Macroeconomic Dynamics Near the ZLB: A Tale of Two Equilibria *

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Abstract

This paper studies the dynamics of a New Keynesian DSGE model near the zero lower bound (ZLB) on nominal interest rates. In addition to the standard targeted-inflation equilibrium, we consider a deflation equilibrium as well as a Markov sunspot equilibrium that switches between a targeted-inflation and a deflation regime. We use the particle filter to estimate the state of the U.S. economy in 2008 under the assumptions that the U.S. economy has been in either the targeted-inflation or the sunspot equilibrium. Under the sunspot equilibrium the observed dynamics in 2008-09 appear more likely than under the targeted-inflation equilibrium. We consider a combination of fiscal policy (calibrated to the American Recovery and Reinvestment Act) and monetary policy (that tries to keep interest rates near zero) and find that the ZLB severely limits the scope for policy interventions in the sunspot equilibrium. JEL CLASSIFICATION: C5, E4, E5

KEY WORDS: DSGE Models, Government Spending Multiplier, Multiple Equilibria, Nonlinear Filtering, Nonlinear Solution Methods, ZLB

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1 Introduction

Since the beginning of 2009 the U.S. Federal Funds rate has been effectively zero. Investors' access to money, which is an asset that in addition to providing transaction services yields a zero nominal return, prevents nominal interest rates from falling below zero and thereby creates a zero lower bound (ZLB). If an economy is at the ZLB, its central bank is unable to stimulate the economy using a conventional monetary policy that reduces interest rates. Traditionally, the ZLB has mostly been ignored in the specification of dynamic stochastic general equilibrium (DSGE) models that are tailored toward the analysis of U.S. monetary and fiscal policy. With the exception of a short period from 2003:Q3 to 2004:Q2 in which the Federal Funds rate dropped to about 1%, the ZLB did not appear to be empirically relevant. Moreover, accounting for the ZLB in a DSGE model complicates the quantitative analysis of the model considerably.

During the Great Recession of 2008-9 the ZLB has become empirically relevant for the U.S. and since then the literature on the analysis DSGE models with a ZLB constraint has been growing rapidly. Our paper contributes to this literature. We solve a small-scale nonlinear New Keynesian DSGE model with an explicit ZLB constraint using a global approximation to the agents' decision rules. Once the ZLB is explicitly included in a monetary model, there typically exist multiple equilibria. In addition to the widely-studied equilibrium in the neighborhood of a steady state in which actual inflation coincides with the central bank's inflation target (targeted-inflation equilibrium), this paper is the first to study two additional equilibria in a nonlinear DSGE model with a full set of structural shocks. Specifically we consider a minimal-state-variable equilibrium in which the endogenous variables fluctuate around a steady state with zero interest rates (deflation equilibrium); and an equilibrium in which the economy alternates between a targeted-inflation regime and a deflation regime according to the realization of a non-fundamental Markov-switching process (sunspot equilibrium). We show analytically, in a special case, and numerically in the more general cases, how these two equilibria behave differently than the targeted-inflation equilibrium, especially near the ZLB.

We use our model to study the following quantitative questions: conditional on the state of the U.S. economy in January 2009, what is the effect of an increase in government spending of the size of the federal contracts, grants, and loans portion of the American Recovery and Reinvestment Act (ARRA)? Does this effect get amplified by a monetary policy that keeps interest rates near the ZLB for an extended period of time? To answer these questions, we parameterize the DSGE model and use an auxiliary particle filter to extract the values for the model's state variables from U.S. data over the period 2000:Q1 to 2008:Q4 conditional on the three equilibria. The deflation equilibrium is empirically not viable because during most of the last decades inflation rates were positive. Thus, conditional on filtered states for 2008:Q4 we conduct policy experiments under the assumption that the economy is either in the targeted-inflation or in the sunspot equilibrium. We find that under the sunspot equilibrium the average government spending multiplier is about 0.6 in the short-run increases to 0.7 for the cumulative effect over three years. In the targeted-inflation equilibrium the short-run multiplier is also about 0.6 and it increases to about 1.0 over a three-year time span.

A monetary policy that keeps the interest rate at zero during 2009 provides hardly any additional stimulating effect in the sunspot equilibrium, because in 2008 the economy switches to the deflation regime, which implies that despite the fiscal stimulus, interest rates are expected to stay close to zero and further cuts are infeasible. Under the targeted-inflation equilibrium, on the other hand, the model predicts that in response to the fiscal stimulus interest rates start to rise and thus an expansionary monetary policy that keeps them at zero generates an additional increase in output that amplifies the fiscal multiplier to 1.5 in the short run and 2.4 over a three-year period. Overall, under the sunspot equilibrium the observed dynamics in 2008-09 appear more likely than under the targeted-inflation equilibrium. Thus, our analysis implies that in 2009 the ZLB severely had severely limited the scope for policy interventions and casts doubts on claims that fiscal multipliers during this period were much larger than one.

It has been well-known that monetary DSGE models with an explicit ZLB constraint deliver multiple equilibria. This issue was discussed, for instance, by Benhabib, Schmitt-Grohé, and Uribe (2001a,b). In a nutshell, the relationship between nominal interest rates and inflation in a DSGE model are characterized by a consumption Euler equation which embodies a version of the Fisher equation, and a monetary policy rule. The kink in the monetary policy rule induced by the ZLB tends to generate two pairs of steady-state interest and inflation rates that solve both equations. Moreover, near the deflation steady state the so-called *Taylor principle* is violated because nominal interest rates cannot aggressively respond to inflation. In turn, the model exhibits local indeterminacy. We discuss the properties of stochastic equilibria that have the targeted-inflation and deflation steady states as their steady state, respectively. Our policy analysis focuses on the targeted-inflation and a sunspot equilibrium. A sunspot equilibrium is also discussed in Mertens and Ravn (2012). However, unlike in our paper, the sunspot shock is the only shock driving their model. Moreover, Mertens and Ravn (2012) do not attempt to fit their sunspot model to actual data.

In terms of solution method, our work is most closely related to the papers by Judd, Maliar, and Maliar (2011) and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), both of which use global projection methods to approximate agents' decision rules. We study a small-scale New Keynesian DSGE model similar to theirs with two endogenous and three exogenous state variables. The solution is based on a variant of the ergodic-set method proposed by Judd, Maliar, and Maliar (2011). However, we consider several important modifications. First, we use a piece-wise smooth approximation with two separate functions characterizing the decisions when the ZLB is binding and when it is not, while the previous papers use smooth approximations with a single function covering the whole state space. This means all our decision rules allow for kinks at points in the state space where the ZLB becomes binding. The location of these points is determined endogenously. This difference in methodology proves to be very important in obtaining accurate approximations, especially for the deflation and sunspot equilibria. Second, since we are interested in fitting U.S. data from 2000 to 2010 and since some of the observations during this period lie far in the tails of the ergodic distribution of our model, we apply the ergodic-set method to state realizations that are in part obtained from simulating the model and in part from applying the particle filter to U.S. data.[1] Finally, both Judd, Maliar, and Maliar (2011) and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez

[1]This procedure is iterative: the simulated data as well as the filtered states are obtained from an initial approximation of the model.

(2012) solely compute what we call the targeted-inflation equilibrium, whereas we use the numerical methods to approximate two alternative equilibria.

Most of the other papers that study DSGE models with ZLB constraints take various shortcuts in their solution methods. Braun and Körber (2011) use a variant of extended shooting to solve a set of nonlinear equilibrium conditions. This method assumes that the system reaches its steady state after a fixed number of periods and at any point in time determines the agents' decision under the assumption of perfect foresight, setting future shocks to zero. Adam and Billi (2007) solve a linear-quadratic optimal policy problem with a linearized Euler equation and Phillips curve subject to a ZLB constraint. While the model is solved nonlinearly, it only contains two exogenous state variables. Eggertsson and Woodford (2003) consider a version of the New Keynesian DSGE model in which both the Euler equation and the Phillips curve are log-linearized and the natural rate of interest follows a two-state Markov process. The economy hits the ZLB when the natural rate turns negative. The subsequent exit from the ZLB is exogenous and occurs with a pre-specified probability. A similar approach is used by Christiano, Eichenbaum, and Rebelo (2011). Unfortunately, some of the DSGE model properties are very sensitive to the approximation technique and to implicit or explicit assumptions about the probability of leaving the ZLB. Detailed analyses are provided in Braun and Körber (2011) and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012).

The effect of an increase in government spending when the economy is at the ZLB is studied by Braun and Körber (2011), Christiano, Eichenbaum, and Rebelo (2011), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), and Mertens and Ravn (2012). Christiano, Eichenbaum, and Rebelo (2011) argue that the fiscal multiplier at the ZLB can be substantially larger than one. A rise in government spending increases output, marginal costs, and expected inflation. If the economy operates according to the Taylor rule, then the central bank will react to the rise in output and inflation by increasing the nominal interest rate. If the economy is at the ZLB and the systematic part of the Taylor rule is not operative, then the expected inflation translates into a fall in expected real rates, which in turn triggers additional consumption in the current period and raises output further.

Thus, the government spending multiplier crucially depends on whether the expansionary fiscal policy triggers an exit from the ZLB. In the target-inflation equilibrium of our model, spells of zero nominal interest rates are short and an expansionary fiscal policy triggers an immediate exit. In turn, the (short-run) fiscal multiplier is substantially less than one. In the sunspot-equilibrium, the economy is in the deflation regime in 2008:Q4 and expected to stay there for several quarters. Although the probability of the fiscal intervention triggering an exit from the ZLB is low the agents' decision rules in the deflation regime imply a government-spending multiplier that is only around 0.6.

The remainder of the paper is organized as follows. Section 2 presents a simple two-equation model that we use to illustrate the multiplicity of equilibria in monetary models with ZLB constraints. We also highlight the types of equilibria studied in this paper. The small-scale New Keynesian model that is used for the quantitative analysis is presented in Section 3. The solution of the model is discussed in Section 4. To fix ideas we first solve a version of the model without endogenous and exogenous persistence in which all equilibrium conditions except for the ZLB constraint are log-linearized. We then proceed with a description of the numerical solution algorithm for the full nonlinear model. Section 5 contains the quantitative analysis. We first illustrate the ergodic distribution of inflation and interest rates under the three equilibria considered in this paper, present some impulse response dynamics of the nonlinear model, and, at last, study the effects of fiscal interventions. Section 6 concludes. Detailed derivations, descriptions of algorithms, and additional quantitative results are summarized in an Online Appendix.

2 A Two-Equation Example

We begin with a simple two-equation example to illustrate the types of equilibria that arise if a ZLB constraint is imposed in a monetary DSGE model. The example is adapted from Benhabib, Schmitt-Grohé, and Uribe (2001a) and Hursey and Wolman (2010). Suppose that the economy can be described by the Fisher relationship

$$R_t = r\mathbb{E}_t[\pi_{t+1}] \tag{1}$$

and the monetary policy rule

$$R_t = \max\left\{1,\ r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\sigma\epsilon_t]\right\},\quad \epsilon_t \sim iidN(0,1),\quad \psi > 1. \tag{2}$$

Here R_t denotes the gross nominal interest rate, π_t is the gross inflation rate, and ϵ_t is a monetary policy shock. The gross nominal interest rate is bounded from below by one. Throughout this paper we refer to this bound as ZLB because it bounds the net interest rate from below by zero.[2] Combining (1) and (2) yields a nonlinear expectational difference equation for inflation

$$\mathbb{E}_t[\pi_{t+1}] = \max\left\{\frac{1}{r},\ \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\sigma\epsilon_t]\right\}. \tag{3}$$

This simple model has two steady states ($\sigma = 0$), which we call targeted-inflation steady state and deflation steady state, respectively. In the targeted-inflation steady state inflation equals π_* and the nominal interest rate is $R = r\pi_*$. In the deflation steady state inflation equals $\pi_D = 1/r$ and the nominal interest is $R_D = 1$. While this paper focuses on nonlinear solutions of DSGE models with ZLB constraints, it is instructive to first take a look at the equilibria that arise in a linear approximation. Taking a piece-wise log-linear approximation around the targeted-inflation steady state and denoting percentage deviation of inflation from this steady state by $\hat{\pi}_t = \ln(\pi_t/\pi_*)$ we obtain

$$\mathbb{E}_t[\hat{\pi}_{t+1}] = \max\left\{-\ln(r\pi_*),\ \psi\hat{\pi}_t + \sigma\epsilon_t\right\}.$$

If the shock standard deviation σ is small then the ZLB is essentially non-binding:

$$\mathbb{E}_t[\hat{\pi}_{t+1}] \approx \psi\hat{\pi}_t + \sigma\epsilon_t \tag{4}$$

and the linearized rational expectations system has the unique stable solution

$$\hat{\pi}_t \approx -\frac{1}{\psi}\sigma\epsilon_t. \tag{5}$$

Alternatively, one can approximate the dynamics of the system near the deflation steady state. Let $\tilde{\pi}_t = \ln(\pi_t/\pi_D)$. Then,

$$\mathbb{E}_t[\tilde{\pi}_{t+1}] = \max\left\{0,\ -(\psi-1)\ln(r\pi_*) + \psi\tilde{\pi}_t + \sigma\epsilon_t\right\}.$$

[2]One needs a model with money to motivate the existence of the ZLB but in this paper, as in many others, we simply assume it.

Since we assumed $\psi > 1$, the ZLB is binding with very probability if σ is small. This leads to

$$\mathbb{E}_t[\tilde{\pi}_{t+1}] \approx 0 \tag{6}$$

This linear rational expectation difference equation has many stable solutions. Following Lubik and Schorfheide (2004), we consider the set of solutions

$$\tilde{\pi}_t = -\frac{1}{\psi}(1+M)\sigma\epsilon_t + \zeta_t, \tag{7}$$

where M is some constant and ζ_t is a sunspot shock. Setting $M = 1$ and $\zeta_t = 0$ highlights that there is an approximate deflation equilibrium that mimics the fluctuations of the targeted-inflation equilibrium.

The multiplicity of solutions to piece-wise linear approximations of the difference equation (3) suggests that the nonlinear difference equation itself also has multiple stable solutions. It is beyond the scope of this paper to consider all of the solutions that arise in DSGE models with ZLB constraints. Instead, we focus on solutions that mimic the fluctuations near the targeted-inflation steady state. Two such solutions are given by

$$\pi_t^{(*)} = \pi_*\gamma_* \exp\left[-\frac{1}{\psi}\sigma\epsilon_t\right], \quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi-1)\psi^2}\right] \tag{8}$$

and

$$\pi_t^{(D)} = \pi_*\gamma_D \exp\left[-\frac{1}{\psi}\sigma\epsilon_t\right], \quad \gamma_D = \frac{1}{\pi_* r}\exp\left[-\frac{\sigma^2}{2\psi^2}\right]. \tag{9}$$

Notice that

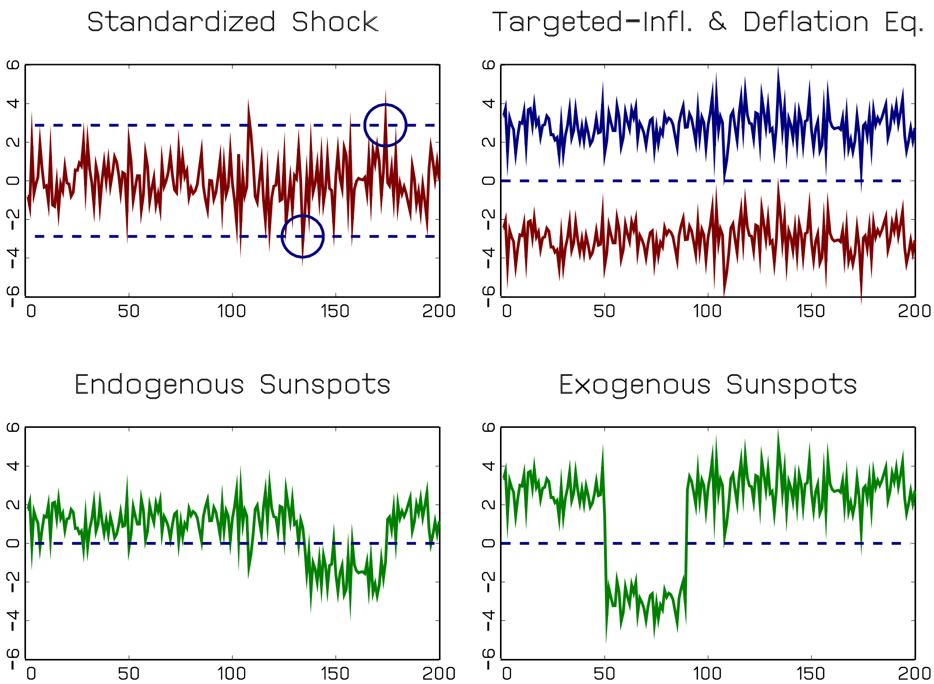
$$\ln\frac{\pi_t^{(*)}}{\pi_*\gamma_*} = \ln\frac{\pi_t^{(D)}}{\pi_*\gamma_D} = -\frac{1}{\psi}\sigma\epsilon_t$$

and equals the term on the right-hand-side of (5). Moreover, for small values of σ the constants $\pi_*\gamma_* \approx \pi_*$ and $\pi_*\gamma_D \approx 1/r_*$. Thus, we refer to $\pi_t^{(*)}$ as the targeted-inflation equilibrium and $\pi_t^{(D)}$ as the deflation equilibrium associated with (3).

In addition to the equilibria (8) and (9) we also consider a equilibria in which a sunspot s_t triggers moves from targeted-inflation to deflation and vice versa:

$$\pi_t^{(s)} = \pi_*\gamma(s_t)\exp\left[-\frac{1}{\psi}\sigma\epsilon_t\right]. \tag{10}$$

Figure 1: Inflation Dynamics in the Two-Equation Model



The sunspot shock $s_t \in \{0, 1\}$ evolves according to a two-state discrete Markov switching process. The constants $\gamma(0)$ and $\gamma(1)$ depend on the transition probabilities of the Markov switching process. The fluctuations of $\pi_t^{(s)}$ around $\pi_* \gamma(s_t)$ are identical to the fluctuations in the targeted-inflation and deflation equilibria. The sunspot process could either evolve independently from the fundamental shock or it could be correlated with ϵ_t.[3] For instance, conditional on $s_{t-1} = 1$ (targeted-inflation regime), suppose that $s_t = 0$, i.e., the economy transitions to the deflation regime, if a large negative shock occurs: $\epsilon_t < \underline{\epsilon}_1$. Similarly, the economy exits the deflation regime, if a large positive shock occurs: $\epsilon_t > \underline{\epsilon}_2$.

A numerical illustration is provided in Figure 1. The upper-left panel depicts the evolution of the shock ϵ_t. The upper-right panel compares the paths of net inflation under the targeted-inflation equilibrium and the deflation equilibrium. The difference between the inflation paths is the level shift due to the constants $\ln \gamma_*$ versus $\ln \gamma_D$. The bottom panel

[3]We thank Mike Woodford for the suggestion to explore equilibria in which the sunspot is triggered by fundamentals.

shows two sunspot equilibria with visible shifts from the targeted-inflation regime to the deflation regime and back. In the right panel the sunspot evolves exogenously, whereas on the left it is endogenous in the sense that it gets triggered by extreme realizations of ϵ_t, which are marked by circles in the top-left panel.

Before proceeding with a more complicated New Keynesian DSGE model we briefly comment on the treatment of fiscal policy in this paper. We will assume that fiscal policy is passive and that lump-sum taxes are used to balance revenues and outlays. Using the convention that B_t denotes the stock of nominal government bonds at the end of period t, a stylized government budget constraint of the two-equation model is given by

$$R_{t-1}B_{t-1} = B_t + T_t.$$

Dividing both sides by the price level P_t and denoting real government debt by $b_t = B_t/P_t$ and real lump-sum taxes by $\vartheta_t = T_t/P_t$ we obtain

$$b_{t-1}\frac{R_{t-1}}{\pi_t} = b_t + \vartheta_t.$$

A concern might be that in the deflation equilibrium the real value of government debt keeps growing. However, notice that the ex-post real return on government bonds is given by

$$\frac{R_{t-1}}{\pi_t^{(D)}} = r\exp\left[\frac{\sigma^2}{2\psi^2} + \frac{1}{\psi}\sigma\epsilon_t\right]$$

Thus, a stationary lump-sum tax process $\vartheta = b_* R_{t-1}/\pi_t^{(D)}$ will keep the real value of government debt at the level b_*.

There exist many other solutions to (3). For instance, one can use the logic of (7) to construct alternative deflation equilibria. Moreover, Benhabib, Schmitt-Grohé, and Uribe (2001a) studied solutions in which the economy transitions from the targeted-inflation regime to a deflation regime and remains in the deflation regime permanently in continuous-time perfect foresight monetary models. Some of these equilibria are discussed further in the Online Appendix. In the remainder of this paper we will focus on equilibria of a small-scale New Keynesian DSGE model that are akin to $\pi_t^{(*)}$, $\pi_t^{(D)}$, and $\pi_t^{(s)}$ with exogenously evolving sunspot shock.

3 A Prototypical New Keynesian DSGE Model

The DSGE model we consider is the small-scale New Keynesian model studied in An and Schorfheide (2007). The model economy consists of perfectly competitive final-goods-producing firms, a continuum of monopolistically competitive intermediate goods producers, a continuum of identical households, and a government that engages in active monetary and passive fiscal policy. This model has been widely studied in the literature and many of its properties are discussed in the textbook by Woodford (2003). To keep the dimension of the state space manageable we abstract from capital accumulation and wage rigidities. We describe the preferences and technologies of the agents in Section 3.1, summarize the equilibrium conditions in Section 3.2, and characterize the steady states of the model in Section 3.3.

3.1 Preferences and Technologies

Households. Households derive utility from consumption C_t relative to an exogenous habit stock and disutility from hours worked H_t. We assume that the habit stock is given by the level of technology A_t, which ensures that the economy evolves along a balanced growth path despite the quasi-linear preferences. We also assume that the households value transaction services from real money balances, detrended by A_t, and include them in the utility function. The households maximize

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \left(\frac{(C_{t+s}/A_{t+s})^{1-\tau} - 1}{1-\tau} - H_{t+s} + \chi V\left(\frac{M_t}{P_t A_t} \right) \right) \right], \tag{11}$$

subject to budget constraint

$$P_t C_t + T_t + M_t + B_t = P_t W_t H_t + M_{t-1} + R_{t-1} B_{t-1} + P_t D_t + P_t SC_t.$$

Here β is the discount factor, $1/\tau$ is the intertemporal elasticity of substitution, and P_t is the price of the final good. The households supply labor services to the firms, taking the real wage W_t as given. At the end of period t households hold money in the amount of M_t. They have access to a bond market where nominal government bonds B_t that pay gross interest R_t are traded. Furthermore, the households receive profits D_t from the firms and pay lump-sum taxes T_t. SC_t is the net cash inflow from trading a full set of state-contingent securities.

Real money balances enter the utility function in an additively separable fashion. An empirical justification of this assumption is provided by Ireland (2004). As a consequence, the equilibrium has a block diagonal structure under the interest-rate feedback rule that we will specify below: the level of output, inflation, and interest rates can be determined independently of the money stock. We assume that the marginal utility $V'(m)$ is decreasing in real money balances m and reaches zero for $m = \bar{m}$, which is the amount of money held in steady state by households if the net nominal interest rate is zero. Since the return on holding money is zero, it provides the rationale for the ZLB on nominal rates. The usual transversality condition on asset accumulation applies.

Firms. The final-goods producers aggregate intermediate goods, indexed by $j \in [0, 1]$, using the technology:

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}.$$

The firms take input prices $P_t(j)$ and output prices P_t as given. Profit maximization implies that the demand for inputs is given by

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/\nu} Y_t.$$

Under the assumption of free entry into the final-goods market, profits are zero in equilibrium and the price of the aggregate good is given by

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}}. \tag{12}$$

We define inflation as $\pi_t = P_t / P_{t-1}$.

Intermediate good j is produced by a monopolist who has access to the following production technology:

$$Y_t(j) = A_t H_t(j), \tag{13}$$

where A_t is an exogenous productivity process that is common to all firms and $H_t(j)$ is the firm-specific labor input. Labor is hired in a perfectly competitive factor market at the real wage W_t. Intermediate-goods-producing firms face quadratic price adjustment costs of the form

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2 Y_t(j),$$

where ϕ governs the price stickiness in the economy and $\bar{\pi}$ is a baseline rate of price change that does not require the payment of any adjustment costs. In our quantitative analysis we set $\bar{\pi} = 1$, that is, it is costless to keep prices constant. Firm j chooses its labor input $N_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}}Y_{t+s}(j) - W_{t+s}H_{t+s} - AC_{t+s}\right)\right]. \tag{14}$$

Here, $Q_{t+s|t}$ is the time t value to the household of a unit of the consumption good in period $t+s$, which is treated as exogenous by the firm.

Government Policies. Monetary policy is described by an interest rate feedback rule of the form

$$R_t = \max\left\{1,\ \left[r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{Y_t}{\gamma Y_{t-1}}\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R\epsilon_{R,t}}\right\}. \tag{15}$$

Here r is the steady state real interest rate, π_* is the target-inflation rate, and $\epsilon_{R,t}$ is a monetary policy shock. The key departure from much of the New Keynesian DSGE literature is the use of the max operator to enforce the ZLB. Provided the ZLB is not binding, the central bank reacts to deviations of inflation from the target rate π_* and deviations of output growth from γ.

The government consumes a stochastic fraction of aggregate output and government spending evolves according to

$$G_t = \left(1 - \frac{1}{g_t}\right)Y_t. \tag{16}$$

The government levies a lump-sum tax T_t (or provides a subsidy if T_t is negative) to finance any shortfalls in government revenues (or to rebate any surplus). Its budget constraint is given by

$$P_t G_t + M_{t-1} + R_{t-1}B_{t-1} = T_t + M_t + B_t. \tag{17}$$

Exogenous shocks. The model economy is perturbed by three exogenous processes. Aggregate productivity evolves according to

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t, \text{ where } \ln z_t = \rho_z \ln z_{t-1} + \sigma_z \epsilon_{z,t}. \tag{18}$$

Thus, on average the economy grows at the rate γ and z_t generates exogenous fluctuations of the technology growth rate. We assume that the government spending shock follows the AR(1) law of motion

$$\ln g_t = (1 - \rho_g) \ln g_* + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t}. \tag{19}$$

The monetary policy shock $\epsilon_{R,t}$ is assumed to be serially uncorrelated. We stack the three innovations into the vector $\epsilon_t = [\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{r,t}]'$ and assume that $\epsilon_t \sim iidN(0, I)$.

3.2 Equilibrium Conditions

Since the exogenous productivity process has a stochastic trend, it is convenient to characterize the equilibrium conditions of the model economy in terms of detrended consumption and output: $c_t = C_t/A_t$ and $y_t = Y_t/A_t$. The consumption Euler equation is given by

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right]. \tag{20}$$

We define

$$\mathcal{E}_t = I\!\!E_t \left[\frac{c_{t+1}^{-\tau}}{\gamma z_{t+1} \pi_{t+1}} \right], \tag{21}$$

which will be useful in the computational algorithm. In a symmetric equilibrium in which all firms set the same price $P_t(j)$ the price-setting decision of the firms leads to the condition

$$\begin{aligned} 1 \quad = \quad & \frac{1}{\nu}(1 - c_t^\tau) + \phi(\pi_t - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\bar{\pi}}{2\nu} \right] \\ & -\phi \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \end{aligned} \tag{22}$$

The aggregate resource constraint can be expressed as

$$c_t = \left[\frac{1}{g_t} - \frac{\phi}{2} (\pi_t - \bar{\pi})^2 \right] y_t. \tag{23}$$

It reflects both government spending as well as the resource cost (in terms of output) caused by price changes. Finally, we reproduce the monetary policy rule

$$R_t = \max\left\{1, \left[r\pi_* \left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{y_t}{y_{t-1}} z_t\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}}\right\}. \tag{24}$$

Before exploring the equilibrium dynamics of the stochastic system, it is instructive to consider the steady states of the model.

3.3 Steady States

As the two-equation model in Section 2, the New Keynesian model with ZLB constraint has two steady states, which we refer to as targeted-inflation and deflation steady state. In the targeted-inflation steady state inflation equals π_*. The real interest rate, nominal interest rate output, and consumption are given by

$$\begin{aligned} r &= \gamma/\beta, \quad R_* = r\pi_*, \quad y_* = \frac{c_*}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_* - \bar{\pi})^2\right]} \\ c_* &= \left[1 - \upsilon - \frac{\phi}{2}(1-2\lambda)\left(\pi_* - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau}, \end{aligned} \tag{25}$$

where $\lambda = \nu(1-\beta)$. In the deflation steady state the nominal interest rate is at the ZLB, that is, $R_D = 1$, and provided that $R_* > 1$ and $\psi_1 > 1$:

$$\begin{aligned} r &= \gamma/\beta, \quad \pi_D = \beta/\gamma, \quad y_D = \frac{c_D}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_D - \bar{\pi})^2\right]} \\ c_D &= \left[1 - \upsilon - \frac{\phi}{2}(1-2\lambda)\left(\pi_D - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau}. \end{aligned} \tag{26}$$

If the real rate $r > 1$ then the economy experiences deflation, which is why we label the steady state as deflation steady state.

The relative consumption and welfare in the two steady states depends on the discrepancy between $\bar{\pi}$ and steady state inflation. If $\bar{\pi} = 1$ and the target-inflation rate π_* is substantially greater than one, then consumption and welfare may be higher in the deflation steady state. If, on the other hand, $\bar{\pi} = \pi_*$ then the targeted-inflation steady state is associated with higher welfare. In the next section, we turn to the analysis of equilibrium dynamics. As in the two-equation model of Section 2, we will construct a targeted-inflation equilibrium in which the economy fluctuates around the targeted-inflation steady state; a deflation equilibrium in which the economy fluctuations near the deflation steady state; and an sunspot equilibrium with a targeted-inflation regime and a deflation regime.

4 Solving the Model Subject to the ZLB Constraint

Most of the existing literature focuses on what we call the targeted-inflation equilibrium. In this equilibrium households choose time t consumption as a function of the exogenous state variables z_t, g_t, and $\epsilon_{R,t}$ and the endogenous state variables R_{t-1} and y_{t-1}. Moreover, firms set prices such that time t inflation depends on the same endogenous and exogenous state variables. As illustrated in the example of Section 2, there exist many equilibria in which the economy enters extended periods of deflation. We consider one equilibrium in which consumption and inflation depend on the same set of state variables as in the targeted-inflation equilibrium. Moreover, we consider an equilibrium in which the agents' decision rules also depend on a Markov-switching sunspot s_t. In sum, the goal is to construct functions

$$
\begin{aligned}
\pi_t &= f_\pi(R_{t-1}, y_{t-1}, z_t, g_t, \epsilon_{R,t}, s_t) \\
c_t &= f_c(R_{t-1}, y_{t-1}, z_t, g_t, \epsilon_{R,t}, s_t)
\end{aligned}
$$

and so on, such that the equilibrium conditions (20) to (24) as well as the laws of motion for the exogenous processes (and the transversality conditions) are satisfied. In general, there exist other, more complex functions $f_\pi(\cdot)$ and $f_c(\cdot)$ that satisfy the equilibrium conditions. We are focusing on "minimal" state-variable solutions. In Section 4.1 we begin by constructing a solution for a piece-wise log-linear approximation of the equilibrium conditions. The numerical procedure to solve the full model without taking linear approximations is presented in Section 4.2.

4.1 Piece-wise Linear Dynamics Near Steady States

We begin by taking log-linear approximations near the targeted-inflation and the deflation steady state. To simplify the exposition we impose the following restrictions on the DSGE model parameters: $\tau = 1$, $\gamma = 1$, $\bar{\pi} = \pi_*$, $\psi_1 = \psi$, $\psi_2 = 0$, $\rho_R = 0$, $\rho_z = 0$, and $\rho_g = 0$. A general approximation of the equilibrium conditions (20) to (24) is presented in the Online Appendix.

Approximating the Targeted-Inflation Equilibrium. Under the parameter restriction $\bar{\pi} = \pi_*$ the steady states of consumption and output are given by $c_* = 1 - \nu$ and $y_* = c_* g_*$. If we let $\kappa_* = c_*/(\nu\phi\pi_*^2)$, then we obtain the familiar linear system:

$$
\begin{aligned}
\hat{R}_t &= \max\left\{ -\ln(r\pi_*),\ \psi\hat{\pi}_t + \sigma_R\epsilon_{R,t} \right\} \\
\hat{c}_t &= \mathbb{E}_t[\hat{c}_{t+1}] - (\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]) \\
\hat{\pi}_t &= \beta\mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa_*\hat{c}_t,
\end{aligned}
\tag{27}
$$

where for each variable $\hat{x}_t = \ln(x_t/x_*)$. The law of motion for output is given by

$$
\hat{y}_t = \hat{c}_t + \sigma_g\epsilon_{g,t}. \tag{28}
$$

It is well known that if the shocks are small enough such that the ZLB is non-binding, the linearized system has a unique stable solution for $\psi > 1$. Since the exogenous shocks are *iid* and the simplified system has no endogenous propagation mechanism, consumption, output, inflation, and interest rates will also be *iid* and can be expressed as a function of $\epsilon_{R,t}$ and $\epsilon_{g,t}$.

We are interested in the case in which the shocks are large enough such that the ZLB is binding with non-negligible probability. In this case the system (27) exhibits piece-wise linear dynamics:

$$
\begin{aligned}
\hat{R}_t(\epsilon_{R,t}) &= \max\left\{-\ln(r\pi_*),\ \frac{1}{1+\kappa\psi}\left[\psi(\kappa+\beta)\mu_\pi^* + \kappa\psi\mu_c^* + \sigma_R\epsilon_{R,t}\right]\right\} \\
\hat{c}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(1-\psi\beta)\mu_\pi^* + \mu_c^* - \sigma_R\epsilon_{R,t}\right] & \text{if } \hat{R}_t \geq -\ln(r\pi_*) \\ \ln(r\pi_*) + \mu_c^* + \mu_\pi^* & \text{otherwise} \end{cases} \\
\hat{\pi}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\mu_\pi^* + \kappa\mu_c^* - \kappa\sigma_R\epsilon_{R,t}\right] & \text{if } \hat{R}_t \geq -\ln(r\pi_*) \\ \kappa\ln(r\pi_*) + (\kappa+\beta)\mu_\pi^* + \kappa\mu_c^* & \text{otherwise} \end{cases} .
\end{aligned} \tag{29}
$$

The constants μ_c^* and μ_π^* are the unconditional expectations of $\hat{c}_t$ and $\hat{\pi}_t$. They can be determined by taking expectations of (29) on both sides of the equalities and solving a nonlinear system of equations.

Approximating a Deflation Equilibrium. In the deflation equilibrium the steady state inflation rate is $\pi_D = \beta$. The log-linearization is more cumbersome because of additional terms that arise from $\pi_D \neq \bar{\pi}$. To ease the expositions assume that $|\pi_D - \bar{\pi}|$ is small and ignore the additional terms from our log-linear approximation. Denote percentage deviations of a variable x_t from its deflation steady state by $\tilde{x}_t = \ln(x_t/x_D)$. If we let $\kappa_D = c_D/(\nu\phi\beta^2)$ and using the steady state relationship $r = 1/\beta$

$$
\begin{aligned}
\tilde{R}_t &= \max\left\{0,\ -(\psi-1)\ln(r\pi_*) + \psi\tilde{\pi}_t + \sigma_R\epsilon_{R,t}\right\} \\
\tilde{c}_t &= \mathbb{E}_t[\tilde{c}_{t+1}] - (\tilde{R}_t - \mathbb{E}_t[\tilde{\pi}_{t+1}]) \\
\tilde{\pi}_t &= \beta\mathbb{E}_t[\tilde{\pi}_{t+1}] + \kappa_D\tilde{c}_t.
\end{aligned} \tag{30}
$$

Provided that $\psi > 1$, the ZLB is binding with high probability if the shock standard deviation σ_R is small. In this case $\tilde{R}_t = 0$. It is well-known that if the central bank does not (or is not able to) react to inflation movements the rational expectation system is indeterminate and has many stable solutions. Throughout this paper we focus on the so-called minimum state variable solution, which, in the context of this simple illustrative model, is the solution in which all variables are *iid* in equilibrium. Taking into account that for some shock realizations the ZLB is not binding we can obtain a deflation equilibrium by adjusting the constants in (29):

$$\begin{aligned}
\tilde{R}_t(\epsilon_{R,t}) &= \max\left\{0, \frac{1}{1+\kappa\psi}\left[\psi(\kappa+\beta)\mu_\pi^D + \kappa\psi\mu_c^D - (\psi-1)\ln(r\pi_*) + \sigma_R\epsilon_{R,t}\right]\right\} \\
\tilde{c}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(1-\psi\beta)\mu_\pi^D + \mu_c^D + (\psi-1)\ln(r\pi_*) - \sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ \mu_c^D + \mu_\pi^D & \text{otherwise} \end{cases} \\
\tilde{\pi}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D + \kappa(\psi-1)\ln(r\pi_*) - \kappa\sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ (\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D & \text{otherwise} \end{cases}.
\end{aligned} \tag{31}$$

Discussion. In the simplified model the government spending shock does not affect interest rates, consumption, and inflation. It simply shifts output according to (28). Moreover, technology growth innovations have no effect on interest rates, inflation, and detrended model variables. A shock $\epsilon_{z,t}$ simply generates a permanent increase in the levels of consumption C_t and Y_t.

The consumption of the household and the pricing decision of the firms depend on the monetary policy shock $\epsilon_{R,t}$. In this simple model, the decision rules have a kink at the point in the state space where the two terms in the max operator of the interest rate equation are equal to each other. In the targeted-inflation equilibrium this point in the state space is given by

$$\bar{\epsilon}_R^* = \frac{1}{\sigma_R}\left[-(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^* - \kappa\psi\mu_c^*\right],$$

whereas in the deflation equilibrium it is

$$\bar{\epsilon}_R^D = \frac{1}{\sigma_R}\left[(\psi-1)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^D - \kappa\psi\mu_c^D\right],$$

Once $\epsilon_{R,t}$ falls below the threshold value $\bar{\epsilon}_R^*$ or $\bar{\epsilon}_R^D$, its marginal effect on the endogenous variables is zero. To the extent that $\bar{\epsilon}_R^D > 0 > \bar{\epsilon}_R^*$, it takes a positive shock in the deflation equilibrium to move away from the ZLB, whereas it takes a large negative monetary shock in the targeted-inflation equilibrium to hit the ZLB. The kink in the decision rules implies that the impulse responses of the endogenous variables to the monetary policy shock $\epsilon_{R,t}$ are highly nonlinear. Motivated by the results so far, when we construct a numerical approximation to the decision rules for the more general DSGE model in Section 4.2, we will use to separate approximations for the regions of the state space in which $R_t = 1$ and the region in which $R_t > 1$.

4.2 Nonlinear Solution

This section discusses the numerical techniques that we use to solve the model described in Section 3.2. The goal is to characterize the various solutions to the system

$$\begin{aligned}
\xi\left(c_t, \pi_t, y_t\right) &= \phi\beta I\!E_t\left[\left(c_{t+1}\right)^{-\tau} y_{t+1}(\pi_{t+1}-\bar{\pi})\pi_{t+1}\right] &(32)\\
c_t^{-\tau} &= \beta R_t \mathcal{E}_t &(33)\\
y_t &= \left[\frac{1}{g_t} - \frac{\phi}{2}\left(\pi_t - \bar{\pi}\right)^2\right]^{-1} c_t &(34)\\
R_t &= \max\left\{1, \left[r_*\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{y_t}{y_{t-1}}z_t\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\epsilon_{R,t}}\right\} &(35)
\end{aligned}$$

where $\mathcal{E}_t$ was defined in (21) and $\xi(.)$ is defined as

$$\xi\left(c, \pi, y\right) = c^{-\tau} y\left\{\frac{1}{\nu}\left(1-c^{\tau}\right) + \phi(\pi-\bar{\pi})\left[\left(1-\frac{1}{2\nu}\right)\pi + \frac{\bar{\pi}}{2\nu}\right] - 1\right\}. \qquad (36)$$

To that end we utilize a global approximation using Chebyshev polynomials following Judd (1992) where all decision rules, $c(.)$, $\pi(.)$, $R(.)$., $y(.)$ and $\mathcal{E}(.)$ are assumed to be functions of the minimum set of state variables $(R_{t-1}, y_{t-1}, g_t, z_t, \epsilon_{R,t})$ and the sunspot variable s_t, where applicable, which we collectively label as $\mathcal{S}_t$. The full solution algorithm is relegated to Section C of the Online Appendix.

Unlike Judd, Maliar, and Maliar (2011) and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), we use a piece-wise smooth approximation of the functions $\pi(\mathcal{S}_t)$ and $\mathcal{E}(\mathcal{S}_t)$ by postulating

$$\begin{aligned}
\pi_t = \pi(\mathcal{S}_t;\Theta) &= \zeta_t f_\pi^1(\mathcal{S}_t;\Theta) + (1-\zeta_t) f_\pi^2(\mathcal{S}_t;\Theta)\\
\mathcal{E}_t = \mathcal{E}(\mathcal{S}_t;\Theta) &= \zeta_t f_{\mathcal{E}}^1(\mathcal{S}_t;\Theta) + (1-\zeta_t) f_{\mathcal{E}}^1(\mathcal{S}_t;\Theta)
\end{aligned}$$

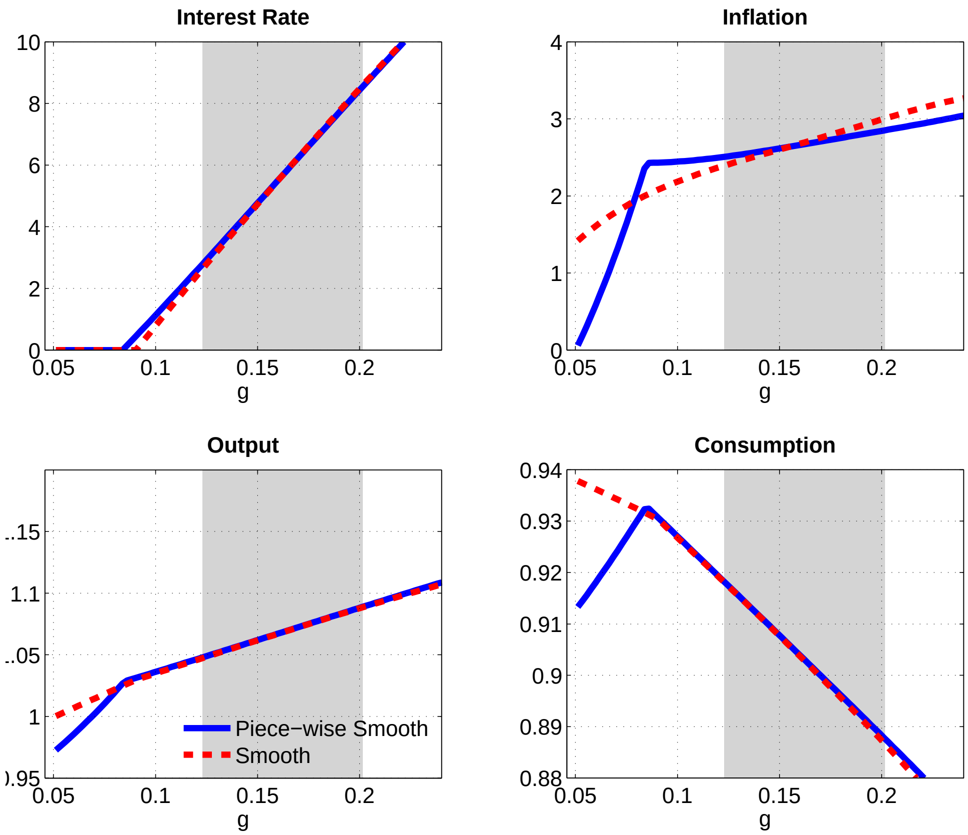
where $\zeta_t = I\{R(\mathcal{S}_t;\Theta) > 1\}$ is an indicator that shows the ZLB is slack. The functions f_j^i are linear combinations of a complete set of Chebyshev polynomials up to fourth order, where the weights are given by a vector Θ. Conditional on $\pi(\mathcal{S}_t)$ and $\mathcal{E}(\mathcal{S}_t)$, the decision rules for consumption, output, and interest rates can be obtained recursively from (33), (34), and (35). Our method is flexible enough to allow for a kink in all decision rules and not just R_t which has a kink by its construction. In our experience this flexibility yields much higher accuracy in the approximated decision rules, especially the inflation decision rule. Figure 2 shows the decision rules as a function of the demand shocks using our piece-wise smooth approximation and a smooth approximation. For $g = 0.09$ the economy hits the ZLB. When approximated smoothly, especially the inflation and consumption decision rules fail to capture the kinks that are apparent in the piece-wise smooth approximation. The decision rule for output illustrates that the (marginal) government-spending multiplier is sensitive to the ZLB - it is noticeably larger in the area of the state space where the ZLB binds.

The solution algorithm amounts to specifying a grid of points $\mathcal{G} = \{\mathcal{S}_1, \ldots, \mathcal{S}_M\}$ in the model's state space and solving for the vector Θ such that the sum of squared residuals associated with (21) and (32) are minimized for $\mathcal{S}_t \in \mathcal{G}$. By construction, (33)-(35) hold exactly. Since the collocation methods, which require the solution to be accurate on a fixed grid typically obtained by the Kronecker product of grids in each dimension, become exceedingly difficult to implement as the number of state variable go above three, we build on the ergodic-set method discussed in Judd, Maliar, and Maliar (2011). This method requires an iteration between solving and simulating the solved model and the approximation is expected to be accurate only over a set of points that characterizes the model's ergodic distribution of $\mathcal{S}_t$. Since our goal is to fit the model to data from the 2008-09 recession and since explaining these data with our model requires realizations of the states that lie far in the tails of the model-implied ergodic distribution, we combine draws from the ergodic distribution with filtered exogenous state variables[4] based on 2009 data on output growth, inflation, and interest rates to generate the grid $\mathcal{G}$. This ensures that our approximation remains accurate in the area of the state space that is relevant for the empirical analysis.

Figure 3 shows the solution grid for the targeted-inflation equilibrium. For each panel

[4]See Section 5.2 for a detailed explanation of the filtering procedure

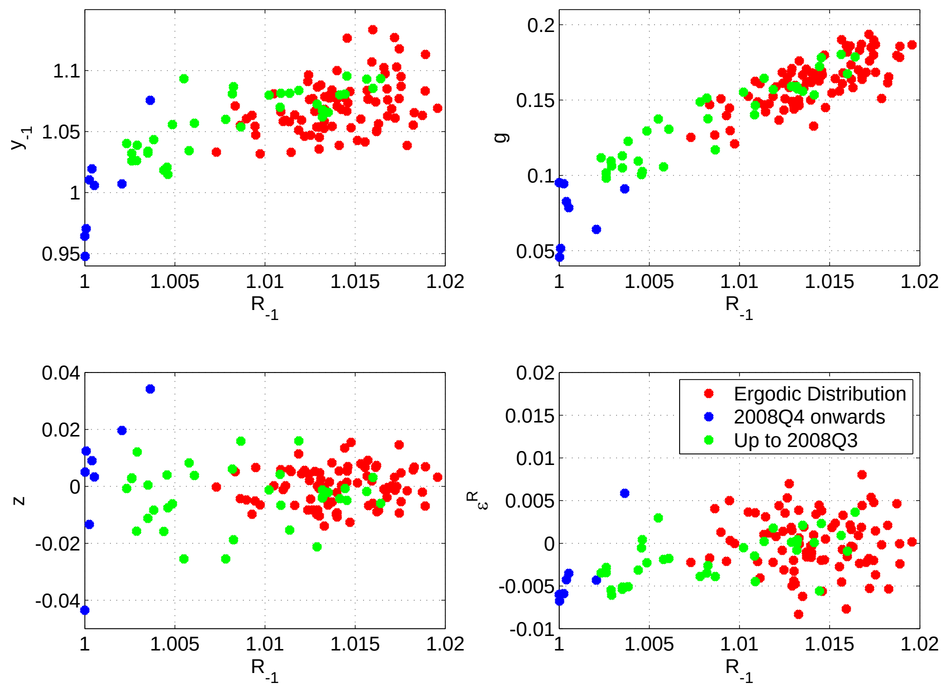
Figure 2: Sample Decision Rules



Notes: The gray shading show 95% coverage of the ergodic distribution in the targeted-inflation equilibrium.

we have R_{t-1} on the x axis and the other state variables on the y axis. The red dots are the grid points that represent the ergodic distribution, the green points are the filtered states from 2000:Q1 to 2008:Q3 and the blue points are the filtered state for the period after 2008:Q3. It is evident that the filtered states lie in the tails of the ergodic distribution of the targeted-inflation equilibrium, which assigns negligible probability to zero interest rates and the exogenous states that push interest rates toward the ZLB.

Figure 3: Solution Grid for the Targeted-Inflation Equilibrium



5 Quantitative Analysis

The quantitative analysis consists of three parts. In Section 5.1 we compare the ergodic distribution of inflation and interest rates under the three equilibria. Moreover, we examine some impulse response functions. In Section 5.2 we use the model to estimate a sequence of historical states for the period 2000:I to 2010:III based on output growth, inflation, and interest rate data. Conditional on the estimated states during the great recession of 2008-09, Section 5.3 assesses the effect of fiscal and monetary policy interventions in the targeted-inflation and the sunspot equilibria. The parameters used for the quantitative analysis in this section are summarized in Table 1. They are obtained by estimating a nonlinear version of the DSGE model, solved using second-order perturbation, based on post-1984 U.S. data of real per capita GDP growth, GDP deflator inflation, and the Federal Funds rate.

In the sunspot model, we assume that the sunspot variable $s_t \in 0, 1$ follows a first-order Markov process with $p_{00} = p_{11} = 0.99$ $p_{ij} \equiv Pr(s_t = s_j | s_{t-1} = s_i)$.

Table 1: DSGE Model Parameters

$\tau = 1.09$	$r = 1.0076$	$\gamma = 1.0059$	$\nu = 0.1$
$g_* = 1/0.85$	$\phi = 221.70$	$\pi_* = 1.007$	$\bar{\pi} = 1$
$\psi_1 = 1.5$	$\psi_2 = 0.8$	$\rho_R = 0.55$	$\sigma_R = 0.0036$
$\rho_g = 0.89$	$\sigma_g = 0.0086$	$\rho_Z = 0.26$	$\sigma_z = 0.0072$

5.1 Equilibrium Dynamics

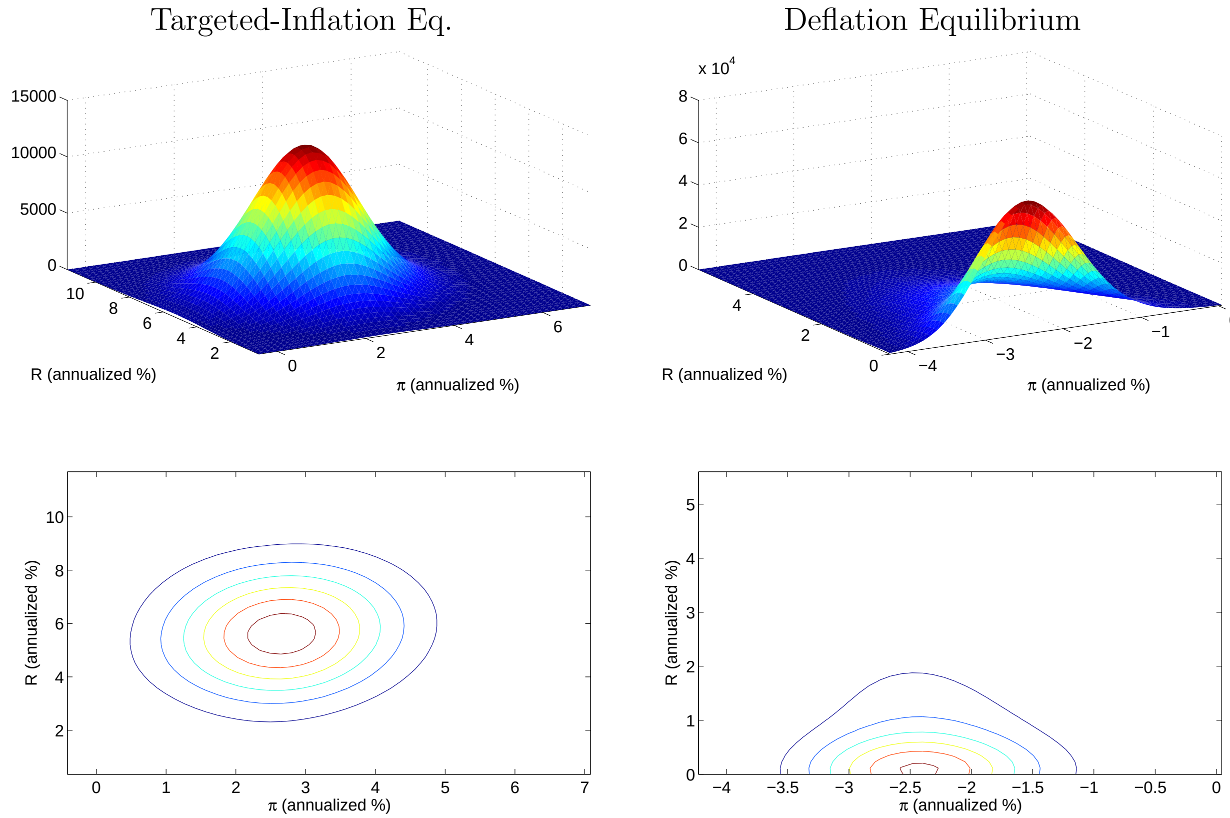
In order to evaluate the ergodic distributions associated with the targeted-inflation, the deflation, and the sunspot equilibria, we simulate a long sequence of draws from each of the equilbria. Figure 4 depicts surface and contour plots of a kernel density estimate of the ergodic distribution of interest and inflation rates in the targeted-inflation and the deflation equilibrium. Inflation and interest rates are reported as annualized net rates, converted in percent. The parameterization of the DSGE model implies that

$$400 \ln \pi_* = 2.8, \quad 400 \ln R_* = 5.88, \quad 400 \ln \pi_D = -3.4, \quad 400 \ln R_D = 0.$$

The ergodic distribution under the targeted-inflation equilibrium is approximately centered at the steady state. Conversely, in the deflation equilibrium the steady state values of inflation and interest rates are substantially lower than the mean values of the ergodic distribution, in part because the steady state coincides with the lower bound of the interest rate.

The left panel of Figure 5 depicts the ergodic distribution of interest and inflation rates, whereas the right panel overlays simulated trajectories of the model variables under the sunspot and the targeted-inflation equilibrium using the same set of shocks. The ergodic distribution is clearly bimodal. The mode associated with the targeted-inflation regime is located slightly to the left of (π_*, R_*), whereas the mode of the deflation regime is associated with an inflation rate of -2.5% and an interest rate of about 0%. The simulated paths show that whenever the economy is in the targeted-inflation regime, the paths of consumption, output, inflation, and interest rates are very similar to the paths that would be observed under the targeted-inflation equilibrium. However, in periods in which the sunspot equilibrium

Figure 4: Ergodic Distributions

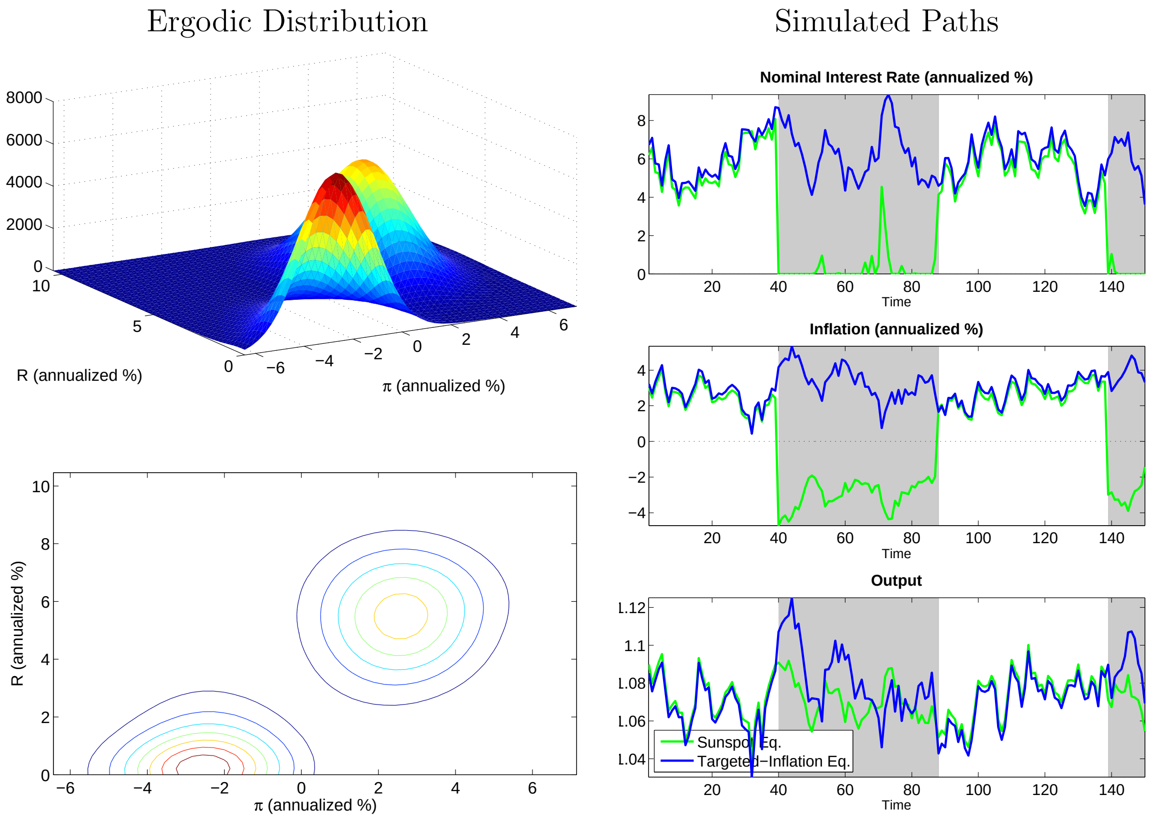


Notes: Figure depicts surface and contour plots of the joint probability density function (kernel density estimate) of interest rates and inflation. Interest rate and inflation are gross rates at a quarterly rate.

shifts to the deflation regime, indicated by gray shading, both inflation and interest rates fall drastically.

Next, we turn to some impulse responses to understand how the equilibria behave. Due to the nonlinearity of the model, there are a number of alternative ways of computing impulse response. We compute impulse responses as the difference between a baseline simulation in which all shocks are drawn from $\epsilon_t \sim N(0, I)$ and an alternative simulation in which the time $t = 1$ monetary policy shock is shifted by δ and all other shocks are identical to their baseline values. The procedure that is used to compute the impulse responses is summarized in Algorithm 1 and a precise definition of the impulse responses depicted in the Figure is

Figure 5: Sunspot Equilibrium



Notes: Figure depicts surface and contour plots of the joint probability density function (kernel density estimate) of interest rates and inflation. Interest rate and inflation are net rates at an annual rate.

provided by (38) below. For a linear model, the algorithm produces the standard impulse responses, which are invariant to the size δ of the shock.

Algorithm 1 (Scaled Impulse Responses) *For $j = 1$ to $j = n_{sim}$ repeat the following steps:*

1. *Draw initial values $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$ from ergodic distribution.*

2. *Generate baseline trajectories based on the innovation sequence $\{\epsilon_t^{(j)}\}_{t=1}^{H}$, where $\epsilon_t^{(j)} = [\epsilon_{z,t}^{(j)}, \epsilon_{g,t}^{(j)}, \epsilon_{R,t}^{(j)}]' \sim N(0, I)$.*

3. *Generate counterfactual trajectories based on the innovation sequence*

$$\begin{aligned} \epsilon^{I(j)}_{z,t} &= \epsilon^{(j)}_{z,t} \quad \textit{and} \quad \epsilon^{I(j)}_{g,t} = \epsilon^{(j)}_{g,t} \quad \textit{for} \quad t = 1, \dots, H; \\ \epsilon^{I}_{R,1} &= \delta + \epsilon_{R,1}; \quad \epsilon^{I}_{R,t} = \epsilon_{R,t} \quad \textit{for} \quad t = 2, \dots, H \end{aligned}$$

4. *Conditional on* $(R^{(j)}_0, y^{(j)}_0, z^{(j)}_0, g^{(j)}_0)$ *compute* $\{R^{(j)}_t, y^{(j)}_t, \pi^{(j)}_t\}_{t=1}^H$ *and* $\{R^{I(j)}_t, y^{I(j)}_t, \pi^{I(j)}_t\}_{t=1}^H$ *based on* $\{\epsilon^{(j)}_t\}$ *and* $\{\epsilon^{I(j)}_t\}$*, respectively, and for a generic variable* x*, let*

$$IRF^{(j)}_\delta(x_t|\epsilon_{R,1}) = (\ln x^{I(j)}_t - \ln x^{(j)}_t)/|\delta|. \tag{37}$$

Compute the mean of $IRF^{(j)}_\delta(x_t|\epsilon_{R,1})$ *across* j*:*

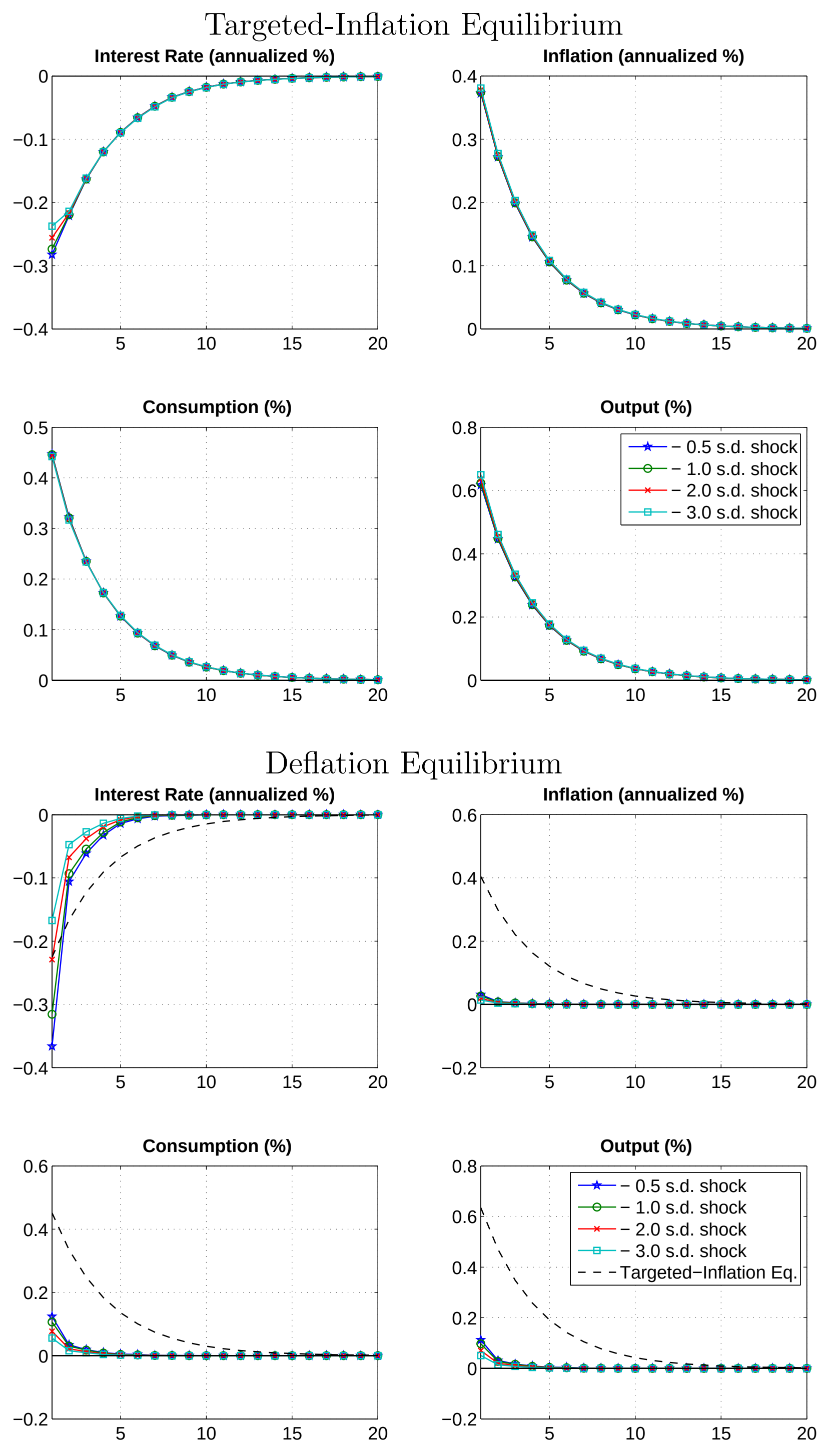
$$IRF_\delta(x_t|\epsilon_{R,1}) = \frac{1}{n_{sim}} \sum_{j=1}^{n_{sim}} IRF^{(j)}_\delta(x_t|\epsilon_{R,1}).\square \tag{38}$$

or any percentile.

The top panels of Figure 6 depict the responses of the economy in the targeted-inflation equilibrium. In response to a monetary policy shock that reduces interest rates by about 25 basis points (bp) annualized, inflation rises by 40bp (annualized) and output grows by about 60bp. The scaling δ of the shock does not alter the shape of the impulse responses. The impulse responses exhibit no apparent nonlinearities and none of the simulated trajectories hits the ZLB. Putting these results in the context of the piece-wise linear approximation of the simplified DSGE model presented in Section 4.1, under the simulated shocks $\hat{R}$ remains greater than $-\ln(r\pi_*)$, even if the monetary policy shocks are shifted by $\delta < 0$ in the initial period.

We now turn to the impulse responses to a negative monetary policy shock in the deflation equilibrium, which are shown in the two bottom rows of Figure 6. The piecewise linear approximation in (29) suggests that for baseline trajectories starting at or near the ZLB an additional reduction of the monetary policy shock by $-|\delta|$ should have little or no effect on interest rates. This truncation effect becomes more severe as $|\delta|$ increases because the interest rate is pushed to the ZLB for a larger fraction of the simulations. Thus, the mean effect (in absolute terms) of the monetary expansion on interest rates should be decreasing in δ. This mechanism is clearly visible in the interest rate response.

Figure 6: Responses to a Monetary Policy Shock



Unlike the piecewise linear specification in Section 4.1 the DSGE model that is used to generate the impulse responses depicted in Figure 6 is dynamic. Even if the economy is not at ZLB in the initial period, an expansionary monetary policy shock increases the probability of hitting the ZLB in the future. The presence of the ZLB leads to a reduction in the expected future nominal interest rate movements compared to an unconstrained environment. Thus, after an expansionary monetary policy intervention agents expect smaller real rate drops in the future. In turn, the increase in consumption in the initial period is lower, because the Euler equation implies that consumption is approximately equal to the discounted sum of expected future real rates. The muted consumption response leads to a muted inflation response via the Phillips curve. Overall, neither inflation nor consumption and output react to the interest rate cut as much as in the targeted-inflation equilibrium. The dampened output and inflation response implies via the interest rate feedback that the interest rate falls more strongly in the deflation equilibrium than in the targeted-inflation equilibrium. Overall, the expansionary monetary policy is less effective in the deflation equilibrium.

5.2 Extracting Historical Shocks for the Period 2000-2010

We now use the DGSGE model to determine the sequence of shocks that lead to the Great Recession in 2008-09. The shocks are extracted under the assumption that the U.S. economy was either in the targeted-inflation equilibrium or in the sunspot equilibrium, both of which are parameterized using the parameters in Table 1. The deflation equilibrium is empirically implausible because in the postwar period the U.S. never experienced a prolonged period of deflation.

The DSGE model can be represented as a state space model. Let y_t be the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates. The vector x_t stacks the continuous state variables which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$ and

$s_t \in \{0, 1\}$ is the Markov-switching process.

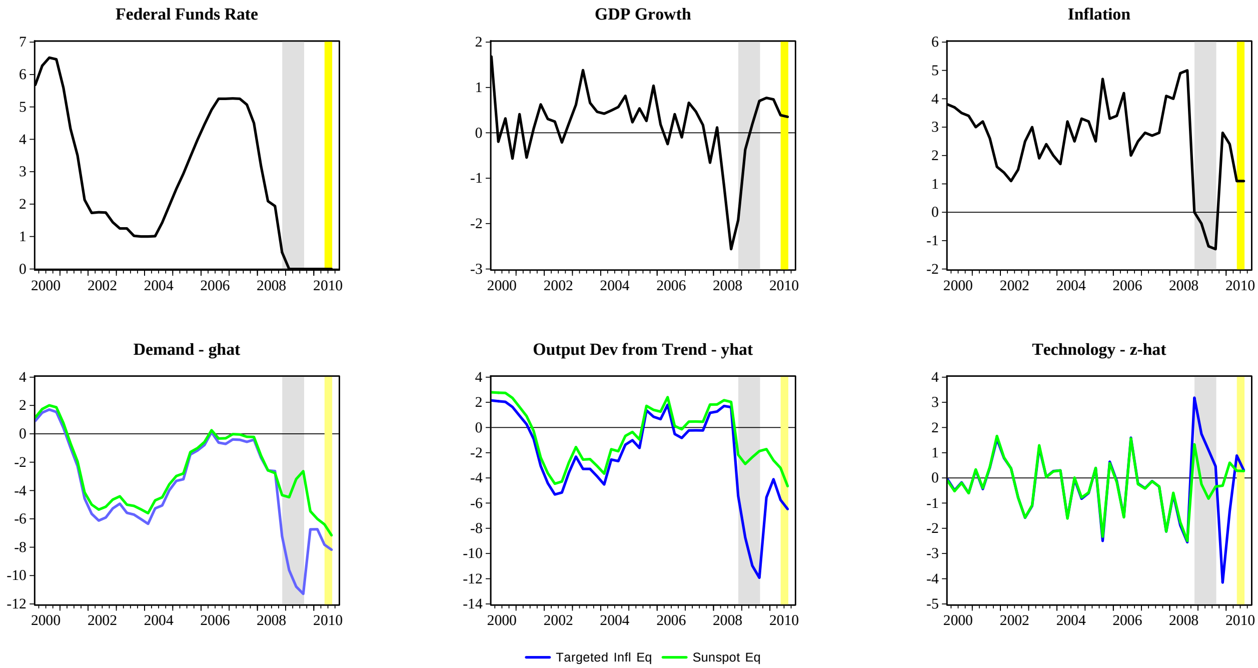
$$\begin{aligned} y_t &= \Psi(x_t) + \nu_t \\ \mathbb{P}\{s_t = 1\} &= \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \\ x_t &= F_{s_t}(x_{t-1}, \epsilon_t) \end{aligned} \tag{39}$$

The first equation in (39) is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. Under the targeted-inflation equilibrium the state-transition equation $x_t = F(x_{t-1}, \epsilon_t)$ is time-invariant.

The state vector x_t is extracted from the observables using a particle filter, also known as sequential Monte Carlo filter. Gordon and Salmond (1993) and Kitagawa (1996) made early contributions to the development of particle filters. In the economics literature the particle filter has been applied to analyze stochastic volatility models, e.g., Pitt and Shephard (1999), and nonlinear DSGE models following Fernández-Villaverde and Rubio-Ramírez (2007). Surveys of sequential Monte Carlo filtering are provided, for instance, in the engineering literature by Arulampalam, Maskell, Gordon, and Clapp (2002) and in the econometrics literature by Giordani, Pitt, and Kohn (2011). A detailed description of the particle filter used in the subsequent quantitative analysis is provided in the Online Appendix.

Figure 7 depicts the time-path of shocks extracted conditional on the two equilibria, as well as the data used. The top panel shows that the period 2008-2010 is characterized by near-zero interest rates, falling output and deflation or low inflation. In order to match this behavior, the targeted-inflation equilibrium needs successively worse demand shocks (as large as 10% below its steady state value), coupled with very large positive technology shocks, followed by a very large reversal. Here, the negative demand shocks are necessary to match the large decline in output but they cannot fully generate the decline in inflation. Hence, the large positive technology shocks are necessary to make up for the difference. In the sunspot model, on the other hand, all that is necessary is a switch in the sunspot variable

Figure 7: Data and Extracted Shocks

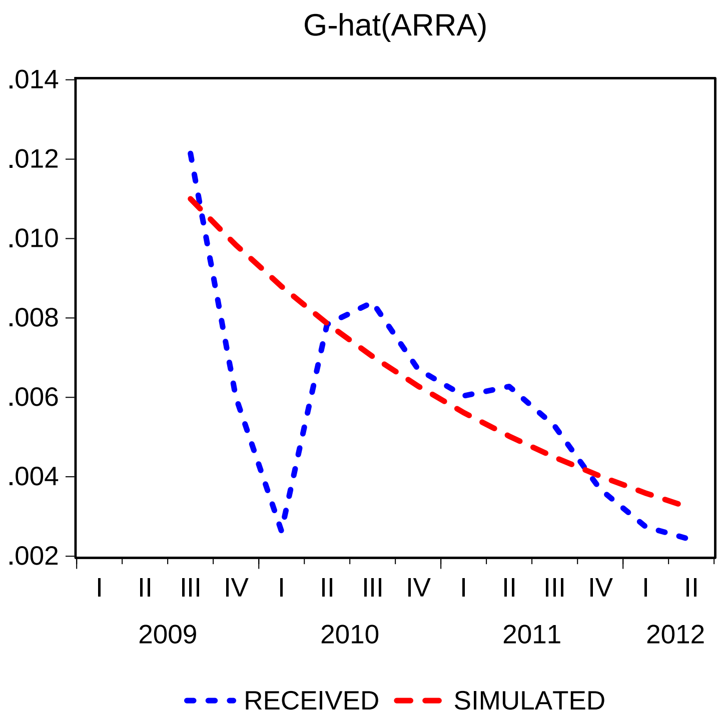


in order to match the decline in inflation and output. As the figure shows, the technology shock remains within 2% of its steady state throughout the period and the demand shock falls more modestly. The results of the filter indicate that the sunspot variable switches for four quarters starting from 2008:Q4 and reverts back in 2009:Q4, which is exactly the period of negative inflation in the data. However in the last two quarters of the sample there is a small probability of a switch again. Overall, under the sunspot equilibrium smaller shocks are needed to rationalize the time path of the observations during the great recession. In other words, the data seem to favor the sunspot equilibrium.

5.3 Policy Experiments

Taking the filtered states from 2009:Q1 as given, we now study the effects of policy interventions during the Great Recession. In particular we consider an increase in government spending that is potentially combined with an expansionary monetary policy. To make the

Figure 8: Calibration of Fiscal Policy Intervention



experiment more realistic, the fiscal policy intervention is calibrated to a portion of the American Recovery and Reinvestment Act (ARRA) of February 2009. ARRA consisted of a combination of tax cuts and benefits; entitlement programs; and funding for federal contracts, grants, and loans. We focus on the third component because it can be interpreted as an increase in g_t. In Section E of the Online Appendix we describe how we map the data in to a demand shock we can use in our model. In sum, we will model as one-period shift δ^{ARRA} in the mean of $\epsilon_{g,1}$ such that for $t = 1, \ldots, H$ the baseline path of the demand shock is given by

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \sigma_g \epsilon_{g,t}$$

and the post-intervention path is given by

$$\hat{g}_t^I = \hat{g}_t + \rho_g^{t-1} \delta^{ARRA}. \tag{40}$$

In Figure 8 we compare $\hat{g}_t^{ARRA}$ constructed from the data in Table A-1 to $(\hat{g}_t^I - \hat{g}_t)$, where $\delta^{ARRA} = 0.011$.[5] While the actual path of the received funds is not perfectly monotone, the calibrated intervention in the DSGE model roughly matches the actual intervention both in terms of magnitude and decay rate.

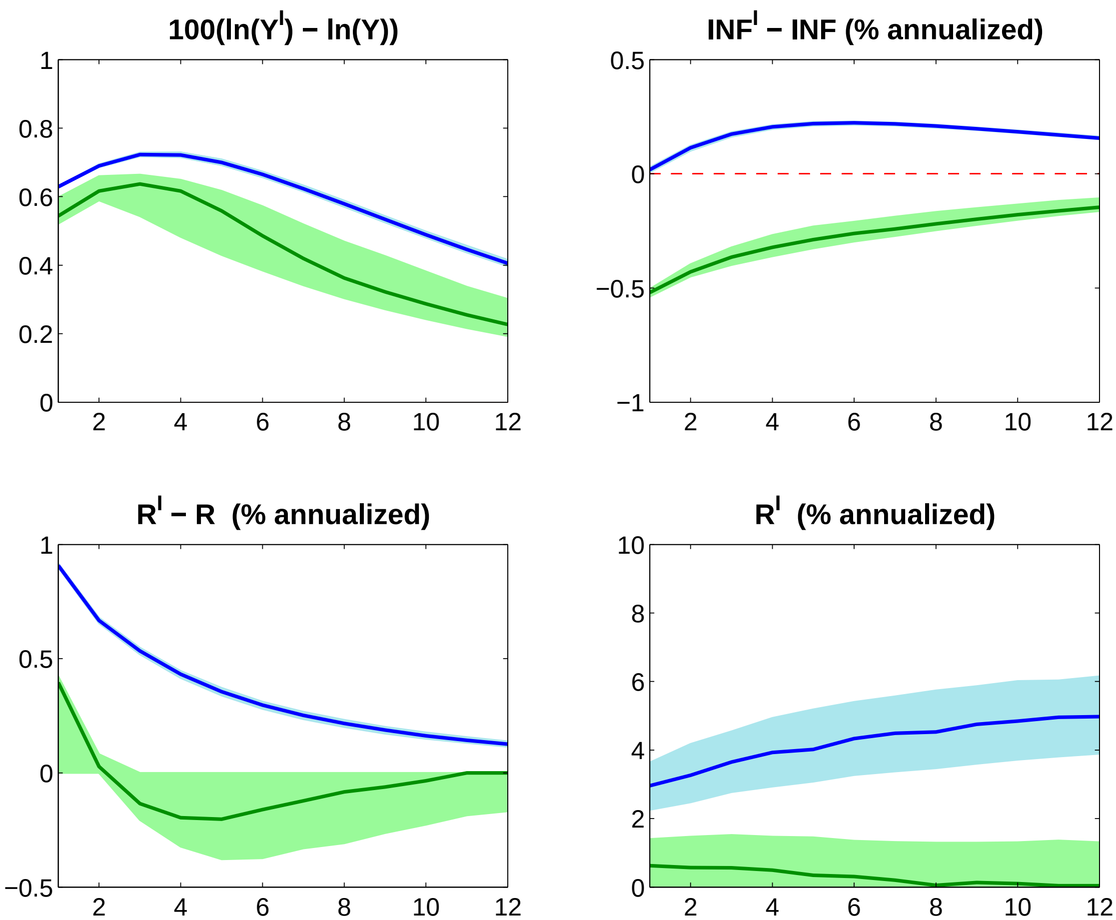
The effect of the fiscal intervention is computed with an algorithm that is similar to Algorithm 1. Three modifications are necessary. First, in Step 1 we start from the expected value of (R_0, y_0, z_0, g_0), and s_1 for the sunspot model, computed with the particle filter. Here period $t = 0$ corresponds to 2008:Q4. Second, in Step 2 we shift the government spending shock (instead of the monetary policy shock) by δ^{ARRA}. Third, in Step 4 we don't scale the difference between the baseline and the counterfactual trajectories in (37) by the size δ of the intervention.

Figure 9 overlays the effects of the fiscal expansion for the sunspot equilibrium and the targeted-inflation equilibrium. The figure shows (pointwise) median responses as well as upper and lower 20% percentiles of the distribution of $IRF^{(j)}(x_t|\epsilon_{g,1})$. In the sunspot equilibrium, the fiscal stimulus raises output by about 0.5% and reduces inflation by about 55bp one period after impact. The central bank ends up responding to these conflicting changes by increasing the interest rate from the level in 2009:Q1, which is near zero. As the effect of the intervention dies out the economy converges towards the mean of the ergodic distribution in deflation regime since with only a 1% transition probability, even after 12 periods, most of the paths still remain in the deflation regime, with close to half of them with interest rates at the ZLB.

In the targeted-inflation equilibrium interest rates in the targeted-inflation equilibrium tend to be significantly higher than in the sunspot equilibrium. While we condition the simulations under both equilibria on the interest rate being near zero in 2009:Q1, the interest rate reverts quickly back to $r\pi_*$ under the targeted-inflation equilibrium, while the economy remains with high probability in the deflation regime and near the ZLB under the sunspot equilibrium. The targeted-inflation equilibrium exhibits fewer nonlinearities than the sunspot equilibrium as evidenced by the very narrow percentile bands. The output effect

[5] Recall that $\sigma_g = 0.0086$

Figure 9: Fiscal Policy Effects in Targeted-Inflation and Sunspot Equilibrium



Notes: Figure compares intervention effects from targeted-inflation equilibrium (blue) and sunspot equilibrium (green): pointwise medians (solid); 20%-80% percentiles (shaded area).

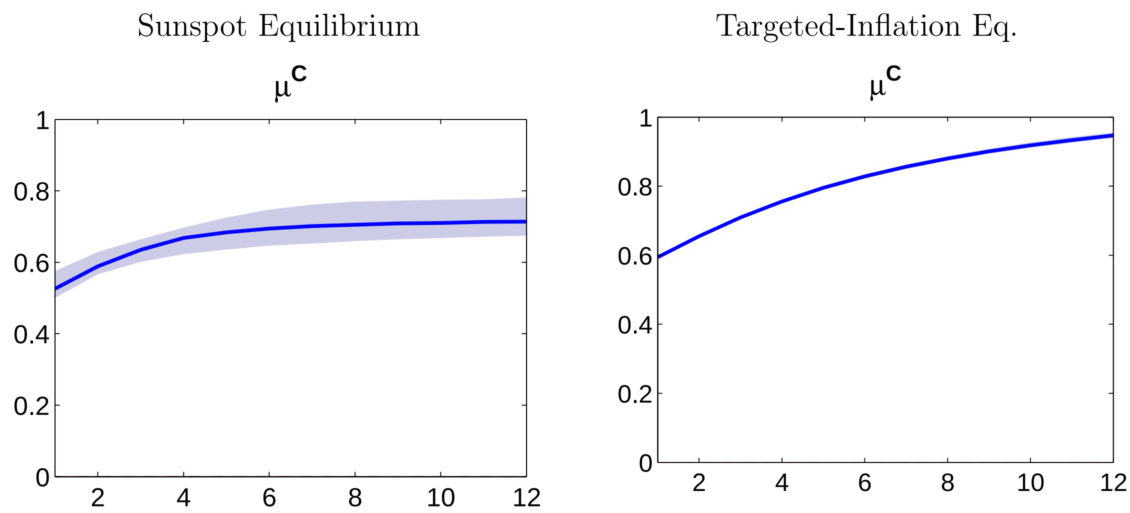
of the fiscal stimulus in the targeted-inflation equilibrium is higher on impact and remains higher throughout the intervention.

Based on the impulse response functions we can calculate government spending multipliers that measure the effect of the fiscal intervention on output relative to the overall increase in government spending. We consider a multiplier defined as

$$\mu_t^c = \frac{\sum_{\tau=1}^{t}(Y_\tau^I - Y_\tau)}{\sum_{\tau=1}^{t}(G_\tau^I - G_\tau)}.$$

Our measure is cumulative over the lifetime of the intervention and are depicted in Figure 10

Figure 10: Government Spending Multipliers



as a function of time. Based on this measure, the fiscal intervention has about the same effect on impact under the two equilibria, while in the medium run, after about three years, the targeted-inflation equilibrium delivers a cumulative multiplier of close to one, roughly a third larger than the sunspot equilibrium.

Since a fiscal expansion creates an upward pressure on the nominal interest rates via the feedback mechanism of the interest rate rule, in principle there is scope for amplifying the effect of the fiscal stimulus by a monetary policy that keeps interest rates near zero. Thus, we now consider a combination of expansionary fiscal and monetary policy. The central bank intervention is assumed to last for four quarters and is implemented using a sequence of unanticipated monetary policy shocks $\epsilon_{R,t}$. A detailed discussion about the advantages and disadvantages of using unanticipated versus anticipated monetary policy shocks to generate predictions conditional on an interest rate path is provided in Del Negro and Schorfheide (2012). We begin with a description of the algorithm that is used to compute the effect of the intervention and will provide an interpretation subsequently.

Algorithm 2 (Effect of Combined Fiscal and Monetary Policy Intervention) *For* $j = 1$ *to* $j = n_{sim}$ *repeat the following steps:*

1. *Initialize the simulation by setting* $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$ *equal to the mean estimate obtained with the particle filter.*

2. *Generate baseline trajectories based on the innovation sequence* $\{\epsilon_t^{(j)}\}_{t=1}^H$ *by letting* $[\epsilon_{z,t}^{(j)}, \epsilon_{g,t}^{(j)}]' \sim N(0, I)$ *and setting* $\epsilon_{R,t} = 0$.

3. *Generate the innovation sequence for the counterfactual trajectories according to*

$$\begin{aligned} \epsilon_{g,1}^{I(j)} &= \delta^{ARRA} + \epsilon_{g,1}^{(j)}; \quad \epsilon_{g,t}^{I(j)} = \epsilon_{g,t}^{(j)} \quad \textit{for} \quad t = 2, \ldots, H; \\ \epsilon_{z,t}^{I(j)} &= \epsilon_{z,t}^{(j)} \quad \textit{for} \quad t = 1, \ldots, H; \\ \epsilon_{R,t}^{I(j)} &= \epsilon_{R,t}^{(j)} = 0 \quad \textit{for} \quad t = 5, \ldots, H; \end{aligned}$$

   In periods $t = 1, \ldots, 4$*, conditional on* $\{\epsilon_{g,t}^{I(j)}, \epsilon_{z,t}^{I(j)}\}_{t=1}^4$*, determine* $\epsilon_{R,t}^{I(j)}$ *by solving for the smallest* $\tilde{\epsilon}_{R,t}$ *such that it is less than* $2\sigma_R$ *that yields either*

$$R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = \tilde{\epsilon}_{R,t}) = 1 \quad \textit{or} \quad 400 \ln\left(R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = 0) - R_t^{I(j)}(\epsilon_{R,t}^{I(j)} = \tilde{\epsilon}_{R,t})\right) = 1.$$

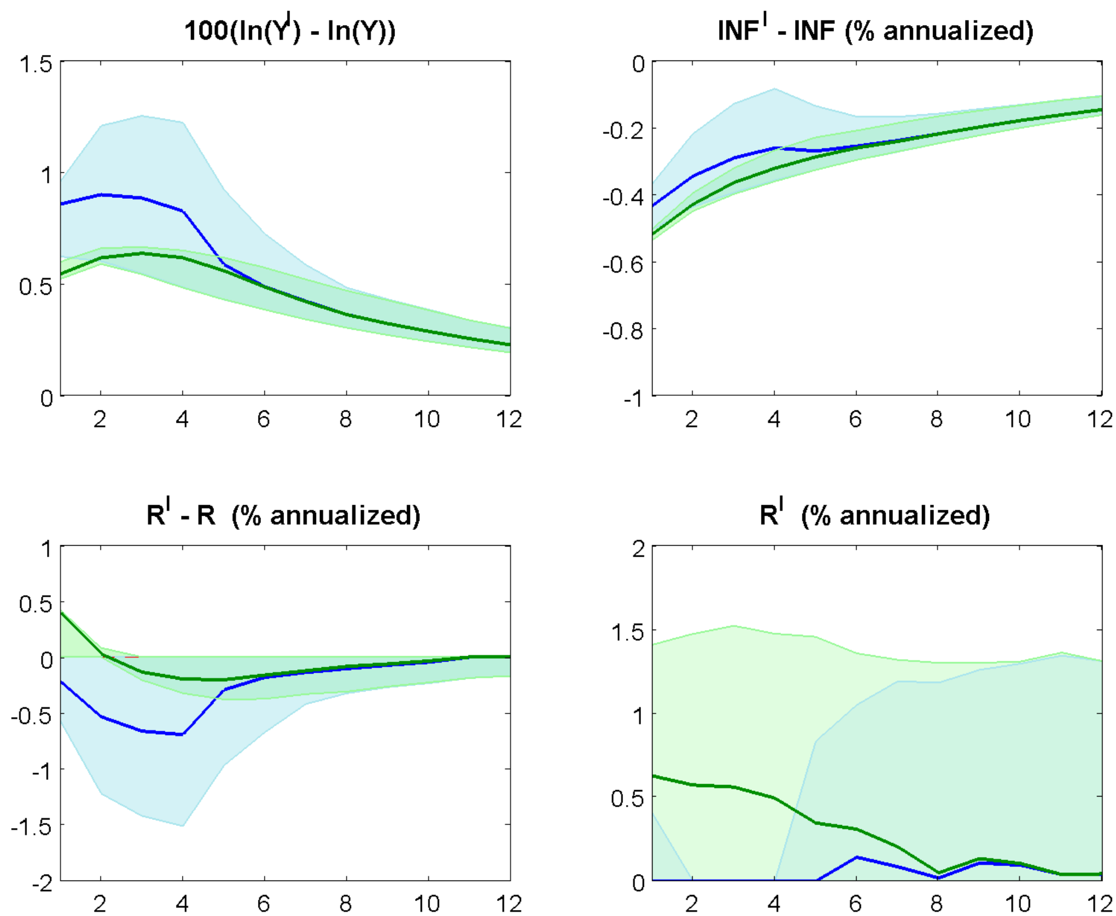
4. *Conditional on* $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$ *compute* $\{R_t^{(j)}, y_t^{(j)}, \pi_t^{(j)}\}_{t=1}^H$ *and* $\{R_t^{I(j)}, y_t^{I(j)}, \pi_t^{I(j)}\}_{t=1}^H$ *based on* $\{\epsilon_t^{(j)}\}$ *and* $\{\epsilon_t^{I(j)}\}$*, respectively, and let*

$$IRF^{(j)}(x_t|\epsilon_{g,1}, \epsilon_{R,1:4}) = (\ln x_t^{I(j)} - \ln x_t^{(j)}). \tag{41}$$

Compute medians and percentile bands based on $IRF^{(j)}(x_t|\epsilon_{g,1}, \epsilon_{R,1:4})$*,* $j = 1, \ldots, n_{sim}$*.* □

The monetary intervention is designed to keep interest rates at or near zero for four quarters. We constrained the size of the intervention by two standard deviations of the monetary policy shock so as not to surprise the agents in the economy beyond what is statistically reasonable. The sequence of monetary policy shock to achieve the ZLB is computed for each trajectory j separately. The model has the implication that along some of the trajectories interest rates quickly rise. In this case, keeping them at zero through an anticipated monetary policy shock corresponds to an implausibly large intervention. To avoid implausibly large interventions, we choose the monetary policy shocks in Step 3 such that the difference between the interest rate the obtains with monetary policy intervention does not fall by more than 100bp below the interest rate that would obtain in the absence of a monetary intervention. Thus, we implicitly assume that the FOMC would reneg on a policy to keep interest rates near zero for an extended period of time in states of the world in which output growth and or inflation turn out to be high. In our experiments, this bound of 100bp was never reached.

Figure 11: Sunspot Equilibrium: Fiscal and Monetary Policy



Notes: Figure depicts pointwise medians (solid); 20%-80% percentiles (shaded area) for only fiscal intervention and both interventions.

Impulse responses for a combined fiscal and monetary stimulus under the sunspot equilibrium are depicted in Figure 11. We overlay the responses to a pure fiscal stimulus, discussed previously. The median interest rate with just the fiscal intervention was about 50bp on impact. The monetary intervention is able to reduce that all the way to the ZLB and keep it there for a total of five quarters. In fact, in periods 2 through 4 almost all simulated paths have interest rates at the ZLB. As a result, the output response goes up from about 50bp on impact to about 80bp with some increased response for the duration of the active monetary intervention. After the four quarter period the additional effect from the expansionary monetary intervention dies out quickly.

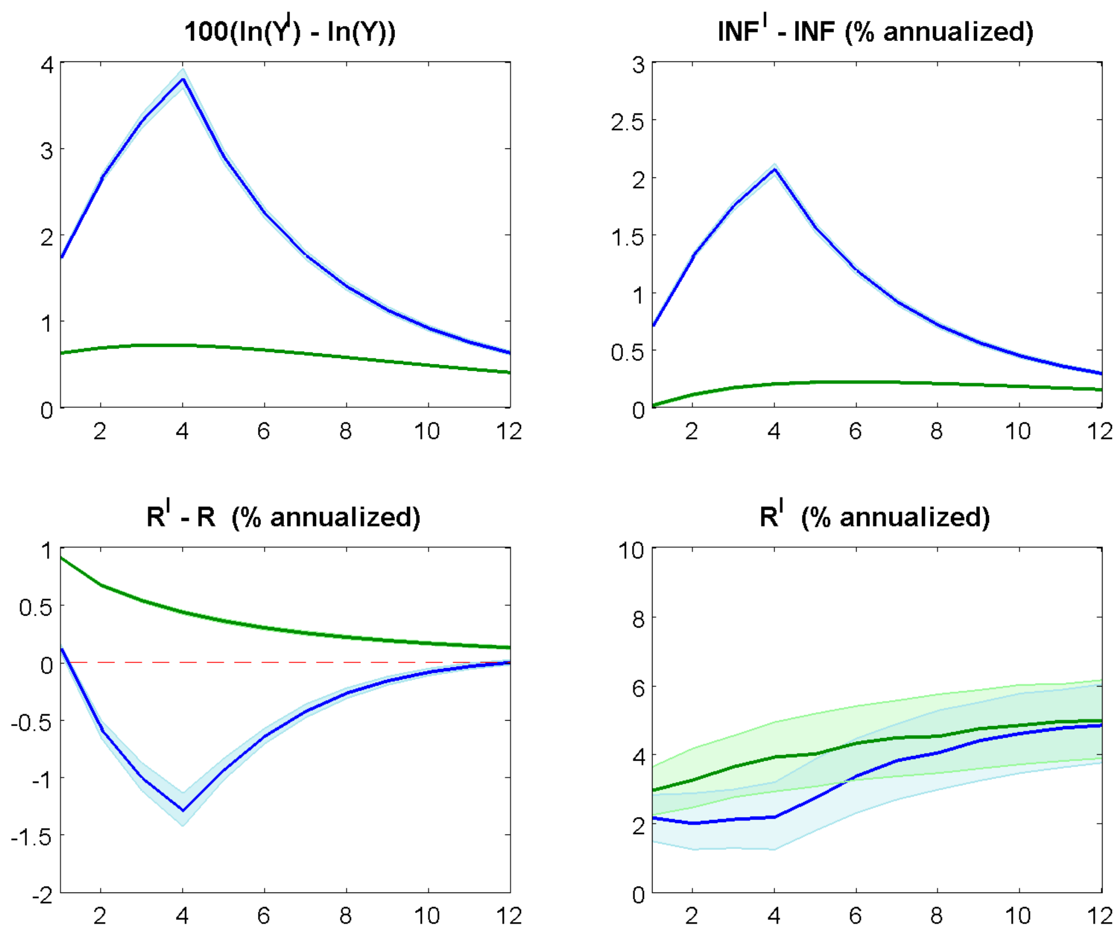
Results for the targeted inflation equilibrium are shown in Figure 12. In the targeted-inflation equilibrium interest rates tend to rise much faster without the monetary intervention because steady state interest rates are high. Thus, there is much more scope for monetary policy interventions because the ZLB poses less of a constraint. The central bank is able to reduce the interest rate by a little under 1% in the first period of the intervention (relative to just the fiscal intervention), reaching about a 2% decrease by the fourth period. This very large monetary intervention leads to an extra 3% increase in output and almost a 2% increase in inflation.

We plot spending multipliers in Figure 13. The multipliers are to be taken with a grain of salt, because we are considering an intervention in which the effect of the fiscal stimulus is amplified by an expansionary monetary policy, which raises the multiplier that we are computing. For the sunspot equilibrium, the multipliers are around 0.9, compared to about 0.7 with just the fiscal intervention. Under the targeted inflation equilibrium since there is much more scope for monetary policy to amplify the fiscal policy, the cumulative multiplier over five periods is about 2.6, relative to 0.8 obtained with just the fiscal intervention. To the extent that the data suggest that the sunspot equilibrium provides a better description of the evolution of the U.S. economy our results indicate that the gains from combining fiscal and monetary policy is modest.

6 Conclusion

We solve a small-scale New Keynesian DSGE model subject to a ZLB constraint on nominal interest rates, considering three equilibria: the standard targeted-inflation equilibrium, a minimum-state-variable deflation equilibrium, and a sunspot equilibrium. We study the

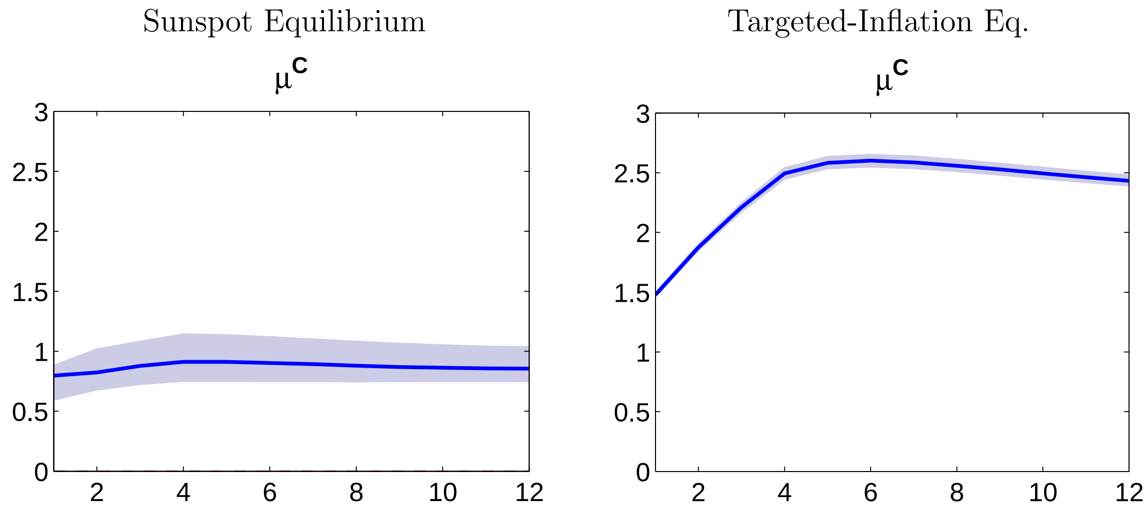
Figure 12: Targeted-Inflation Equilibrium: Fiscal and Monetary Policy



Notes: Figure depicts pointwise medians (solid); 20%-80% percentiles (shaded area) for only fiscal intervention and both interventions.

characteristics of these three equilibria. In terms of its ability to fit U.S. output growth, inflation, and interest rate data between 2000 and 2009, the deflation equilibrium is the least compelling, and the sunspot equilibrium the most compelling equilibrium. For a fiscal expansion that matches the size of the federal contracts, grants, and loans portion of ARRA, we find that the government spending multiplier is about one. According to our analysis, there was no scope in 2009 to amplify the fiscal policy by a conventional expansionary monetary policy, because the fiscal expansion was not strong enough to move the economy away from the ZLB.

Figure 13: Government Spending Multipliers



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Appendix to "Macroeconomic Dynamics Near the ZLB: A Tale of Two Equilibria"

A Solving the Two-Equation Model

The model is characterized by the nonlinear difference equation

$$\mathbb{E}_t[\pi_{t+1}] = \max\left\{\frac{1}{r},\ \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\epsilon_t]\right\}. \tag{A.1}$$

We assume that $r\pi_* \geq 1$ and $\psi > 1$.

The Targeted-Inflation Equilibrium and Deflation Equilibrium. Consider a solution to (A.1) that takes the following form

$$\pi_t = \pi_*\gamma\exp[\lambda\epsilon_t]. \tag{A.2}$$

We now determine values of γ and λ such that (A.1) is satisfied. We begin by calculating the following expectation

$$\begin{aligned}
\mathbb{E}_t[\pi_{t+1}] &= \pi_*\gamma\frac{1}{\sqrt{2\pi\sigma^2}}\int\exp[\lambda\epsilon]\exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right]d\epsilon \\
&= \pi_*\gamma\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{1}{2}\lambda^2\sigma^2\right]\int\exp\left[-\frac{1}{2\sigma^2}(\epsilon-\lambda\sigma^2)^2\right]d\epsilon \\
&= \pi_*\gamma\exp\left[\frac{1}{2}\lambda^2\sigma^2\right].
\end{aligned}$$

Combining this expression with (A.1) yields

$$\gamma\exp[\lambda^2\sigma^2/2] = \max\left\{\frac{1}{r\pi_*},\ \gamma^{\psi}\exp[(\psi\lambda+1)\epsilon_t]\right\}. \tag{A.3}$$

By choosing $\lambda = -1/\psi$ we ensure that the right-hand-side of (A.3) is always constant. Thus, (A.3) reduces to

$$\gamma\exp[\sigma^2/(2\psi^2)] = \max\left\{\frac{1}{r\pi_*},\ \gamma^{\psi}\right\} \tag{A.4}$$

Depending on whether the nominal interest rate is at the ZLB ($R_t = 1$) or not, we obtain two solutions for γ by equating the left-hand-side of (A.4) with either the first or the second term in the max operator:

$$\gamma_D = \frac{1}{r\pi_*}\exp\left[-\frac{\sigma^2}{2\psi^2}\right] \quad \text{and} \quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi-1)\psi^2}\right]. \tag{A.5}$$

The derivation is completed by noting that

$$\begin{aligned}
\gamma_D^{\psi} &= \frac{1}{r\pi_*}\exp\left[-\frac{\sigma^2}{2\psi}\right] \le \frac{1}{r\pi_*} \\
\gamma_*^{\psi} &= \exp\left[\frac{\sigma^2}{2(\psi-1)\psi}\right] \ge 1 \ge \frac{1}{r\pi_*}.
\end{aligned}$$

A Sunspot Equilibrium. Let $s_t \in \{0,1\}$ denote the Markov-switching sunspot process. Assume that the system is in the targeted-inflation regime if $s_t = 1$ and that it is in the deflation regime if $s_t = 0$ (the 0 is used to indicate that the system is near the ZLB). The probabilities of staying in state 0 and 1, respectively, are denoted by ψ_{00} and ψ_{11}. We conjecture that the inflation dynamics follow the process

$$\pi_t^{(s)} = \pi_* \gamma(s_t)\exp[-\epsilon_t/\psi] \tag{A.6}$$

In this case condition (A.4) turns into

$$\begin{aligned}
\mathbb{E}_t[\pi_{t+1}|s_t=0]/\pi_* &= \big(\psi_{00}\gamma(0) + (1-\psi_{00})\gamma(1)\big)\exp[\sigma^2/(2\psi^2)] = \frac{1}{r\pi_*} \\
\mathbb{E}_t[\pi_{t+1}|s_t=1]/\pi_* &= \big(\psi_{11}\gamma(1) + (1-\psi_{11})\gamma(0)\big)\exp[\sigma^2/(2\psi^2)] = [\gamma(1)]^{\psi}.
\end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the Markov-transition probabilities ψ_{00} and ψ_{11}. Then (A.6) is a stable solution of (A.1) provided that

$$[\gamma(0)]^{\psi} \le \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^{\psi} \ge \frac{1}{r\pi_*}.$$

Sunspot Shock is Correlated with Fundamentals. As before, let $s_t \in \{0,1\}$ be a Markov-switching sunspot process. However, now assume that a state transition is triggered by certain realizations of the monetary policy shock ϵ_t. In particular, if $s_t = 0$, then suppose $s_{t+1} = 0$ whenever $\epsilon_{t+1} \le \underline{\epsilon}_0$, such that

$$\psi_{00} = \Phi(\underline{\epsilon}_0),$$

where $\Phi(\cdot)$ is the cumulative density function of a $N(0,1)$. Likewise, if $s_t = 1$, then let $s_{t+1} = 0$ whenever $\epsilon_{t+1} > \underline{\epsilon}_0$, such that

$$\psi_{11} = 1 - \Phi(\underline{\epsilon}_1).$$

To find the constants $\gamma(0)$ and $\gamma(1)$, we need to evaluate

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\underline{\epsilon}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon+\sigma^2/\psi)^2\right] d\epsilon \\
&= \ \mathbb{P}\left\{\frac{\epsilon+\sigma^2/\psi}{\sigma} \leq \frac{\underline{\epsilon}+\sigma^2/\psi}{\sigma}\right\} = \Phi\left(\frac{\underline{\epsilon}+\sigma^2/\psi}{\sigma}\right).
\end{aligned}
$$

Thus, condition (A.4) turns into

$$
\begin{aligned}
\frac{1}{r\pi_*} &= \left[\gamma(0)\Phi(\underline{\epsilon}_0)\Phi\left(\frac{\underline{\epsilon}_0+\sigma^2/\psi}{\sigma}\right)+\gamma(1)(1-\Phi(\underline{\epsilon}_0))\left(1-\Phi\left(\frac{\underline{\epsilon}_0+\sigma^2/\psi}{\sigma}\right)\right)\right]\exp[\sigma^2/(2\psi^2)] \\
\gamma^{\psi}(1) &= \left[\gamma(1)(1-\Phi(\underline{\epsilon}_1))\left(1-\Phi\left(\frac{\underline{\epsilon}_1+\sigma^2/\psi}{\sigma}\right)\right)+\gamma(0)\Phi(\underline{\epsilon}_1)\Phi\left(\frac{\underline{\epsilon}_1+\sigma^2/\psi}{\sigma}\right)\right]\exp[\sigma^2/(2\psi^2)].
\end{aligned}
$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the thresholds $\underline{\epsilon}_0$ and $\underline{\epsilon}_1$. Then (A.6) is a stable solution of (A.1) provided that

$$
[\gamma(0)]^{\psi} \leq \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^{\psi} \geq \frac{1}{r\pi_*}.
$$

Benhabib, Schmitt-Grohé, and Uribe (2001a) Dynamics. BSGU constructed equilibria in which the economy transitioned from the targeted-inflation equilibrium to the deflation equilibrium. Consider the following law of motion for inflation

$$
\pi_t^{(BGSU)} = \pi_*\gamma_* \exp[-\epsilon_t/\psi]\exp\big[-\psi^{t-t_0}\big]. \tag{A.7}
$$

Here, γ_* was defined in (A.5) and $-t_0$ can be viewed as the initialization period for the inflation process. We need to verify that $\pi_t^{(BGSU)}$ satisfies (A.1). From the derivations that lead to (A.4) we deduce that

$$
\gamma_*\mathbb{E}_{t+1}\big[\exp[-\epsilon_{t+1}/\psi]\big] = \gamma_*^{\psi}.
$$

Since

$$
\exp\big[-\psi^{t+1-t_0}\big] = \big(\exp\big[-\psi^{t-t_0}\big]\big)^{\psi},
$$

we deduce that the law of motion for $\pi_t^{(BGSU)}$ in (A.7) satisfies the relationship

$$
\mathbb{E}_t[\pi_{t+1}] = \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\epsilon_t].
$$

Moreover, since $\psi > 1$ the term $\exp\big[-\psi^{t-t_0}\psi\big] \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, the economy will move away from the targeted-inflation equilibrium and at some suitably defined t_* reach the deflation equilibrium and remain there permanently. Overall the inflation dynamics take the form

$$\pi_t = \pi_* \begin{cases} \gamma_* \exp[-\epsilon_t/\psi] \exp\big[-\psi^{t-t_0}\big] & \text{if } t \leq t_* \\ \gamma_D \exp[-\epsilon_t/\psi] & \text{otherwise} \end{cases}, \tag{A.8}$$

where γ_* and γ_D were defined in (A.5).

Alternative Deflation Equilibria. Around the deflation steady state the system is locally indeterminate. This suggests that we might be able to construct alternative solutions to (A.1). Consider the following conjecture for inflation

$$\pi_t = \pi_* \gamma \min\big\{ \exp[-c/\psi],\, \exp[-\epsilon/\psi]\big\}, \tag{A.9}$$

where c is a cutoff value. The intuition for this solution is the following. Large positive shocks ϵ that could push the nominal interest rate above one, are off-set by downward movements in inflation. Negative shocks do not need to be off-set, because they push the desired gross interest rate below one and the max operator in the policy rule keeps the interest rate at one. Formally, we can compute the expected value of inflation as follows:

$$\begin{aligned} \mathbb{E}_t[\pi_{t+1}] &= \pi_* \gamma \Bigg[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{c} \exp[-c/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right] d\epsilon \\ &\quad \frac{1}{\sqrt{2\pi\sigma^2}} \int_{c}^{\infty} \exp[-\epsilon/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right] d\epsilon \\ &= \pi_* \gamma \Bigg[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \int_c^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon + \sigma^2/\psi)^2\right] d\epsilon \Bigg] \\ &= \pi_* \gamma \Bigg[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi}\right)\right) \Bigg] \end{aligned} \tag{A.10}$$

Here $\Phi(\cdot)$ denotes the cdf of a standard Normal random variable. Now define

$$f(c,\psi,\sigma) = \left[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi}\right)\right) \right].$$

Then another solution for which interest rates stay at the ZLB is given by

$$\bar{\gamma} = \frac{1}{r_* \pi_* f(c,\psi,\sigma)}$$

It can be verified that for c small enough the condition

$$\frac{1}{r_* \pi_*} \geq \bar{\gamma}^{\psi} \min\Big\{ \exp[-c+\epsilon],\, 1 \Big\}$$

is satisfied.

B Model Solution

The equilibrium conditions (in terms of detrended variables, i.e., $c_t = C_t/A_t$ and $y_t = Y_t/A_t$) take the form

$$\begin{aligned}
1 &= \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right] && \text{(A.11)} \\
1 &= \frac{1}{\nu} (1 - c_t^\tau) + \phi(\pi_t - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\bar{\pi}}{2\nu} \right] && \text{(A.12)} \\
&\quad - \phi \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \\
c_t &= \left[\frac{1}{g_t} - \frac{\phi}{2} (\pi_t - \bar{\pi})^2 \right] y_t && \text{(A.13)} \\
R_t &= \max \left\{ 1, \left[r \pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{y_t}{y_{t-1}} z_t \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}. && \text{(A.14)}
\end{aligned}$$

B.1 Approximation Near the Targeted-Inflation Steady State

Steady State. Steady state inflation equals π_*. Let $\lambda = \nu(1-\beta)$, then

$$\begin{aligned}
r &= \gamma/\beta \\
R_* &= r\pi_* \\
c_* &= \left[1 - v - \frac{\phi}{2}(1-2\lambda) \left(\pi_* - \frac{1-\lambda}{1-2\lambda} \bar{\pi} \right)^2 + \frac{\phi}{2} \frac{\lambda^2}{1-2\lambda} \bar{\pi}^2 \right]^{1/\tau} \\
y_* &= \frac{c_*}{\left[\frac{1}{g_*} - \frac{\phi}{2} (\pi_* - \bar{\pi})^2 \right]}.
\end{aligned}$$

Log-linearization. We omit the hats from variables that capture deviations from the targeted-inflation steady state. The linearized consumption Euler equation (A.11) is

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau} (R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).$$

The price setting equation (A.12) takes the form

$$\begin{aligned}
0 &= -\frac{\tau c_*^\tau}{\nu} c_t + \phi \pi_* \left[\left(1 - \frac{1}{2\nu} \right) \pi_* + \frac{\bar{\pi}}{2\nu} \right] \pi_t + \phi \pi_* (\pi_* - \bar{\pi}) \left(1 - \frac{1}{2\nu} \right) \pi_t \\
&\quad - \phi \beta \pi_* (\pi_* - \bar{\pi}) \Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}] \Big) - \phi \beta \pi_*^2 \mathbb{E}_t[\pi_{t+1}].
\end{aligned}$$

Log-linearizing the aggregate resource constraint (A.12) yields

$$c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\pi_* - \bar{\pi})^2} g_t - \frac{\phi \pi_* (\pi_* - \bar{\pi})}{1/g_* - \phi(\pi_* - \bar{\pi})^2} \pi_t$$

Finally, the monetary policy rule becomes

$$R_t = \max \left\{ -\ln(r\pi_*),\ (1-\rho_R)\psi_1 \pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R \epsilon_{R,t} \right\}.$$

B.2 Approximation Near the Deflation Steady State

Steady State. As before, let $\lambda = \nu(1-\beta)$. The steady state nominal interest rate is $R_D = 1$ and provided that $\beta/(\gamma\pi_*) < 1$ and $\psi_1 > 1$:

$$\begin{aligned} r &= \gamma/\beta \\ \pi_D &= \beta/\gamma \\ c_D &= \left[1 - \upsilon - \frac{\phi}{2}(1-2\lambda)\left(\pi_D - \frac{1-\lambda}{1-2\lambda}\bar{\pi} \right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2 \right]^{1/\tau} \\ y_D &= \frac{c_D}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_D - \bar{\pi})^2 \right]}. \end{aligned}$$

Log-linearization. We omit the tildes from variables that capture deviations from the deflation steady state. The linearized consumption Euler equation (A.11) is

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).$$

The price setting equation (A.12) takes the form

$$\begin{aligned} 0 &= -\frac{\tau c_D^\tau}{\nu} c_t + \phi\beta \left[\left(1 - \frac{1}{2\nu} \right) \beta + \frac{\bar{\pi}}{2\nu} \right] \pi_t + \phi\beta(\beta - \bar{\pi}) \left(1 - \frac{1}{2\nu} \right) \pi_t \\ &\quad -\phi\beta^2(\beta - \bar{\pi}) \Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}] \Big) - \phi\beta^3 \mathbb{E}_t[\pi_{t+1}]. \end{aligned}$$

Log-linearizing the aggregate resource constraint (A.12) yields

$$c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\beta - \bar{\pi})^2} g_t - \frac{\phi\beta(\beta - \bar{\pi})}{1/g_* - \phi(\beta - \bar{\pi})^2} \pi_t$$

Finally, the monetary policy rule becomes

$$\begin{aligned} R_t &= \max \Big\{ 0,\ -(1-\rho_R)\ln(r\pi_*) - (1-\rho_R)\psi_1 \ln(\pi_*/\beta) \\ &\quad +(1-\rho_R)\psi_1\pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R \epsilon_{R,t} \Big\}. \end{aligned}$$

B.3 Targeted-Inflation Equilibrium in Simplified Model

After imposing the parameter restrictions discussed in the main text we obtain the system (omitting hats)

$$\begin{aligned} R_t &= \max\left\{ -\ln(r\pi_*),\ \psi\pi_t + \sigma_R\epsilon_{R,t} \right\} \\ c_t &= \mathbb{E}_t[c_{t+1}] - (R_t - \mathbb{E}_t[\pi_{t+1}]) \\ \pi_t &= \beta\mathbb{E}_t[\pi_{t+1}] + \kappa c_t \end{aligned} \tag{A.15}$$

Since the conjectured law of motion is *iid*, the conditional expectations of inflation and consumption equal their unconditional means which we denote by μ_π and μ_c, respectively. In turn, the Euler equation in (A.15) simplifies to the static relationship

$$c_t = -R_t + \mu_c + \mu_\pi. \tag{A.16}$$

Similarly, the Phillips curve in (A.15) becomes

$$\pi_t = \kappa c_t + \beta\mu_\pi. \tag{A.17}$$

Combining (A.16) and (A.17) yields

$$\pi_t = -\kappa R_t + (\kappa + \beta)\mu_\pi + \kappa\mu_c. \tag{A.18}$$

We now can use (A.18) to eliminate inflation from the monetary policy rule:

$$R_t = \max\left\{ -\ln(r\pi_*),\ -\kappa\psi R_t + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \right\} \tag{A.19}$$

Define

$$R_t^{(1)} = -\ln(r\pi_*) \quad \text{and} \quad R_t^{(2)} = \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big].$$

Let $\bar{\epsilon}_{R,t}$ be the value of the monetary policy shock for which $R_t = -\ln(r\pi_*)$ and the two terms in the max operator of (A.19) are equal

$$\sigma_R\bar{\epsilon}_{R,t} = -(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi - \kappa\psi\mu_c.$$

To complete the derivation of the equilibrium interest rate it is useful to distinguish the following two cases. Case (i): suppose that $\epsilon_{R,t} < \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(1)}$ is consistent with (A.19). If the monetary policy shock is less than the threshold value, then

$$(\kappa + \beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} < -(1 + \kappa\psi)\ln(r\pi_*).$$

Thus,

$$-\kappa\psi R_t^{(1)} + (\kappa + \beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} < -\kappa\psi R_t^{(1)} - (1 + \kappa\psi)\ln(r\pi_*) = -\ln(r\pi_*),$$

which confirms that (A.19) is satisfied.

Case (ii): suppose that $\epsilon_{R,t} > \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(2)}$ is consistent with (A.19). If the monetary policy shock is greater than the threshold value, then

$$(\kappa + \beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} > -(1 + \kappa\psi)\ln(r\pi_*).$$

In turn,

$$\begin{aligned}
&-\kappa\psi R_t^{(2)} + (\kappa + \beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&= \; -\frac{\kappa\psi}{1 + \kappa\psi}\left[(\kappa + \beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right] + (\kappa + \beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&= \; \frac{1}{1 + \kappa\psi}\left[(\kappa + \beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right] \\
&> \; -\ln(r\pi_*),
\end{aligned}$$

which confirms that (A.19) is satisfied.

We can now deduce that

$$R_t = \max\left\{-\ln(r\pi_*),\ \frac{1}{1 + \kappa\psi}\left[\psi(\kappa + \beta)\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right]\right\}. \tag{A.20}$$

Combining (A.16) and (A.20) yields equilibrium consumption

$$c_t = \begin{cases} \frac{1}{1+\kappa\psi}\left[(1 - \psi\beta)\mu_\pi + \mu_c - \sigma_R\epsilon_{R,t}\right] & \text{if } R_t \geq -\ln(r\pi_*) \\ \ln(r\pi_*) + \mu_c + \mu_\pi & \text{otherwise} \end{cases}. \tag{A.21}$$

Likewise, combining (A.17) and (A.20) delivers equilibrium inflation

$$\pi_t = \begin{cases} \frac{1}{1+\kappa\psi}\left[(\kappa + \beta)\mu_\pi + \kappa\mu_c - \kappa\sigma_R\epsilon_{R,t}\right] & \text{if } R_t \geq -\ln(r\pi_*) \\ \kappa\ln(r\pi_*) + (\kappa + \beta)\mu_\pi + \kappa\mu_c & \text{otherwise} \end{cases}. \tag{A.22}$$

Equations (A.20), (A.21), and (A.22) appear in the main text.

If $X \sim N(\mu, \sigma^2)$ and C is a truncation constant, then

$$\mathbb{E}[X|X \geq C] = \mu + \frac{\sigma \phi_N(\alpha)}{1 - \Phi_N(\alpha)},$$

where $\alpha = (C - \mu)/\sigma$, $\phi_N(x)$ and $\Phi_N(\alpha)$ are the probability density function (pdf) and the cumulative density function (cdf) of a $N(0, 1)$. Define the cutoff value

$$C = -(1 + \kappa\psi) \ln(r\pi_*) - (\kappa + \beta)\psi\mu_\pi - \kappa\psi\mu_c. \tag{A.23}$$

Using the definition of a cdf and the formula for the mean of a truncated Normal random variable, we obtain that

$$\begin{aligned} \mathbb{P}[\epsilon_{R,t} \geq C/\sigma_R] &= 1 - \Phi_N(C_y/\sigma_R) \\ \mathbb{E}[\epsilon_{R,t} \,|\, \epsilon_{R,t} \geq C/\sigma_R] &= \frac{\sigma_R \phi_N(C/\sigma_R)}{1 - \Phi_N(C/\sigma_R)}. \end{aligned}$$

Thus,

$$\begin{aligned} \mu_c &= \frac{1 - \Phi_N(C_y/\sigma_R)}{1 + \kappa\psi} \left[(1 - \psi\beta)\mu_\pi + \mu_c \right] - \frac{\sigma_R \phi_N(C_y/\sigma_R)}{(1 + \kappa\psi)(1 - \Phi_N(C_y/\sigma_R))} \qquad \text{(A.24)} \\ &\quad + \Phi_N(C_y/\sigma_R) \left[\ln(r\pi_*) + \mu_c + \mu_\pi \right] \\ \mu_\pi &= \frac{1 - \Phi_N(C_y/\sigma_R)}{1 + \kappa\psi} \left[(\kappa + \beta)\mu_\pi + \kappa\mu_c \right] - \frac{\kappa\sigma_R \phi_N(C_y/\sigma_R)}{(1 + \kappa\psi)(1 - \Phi_N(C_y/\sigma_R))} \qquad \text{(A.25)} \\ &\quad + \Phi_N(C_y/\sigma_R) \left[\kappa \ln(r\pi_*) + (\kappa + \beta)\mu_\pi + \kappa\mu_c \right] \end{aligned}$$

The constants C, μ_c and μ_π can be obtained by solving the system of nonlinear equations comprised of (A.23) to (A.25).

C Details of the Solution Algorithm

Algorithm 3 (Solution Algorithm) *1. Start with a guess for Θ. For the targeted-inflation equilibrium, this guess is obtained from a linear approximation around the inflation target. For the deflation equilibrium, it is obtained by assuming constant decision rules at the deflation steady state. For the sunspot equilibrium it is obtained* *by letting the $s_t = 1$ decision rules come from the targeted-inflation equilibrium and the $s_t = 0$ decision rules come from the deflation equilibrium.*

2. *Given this guess, simulate the model for a large number of periods.*

3. *Given the simulated path, obtain the grid for the state variables over which the approximation needs to be accurate. For the targeted-inflation equilibrium, half of these grid points come from the ergodic distribution, obtained using a cluster-grid algorithm as in Judd, Maliar, and Maliar (2011). The other half come from the filtered exogenous state variables from 2009. Label these grid points as $\{\mathcal{S}_1, ..., \mathcal{S}_M\}$.*[6]

4. *Solve for the Θ by minimizing the sum of squared residuals obtained following the steps below, using a variant of a Newton algorithm.*

   (a) *For a generic grid point $\mathcal{S}_i$, and the current value for Θ, compute $f_\pi^1(\mathcal{S}_i;\Theta)$, $f_\pi^2(\mathcal{S}_i;\Theta)$, $f_\mathcal{E}^1(\mathcal{S}_i;\Theta)$ and $f_\mathcal{E}^2(\mathcal{S}_i;\Theta)$.*

   (b) *Assume $\zeta_i \equiv I\{R(\mathcal{S}_i,\Theta) > 1\} = 1$ and compute π_i, $\mathcal{E}_i$, as well as y_i and c_i using (33) and (34), substituting in (35).*

   (c) *If R_i that follows from (35) using π_i and y_i obtained in (b) is greater than unity, then ζ_i is indeed equal to one. Otherwise, set $\zeta_i = 0$ (and thus $R_i = 1$) and recompute all other objects.*

   (d) *The final step is to compute the residual functions. There are four residuals, corresponding to the four functions being approximated. For a given set of state variables $\mathcal{S}_i$, only two of them will be relevant since we either need the constrained decision rules or the unconstrained ones. Regardless, the relevant residual functions will be given by*

$$\mathcal{R}^1(\mathcal{S}_i) = \mathcal{E}_i - \left[\int\int\int \frac{c(\mathcal{S}')^{-\tau}}{\gamma z'\pi(\mathcal{S}')} dF(z')\,dF(g')\,dF(\epsilon_R')\right] \tag{A.26}$$

$$\mathcal{R}^2(\mathcal{S}_i) = f(c_i,\pi_i,y_i) - \phi\beta\int\int\int c(\mathcal{S}')^{-\tau} y(\mathcal{S}')\left[\pi(\mathcal{S}') - \bar{\pi}\right]\pi(\mathcal{S}')dF(z')\,dF(g')\,dF(\epsilon_R') \tag{A.27}$$

[6] For the deflation equilibrium, we use a time-separated grid algorithm which suits the behavior of this equilibrium better. For the sunspot equilibrium, we use the same time-separated grid algorithm.

Note that this step involves computing $\pi(\mathcal{S}')$, $y(\mathcal{S}')$, $c(\mathcal{S}')$ and $R(\mathcal{S}')$ which is done following steps (a)-(c) above for each value of $\mathcal{S}'$. We use a non-product monomial integration rule to evaluate these integrals.

(e) *The objective function to be minimized is the sum of squared residuals obtained in (d).*

5. *Repeat steps 2-4 sufficient number of times so that the ergodic distribution remains unchanged from one iteration to the next. For the targeted-inflation equilibrium, we also iterate between solution and filtering to make sure the filtered states used in the solution grid remain unchanged.*

D Particle Filter

The particle filter is used to extract information about the state variables of the model from data on output growth, inflation, and nominal interest rates over the period 2000:Q1 to 2010:Q4. Throughout this section we focus on the particle filter for the sunspot equilibrium because it involves an additional state variable. The analysis for the targeted-inflation equilibrium and the deflation equilibrium is a special case in which the discrete state s_t is constant.

D.1 State-Space Representation

Let y_t be the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates. The vector x_t stacks the continuous state variables which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$ and $s_t \in \{0, 1\}$ is the Markov-switching process.

$$y_t = \Psi(x_t) + \nu_t \tag{A.28}$$

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \tag{A.29}$$

$$x_t = F_{s_t}(x_{t-1}, \epsilon_t) \tag{A.30}$$

The first equation is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. We subsequently use the densities $p(y_t|s_t)$, $p(s_t|s_{t-1})$, and $p(x_t|x_{t-1}, s_t)$ to summarize the measurement and the state transition equations.

Let $z_t = [x_t', s_t]'$ and $Y_{t_0:t_1} = \{y_{t_0}, \ldots, y_{t_1}\}$. The distribution $p(z_t|Y_{1:t})$ is approximated by a set of pairs $\{(z_t^{(i)}, \pi_t^{(i)})\}_{i=1}^N$, where $z_t^{(i)}$ is the i'th particle, $\pi_t^{(i)}$ is its weight, and N is the number of particles. The particles $z_t^{(i)}$ are generated from some proposal density and the $\pi_t^{(i)}$'s correspond to normalized weights in an importance sampling approximation:

$$
\begin{aligned}
\mathbb{E}[f(z_t)|Y_{1:t}] &= \int_{z_t} f(z_t) \frac{p(y_t|z_t)p(z_t|Y_{1:t-1})}{p(y_t|Y_{1:t-1})} dz_t \\
&= \int_{z_{t-1:t}} f(z_t) \frac{p(y_t|z_t)p(z_t|z_{t-1})p(z_{t-1}|Y_{1:t-1})}{p(y_t|Y_{1:t-1})} dz_{t-1:t} \\
&\approx \frac{\sum_{i=1}^N f(z_t^{(i)}) \left(\frac{1}{N} \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})p(z_{t-1}^{(i)}|Y_{1:t-1})}{g(z_{t-1:t}^{(i)}|Y_{1:t})} \right)}{\sum_{j=1}^N \left(\frac{1}{N} \frac{p(y_t|z_t^{(j)})p(z_t^{(j)}|z_{t-1}^{(j)})p(z_{t-1}^{(j)}|Y_{1:t-1})}{g(z_{t-1:t}^{(j)}|Y_{1:T})} \right)} \\
&= \sum_{i=1}^N f(z_t^{(i)}) \frac{\tilde{\pi}_t^{(i)}}{\sum_{j=1}^N \tilde{\pi}_t^{(j)}} = \sum_{i=1}^N f(z_t^{(i)}) \pi_t^{(i)},
\end{aligned}
$$

where the un-normalized and normalized probability weights are given by

$$
\tilde{\pi}_t^{(i)} = \frac{1}{N} \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})p(z_{t-1}^{(i)}|Y_{1:t-1})}{g(z_{t-1:t}^{(i)}|Y_{1:T})} \quad \text{and} \quad \pi_t^{(i)} = \frac{\tilde{\pi}_t^{(i)}}{\sum_{j=1}^N \tilde{\pi}_t^{(j)}},
$$

respectively, and the $z_{t-1:t}^{(i)}$'s are drawn from a probability distribution with a density that is proportional to $g(z_{t-1:t}^{(i)}|Y_{1:t})$. In particular, we adopt an approach known as auxiliary particle filtering, e.g. Pitt and Shephard (1999), and consider proposal densities of the form

$$
g(z_{t-1:t}|Y_{1:t}) \propto p(z_{t-1}|Y_{1:t-1}) q(z_t|z_{t-1}, y_t)
$$

such that

$$
\tilde{\pi}_t^{(i)} = \frac{1}{N} \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})}{q(z_t^{(i)}|z_{t-1}^{(i)}, y_t)}. \tag{A.31}
$$

Since our model has discrete and continuous state variables, we write

$$p(z_t|z_{t-1}) = \begin{cases} p_0(x_t|x_{t-1}, s_t = 0)\mathbb{P}\{s_t = 0|s_{t-1}\} & \text{if } s_t = 0 \\ p_1(x_t|x_{t-1}, s_t = 1)\mathbb{P}\{s_t = 1|s_{t-1}\} & \text{if } s_t = 1 \end{cases}$$

and consider proposal densities of the form

$$q(z_t|z_{t-1}, y_t) = \begin{cases} q_0(x_t|x_{t-1}, y_t, s_t = 0)\lambda(z_{t-1}, y_t) & \text{if } s_t = 0 \\ q_1(x_t|x_{t-1}, y_t, s_t = 1)(1 - \lambda(z_{t-1}, y_t)) & \text{if } s_t = 1 \end{cases},$$

where $\lambda(x_{t-1}, y_t)$ is the probability that $s_t = 0$ under the proposal distribution.

D.2 Filtering

The particle filter generates the importance sampling approximation of $p(z_t|Y_{1:t})$ sequentially for $t = 1, \ldots, T$.

Initialization. To generate the initial set of particles $\{(z_0^{(i)}, \pi_0^{(i)}\}_{i=1}^N$, for each i simulate the DSGE model for T_0 periods, starting from the targeted-inflation steady state, and set $\pi_0^{(i)} = 1/N$.

Sequential Importance Sampling. For $t = 1$ to T:

1. $\{z_{t-1}^{(i)}, \pi_{t-1}^{(i)}\}_{i=1}^N$ is the particle approximation of $p(z_{t-1}|Y_{1:t-1})$. For $i = 1$ to N:

   (a) Draw $z_t^{(i)}$ conditional on $z_{t-1}^{(i)}$ from $q(z_t|z_{t-1}^{(i)}, y_t)$.

   (b) Compute the unnormalized particle weights $\tilde{\pi}_t^{(i)}$. Period $t-1$ particles were effectively sampled with probability $1/N$ instead of drawn from the mixture

   $$\hat{p}(z_{t-1}|Y_{1:t-1}) = \sum_{j=1}^N \pi_{t-1}^{(j)} \delta(z_{t-1} - z_{t-1}^{(j)}),$$

   where $\delta(x)$ is the dirac function with the properties $\delta(0) = \infty$, $\delta(x) = 0$ if $x \neq 0$ and $\int \delta(x)dx = 1$. Thus, (A.31)needs to be adjusted by $\pi_{t-1}^{(i)}$:

   $$\tilde{\pi}_t^{(i)} = \frac{1}{N} \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})}{q(z_t^{(i)}|z_{t-1}^{(i)}, y_t)} \pi_{t-1}^{(i)}.$$

2. Compute the normalized particle weights $\pi_t^{(i)}$ and the effective sample size $ESS_t = 1/\sum_{i=1}^N (\pi_t^{(i)})^2$.

3. Resample the particles via deterministic resampling (see Kitagawa (1996)). Reset weights to be $\pi_t^{(i)} = 1/N$ and approximate $p(z_t|Y_{1:t})$ by $\{(z_t^{(i)}, \pi_t^{(i)})\}_{i=1}^n$.

D.3 Tuning of the Filter

In the empirical analysis we set $T_0 =$ and $N =$. The mean of ESS_t is; the max is ???; and the min is ???. The particle filter approximation deteriorates in periods ???

Targeted-Inflation Equilibrium. Since the discrete state s_t is irrelevant in this equilibrium, let $z_t = x_t$. Due to the nonlinear state transition it is difficult to evaluate $p(z_t|z_{t-1})$ directly. Recall that $x_t = F(x_{t-1}, \epsilon_t)$. Let $p_\epsilon(\epsilon_t)$ be the DSGE model-implied density of the innovation distribution, which is $\epsilon_t \sim N(0, I)$. Conditional on $x_{t-1}^{(i)}$ and y_t we apply the Kalman-filter updating equations to a log-linearized version of the DSGE model to obtain a preliminary estimate $\hat{\epsilon}_{t|t}$. We then generate a draw from $\epsilon_t^{(i)} \sim N(\hat{\epsilon}_{t|t}, I)$, denoting the density associated with this distribution by $q_\epsilon(\epsilon_t)$ and let $x_t^{(i)} = F(x_{t-1}^{(i)}, \epsilon_t^{(i)})$. In slight abuse of notation (ignoring that the dimension of x_t is larger than the dimension of ϵ_t and that its distribution is singular), we can apply the change of variable formula to obtain a representation of the proposal density

$$q(x_t^{(i)}|x_{t-1}^{(i)}) = q_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

Using the same change-of-variable formula, we can represent

$$p(x_t^{(i)}|x_{t-1}^{(i)}) = p_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

By construction, the Jacobian terms cancel and the ratio that is needed to calculate the unnormalized particle weights for period t in (A.31) simplifies to

$$\frac{p(x_t^{(i)}|x_{t-1}^{(i)})}{q(x_t^{(i)}|x_{t-1}^{(i)})} = \frac{p_\epsilon(\epsilon_t^{(i)})}{q_\epsilon(\epsilon_t^{(i)})}.$$

Sunspot Equilibrium. Discuss choice of $\lambda(z_{t-1}, y_t)$.

E Calibration of the Policy Experiment

Table A-1 summarizes the award and disbursements of funds for federal contracts, grants, and loans. We translate the numbers in the table into a one-period location shift of the distribution of $\epsilon_{g,t}$. In our model total government spending is a fraction ζ_t of aggregate output, where ζ_t evolves according to an exogenous process:

$$G_t = \zeta_t Y_t; \quad \zeta_t = 1 - \frac{1}{g_t}; \quad \ln(g_t/g_*) = \rho_g \ln(g_{t-1}/g_*) + \sigma_g \epsilon_{g,t}$$

For the subsequent calibration of the fiscal intervention it is convenient to define the percentage deviations of g_t and ζ_t from their respective steady states: $\hat{g}_t = \ln(g_t/g_*)$ and $\hat{\zeta}_t = \ln(\zeta_t/\zeta_*)$. According to the parameterization of the DSGE model in Table 1 $\zeta_* = 0.15$ and $g_* = 1.177$. Thus, government spending is approximately 15% of GDP. We assume that the fiscal expansion approximately shifts $\hat{\zeta}_t$ to $\hat{\zeta}_t^I = \hat{\zeta}_t + \hat{\zeta}_t^{ARRA}$.

We construct $\hat{\zeta}_t^{ARRA}$ as follows. Let G_t^{ARRA} correspond to the additional government spending stipulated by ARRA. Since we focus on received rather than awarded funds, G_t^{ARRA} corresponds to the third column of Table A-1. The size of the fiscal expansion as a fraction of GDP is

$$\zeta_t^{ARRA} = G_t^{ARRA}/Y_t,$$

where Y_t here corresponds to the GDP data reported in the last column of Table A-1. We then divide by ζ_* to convert it into deviations from the steady state level: $\hat{\zeta}_t^{ARRA} = \zeta_t^{ARRA}/\zeta_*$. Taking a log-linear approximation of the relationship between g_t and ζ_t leads to

$$\hat{g}_t^{ARRA} = 0.177 \cdot G_t^{ARRA}/(\zeta_* Y_t).$$

Table A-1: ARRA Funds for Contracts, Grant, and Loans

	Awarded	Received	Nom. GDP
2009:3	158	36	3488
2009:4	17	18	3533
2010:1	26	8	3568
2010:2	16	24	3603
2010:3	33	26	3644
2010:4	9	21	3684
2011:1	4	19	3704
2011:2	4	20	3751
2011:3	8	17	3791
2011:4	0	12	3830
2012:1	3	9	3870
2012:2	0	8	3899

Source: www.recovery.org.

Table A-2: Fiscal Multipliers

Horizon	Fiscal and Monetary		Fiscal Only	
	π_* Eq.	Sunspot Eq.	π_* Eq.	Sunspot Eq.
1	1.48	0.80	0.59	0.53
2	1.87	0.82	0.65	0.59
3	2.21	0.88	0.71	0.63
4	2.49	0.91	0.76	0.67
5	2.58	0.91	0.80	0.68
6	2.60	0.90	0.83	0.69
7	2.59	0.89	0.86	0.70
8	2.56	0.88	0.88	0.71
9	2.53	0.87	0.90	0.71
10	2.49	0.86	0.92	0.71
11	2.46	0.86	0.93	0.71
12	2.43	0.85	0.95	0.71