LECTURE 4

Rank of a matrix. Calculation of the rank of a matrix by elementary transformations.

Consider a matrix of the dimension $m \times n$: $A(m;n) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$

Take a natural number k such that $k \le m$ and $k \le n$. Choose in A arbitrary k rows and k columns. As a result we obtain a square matrix of the k-th order. The determinant of this matrix is called *minor* of the k-th order of the matrix A. The number of all minors of different orders can be great even for a matrix of non-large dimension. For example, a matrix of dimension 4 x 5 has 5 minors of the fourth order, 40 minors of the third order, 60 minors of the second order and 20 minors of the first order (minors of the first order coincide with elements of a matrix).

We say a minor is *non-zero* if it isn't equal to zero.

The rank of a matrix A is the greatest order of its non-zero minors. The rank of a matrix is denoted by *Rank A* or *r(A)*.

This definition implies that for calculation of the rank of a matrix it is necessary to compute all minors of all orders of a matrix A, choose non-zero minors and determine which one of them has the greatest order. However such a method of calculating the rank of a matrix is useless due to a great number of calculations because there can be a lot of minors at a matrix. For example, a matrix *A(4; 5)* has 125 minors.

Thus, a question on a shorter method of calculating the rank of a matrix emerges. One of such methods is based on the theorem on bordering:

Theorem. If there is a non-zero minor of the r-th order in a matrix A and all its bordering minors of the $r+1$-th order are equal to zero then the rank of A is equal to r, i.e. $r(A)=r$.

The theorem implies a practical rule of calculating the rank of a matrix: find a minor $M \neq 0$ in a matrix A and compute all its bordering minors. If all of them are equal to zero then the rank of the matrix is equal to the order of minor M. If for calculating process of bordering minors we find a non-zero minor M^* then we interrupt further calculating the bordering minors and pass to a bordering the minor M^*, i.e. repeat the described cycle of calculations. There will be finitely many such cycles and the number of such cycles doesn't exceed maximal of m and n, i.e. the number of rows and the number of columns.

Example 1. Calculate the rank of matrix $A(4;5) = \begin{pmatrix} 1 & 2 & -3 & 0 & 9 \\ -3 & 4 & 0 & 5 & 4 \\ -2 & 1 & -3 & 2 & -1 \\ 5 & 3 & 2 & -3 & 0 \end{pmatrix}$.

Solution: Consider, for example, the minor $M = \begin{vmatrix} -3 & 0 \\ 0 & 5 \end{vmatrix} = -15 \neq 0.$ This implies *Rank A* ≥ 2. Further take three minors of the third order M_1, M_2 and M_3 which are bordering the minor M:

$$\begin{vmatrix} 1 & -3 & 0 \\ -3 & 0 & 5 \\ -2 & -3 & 2 \end{vmatrix}; \quad \begin{vmatrix} 2 & -3 & 0 \\ 4 & 0 & 5 \\ 1 & -3 & 2 \end{vmatrix}; \quad \begin{vmatrix} -3 & 0 & 9 \\ 0 & 5 & 4 \\ -3 & 2 & -1 \end{vmatrix}.$$

Calculate $M_1 = \begin{vmatrix} 1 & -3 & 0 \\ -3 & 0 & 5 \\ -2 & -3 & 2 \end{vmatrix} = 30 - 18 + 15 = 27 \neq 0.$

Consequently, *Rank A* ≥ 3. Since $M_1 \neq 0$ it isn't necessary to calculate the rest two bordering minors. Now we will consider bordering minors of the fourth order for the minor M_1:

$$N_1 = \begin{vmatrix} 1 & 2 & -3 & 0 \\ -3 & 4 & 0 & 5 \\ -2 & 1 & -3 & 2 \\ 5 & 3 & 2 & -3 \end{vmatrix} \text{ and } N_2 = \begin{vmatrix} 1 & -3 & 0 & 9 \\ -3 & 0 & 5 & 4 \\ -2 & -3 & 2 & -1 \\ 5 & 2 & -3 & 0 \end{vmatrix}.$$

Since $N_1 = 31 \neq 0$ then *Rank* $A \geq 4$ and it isn't necessary to calculate the minor N_2. And since there is no minor of the fifth order in the matrix A then *Rank* $A = 4$.

The considered methods of calculating the rank of a matrix (based just on the definition and the theorem on bordering) are practically useless for matrices of higher orders due to a more number (complexity) of calculations. Therefore, we formulate a theorem permitting to calculate the rank of a matrix of an arbitrary order by a universal method.

Theorem. The rank of a matrix doesn't change if:

1) All the rows are replaced by the corresponding columns and vice versa;
2) Replace two arbitrary rows (columns);
3) Multiply (divide) each element of a row (column) on the same non-zero number;
4) Add to (subtract from) elements of a row (column) the corresponding elements of any other row (column) multiplied on the same non-zero number.

Example 2. Calculate the rank of a matrix $A = \begin{pmatrix} 3 & 2 & 1 & 3 & -1 \\ 2 & 3 & 5 & 1 & 0 \\ 8 & 5 & 6 & 4 & -1 \\ 13 & 10 & 12 & 8 & -2 \end{pmatrix}$.

Solution: 1) Multiply all the elements of the last column on (-1) and replace the last and the first columns, and we obtain: $\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 3 & 5 & 1 & 2 \\ 1 & 5 & 6 & 4 & 8 \\ 2 & 10 & 12 & 8 & 13 \end{pmatrix}$. 2) Add to all the elements of the third row the corresponding elements of the first row multiplied on (-1), and we obtain: $\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 3 & 5 & 1 & 2 \\ 0 & 3 & 5 & 1 & 5 \\ 2 & 10 & 12 & 8 & 13 \end{pmatrix}$.

3) Add to all the elements of the fourth row the corresponding elements of the first row multiplied on (-2), and we obtain: $\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 3 & 5 & 1 & 2 \\ 0 & 3 & 5 & 1 & 5 \\ 0 & 6 & 10 & 2 & 7 \end{pmatrix}$. As a result, we obtained "good" first column (it consist of one 1 and zeros). Further we make zeros in the first row by this column.

4) Add to all the elements of the second column the corresponding elements of the first column multiplied on (-2); subtract from the elements of the third column the corresponding elements of the first column; add to all the elements of the fourth column the corresponding elements of the first column multiplied on (-3); add to all the elements of the fifth column the corresponding elements of the first column multiplied on (-3). The rank of the matrix A will not change after these transformations. As a result, we have: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 5 & 1 & 2 \\ 0 & 3 & 5 & 1 & 5 \\ 0 & 6 & 10 & 2 & 7 \end{pmatrix}$. 5) The elements of the second and the third columns can be reduced on common for them non-zero multipliers (3 and 5 respectively). We have: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 2 & 7 \end{pmatrix}$. We see that there are three identical columns in the matrix (second, third and fourth). Subtract from the elements of the third and the fourth columns the corresponding

elements of the second column, and we obtain: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 & 7 \end{pmatrix}$. 6) Add to all the elements of the fifth column the corresponding elements of the second column multiplied on (-2), and we obtain: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 3 \end{pmatrix}$. As a result, we have "good" second row (it consists of one 1 and zeros). Further, subtract from the elements of the third row the corresponding elements of the second row; add to all the elements of the fourth row the corresponding elements of the second row multiplied on (-2), and we obtain: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$. We see that there are two identical rows in the matrix. Subtract from the elements of the fourth row the corresponding elements of the third row, and we obtain: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Interchange the third and the fifth columns and reduce the elements of the third column on common multiplier 3, and finally we obtain: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Consequently, $r(A) = 3$.

Criterion for compatibility of a system of linear equations.

The notion of the rank of a matrix is used for investigation of compatibility of a system of linear equations.

Consider a system of m linear equations with n variables $x_1, x_2, x_3, \dots, x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m. \end{cases}$$

The basic (coefficient) and extended (augmented) matrices of the system are the following:

$$A(m;n) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad C(m;n+1) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}.$$

A criterion for compatibility of a system of linear equations is expressed by the Kronecker-Capelli theorem. This theorem gives an effective method permitting in finitely many steps to determine whether the system is compatible or not.

Theorem of Kronecker-Capelli. A system of linear equations is compatible iff the rank of the basic matrix A equals the rank of the extended matrix C, i.e. $Rank\ A = Rank\ C$. Moreover:

1) If $Rank\ A = Rank\ C = n$ (where n is the number of variables in the system) then the system has a unique solution.

2) If $Rank\ A = Rank\ C < n$ then the system has infinitely many solutions.

Recall that a system of linear equations is said to be *homogeneous* if all the constant terms are zero. Thus, a homogeneous system has the form $AX = 0$. Clearly, such a system ha the zero vector $0 = (0, 0, ..., 0)$ as a solution, called the *zero* or *trivial* solution. Accordingly, we are usually interested in whether or not the system has a non-zero solution.

Corollary. A homogeneous system of linear equations has a nonzero solution iff $Rank\, A = Rank\, C < n.$

Systems in Triangular and Echelon forms

Here we consider two simple types of systems of liner equations: systems in triangular form and the more general systems in echelon forms.

Consider the following system of linear equations, which is in *triangular form*:

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 - 2x_4 = 9 \\ \quad\;\; 5x_2 - \;\; x_3 + 3x_4 = 1 \\ \qquad\quad\; 7x_3 - \;\; x_4 = 3 \\ \qquad\qquad\quad\;\; 2x_4 = 8 \end{cases}$$

Recall that by the *leading unknown* of a linear equation we mean the first unknown in the equation with a nonzero coefficient. That is, the first unknown x_1 is the leading unknown in the first equation, the second unknown x_2 is the leading unknown in the second equation, and so on. Thus, in particular, the system is square and each leading unknown is *directly* to the right of the leading unknown in the preceding equation.

Obviously, such a triangular system always has a unique solution, which may be obtained by *back-substitution*. That is,

(1) First solve the last equation for the last unknown to get $x_4 = 4.$

(2) Then substitute this value $x_4 = 4$ in the next-to-last equation, and solve for the next-to-last unknown x_3 as follows: $7x_3 - 4 = 3$ or $7x_3 = 7$ or $x_3 = 1.$

(3) Now substitute $x_3 = 1$ and $x_4 = 4$ in the second equation, and solve for the second unknown x_2 as follows: $5x_2 - 1 + 12 = 1$ or $5x_2 + 11 = 1$ or $5x_2 = -10$ or $x_2 = -2.$

(4) Finally, substitute $x_2 = -2, x_3 = 1, x_4 = 4$ in the first equation, and solve for the first unknown x_1 as follows: $2x_1 + 6 + 5 - 8 = 9$ or $x_1 = 3.$

Thus, $x_1 = 3, x_2 = -2, x_3 = 1, x_4 = 4$ is the unique solution of the system.

The following system of linear equations is said to be in *echelon form*:

$$\begin{cases} 2x_1 + 6x_2 - x_3 + 4x_4 - 2x_5 = 15 \\ \qquad\qquad\;\; x_3 + 2x_4 + 2x_5 = 5 \\ \qquad\qquad\qquad\; 3x_4 - 9x_5 = 6 \end{cases}$$

That is, the leading unknown in each equation other than the first is to the right of the leading unknown in the preceding equation. The leading unknowns in the system, $x_1, x_3, x_4,$ are called *pivot* variables, and the other unknowns, x_2 and $x_5,$ are called *free* variables.

Generally speaking, an *echelon system* or a *system in echelon form* has the following form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + ... + a_{1n}x_n = b_1 \\ \qquad\quad a_{2j_2}x_{j_2} + a_{2j_2+1}x_{j_2+1} + ... + a_{2n}x_n = b_2 \\ \qquad\qquad \text{..} \\ \qquad\qquad\qquad a_{rj_r}x_{j_r} + ... + a_{rn}x_n = b_r \end{cases}$$

where $1 < j_2 < ... < j_r,$ and $a_{11}, a_{2j_2}, ..., a_{rj_r}$ are not zero. The *pivot* variables are $x_1, x_{j_2}, ..., x_{j_r}.$ Note that $r \le n.$

Theorem. Consider a system of linear equations in echelon form, say with r equations in n unknowns, There are two cases:

(i) $r = n.$ That is, there are as many equations as unknowns (triangular form). Then the system has a unique solution.

(ii) $r < n.$ That is, there are more unknowns than equations. Then we can arbitrarily assign values to the $n-r$ free variables and solve uniquely for the r pivot variables, obtaining a solution of the system.

Solving a system of linear equations by the Gaussian Elimination Method

It is convenient to produce a numerical solution of linear algebraic equations by determinants for systems of two and three equations. And in case of systems of a greater number of equations it is much more profitable to use the *Gauss method* which consists of a sequential (successive) elimination of unknowns.

Let's explain the sense of this method on a system of fours equations with fours unknowns:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z + a_{14}u = a_{15} & (a) \\ a_{21}x + a_{22}y + a_{23}z + a_{24}u = a_{25} & (b) \\ a_{31}x + a_{32}y + a_{33}z + a_{34}u = a_{35} & (c) \\ a_{41}x + a_{42}y + a_{43}z + a_{44}u = a_{45} & (d) \end{cases}$$

Suppose that $a_{11} \neq 0$ (if $a_{11} = 0$ then we change the order of equations by choosing as the first equation such an equation in which the coefficient of x is not equal to zero).

I step: divide the equation *(a)* on a_{11}, then multiply the obtained equation on a_{21} and subtract from *(b)*; further multiply $(a)/a_{11}$ on a_{31} and subtract from *(c)*; at last, miltiply $(a)/a_{11}$ on a_{41} and subtract from *(d)*.

As a result of I step, we come to the system $\begin{cases} x + b_{12}y + b_{13}z + b_{14}u = b_{15} & (e) \\ b_{22}y + b_{23}z + b_{24}u = b_{25} & (f) \\ b_{32}y + b_{33}z + b_{34}u = b_{35} & (g) \\ b_{42}y + b_{43}z + b_{44}u = b_{45} & (i) \end{cases}$

where b_{ij} are obtained from a_{ij} by the following formulas: $b_{1j} = a_{1j} / a_{11}$ (j = 2, 3, 4, 5);

$$b_{ij} = a_{ij} - a_{i1} \cdot b_{1j} \quad (i = 2, 3, 4;\ j = 2, 3, 4, 5).$$

II step: do the same actions with *(f), (g), (i)* (as with *(a), (b), (c), (d))* and etc.

As a final result the initial system will be transformed to a so-called step form:

$$\begin{cases} x + b_{12}y + b_{13}z + b_{14}u = b_{15} \\ \quad y + c_{23}z + c_{24}u = c_{25} \\ \quad\quad z + d_{34}u = d_{35} \\ \quad\quad\quad u = e_{45} \end{cases}$$

From a transformed system all unknowns are determined sequentially without difficulty. If the system has a unique solution, the step system of equations will be reduced to *triangular* in which the last equation contains one unknown. In case of an indeterminate system, i.e. in which the number of unknowns is more than the number of linearly independent equations and therefore permitting infinitely many solutions, a triangular system is impossible because the last equation contains more than one unknown. If the set of equations is incompatible then after reduction to a step form it contains at least one equation of the kind 0 = 1, i.e. the equation in which all unknowns have zero factors, and the right part is not equal to zero. Such system has no solutions.

Example 1. $\begin{cases} 3x + 2y + z = 5 \\ x + y - z = 0 \\ 4x - y + 5z = 3 \end{cases}$

Interchange the first and the second equations of the system: $\begin{cases} x + y - z = 0 \\ 3x + 2y + z = 5 \\ 4x - y + 5z = 3 \end{cases}$. Subtract from the second equation the first equation multiplied on 3; also subtract from the third equation the first equation multiplied on 4. We obtain: $\begin{cases} x + y - z = 0 \\ -y + 4z = 5 \\ -5y + 9z = 3 \end{cases}$. Further subtract from the third equation the second equation multiplied on 5: $\begin{cases} x + y - z = 0 \\ -y + 4z = 5 \\ -11z = -22 \end{cases}$. Multiply the second equation on (-2), and the third – divide on (-11): $\begin{cases} x + y - z = 0 \\ y - 4z = -5 \\ z = 2 \end{cases}$. The system of equations has a triangular form, and consequently it has a unique solution. From the last equation we have z = 2; substituting this value in the second equation, we receive y = 3 and, at last from the first equation we find x = -1.

Example 2. $\begin{cases} x + 2y + 3z = 4 \\ 2x + y - z = 3 \\ 3x + 3y + 2z = 0 \end{cases}$

Subtract from the second equation the first equation multiplied on 2; also subtract from the third equation the first equation multiplied on 3: $\begin{cases} x + 2y + 3z = 4 \\ -3y - 7z = -5 \\ -3y - 7z = -12 \end{cases}$. Further subtract from the third equation the second equation and we obtain: $\begin{cases} x + 2y + 3z = 4 \\ -3y - 7z = -5 \\ 0 = -7 \end{cases}$. We have the equation 0 = -7. Consequently the system has no solution.

Example 3. $\begin{cases} x + 2y + 3z = 4 \\ 2x + y - z = 3 \\ 3x + 3y + 2z = 7 \end{cases}$

Subtract from the second equation the first equation multiplied on 2; also subtract from the third equation the first equation multiplied on 3: $\begin{cases} x + 2y + 3z = 4 \\ -3y - 7z = -5 \\ -3y - 7z = -5 \end{cases}$. Further subtract from the third equation the second equation and we obtain: $\begin{cases} x + 2y + 3z = 4 \\ -3y - 7z = -5 \\ 0 = 0 \end{cases}$. We have that unknowns are more than equations. Express x and y through the variable z: $3y + 7z = 5 \Rightarrow 3y = 5 - 7z \Rightarrow y = \dfrac{5 - 7z}{3}$

$x = 4 - 2y - 3z = 4 - \dfrac{2(5 - 7z)}{3} - 3z = \dfrac{2 + 5z}{3}$. As a result we have: $\begin{cases} x = \dfrac{2 + 5z}{3} \\ y = \dfrac{5 - 7z}{3} \end{cases}$, where z is an arbitrary number. Thus, the system has infinitely many solutions.

Glossary

rank – ранг; **to border** – окаймлять; **triangular** – треугольный
sequential (**successive**) – последовательный; **echelon** – эшелон, звено
pivot – стержень, основа

Exercises for Seminar 4

4.1. Find the ranks of matrices:

a) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$; b) $\begin{pmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$; c) $\begin{pmatrix} 4 & 3 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix}$; d) $\begin{pmatrix} 0 & -2 & 3 & 0 & 2 \\ -4 & 5 & 7 & -10 & 0 \\ 3 & -4 & 5 & 3 & -5 \\ 2 & -3 & 2 & 3 & -1 \end{pmatrix}$.

4.2. Investigate the following systems, i.e. determine the compatibility of the systems by the Kronecker-Capelli theorem: a) $\begin{cases} 2x+y-5z=-1 \\ x+2y-4z=1 \\ x-y+z=-2 \end{cases}$; b) $\begin{cases} x+2y+3z=14 \\ 3x+2y+z=10 \\ x+y+z=6 \\ 2x+3y-z=5 \\ x+y=3 \end{cases}$

4.3. Solve the systems of equations by the Gauss method:

a) $\begin{cases} 2x+4y+6z=8 \\ 2x+y-z=3 \\ 3x+3y+2z=7 \end{cases}$; b) $\begin{cases} x_1+x_2-x_3+x_4=4 \\ 2x_1-x_2+3x_3-2x_4=1 \\ x_1-x_3+2x_4=6 \\ 3x_1-x_2+x_3-x_4=0 \end{cases}$; c) $\begin{cases} 0{,}04x-0{,}08y+4z=20 \\ 4x+0{,}24y-0{,}08z=8 \\ 0{,}09x+3y-0{,}15z=9 \end{cases}$.

4.4. Find such values of parameter a for which the following system is compatible: $\begin{cases} x+y=a \\ ax-2y=4 \\ x+ay=2 \end{cases}$.

4.5. Solve the systems of equations:

a) $\begin{cases} x_1+x_2+x_3+x_4+x_5=0 \\ x_1-x_2+2x_3-2x_4+3x_5=0 \\ x_1+x_2+4x_3+4x_4+9x_5=0 \\ x_1-x_2+8x_3-8x_4+27x_5=0 \\ x_1+x_2+16x_3+16x_4+81x_5=0 \end{cases}$; b) $\begin{cases} x_1-2x_2+x_3+x_4=1 \\ x_1-2x_2+x_3-x_4=-1 \\ x_1-2x_2+x_3+5x_4=5 \end{cases}$.

Exercises for Homework 4

4.6. Solve the systems of equations by the Gauss method:

a) $\begin{cases} 2x+y-z=5 \\ x-2y+3z=-3 \\ 7x+y-z=10 \end{cases}$; b) $\begin{cases} 4x+2y+3z=-2 \\ 2x+8y-z=8 \\ 9x+y+8z=0 \end{cases}$; c) $\begin{cases} 3x_1-x_2+x_3+2x_5=18 \\ 2x_1-5x_2+x_4+x_5=-7 \\ x_1-x_4+2x_5=8 \\ 2x_2+x_3+x_4-x_5=10 \\ x_1+x_2-3x_3+x_4=1 \end{cases}$.

4.7. Find the ranks of matrices:

a) $\begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 1 & 6 \\ 3 & 1 & 2 & 6 \end{pmatrix}$; b) $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 3 & 0 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 6 & 8 \\ 1 & 2 & 1 & 3 & 4 \end{pmatrix}$; d) $\begin{pmatrix} 1 & -2 & 3 & -1 & 2 \\ 3 & -1 & 5 & -3 & -1 \\ 2 & 1 & 2 & -2 & -3 \end{pmatrix}$.

4.8. Investigate the following systems, i.e. determine the compatibility of the systems by the Kronecker-Capelli theorem: a) $\begin{cases} 2x+y-5z=-1 \\ x+2y-4z=1 \\ x-y-z=-2 \end{cases}$; b) $\begin{cases} 3x+2y=4 \\ x-4y=-1 \\ 7x+10y=12 \\ 5x+6y=8 \\ 3x-16y=-5 \end{cases}$; c) $\begin{cases} x+5y+4z=1 \\ 2x+10y+8z=3 \\ 3x+15y+12z=5 \end{cases}$.

4.9. Solve the systems of equations:

a) $\begin{cases} x_1+3x_2+2x_3=0 \\ 2x_1-x_2+3x_3=0 \\ 3x_1-5x_2+4x_3=0 \\ x_1+17x_2+4x_3=0 \end{cases}$; b) $\begin{cases} x_1+x_2-3x_3=-1 \\ 2x_1+x_2-2x_3=1 \\ x_1+x_2+x_3=3 \\ x_1+2x_2-3x_3=1 \end{cases}$.