Autocovariance generating function:

This is a function constructed by taking the j^{th} autocovariance, multiplying by some number z raised to the j^{th} power, and adding all the terms together:

$g_y(z) = \sum_{-\infty}^{\infty} \gamma_j z^j$

Ex. MA(1): $y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$

$\gamma_0 = (1+\theta_1^2)\sigma^2$

$\gamma_{-1} = \gamma_1 = \theta_1 \sigma^2$

$\gamma_2 = \gamma_{-2} = \gamma_3 = \gamma_{-3} = ... = 0$

$g_y(z) = \gamma_0 z^0 + \gamma_{-1} z^{-1} + \gamma_1 z^1 = (1+\theta_1^2)\sigma^2 + \theta_1^2 \sigma^2 z^{-1} + \theta_1^2 \sigma^2 z = ((1+\theta_1^2) + \theta_1 z^{-1} + \theta_1 z)\sigma^2$

$= \sigma^2 (1+\theta_1 z)(1+\theta_1 z^{-1})$

MA(q):

$g_y z = \sigma^2 (1 + \theta_1 z^1 + \theta_2 z^2 + ... + \theta_q z^q)(1 + \theta_1 z^{-1} + \theta_2 z^{-2} + ... + \theta_q z^{-q})$

$MA(\infty):$

$q_t(z) = \sigma^2 \theta(z)\theta(z^{-1}) = (\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + ...)z^1 + (\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + ...)z^2 + ... + (\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + ...)z^j + ...$

$\implies \gamma_1 = (\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + ...)$

AR(1): Suppose stationarity

$\phi(L)(y_t - \mu) = \varepsilon_t \qquad (\phi(L) = (1 - \phi L))$

Stationarity $\implies \phi^{-1}(L)$ exists

$\implies y_t - \mu = \phi^{-1}(L)\varepsilon_t = (\sum_{j=0}^{\infty} \phi^j L^j)\varepsilon_t$

$g_y(z) = \sigma^2 \phi^{-1}(z)\phi^{-1}(z^{-1}) = \sigma^2 (\sum_{j=0}^{\infty} \phi^j z^j)(\sum_{j=0}^{\infty} \phi^j z^{-j})$

$= \sigma^2 (1 + \phi^1 z^1 + \phi^2 z^2 + ...)(1 + \phi^1 z^{-1} + \phi^2 z^{-2} + ...)$

$\implies \gamma_j = \sigma^2 (\phi^j + \phi^1 \phi^{j+1} + \phi^2 \phi^{j+2} + ...)$

$= \sigma^2 \phi^j (1 + \phi^2 + \phi^4 + \phi^6 + \phi^8 + ...)$

$= \sigma^2 \phi^j \sum_{j=0}^{\infty} (\phi^2)^j = \frac{\sigma^2 \phi^j}{1-\phi^2}$

Recall MA(q) process:

$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$ is stationary for all values$\{\theta_j\}_{j=1}^{q}$

- $\theta(L)$ can be factored as: $\theta(L) = (1 + \theta_1 L + ... + \theta_q L) = (1 - \eta_1 L)(1 - \eta_2 L)...(1 - \eta_q L)$
- $\{\frac{1}{\eta_i}\}_{i=1}^{q}$ are the roots of $\theta(L)$
- Unlike AR process we don't need restrictions on $\theta(L)$ to ensure stationarity

Suppose $|\eta_i| < 1 \forall$ i. Then $(1 - \eta_i L)^{-1} = \sum_{j=0}^{\infty} \eta_i^j L^j \forall$ i.

Then: $\theta(L)^{-1} = (\sum_{j=0}^{\infty} \eta_1^j L^j)(\sum_{j=0}^{\infty} \eta_2^j L^j)...(\sum_{j=0}^{\infty} \eta_q^j L^j)$

Begin with $y_t = \theta(L)\varepsilon_t$ if all roots of $\theta(L)$ lie outside the unit circle then $\theta^{-1}(L)$ exists and we can write: $\theta^{-1}(L)y_t = \varepsilon_t \implies \varepsilon_t = \theta^{-1}(L)y_t$

<u>Motivation for Causality:</u>

Suppose $y_t = \phi y_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim WN(0, \sigma^2)$

What if $\phi > 1$?

Run the recursion forward:

$\phi y_{t-1} = y_t - \varepsilon_t \implies y_{t-1} = \frac{1}{\theta} y_t - \frac{1}{\theta}\varepsilon_t$

$= \frac{1}{\phi}(\frac{1}{\phi}y_{t+1} - \frac{1}{\phi}\varepsilon_{t+1}) - \frac{1}{\phi}\varepsilon_t = (\frac{1}{\phi})^2 y_{t+1} - (\frac{1}{\phi})^2 \varepsilon_{t+1} - \frac{1}{\phi}\varepsilon_t$

$= \sum_{j=0}^{\infty}(\frac{1}{\phi})^j \varepsilon_{t+j}$

Stationary, but involves future $\varepsilon \implies$ but this isn't causal.

So need inside unit circle for causality (i.e. inly passed ε_s).

Invertability allows for causality.