L E C T U R E 8
Linear transformations (operators)

Let every element x of a linear space V be corresponded a unique element y of a linear space V^*. In this case we say an *operator* A is acting in V and having values in V^* with $Ax = y$. Operators are subdivided on *mappings* if $V^* \not\subset V$ and *transformations* if $V^* \subseteq V$. Furthermore we consider transformations acting in V.

We say that *a transformation* A is determined in a linear space V if each vector $x \in V$ is corresponded the vector $Ax \in V$ by some rule. A transformation A is *linear* if for any vectors x and y and for any real number λ the following equalities hold:

$$A(x + y) = Ax + Ay, \quad A(\lambda x) = \lambda Ax.$$

A linear transformation is called *identity* if it transforms each vector x to itself. An identity linear transformation is denoted by E. Thus, $Ex = x$.

Example. Show that the transformation $Ax = \alpha x$ (where α is a real number) is linear.

Solution. We have $A(x+y) = \alpha(x+y) = \alpha x + \alpha y = Ax + Ay$,

$$A(\lambda x) = \alpha(\lambda x) = \lambda(\alpha x) = \lambda Ax.$$

Thus, both conditions determining a linear transformation hold.

The considered transformation A is called a *transformation of similarity*.

Example. A transformation A is determined by the equality $Ax = x + x_0$ (where $x_0 \in V$ – is a fixed non-zero vector) in a linear space. Is the transformation A linear?

Solution. By the equalities $Ax = x + x_0$, $Ay = y + x_0$, $A(x + y) = x + y + x_0$, $A(x + y) = Ax + Ay$ we conclude that $x + y + x_0 = (x + x_0) + (y + x_0)$. It implies that $x_0 = 0$ contradicting the hypothesis. Consequently the transformation A is not linear.

Example. In the space R^2 the transformation $Ax = y$ where $x, y \in R^2$ and A is an arbitrary square matrix of the second order is linear.

Example. In the space of infinitely many differentiable functions the operation of differentiation assigning for every element of the space its derivative is a linear transformation.

Example. In the space of all polynomials $P(t)$ of one independent variable t the operation of multiplication of a polynomial on the independent variable t is a linear transformation.

Actions over linear transformations

Let A and B be arbitrary linear transformations in a linear space V, λ be an arbitrary real number, and $x \in V$ be an element.

The sum of linear transformations A and B is called the transformation C_1 determined by the equality $C_1 x = Ax + Bx$. Notation: $C_1 = A + B$.

Lemma. The sum of two linear transformations is a linear transformation.

Proof: Let $x, y, z \in V$ and $x = \lambda y + \mu z$, and let A and B be linear transformations, $C_1 = A + B$. Then $C_1(\lambda y + \mu z) = A(\lambda y + \mu z) + B(\lambda y + \mu z) = \lambda Ay + \mu Az + \lambda By + \mu Bz =$

$$= \lambda(Ay + By) + \mu(Az + Bz) = \lambda Cy + \mu Cz. \quad \square$$

Zero transformation O is a transformation of a linear space V such that every element $x \in V$ is corresponded the zero-element of this linear space.

A transformation being *opposite* for a transformation A is a transformation denoted by $(-A)$ such that every element $x \in V$ is corresponded the element $(-Ax)$.

Obviously, zero and opposite transformations are linear.

The product of a linear transformation A on number λ is the transformation C_2 determined by the equality $C_2 x = \lambda Ax$. Notation: $C_2 = \lambda A$.

Lemma. The product of a linear transformation on number is a linear transformation for which the following holds: $\alpha(\beta A) = (\alpha\beta)A$; $1 \cdot A = A$; $(\alpha + \beta)A = \alpha A + \beta A$.

Theorem. The set of all linear transformations acting in a linear space V is a linear space.

The product of a linear transformation A on a linear transformation B is called the transformation C_3 determined by the equality $C_3 x = A(Bx)$. Notation: $C_3 = AB$.

Lemma. The product of linear transformations is a linear transformation.

Proof: Let A and B be linear transformations. Then

$$AB(\alpha x+\beta y)=A(\alpha Bx+\beta By)=\alpha A(Bx)+\beta A(By)=\alpha(AB)x+\beta(AB)y.$$
At addition of linear transformations the commutative law holds, i.e. $A+B=B+A$. The product *AB* differs from the product *BA* in general.

Let *A* and *B* be linear transformations. The transformation $AB-BA$ is called the *commutator* of *A* and *B*. If *A* and *B* are commuting, i.e. $AB = BA$, then its commutator is zero transformation.

<u>*Exercise.*</u> Find the commutator for transformations *A* and *B* in the linear space of polynomials $P(t)$ of one independent variable *t*, where *A* is the operation of differentiation assigning for every element of the space its derivative, and *B* is the operation of multiplication of a polynomial on the independent variable *t*.

Solution: Construct the transformation $AB-BA$. Let $P_n(t)=\sum_{k=0}^{n}\alpha_k t^k$. We have:

$$A(P_n(t))=\frac{d}{dt}(P_n(t))=\frac{d}{dt}\left(\sum_{k=0}^{n}\alpha_k t^k\right)=\sum_{k=1}^{n}k\alpha_k t^{k-1}$$
$$B(P_n(t))=t\cdot(P_n(t))=t\cdot\left(\sum_{k=0}^{n}\alpha_k t^k\right)=\sum_{k=0}^{n}\alpha_k t^{k+1}$$

Then we obtain the following:

$$B(A(P_n(t)))=t\cdot\left(\sum_{k=1}^{n}k\alpha_k t^{k-1}\right)=\sum_{k=1}^{n}k\alpha_k t^k=\sum_{k=0}^{n}k\alpha_k t^k$$
$$A(B(P_n(t)))=\frac{d}{dt}\left(\sum_{k=0}^{n}\alpha_k t^{k+1}\right)=\sum_{k=0}^{n}(k+1)\alpha_k t^k$$

Then $(AB-BA)(P_n(t))=\left(\sum_{k=0}^{n}(k+1)\alpha_k t^k-\sum_{k=0}^{n}k\alpha_k t^k\right)=\sum_{k=0}^{n}\alpha_k t^k=P_n(t)$.

Thus, we see that A and B aren't commuting and $AB-BA$ is identity, i.e. $AB-BA=E$.

Observe that $AE=EA=A$ for every transformation of a linear space.

List some properties of operations over linear transformations in the space V:

$$A(BC)=(AB)C;\ AE=EA=A;\ (A+B)C=AC+BC;\ C(A+B)=CA+CB.$$

If for a linear transformation *A* there are such linear transformations *B* and *C* that $BA = E$, $AC = E$ then $B = C$. In this case we denote $B = C = A^{-1}$, and the linear transformation A^{-1} is called an *inverse linear transformation* according to the linear transformation *A*. Thus, $A^{-1}A=AA^{-1}=E$.

Example. In the linear space V of functions $f(t)$ having on $[\alpha,\beta]$ the derivative of any order and satisfying the conditions $f^{(k)}(\alpha)=0,\ k=0, 1, 2,\ldots$ the operation of differentiation $Af=\frac{df}{dt}$ and the operation of integration with variable upper limit $Bf=\int_{\alpha}^{t}f(v)dv$ are mutually inverse linear transformations.

Indeed, $ABf=\frac{d}{dt}\int_{\alpha}^{t}f(v)dv=f(t)=Ef$ and $BAf=\int_{\alpha}^{t}\frac{df}{dv}dv=f(t)-f(\alpha)=f(t)=Ef$.

The coordinate representation of linear transformations

Let a linear transformation *A* be given in a *n*–dimensional space V of which vectors $e_1, e_2, \ldots, e_n$ form a basis. Since $Ae_1, Ae_2, \ldots, Ae_n$ are vectors of the space V, then each of them can be expressed by a unique way through vectors of the basis:

$$Ae_1 = a_{11}e_1 + a_{21}e_2 + \ldots + a_{n1}e_n,$$
$$Ae_2 = a_{12}e_1 + a_{22}e_2 + \ldots + a_{n2}e_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$Ae_n = a_{1n}e_1 + a_{2n}e_2 + \ldots + a_{nn}e_n.$$

The matrix $A_e = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ is called the *matrix of the linear transformation A* in the basis e_1, e_2, …, e_n. The columns of the matrix are composed from coefficients in formulas of transformation of the basis vectors. Consider an arbitrary vector $x = x_1e_1 + x_2e_2 + \dots + x_ne_n$ in the space V . Since $Ax \in V$, then the vector Ax can be expressed through the basis vectors:

$$Ax = x_1'e_1 + x_2'e_2 + \dots + x_n'e_n$$

The coordinates $(x_1', x_2', \dots, x_n')$ of the vector Ax are expressed through coordinates $(x_1, x_2, \dots, x_n)$ of the vector x by the formulas:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n. \end{aligned}$$

We can name these n equalities as a linear transformation A in the basis e_1, e_2, …, e_n. The coefficients in formulas of this linear transformation are elements of rows of the matrix A_e .

In matrix form: $\begin{pmatrix} Ae_1 \\ Ae_2 \\ \dots \\ Ae_n \end{pmatrix} = A_e^T \begin{pmatrix} e_1 \\ e_2 \\ \dots \\ e_n \end{pmatrix}$; $\begin{pmatrix} x_1' \\ x_2' \\ \dots \\ x_n' \end{pmatrix} = A_e \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$.

Theorem. There is a one-to-one correspondence between the set of all linear transformations of a n-dimensional linear space V and the set of all matrices of dimension $n \times n$.

Proof. It has been above showed that for every linear transformation in V is corresponded the matrix of dimension $n \times n$. On other hand, the expression $\begin{pmatrix} x_1' \\ x_2' \\ \dots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ can be accepted as a definition of some transformation. Its linearity follows from the rules of operations with matrices. □

Example. Find the matrix of an identity transformation E in a n-dimensional space.

Solution: An identity transformation does not change the basis vectors: $e_1' = e_1$, $e_2' = e_2$, $e_3' = e_3$, …, $e_n' = e_n$, i.e.

$$\begin{aligned} e_1' &= 1 \cdot e_1 + 0 \cdot e_2 + \dots + 0 \cdot e_n, \\ e_2' &= 0 \cdot e_1 + 1 \cdot e_2 + \dots + 0 \cdot e_n, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ e_n' &= 0 \cdot e_1 + 0 \cdot e_2 + \dots + 1 \cdot e_n. \end{aligned}$$

Consequently, the matrix of an identity transformation is identity matrix:

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

A linear transformation A in a finitely dimensional space is called *regular* (*nonsingular*) if the determinant of the matrix of this transformation differs from zero.

Every regular linear transformation A has an inverse transformation A^{-1} and only one.

If a regular linear transformation A in the coordinate form is determined by the following equalities:

$$\begin{aligned} x' &= a_{11}x + a_{12}y + ... + a_{1n}u, \\ y' &= a_{21}x + a_{22}y + ... + a_{2n}u, \\ &\dots\dots\dots\dots\dots\dots \\ u' &= a_{n1}x + a_{n2}y + ... + a_{nn}u, \end{aligned}$$

then the inverse linear transformation A^{-1} is determined as follows:

$$\begin{aligned} x &= \frac{A_{11}}{|A|}x' + \frac{A_{21}}{|A|}y' + ... + \frac{A_{n1}}{|A|}u', \\ y &= \frac{A_{12}}{|A|}x' + \frac{A_{22}}{|A|}y' + ... + \frac{A_{n2}}{|A|}u', \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ u &= \frac{A_{1n}}{|A|}x' + \frac{A_{2n}}{|A|}y' + ... + \frac{A_{nn}}{|A|}u', \end{aligned}$$

Here A_{ij} is the cofactor of the element a_{ij} of the matrix A, $|A|$ is the determinant of the matrix A. The matrix of the linear transformation A^{-1} is inverse according to the matrix A and is determined by the equality: $A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & ... & A_{n1} \\ A_{12} & A_{22} & ... & A_{n2} \\ ... & ... & ... & ... \\ A_{1n} & A_{2n} & ... & A_{nn} \end{pmatrix}$.

Example. Let A_k be a transformation of turning every vector on angle ϕ_k in the space V_2 of vectors on a plane. Find matrix of the following transformations: 1) A_1A_2; 2) A_1^{-1}.

Solution: 1) The matrix of the transformation A_k is the following: $A_k = \begin{pmatrix} \cos\phi_k & -\sin\phi_k \\ \sin\phi_k & \cos\phi_k \end{pmatrix}$

The matrix of the product of transformations is equal to the product of matrices of these transformations:

$$A_1A_2 = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix} \cdot \begin{pmatrix} \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 \end{pmatrix} = \begin{pmatrix} \cos(\phi_1+\phi_2) & -\sin(\phi_1+\phi_2) \\ \sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) \end{pmatrix}.$$

2) The matrix of inverse transformation A_1^{-1} is the inverse matrix for A_1:

$$A_1^{-1} = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1 \end{pmatrix}.$$

Image and Kernel of a linear transformation

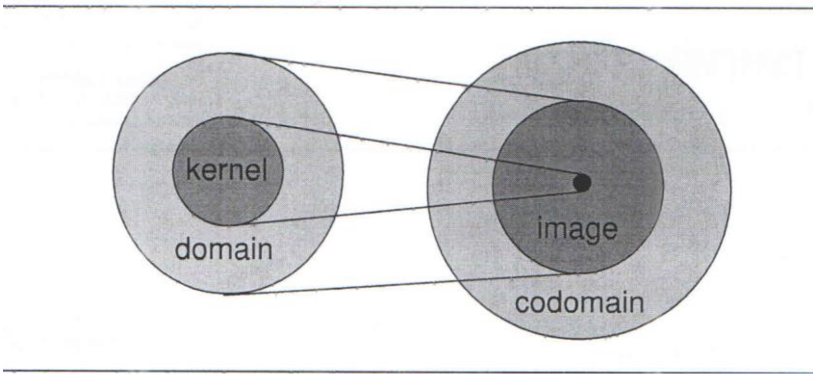
Let A be a linear transformation of a linear space V. The *image* of A, also called the *range* of A, is the set of values of A, i.e. $im(A) := \{Ax \mid x \in V\}$.

The *kernel* of A, also called the *null space* of A, is the inverse image of the zero vector $0 \in V$, i.e. $\ker(A) := A^{-1}(0) = \{x \in V \mid Ax = 0\}$.

Theorem. The image of a linear subspace of V regarding to a linear transformation is a linear subspace.

The dimensions of the image and the kernel of a linear transformation A are called the *rank* and the *nullity* of this transformation and they are denoted by $rank(A)$ and $nullity(A)$ respectively.

Theorem. The sum of the rank and the nullity of a linear transformation of a finitely dimensional linear space V is equal to the dimension of V.



Example. Consider the arithmetic 3-dimensional linear space R^3, and let A be a linear transformation with the matrix $A=\begin{pmatrix}2&0&1\\1&1&3\\3&1&4\end{pmatrix}$. Find the rank, kernel and nullity of A.

Solution: $\begin{pmatrix}2&0&1\\1&1&3\\3&1&4\end{pmatrix}\sim\begin{pmatrix}1&1&3\\2&0&1\\3&1&4\end{pmatrix}\sim\begin{pmatrix}1&1&3\\0&-2&-5\\0&-2&-5\end{pmatrix}\sim\begin{pmatrix}1&1&3\\0&2&5\\0&0&0\end{pmatrix}$. Thus, $rank(A)=2$.

Find the kernel of A: $Ax=0 \Leftrightarrow \begin{cases}2x_1+x_3=0\\ x_1+x_2+3x_3=0\\ 3x_1+x_2+4x_3=0\end{cases} \Leftrightarrow \begin{cases}x_1+x_2+3x_3=0\\ 2x_2+5x_3=0\end{cases}$.

We have: $x_2=-5/2x_3, x_1=-x_2-3x_3=5/2x_3-3x_3=-1/2x_3$.

Let $x_3=2$. Then $x_1=-1, x_2=-5$. Let $u=(-1,-5,\ 2)$. Thus, $\ker(A)=L(u)$, i.e. the linear hull of the vector $u=(-1,-5,\ 2)$. Consequently, $nullity(A)=1$.

Glossary

regular (nonsingular) transformation – невырожденное преобразование
similarity – подобие; **homogeneous** – однородный
commutator – коммутатор; **mapping** – отображение; **opposite** – противоположный
kernel – ядро; **nullity** – дефект

Exercises for Seminar 8

8.1. Let a linear space of vectors $x=\xi_1e_1+\xi_2e_2+\xi_3e_3+\xi_4e_4$ where $\xi_1,\ \xi_2,\ \xi_3,\ \xi_4$ are all possible real numbers be given. A transformation *A* is only the replacement the second and the third coordinates for each vector, i.e. $Ax=\xi_1e_1+\xi_3e_2+\xi_2e_3+\xi_4e_4$. Is the transformation *A* linear?

8.2. Let *A* be a linear transformation. Prove that a transformation *B* defined by the equality *Bx* = *Ax – 2x* is linear.

8.3. When a transformation *A* is linear if $Ax=x_0$ where *x* is an arbitrary vector of a linear space V, and x_0 is a fixed vector?

8.4. Find the matrix of a transformation of similarity $Ax=\alpha x$ in a *n*-dimensional space.

8.5. A linear transformation *A* is considered in a 4-dimensional linear space. Write this transformation in the coordinate form if $Ae_1=e_3+e_4$, $Ae_2=e_1+e_4$, $Ae_3=e_4+e_2$, $Ae_4=e_2+e_3$.

8.6. Find the rank, kernel and nullity of a linear transformation A:

a) $A=\begin{pmatrix}1&1&1\\-1&1&1\\-1&-1&1\end{pmatrix}$; b) $A=\begin{pmatrix}1&2&1&-1\\-1&1&2&2\\1&3&-1&0\\0&3&3&1\end{pmatrix}$; c) $A=\begin{pmatrix}2&-1&2&0\\-1&1&1&1\\1&-2&-5&-3\\1&2&11&5\end{pmatrix}$.

8.7. The transformation A is turning each vector of the plane xOy on angle $\alpha = \pi/4$. Find the transformation $A + E$ in the coordinate form.
8.8. Let the following linear transformations be given:

$$\begin{aligned} x' &= x+2y+3z, \\ y' &= 4x+5y+6z, \\ z' &= 7x+8y+9z \end{aligned} \quad (A) \text{ and } \quad \begin{aligned} x' &= x+3y+4{,}5z, \\ y' &= 6x+7y+9z, \\ z' &= 10{,}5x+12y+13z. \end{aligned} \quad (B)$$

Find $3A - 2B$.
8.9. The transformation A is turning every vector of the plane xOy on angle α. Find the matrix of the transformation A^2 (i.e. $A \cdot A$).
8.10. Let the following linear transformation A be given: $x' = -0{,}5(y+z)$, $y' = -0{,}5(x+z)$, $z' = -0{,}5(x+y)$. Find the matrix of the inverse linear transformation.
8.11. A linear transformation A is given in a linear space with the basis e_1, e_2. Find the matrix of the inverse transformation if $Ae_1 = e_2$, $Ae_2 = e_1$.
8.12. Which value λ does not the linear transformation $x' = -2x+y+z$, $y' = x-2y+z$, $z' = x+y+\lambda z$ have the inverse one at?

Exercises for Homework 8

8.13. A linear transformation of the set of all vectors on plane Oxy consists in turning each vector on angle $\alpha = \pi/6$. Find this transformation in the coordinate form.
8.14. Let a linear space of vectors $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4$ where ξ_1, ξ_2, ξ_3, ξ_4 are all possible real numbers be given. Prove that a transformation A defined by the equality $Ax = \xi_2 e_1 + \xi_3 e_2 + \xi_4 e_3 + \xi_1 e_4$ is linear and find its matrix.
8.15. Let the following linear transformations be given:

$$\begin{aligned} x' &= x+y, \\ y' &= y+z, \\ z' &= x+z \end{aligned} \quad (A) \text{ and } \quad \begin{aligned} x' &= y+z, \\ y' &= x+z, \\ z' &= x+y. \end{aligned} \quad (B)$$

Find the transformations AB and BA.
8.16. The linear transformation A is turning every vector of the plane xOy on angle $\alpha = \pi/4$. Find the matrix of the linear transformation $B = A^2 + A\sqrt{2} + E$.
8.17. Find the rank, kernel and nullity of a linear transformation A:

a) $A = \begin{pmatrix} 4 & 3 & 1 \\ -3 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$; b) $A = \begin{pmatrix} -1 & 0 & 1 \\ -1 & -2 & 3 \\ -1 & -3 & 4 \end{pmatrix}$; c) $A = \begin{pmatrix} 2 & -1 & 1 & 1 \\ -1 & 1 & -2 & 2 \\ 2 & 1 & -5 & 11 \\ 1 & 2 & -7 & 13 \end{pmatrix}$.

8.18. The linear transformation A is turning every vector of the plane xOy on angle α. Find the matrix $B = A + A^{-1}$.
8.19. Let the following linear transformation A be given: $x' = x+y$, $y' = 2(x+y)$. Find the inverse linear transformation.
8.20. The linear transformation A is turning every vector of the plane xOy on angle $\pi/4$. Find the matrix A^{-2}.