The Kalman Filter

Recall the basic state-space representation

$$\underset{(rx1)}{\vec{\xi}_{t+1}} = \underset{(rxr)}{F}\underset{(rx1)}{\vec{\xi}_t} + \underset{(rx1)}{\vec{v}_{t+1}} \tag{1}$$

$$\underset{(nx1)}{\vec{y}_t} = \underset{(nxk)}{A^{'}}\underset{(kx1)}{\vec{x}_t} + \underset{(nxr)}{H^{'}}\underset{(rx1)}{\vec{\xi}_t} + \underset{(nx1)}{\vec{w}_t} \tag{2}$$

$$E[\vec{v}_t\vec{v}_\tau^{'}] = \begin{cases} \underset{(nxn)}{Q} & t=\tau \\ 0 & \text{o/w} \end{cases} \tag{3}$$

$$E[\vec{w}_t\vec{w}_\tau^{'}] = \begin{cases} \underset{(nxn)}{R} & t=\tau \\ 0 & \text{o/w} \end{cases} \tag{4}$$

$$E[\vec{v}_t\vec{w}_\tau^{'}] = 0 \ \forall \ t,\tau \tag{5}$$

Collect all known information at time t into a vector:

$\underset{((n+k)tx1)}{\vec{\mathcal{Y}}_t} = (\vec{y_t}^{'}, \vec{y_{t-1}}^{'}, ..., \vec{y_1}^{'}, \vec{x_t}^{'}, \vec{x_{t-1}}^{'}, ..., \vec{x_1}^{'})$

The Kalman Filter computes:

$\hat{\vec{\xi}}_{t+1|t} = \hat{E}[\vec{\xi}_t|\vec{\mathscr{Y}}_t]$

$\underset{(kxk)}{P_{t+1|t}} = E[(\vec{\xi}_{t+1} - \hat{\vec{\xi}}_{t+1|t})(\hat{\xi}_{t+1} - \hat{\vec{\xi}}_{t+1|t})^{'}]$

where $P_{t+1|t}$ is the MSE matrix for $\hat{\vec{\xi}}_{t+1|t}$

Starting the Recursion

$\hat{\vec{\xi}}_{1|0} = E[\vec{\xi}_1|\underset{\emptyset}{\underset{\parallel}{\mathscr{Y}_0}}] = E[\vec{\xi}_1]$

$P_{1|0} = E[(\vec{\xi}_1 - E[\vec{\xi}_1])(\vec{\xi}_1 - E[\vec{\xi}_1])^{'}]$

By (1), the unconditional expectation of $\vec{\xi}_t$ is:

$E[\vec{\xi}_{t+1}] = FE[\vec{\xi}_t]$

$\implies E[\xi_t] = FE[\xi_t]$

$\implies (I_t - F)E[\xi_t] = 0$

$\implies E[\xi_t] = 0$

Further by (1):

$\underbrace{E[\vec{\xi}_{t+1}\vec{\xi}_{t+1}^{'}]}_{\Sigma} = E[(F\vec{\xi}_t + \vec{v}_{t+1})^{'}] = F\underbrace{E[\vec{\xi}_t\vec{\xi}_t^{'}]}_{\Sigma}F^{'} + F\underbrace{E[\vec{\xi}_t\vec{v}_{t+1}^{'}]}_{0} + \underbrace{E[\vec{v}_{t+1}\vec{\xi}_t^{'}]}_{0}F^{'} + \underbrace{E[\vec{v}_{t+1}\vec{v}_{t+1}^{'}]}_{Q}$

$\implies \Sigma = F\Sigma F^{'} + Q$

$\text{Vec}(\Sigma) = [I_{r^2} - (F \otimes F)]^{-1}\text{Vec}(Q)$

In this case, $P_{1|0} = \Sigma$

Forecasting y_t

Given values for $\hat{\vec{\xi}}_{t|t-1}$ and $P_{t|t-1}$, our objective will be to obtain $\hat{\vec{\xi}}_{t+1|t}$ and $P_{t+1|t}$. Since $\vec{x}_t$ contains no information about $\vec{\xi}_t$ beyond what is contained in $\vec{\mathscr{Y}}_{t-1}$:

$E[\vec{\xi}_t|\vec{x}_t, \vec{\mathscr{Y}}_{t-1}] = E[\vec{\xi}_t|\vec{\mathscr{Y}}_{t-1}] = \hat{\vec{\xi}}_{t|t-1}$

Defining $\hat{\vec{y}}_{t|t-1} = \hat{E}[\vec{y}_t|\vec{x}_t, \vec{\mathscr{Y}}_{t-1}]$, by (2)

$$\hat{\vec{y}}_{t|t-1} = A'\vec{x}_t + H'E[\vec{\xi}_t|\vec{x}_t, \vec{\mathscr{Y}}_{t-1}] + \underbrace{E[\vec{w}_t|\vec{x}_t, \vec{\mathscr{Y}}_{t-1}]}_{0} = A'\vec{x}_t + H'\hat{\vec{\xi}}_{t|t-1}$$

The forecast error is:

$$\vec{y}_t - \hat{\vec{y}}_{t|t-1} = A'\cancel{\vec{x}_t} + H'\vec{\xi}_t + \vec{w}_t - A'\cancel{\vec{x}_t} - H'\hat{\vec{\xi}}_{t|t-1} = H'(\vec{\xi}_t - \hat{\vec{\xi}}_{t|t-1}) + \vec{w}_t$$

Which has the MSE matrix:

$$E[(\vec{y}_t - \hat{\vec{y}}_{t|t-1})(\vec{y}_t - \hat{\vec{y}}_{t|t-1})'] = H'\underbrace{E[(\vec{\xi}_t - \hat{\vec{\xi}}_{t|t-1})(\vec{\xi}_t - \hat{\vec{\xi}}_{t|t-1})']}_{P_{t|t-1}}H + \underbrace{E[\vec{w}_t\vec{w}_t']}_{R} = H'P_{t|t-1}H + R$$

Where we have used the fact that:

$E[\vec{w}_t(\vec{\xi}_t - \hat{\vec{\xi}}_{t|t-1})] = 0$ because $E[\vec{w}_t\vec{\xi}_t'] = 0$ and because

$E[\vec{w}_t\hat{\vec{\xi}}_{t|t-1}'] = E[\vec{w}_t(F\xi_{t-1})'] = E[\vec{w}_t\vec{\xi}_{t-1}']F' = 0$

Update the forecast of ξ_t

After wer observe $\vec{y}_t$, we can obtain a new forecast of $\vec{\xi}_t$:

$\hat{\vec{\xi}}_{t|t} = E[\vec{\xi}_t|\vec{y}_t, \vec{x}_t, \vec{\mathscr{Y}}_t] = E[\vec{\xi}_t|\vec{\mathscr{Y}}_t]$

The formula for updating a linear projection in this fashion is ([4.5.30] in Hamilton):

$$\begin{aligned}\hat{\vec{\xi}}_{t|t} &= \hat{\vec{\xi}}_{t|t-1} + E[(\vec{\xi}_t - \hat{\vec{\xi}}_{t|t-1})(\vec{y}_t - \hat{\vec{y}}_{t|t-1})'] \text{ x } E[(\vec{y}_t - \hat{\vec{y}}_{t|t-1})(\vec{y}_t - \hat{\vec{y}}_{t|t-1})']^{-1} \text{ x } (\vec{y}_t - \hat{\vec{y}}_{t|t-1}) \\ &= \hat{\vec{\xi}}_{t|t-1} + E[(\vec{\xi}_t - \hat{\vec{\xi}}_{t|t-1})(\vec{w}_t' + (\vec{\xi}_t - \hat{\vec{\xi}}_{t|t-1})'H)] \text{ x } (H'P_{t|t-1}H + R)^{-1}(\vec{y}_t - A'\vec{x}_t + H'\hat{\vec{\xi}}_{t|t-1}) \\ &= \hat{\vec{\xi}}_{t|t-1} + \overbrace{P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}}^{K_t} \text{ x } (\vec{y}_t - A'\vec{x}_t + H'\hat{\vec{\xi}}_{t|t-1})\end{aligned}$$

The associated MSE is:

$$P_{t|t} = E[(\vec{\xi}_t - \hat{\vec{\xi}}_{t|t})(\vec{\xi}_t - \hat{\vec{\xi}}_{t|t})'] = P_{t|t-1} - P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}H'P_{t|t-1}$$

Forecasting $\vec{\xi}_{t+1}$

$$\hat{\vec{\xi}}_{t+1|t} = \hat{E}[\vec{\xi}_{t+1}|\vec{\mathscr{Y}}_t] = F\underbrace{\hat{E}[\vec{\xi}_t|\vec{\mathscr{Y}}_t]} + E[\vec{v}_{t+1}|\vec{\mathscr{Y}}_t] = F\hat{\vec{\xi}}_{t|t} = F\hat{\vec{\xi}}_{t|t-1} + FK_t(\vec{y}_t - A'\vec{x}_t + H'\hat{\vec{\xi}}_{t|t-1})$$

$$\begin{aligned}P_{t+1|t} &= E[(\vec{\xi}_{t+1} - \hat{\vec{\xi}}_{t+1|t})(\vec{\xi}_{t+1} - \hat{\vec{\xi}}_{t+1|t})'] = E[(F\vec{\xi}_t + \vec{v}_{t+1} - F\hat{\vec{\xi}}_{t|t})(F\vec{\xi}_t + \vec{v}_{t+1} - F\hat{\vec{\xi}}_{t|t})'] \\ &= FP_{t|t}F' + Q = F(P_{t|t-1} - P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1} \text{ x } H'P_{t|t-1})F' + Q\end{aligned}$$

Forecast $\boldsymbol{y}_{t+1}$

$\boldsymbol{y}_{t+1|t} = E[\boldsymbol{y}_{t+1}|\boldsymbol{x}_{t+1}, \mathscr{Y}_t] = A'\boldsymbol{x}_{t+1}$

which has associated MSE:

$E[\boldsymbol{y}_{t+1} - \hat{\boldsymbol{y}}_{t+1|t})(\boldsymbol{y}_{t+1} - \hat{\boldsymbol{y}}_{t+1|t})'] = H'P_{t+1|t}H + R$

Forecast $\boldsymbol{y}_{t+s}$

Iterating on the state equation:

$\boldsymbol{\xi}_{t+s} = F^s\boldsymbol{\xi}_t + F^{s-1}\boldsymbol{v}_{t+1} + F^{s-2}\boldsymbol{v}_{t+2} + \ldots + F\boldsymbol{v}_{t+s-1} + \boldsymbol{v}_{t+s}$

$\implies E[\boldsymbol{\xi}_{t+s}|\boldsymbol{\xi}_t, \mathscr{Y}_t] = F^s\boldsymbol{\xi}_t$

$\hat{\boldsymbol{\xi}}_{t+s|t} = E[\boldsymbol{\xi}_{t+s}|\mathscr{Y}_t] = E[E[\boldsymbol{\xi}_{t+s}|\boldsymbol{\xi}_t, \mathscr{Y}_t]|\mathscr{Y}_t] = E[F^s \boldsymbol{\xi}_t|\mathscr{Y}_t] = F^s \hat{\boldsymbol{\xi}}_{t|t}$

$E[X|I_1] = E[E[X|I_2]|I_1]$ if $I_1 \subseteq I_2$

$E_t[E_{t+1}[X]] = E_t[X]$

The forecast error is:

$\boldsymbol{\xi}_{t+s} - \hat{\boldsymbol{\xi}}_{t+s|t} = F^s(\boldsymbol{\xi}_t - \hat{\boldsymbol{\xi}}_{t|t}) + F^{s-1}\boldsymbol{v}_{t+1} + \ldots + F\boldsymbol{v}_{t+s-1} + \boldsymbol{v}_{t+s}$

with MSE:

$P_{t+s|t} = F^s P_{t|t}(F^{'})^s + F^{s-1}Q(F^{'})^{s-1} + \ldots + FQF^{'} + Q$

Rewrite the observation equation:

$\boldsymbol{y}_{t+s} = A^{'}\boldsymbol{x}_{t+s} + H^{'}\boldsymbol{\xi}_{t+s} + \boldsymbol{w}_{t+s}$

Thus,

$\hat{\boldsymbol{y}}_{t+s|t} = E[\boldsymbol{y}_{t+s}|\boldsymbol{x}_{t+s}, \mathscr{Y}_t] = A^{'}\boldsymbol{x}_{t+s} + H^{'}\hat{\boldsymbol{\xi}}_{t+s|t}$

The forecast error is:

$\boldsymbol{y}_{t+s} - \hat{\boldsymbol{y}}_{t+s|t} = \cancel{A^{'}\boldsymbol{x}_{t+s}} + H^{'}\boldsymbol{\xi}_{t+s} + \boldsymbol{w}_{t+s} - \cancel{A^{'}\boldsymbol{x}_{t+s}} - H^{'}\hat{\boldsymbol{\xi}}_{t+s|t} = H^{'}(\boldsymbol{\xi}_{t+s} - \hat{\boldsymbol{\xi}}_{t+s|t}) + \boldsymbol{w}_{t+s}$

with MSE:

$E[(\boldsymbol{y}_{t+s} - \hat{\boldsymbol{y}}_{t+s|t})(\boldsymbol{y}_{t+s} - \hat{\boldsymbol{y}}_{t+s|t})^{'}] = H^{'}P_{t+s|t}H + R$

Summary of Kalman Filter Steps

1. Start with forecast $\hat{\boldsymbol{\xi}}_{1|0}$ and associated MSE matrix $P_{1|0}$

2. Given some forecast $\hat{\boldsymbol{\xi}}_{t|t-1}$ and MSE $P_{t|t-1}$ compute

   $\hat{\boldsymbol{\xi}}_{t|t-1} = E[\boldsymbol{\xi}_t|\mathscr{Y}_t]$

   $P_{t|t} = E[(\hat{\boldsymbol{\xi}}_t - \hat{\boldsymbol{\xi}}_{t|t})(\hat{\boldsymbol{\xi}}_t - \hat{\boldsymbol{\xi}}_{t|t})^{'}]$

3. Given $\hat{\boldsymbol{\xi}}_{t|t}$ and MSE $P_{t|t}$, compute

   $\hat{\boldsymbol{\xi}}_{t+1|t} = E[\boldsymbol{\xi}_{t+1}|\mathscr{Y}_t]$

   $P_{t+1|t} = E[(\hat{\boldsymbol{\xi}}_{t+1} - \hat{\boldsymbol{\xi}}_{t+1|t})(\hat{\boldsymbol{\xi}}_{t+1} - \hat{\boldsymbol{\xi}}_{t+1|t})^{'}]$

4. Given $\hat{\boldsymbol{\xi}}_{t+1|t}$ and MSE $P_{t+1|t}$, compute

   $\hat{\boldsymbol{y}}_{t+1|t} = E[\boldsymbol{y}_{t+1}|\boldsymbol{x}_{t+1}, \mathscr{Y}_t]$

   $E[(\boldsymbol{y}_{t+1} - \hat{\boldsymbol{y}}_{t+1|t})(\boldsymbol{y}_{t+1} - \hat{\boldsymbol{y}}_{t+1|t})^{'}]$

Example: Long-Run Risks

$x_{t+1} = \rho x_t + \varphi_e \sigma e_{t+1}$

$g_{t+1} = \mu + x_t + \sigma\eta_{t+1}$

$g_{d,t+1} = \mu_d + \phi x_t + \varphi_d \sigma u_{t+1}$

$\varphi_{t+1}, u_{t+1}, \eta_{t+1} \overset{i.i.d.}{\sim} N(0,1)$

$g_t = log\left(\frac{C_t}{C_{t-1}}\right)$, where C_t is aggregate consumption

$g_{d,t} = log\left(\frac{D_t}{D_{t-1}}\right)$ where D_t is dividend of market asset.