LECTURE 6
Linear space

Linear (vector) space. Consider such a set V of elements $x, y, z, \ldots$, in which the *sum* $x + y \in V$ for every $x, y \in V$ and the *product* $\lambda x \in V$ for every $x \in V$ and every real number λ are determined. If an addition of elements of V and a multiplication of an element of V on a real number satisfy the following conditions:

(1) $x + y = y + x$;

(2) $(x + y) + z = x + (y + z)$;

(3) there is such an element $0 \in V$ (*zero-element*) so that $x + 0 = x$ for every $x \in V$;

(4) for every element $x \in V$ there is an element $y \in V$ (*opposite element*) such that $x + y = 0$ (further we shall write $y = -x$, i.e. $x + (-x) = 0$);

(5) $1 \cdot x = x$;

(6) $\lambda(\mu x) = (\lambda\mu) x$;

(7) $(\lambda + \mu) x = \lambda x + \mu x$;

(8) $\lambda(x + y) = \lambda x + \lambda y$,

then the set V is said to be a *linear* (or *vector*) *space*, and elements $x, y, z, \ldots$ of the space – *vectors*.
The above defined linear space is called *real* since an operation of multiplication of elements on real numbers is determined. If an operation of multiplication of elements on complex numbers is determined for elements of V and all eight axioms (1)-(8) are true then V is called a *complex* linear space.
Examples of linear spaces: the set of real numbers; the set of geometric vectors; the set of vectors which are parallel to some plane (line); the set of matrices of dimension $m \times n$ with real coefficients; the set of polynomials of degree $\leq n$.
The difference of vectors x and y of a linear space is such a vector v of the space so that $x - y = v$. The difference of vectors x and y is denoted by $x - y$, i.e. $x - y = v$. It is easy to see $x - y = x + (-y)$.
Example. Is the set of positive real numbers a linear space if the operations of addition and multiplication on number for elements of the set are determined by "ordinary" way?
Solution: The set of positive real numbers doesn't form a linear space since for example there is no zero-element.
Theorem 1. For every linear space there is only one zero-element.
Proof: Let there be two different zero-elements 0_1 and 0_2. Then by axiom (3) the following holds: $0_1 + 0_2 = 0_1$, $0_2 + 0_1 = 0_2$. Then by commutability of the operation of addition we have $0_1 = 0_2$.
Theorem 2. For every element $x \in V$ the equality $0 \cdot x = 0$ holds.
Proof: By axioms of a linear space we have $x = 1 \cdot x = (0+1) \cdot x = 0 \cdot x + 1 \cdot x = 0 \cdot x + x$. Adding to both parts of the equality the element y that is opposite to x, we obtain that $0 \cdot x = 0$.
Theorem 3. For every element of a linear space there is only one opposite element.
Proof: Let for an element x there be two different opposite elements y_1 and y_2. Then by axiom (4) the following holds: $x + y_1 = 0$, $x + y_2 = 0$. Add to both parts of the first equality the element y_2 and we obtain $y_2 + (x + y_1) = y_2$.
On the other hand we have $y_2 + (x + y_1) = (y_2 + x) + y_1 = 0 + y_1 = y_1$, i.e. $y_2 = y_1$.
Theorem 4. The element $(-1) \cdot x$ is opposite for an element x.
Proof: By axioms of a linear space and Theorem 2 we have

$$0 = 0 \cdot x = (1-1) \cdot x = 1 \cdot x + (-1) \cdot x = x + (-1) \cdot x$$

This equality means that the opposite to x element is $(-1) \cdot x$.
Example. Let $\{(\xi_1, \xi_2, \ldots, \xi_n), (\eta_1, \eta_2, \ldots, \eta_n), (\zeta_1, \zeta_2, \ldots, \zeta_n), \ldots\}$ – the set of all n–tuples of real numbers so that the sum of every two elements is determined by $(\xi_1, \xi_2, \ldots, \xi_n) + (\eta_1, \eta_2, \ldots, \eta_n) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots, \xi_n + \eta_n)$, and the product of every element on a real number is determined by $\lambda(\xi_1, \xi_2, \ldots, \xi_n) = (\lambda\xi_1, \lambda\xi_2, \ldots, \lambda\xi_n)$. Prove that this set is a linear space.
Solution: Denote $x = (\xi_1, \xi_2, \ldots, \xi_n)$, $y = (\eta_1, \eta_2, \ldots, \eta_n)$, $z = (\zeta_1, \zeta_2, \ldots, \zeta_n)$. Check the formulated above properties (1) – (8).

(1) $x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots, \xi_n + \eta_n)$, $y + x = (\eta_1 + \xi_1, \eta_2 + \xi_2, \ldots, \eta_n + \xi_n)$,
i.e. $x + y = y + x$.

(2) $x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots, \xi_n + \eta_n)$, $y + z = (\eta_1 + \zeta_1, \eta_2 + \zeta_2, \ldots, \eta_n + \zeta_n)$,
$(x + y) + z = (\xi_1 + \eta_1 + \zeta_1, \xi_2 + \eta_2 + \zeta_2, \ldots, \xi_n + \eta_n + \zeta_n)$,

$x + (y + z) = (\xi_1 + \eta_1 + \zeta_1, \xi_2 + \eta_2 + \zeta_2, \ldots, \xi_n + \eta_n + \zeta_n)$.
Thus, $(x + y) + z = x + (y + z)$.
(3) The zero-element is $0 = (0, 0, \ldots, 0)$. Indeed, $x + 0 = (\xi_1 + 0, \xi_2 + 0, \ldots, \xi_n + 0) = x$.
(4) The element $(-\xi_1, -\xi_2, \ldots, -\xi_n)$ is opposite to an element $(\xi_1, \xi_2, \ldots, \xi_n)$ because $(\xi_1, \xi_2, \ldots, \xi_n) + (-\xi_1, -\xi_2, \ldots, -\xi_n) = (0, 0, ..., 0) = 0$.
(5) $1 \cdot x = (1 \cdot \xi_1, 1 \cdot \xi_2, \ldots, 1 \cdot \xi_n) = x$.
(6) $\lambda(\mu x) = \lambda(\mu \xi_1, \mu \xi_2, \ldots, \mu \xi_n) = (\lambda\mu \xi_1, \lambda\mu \xi_2, \ldots, \lambda\mu \xi_n) = (\lambda\mu) x$.
(7) $(\lambda + \mu)x = ((\lambda + \mu)\xi_1, (\lambda + \mu)\xi_2, \ldots, (\lambda + \mu)\xi_n) = (\lambda\xi_1 + \mu\xi_1, \lambda\xi_2 + \mu\xi_2, \ldots, \lambda\xi_n + \mu\xi_n) = (\lambda\xi_1, \lambda\xi_2, \ldots, \lambda\xi_n) + (\mu \xi_1, \mu \xi_2, \ldots, \mu \xi_n) = \lambda(\xi_1, \xi_2, \ldots, \xi_n) + \mu (\xi_1, \xi_2, \ldots, \xi_n) = \lambda x + \mu x$;
(8) $\lambda(x + y) = \lambda(\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots, \xi_n + \eta_n) = (\lambda\xi_1 + \lambda\eta_1, \lambda\xi_2 + \lambda\eta_2, \ldots, \lambda\xi_n + \lambda\eta_n) = (\lambda\xi_1, \lambda\xi_2, \ldots, \lambda\xi_n) + (\lambda\eta_1, \lambda\eta_2, \ldots, \lambda\eta_n) = \lambda(\xi_1, \xi_2, \ldots, \xi_n) + \lambda(\eta_1, \eta_2, \ldots, \eta_n) = \lambda x + \lambda y$.
This space is called *the arithmetic linear space* over the field of real numbers and is denoted by R^n.

Linearly independent vectors. Let $x, y, z, \ldots, u$ be vectors of a linear space V.
A vector $v = \alpha x + \beta y + \gamma z + \ldots + \lambda u$, where $\alpha, \beta, \gamma, \ldots, \lambda$ – real numbers, also belongs to V. It is called a *linear combination* of vectors $x, y, z, \ldots, u$.
Let a linear combination of vectors $x, y, z, \ldots, u$ be zero-vector, i.e.

$$\alpha x + \beta y + \gamma z + \ldots + \lambda u = 0 \qquad (1)$$

The vectors $x, y, z, \ldots, u$ are called *linearly independent* if the equality (1) holds only for $\alpha = \beta = \gamma = \ldots = \lambda = 0$. If (1) can also hold when not all numbers $\alpha, \beta, \gamma, \ldots, \lambda$ are equal to zero then the vectors $x, y, z, \ldots, u$ are called *linearly dependent*.
It is easy to see that vectors $x, y, z, \ldots, u$ are linearly dependent iff one of these vectors can be represented as a linear combination of the rest vectors.

Example. Show that if there is zero-vector among vectors $x, y, z, \ldots, u$ then these vectors are linearly dependent.
Solution: Let $x = 0$. Since the equality $\alpha x + \beta y + \gamma z + \ldots + \lambda u = 0$ can hold for $\alpha \neq 0, \beta = \gamma = \ldots = \lambda = 0$, the vectors are linearly dependent.
Example. Elements of a linear space are ordered n–tuples of real numbers $x_i = (\xi_{1i}, \xi_{2i}, \ldots, \xi_{ni})$, $i = 1, 2, 3, \ldots$. Which condition must the numbers ξ_{ik} ($i = 1, \ldots, n$; $k = 1, \ldots, n$) satisfy for linear independence of vectors $x_1, x_2, \ldots, x_n$ if the sum of vectors and the product of a vector on number are determined by $x_i + x_k = (\xi_{1i} + \xi_{1k}, \xi_{2i} + \xi_{2k}, \ldots, \xi_{ni} + \xi_{nk})$, $\lambda x_i = (\lambda\xi_{1i}, \lambda\xi_{2i}, \ldots, \lambda\xi_{ni})$?
Solution: Consider the equality $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0$. It is equivalent to the system of equations

$$\begin{cases} \alpha_1\xi_{11} + \alpha_2\xi_{12} + \ldots + \alpha_n\xi_{1n} = 0, \\ \alpha_1\xi_{21} + \alpha_2\xi_{22} + \ldots + \alpha_n\xi_{2n} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_1\xi_{n1} + \alpha_2\xi_{n2} + \ldots + \alpha_n\xi_{nn} = 0. \end{cases}$$

In case of linear independence of vectors $x_1, x_2, \ldots, x_n$ the system must have a unique solution $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, i.e.

$$\begin{vmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{vmatrix} \neq 0.$$

In particular, vectors (ξ_{11}, ξ_{21}) and (ξ_{12}, ξ_{22}) are linearly independent iff $\xi_{11}\xi_{22} - \xi_{12}\xi_{21} \neq 0$.
Example. Prove that in the space P_2 of polynomials of degree ≤ 2 three polynomials $1, x, x^2$ are linearly independent and every element of the space P_2 is a linear combination of these elements.
Solution: The zero-element in the space P_2 is polynomial which is identically equal to 0. Compose a linear combination of these polynomials and equate it to zero-element:

$$a \cdot 1 + b \cdot x + c \cdot x^2 \equiv 0$$

This equality is true for all x only if $a = b = c = 0$. In fact if there is at least one of coefficients a, b and c is not equal to 0 then there is a polynomial of degree ≤ 2 in the left part of the equality, and this polynomial has at most two roots; and consequently it isn't equal to 0 for all x. Consequently the polynomials 1, x, x^2 are linearly independent. Obviously, any polynomial $a+bx+cx^2$ of P_2 is a linear combination of the polynomials 1, x, x^2 with coefficients a, b and c.

Dimension and basis of a linear space. If there are n linearly independent vectors in a linear space V and any $n+1$ vectors of the space are linearly dependent then the space V is said to be *n–dimensional*. We also say the dimension of V is equal to n and write $d(V) = n$. A space in which one can find arbitrarily many linearly independent vectors is said to be *infinitely dimensional*. If V is an infinitely dimensional space, $d(V) = \infty$.

A set of n linearly independent vectors of a n–dimensional space is said to be a *basis*.

Theorem. *Every vector of a linear n–dimensional space can be uniquely represented as a linear combination of the vectors of a basis.*

Thus, if $e_1, e_2, \dots, e_n$ – a basis of n–dimensional linear space V, every vector $x \in V$ can be uniquely represented as $x = \xi_1e_1 + \xi_2e_2 + \dots + \xi_ne_n$.

Consequently, the vector x in the basis $e_1, e_2, \dots, e_n$ is uniquely determined by numbers $\xi_1, \xi_2, \dots, \xi_n$. These numbers are said to be the *coordinates* of the vector x in the basis.

If $x = \xi_1e_1 + \xi_2e_2 + \dots + \xi_ne_n$, $y = \eta_1e_1 + \eta_2e_2 + \dots + \eta_ne_n$, then $x + y = (\xi_1 + \eta_1)e_1 + (\xi_2 + \eta_2)e_2 + \dots + (\xi_n + \eta_n)e_n$, $\lambda x = \lambda\xi_1e_1 + \lambda\xi_2e_2 + \dots + \lambda\xi_ne_n$.

To determine the dimension of a linear space is useful to use the following theorem:

Theorem. *If every vector of a linear space V can be represented as a linear combination of linearly independent vectors $e_1, e_2, \dots, e_n$, then $d(V) = n$ (and consequently the vectors $e_1, e_2, \dots, e_n$ form a basis in the space V).*

Examples: 1) Any ordered triple of non-coplanar vectors is a basis in the space V_3 (the space of geometric vectors) and $d(V_3) = 3$; any ordered pair of non-collinear vectors of V_2 (plane) is a basis in V_2 and $d(V_2) = 2$; any non-zero vector of V_1 (line) is a basis in V_1 and $d(V_1) = 1$.

2) The polynomials $p_0(x) = 1,\ p_1(x) = x,\ p_2(x) = x^2$ form a basis in the space P_2. In fact, these polynomials are linearly independent and any polynomial $p(x) = a + bx + cx^2$ of P_2 can be presented as a linear combination of polynomials p_0, p_1, p_2. Observe that the coordinates of the element $p(x)$ in the basis p_0, p_1, p_2 are the numbers a, b, c and $d(P_2) = 3$.

Example. Consider the linear space of all pairs of ordered real numbers $x_1 = (\xi_{11}, \xi_{21})$, $x_2 = (\xi_{12}, \xi_{22})$, $x_3 = (\xi_{13}, \xi_{23})$, …, where the sum of vectors and the product of vector on a number are determined by $x_i + x_k = (\xi_{1i} + \xi_{1k}, \xi_{2i} + \xi_{2k})$; $\lambda x_i = (\lambda\xi_{1i}, \lambda\xi_{2i})$. Prove that the vectors $e_1 = (1, 2)$ and $e_2 = (3, 4)$ form a basis of the given linear space. Find the coordinates of the vector $x = (7, 10)$ in this basis.

Solution: Show firstly that the vectors e_1 and e_2 are linearly independent, i.e. the equality $\alpha_1e_1 + \alpha_2e_2 = 0$ holds only for $\alpha_1 = \alpha_2 = 0$.

$$\alpha_1e_1 + \alpha_2e_2 = 0 \Leftrightarrow \alpha_1(1, 2) + \alpha_2(3, 4) = 0 \Leftrightarrow \begin{cases} \alpha_1 + 3\alpha_2 = 0 \\ 2\alpha_1 + 4\alpha_2 = 0 \end{cases}.$$

The last system has a unique solution $\alpha_1 = \alpha_2 = 0$ iff the basic determinant of the system is not equal to zero.

Indeed, $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2 \neq 0$. Consequently, the vectors e_1 and e_2 are linearly independent.

Consider a vector $y = (\eta_1, \eta_2)$. Show that for all η_1 and η_2 one can determine the numbers λ and μ so that the equality $y = \lambda e_1 + \mu e_2$ holds, i.e. $(\eta_1, \eta_2) = (\lambda + 3\mu, 2\lambda + 4\mu)$. Since the system of equations $\begin{cases} \lambda + 3\mu = \eta_1, \\ 2\lambda + 4\mu = \eta_2 \end{cases}$ is determinate, there is a unique pair of values (λ, μ) for which the equality holds. Thus, the vectors e_1 and e_2 form a basis. Find the coordinates of vector $x = (7, 10)$ in this basis. The problem is reduced to a determination of λ and μ from the system $\begin{cases} \lambda + 3\mu = 7, \\ 2\lambda + 4\mu = 10. \end{cases}$

We find $\lambda = 1$, $\mu = 2$, i.e. $x = e_1 + 2e_2$.

Isomorphism of linear spaces. Consider two linear spaces V and V'. We denote elements of the space V by $x, y, z, \dots$, and elements of the space V' by $x', y', z', \dots$.

Recall that a mapping $F: V \to V'$ is a *one-to-one correspondence* if:
(1) for all elements $x, y \in V$ if $x \neq y$ then $F(x) \neq F(y)$ (i.e. F is an *injection*);
(2) for every element $x' \in V'$ there is an element $x \in V$ such that $F(x) = x'$ (i.e. F is an *surjection*).
The spaces V and V' are called *isomorphic* if there is such a one-to-one correspondence $F: V \to V'$ such that for all elements $x, y \in V$ and for every number λ $F(x+y) = F(x) + F(y)$ and $F(\lambda x) = \lambda F(x)$. The mapping F is called an *isomorphism*.

Theorem. Two finitely dimensional linear spaces V and V' are isomorphic if and only if their dimensions are equal.

Proof: ($\Leftarrow$) Let $\dim(V) = \dim(V') = n$ for some natural n. Consequently, there are $\{e_1, e_2, ..., e_n\}$ and $\{e'_1, e'_2, ..., e'_n\}$ being bases of V and V' respectively. Take an arbitrary $x \in V$. Then for some numbers $\xi_1, \xi_2, ..., \xi_n$ $x = \xi_1 e_1 + \xi_2 e_2 + ... + \xi_n e_n$. Then let $F: V \to V'$ be a mapping such that $F(x) = x'$ where $x' = \xi_1 e'_1 + \xi_2 e'_2 + ... + \xi_n e'_n$ for every $x \in V$. We assert that F is an isomorphism between V and V'.

($\Rightarrow$) Let V and V' be isomorphic. Consequently, there is an isomorphism F between them. Assume the contrary: $n = \dim(V) > \dim(V') = m$. Consequently, there are elements $e_1, e_2, ..., e_n \in V$ being linearly independent. There exist elements $e'_1, e'_2, ..., e'_n \in V'$ such that $F(e_i) = e'_i$ for all $i \leq n$. Since $\dim(V') = m$, the elements $e'_1, e'_2, ..., e'_n$ are linearly dependent, and there are some numbers $\xi_1, \xi_2, ..., \xi_n$ non-simultaneously equal to zero such that $\xi_1 e'_1 + \xi_2 e'_2 + ... + \xi_n e'_n = 0$. Observe that for any isomorphism the image of zero-element of a linear space is zero-element of other linear space. Therefore we have that $\xi_1 e_1 + \xi_2 e_2 + ... + \xi_n e_n = 0$ contradicting their linear independence.

Example. Find a correspondence between elements of linear spaces T_3 (the set of matrices of dimension 3×1) and P_2 being isomorphism.

Solution: It can see that the elements $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ form a basis in the space T_3. We know that the polynomials $p_0(x) = 1,\ p_1(x) = x,\ p_2(x) = x^2$ form a basis in the space P_2.

An element $y = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ of T_3 in the basis e_1, e_2, e_3 has the coordinates c_1, c_2, c_3 since $y = c_1 e_1 + c_2 e_2 + c_3 e_3$.

For the element y of T_3 put in correspondence the element y' of P_2 having in the basis p_0, p_1, p_2 the same coordinates c_1, c_2, c_3, i.e. $y' = c_1 + c_2 x + c_3 x^2$. This correspondence is one-to-one since every element is determined by its coordinates in the basis and the following holds: $y_1 + y_2 \leftrightarrow y'_1 + y'_2,\ \lambda y \leftrightarrow \lambda y'$. Consequently, this correspondence is isomorphism.

Glossary

linearly independent – линейно независимый; **basis** – базис
dimension – размерность; **transformation** – преобразование
coplanar – компланарный; **commutability** - коммутативность
one-to-one correspondence – взаимно однозначное соответствие

Exercises for Seminar 6

6.1. Let the set of systems of four numbers $\{(\xi_1,\ \xi_2,\ 0,\ 0) \mid \xi_1,\ \xi_2$ are real numbers$\}$ be given. The sum of every two elements is determined by $(\xi_1, \xi_2, \xi_3, \xi_4) + (\eta_1, \eta_2, \eta_3, \eta_4) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3, \xi_4 + \eta_4)$, and the product of every element on a real number is determined by $\lambda(\xi_1, \xi_2, \xi_3, \xi_4) = (\lambda\xi_1, \lambda\xi_2, \lambda\xi_3, \lambda\xi_4)$. Is the set a linear space?

6.2. Let the set of systems of four numbers $\{(\xi_1,\ \xi_2, 1, 1) \mid \xi_1,\ \xi_2$ are real numbers$\}$ be given. The sum of elements and the product of an element on a real number are determined as in 6.1. Does the set form a linear space?

6.3. Let $P_2^*(t) := \{\alpha_0 + \alpha_1 t + \alpha_2 t^2 \mid \alpha_0, \alpha_1, \alpha_2$ are real numbers, $\alpha_2 \neq 0\}$. Is $P_2^*(t)$ a linear space?

6.4. Let the set of all polynomials with degree ≤ 2 $\{\alpha_0 + \alpha_1 t + \alpha_2 t^2 \mid \alpha_0, \alpha_1, \alpha_2$ are real numbers$\}$ be given. Prove that the set is a linear space and the vectors $P_1 = 1 + 2t + 3t^2$, $P_2 = 2 + 3t + 4t^2$ and $P_3 = 3 + 5t + 7t^2$ are linearly dependent.

6.5. Prove that if *n* vectors of a linear space *x, y, z, …, u* are linearly dependent then *n* + 1 vectors of the linear space *x, y, z, …, u, v* are also linearly dependent.

6.6. Are the vectors $a = (2;3;-2)$, $b = (-1;4;5)$ and $c = (0;1;2)$ linearly independent?

6.7. Consider the linear space of all triples of ordered real numbers $x_1 = (\xi_{11}, \xi_{21}, \xi_{31})$, $x_2 = (\xi_{12}, \xi_{22}, \xi_{32})$, $x_3 = (\xi_{13}, \xi_{23}, \xi_{33})$, …, where the sum of vectors and the product of vector on a number are determined by $x_i + x_k = (\xi_{1i} + \xi_{1k}, \xi_{2i} + \xi_{2k}, \xi_{3i} + \xi_{3k})$; $\lambda x_i = (\lambda\xi_{1i}, \lambda\xi_{2i}, \lambda\xi_{3i})$. Prove that the vectors $a = (1;\, 1;\, 1)$, $b = (2;\, 0;\, -1)$ and $c = (3;-1;0)$ form a basis of the linear space. Find the coordinates of the vector $x = (5, -1, 2)$ in this basis.

6.8. Can a linear space consist of: 1) one vector; 2) two different vectors?

6.9. Prove that three coplanar vectors *a, b* and *c* are linearly dependent (Vectors are *coplanar* if they are parallel to the same plane).

6.10. Prove that three non-coplanar vectors *a, b* and *c* are linearly independent.

Exercises for Homework 6

6.11. Let the set of all pairs of real numbers $\{x = (\xi_1, \xi_2) \mid \xi_1, \xi_2$ are positive numbers$\}$ be given. The sum of every two elements $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ is determined by $x + y = (\xi_1\eta_1, \xi_2\eta_2)$, and the product of every element $x = (\xi_1, \xi_2)$ on a real number λ is determined by $\lambda x = (\xi_1^\lambda, \xi_2^\lambda)$. Is the set a linear space?

6.12. Let the set of all triples of integers $\{(\xi_1, \xi_2, \xi_3) \mid \xi_1, \xi_2, \xi_3$ are integers$\}$ be given. The sum of every two elements is determined by $(\xi_1, \xi_2, \xi_3) + (\eta_1, \eta_2, \eta_3) = (\xi_1 + \eta_1, \xi_2 + \eta_2, \xi_3 + \eta_3)$, and the product of every element on a real number λ is determined by $\lambda(\xi_1, \xi_2, \xi_3) = (\lambda\xi_1, \lambda\xi_2, \lambda\xi_3)$. Is the set a linear space?

6.13. When vectors $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ defined in 6.11 are linearly dependent?

6.14. Is the set of all the geometric vectors having the beginning in the origin of coordinates and being in the first octant a linear space?

6.15. Consider the linear space of all triples of ordered real numbers $x_1 = (\xi_{11}, \xi_{21}, \xi_{31})$, $x_2 = (\xi_{12}, \xi_{22}, \xi_{32})$, $x_3 = (\xi_{13}, \xi_{23}, \xi_{33})$, …, where the sum of vectors and the product of vector on a number are determined by $x_i + x_k = (\xi_{1i} + \xi_{1k}, \xi_{2i} + \xi_{2k}, \xi_{3i} + \xi_{3k})$; $\lambda x_i = (\lambda\xi_{1i}, \lambda\xi_{2i}, \lambda\xi_{3i})$. Prove that the vectors $a = (0;1;1)$, $b = (1;0;1)$ and $c = (1;1;0)$ form a basis of the linear space. Find the coordinates of the vector $x = (3;5;-2)$ for this basis.

6.16. Are the vectors $a_1 = (1;6;-3)$, $a_2 = (-2;-5;7)$ and $a_3 = (-1;1;4)$ linearly independent?

6.17. Show that elements $e_1 = (1;10)$ and $e_2 = (10;1)$ of the linear space introduced in 6.11 form a basis. Find the coordinates of the vector $x = (2;\, 3)$ in this basis.

6.18. Show that the set of all matrices of the second order is a 4-dimensional linear space.

6.19. Show that the matrices $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ form a basis of the linear space introduced in 6.18.

6.20. Prove that any four vectors *a, b, c* and *d* (of the geometric space) are linearly dependent.