Over Identifying Restrictions

Since

$\sqrt{T}\boldsymbol{g}_T(\boldsymbol{\theta}_0) \longrightarrow N(0, S)$

$(\sqrt{T}\boldsymbol{g}_T(\boldsymbol{\theta}_0)')S^{-1}(\sqrt{T}\boldsymbol{g}_T(\boldsymbol{\theta}_0)) = T\boldsymbol{g}_T(\boldsymbol{\theta}_0)'S^{-1}\boldsymbol{g}_T(\boldsymbol{\theta}_0) \xrightarrow{d} \chi^2(r)$

where $r > k$ is the number of moment conditions.

It turns out that

$T\boldsymbol{g}_T(\hat{\theta})'\hat{S}^{-1}\boldsymbol{g}_T(\hat{\theta}) \cancel{\xrightarrow{d}} \chi^2(r)$

This is because k moment conditions will be set to zero exactly.

Consider $r = k$. In this case $\boldsymbol{g}_T(\hat{\theta}) = 0$ exactly and $T\boldsymbol{g}_T(\hat{\theta})'\hat{S}^{-1}\boldsymbol{g}_T(\hat{\theta})$

$r - k$ of the moment conditions will be non-zero.

Thus,

$J_T(\hat{\theta}) = T\boldsymbol{g}_T(\hat{\theta})'\hat{S}^{-1}\boldsymbol{g}_T(\hat{\theta}) \xrightarrow{d} \chi^2(r-k)$

To test if our moment conditions are close to zero, we compute $J_T(\hat{\theta})$ and compare with a $\chi^2(r-k)$ distribution

If $J_T(\hat{\theta})$ is far in the tail of the $\chi^2(r-k)$ distribution, we might conclude that the model is misspecified.

Asset Pricing with GMM

Suppose an agent derives utility from consumption, c_t, and seeks to maximze the discounted sum of expected utility.

$\sum_{\tau=0}^{\infty} \beta^\tau E[u(c_{t+\tau})|\Omega_t]$

where $u(c_t)$ is the period utility function and satisfies:

$\frac{\partial u(c_t)}{\partial c_t} > 0$ and $\frac{\partial^2 u(c_t)}{\partial c_t^2} < 0$

Suppose that the agent can purchase m assets paying gross returns $(1 + r_{i,t+1})$ between periods t and $t+1$ for $i = 1, ...m$.

The agent's portfolio must satisfy

$u'(c_t) = \beta E[(1 + r_{i,t+1})u'(c_{t+1})|\Omega_t]$ for $i = 1, ...m$

These conditions say that marginal utility of consuming an extra unit today should be equivalent to the expected marginal consumption gained by purchasing a unit of any asset.

If these conditions didn't hold, the agent wouldn't be at an optimum.

The portfolio conditions can be rewritten as:

$E\left[\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}(1 + r_{i,t+1}) - 1\right)\Big|\Omega_t\right] = 0$ for $i = 1, ..., m$.

Given a vector $\boldsymbol{x}_t \in \Omega_t$, by the law of iterated expectations

$$E\left[E\left[\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}(1+r_{i,t+1})-1\right)\boldsymbol{x}_t\middle|\Omega_t\right]\right]=E\left[\underbrace{\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}(1+r_{i,t+1})-1\right)\boldsymbol{x}_t}_{\boldsymbol{h}(\boldsymbol{\theta},\boldsymbol{y}_t)}\right]=0 \text{ for } i=1,...m$$

Economic theory says that all returns discounted by $\beta\frac{u'(c_{t+1})}{u'(c_t)}$ should be identical:

$$E\left[\underbrace{\beta\frac{u'(c_{t+1})}{u'(c_t)}}_{m_{t,t+1}}(1+r_{i,t+1})\right]=1 \implies E[m_{t,t+1}(1+r_{i,t+1})]=1$$

$\beta\frac{u'(c_{t+1})}{u'(c_t)}(1+r_{i,t+1})-1$ is a forecast error and should be uncorrelated with any variable $\boldsymbol{x}_t\in\Omega_t$

This problem maps easily into GMM where

$\boldsymbol{y}_t=(r_{1,+1},...,r_{m,t+1},c_t,c_{t+1},\boldsymbol{x}_t')'$

$$\boldsymbol{h}(\boldsymbol{\theta},\boldsymbol{y}_t)=\begin{bmatrix}\left(1-\beta\frac{u'(c_{t+1})}{u'(c_t)}(1+r_{i,t+1})\right)\boldsymbol{x}_t\\ \vdots\\ \left(1-\beta\frac{u'(c_{t+1})}{u'(c_t)}(1+r_{m,t+1})\right)\boldsymbol{x}_t\end{bmatrix}$$

$$\boldsymbol{g}_T(\boldsymbol{\theta})=\frac{1}{T}\sum_{t=0}^{T}\boldsymbol{h}(\boldsymbol{\theta},\boldsymbol{y}_t)$$

$\theta=\beta$ and utility function parameters

Since the forecast errors in $\boldsymbol{h}(\boldsymbol{\theta},\boldsymbol{y}_t)$ are unpredictable, they exhibit no serial correlation.

Thus, $\boldsymbol{h}(\boldsymbol{\theta},\boldsymbol{y}_t)$ exhibits no serial correlation.

This means S can be simply be estimated by

$$\hat{S}_T=\frac{1}{T}\sum_{t=0}^{T}\boldsymbol{h}(\hat{\theta},\boldsymbol{y}_t)\boldsymbol{h}(\hat{\theta},\boldsymbol{y}_t)'$$

Hansen and Singleton (1982) used GMM to estimate parameters of a model where

$$u(c_t)=\begin{cases}\frac{c_t^{1-\gamma}}{1-\gamma} & \text{for } \gamma>0 \text{ and } \gamma\neq 1\\ log(c_t) & \text{for } \gamma=1\end{cases}$$

In this case, $\boldsymbol{\theta}=(\beta,\gamma)'$.

Since forecast errors are uncorrelated with past returns and consumption, the lagged values of asset returns and aggregate consumption in $\boldsymbol{x}_t$