LECTURE 1
Matrices. Operations over matrices.

Definition of a numerical matrix. Classification of matrices.

A numerical matrix of dimension $m \times n$ is a rectangular table of numbers consisting of «m» horizontal lines (rows) and «n» vertical lines (columns).
It has the following form:

$$A(m;n)=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

The numbers a_{ij} $(i = 1, 2, \dots, m; \ j = 1, 2, \dots, n)$ are called *elements* of the matrix. The index «i» denotes the number of a matrix row, the index «j» – the number of a matrix column.
In general, objects of an arbitrary nature can be elements of a matrix.

Examples: $A(3; 2) = \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$; $A(2; 4) = \begin{pmatrix} -1 & 2 & 5 & 6 \\ 3 & 1 & 4 & -7 \end{pmatrix}$.

If $n = 1$, a matrix $A(m; 1)$ is called a *column matrix* (or *a matrix-column*).

It has the following form: $A(m; 1) = \begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{pmatrix}$. For example, $A(3;1) = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$.

If $m = 1$, a matrix $A(1; n)$ is called a *row matrix* (or *a matrix-row*).
It has the following form: $A(1;\ n) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix}$.
For example, $A(1;\ 3) = \begin{pmatrix} -1 & 4 & 2 \end{pmatrix}$.
If we replace all the rows by columns and vice versa in a matrix $A(m; n)$ then the changed matrix is called the *transposed* matrix to the matrix A and it is denoted by A^T:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

If $m = n$, then a matrix $A(n; n)$ is called a *square* matrix of the n-th order (or an n -square matrix).

It has the following form: $A\ (n;\ n) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$.

In particular, for $n = 2$: $A(2; 2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

for $n = 3$: $A(3; 3) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

– square matrices of the second and the third order respectively.

The *main diagonal* of a square matrix *A(n; n)* is the diagonal consisting of the elements a_{11}, a_{22}, a_{33}, …, a_{nn}.

A *diagonal* matrix is a square matrix *D*(*n; n)* of which all the elements non-lying on the main diagonal are equal to zero.

It has the following form: $D(n;n) = \begin{pmatrix} a_{11} & 0 & ... & 0 \\ 0 & a_{22} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & a_{nn} \end{pmatrix}$.

A square matrix $A=[a_{ij}]$ is *upper triangular* or simply *triangular* if all entries below the main diagonal are equal to 0 – that is, if $a_{ij}=0$ for $i > j$. A *lower triangular matrix* is a square matrix whose entries above the main diagonal are all zero.

An *identity* (or *unit*) matrix is a diagonal matrix of which all the diagonal elements are equal to 1. It is denoted by E (or I).

The n-square identity matrix has the following form: $E(n;n) = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{pmatrix}$.

For example, $E(2;2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $E(3;3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

A *zero* matrix is a matrix of which all the elements are equal to zero.

It has the following form: $0 = \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix}$.

A *symmetric* matrix is a square matrix of which the elements located symmetrically according to the main diagonal are equal each other, i.e. $a_{ik} = a_{ki}$ $(i = 1, 2,..., n;\ k = 1, 2, ..., n)$.

It has the following form: $C(n;n) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{12} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{1n} & a_{2n} & ... & a_{nn} \end{pmatrix}$.

In particular, $C(2;2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$.

Two matrices *A(m; n)* and *B(m; n)* of the same dimension are *equal* if all their corresponding elements are equal, i.e. $a_{ik} = b_{ik}$ $(i = 1, 2, ..., m;\ k = 1, 2,..., n)$.

Matrices of different dimensions aren't compared among themselves.

Operations over matrices

Linear operations over matrices are addition, subtraction of matrices and multiplication of matrices on a number.

a) *Addition* and *subtraction* of matrices are only defined for matrices of the same dimension, i.e. for matrices of the form:

$A(m;n) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$ and $B(m;n) = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ ... & ... & ... & ... \\ b_{m1} & b_{m2} & ... & b_{mn} \end{pmatrix}$.

The *sum* (*difference*) of two matrices is a matrix $C(m; n) = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{pmatrix}$ of which elements c_{ik} are equal to the sum (difference) of the corresponding elements a_{ik} and b_{ik}, i.e. $c_{ik} = a_{ik} \pm b_{ik}$ *(i = 1, 2, …, m; k = 1, 2, …, n).* The sum (difference) of matrices is denoted by $A \pm B$, i.e. $C(m; n) = A(m; n) \pm B(m; n)$, or in a developed form:

$$C(m; n) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \dots & a_{2n} \pm b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \dots & a_{mn} \pm b_{mn} \end{pmatrix}.$$

Thus, the sum (difference) of two matrices is determined elementwise.

Example: $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 8 \end{pmatrix} + \begin{pmatrix} -6 & 3 & 21 \\ 7 & 0 & -40 \end{pmatrix} = \begin{pmatrix} 2+(-6) & 1+3 & 3+21 \\ 1+7 & 3+0 & 8+(-40) \end{pmatrix} = \begin{pmatrix} -4 & 4 & 24 \\ 8 & 3 & -32 \end{pmatrix}.$

b) The *product of a matrix A(m; n) by a number (scalar)* λ is the matrix obtained from the matrix *A(m; n)* by multiplying all its elements on λ, i.e. the elements b_{ik} of the matrix *B(m; n)* are determined by the following formula: $b_{ik} = \lambda \cdot a_{ik}$ *(i = 1, 2,…, m; k = 1, 2,…, n).*
The product of a matrix *A(m; n)* on a number λ is denoted by λ*A*.
Thus: $B(m; n) = \lambda \cdot A(m; n)$ or in a developed form:

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} = \lambda \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{pmatrix}.$$

Example: $5 \cdot \begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix} \cdot 5 = \begin{pmatrix} 10 & 15 \\ 35 & -5 \end{pmatrix}.$

Theorem 1. Consider any matrices A, B, C (with the same size) and any scalars k and k'. Then
(i) $(A+B)+C = A+(B+C)$; (ii) $A+0 = 0+A = A$; (iii) $A+(-A) = (-A)+A = 0$;
(iv) $A+B = B+A$; (v) $k(A+B) = kA+kB$; (vi) $(k+k')A = kA+k'A$; (vii) $(kk')A = k(k'A)$;
(viii) $1 \cdot A = A$.

Summation Symbol

Before we define matrix multiplication, it will be instructive firstly to introduce the *summation symbol* Σ (the Greek capital letter sigma). Suppose $f(k)$ is an algebraic expression involving the letter k. Then the expression $\sum_{k=1}^{n} f(k)$ has the following meaning. First we set $k = 1$ in $f(k)$, obtaining $f(1)$. Then we set $k = 2$ in $f(k)$, obtaining $f(2)$, and add this to $f(1)$, obtaining $f(1) + f(2)$. Then we set $k = 3$ in $f(k)$, obtaining $f(3)$, and add this to the previous sum, obtaining $f(1) + f(2) + f(3)$. We continue this process until we obtain the sum $f(1) + f(2) + \dots + f(n)$.

Observe that at each step we increase the value of k by 1 until we reach n. The letter k is called the *index*, and 1 and n are called, respectively, the *lower* and *upper* limits.

We also generalize our definition by allowing the sum to range from any integer n_1 to any integer n_2. That is, we define $\sum_{k=n_1}^{n_2} f(k) = f(n_1) + f(n_1+1) + f(n_1+2) + \ldots + f(n_2)$.

Example. (a) $\sum_{k=1}^{5} x_k = x_1 + x_2 + x_3 + x_4 + x_5$ and $\sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

(b) $\sum_{j=2}^{5} j^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54$ and $\sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

(c) $\sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \ldots + a_{ip} b_{pj}$

Multiplication of matrices

As against the operations of addition (subtraction) the operation of multiplication of a matrix on a matrix is determined by more complicated way.

We can speak on product of rectangular matrices *A* and *B* only if the number of columns of the first matrix *A* is equal to the number of rows of the second matrix *B*, and the number of rows of the matrix *A·B* is equal to the number of rows of the matrix *A*, the number of columns of the matrix *A·B* is equal to the number of columns of the matrix *B*.

The rule of multiplication of matrices can be formulated as follows: to receive an element standing in the *i*-th row and the *k*-th column of the product of two matrices, it is necessary the elements of the *i*-th row of the first matrix multiply on the corresponding elements of the *k*-th column of the second matrix and add the obtained products.

Let $A(m;n) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$; $B(n;k) = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1k} \\ b_{21} & b_{22} & \ldots & b_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & \ldots & b_{nk} \end{pmatrix}$.

Then $C(m;k) = A(m;n) \cdot B(n;k) = \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1k} \\ c_{21} & c_{22} & \ldots & c_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ c_{m1} & c_{m2} & \ldots & c_{mk} \end{pmatrix}$, where c_{ij} are defined as follows:

$$c_{11} = a_{11} b_{11} + a_{12} b_{21} + \ldots + a_{1n} b_{n1} = \sum_{i=1}^{n} a_{1i} b_{i1}$$

$$c_{12} = a_{11} b_{12} + a_{12} b_{22} + \ldots + a_{1n} b_{n2} = \sum_{i=1}^{n} a_{1i} b_{i2}$$

$$\ldots \quad \ldots \quad \ldots \ \ldots \ \ldots \ \ldots \quad \ldots \ \ldots. \ \ldots. \ \ldots$$

$$c_{1k} = a_{11} b_{1k} + a_{12} b_{2k} + \ldots + a_{1n} b_{nk} = \sum_{i=1}^{n} a_{1i} b_{ik}$$

$$c_{21} = a_{21} b_{11} + a_{22} b_{21} + \ldots + a_{2n} b_{n1}$$

$$c_{22} = a_{21} b_{12} + a_{22} b_{22} + \ldots + a_{2n} b_{n2}$$

$$\ldots \quad \ldots \quad \ldots \ \ldots \ \ldots \ \ldots \quad \ldots \ \ldots. \ \ldots. \ \ldots$$

$$c_{2k} = a_{21} b_{1k} + a_{22} b_{2k} + \ldots + a_{2n} b_{nk}$$

$$\ldots \ \ldots \ \ldots \ \ldots \ \ldots \quad \ldots \ \ldots \ \ldots \ \ldots \ \ldots$$

$$c_{m1} = a_{m1} b_{11} + a_{m2} b_{21} + \ldots + a_{mn} b_{n1}$$

$$c_{m2} = a_{m1}b_{12} + a_{m2}b_{22} + \ldots + a_{mn}b_{n2}$$

$$\ldots \quad \ldots \quad \ldots \ \ldots \ \ldots \quad \ldots \quad \ldots \ \ldots . \ \ldots . \ \ldots$$

$$c_{mk} = a_{m1}b_{1k} + a_{m2}b_{2k} + \ldots + a_{mn}b_{nk} = \sum_{i=1}^{n} a_{mi}b_{ik}$$

Examples:

1) $\begin{pmatrix} 5 & -1 & 3 \\ 2 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 \\ -2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 5\cdot(-1)+(-1)\cdot(-2)+3\cdot 3 & 5\cdot 3+(-1)\cdot 1+3\cdot 0 \\ 2\cdot(-1)+0\cdot(-2)+(-1)\cdot 3 & 2\cdot 3+0\cdot 1+(-1)\cdot 0 \end{pmatrix} = \begin{pmatrix} 6 & 14 \\ -5 & 6 \end{pmatrix};$

2) $\begin{pmatrix} 0 & -3 & 1 \\ 2 & 1 & 5 \\ -4 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0\cdot 3+(-3)\cdot(-2)+1\cdot 2 \\ 2\cdot 3+1\cdot(-2)+5\cdot 2 \\ -4\cdot 3+0\cdot(-2)+(-2)\cdot 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \\ -16 \end{pmatrix};$

3) $\begin{pmatrix} 5 & 1 & 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & -4 \\ 3 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 5\cdot 2+1\cdot 1+0\cdot 3+(-3)\cdot 0 & 5\cdot 0+1\cdot(-4)+0\cdot 1+(-3)\cdot(-1) \end{pmatrix} =$

$= \begin{pmatrix} 11 & -1 \end{pmatrix}.$

The product of two matrices, generally speaking, depends on the order of multiplicands. It can even happen that the product of two matrices taken in one order will have sense, and the product of the same matrices taken in the opposite order will not have any sense.

Example:

1) $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 0\cdot 5+(-1)\cdot 0 & 0\cdot 1+(-1)\cdot 7 \\ 2\cdot 5+3\cdot 0 & 2\cdot 1+3\cdot 7 \end{pmatrix} = \begin{pmatrix} 0 & -7 \\ 10 & 23 \end{pmatrix}.$

2) $\begin{pmatrix} 5 & 1 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 5\cdot 0+1\cdot 2 & 5\cdot(-1)+1\cdot 3 \\ 0\cdot 0+7\cdot 2 & 0\cdot(-1)+7\cdot 3 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 14 & 21 \end{pmatrix}.$

The above example shows that matrix multiplication is not commutative – that is, in general, $AB \neq BA.$

The identity matrix E doesn't change any elements of a matrix A by multiplying on the matrix A (if this multiplication is possible), i.e. $A\cdot E = A$ or $E \cdot A = A$. If a matrix A is square and has the same dimension with E then $A \cdot E = E \cdot A = A$.

Examples:

1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix};$ $\qquad \begin{pmatrix} 3 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 7 \end{pmatrix};$

2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ -3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ -3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ -3 & 2 & 1 \end{pmatrix}.$

Theorem 2. Let A, B, C be matrices. Then, whenever the products and sums are defined,

(i) $(AB)C = A(BC)$ (associative law),

(ii) $A(B+C) = AB + AC$ (left distributive law),

(iii) $(B+C)A = BA + CA$ (right distributive law),

(iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar.

We note that $0A = 0$ and $B0 = 0$, where 0 is the zero matrix.

Theorem 3. Let A and B be matrices and let k be a scalar. Then, whenever the sum and product are defined, (i) $(A+B)^T = A^T + B^T$, (ii) $(A^T)^T = A$, (iii) $(kA)^T = kA^T$, (iv) $(AB)^T = B^T A^T$.

We emphasize that, by (iv), the transpose of a product is the product of the transposes, but in the reverse order.

Let A be an n-square matrix. *Powers* of A are defined as follows:
$A^2 = AA,\ A^3 = A^2A,\ ...,\ A^{n+1} = A^nA,\ ...,$ and $A^0 = E.$
Polynomials in the matrix A are also defined. Specifically, for any polynomial $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where the a_i are scalars, $f(A)$ is defined to be the following matrix: $f(A) = a_0E + a_1A + a_2A^2 + ... + a_nA^n$.
Note that $f(A)$ is obtained from $f(x)$ by substituting the matrix A for the variable x and substituting the matrix a_0E for the scalar a_0. If $f(A)$ is the zero matrix, then A is called a *zero* or *root* of $f(x)$.

Glossary

matrix (*plural* – **matrices**) – матрица; **numerical** – числовой; **row** – строка; **column** – столбец
dimension – размерность; **rectangular** – прямоугольный; **element** – элемент
transposed matrix – транспонированная матрица; **square matrix** – квадратная матрица
main diagonal – главная диагональ; **diagonal matrix** – диагональная матрица
identity matrix – единичная матрица; **zero matrix** – нулевая матрица
symmetric matrix – симметрическая матрица; **addition** – сложение; **subtraction** – вычитание
to obtain – получать

Exercises for Seminar 1

1.1. Let $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & -4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 2 & 2 \end{pmatrix}$. Find *C = 3A + 2B* and *D = 4B – 5A*.

1.2. Let $A = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 1 & 3 \\ -1 & 3 \end{pmatrix}$. Find *AB* and *BA* if possible.

1.3. Find A^3 if $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.

1.4. Let $A = \begin{pmatrix} 5 & 1 & 0 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 1 & -4 \\ 3 & 1 \\ 0 & -1 \end{pmatrix}$. Find *AB* and *BA* if possible.

1.5. Compute: a) $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 5 & 1 \\ 0 & 7 \end{pmatrix}$; b) $\begin{pmatrix} 5 & 1 \\ 0 & 7 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix}$;

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ -3 & 2 & 1 \end{pmatrix}$; e) $\begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 3 \\ 3 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

1.6. Find the matrix *X* for which *A + 2X = 3B* if

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 3 & -1 \\ 2 & 1 & 0 \end{pmatrix},\quad B = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 3 & -1 \\ 4 & 1 & 2 \end{pmatrix}.$$

1.7. Show that the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix}$ is a root of the following polynomial:

P(X) = X ^{3} *– 6X* ^{2} *+ 8X – 9.*

1.8. Multiply the following matrices:

a) $\begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix}$; b) $\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -f & e & -d \\ f & 0 & -c & b \\ -e & c & 0 & -a \\ d & -b & a & 0 \end{pmatrix}$.

1.9. Find the following matrices: a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$; b) $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}^n$.

1.10. Compute AA^T where $A = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}$, and A^T is the matrix transposed to A.

1.11. Check that $\begin{pmatrix} c_1 & c_2 & \dots & c_k \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{km} \end{pmatrix} = c_1\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{pmatrix} +$

$+ c_2\begin{pmatrix} a_{21} & a_{22} & \dots & a_{2m} \end{pmatrix} + \dots + c_k\begin{pmatrix} a_{k1} & a_{k2} & \dots & a_{km} \end{pmatrix}$.

Exercises for Homework 1

1.12. Find $2A + 5B$ and $A - 3B$ if $A = \begin{pmatrix} 3 & 5 \\ 4 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$.

1.13. Find AB and BA (if possible), where $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ 1 & 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

1.14. Find $2A^2 + 3A + 5E$, where $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix}$, E is the identity matrix of the third order.

1.15. Show that $S = 3A - 2B$ is symmetric if $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 3 & 6 \\ 2 & -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -4 & -3 & 5 \\ 3 & -1 & 4 \\ 5 & -8 & -1 \end{pmatrix}$.

1.16. Let $A = \begin{pmatrix} 3 \\ -1 \\ 7 \\ 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & -2 & 3 \\ 4 & -1 & 0 & 2 \\ 3 & 1 & -1 & 2 \end{pmatrix}$. Find AB and BA if possible.

1.17. Compute: a) $\begin{pmatrix} 6 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -5 & 4 & -3 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix}$;

c) $\begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix} \cdot \begin{pmatrix} -5 & 0 \\ -2 & 1 \end{pmatrix}$; d) $\begin{pmatrix} -5 & 0 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$.

1.18. Find the value of the polynomial $P(X)$ of the matrix A:

a) $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$; $P(X) = X^3 - 3X + 1$.

b) $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & -1 \\ -1 & 0 & -3 \end{pmatrix}$; $P(X) = X^3 - 3X$.