# Moving Average Processes

## Econ 211C – Unit 1, Section 3

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# White Noise Revisited

White noise, $\boldsymbol{\varepsilon}_{\mathcal{T}}$ is a fundamental building block of canonical time series processes.

- For most of this course we will assume $\mathcal{T} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- That is, $\boldsymbol{\varepsilon}_{\mathcal{T}} = \{\varepsilon_t\}_{t=-\infty}^{\infty}$.
- We will often use the abbreviation $\{\varepsilon_t\}$.

# $MA(1)$

Given white noise $\{\varepsilon_t\}$, consider the process

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1},$$

where $\mu$ and $\theta$ are constants.

- This is a *first-order moving average* or $MA(1)$ process.
- We can rewrite in terms of the lag operator:

  $$Y_t = c + \theta(L)\varepsilon_t, .$$

  where $\theta(L) = (1 + \theta L)$.

# $MA(1)$ Mean and Variance

The mean of the first-order moving average process is

$$\begin{aligned}
\mathrm{E}\left[Y_t\right] &= \mathrm{E}\left[\mu + \varepsilon_t + \theta\varepsilon_{t-1}\right] \\
&= \mu + \mathrm{E}\left[\varepsilon_t\right] + \theta\mathrm{E}\left[\varepsilon_{t-1}\right] \\
&= \mu.
\end{aligned}$$

# $MA(1)$ Autocovariances

$$
\begin{aligned}
\gamma_j &= \mathrm{E}\left[(Y_t-\mu)(Y_{t-j}-\mu)\right] \\
&= \mathrm{E}\left[(\varepsilon_t+\theta\varepsilon_{t-1})(\varepsilon_{t-j}+\theta\varepsilon_{t-j-1})\right] \\
&= \mathrm{E}\left[\varepsilon_t\varepsilon_{t-j}+\theta\varepsilon_t\varepsilon_{t-j-1}+\theta\varepsilon_{t-1}\varepsilon_{t-j}+\theta^2\varepsilon_{t-1}\varepsilon_{t-j-1}\right] \\
&= \mathrm{E}\left[\varepsilon_t\varepsilon_{t-j}\right]+\theta\mathrm{E}\left[\varepsilon_t\varepsilon_{t-j-1}\right]+\theta\mathrm{E}\left[\varepsilon_{t-1}\varepsilon_{t-j}\right]+\theta^2\mathrm{E}\left[\varepsilon_{t-1}\varepsilon_{t-j-1}\right].
\end{aligned}
$$

- If $j=0$

  $$\gamma_0=\mathrm{E}\left[\varepsilon_t^2\right]+\theta\mathrm{E}\left[\varepsilon_t\varepsilon_{t-1}\right]+\theta\mathrm{E}\left[\varepsilon_{t-1}\varepsilon_t\right]+\theta^2\mathrm{E}\left[\varepsilon_{t-1}^2\right]=(1+\theta^2)\sigma^2.$$

- If $j=1$

  $$\gamma_1=\mathrm{E}\left[\varepsilon_t\varepsilon_{t-1}\right]+\theta\mathrm{E}\left[\varepsilon_t\varepsilon_{t-2}\right]+\theta\mathrm{E}\left[\varepsilon_{t-1}^2\right]+\theta^2\mathrm{E}\left[\varepsilon_{t-1}\varepsilon_{t-2}\right]=\theta\sigma^2.$$

- If $j>1$, all of the expectations are zero: $\gamma_j=0$.

# $MA(1)$ Stationarity and Ergodicity

Since the mean and autocovariances are independent of time, an $MA(1)$ is weakly stationary.

- This is true *for all values of* $\theta$

The condition for ergodicity of the mean also holds:

$$\begin{aligned}\sum_{j=0}^{\infty}|\gamma_j| &= \gamma_0 + \gamma_1 \\ &= (1+\theta^2)\sigma^2 + \left|\theta\sigma^2\right| < \infty\end{aligned}$$

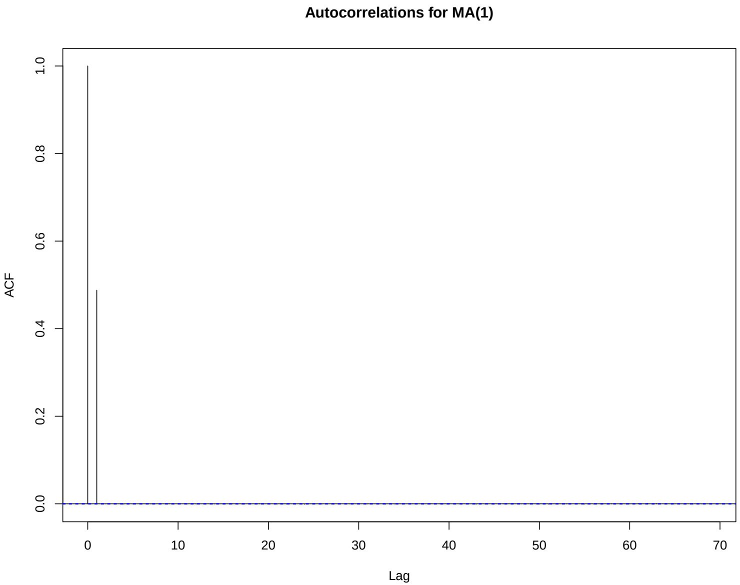
- If $\{\varepsilon_t\}$ is *Gaussian* then $\{Y_t\}$ is also ergodic *for all moments.*

# $MA(1)$ Autocorrelations

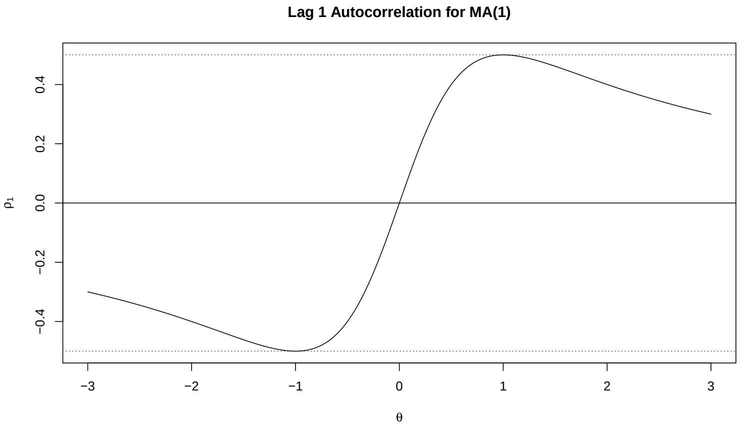
The autocorrelations of an $MA(1)$ are

- $j = 0$: $\rho_0 = 1$ (*always*).
- $j = 1$:
$$\rho_1 = \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2}$$
- $j > 1$: $\rho_j = 0$.
- If $\theta > 0$, first-order lags of $Y_t$ are *positively* autocorrelated.
- If $\theta < 0$, first-order lags of $Y_t$ are *negatively* autocorrelated.
- $\max\{\rho_1\} = 0.5$ and occurs when $\theta = 1$.
- $\min\{\rho_1\} = -0.5$ and occurs when $\theta = -1$.

# $MA(1)$ Autocorrelations



# $MA(1)$ Autocorrelations



# $MA(1)$ Autocorrelations

From the figure above we see that there are two values of $\theta$ that generate each value of $\rho_1$.

- In fact, $\theta$ and $1/\theta$ correspond to the same $\rho_1$:

$$\rho_1 = \frac{1/\theta}{1 + (1/\theta)^2} = \frac{\theta^2}{\theta^2} \frac{1/\theta}{1 + (1/\theta)^2} = \frac{\theta}{1 + \theta^2}.$$

- Consider:

$$Y_t = \varepsilon_t + 0.5\varepsilon_{t-1}$$
$$Y_t = \varepsilon_t + 2\varepsilon_{t-1}$$

- Then:

$$\rho_1 = \frac{0.5}{1 + 0.5^2} = \frac{2}{1 + 2^2} = 0.4.$$

# $MA(q)$

A *qth-order moving average* or $MA(q)$ process is

$$Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q},$$

where $\mu, \theta_1, \ldots, \theta_q$ are any real numbers.

- We can rewrite in terms of the lag operator:

  $$Y_t = \mu + \theta(L)\varepsilon_t,$$

  where $\theta(L) = (1 + \theta_1 L^1 + \ldots + \theta_q L^q)$.

# $MA(q)$ Mean

As with the $MA(1)$:

$$\begin{aligned} \mathrm{E}[Y_t] &= \mathrm{E}[\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}] \\ &= \mu + \mathrm{E}[\varepsilon_t] + \theta_1 \mathrm{E}[\varepsilon_{t-1}] + \ldots + \theta_q \mathrm{E}[\varepsilon_{t-q}] \\ &= \mu. \end{aligned}$$

# $MA(q)$ Autocovariances

$$\begin{aligned}\gamma_j &= \mathrm{E}\left[(Y_t - \mu)(Y_{t-j} - \mu)\right] \\ &= \mathrm{E}\Big[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}) \\ &\qquad\qquad\qquad \times (\varepsilon_{t-j} + \theta_1\varepsilon_{t-j-1} + \ldots + \theta_q\varepsilon_{t-j-q})\Big].\end{aligned}$$

- For $j > q$, all of the products result in zero expectations: $\gamma_j = 0$, for $j > q$.
- For $j = 0$, the squared terms result in nonzero expectations, while the cross products lead to zero expectations:

$$\gamma_0 = \mathrm{E}\left[\varepsilon_t^2\right] + \theta_1^2\mathrm{E}\left[\varepsilon_{t-1}^2\right] + \ldots + \theta_q^2\mathrm{E}\left[\varepsilon_{t-q}^2\right] = \left(1 + \sum_{j=1}^{q}\theta_j^2\right)\sigma^2.$$

# $MA(q)$ Autocovariances

- For $j = \{1, 2, \ldots, q\}$, the nonzero expectation terms are

$$\begin{aligned}\gamma_j &= \theta_j \mathrm{E}\left[\varepsilon_{t-j}^2\right] + \theta_{j+1}\theta_1 \mathrm{E}\left[\varepsilon_{t-j-1}^2\right] \\ &\qquad\qquad + \theta_{j+2}\theta_2 \mathrm{E}\left[\varepsilon_{t-j-2}^2\right] + \ldots + \theta_q \theta_{q-j} \mathrm{E}\left[\varepsilon_{t-q}^2\right] \\ &= (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \ldots + \theta_q\theta_{q-j})\sigma^2.\end{aligned}$$

The autocovariances can be stated concisely as

$$\gamma_j = \begin{cases} (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \ldots + \theta_q\theta_{t-q})\sigma^2 & \text{for } j = 0, 1, \ldots, q \\ 0 & \text{for } j > q. \end{cases}$$

where $\theta_0 = 1$.

# $MA(q)$ Autocorrelations

The autocorrelations can be stated concisely as

$$\rho_j = \begin{cases} \frac{\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + \ldots + \theta_q \theta_{t-q}}{\theta_0^2 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2} & \text{for } j = 0, 1, \ldots, q \\ 0 & \text{for } j > q. \end{cases}$$

where $\theta_0 = 1$.

# $MA(2)$ Example

For an $MA(2)$ process

$$\begin{aligned}
\gamma_0 &= (1 + \theta_1^2 + \theta_2^2)\sigma^2 \\
\gamma_1 &= (\theta_1 + \theta_2\theta_1)\sigma^2 \\
\gamma_2 &= \theta_2\sigma^2 \\
\gamma_3 &= \gamma_4 = \ldots = 0.
\end{aligned}$$

# $MA(q)$ Stationarity and Ergodicity

Since the mean and autocovariances are independent of time, an $MA(q)$ is weakly stationary.

- This is true *for all values of* $\{\theta_j\}_{j=1}^{q}$.

The condition for ergodicity of the mean also holds:

$$\sum_{j=0}^{\infty} |\gamma_j| = \sum_{j=0}^{q} |\gamma_j| < \infty.$$

- If $\{\varepsilon_t\}$ is *Gaussian* then $\{Y_t\}$ is also ergodic *for all moments*.

# $MA(\infty)$

If $\theta_0 = 1$, the $MA(q)$ process can be written as

$$Y_t = \mu + \sum_{j=0}^{q} \theta_j \varepsilon_{t-j}.$$

- If we take the limit $q \to \infty$:

$$Y_t = \mu + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} = \mu + \theta(L)\varepsilon_t,$$

where $\theta(L) = \sum_{j=0}^{\infty} \theta_j L^j$.

- It can be shown that an $MA(\infty)$ process is weakly stationary if

$$\sum_{j=0}^{\infty} \theta_j^2 < \infty.$$

# $MA(\infty)$

Since absolute summability implies square summability

$$\sum_{j=0}^{\infty} |\theta_j| \Rightarrow \sum_{j=0}^{\infty} \theta_j^2,$$

an $MA(\infty)$ process satisfying absolute summability is also weakly stationary.

- In general

$$\sum_{j=0}^{\infty} \theta_j^2 \nRightarrow \sum_{j=0}^{\infty} |\theta_j|.$$

# $MA(\infty)$ Moments

Following the same reasoning as above,

$$\mathrm{E}\left[Y_t\right] = \mu$$

$$\gamma_j = \sigma^2 \sum_{i=0}^{\infty} \theta_{j+i}\theta_i.$$

- $\sum_{j=0}^{\infty} |\theta_j| \Rightarrow \sum_{j=0}^{\infty} |\gamma_j|$.
- So if the $MA(\infty)$ has absolutely summable coefficients, it is ergodic for the mean.
- Further, if $\{\varepsilon_t\}$ is *Gaussian* then $\{Y_t\}$ is also ergodic *for all moments*.