LECTURE 3

Systems of linear equations, its classification. Matrix representation of a system of linear equations.

Consider a system of m linear equations with n variables $x_1, x_2, x_3, \dots, x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m. \end{cases} \quad (1)$$

The coefficients a_{ik} ($i = 1, 2, \dots, m;\ k = 1, 2, \dots, n$) of unknowns are enumerated by two indices: the first index i indicates the number of the equation, and the second j – the number of the variable. This numeration of coefficients of the system (1) by two indices is analogous to the numeration of matrix elements. According to the system (1) consider the following matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad C = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix A is called the *basic matrix* (the *coefficient matrix*) of the system (1), and the matrix C – the *extended* one (the *augmented matrix*). Multiplying the matrices A and X we obtain:

$$A \cdot X = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}.$$

Thus, the system (1) in a matrix representation is written as: $A \cdot X = B$ (2)

The system (1) is said to be *homogeneous* if all the constant terms are zero – that is, if $b_1 = 0, b_2 = 0$, $\dots$, $b_m = 0$. Otherwise the system is said to be *nonhomogeneous*.

A *solution* of the system (1) is an ordered set of real numbers ($\alpha_1, \alpha_2, \dots, \alpha_n$) which inverts each equation of the system in a true numeric identity.

The system of equations (1) is called *compatible (consistent)* if it has at least one solution and is called *non-compatible (inconsistent)* if it has no solution.

The system of equations (1) is called *determinate* if it is compatible and has a unique solution, and is called *indeterminate* if it is compatible and has infinitely many solutions.

To find a solution of the system of linear equations (1) firstly it is necessary to conduct its investigation that is the following:

1. To clarify whether the given system (1) is compatible or not;
2. If the system is compatible then determine whether it is determinate or indeterminate;
3. If the system is determinate then find its unique solution;
4. If the system is indeterminate then describe an infinite set of its solutions.

Solving a system of linear equations by means of determinants (Cramer rule)

Consider a system of two equations with two unknowns: $\begin{cases} a_{11}x + a_{12}y = b_1, \\ a_{21}x + a_{22}y = b_2. \end{cases}$ (3).

The determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called the *basic* determinant of the system of equations (3), and the determinants $\Delta_x = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$ and $\Delta_y = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$ are called the *auxiliary* determinants of the system (3).

To exclude the variable y, multiply both parts of the first equation on a_{22}, and both parts of the second one on ($-a_{12}$) and then add these equations termwise. As a result we obtain:

$$(a_{11}a_{22} - a_{21}a_{12}) \cdot x = b_1a_{22} - b_2a_{12}, \text{ i.e. } \Delta \cdot x = \Delta_x.$$

To exclude the variable x, multiply both parts of the first equation on ($-a_{21}$), and the second equation – on a_{11} and then add the obtained equations termwise. As a result we obtain:

$$(a_{11}a_{22} - a_{21}a_{12}) \cdot y = b_2 a_{11} - b_1 a_{21}, \text{ i.e. } \Delta \cdot y = \Delta_y.$$

Thus, the original system (3) is equivalent to the following: $\begin{cases} \Delta \cdot x = \Delta_x, \\ \Delta \cdot y = \Delta_y. \end{cases}$ (4)

The following cases are possible:

1) $\Delta \neq 0$.

In this case each equation of the system (4) has a unique solution: $\begin{cases} x = \dfrac{\Delta_x}{\Delta}, \\ y = \dfrac{\Delta_y}{\Delta}. \end{cases}$ (5)

Consequently, the original system (3) is compatible and determinate. Its unique solution is founded by the formulas (5) which are called the *Cramer formulas*.

2) $\Delta = 0$, but at least one of Δ_x, Δ_y is not equal to zero.

In this case there is at least one equation of the system (4) which has no solution. Consequently, the original system (3) has no solution, i.e. is non-compatible.

3) $\Delta = \Delta_x = \Delta_y = 0$.

In this case each equation of the system (4) has infinitely many solutions. Consequently, the original system (3) is compatible and indeterminate.

Example 1. Solve the system $\begin{cases} 2x - 3y = -7, \\ 3x + 4y = -2. \end{cases}$

Solution: 1) Compute the basic determinant of the system $\Delta = \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = 8 + 9 = 17.$ 2) Compute the auxiliary determinants of the system:

$$\Delta_x = \begin{vmatrix} -7 & -3 \\ -2 & 4 \end{vmatrix} = -28 - 6 = -34; \ \Delta_y = \begin{vmatrix} 2 & -7 \\ 3 & -2 \end{vmatrix} = -4 + 21 = 17.$$

Since $\Delta = 17 \neq 0$ then $x = \dfrac{\Delta_x}{\Delta} = \dfrac{-34}{17} = -2; \quad y = \dfrac{\Delta_y}{\Delta} = \dfrac{17}{17} = 1.$

Checking: $\begin{cases} 2 \cdot (-2) - 3 \cdot 1 = -7 \\ 3 \cdot (-2) + 4 \cdot 1 = -2 \end{cases}$ It is true. The answer: *$x = -2$; $y = 1$.*

The Cramer formulas are fair for such systems of linear equations at which the basic determinant of a system is not equal to zero and the number of variables is equal to the number of equations.

For example, for a system of three linear equations with three variables *x, y, z*:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

the Cramer formulas are deduced analogously and have the following form:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta} \qquad (6)$$

where $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \ \Delta_x = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \ \Delta_y = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \ \Delta_z = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}.$

Example 2. Solve the system of equations: $\begin{cases} 2x - 3y + z = -5, \\ -x + 2y - 3z = -4, \\ 3x - y + 2z = 1. \end{cases}$

Solution. Compute the basic and auxiliary determinants of the system:

$$\Delta = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 2 & -3 \\ 3 & 1 & 2 \end{vmatrix} = 8+27+1-6-6-6 = 18 \neq 0$$ – the system has a unique solution.

$$\Delta_x = \begin{vmatrix} -5 & -3 & 1 \\ -4 & 2 & -3 \\ 1 & -1 & 2 \end{vmatrix} = -20+9+4-2-24+1 = -18,$$

$$\Delta_y = \begin{vmatrix} 2 & -5 & 1 \\ -1 & -4 & -3 \\ 3 & 1 & 2 \end{vmatrix} = -16+45-1+12-10+6 = 36,$$

$$\Delta_z = \begin{vmatrix} 2 & -3 & -5 \\ -1 & 2 & -4 \\ 3 & -1 & 1 \end{vmatrix} = 4+36-5+30-3-8 = 54.$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-18}{18} = -1; \quad y = \frac{\Delta_y}{\Delta} = \frac{36}{18} = 2; \quad z = \frac{\Delta_z}{\Delta} = \frac{54}{18} = 3.$$

Checking: $\begin{cases} 2\cdot(-1) - 3\cdot 2 + 3 = -5 \\ 1 + 2\cdot 2 - 3\cdot 3 = -4 \\ 3\cdot(-1) - 2 + 2\cdot 3 = 1 \end{cases}$ – it is true. *The answer*: *x = -1; y = 2; z = 3.*

In general case for a system of *n* linear equations with *n* variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases}$$

the Cramer formulas are the following: $x_i = \dfrac{\Delta_{x_i}}{\Delta}, i = 1,2,...,n$ (7)

where $\Delta = \begin{vmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{vmatrix} \neq 0$ – the basic determinant of the system, Δ_{x_i} – the determinant obtained from the basic determinant Δ by replacement of the *i*–th column on the column of constant terms.

Inverse matrix

Consider a system of *n* linear equations with *n* variables ($x_1, x_2, \ldots, x_n$):

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases} \quad (8)$$

Consider $A(n;n) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$, $X(n;1) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B(n;1) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

Then the system (8) is written in a matrix representation:

$$A(n;n)\cdot X(n;1) = B(n;1) \quad (9)$$

A matrix *A(n; n)* is called *regular* (*invertible* or *nonsingular*) if its determinant is not equal to zero, i.e. $\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \neq 0.$

A matrix $A^{-1}(n; n)$ is called *inverse* to a matrix *A(n; n)* if the following holds:

$$A(n; n) \cdot A^{-1}(n; n) = A^{-1}(n; n) \cdot A(n; n) = E(n; n),$$

where $E(n;n) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$ is the identity matrix.

Find a solution of the matrix equation (9), i.e. find a matrix-column of variables *X(n; 1)*. Multiply both parts of the matrix equation (9) on the left (since the order of multiplying matrices is important) on matrix $A^{-1}(n; n)$. As a result we obtain: $A^{-1} \cdot (A \cdot X) = A^{-1} \cdot B$ or $(A^{-1} \cdot A) \cdot X = A^{-1} \cdot B$; $E \cdot X = A^{-1} \cdot B$, but $E \cdot X = X$, consequently $X = A^{-1} \cdot B$. (10)

Thus, we find a solution (matrix *X(n; 1)*) of the matrix equation by formula (10), i.e. to find matrix *X(n; 1)* (and variables $x_1, x_2, \dots, x_n$) it is necessary to find an inverse matrix $A^{-1}(n; n)$ and multiply it on matrix *B(n; 1)*. Consequently all is reduced to a determination of an inverse matrix $A^{-1}(n; n)$.

Show a finding the inverse matrix A^{-1} on example of a system of three linear equations with three unknowns: $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}.$

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$ then the system has a unique solution which is computed by the Cramer formulas: $x_1 = \frac{\Delta_{x_1}}{\Delta},\ x_2 = \frac{\Delta_{x_2}}{\Delta},\ x_3 = \frac{\Delta_{x_3}}{\Delta}.$

Transform the auxiliary determinants of the system:

$$\Delta_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = b_1 \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_2 \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + b_3 \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = A_{11} \cdot b_1 + A_{21} \cdot b_2 + A_{31} \cdot b_3,$$

where A_{11}, A_{21}, A_{31} – the cofactors of the elements a_{11}, a_{21}, a_{31} of matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$

Transforming analogously the second and the third formulas, we finally obtain:

$$\begin{cases} x_1 = \frac{A_{11}}{\Delta} \cdot b_1 + \frac{A_{21}}{\Delta} \cdot b_2 + \frac{A_{31}}{\Delta} \cdot b_3 \\ x_2 = \frac{A_{12}}{\Delta} \cdot b_1 + \frac{A_{22}}{\Delta} \cdot b_2 + \frac{A_{32}}{\Delta} \cdot b_3 \\ x_3 = \frac{A_{13}}{\Delta} \cdot b_1 + \frac{A_{23}}{\Delta} \cdot b_2 + \frac{A_{33}}{\Delta} \cdot b_3 \end{cases}. \text{ This implies: } X(n;1) = \begin{pmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta} & \frac{A_{31}}{\Delta} \\ \frac{A_{12}}{\Delta} & \frac{A_{22}}{\Delta} & \frac{A_{32}}{\Delta} \\ \frac{A_{13}}{\Delta} & \frac{A_{23}}{\Delta} & \frac{A_{33}}{\Delta} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad (11)$$

We obtain the rule of finding the inverse matrix by formulas (10) and (11):

$$A^{-1}(3;3) = \begin{pmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta} & \frac{A_{31}}{\Delta} \\ \frac{A_{12}}{\Delta} & \frac{A_{22}}{\Delta} & \frac{A_{32}}{\Delta} \\ \frac{A_{13}}{\Delta} & \frac{A_{23}}{\Delta} & \frac{A_{33}}{\Delta} \end{pmatrix} = \frac{1}{\Delta} \cdot \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}. \qquad (12)$$

It is necessary to mark that the inverse matrix exists only for regular matrixes (i.e. for square matrixes at which the determinant is not equal to zero).

The order of actions for finding the inverse matrix A^{-1} to a matrix A is the following:

1) Compute the determinant Δ of the matrix A (if it equals zero then there is no inverse matrix).
2) Compute the cofactors A_{ij} to all the elements a_{ij} of the matrix A.
3) Compose the matrix A^* of the cofactors.
4) Compose the matrix A^{*T} – transposed to the matrix A^* (a matrix obtained from the matrix A^* by interchanging rows and columns is called *transposed* to the matrix A^*).
5) Each element of the matrix A^{*T} is divided on the determinant $\Delta \neq 0$.

Thus, finally we have $A^{-1} = \frac{1}{\Delta} \cdot \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}.$

Example 3. Find the inverse matrix to the matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}.$

Solution. 1) $\Delta = \begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} = 9 - 4 = 5 \neq 0.$ 2) $A_{11} = 3;\ A_{12} = -4;\ A_{21} = -1;\ A_{22} = 3.$

3) $A^* = \begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}.$ 4) $A^{*T} = \begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}.$ 5) $A^{-1} = \frac{1}{5} \cdot \begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix}.$

Checking:

$$A \cdot A^{-1} = \frac{1}{5} \cdot \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ -4 & 3 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 3 \cdot 3 + 1 \cdot (-4) & 3 \cdot (-1) + 1 \cdot 3 \\ 4 \cdot 3 + 3 \cdot (-4) & 4 \cdot (-1) + 3 \cdot 3 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E.$$

Example 4. Find the inverse matrix to the matrix $A = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix}.$

Solution: 1) $\Delta = \begin{vmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{vmatrix} = 12 - 2 + 0 - 0 - 8 + 3 = 5 \neq 0.$

2) $A_{11} = 5$; $A_{12} = 10$; $A_{13} = 0$; $A_{21} = 4$; $A_{22} = 12$; $A_{23} = 1$; $A_{31} = -1$; $A_{32} = -3$; $A_{33} = 1$.

3) $A^* = \begin{pmatrix} 5 & 10 & 0 \\ 4 & 12 & 1 \\ -1 & -3 & 1 \end{pmatrix}.$ 4) $A^{*T} = \begin{pmatrix} 5 & 4 & -1 \\ 10 & 12 & -3 \\ 0 & 1 & 1 \end{pmatrix}.$ 5) $A^{-1} = \frac{1}{5} \cdot \begin{pmatrix} 5 & 4 & -1 \\ 10 & 12 & -3 \\ 0 & 1 & 1 \end{pmatrix}.$

Checking:

$$A \cdot A^{-1} = \frac{1}{5} \cdot \begin{pmatrix} 3 & -1 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 4 & -1 \\ 10 & 12 & -3 \\ 0 & 1 & 1 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem. Let A and B be invertible. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. More generally, if $A_1, A_2, \dots, A_k$ are invertible, then their product is invertible and $(A_1 A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1} A_1^{-1}$, i.e. the inverse of the product is equal to the product of the inverses in the reverse order.

Glossary

compatible – совместный; **determinate** – определенный
basic – основной; **auxiliary** – вспомогательный; **augmented** – расширенная
inverse matrix – обратная матрица; **regular matrix** – невырожденная матрица
invertible matrix – обратимая матрица

Exercises for Seminar 3

3.1. Solve systems of linear equations:

a) $\begin{cases} 5x-3y=1 \\ x+11y=6 \end{cases}$; b) $\begin{cases} 2x-3y=6 \\ 4x-6y=5 \end{cases}$; c) $\begin{cases} 3x+2y=\dfrac{1}{6} \\ 9x+6y=\dfrac{1}{2} \end{cases}$.

3.2. Solve the following systems:

a) $\begin{cases} 2x-3y+z=-5 \\ -x+2y-3z=-4 \\ 3x-y+2z=1 \end{cases}$; b) $\begin{cases} x+2y+3z=4 \\ 2x+y-z=3 \\ 3x+3y+2z=0 \end{cases}$; c) $\begin{cases} x+2y+3z=4 \\ 2x+y-z=3 \\ 3x+3y+2z=7 \end{cases}$.

3.3. Find the inverse matrix A^{-1} to the matrix $A=\begin{pmatrix} 5 & 7 \\ 3 & 4 \end{pmatrix}$.

3.4. Find the inverse matrix A^{-1} to the matrix $A=\begin{pmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

3.5. Find the inverse matrix B^{-1} to the matrix $B=\begin{pmatrix} 4 & -8 & -5 \\ -4 & 7 & -1 \\ -3 & 5 & 1 \end{pmatrix}$.

3.6. Solve the system of equations by the method of inverse matrix: $\begin{cases} 2x+y-z=5 \\ x-2y+3z=-3 \\ 7x+y-z=10 \end{cases}$;

3.7. Find the matrix X from the equation:

a) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} X = \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix}$;

b) $X \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \\ 1 & -2 & 5 \end{pmatrix}$;

c) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} X \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix}$.

3.8. Solve the system: $\begin{cases} x_1+x_2+2x_3+3x_4=1 \\ 3x_1-x_2-x_3-2x_4=-4 \\ 2x_1+3x_2-x_3-x_4=-6 \\ x_1+2x_2+3x_3-x_4=-4 \end{cases}$.

Exercises for Homework 3

3.9. Solve systems of linear equations:

a) $\begin{cases} 2x-3y=-7 \\ 3x+4y=-2 \end{cases}$; b) $\begin{cases} 2x+y=\dfrac{1}{5} \\ 4x+2y=\dfrac{1}{3} \end{cases}$; c) $\begin{cases} 2x+y=3 \\ 4x+2y=6 \end{cases}$.

3.10. Solve the following systems:

a) $\begin{cases} 2x+y=5 \\ x+3z=16 \\ 5y-z=10 \end{cases}$; b) $\begin{cases} -5x+y+z=0 \\ x-6y+z=0 \\ x+y-7z=0 \end{cases}$; c) $\begin{cases} x+y+z=0 \\ 3x+6y+5z=0 \\ x+4y+3z=0 \end{cases}$.

3.11. Find the inverse matrix A^{-1} to the matrix A:

a) $A=\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$; b) $A=\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$; c) $A=\begin{pmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{pmatrix}$.

3.12. Solve the system of equations by the method of inverse matrix: $\begin{cases} 5x-y-z=0 \\ x+2y+3z=14 \\ 4x+3y+2z=16 \end{cases}$.

3.13. Solve the system: $\begin{cases} x_1+2x_2+3x_3-2x_4=6 \\ 2x_1-x_2-2x_3-3x_4=8 \\ 3x_1+2x_2-x_3+2x_4=4 \\ 2x_1-3x_2+2x_3+x_4=-8 \end{cases}$.