Vector Autoregression

A pth order vector autoregression generalizes a scalar AR(p):

$\vec{y}_t = \vec{c}_t + \Phi_1 \vec{Y}_{t-1} + \Phi_2 \vec{Y}_{t-2} + \ldots + +\Phi_p \vec{Y}_{p-1} + \vec{\varepsilon}_t$

where $y_t = (y_{1t}, y_{2t}, \ldots, y_{nt})'$ is an n x 1 vector of variables, $\vec{c} = (c, c, \ldots, c)'$ is an n x 1 vector of constants Φ_j is an n x n matrix of autoregressive coefficients for $j = 1, \ldots, p$

and $\vec{\varepsilon}_t = (\varepsilon_{1t}, \ldots, \varepsilon_{nt})'$ s.t.

$E[\vec{\varepsilon}_t] = \vec{0}$ and $E[\vec{\varepsilon}_t \vec{\varepsilon}_\tau] = \begin{cases} \Omega & t = \tau \\ 0 & \text{o/w} \end{cases}$

$\vec{\varepsilon}_t$ is a vector white noise process

$\vec{y}_t$ is referrred to as a VAR(p)

Recall, an AR(p) can be written as a VAR(1)

$\vec{y}_t = \Phi \vec{y}_{t-1} + \vec{v}_t$

$$\text{where } \vec{y}_t = \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}_{px1}, \quad \Phi = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{pxp} \quad \vec{v}_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{px1}$$

$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \varepsilon_t$

In lag operator notation,

$\left[I_n - \Phi_1 L - \Phi_2 L^2 - \ldots - \Phi_p L^p\right] \vec{y}_t = \vec{c} + \vec{\varepsilon}_t$

Φ is a matrix where each component is a scalar log polynomial.

The concept of weak stationarity is unchanged:

$\vec{y}_t$ is weakly stationary if $E[\vec{y}_t] + E[\vec{y}_t \vec{y}'_{t-j}]$ are independent of $t \ \forall \ j$

Weak stationarity results in:

$E[\vec{y}_t] = \vec{\mu} = \vec{c} + \Phi_1 \vec{\mu} + \ldots + \Phi_p \vec{\mu}$

$\implies \vec{\mu} = [I_n - \Phi_1 - \ldots - \Phi_p]' \vec{c}_1$

where $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)' \quad \overbrace{\neq}^{\text{not necessarily}} \quad (\mu, \mu, \ldots, \mu)'$

Alternatively,

$(\vec{y}_t - \vec{\mu}) = \Phi_1(\vec{y}_{t-1} - \vec{\mu}) + \ldots + \Phi_p(\vec{y}_{t-p} - \vec{\mu}) + \vec{\varepsilon}_t$

We can write a VAR(p) as a VAR(1):

$\vec{\xi}_t = F \vec{\xi}_{t-1} + \vec{v}_t$

where

$$\vec{\xi}_t = \begin{bmatrix} \vec{y}_t - \vec{\mu} \\ \vec{y}_{t-1} - \vec{\mu} \\ \vdots \\ \vec{y}_{t-p+1} - \vec{\mu} \end{bmatrix}_{npx1}$$

$$F = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_{p-1} & \Phi_p \\ I_n & 0 & 0\dots & 0 & 0 & \\ 0 & I_n & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & I_n \end{bmatrix}_{npxnp}$$

$$\vec{v}_t = \begin{bmatrix} \vec{\varepsilon}_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{px1}$$

Clearly,

$$E[\vec{v}_t\vec{v}_\tau'] = \begin{cases} Q & t = \tau \\ 0 & \text{o/w} \end{cases}$$

where

$$Q = \begin{bmatrix} \Omega & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{npxnp}$$

Recursively iterating on the VAR(1):

$\vec{\xi}_{t+s} = \vec{v}_{t+s} + F\vec{v}_{t+s-1} + F^2\vec{v}_{t+s-2} + \dots + F^{s-1}\vec{v}_{t+1} + F^s\xi_t$

Assuming F is nonsingular, it can be decomposed as

$F = T\Lambda T^{-1}$

where Λ is a diagonal matrix comprised of the np eigenvalues of F and T is a matrix of eigenvalues as columns

Thus,

$F = FF = (T\Lambda T^{-1})T\Lambda T^{-1} = T\Lambda^2 T^{-1}$

$\implies F^s = T\Lambda^s T^{-1} \to 0 \quad \text{if } |\lambda_k| < 1 \text{ for } k = 1, \dots, np$

If $F^s \to 0$ as $s \to \infty$, the effect of $\vec{\varepsilon}_t$ on $\vec{\xi}_{t+s}$ dies out as $s \to \infty$, which is necessary for stationarity and causality

Alternatively, $\vec{y}_t$ is stationary and causal if the roots of $[I_n - \Phi_1 z - \Phi_2 z^2 - \dots - \Phi_p z^p]$ all lie outside the unit circle.

<u>Vector MA(∞) Representation</u>

If $F^s \to 0$ as $s \to 0$, then

$\vec{\xi}_{t+s} = \vec{v}_{t+s} + F\vec{v}_{t+s-1} + F^2\vec{v}_{t+s-2} + \dots + F^{s-1}\vec{v}_{t+1} + F^s\xi_t$

which is a vector of MA(∞) process

We can also write $\vec{y}_t$ alone as a vector MA(∞)

First, recognize

$\vec{y}_{t+s} = \vec{\mu}+\vec{\varepsilon}_{t+s}+\Psi_1\vec{\varepsilon}_{t+s-1}+\Psi_2\vec{\varepsilon}_{t+s-2}+\ldots+\Psi_{s-1}\vec{\varepsilon}_{t+1}+F_{11}^{(s)}(\vec{y}_t-\vec{\mu})+F_{12}^{(s)}(\vec{y}_{t-1}-\vec{\mu})+\ldots+F_{1p}^{(s)}(\vec{y}_{t-p+1}-\vec{\mu})$

where $\Psi_j = F_{11}^{(j)}$ and $F_{1k}^{(j)}$ is comprised of rows 1 to n and columns (k-1)n+1 + 1 to kn of matrix F^j

Side note:

$$\left.\begin{array}{c}(FxF)[1:n,1:n]\\ F[1:n,1:n]xF[1:n,1:n]\end{array}\right\}\text{not the same}$$

If all eigenvalues of F are inside the unit circle, $F^s \to 0$ as $s \to \infty$, which means $F_{1k}^{(s)} \to 0$ as $s \to \infty$.

In the limit

$\vec{y}_{t+s} = \vec{\mu} + \vec{\varepsilon}_{t+s} + \Psi_1\vec{\varepsilon}_{t+s-1} + \Psi_2\vec{\varepsilon}_{t+s-2} + \ldots = \vec{\mu} + \Psi(L)\varepsilon_t$

In this case $\Psi(L) = \Phi(L)^{-1}$ or $[1 - \Phi_1 L - \Phi_2 L^2 - \ldots - \Phi_p L^p][1 + \Psi_1 L + \Psi_2 L^2 + \ldots] = I_n$

$E[\varepsilon_t] = 0$

$$E[\varepsilon_t\varepsilon_\tau^{'}] = \begin{cases}\Omega & t=\tau\\ 0 & \text{o/w}\end{cases}$$

$$\varepsilon_t = \begin{bmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\\ \varepsilon_{3t}\end{bmatrix}$$

Note that we can always write a stationary and causal VAR(p) as a vector MA(∞) with a mutually uncorrelated white noise vector:

Define $\vec{u}_t = H\vec{\varepsilon}_t$ s.t. $H\Lambda H^{'} = D$

Then

$\vec{y}_t = \vec{\mu} + H^{-1}H\vec{\varepsilon}_t + \Psi_1(H^{-1}H)\vec{\varepsilon}_{t-1} + \ldots = \vec{\mu} + J_0\vec{u}_t + J_1\vec{u}_{t-1} + J_2\vec{u}_{t-2} + \ldots$ where $J_s = \Psi_s H^{-1}$

In this case the leading matrix $J_0 \neq I_n$

$E[u_t u_t^{'}] = E[H\boldsymbol{\varepsilon}_t]\boldsymbol{\varepsilon}_t^{'}H^{'}] = HE[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t]H^{'} = H\Omega H^{'}$

$H^{'}\Lambda H = \cancel{H^{'}}\cancel{H}\Omega\cancel{H^{'}}\cancel{H}$