Lag Operators

Econ 211C – Unit 1, Section 2

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Lag Operator

Given a sequence of values, $y_1, y_2, \ldots, y_t$, indexed by time, the lag operator, L, is defined as

$$Ly_t = y_{t-1}.$$

- The lag operator shifts a time value y_t back by one period.
- y_t can be thought of as the input of the operator and y_{t-1} as the output.
- The lag operator can be applied to all values in a series $\{y_t\}_{t=1}^{T}$ and the result is a new series shifted back by one period: $\{y_t\}_{t=0}^{T-1}$.

Lag Operator

Applying the lag operator twice:

$$L(Ly_t) = Ly_{t-1} = y_{t-2}.$$

- We write $L(Ly_t)$ as $L^2 y_t$.
- Applying recursively:

$$L^k y_t = y_{t-k}.$$

- We will define $L^0 = 1$.

Useful Properties of the Lag Operator

- The lag operator is commutative:

$$L(\beta y_t) = \beta L y_t.$$

- The lag operator is distributive:

$$L(x_t + y_t) = L z_t = z_{t-1} = x_{t-1} + y_{t-1} = L x_t + L y_t,$$

  where $z_t = x_t + y_t$.

- The lag of a constant is the same constant:

$$Lc = c.$$

First-Order Difference Equation

Suppose we have a first-order difference equation:

$$y_t = \phi y_{t-1} + w_t.$$

In terms of the lag operator

$$(1 - \phi L) y_t = w_t.$$

We can write

$$\phi(L) y_t = w_t,$$

where $\phi(L) = (1 - \phi L)$.

First-Order Difference Equation

Suppose the operator $\phi(L) = (1 - \phi L)$ has an inverse:

$$\phi(L)^{-1} = (1 - \phi L)^{-1}.$$

- The inverse is the operator such that

$$(1 - \phi L)^{-1}(1 - \phi L) = 1.$$

- If an inverse operator exists,

$$y_t = \phi(L)^{-1} w_t = (1 - \phi L)^{-1} w_t.$$

Recursive Substitution of First-Order Difference Equation

Applying recursive substitution to the first-order difference equation:

$$
\begin{aligned}
y_t &= \phi y_{t-1} + w_t \\
&= \phi(\phi y_{t-2} + w_{t-1}) + w_t \\
&= w_t + \phi w_{t-1} + \phi^2 y_{t-2} \\
&= w_t + \phi w_{t-1} + \phi^2(\phi y_{t-3} + w_{t-2}) \\
&= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 y_{t-3} \\
&\vdots \\
&= \sum_{i=0}^{\infty} \phi^i w_{t-i} = \sum_{i=0}^{\infty} \phi^i L^i w_t.
\end{aligned}
$$

- The infinite recursive substitution can only be performed if $|\phi| < 1$.

Inverse of Lag Operator

Restating the previous result, for $|\phi| < 1$:

$$y_t = \left(\sum_{i=0}^{\infty} \phi^i L^i \right) w_t.$$

Substituting:

$$w_t = (1 - \phi L) y_t = (1 - \phi L) \left(\sum_{i=0}^{\infty} \phi^i L^i \right) w_t.$$

So when $|\phi| < 1$:

$$(1 - \phi L)^{-1} = \sum_{i=0}^{\infty} \phi^i L^i.$$

That is, $\sum_{i=0}^{\infty} \phi^i L^i$ is the inverse operator of $(1 - \phi L)$.

pth-Order Difference Equation

Suppose we have a pth-order difference equation:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + w_t.$$

In terms of the lag operator

$$(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p) y_t = w_t.$$

We can write

$$\phi(L) y_t = w_t,$$

where

$$\phi(L) = (1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p).$$

Factoring Polynomials

In general, a pth-order, real-valued polynomial can be factored as

$$1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = (1 - \lambda_1 z)(1 - \lambda_2 z) \cdots (1 - \lambda_p z).$$

- $\left\{ \frac{1}{\lambda_i} \right\}_{i=1}^{p}$ are the p roots of the polynomial.
- Some of the roots may be complex and some may be identical.

Factoring pth-Order Difference Equation

If we factor the pth-order lag polynomial in the same way as a real-valued polynomial:

$$\begin{aligned}(1-\phi_1 L-\phi_2 L^2-\ldots-\phi_p L^p)y_t \\ =(1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_p L)y_t=w_t.\end{aligned}$$

If $|\lambda_i|<1$,

$$(1-\lambda_i L)^{-1}=\sum_{j=0}^{\infty}\lambda_i^j L^j, \quad \forall i.$$

In this case,

$$\begin{aligned}y_t &= (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}\cdots(1-\lambda_p L)^{-1}w_t \\ &= \left(\sum_{j=0}^{\infty}\lambda_1^j L^j\right)\left(\sum_{j=0}^{\infty}\lambda_2^j L^j\right)\cdots\left(\sum_{j=0}^{\infty}\lambda_p^j L^j\right)w_t.\end{aligned}$$

Factoring pth-Order Difference Equation

If we define

$$\theta(L) = \left(\sum_{j=0}^{\infty} \lambda_1^j L^j\right) \left(\sum_{j=0}^{\infty} \lambda_2^j L^j\right) \cdots \left(\sum_{j=0}^{\infty} \lambda_p^j L^j\right)$$

then

$$y_t = \theta(L) w_t.$$

- Clearly, $\phi(L)^{-1} = \theta(L)$.
- Note that the inverse only exists when $|\lambda_i| < 1, \forall i$.
- This can also be stated as: the inverse only exists when the roots of $\phi(L)$ are greater than unity: $\frac{1}{|\lambda_i|} > 1, \forall i$.

Vector Difference Equation

We can rewrite the pth-order difference equation as

$$\boldsymbol{y}_t = \Phi \boldsymbol{y}_{t-1} + \boldsymbol{w}_t,$$

where

$$\boldsymbol{y}_t = \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} \quad \Phi = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad \boldsymbol{w}_t = \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Vector Difference Equation

It turns out that the values $\{\lambda_i\}_{i=1}^{p}$ are the p eigenvalues of Φ.

- So the eigenvalues of Φ are the inverses of the roots of the lag polynomial $\phi(L)$.
- Hence, $\phi(L)^{-1}$ exists if all p roots of $\phi(L)$ lie *outside* the unit circle or all p eigenvalues of Φ lie *inside* the unit circle.