State Space Representation

A state space model is a dynamic system of equations

$$\vec{\xi}_{t+1} = F\vec{\xi}_t + \vec{v}_{t+1} \tag{1}$$

$$\vec{y}_t = A^{'}\vec{x}_t + H^{'}\vec{\xi}_t + \vec{w}_t \tag{2}$$

$$E[\vec{v}_t\vec{v}_\tau^{'}] = \begin{cases} Q & t=\tau \\ 0 & \text{o/w} \end{cases} \tag{3}$$

$$E[\vec{w}_t\vec{w}_\tau^{'}] = \begin{cases} R & t=\tau \\ 0 & \text{o/w} \end{cases} \tag{4}$$

$$E[\vec{v}_t\vec{w}_\tau^{'}] = 0 \ \forall\ t,\tau \tag{5}$$

Where $\vec{y}_t$ is a vector of n variables observed at t, $\vec{\xi}_t$ is a vector of r unobserved variables at t and $\vec{x}_t$ is a vector of exogenous or predetermined variables at t.

- (1) is the state equation
- (2) is the observation equation
- $\vec{v}_t$ and $\vec{w}_t$ are vector WN processes and mutually uncorrelated at all lags.

If we assume $E[\vec{v}_t\vec{\xi}_t^{'}] = E[\vec{w}_t\vec{\xi}_t^{'}] = 0$,
$E[\vec{v}_t\vec{\xi}_t^{'}] = E[\vec{v}_t(\vec{v}_\tau^{'} + \vec{v}_{\tau-1}^{'}F^{'} + \ldots + \vec{v}_2^{'}F^{t-2\,'} + \vec{\xi}_1^{'}F^{t-1\,'})] = 0\ \forall\ \tau < t$
Similarly, $E[\vec{w}_t\vec{\xi}_t^{'}] = 0\ \forall\ \tau < t$
$E[\vec{v}_t\vec{y}_\tau^{'}] = E[\vec{v}_t(A^{'}\vec{x}_\tau + H^{'}\vec{\xi}_t + \vec{w}_\tau)^{'}] = 0\ \forall\ \tau < t$
Similarly, $E[\vec{w}_t\vec{y}_\tau^{'}] = 0\ \forall\ \tau < t$

Example AR(p)

$y_{t+1} - \mu = \phi_1(y_t - \mu) + \phi_2(y_{t-1} - \mu) + \ldots + \phi_p(y_{t-p+1} - \mu) + \varepsilon_{t+1}$

$$\vec{\xi}_t = \begin{bmatrix} y_t - \mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t-p+1} - \mu \end{bmatrix}, \quad F = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \quad \vec{v}_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{y}_t = y_t \quad A^{'} = \mu \quad \vec{x}_t = 1 \quad H^{'} = [1\ 0\ \ldots\ 0] \quad \vec{w}_t = 0 \quad R = 0$

$$Q = \begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

<u>Example: ARMA(p,q)</u>

$y_{t+1} - \mu = \phi_1(y_t - \mu) + \ldots + \phi_p(y_{t-p+1} - \mu) + \varepsilon_{t+1} + \theta_1\varepsilon_t + \ldots + \theta_{r-1}\varepsilon_{t-r+2}$

where $r = \max\{p, q+1\}$ and $\phi_j = 0$ for $j > p$ and $\theta_j = 0$ for $j > q$

$$\vec{\xi}_t = \begin{bmatrix} y_t - \mu \\ \phi_2(y_{t-1} - \mu) + \ldots + \phi_p(y_{t-p+1} - \mu) + \theta_1\varepsilon_t + \ldots + \theta_{r-1}\varepsilon_{t-r+1} \\ \vdots \\ \phi_r y_{t-1} + \theta_{r-1}\varepsilon_t \end{bmatrix}$$

$$F = \begin{bmatrix} \phi_1 & 1 & 0 & \dots & 0 \\ \phi_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \phi_{r-1} & 0 & 0 & \dots & 1 \\ \phi_r & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\vec{v}_t = \begin{bmatrix} \varepsilon_t \\ \theta_1\varepsilon_t \\ \vdots \\ \theta_{r-2}\varepsilon_t \\ \theta_{r-1}\varepsilon_t \end{bmatrix}$$

$\vec{y}_t = y_t \quad A' = \mu \quad \vec{x}_t = 1 \quad H' = [1\ 0\ \ldots\ 0] \quad \vec{w}_t = 0 \quad R = 0$

$\vec{\xi}_{t+1} = F\vec{\xi}_t + \vec{v}_{t+1}$