A Compound-Multifractal Model for High-Frequency Asset Returns

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Motivation

In an ideal world, asset returns would follow a Gaussian distribution.

- Unfortunately, the do not.
- Almost all asset returns are better characterized by a fat-tailed distribution.
- This was first observed in Mandelbrot (1963).
- After decades of research, there is still no consensus regarding which family of fat-tailed distributions best characterizes asset returns.

Contribution

We carry out a ground-level re-examination of the process that generates short-period returns.

- We do this in the context of high-frequency, trade-by-trade data.

En route to our final model we highlight two insights:

1. Returns distributions are effectively categorized into two groups: those occurring during prescheduled news announcement periods, and those that do not.
2. Outside of news announcement periods, returns measured in trade-time (not clock-time) follow a Gaussian distribution.

Contribution

To arrive at a final model of clock-time returns, we pair a Gaussian distribution for trade-time returns with a high-quality model of inter-trade durations.

- We use the model Markov-Switching Multifractal Duration (MSMD) model of Chen, Diebold and Schorfheide (2013) to characterize trade durations.
- This model is based on the Multifractal Model of Asset Returns developed by Mandelbrot, Calvet and Fisher (1997) as well as subsequent work by Calvet and Fisher.
- Our model can be characterized as a mixture of Gaussian distributions.

We show that our resulting mixture of Gaussians provides a very good fit to the data.

Data

We focus our analysis on the CME E-mini S&P 500 Futures contract.

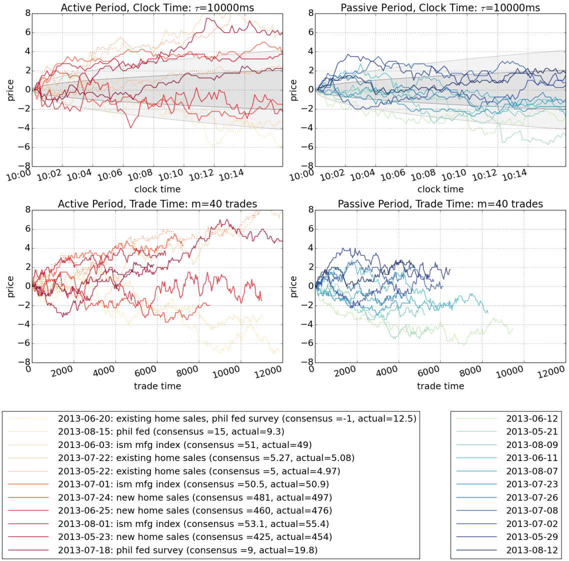
- This is a futures contract traded on the value of the S&P 500 Index.
- We obtained the full record of tick-by-tick trades for the period 18 May 2013 to 18 August 2013.
- It is exemplary of a large class of assets (including equities).
    - This is attributed to its liquidity and the relationship of price formation between the futures and equities exchanges (Laughlin, Aguirre and Grundfest (2013)).

Data

Our data sample contains 6,832,305 trade records (no quotes).

- We subsample trades that occurred in the 1000 seconds following 8:30 am and 10:00 am on each day of our sample (common news announcement times).
- We used the EconoDay calendar to determine days with news announcements scheduled for 8:30 am or 10:00 am.
- If a news announcement was scheduled for either time, the ensuing trades were classified as news affected (active) or not news affected (passive).
- The resulting sorted subsamples are roughly equal in size: 191,127 active trades and 174,041 passive trades.

Data



Active Period, Clock Time: τ=10000ms
Passive Period, Clock Time: τ=10000ms
Active Period, Trade Time: m=40 trades
Passive Period, Trade Time: m=40 trades
price
clock time
trade time
2013-06-20: existing home sales, phil fed survey (consensus =-1, actual=12.5)
2013-08-15: phil fed (consensus =15, actual=9.3)
2013-06-03: ism mfg index (consensus =51, actual=49)
2013-07-22: existing home sales (consensus =5.27, actual=5.08)
2013-05-22: existing home sales (consensus =5, actual=4.97)
2013-07-01: ism mfg index (consensus =50.5, actual=50.9)
2013-07-24: new home sales (consensus =481, actual=497)
2013-06-25: new home sales (consensus =460, actual=476)
2013-08-01: ism mfg index (consensus =53.1, actual=55.4)
2013-05-23: new home sales (consensus =425, actual=454)
2013-07-18: phil fed survey (consensus =9, actual=19.8)
2013-06-12
2013-05-21
2013-08-09
2013-06-11
2013-08-07
2013-07-23
2013-07-26
2013-07-08
2013-07-02
2013-05-29
2013-08-12

Returns

We define returns in two ways.

- Clock-time returns:

$$r_\tau(t) = p(t) - p(t - \tau)$$

where τ is the clock-time duration.

- Trade-time returns:

$$r_m(n) = p(n) - p(n - m),$$

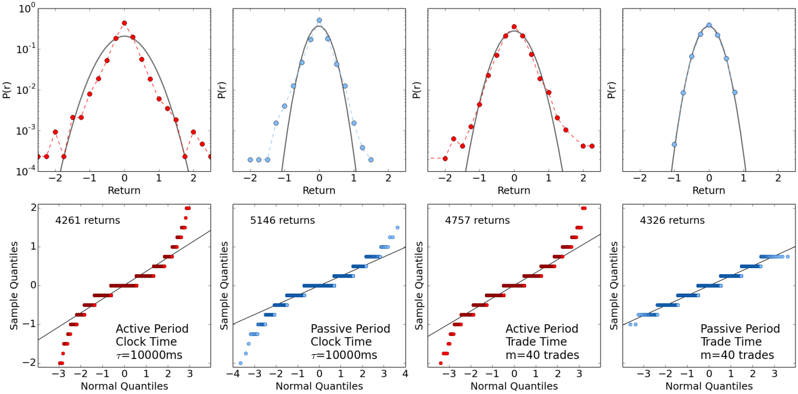
where n denotes the n-th trade and m is the number of trades in a unit of time.

Trade Time

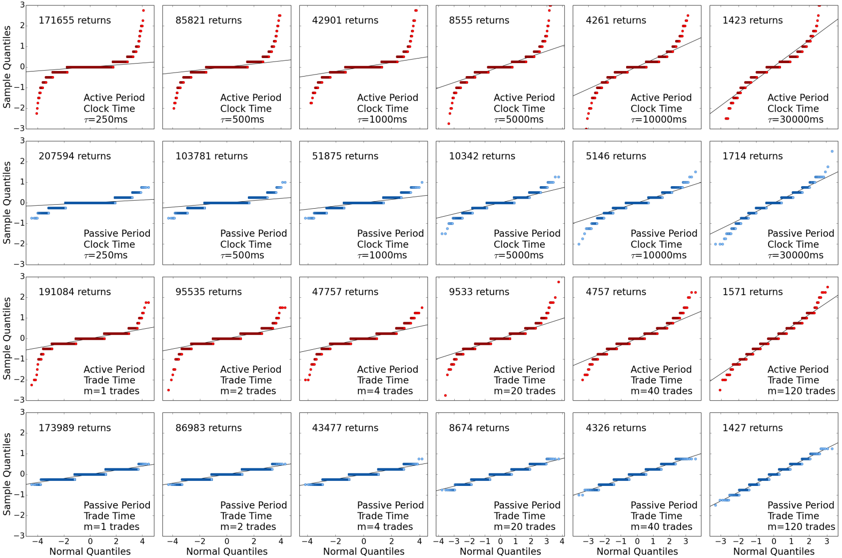
Trade time fixes a certain number of trades as a unit of time.

- The clock time between trade-time intervals may be variable.
- In our data there are 54 passive 1000-second intervals with 174,041 observations.
  - This is an average of 3.22 trades per second.
- There are 43 passive 1000-second intervals with 191,127 observations.
  - This is an average of 4.44 trades per second.
- So we roughly equate a trade-time interval of $m = 1$ trade to a clock-time interval of $\tau = 0.25$ seconds.

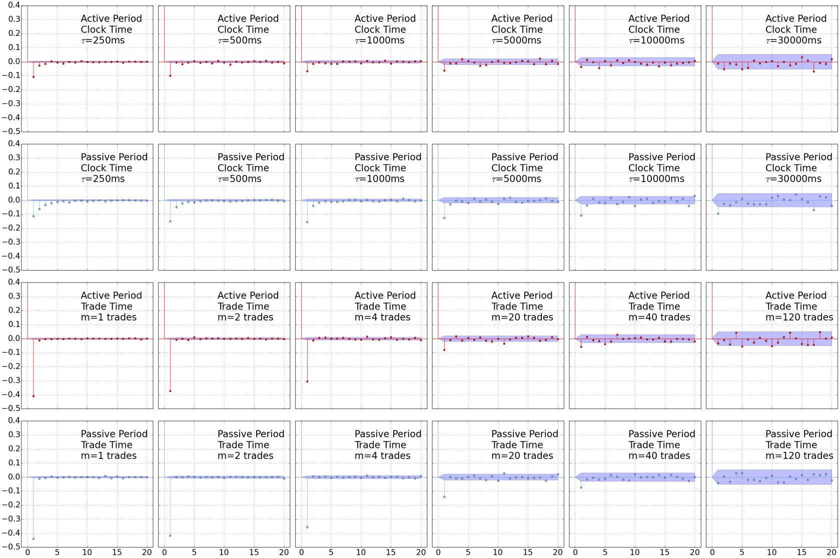
Empirical Densities, $\tau = 10,000$ ms and $m = 40$



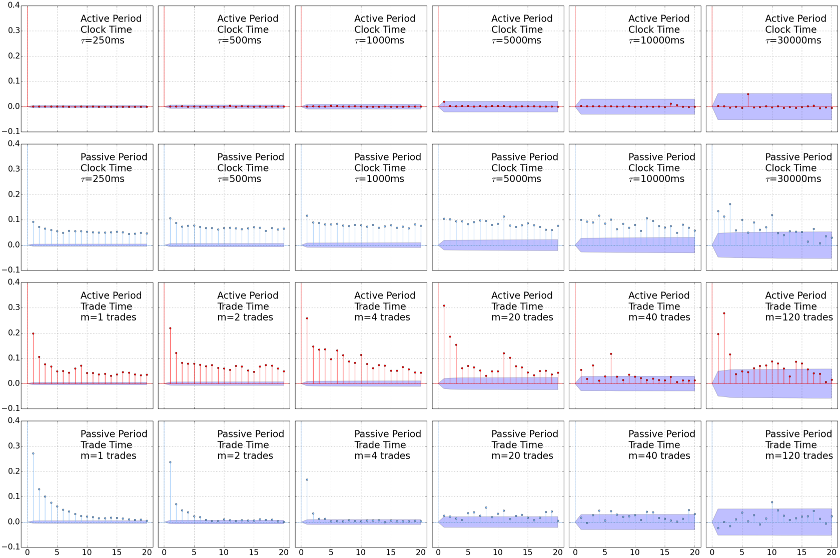
Q-Q Plots of Clock-Time and Trade-Time Returns



ACFs of Clock-Time and Trade-Time Returns



ACFs of Clock-Time and Trade-Time Squared Returns



General Model

We develop a hierarchical model of clock-time returns that mixes a distribution of trade-time returns with a distribution of trade arrival.

- Given m, assume trade-time returns are i.i.d. Gaussian:

$$r_m(n) \overset{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma), \forall n.$$

Trade arrivals will be distributed according to some counting process.

- For clock-time duration τ, denote the number of m-period executed trades as $N_m(\tau)$ with probability $P(N_m(\tau) = k)$.

General Model

We are interested in the random variable

$$r_\tau(t) = \sum_{i=1}^{N_m(\tau)} r_m(n).$$

The probability density function of $r_\tau(t)$ is,

$$p(r_\tau(t)|\mu,\sigma) = \sum_{k=1}^{\infty} p\left(\sum_{i=1}^{k} r_m(n) \middle| N_m(\tau) = k, \mu, \sigma\right) P(N_m(\tau) = k).$$

- This is a finite Normal mixture model.
- The mixture weights vary according to the probability distribution of $N_m(\tau)$.

General Model

The mixture model can also be viewed as a hierarchical model.

- In the first stage the number of trades is drawn from the distribution of $N_m(\tau)$.

- In the second stage a single τ-period return is drawn from the normal distribution:

$$r_\tau(t) = \sum_{i=1}^{N_m(\tau)} r_m(n) \sim \mathcal{N}\left(N_m(\tau)\mu, \sqrt{N_m(\tau)}\sigma\right)$$

- The resulting distribution of $r_\tau(t)$ will have fatter tails than a Gaussian.

- The tail fatness will be intimately related to the distribution of $N_m(\tau)$.

Poisson Trade Arrival

A starting point for modeling trade arrivals would be to assume they follow a Poisson process:

$$N_m(\tau) \sim \text{Poisson}(\lambda\tau)$$

or

$$P\left(N_m(\tau) = k\right) = \frac{(\lambda\tau)^k}{k!} \exp -\lambda\tau,$$

where λ is the arrival intensity parameter.

Poisson Trade Arrival

The probability density function $r_\tau(t)$ is

$$p(r_\tau(t)|\mu,\sigma) = \sum_{k=1}^{\infty} \frac{1}{\sigma\sqrt{2\pi k}} \exp\left\{-\frac{1}{2}\frac{\left(\sum_{i=1}^{k} r_m(n) - k\mu\right)^2}{k\sigma^2}\right\} \times \exp\{-\lambda\tau\}\frac{(\lambda\tau)^k}{k!}.$$

- The density function cannot be obtained in closed form, but we can approximate it via Monte Carlo simulation.
- Poisson trade arrivals are associated with Exponential inter-trade arrival times.

MSMD Model

As an alternative, we use the Markov-Switching Multifractal Duration (MSMD) Model for inter-trade durations.

- This model is due to Chen, Diebold and Schorfheide (2013).

The core components of the MSMD model are:

- $\bar{k}$ latent state variables, $M_{k,i}$, that obey two-state Markov-switching processes.
- Persistence parameters, γ_k, for each latent variable $M_{k,i}$, $k = 1, 2, \ldots, \bar{k}$.

MSMD Model

The distribution of MSMD trade durations, d_i, is governed by the equations:

$$d_i = \frac{\varepsilon_i}{\lambda_i}$$

$$\varepsilon_i \sim Exp(1)$$

$$\lambda_i = \lambda \prod_{k=1}^{\bar{k}} M_{k,i}$$

$$M_{k,i} = \begin{cases} M & \text{with probability } \gamma_k \\ M_{k,i-1} & \text{otherwise} \end{cases}$$

$$\gamma_k = 1 - (1 - \gamma_{\bar{k}})^{b^{k-\bar{k}}}$$

$$M = \begin{cases} m_0 & \text{with probability } 1/2 \\ 2 - m_0 & \text{otherwise.} \end{cases}$$

MSMD Model

- The MSMD model is characterized by five parameters: $\bar{k} \in \mathbb{N}$, $\lambda > 0$, $\gamma_{\bar{k}} \in (0, 1)$, $b \in (1, \infty)$ and $m_0 \in (0, 2]$.
- Conditional on knowing the values of the latent state variables, inter-trade durations are Exponentially distributed with rate parameter λ_i.
- As time evolves the latent states, $M_{k,i}$, switch values with varying degrees of persistence, γ_k.
- This causes the actual distribution of intra-day trade durations to be a mixture of Exponentials.
- The latent states can be interpreted as shocks that have varying impacts over diverse timescales.

MSMD Counting Process

The MSMD model is associated with a counting process $N_m(\tau)$.

- The density of the counting process cannot be obtained in closed form.
- We can simulate from the distribution of the counting process.
- We can pair simulations from the counting density with Gaussian random variables to obtain simulations for clock-time returns $r_\tau(t)$ associated with the MSMD model.

Estimation

We use the passive-market E-mini data to estimate the Exponential and MSMD duration models.

- Estimation is done via maximum likelihood.
- Since the MSMD density cannot be obtained in closed form, we resort to the nonlinear filtering method of Hamilton (1989) to obtain MLEs.
- We estimate the model using trade-time unit $m = 4$.
- Estimates were not stable for $m < 4$.
- Following Chen et al. (2013), we fix $\bar{k} = 7$ and estimate the other four MSMD parameters.

Estimation

We assume that the trade-time returns, $r_m(n)$, follow a Gaussian density.

- As a result, MLEs are simply the sample mean and sample standard deviation of observed trade-time returns in the data.
- As mentioned above, we fix $m = 4$ for trade-time returns.

Estimates

λ	$\gamma_{\bar{k}}$	b	m_0	ν	λ	μ	σ
15.71	0.1906	3.039	1.623	1.189	0.8408	-0.0006324	0.1393

Simulations of Clock-Time Returns

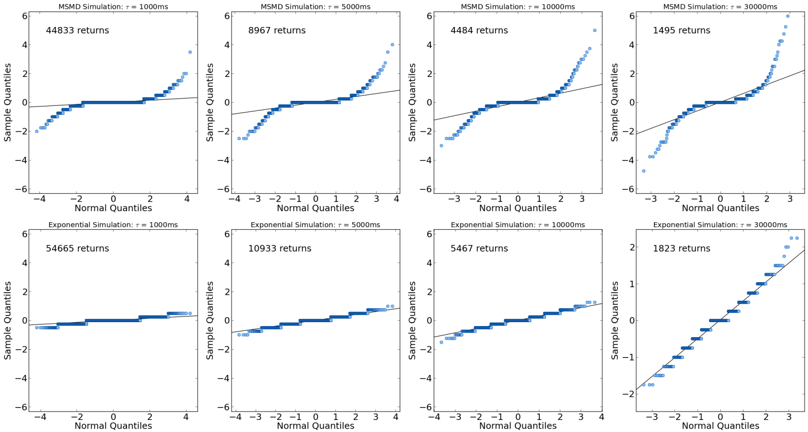
With estimates of the component distributions in hand, we can simulate clock-time returns.

- First, simulate inter-trade durations for $m = 4$ from the MSMD or Exponential models.
- Second, pair the durations with independent draws of trade-time returns from the estimated Gaussian density.
- Third, aggregate individual returns within a fixed clock time interval.

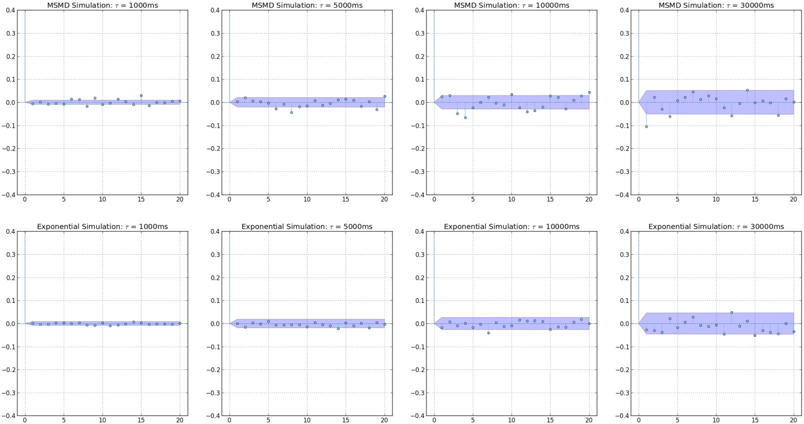
Simulations of Clock-Time Returns

- We separately simulate 25,000 MSMD and Exponential durations.
- We pair each set of durations with the same simulation of 25,000 trade-time returns.
- We aggregate for clock-time intervals $\tau = \{1000, 5000, 10000, 30000\}$ ms.
- The resulting number of clock-time returns, under both models roughly corresponds to the number of observations in our passive-period data for corresponding values of τ.

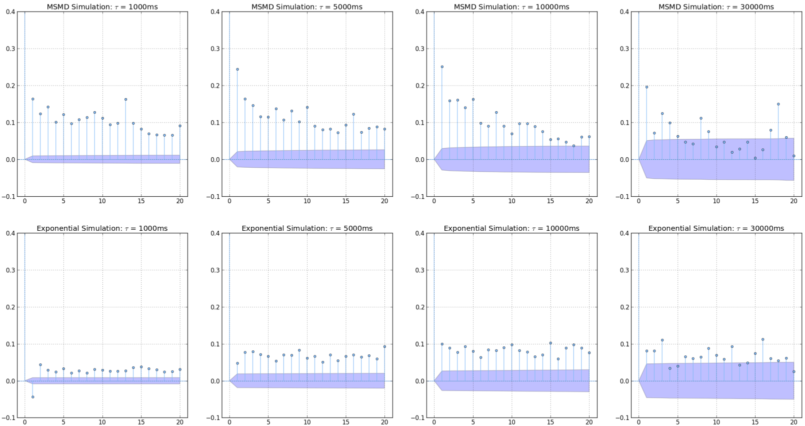
Q-Q Plots of Simulated Clock-Time Returns



ACFs of Simulated Clock-Time Returns



ACFs of Simulated Clock-Time Squared Returns

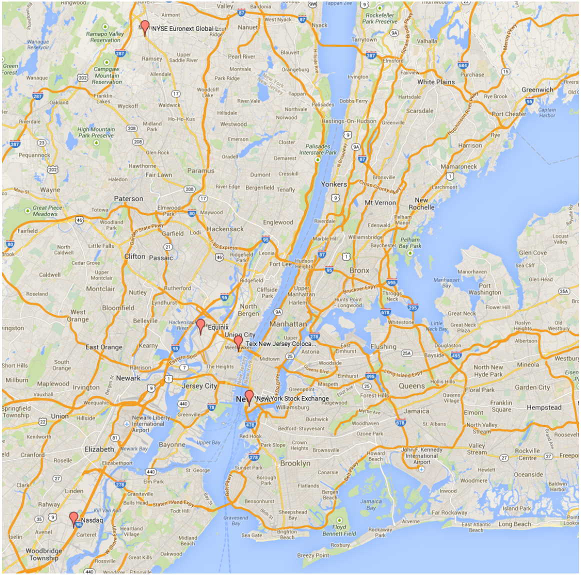


Market Fragmentation

Our results will inform the discussion of market fragmentation.

- The equities markets consist of 13 exchanges at 4 locations in metro New York.
  1. Weehawken, NJ (BATS)
  2. Secaucus, NJ (DirectEdge)
  3. Mahwah, NJ (NYSE)
  4. Carteret, NJ (Nasdaq)
- Futures trade at the CME Globex exchange in Aurora, IL.
- Options trade at the CBOE exchange in Secaucus, NJ.
- Currencies trade primarily in Secaucus, NJ.

Exchanges in Metro NY



Exchanges in IL and NY

Exchange Latencies and Market Influence

- The round-trip line time between Chicago and NY is roughly 10 ms.
- The longest round-trip line time between metro NY exchanges is no longer that 666 microseconds (2/3 ms).
- A rough estimate of market influence on price formation: 70% futures (IL), 30% spot (NJ).
- We approximate aggregate round trip market latency as

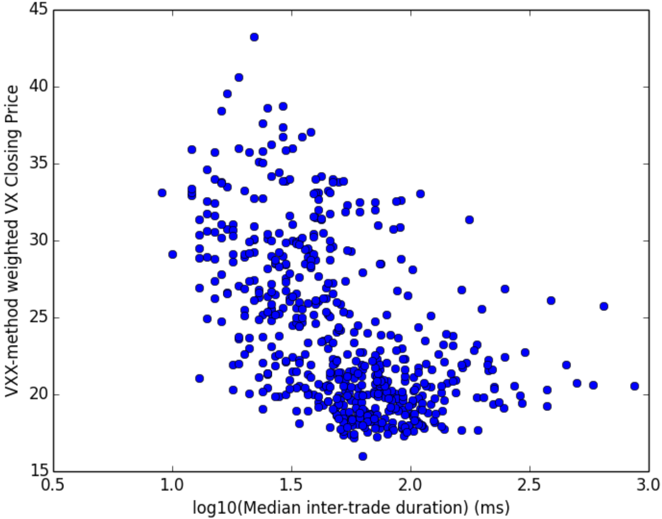
$$0.7 \times 10 \text{ ms} + 0.3 \times 0.66 \text{ ms} \approx 7 \text{ ms}.$$

Latency/Volatility Relationship

There is a relationship between volatility and trade latency in market data.

- Aug 9, 2011 was a heavy trading day for E-mini: roughly 5M shares traded with median inter-trade duration of 18 ms.
- Closing VIX on Aug 9, 2011 was 48.00.
- Jul 11, 2011 was a more typical day: roughly 1M shares traded with median inter-trade duration of 80 ms.
- Closing VIX on Jul 11, 2011 was 18.39.

Latency/Volatility Relationship



Latency/Volatility Relationship

These values suggest that in a stressed market (where latencies approach 7 ms) the maximum attainable VIX would be between 60 and 80.

- The highest observed closing VIX value observed to date is 81.65 on Oct 27, 2008.
- Our calculation assumes that VIX scales with the square root of inter-trade duration.
- We are working on generating model implied volatilities that are associated with inter-trade durations.

Flash Boys

In his book, Flash Boys, Michael Lewis suggests that market fragmentation is bad.

- The book features a new exchange, IEX, co-located in Weehawken, that is claims to promote fairness.
- If market influence were to shift entirely to the New York metro area, aggregate market latency would reduce to something of order 1 ms.
- This would correspond to a VIX upper bound of 200.
- Further, co-locating all NY exchanges would reduce the total market latency to roughly 1 microsecond, or a VIX upper bound of 6500.
- Market fragmentation forces an implicit threshold which bounds market volatility.

Conclusion

We develop a model that approximates the distribution of high-frequency asset returns quite well.

- A striking result of our analysis is that when measured in trade-time, outside of scheduled news announcements, trade-time returns follow a Gaussian distribution.
- When pairing Gaussian trade-time returns with an accurate model of inter-trade duration, we obtain a good approximation of the returns distribution in clock time.
- The clock-time returns distribution has fat tails that are commensurate with the data and also exhibits volatility clustering.
- Finally, our work with duration data suggests that market fragmentation is desirable from the perspective of placing an implicit bound on market volatility.