Autocovariances of vector processes

Given an n-dimensional, weakly stationary vector process the jth autocovariance matix is defined as:

$\Gamma_{j,t} = E[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_{t-j} - \boldsymbol{\mu})^{'}]$

This is an n x n matrix

In general, $\Gamma_j \neq \Gamma_{-j}$

The (1,2) element of Γ_j is : Cov$(y_{1,t}, y_{2,t-j})$

The (1,2) element of Γ_{-j} is: Cov$(y_{1,t}, y_{2,t+j})$

Since $y_{1,t}$ is different from $y_{2,t}$, there is no reason these covariances should be identical.

What is true? $\Gamma_j = \Gamma^{'}_{-j}$

The (1,2) element of $\Gamma^{'}_{-j}$ is the (2,1) element of Γ_{-j}: Cov$(y_{2,t}, y_{1,t+j})$

Stationarity does impose: Cov$(y_{1,t}, y_{2,t-j}) = Cov(y_{1,t+j}, y_{2,t})$

Vector MA(q) Processes

A vector moving average process of order q is

$\boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \Theta_1 \boldsymbol{\varepsilon}_{t-1} + \Theta_2 \boldsymbol{\varepsilon}_{t-2} + \ldots + \Theta_q \boldsymbol{\varepsilon}_{t-q}$

where $\boldsymbol{\varepsilon}_t \overset{i.i.d}{\sim} WN(\mathbf{0}, \Omega)$ and Θ_j is a N x N matrix of MA coefficients for $j = 1, \ldots, q$

We can define $\Theta_0 = I_n$

Clearly E$[\boldsymbol{y}_t] = \boldsymbol{\mu} \; \forall \; t$

The jth autocovariance matrix is:

$\Gamma_j = E[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_{t-j} - \boldsymbol{\mu})^{'}] = E[(\Theta_0 \boldsymbol{\varepsilon}_t + \Theta_1 \boldsymbol{\varepsilon}_{t-1} + \ldots + \Theta_q \boldsymbol{\varepsilon}_{t-q})(\Theta_0 \boldsymbol{\varepsilon}_{t-j} + \Theta_1 \boldsymbol{\varepsilon}_{t-j-1} + \ldots + \Theta_q \boldsymbol{\varepsilon}_{t-j-q}]$

For $|j| > q : \Gamma_j = 0_{NxN}$

For $j = 0 : \Gamma_j = \Theta_0 \Omega \Theta^{'}_0 + \Theta_1 \Omega \Theta^{'}_1 + \ldots + \Theta_q \Omega \Theta^{'}_q = \sum_{i=1}^{q} \Theta_i \Omega \Theta^{'}_i$

For $j = 1, \ldots, q$:

$\Gamma_j = \Theta_j \Omega \Theta^{'}_0 + \Theta_{j+1} \Omega \Theta^{'}_1 + \ldots + \Theta_q \Omega \Theta^{'}_{q-j} = \sum_{i=0}^{q-j} \Theta_{j+i} \Omega \Theta^{'}_i$

For $j = 1, \ldots, -q$:

$\Gamma_j = \Theta_0 \Omega \Theta^{'}_{-j} + \Theta_1 \Omega \Theta^{'}_{-j+1} + \ldots + \Theta_{q+j} \Omega \Theta^{'}_q = \sum_{i=0}^{q+j} \Theta_i \Omega \Theta^{'}_{j+i}$

$\Gamma^{'}_j = \Gamma_{-j}$

Because 1st and 2nd moments of $\boldsymbol{y}_t$ are independent of time, the vector MA(q) process is weakly stationary.

Vector MA(∞) Processes

The vector MA(∞) is the limit of the vector MA(q):

$\boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \Theta_1 \boldsymbol{\varepsilon}_{t-1} + \Theta_2 \boldsymbol{\varepsilon}_{t-2} + \ldots$

The sequence of matrices $\{\Theta_s\}_{s=0}^{\infty}$ is absolutely summable if each component sequence is absolutely summable.

If $\{\Theta_s\}_{s=0}^{\infty}$ are absolutely summable:

$E[\boldsymbol{y}_t] = \boldsymbol{\mu}$

$\Gamma_j = \sum_{i=0}^{\infty} \Theta_{j+i}\Omega\Theta_i^{'},\ j = 0, 1, 2, ...$

$\boldsymbol{y}_t$ is ergodic for 1st and 2nd moments

Clearly, this means $\boldsymbol{y}_t$ is stationary

When a stationary VAR(p) is expressed as a vector MA(∞), it satisfies the absolute summability condition.

$\Theta_s = F^s = T\Lambda^s T^{-1}$

The component-wise sum of absolute values over s = 0,1,2,... will be a weighted average of absolute values of eigenvalues raised to power s

Because of stationarity, $|\lambda_i| < 1,\ i = 1, ..., np$

Which means $\{F^s\}_{s=0}^{\infty}$ is absolutely summable

Autocovariance of VAR(p)

Recall that a VAR(p) can be expressed as:

$\boldsymbol{\xi}_t = F\boldsymbol{\xi}_{t-1} + \boldsymbol{v}_t$

$$\sum = E[\boldsymbol{\xi}_t\boldsymbol{\xi}_t^{'}] = \begin{bmatrix} \Gamma_0 & \Gamma_1 & \dots & \Gamma_{p-1} \\ \Gamma_1^{'} & \Gamma_0 & \dots & \Gamma_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{p-1}^{'} & \Gamma_{p-2}^{'} & \dots & \Gamma_0 \end{bmatrix}$$

By the definition of $\vec{\xi_t}$

$$\sum = E[\boldsymbol{\xi}_t\boldsymbol{\xi}_t^{'}] = E\left[(F\vec{\xi}_{t-1} + \vec{t})(F\vec{\xi}_{t-1} + \vec{v}_t)^{'}\right] = F\underbrace{E[\vec{\xi}_{t-1}\vec{\xi}_{t-1}^{'}]}_{\Sigma}F^{'} + \underbrace{E[\vec{v}_t\vec{v}_t^{'}]}_{Q} = F\Sigma F^{'} + Q$$

$Vec(ABC) = C^{'} \bigotimes A \cdot Vec(B)$

$Vec(\Sigma) = F \bigotimes F \cdot Vec(\Sigma) + Vec(Q)$

$\implies Vec(\Sigma) = [I - F \bigotimes F]^{-1} \cdot Vec(Q)$

$F \bigotimes F$ is an $(np)^2 x (np)^2$ matrix

Because all eigenvalues of F lie inside the unit circle, so do all eigenvalues of $F \bigotimes F$, which means $F \bigotimes F$ is invertible

$\Sigma_j = E[\vec{\xi}_t t \vec{-} j^{'}] = FE[\vec{\xi}_{t-1}\vec{\xi}_{t-1}^{'}] = F\Sigma_{j-1}, j = 1, 2, 3, ...$

$\Sigma_j = F^j\Sigma$