mu(1 - \phi_1 - \ldots - \phi_p) + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\
&\qquad + \varepsilon_t + \theta_1 \varepsilon_1 + \ldots \theta_q \varepsilon_{t-q} \\
\Rightarrow (Y_t - \mu) &= \phi_1(Y_{t-1} - \mu) + \ldots + \phi_p(Y_{t-p} - \mu) \\
&\qquad + \varepsilon_t + \theta_1 \varepsilon_1 + \ldots \theta_q \varepsilon_{t-q}.
\end{aligned}
$$

$$
\begin{aligned}
\gamma_j &= \mathrm{E}\left[(Y_t - \mu)(Y_{t-j} - \mu)\right] \\
&= \phi_1 \mathrm{E}\left[(Y_{t-1} - \mu)(Y_{t-j} - \mu)\right] + \ldots \\
&\qquad + \phi_p \mathrm{E}\left[(Y_{t-p} - \mu)(Y_{t-j} - \mu)\right] \\
&\qquad\qquad + \mathrm{E}\left[\varepsilon_t (Y_{t-j} - \mu)\right] + \theta_1 \mathrm{E}\left[\varepsilon_{t-1}(Y_{t-j} - \mu)\right] \\
&\qquad\qquad\qquad + \ldots + \theta_q \mathrm{E}\left[\varepsilon_{t-q}(Y_{t-j} - \mu)\right]
\end{aligned}
$$

Autocovariances of $ARMA(p,q)$

- For $j > q$, γ_j will follow the same law of motion as for an $AR(p)$ process:

$$\gamma_j = \phi_1 \gamma_{j-1} + \ldots + \phi_p \gamma_{j-p} \quad \text{for } j = q+1, \ldots$$

- These values will not be the same as the $AR(p)$ values for $j = q+1, \ldots$, since the initial $\gamma_0, \ldots, \gamma_q$ will differ.

- The first q autocovariances, $\gamma_0, \ldots, \gamma_q$, of an $ARMA(p,q)$ will be more complicated than those of an $AR(p)$.

Redundancy of $ARMA(p, q)$

Factoring the polynomials $\phi(L)$ and $\theta(L)$, an $ARMA(p, q)$ can be written as

$$(1 - \lambda_1 L) \cdots (1 - \lambda_p L)(Y_t - \mu) = (1 - \eta_1 L) \cdots (1 - \eta_q L)\varepsilon_t.$$

- If two of the roots are identical, $\lambda_i = \eta_j$, both polynomials can be divided by $(1 - \lambda_i L)$.
- The result would be an $ARMA(p - 1, q - 1)$:

$$(1 - \phi_1^* L - \ldots - \phi_{p-1}^* L^p)(Y_t - \mu) = (1 + \theta_1^* L + \ldots + \theta_{q-1}^* L^q)\varepsilon_t.$$