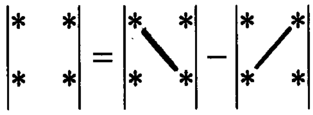
LECTURE 2
Determinants, minors and cofactors.

Determinants of the second and third order.

The determinant of the second order which corresponds to a square matrix of the second order $A(2;\,2)=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the number denoted by the following symbol $\Delta=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ and which is equal to: $\Delta = a_{11}\cdot a_{22} - a_{21}\cdot a_{12}$ (1)

This is illustrated by the following scheme:



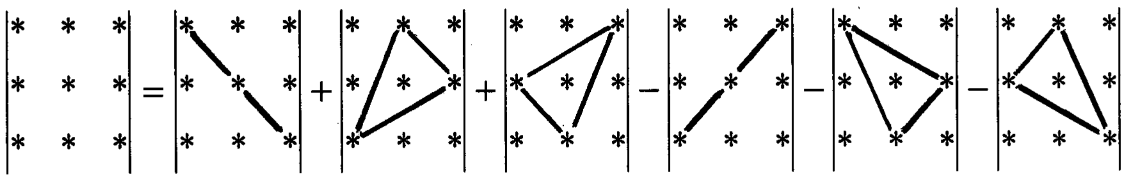
Example: $\Delta = \begin{vmatrix} 2 & -3 \\ 4 & -1 \end{vmatrix} = 2\cdot(-1) - 4\cdot(-3) = -2 + 12 = 10.$

Observe that there is no determinant corresponding to a rectangular (non-square) matrix.

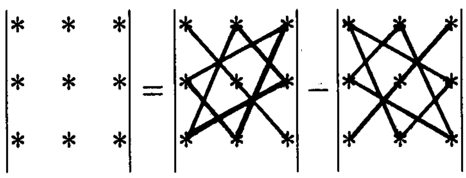
The determinant of the third order which corresponds to a square matrix of the third order $A(3;3)=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is the number denoted by the following symbol $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and which is equal to

$$\Delta = a_{11}\cdot a_{22}\cdot a_{33} + a_{12}\cdot a_{23}\cdot a_{31} + a_{21}\cdot a_{32}\cdot a_{13} - a_{31}\cdot a_{22}\cdot a_{13} - a_{21}\cdot a_{12}\cdot a_{33} - a_{32}\cdot a_{23}\cdot a_{11} \quad (2)$$

For calculation of determinants of the third order is convenient to use «the triangle rule» which follows from the formula (2) and symbolically is written as:



or



Example:

$$\begin{vmatrix} -1 & 2 & 3 \\ -2 & 1 & 4 \\ 5 & -3 & 2 \end{vmatrix} = (-1)\cdot1\cdot2 + 2\cdot4\cdot5 + (-2)\cdot(-3)\cdot3 - 5\cdot1\cdot3 - (-2)\cdot2\cdot2 - (-3)\cdot4\cdot(-1) =$$

$$= -2 + 40 + 18 - 15 + 8 - 12 = 37.$$

Minors and cofactors (signed minors) of elements of a determinant.

The *minor* M_{ij} of the element a_{ij} of a square matrix *A(n; n)* is the determinant obtained from the given one by deleting both the *i*-th row and the *j*-th column.

For a determinant of the second order $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$: $\begin{matrix} M_{11} = a_{22}; & M_{12} = a_{21}; \\ M_{21} = a_{12}; & M_{22} = a_{11}. \end{matrix}$

Example: $\Delta = \begin{vmatrix} -2 & 1 \\ 3 & -4 \end{vmatrix}$. $\begin{matrix} M_{11} = -4; & M_{12} = 3; \\ M_{21} = 1; & M_{22} = -2. \end{matrix}$

For a determinant of the third order $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$: $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$;

$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and etc.

Example: Find M_{13} and M_{32} for the determinant of the third order

$\Delta = \begin{vmatrix} -4 & 3 & 2 \\ 0 & -1 & 4 \\ 5 & 2 & -3 \end{vmatrix}$.

Solution: $M_{13} = \begin{vmatrix} 0 & -1 \\ 5 & 2 \end{vmatrix} = 0 + 5 = 5$; $M_{32} = \begin{vmatrix} -4 & 2 \\ 0 & 4 \end{vmatrix} = -16 - 0 = -16$.

The *cofactor* A_{ij} of the element a_{ij} of a square matrix $A(n; n)$ is the minor M_{ij} multiplied on the number $(-1)^{i+j}$, i.e. $A_{ij} = (-1)^{i+j} \cdot M_{ij}$.

Thus, if the sum of indices $i + j$ of the element a_{ij} is even then the cofactor of this element coincides with its minor by sign, i.e. $A_{ij} = M_{ij}$; if the sum of indices $i + j$ of the element a_{ij} is odd then the cofactor of this element is opposite to its minor by sign, i.e. $A_{ij} = -M_{ij}$.

Example: Find A_{11} and A_{21} for the determinant $\Delta = \begin{vmatrix} -2 & -3 \\ 4 & 5 \end{vmatrix}$.

Solution: $A_{11} = M_{11} = 5$; $A_{21} = -M_{21} = -(-3) = 3$.

Example: Find A_{23} and A_{31} for the determinant $\Delta = \begin{vmatrix} -5 & 4 & 3 \\ 2 & -1 & 5 \\ 0 & -2 & -4 \end{vmatrix}$.

Solution:

$A_{23} = -M_{23} = -\begin{vmatrix} -5 & 4 \\ 0 & -2 \end{vmatrix} = -(10 - 0) = -10$;

$A_{31} = M_{31} = \begin{vmatrix} 4 & 3 \\ -1 & 5 \end{vmatrix} = 20 + 3 = 23$.

Properties of determinants

(we consider them on examples)

Property 1. The value of a determinant doesn't change if we interchange (replace) all its rows by the corresponding columns and vice versa.

Example: $\begin{vmatrix} -1 & 2 & 1 \\ 3 & 0 & -2 \\ 2 & -4 & 5 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 2 \\ 2 & 0 & -4 \\ 1 & -2 & 5 \end{vmatrix} = -42.$

Property 2. If we interchange (swap) any two rows (columns) then the determinant will change a sign.

Examples: 1) $\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & -3 \\ 4 & 5 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 1 \\ 4 & 5 & 0 \\ 2 & 1 & -3 \end{vmatrix}$,

2) $\begin{vmatrix} -1 & 0 & 1 \\ 2 & 1 & -3 \\ 4 & 5 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 \\ -3 & 1 & 2 \\ 0 & 5 & 4 \end{vmatrix}$.

Corollary. A determinant having two identical rows (columns) is equal to zero.

Examples: $\begin{vmatrix} -3 & 1 & 2 \\ 5 & 6 & 7 \\ -3 & 1 & 2 \end{vmatrix} = 0;$ $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ -7 & 4 & -7 \end{vmatrix} = 0.$

Property 3. If all the elements of a row (column) of a determinant are multiplied on the same non-zero number «*m*» then the determinant value increases (decreases) in «*m*» times.

Example: $\Delta_1 = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = -7.$ For example, multiply all the elements of the second column on 4.

We obtain: $\Delta_2 = \begin{vmatrix} -1 & 8 \\ 3 & 4 \end{vmatrix} = -28 = 4 \cdot (-7) = 4 \cdot \Delta_1.$

Corollary 1. If all the elements of a row (column) of a determinant have a non-zero common multiplier, it can be taken out the determinant sign.

Example: $\begin{vmatrix} -3 & 5 & 4 \\ 1 & 0 & 6 \\ 2 & -1 & 8 \end{vmatrix} = 2 \cdot \begin{vmatrix} -3 & 5 & 2 \\ 1 & 0 & 3 \\ 2 & -1 & 4 \end{vmatrix}$ (the common multiplier 2 was taken out from the last column).

Corollary 2. A determinant at which elements of two arbitrary rows (columns) are proportional respectively is equal to zero.

Example: $\begin{vmatrix} -1 & 4 & 2 \\ 2 & 0 & -4 \\ 3 & 1 & -6 \end{vmatrix} = 0$ (the elements of the first and the third columns are proportional respectively, i.e. $\begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix} = -2 \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$).

Property 4. Let each element of an arbitrary row (column) be the sum of two addends. Then the determinant is equal to the sum of two determinants such that the corresponding row (column) of the first determinant consists of the first addends, and the corresponding row (column) of the second determinant consists of the second addends.

Example: For example, let the first column of the determinant be the sum of two addends. Then

$$\begin{vmatrix} -1+2 & 3 & 4 \\ 1-3 & 0 & -2 \\ 4+5 & 7 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 4 \\ 1 & 0 & -2 \\ 4 & 7 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 4 \\ -3 & 0 & -2 \\ 5 & 7 & -3 \end{vmatrix}.$$

Corollary. A determinant doesn't change its value if all the elements of a row (column) are added to (subtracted from) the corresponding elements of an arbitrary other row (column) multiplied on the same non-zero number.

Example: Compute the determinant by using its properties: $\begin{vmatrix} -1 & 2 & 3 \\ 4 & 0 & 1 \\ 2 & 4 & 7 \end{vmatrix}$.

1) Observe that the second column has the common multiplier 2. Take out it for the determinant sign. We obtain: $2 \cdot \begin{vmatrix} -1 & 1 & 3 \\ 4 & 0 & 1 \\ 2 & 2 & 7 \end{vmatrix}$.

2) If we add the second column to the first one, the obtained first column will have a common multiplier. Let's do these two operations:

$$2 \cdot \begin{vmatrix} 0 & 1 & 3 \\ 4 & 0 & 1 \\ 4 & 2 & 7 \end{vmatrix} = 8 \cdot \begin{vmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 7 \end{vmatrix}.$$

3) If we subtract the second row from the third one, the obtained third row will also have a common multiplier. Let's do these two operations:

$$8 \cdot \begin{vmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 6 \end{vmatrix} = 16 \cdot \begin{vmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix}.$$

4) Observe that the first and the third rows of the determinant are identical. Consequently, it is equal to zero.

Let's give statements of two theorems (without proof) permitting to calculate values of determinants of an *arbitrary* order by sequential reducing them to determinants of lower order.

Theorem 1. If all the elements of an arbitrary *i–th* row (either *j–th* column) of a determinant but a_{ij} are equal to zero, the determinant is equal to the product of this non-zero element a_{ij} on its cofactor A_{ij}, i.e. $\Delta = a_{ij} \cdot A_{ij}$.

Example: For example, let all the elements of the second column of the determinant but one are equal to zero. Then:

$$\Delta = \begin{vmatrix} -1 & 0 & 5 \\ 4 & 0 & 3 \\ 7 & 2 & 4 \end{vmatrix} = a_{32} \cdot A_{32} = 2 \cdot (-M_{32}) = -2 \cdot \begin{vmatrix} -1 & 5 \\ 4 & 3 \end{vmatrix} = -2 \cdot (-3 - 2) = 46.$$

By using Theorem 1 we have reduced the determinant of the third order to a determinant of the second order.

Theorem 2. A determinant is equal to the sum of products of elements of an arbitrary row (column) on their cofactors. This operation is called a *decomposition* of a determinant on a row (column).

Example: Compute the determinant of the fourth order $\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 \\ 3 & -1 & -1 & 0 \\ 1 & 2 & 0 & -5 \end{vmatrix}$.

Solution: For example, decompose the determinant on elements of the second row (since it is the simplest). Then:

$$\Delta = a_{21}\cdot A_{21} + a_{22}\cdot A_{22} + a_{23}\cdot A_{23} + a_{24}\cdot A_{24} = 1\cdot(-M_{21}) + 0\cdot M_{22} + 1\cdot(-M_{23}) + 2\cdot M_{24} =$$

$$= -\begin{vmatrix} 2 & 3 & 4 \\ -1 & -1 & 0 \\ 2 & 1 & -5 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & -5 \end{vmatrix} + 2\cdot\begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & -1 \\ 1 & 2 & 0 \end{vmatrix} = -3 - 63 + 2\cdot 21 = -24.$$

Theorem 2 allowed us to reduce the determinant order, i.e. to reduce the determinant of the fourth order to an algebraic sum of determinants of the third order.

Theorem 3. Let A and B be square matrices of the $n-$th order. Then $\det AB = \det A \cdot \det B$.

Permutations. Sign (Parity) of a permutation.

A *permutation* σ of the set $\{1,2,...,n\}$ is a one-to-one mapping of the set onto itself or, equivalently, a rearrangement of the numbers 1,2, …, n. Such a permutation σ is denoted by

$$\sigma = \begin{pmatrix} 1 & 2 & ... & n \\ j_1 & j_2 & ... & j_n \end{pmatrix} \text{ or } \sigma = j_1 j_2 ... j_n, \text{ where } j_i = \sigma(i)$$

The set of all such permutations is denoted by S_n, and the number of such permutations is $n!$. If $\sigma \in S_n$, then the inverse mapping $\sigma^{-1} \in S_n$; and if $\sigma, \tau \in S_n$, then the composition mapping $\sigma \circ \tau \in S_n$. Also, the identity mapping $\varepsilon = \sigma \circ \sigma^{-1} \in S_n$.

Consider an arbitrary permutation $\sigma \in S_n$, say $\sigma = j_1 j_2 ... j_n$. We say σ is even or odd permutation according to whether there is an even or odd number of inversions in σ. By an *inversion* in σ we mean a pair of integers (i,k) such that $i > k$, but i precedes k in σ. We then define the sign or parity of σ, written $\operatorname{sgn}\sigma$, by $\operatorname{sgn}\sigma = \begin{cases} 1 & if\, \sigma\, is\ even \\ -1 & if\, \sigma\, is\ odd \end{cases}$.

Example. (a) Find the sign of $\sigma = 35142$ in S_5. For each element k, we count the number of elements i such that $i > k$ and i precedes k in σ. There are 2 numbers (3 and 5) greater than and preceding 1; 3 numbers (3, 5 and 4) greater than and preceding 2; 1 number (5) greater than and preceding 4. There are no numbers greater than and preceding either 3 or 5. Because there are, in all, six inversions, σ is even and $\operatorname{sgn}\sigma = 1$.

(b) The identity permutation $\varepsilon = 123...n$ is even because there are no inversions in ε.

(c) In S_2, the permutation 12 is even and 21 is odd. In S_3, the permutations 123, 231, 312 are ven and the permutations 132, 213, 321 are odd.

(d) Let τ be the permutation that interchanges two numbers i and j and leaves the other numbers fixed. That is, $\tau(i) = j,\ \tau(j) = i,\ \tau(k) = k,$ where $k \neq i, j$. We call τ a *transposition*. Obviously, any transposition is odd.

Remark: One can show that, for any n, half of the permutations in S_n are even and half of them are odd. For example, 3 of the 6 permutations in S_3 are even, and 3 are odd.

Notion of a determinant of the *n*-th order

Let A be a square matrix of order n. Consider a product of n elements of A such that one and only one element comes from each row and one and only one element cones from each column. Such a product can be written in the form $a_{1j_1} a_{2j_2} ... a_{nj_n}$, that is, where the factors come from successive rows, and so the first subscripts are in the natural order 1,2 , …, n. Now because the factors come from different columns, the sequence of second subscripts forms a permutation $\sigma = j_1 j_2 ... j_n$ in S_n. Conversely, each permutation in S_n determines a product of the above form. Thus, the matrix A contains $n!$ such products.

Definition: The determinant of A, denoted by $\det(A)$ or $|A|$, is the sum of all the above $n!$ products, where each such product is multiplied by $\operatorname{sgn}\sigma$. That is, $\det(A) = \sum_{\sigma} (\operatorname{sgn}\sigma) a_{1j_1} a_{2j_2} ... a_{nj_n}$

or $\det(A) = \sum_{\sigma \in S_n} (\operatorname{sgn}\sigma) a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)}$.

Theorem 4 (Laplace Expansion) The determinant of the *n*–th order corresponding to a square matrix of the *n*–th order *A(n; n)* is equal to the sum of products of elements of an arbitrary row (column) on its cofactors, i.e.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = a_{i1} \cdot A_{i1} + a_{i2} \cdot A_{i2} + \dots + a_{ij} \cdot A_{ij} + \dots + a_{in} \cdot A_{in}$$

or $\Delta = a_{1j} \cdot A_{1j} + a_{2j} \cdot A_{2j} + \dots + a_{ij} \cdot A_{ij} + \dots a_{nj} \cdot A_{nj}$.

The above formulas for Δ are called the *Laplace expansions* of the determinant of A by the i th row and the j th column.

Glossary

determinant – определитель, детерминант; **decomposition** – разложение
minor – минор; **cofactor** (**signed minor**) – алгебраическое дополнение
a triangle rule – правило треугольника; **to coincide** – совпадать
even – четный; **odd** – нечетный; **neighbouring** – соседний; **parity** – четность

Exercises for Seminar 2

2.1. Compute determinants of the second order:

a) $\Delta = \begin{vmatrix} -4 & 2 \\ 3 & 1 \end{vmatrix}$; b) $\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix}$.

2.2. Compute determinants of the third order:

a) $\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{vmatrix}$; b) $\begin{vmatrix} -1 & 5 & 2 \\ 0 & 7 & 0 \\ 1 & 2 & 1 \end{vmatrix}$; c) $\begin{vmatrix} 1 & 3 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 0 \end{vmatrix}$; d) $\begin{vmatrix} 1 & 1 & 0 \\ 3 & 3 & -1 \\ 4 & 1 & 2 \end{vmatrix}$.

2.3. Find M_{11}, M_{23}, A_{31} and A_{12} if $\Delta = \begin{vmatrix} -1 & 2 & 1 \\ 3 & 0 & -2 \\ 2 & -4 & 5 \end{vmatrix}$.

2.4. Compute the determinant of the fourth order: $\Delta = \begin{vmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 2 & 3 \\ 3 & 1 & 6 & 1 \end{vmatrix}$.

2.5. Solve the equation: $\begin{vmatrix} 3x & -1 \\ x & 2x-3 \end{vmatrix} = \frac{3}{2}$.

2.6. Solve the inequality: $\begin{vmatrix} x & 3x \\ 4 & 2x \end{vmatrix} < 14$.

2.7. Compute the determinants by decomposing them on a row (a column) containing letters:

a) $\begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ a & b & c & d \\ -1 & -1 & 1 & 0 \end{vmatrix}$; b) $\begin{vmatrix} 2 & 1 & 1 & x \\ 1 & 2 & 1 & y \\ 1 & 1 & 2 & z \\ 1 & 1 & 1 & t \end{vmatrix}$; c) $\begin{vmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{vmatrix}$.

2.8. Compute the determinants:

a) $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$; b) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$; c) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$.

2.9. Prove that $\begin{vmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$.

Exercises for Homework 2

2.10. Compute determinants of the second order:

a) $\Delta = \begin{vmatrix} -3 & -5 \\ 1 & -2 \end{vmatrix}$; b) $\Delta = \begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix}$.

2.11. Compute determinants of the third order:

a) $\begin{vmatrix} 5 & 3 & 2 \\ -1 & 2 & 4 \\ 7 & 3 & 6 \end{vmatrix}$; b) $\begin{vmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix}$; c) $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix}$.

2.12. Find M_{22}, M_{32}, A_{33} and A_{21} if $\Delta = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 2 & 9 \\ 1 & 1 & 2 \end{vmatrix}$.

2.13. Compute the determinant of the fourth order: $\Delta = \begin{vmatrix} 3 & -1 & 4 & 2 \\ 5 & 2 & 0 & 1 \\ 0 & 2 & 1 & -3 \\ 6 & -2 & 9 & 8 \end{vmatrix}$.

2.14. Solve the equation: $\begin{vmatrix} 1 & 3 & x \\ 4 & 5 & -1 \\ 2 & -1 & 5 \end{vmatrix} = 0$.

2.15. Solve the inequality: $\begin{vmatrix} 2 & x+2 & -1 \\ 1 & 1 & -2 \\ 5 & -3 & x \end{vmatrix} > 0$.

2.16. Compute the determinants:

a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix}$; b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix}$; c) $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$.