Invertability

MA(q):

$y_t - \mu = \theta(L)\varepsilon_t$

$\theta(L) = (1 + \theta_1 L^1 + \theta_2 L^2 + \ldots + \theta_q L^q) = (1 - \eta_1 L)(1 - \eta_2 L)\ldots(1 - \eta_q L)$

$\theta(L)^{-1} = (\sum_{j=0}^{\infty} \eta_1^j L^j)\ldots(\sum_{j=0}^{\infty} \eta_q^j L^j)$

In this case, the process $\{y_t\}$ is invertible: $\varepsilon_t = \theta)(L)^{-1}(y_t - \mu)$

or $\{\varepsilon_t\}$ can be expressed as a causal function of $\{y_t\}$.

What if some of $|\eta_i| > 1$?

Suppose WLOG (without loss of generality):

$|\eta_i| < 1$ for i =1,...,m

$|\eta_i| > 1$ for i =m+1,...,q

Create a new process:

$y_t - \mu = \tilde{\theta}(L)\tilde{\varepsilon_t}$

where:

$\tilde{\theta}(L) = (1 + \tilde{\theta_1} L^1 + \ldots + \tilde{\theta_q} L^q) = (1 - \eta_1 L)\cdots(1 - \eta_m L)\cdots(1 - \frac{1}{\eta_{m+1}}L)\cdots(1 - \frac{1}{\eta_q}L)$

$\tilde{\varepsilon_t} \sim WN(0, \sigma^2 \eta_{m+1}^2 \cdots \eta_q^2)$

Reminder:

ACGF of an MA(q): $y_t - \mu = \theta(L)\varepsilon_t$ ($\theta(L)$ is order q polynomial)

$g_y(z) = \sigma^2 \theta(z)\theta(z^{-1})$

So ACGF of $\tilde{y_t}$ should be:

$g_{\tilde{y}}(z) = \tilde{\sigma^2}\tilde{\theta}(z)\tilde{\theta}(z^{-1})$

$= \sigma^2 \eta_{m+1}^2 \eta_{m+2}^2 \cdots \eta_q^2 \cdot \{\prod_{j=1}^{m}(1 - \eta_j z)\prod_{j=m+1}^{q}(1 - \frac{1}{\eta_j}z)\} \cdot \{\prod_{j=1}^{m}(1 - \eta_j z^{-1})\prod_{j=m+1}^{q}(1 - \frac{1}{\eta_j}z^{-1})\}$

$= \sigma^2 \eta_{m+1}^2 \eta_{m+2}^2 \cdots \eta_q^2 \cdot \tilde{\theta(z)} \cdot \tilde{\theta}(z^{-1})$

$= \sigma^2 \{\prod_{j=1}^{m}(1 - \eta_j z)\prod_{j=m+1}^{q}(\eta_j z^{-1})(1 - \eta_j z^{-1})\} \cdot \{\prod_{j=1}^{m}(1 - \eta_j z^{-1})\prod_{j=m+1}^{q}(\eta_j z)(1 - \eta_j^{-1} z^{-1})\}$

$= \sigma^2 \{\prod_{j=1}^{m}(1 - \eta_j z)\prod_{j=m+1}^{q}(\eta_j z^{-1} - 1) \cdot \{\prod_{j=1}^{m}(1 - \eta_j z^{-1})\prod_{j=m+1}^{q}(\eta_j z - 1)\}$

$= \sigma^2 \prod_{j=1}^{q}(1 - \eta_j z^{-1})\prod_{j=1}^{q}(1 - \eta_j z) = \sigma^2 \theta(z^{-1})\theta(z) = g_y(z)$

So $\{\tilde{y_t}\}$ and $\{y_t\}$ have the same ACGFs.

- This means $\{\tilde{y_t}\}$ and $\{y_t\}$ are indentical up to 2nd moments.
- The implication is that every MA(q) has many equivalent (2nd moment) representations. How many? $\Longrightarrow$ 2^q
- If ε_t and $\tilde{\varepsilon_t}$ are gaussian then $\{\tilde{y_t}\}$ and $\{y_t\}$ are identical.
- Only one of these representations will be invertible (only if $|\eta_j| \neq 1 \forall$ j).

$(1-\eta_j L)^{-1} = \sum_{j=0}^{\infty} \eta_j^i L^i \not< \infty$ if $\eta_j \geq 1$

Causality

Recall from last class an AR(1) $(1-\phi L)(y_t - \mu) = \varepsilon_t$ is stationary even if $|\phi| > 1$. In this case the stationary representation is $y_t - \mu = \sum_{j=0}^{\infty} (\frac{1}{\phi})^j \varepsilon_{t+j}$

We already knew $\{y_t\}$ is stationary for $|\phi| < 1$:

$y_t - \mu = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$

Then only case where an AR(1) is not stationary is when $|\phi| = 1$.

This extends to an AR(p):

- If any root of $\phi(L)$ is on the unit circle, $\phi(L)(y_t - \mu) = \varepsilon_t$ is not stationary.
- If all roots $\phi(L)$ are above or below $\{y_t\}$ is stationary.
- But $\phi(L)$ may not have an inverse.
- If $\phi(L)^{-1}$ exists (all roots outside unit circle). Then $\{y_t\}$ is stationary <u>and</u> causal (formally a causal function of $\{\varepsilon_t\}$).

<u>Definition:</u>

A process $\{x_t\}$ is a causal function of $\{w_t\}$ if $\exists \psi(L) = \psi_0 + \psi_1 L^1 + \ldots$

s.t. $x_t = \psi(L) w_t$ and $\sum_{j=0}^{\infty} |\psi_j| < \infty$

<u>Causality of MA(q):</u>

MA(q0 is always stationary. For an MA(q), $\{\varepsilon_t\}$ is a causal fucntion of $\{y_t\}$ if all roots of $\theta(L)$ lie outside the unit circle.

- In this case the MA(q) is invertible: $\theta(L)^{-1}$ exists.

  $\varepsilon_t = \theta(L)^{-1}(y_t - \mu)$

  $y_t = \mu + \phi y_{t-1} + \varepsilon_t$.