Autoregressive Processes

Econ 211C – Unit 1, Section 4

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$AR(1)$ Process

Given white noise $\{\varepsilon_t\}$, consider the process

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t,$$

where c and ϕ are constants.

- This is a *first-order autoregressive* or $AR(1)$ process.
- We can rewrite in terms of the lag operator:

$$(1 - \phi L) Y_t = c + \varepsilon_t.$$

$AR(1)$ as $MA(\infty)$

From our discussion of lag operators, we know that if $|\phi| < 1$

$$
\begin{aligned}
Y_t &= (1-\phi L)^{-1}c + (1-\phi L)^{-1}\varepsilon_t \\
&= \left(\sum_{i=0}^{\infty} \phi^i L^i\right) c + \left(\sum_{i=0}^{\infty} \phi^i L^i\right) \varepsilon_t \\
&= \left(\sum_{i=0}^{\infty} \phi^i\right) c + \left(\sum_{i=0}^{\infty} \phi^i L^i\right) \varepsilon_t \\
&= \frac{c}{1-\phi} + \theta(L)\varepsilon_t,
\end{aligned}
$$

where

$$
\theta(L) = \sum_{i=0}^{\infty} \theta_i L^i = \sum_{i=0}^{\infty} \phi^i L^i = \phi(L)^{-1}.
$$

$AR(1)$ as $MA(\infty)$

Restating, when $|\phi| < 1$

$$Y_t = \frac{c}{1-\phi} + \theta(L)\varepsilon_t = \frac{c}{1-\phi} + \sum_{i=0}^{\infty} \phi^i \varepsilon_{t-i}.$$

- This is an $MA(\infty)$ with $\mu = c/(1-\phi)$ and $\theta_i = \phi^i$.
- Note that $|\phi| < 1$ implies

$$\sum_{j=0}^{\infty} |\theta_j| = \sum_{j=0}^{\infty} |\phi|^j < \infty,$$

  which means that Y_t is weakly stationary.

Expectation of $AR(1)$

Assume Y_t is weakly stationary: $|\phi| < 1$.

$$\begin{aligned} \mathrm{E}\left[Y_t\right] &= c + \phi \mathrm{E}\left[Y_{t-1}\right] + \mathrm{E}\left[\varepsilon_t\right] \\ &= c + \phi \mathrm{E}\left[Y_t\right] \\ \Rightarrow \mathrm{E}\left[Y_t\right] &= \frac{c}{1-\phi}. \end{aligned}$$

A Useful Property

If Y_t is weakly stationary,

$$Y_{t-j} - \mu = \sum_{i=0}^{\infty} \phi^i \varepsilon_{t-j-i}.$$

- That is, for $j \geq 1$, Y_{t-j} is a function of lagged values of ε_t and not ε_t itself.
- As a result, for $j \geq 1$

$$\mathrm{E}\left[(Y_{t-j} - \mu)\varepsilon_t\right] = \sum_{i=0}^{\infty} \phi^i \mathrm{E}\left[\varepsilon_t \varepsilon_{t-j-i}\right] = 0.$$

Variance of $AR(1)$

Given that $\mu = c/(1-\phi)$ for weakly stationary Y_t:

$$\begin{aligned} Y_t &= \mu(1-\phi) + \phi Y_{t-1} + \varepsilon_t \\ \Rightarrow (Y_t - \mu) &= \phi(Y_{t-1} - \mu) + \varepsilon_t. \end{aligned}$$

Squaring both sides and taking expectations:

$$\begin{aligned} \mathrm{E}\left[(Y_t-\mu)^2\right] &= \phi^2 \mathrm{E}\left[(Y_{t-1}-\mu)^2\right] + 2\phi \mathrm{E}\left[(Y_{t-1}-\mu)\varepsilon_t\right] + \mathrm{E}\left[\varepsilon_t^2\right] \\ &= \phi^2 \mathrm{E}\left[(Y_t-\mu)^2\right] + \sigma^2 \end{aligned}$$

$$\begin{aligned} &\Rightarrow (1-\phi^2)\gamma_0 = \sigma^2 \\ &\Rightarrow \gamma_0 = \frac{\sigma^2}{1-\phi^2} \end{aligned}$$

Autocovariances of $AR(1)$

For $j \geq 1$,

$$\begin{aligned}
\gamma_j &= \mathrm{E}\left[(Y_t - \mu)(Y_{t-j} - \mu)\right] \\
&= \phi \mathrm{E}\left[(Y_{t-1} - \mu)(Y_{t-j} - \mu)\right] + \mathrm{E}\left[\varepsilon_t (Y_{t-j} - \mu)\right] \\
&= \phi \gamma_{j-1} \\
&\vdots \\
&= \phi^j \gamma_0 .
\end{aligned}$$

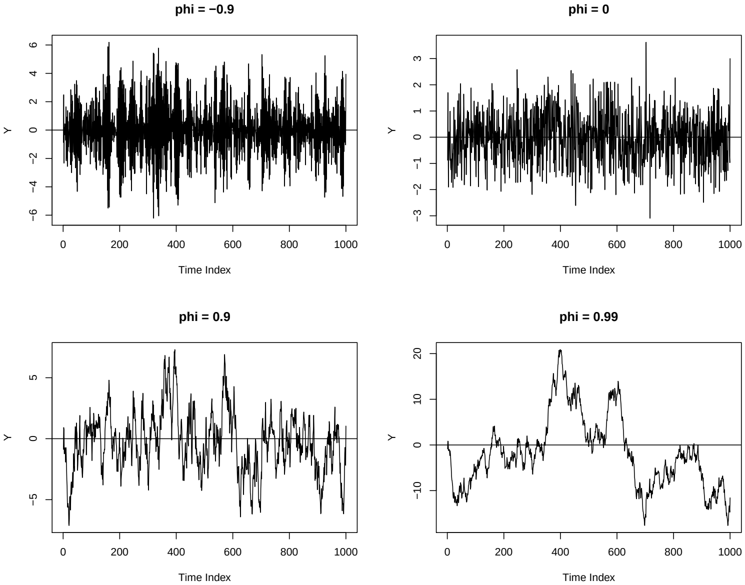
Autocorrelations of $AR(1)$

The autocorrelations of an $AR(1)$ are

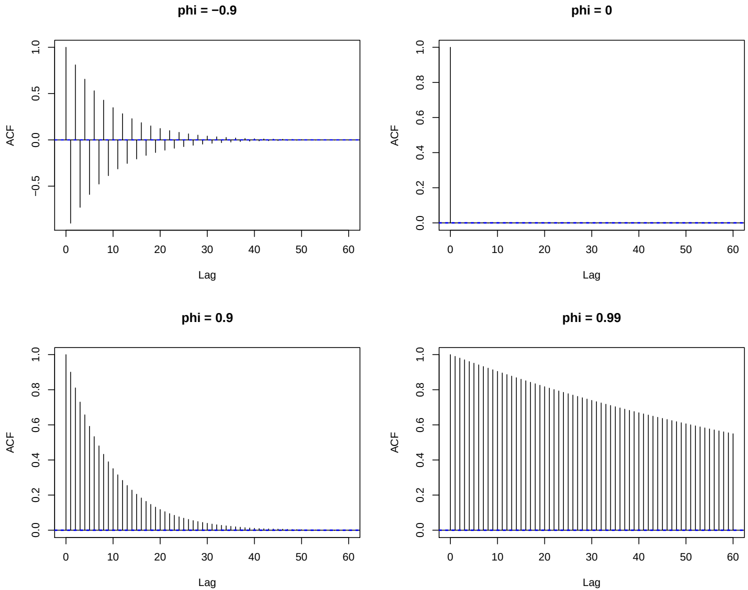
$$\rho_j = \frac{\gamma_j}{\gamma_0} = \phi^j, \quad \forall j \geq 0.$$

- Since we assumed $|\phi| < 1$, the autocorrelations decay exponentially as j increases.
- Note that if $\phi \in (-1, 0)$, the autocorrelations decay in an oscillatory fashion.

Examples of Realizations of $AR(1)$ Processes



$AR(1)$ Autocorrelations



$AR(p)$ Process

Given white noise $\{\varepsilon_t\}$, consider the process

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t,$$

where c and $\{\phi\}_{i=1}^{p}$ are constants.

- This is a *pth-order autoregressive* or $AR(p)$ process.
- We can rewrite in terms of the lag operator:

$$\phi(L) Y_t = c + \varepsilon_t.$$

  where

$$\phi(L) = (1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p).$$

$AR(p)$ as $MA(\infty)$

From our discussion of lag operators,

$$Y_t = \phi(L)^{-1}c + \phi(L)^{-1}\varepsilon_t,$$

if the roots of $\phi(L)$ all lie outside the unit circle.

- In this case, $\phi(L) = (1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_p L)$.
- If the roots, $\frac{1}{|\lambda_i|} > 1,\ \forall i$ then $|\lambda_i| < 1,\ \forall i$ and

$$\begin{aligned}\phi(L)^{-1} &= (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}\cdots(1-\lambda_p L)^{-1}\\ &= \left(\sum_{j=0}^{\infty}\lambda_1^j L^j\right)\left(\sum_{j=0}^{\infty}\lambda_2^j L^j\right)\cdots\left(\sum_{j=0}^{\infty}\lambda_p^j L^j\right).\end{aligned}$$

$AR(p)$ as $MA(\infty)$

For $|\lambda_i| < 1$, $\forall i$

- Y_t is an $MA(\infty)$ with $\mu = \phi(L)^{-1}c$ and $\theta(L) = \phi(L)^{-1}$.
- It can be shown that $\sum_{i=1}^{\infty} |\theta_i| < \infty$.
- As a result, Y_t is weakly stationary.

Vector Autoregressive Process

We can rewrite the $AR(p)$ as

$$\boldsymbol{Y}_t = \boldsymbol{c} + \Phi \boldsymbol{Y}_{t-1} + \boldsymbol{\varepsilon}_t,$$

where

$$\boldsymbol{Y}_t = \begin{bmatrix} Y_t \\ Y_{t-1} \\ Y_{t-2} \\ \vdots \\ Y_{t-p+1} \end{bmatrix} \quad \Phi = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad \boldsymbol{\varepsilon}_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and $\boldsymbol{c} = (c, c, \dots, c)'_{1\times p}$.

Vector Autoregressive Process

It turns out that the values $\{\lambda_i\}_{i=1}^{p}$ are the p eigenvalues of Φ.

- So the eigenvalues of Φ are the inverses of the roots of the lag polynomial $\phi(L)$.
- Hence, $\phi(L)^{-1}$ exists if all p roots of $\phi(L)$ lie *outside* the unit circle or all p eigenvalues of Φ lie *inside* the unit circle.
- These conditions ensure weak stationarity of the $AR(p)$ process.

Expectation of $AR(p)$

Assume Y_t is weakly stationary: the roots of $\phi(L)$ lie outside the unit circle.

$$
\begin{aligned}
\mathrm{E}\left[Y_t\right] &= c + \phi_1 \mathrm{E}\left[Y_{t-1}\right] + \ldots + \phi_p \mathrm{E}\left[Y_{t-p}\right] + \mathrm{E}\left[\varepsilon_t\right] \\
&= c + \phi_1 \mathrm{E}\left[Y_t\right] + \ldots + \phi_p \mathrm{E}\left[Y_t\right] \\
\Rightarrow \mathrm{E}\left[Y_t\right] &= \frac{c}{1 - \phi_1 - \ldots - \phi_p} = \mu.
\end{aligned}
$$

Autocovariances of $AR(p)$

Given that $\mu = c/(1-\phi_1-\ldots-\phi_p)$ for weakly stationary Y_t:

$$\begin{aligned} Y_t &= \mu(1-\phi_1-\ldots-\phi_p)+\phi_1 Y_{t-1}+\ldots+\phi_p Y_{t-p}+\varepsilon_t \\ \Rightarrow (Y_t-\mu) &= \phi_1(Y_{t-1}-\mu)+\ldots+\phi_p(Y_{t-p}-\mu)+\varepsilon_t. \end{aligned}$$

Thus,

$$\begin{aligned} \gamma_j &= \mathrm{E}\left[(Y_t-\mu)(Y_{t-j}-\mu)\right] \\ &= \phi_1 \mathrm{E}\left[(Y_{t-1}-\mu)(Y_{t-j}-\mu)\right]+\ldots \\ &\qquad +\phi_p \mathrm{E}\left[(Y_{t-p}-\mu)(Y_{t-j}-\mu)\right]+\mathrm{E}\left[\varepsilon_t(Y_{t-j}-\mu)\right] \\ &= \begin{cases} \phi_1\gamma_{j-1}+\ldots+\phi_p\gamma_{j-p} & \text{for } j=1,\ldots \\ \phi_1\gamma_1+\ldots+\phi_p\gamma_p+\sigma^2 & \text{for } j=0. \end{cases} \end{aligned} \tag{1}$$

Autocovariances of $AR(p)$

For $j = 0, 1, \ldots, p$, System (1) is a system of $p + 1$ equations with $p + 1$ unknowns: $\{\gamma_j\}_{j=0}^{p}$.

- $\{\gamma_j\}_{j=0}^{p}$ can be solved for as functions of $\{\phi_j\}_{j=1}^{p}$ and σ^2.
- It can be shown that $\{\gamma_j\}_{j=0}^{p}$ are the first p elements of the first column of $\sigma^2[I_{p^2} - \Phi \otimes \Phi]^{-1}$, where $\otimes$ denotes the Kronecker product.
- $\{\gamma_j\}_{j=p+1}^{\infty}$ can then be determined using prior values of γ_j and $\{\phi_j\}_{j=1}^{p}$.

Autocorrelations of $AR(p)$

Dividing the autocovariances by γ_0,

$$\rho_j = \phi_1 \rho_{j-1} + \ldots + \phi_p \rho_{j-p} \quad \text{for } j = 1, \ldots$$