LECTURE 7

Transformation of the coordinates at transition to a new basis

In the linear space R^n we can choose a basis by non-unique way, and therefore the rule of changing the coordinates of an element of the space at transition from one basis to another has a practical interest.

Let there be two bases: e_1, e_2, e_3, … (old) and $e_1', e_2', e_3', ...$ (new) in a n-dimensional linear space R^n, and dependences expressing each vector of the new basis through vectors of the old basis are the following:

$$\begin{aligned} e_1' &= a_{11}e_1 + a_{21}e_2 + ... + a_{n1}e_n, \\ e_2' &= a_{12}e_1 + a_{22}e_2 + ... + a_{n2}e_n, \qquad (*) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ e_n' &= a_{1n}e_1 + a_{2n}e_2 + ... + a_{nn}e_n. \end{aligned}$$

The matrix $A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$ is called the *transition matrix* from the old basis to the new one.

Theorem. Every transition matrix A is regular, i.e. $\det A \neq 0$.

Proof: Prove the theorem for n = 2. Let $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ be "old" and "new" bases of a linear space V. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be the transition matrix from $\{e_1, e_2\}$ to $\{e_1', e_2'\}$, i.e.

$$\begin{cases} e_1' = a_{11}e_1 + a_{21}e_2 \\ e_2' = a_{12}e_1 + a_{22}e_2 \end{cases} \quad (**)$$

Assume the contrary: $\det A = 0$, i.e. $a_{11}a_{22} - a_{12}a_{21} = 0$. Suppose that $a_{21} \neq 0$ or $a_{22} \neq 0$ (if both a_{21} or a_{22} are equal to zero, then obviously e_1' and e_2' are linearly dependent). Multiply both parts of the first equation of (**) on a_{22}, and multiply both parts of the second equation of (**) on $(-a_{21})$. Then add the obtained expressions. We have $a_{22}e_1' - a_{21}e_2' = 0$, i.e. e_1' and e_2' are linearly dependent, contradicting the hypothesis that they form a basis. □

Take an arbitrary vector x. Let $(\xi_1; \xi_2; ...; \xi_n)$ be the coordinates of this vector in the old basis, and $(\xi_1'; \xi_2'; ...; \xi_n')$ be its coordinates in the new basis.

Theorem. The coordinates $\xi_1, \xi_2, ..., \xi_n$ and $\xi_1', \xi_2', ..., \xi_n'$ are connected by the following formulas:

$$\begin{aligned} \xi_1 &= a_{11}\xi_1' + a_{12}\xi_2' + ... + a_{1n}\xi_n', \\ \xi_2 &= a_{21}\xi_1' + a_{22}\xi_2' + ... + a_{2n}\xi_n', \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \xi_n &= a_{n1}\xi_1' + a_{n2}\xi_2' + ... + a_{nn}\xi_n'. \end{aligned}$$

which are called *formulas of transformation of coordinates*.

It is easily to see that the columns of the matrix A are the coordinates in formulas of transition from the old basis to new, and the rows of this matrix are the coordinates in formulas of transformation of old coordinates through new.

Proof. By (*) we have $\sum_{i=1}^{n} \xi_i e_i = x = \sum_{j=1}^{n} \xi_j' e_j' = \sum_{j=1}^{n} \xi_j' \sum_{i=1}^{n} a_{ij} e_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} \xi_j' \right) e_i$

Consequently, $\sum_{i=1}^{n} \left(\xi_i - \sum_{j=1}^{n} a_{ij} \xi_j' \right) e_i = 0$. We have a linear combination of linearly independent elements that is equal to zero. It must be trivial. Then we obtain: $\xi_i = \sum_{j=1}^{n} a_{ij} \xi_j'$, $i = 1,2,..., n$.

In a matrix form: If $\begin{pmatrix} e_1' \\ e_2' \\ \dots \\ e_n' \end{pmatrix} = A^T \cdot \begin{pmatrix} e_1 \\ e_2 \\ \dots \\ e_n \end{pmatrix}$, then $\begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} = A \cdot \begin{pmatrix} \xi_1' \\ \xi_2' \\ \dots \\ \xi_n' \end{pmatrix}$.

Example. Let a vector $x = e_1 + e_2 + e_3 + e_4$ be given. Express this vector through the new basis $e_1',\ e_2',\ e_3',\ e_4'$ if $e_1' = e_2 + e_3 + e_4,\ e_2' = e_1 + e_3 + e_4,\ e_3' = e_1 + e_2 + e_4,\ e_4' = e_1 + e_2 + e_3$.

Solution:

I *way*. Write the transition matrix from the old basis to new:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The rows of the matrix are coefficients in formulas of transformation of coordinates:

$$\xi_1 = \xi_2' + \xi_3' + \xi_4',\ \xi_2 = \xi_1' + \xi_3' + \xi_4',\ \xi_3 = \xi_1' + \xi_2' + \xi_4',\ \xi_4 = \xi_1' + \xi_2' + \xi_3'$$

Since $\xi_1 = \xi_2 = \xi_3 = \xi_4 = 1$ then by solving the system of equations we find $\xi_1' = \xi_2' = \xi_3' = \xi_4' = 1/3$ and $x = \frac{1}{3}(e_1' + e_2' + e_3' + e_4')$.

II *way*. Since $e_1' + e_2' + e_3' + e_4' = 3e_1 + 3e_2 + 3e_3 + 3e_4$, then

$$e_1 + e_2 + e_3 + e_4 = \frac{1}{3}(e_1' + e_2' + e_3' + e_4').$$

Consequently, $x = \frac{1}{3}(e_1' + e_2' + e_3' + e_4')$.

Example. The system of coordinates *xOy* has been turned around the origin of coordinates on angle α. Express the coordinates of a vector $a = xi + yj$ in the new system through its coordinates in the old system.

Solution: Decompose the vectors i' and j' on orts *i* and *j*:

$$i' = i\cos\alpha + j\sin\alpha,$$

$$j' = i\cos\left(\frac{\pi}{2} + \alpha\right) + j\sin\left(\frac{\pi}{2} + \alpha\right).$$

Since $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha,\ \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$ then the matrix of transition from the old basis {*i, j*} to the new basis { i', j' } is the following:

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Then we obtain $x = x'\cos\alpha - y'\sin\alpha,\ y = x'\sin\alpha + y'\cos\alpha,$

i.e. $x' = x\cos\alpha + y\sin\alpha,\ y' = -x\sin\alpha + y\cos\alpha.$

Subspace of a linear space

A non-empty set V' formed of elements of a linear space V is called a *subspace* of the linear space V if for all $x, y \in V'$ and every number λ $x + y \in V'$ and $\lambda x \in V'$.

Remark. Obviously, the set V' is a linear space itself since all the axioms of a linear space hold.

Example. The set of all vectors that are parallel to the same plane is a subspace of all geometric vectors of the space.

Example. Determine whether the set V' of square matrices of the *n*-th order with zero first row in the linear space V of all square matrices of the *n*-th order is a linear subspace, and if yes then find its dimension.

Solution: Indeed, if we take two matrices with zero first row $A, B \in V'$:

$$A=\begin{pmatrix} 0 & 0 & ... & 0 \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} \text{ and } B=\begin{pmatrix} 0 & 0 & ... & 0 \\ b_{21} & b_{22} & ... & b_{2n} \\ ... & ... & ... & ... \\ b_{n1} & b_{n2} & ... & b_{nn} \end{pmatrix}, \text{ we have}$$

$$A+B=\begin{pmatrix} 0 & 0 & ... & 0 \\ a_{21}+b_{21} & a_{22}+b_{22} & ... & a_{2n}+b_{2n} \\ ... & ... & ... & ... \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & ... & a_{nn}+b_{nn} \end{pmatrix} \in V' \text{ and } \lambda A=\begin{pmatrix} 0 & 0 & ... & 0 \\ \lambda a_{21} & \lambda a_{22} & ... & \lambda a_{2n} \\ ... & ... & ... & ... \\ \lambda a_{n1} & \lambda a_{n2} & ... & \lambda a_{nn} \end{pmatrix} \in V'$$

Thus, V' is a subspace of V. Obviously, $d(V')=n^2-n$.

Example. Determine whether the set V' of singular square matrices of the 2-nd order (i.e. having zero determinant) in the linear space V of all square matrices of the 2-nd order is a linear subspace.

Solution: Let $A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously, $A, B \in V'$ since det A = 0, det B = 0.

But $A+B=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=E \notin V'$, i.e. V' is not a subspace of V.

If $x, y, z, \ldots, u$ are vectors of a linear space V then all vectors $\alpha x+\beta y+\gamma z+...+\lambda u$, where α, β, γ, …, λ are all possible real numbers, form a subspace of the space V. The set of all linear combinations of vectors $\alpha x+\beta y+\gamma z+...+\lambda u$ is called a *linear hull* of the vectors $x, y, z, \ldots, u$ and denoted by $L(x, y, z, \ldots, u)$.

Example. Find the dimension and a basis of the linear hull of vectors $c_1=(1,-1), c_2=(-1,6)$ and $c_3=(-1,1)$ in the space R^2.

Solution: Let $L=\{\alpha c_1+\beta c_2+\gamma c_3 \mid \alpha,\beta,\gamma \in R\}$. Obviously, $c_3=-c_1$, i.e. these vectors are linearly dependent. Since $\begin{vmatrix} 1 & -1 \\ -1 & 6 \end{vmatrix} \neq 0$, vectors c_1 and c_2 are linearly independent. Then $d(L)=2$ and $\{c_1, c_2\}$ form a basis of L.

If V_1 is a subspace of a linear space V then $d(V_1) \le d(V)$.

Let V_1 and V_2 be subspaces of a linear space V.

The *union* of V_1 and V_2 is called the set of elements $x \in V$ such that $x \in V_1$ or $x \in V_2$. The union of V_1 and V_2 is denoted by $V_1 \cup V_2$.

The *intersection* of V_1 and V_2 is called the set of all elements simultaneously belonging to V_1 and V_2. The intersection of V_1 and V_2 is denoted by $V_1 \cap V_2$.

The *sum* of V_1 and V_2 is called the set of all elements of kind $x + y$ where $x \in V_1$ and $y \in V_2$. The sum of V_1 and V_2 is denoted by V_1+V_2.

The *direct sum* of V_1 and V_2 is called the set of all elements of kind $x + y$ where $x \in V_1$, $y \in V_2$ and $V_1 \cap V_2=\{0\}$. The direct sum of V_1 and V_2 is denoted by $V_1 \oplus V_2$.

Theorem. Both the intersection and the sum of subspaces V_1 and V_2 are subspaces of V.

Theorem. The dimension of the sum of subspaces V_1 and V_2 is equal to:

$$\dim(V_1+V_2)=\dim(V_1)+\dim(V_2)-\dim(V_1 \cap V_2)$$

Proof. Let the subspace $V_1 \cap V_2$ has a basis $\{e_1, e_2, ..., e_k\}$, i.e. $\dim(V_1 \cap V_2)=k$. Complete this basis by elements $\{e_1', e_2', ..., e_l'\}$ to a basis in V_1 and by elements $\{e_1'', e_2'', ..., e_m''\}$ to a basis in V_2. Then every element $x \in V_1+V_2$ can be decomposed on the system of elements

$$\{e_1, e_2, ..., e_k, e_1', e_2', ..., e_l', e_1'', e_2'', ..., e_m''\}$$

Show that this system is linearly independent in V. Consider some linear combination of these elements that is equal to zero: $\sum_{i=1}^{l} \xi_i' e_i' + \sum_{j=1}^{k} \xi_j e_j + \sum_{p=1}^{m} \xi_p'' e_p'' = 0 \qquad (**)$.

Observe that by our construction $\tilde{x} = \sum_{p=1}^{m} \xi''_p e''_p \in V_2$.

But on the other hand $\tilde{x} = \sum_{p=1}^{m} \xi''_p e''_p = -\left(\sum_{i=1}^{l} \xi'_i e'_i + \sum_{j=1}^{k} \xi_j e_j \right) \in V_1$. This means that $\tilde{x} \in V_1 \cap V_2$ and consequently all $\xi'_i = 0,\ i = 1, 2, ..., l$ and $\xi''_p = 0,\ p = 1, 2, ..., m$ in (**). And since $\{e_1, e_2, ..., e_k\}$ is a basis, all $\xi_j = 0,\ j = 1, 2, ..., k$, and the linear combination of (**) is trivial. Consequently, $\{e_1, e_2, ..., e_k, e'_1, e'_2, ..., e'_l, e''_1, e''_2, ..., e''_m\}$ is a linearly independent system of elements. Then it is a basis in $V_1 + V_2$.

Thus, $\dim(V_1 + V_2) = l + k + m = (k + l) + (k + m) - k = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$.

Subspaces formed by solutions of a homogeneous linear system of equations

Consider a homogeneous linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0 \end{cases} \qquad (1)$$

Let $x_1 = \lambda_1, x_2 = \lambda_2, ..., x_n = \lambda_n$ be a solution of the system. Write this solution as the vector $f = (\lambda_1; \lambda_2; ...; \lambda_n)$. The collection of linearly independent solutions $f_1, f_2, ..., f_n$ of the system of equations (1) is called the *fundamental system of solutions* if any solution of the system of equations (1) can be represented in form of linear combination of vectors $f_1, f_2, ..., f_n$.

Theorem (on existence of fundamental system of solutions). If the rank of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

is less than *n* then the system (1) has non-zero solutions. The number of vectors determining the fundamental system of solutions is found by the formula $k = n - r$ where *r* is the rank of the matrix. Thus, if we consider the linear space R^n of which vectors are all possible systems of *n* real numbers then the collection of all solutions of the system (1) is a subspace of the space R^n. The dimension of this subspace is equal to *k*.

Example. Find a basis and the dimension of the subspace of solutions of the following linear homogeneous system of equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 0, \\ (1/2)x_1 + x_2 + (3/2)x_3 + 2x_4 = 0, \\ (1/3)x_1 + (2/3)x_2 + x_3 + (4/3)x_4 = 0, \\ (1/4)x_1 + (1/2)x_2 + (3/4)x_3 + x_4 = 0. \end{cases}$$

Solution: The rank of the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1/2 & 1 & 3/2 & 2 \\ 1/3 & 2/3 & 1 & 4/3 \\ 1/4 & 1/2 & 3/4 & 1 \end{pmatrix}$ is equal to 1 since all the minors of the matrix but minors of the first order are equal to zero. The number of unknowns is 4, therefore the dimension of the subspace of solutions $k = n - r = 4 - 1 = 3$, i.e. this subspace is three-dimensional. Since $r = 1$ then it is sufficiently to take one equation from this system. Take the first equation and write it as follows: $x_1 = -2x_2 - 3x_3 - 4x_4$. If $x_2 = 1, x_3 = 0, x_4 = 0$ then $x_1 = -2$; if $x_2 = 0, x_3 = 1, x_4 = 0$ then $x_1 = -3$; if $x_2 = 0, x_3 = 0, x_4 = 1$ then $x_1 = -4$. Thus, we obtain linearly independent vectors $f_1 = (-2;\,1;\,0;\,0)$, $f_2 = (-3;\,0;\,1;\,0)$, $f_3 = (-4;\,0;\,0;\,1)$ which form a basis of the three-dimensional subspace of solutions of the system.

Glossary

transformation – преобразование; **to turn** – повернуть; **coplanar –** компланарный
around – вокруг; **transition –** переход; **linear hull** – линейная оболочка

Exercises for Seminar 7

7.1. Let a vector $x = 8e_1 + 6e_2 + 4e_3 - 14e_4$ be given. Express this vector through the new basis $e_1',\ e_2',\ e_3',\ e_4'$ if $e_1' = -3e_1 + e_2 + e_3 + e_4$, $e_2' = 2e_1 - 4e_2 + e_3 + e_4$, $e_3' = e_1 + 3e_2 - 5e_3 + e_4$, $e_4' = e_1 + e_2 + 4e_3 - 5e_4$.

7.2. Let a vector $x = 2(e_1 + e_2 + e_3 + ... + e_n)$ be given, where n is odd. Express this vector through the new basis $e_1',\ e_2',\ ...,\ e_n'$ if $e_1' = e_1 + e_2$, $e_2' = e_2 + e_3$, $e_3' = e_3 + e_4$, …, $e_{n-1}' = e_{n-1} + e_n$, $e_n' = e_n + e_1$. Check that if n is even then $e_1',\ e_2',\ ...,\ e_n'$ is not a basis.

7.3. Can a subspace of a linear space V consist of one element?

7.4. Consider the linear space V of which elements are all possible systems of real numbers: $x = (\xi_1, \xi_2, \xi_3, \xi_4),\ y = (\eta_1, \eta_2, \eta_3, \eta_4), \ldots$. Addition of two elements and multiplication of an element on number are determined by the following equalities:

$$x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3, \xi_4 + \eta_4),\ \lambda x = (\lambda\xi_1, \lambda\xi_2, \lambda\xi_3, \lambda\xi_4)$$

Prove that the set V_1 of elements $x = (0, \xi_2, \xi_3, \xi_4),\ y = (0, \eta_2, \eta_3, \eta_4), \ldots$ and the set V_2 of elements $x = (\xi_1, 0, \xi_3, \xi_4),\ y = (\eta_1, 0, \eta_3, \eta_4), \ldots$ are subspaces of the linear space V.

7.5. Find the intersection $V_1 \cap V_2$ and the sum $V_1 + V_2$ of the subspaces V_1 and V_2 for the linear space V considered in Ex. 7.4.

7.6. Find a basis and the dimension of the subspace of solutions of the following system of equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 0, \\ 2x_1 - x_2 - x_3 = 0, \\ -2x_1 + 4x_2 - 2x_3 = 0. \end{cases}$$

7.7. Find a basis and the dimension of the subspace of solutions of the following system of equations

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 0, \\ x_1 - x_2 + x_3 - x_4 = 0, \\ 3x_1 + x_2 - x_3 + x_4 = 0, \\ 3x_1 - x_2 + x_3 - x_4 = 0. \end{cases}$$

7.8. Determine whether is a given set of vectors in the arithmetic n-dimensional linear space a linear subspace, and if yes then find its dimension:
1) the set of vectors of which all the coordinates are equal each other;
2) the set of vectors of which the sum of coordinates is equal to 0.

7.9. Determine whether is a given set of vectors of the geometric space a linear subspace, and if yes then find its dimension:
1) the set of vectors of a plane which are parallel to a given line;
2) the set of vectors of a plane which don't exceed 1 by module.

7.10. Determine the dimension of the subspace of solutions, a basis and general solution of the following system of equations

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 0, \\ x_1 - 2x_2 + x_3 + x_4 - x_5 = 0. \end{cases}$$

7.11. Find the dimension and a basis of the linear hull of the vectors $c_1 = (-3,\ 2,\ 0)$, $c_2 = (-3,\ 6, -15)$ and $c_3 = (0,\ -4,\ 15)$.

Exercises for Homework 7

7.12. Let a vector $x = e_1 - 2e_2 + e_3$ be given. Express this vector through the new basis e_1', e_2', e_3' if $e_1' = -2e_1 + e_2 - 2e_3$, $e_2' = e_1 - 3e_2 + e_3$, $e_3' = 3e_1 + e_2 + 2e_3$.

7.13. Consider the linear space V of polynomials with degree ≤ 5. Prove that the set V_1 of polynomials of kind $a_0 t + a_1$ and the set V_2 of polynomials $b_0 t^4 + b_1 t^2 + b_2$ are subspaces of the space V if the addition of elements and the multiplication of an element on number are usual.

7.14. Find the subspaces $V_1 \cap V_2$ and $V_1 + V_2$ for the linear space V considered in Ex. 7.13.

7.15. Determine whether is a given set of vectors in the arithmetic n-dimensional linear space a linear subspace, and if yes then find its dimension:

1) the set of vectors of which the first coordinate is equal to 0;
2) the set of vectors of which the sum of coordinates is equal to 1.

7.16. Determine whether is a given set of square matrices of the n-th order in the linear space of all square matrices of the n-th order a linear subspace, and if yes then find its dimension:

1) the set of diagonal matrices;
2) the set of symmetrical matrices.

7.17. Let a vector $x = 3e_1 - 2e_2 - e_3 + 4e_4$ be given. Express this vector through the new basis e_1', e_2', e_3', e_4' if $e_1' = e_1 + e_2 - e_3 + e_4$, $e_2' = 2e_1 - e_2 + 3e_3 - 2e_4$, $e_3' = e_1 - e_3 + 2e_4$, $e_4' = 3e_1 - e_2 + e_3 - e_4$.

7.18. Find a basis and the dimension of the subspace of solutions of the following system of equations

$$\begin{cases} x_1 - x_2 + 2x_3 = 0, \\ 3x_1 + x_2 - x_3 = 0, \\ -2x_1 + 2x_2 - 4x_3 = 0. \end{cases}$$

7.19. Find a basis and the dimension of the subspace of solutions of the following system of equations

$$\begin{cases} x_1 - x_2 - 2x_3 + 2x_4 = 0, \\ -2x_1 + x_2 + 4x_3 - 2x_4 = 0, \\ -x_1 + 2x_2 + 2x_3 - 4x_4 = 0, \\ 2x_1 - 3x_2 - 4x_3 + 6x_4 = 0. \end{cases}$$

7.20. Determine the dimension of the subspace of solutions, a basis and general solution of the following system of equations

$$\begin{cases} 3x_1 - x_2 - 2x_3 + x_4 + 3x_5 = 0, \\ 2x_1 + 2x_2 - 3x_3 - x_4 + x_5 = 0. \end{cases}$$

7.21. Find the dimension and a basis of the linear hull of the vectors $c_1 = (1, 2, 1, 3)$, $c_2 = (1, 1, 1, 3)$ and $c_3 = (1, 0, 1, 3)$.