LECTURE 11
Orthogonal transformations

A linear transformation A of a Euclidean space is called *orthogonal* if it preserves the scalar product of any two vectors x and y of the space, i.e. $(Ax, Ay) = (x, y)$.

The length of a vector x for an orthogonal transformation is not changed, i.e. $|Ax| = |x|$.

Thus, $\dfrac{(x,y)}{|x||y|} = \dfrac{(Ax,Ay)}{|Ax|\cdot|Ay|}$.

The last equality implies that an orthogonal transformation A does not change the angle between any two vectors x and y.

An orthogonal transformation transfers any orthonormal basis in orthonormal. Conversely, if a linear transformation transfers an orthonormal basis in orthonormal then it is orthogonal.

Example. The scalar product of elements $f_1 = A_1\sin x + B_1\cos x$ and $f_2 = A_2\sin x + B_2\cos x$ in the linear hull $L = L(\sin x, \cos x)$ has been introduced by the formula: $(f_1, f_2) = A_1A_2 + B_1B_2$.

Prove that a linear transformation D (differentiation) acting in L is orthogonal.

Solution: Find Df_1 and Df_2:

$$Df_1 = D(A_1\sin x + B_1\cos x) = A_1\cos x - B_1\sin x = -B_1\sin x + A_1\cos x$$
$$Df_2 = D(A_2\sin x + B_2\cos x) = A_2\cos x - B_2\sin x = -B_2\sin x + A_2\cos x$$

By definition of scalar product $(f_1, f_2) = A_1A_2 + B_1B_2$.

Then $(Df_1, Df_2) = (-B_1)(-B_2) + A_1A_2 = A_1A_2 + B_1B_2$. Thus, $(Df_1, Df_2) = (f_1, f_2)$, i.e. D is an orthogonal transformation in L.

Quadratic forms

A *quadratic form* of real variables $x_1, x_2, ..., x_n$ is a polynomial of the second degree according to these variables which does not contain a free term and terms of the first degree.

If $f(x_1, x_2, ..., x_n)$ is a quadratic form of variables $x_1, x_2, ..., x_n$ and λ is a real number then

$$f(\lambda x_1, \lambda x_2, ..., \lambda x_n) = \lambda^2 f(x_1, x_2, ..., x_n).$$

If $n = 2$ then $f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$.

If $n = 3$ then $f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$.

Further we give all necessary formulations and definitions for a quadratic form of three variables.

A matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ with $a_{ik} = a_{ki}$ for all $i, k = 1, 2, 3$ is called *matrix of quadratic form* $f(x_1, x_2, x_3)$, and the corresponding determinant – *determinant* of this *quadratic form*.

Since A is a symmetric matrix then the roots λ_1, λ_2 and λ_3 of the characteristic equation $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$ are real numbers.

Let

$$e_1' = b_{11}e_1 + b_{21}e_2 + b_{31}e_3,$$
$$e_2' = b_{12}e_1 + b_{22}e_2 + b_{32}e_3,$$
$$e_3' = b_{13}e_1 + b_{23}e_2 + b_{33}e_3$$

be normalized eigenvectors corresponding to characteristic numbers $\lambda_1, \lambda_2, \lambda_3$ in an orthonormal basis e_1, e_2, e_3. Then the vectors e_1', e_2', e_3' form an orthonormal basis.

The matrix $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ is the transition matrix from the basis e_1, e_2, e_3 to the basis e_1', e_2', e_3'.

The formulas of transformation of coordinates at transition to the new orthonormal basis have the following form:

$$x_1 = b_{11}x'_1 + b_{12}x'_2 + b_{13}x'_3,$$
$$x_2 = b_{21}x'_1 + b_{22}x'_2 + b_{23}x'_3,$$
$$x_3 = b_{31}x'_1 + b_{32}x'_2 + b_{33}x'_3.$$

Transforming by these formulas the quadratic form $f(x_1, x_2, x_3)$ we obtain the following quadratic form: $f(x'_1, x'_2, x'_3) = \lambda_1 x_1'^2 + \lambda_2 x_2'^2 + \lambda_3 x_3'^2$. It does not contain terms with $x'_1x'_2,\ x'_1x'_3,\ x'_2x'_3$.

We say that a quadratic form $f(x_1, x_2, x_3)$ has been reduced to a *canonical type* by an orthogonal transformation *B*.

Here arguments were performed in assumption that characteristic numbers $\lambda_1, \lambda_2, \lambda_3$ are different. At solving exercises we show how it should act if there are identical numbers among characteristic ones.

Example. Reduce the quadratic form $f = 27x_1^2 - 10x_1x_2 + 3x_2^2$ to a canonical type.

Solution: Here $a_{11} = 27, a_{12} = -5, a_{22} = 3.$ Compose the characteristic equation:

$$\begin{vmatrix} 27-\lambda & -5 \\ -5 & 3-\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - 30\lambda + 56 = 0$$

Solving this equation we obtain the characteristic numbers $\lambda_1 = 2, \lambda_2 = 28.$

Find eigenvectors. If $\lambda = 2$ we obtain the following system of equations: $\begin{cases} 25\xi_1 - 5\xi_2 = 0 \\ -5\xi_1 + \xi_2 = 0 \end{cases}$.

Thus, $\xi_2 = 5\xi_1$. Assuming $\xi_1 = c$ we have $\xi_2 = 5c$, i.e. an eigenvector corresponding to the characteristic number $\lambda = 2$ is $u = c(e_1 + 5e_2)$.

If $\lambda = 28$ we obtain the following system of equations: $\begin{cases} -\xi_1 - 5\xi_2 = 0 \\ -5\xi_1 - 25\xi_2 = 0 \end{cases}$.

And consequently we obtain an eigenvector $v = c(-5e_1 + e_2)$ corresponding to $\lambda = 28.$

In order to normalize the vectors *u* and *v* we should take $c = \dfrac{1}{\sqrt{1^2 + 5^2}} = \dfrac{1}{\sqrt{26}}$.

Thus, we have found normalized eigenvectors: $e'_1 = \dfrac{1}{\sqrt{26}}e_1 + \dfrac{5}{\sqrt{26}}e_2,\ e'_2 = -\dfrac{5}{\sqrt{26}}e_1 + \dfrac{1}{\sqrt{26}}e_2.$

The transition matrix from the orthonormal basis e_1, e_2 to the orthonormal basis e'_1, e'_2 is the following: $B = \begin{pmatrix} 1/\sqrt{26} & -5/\sqrt{26} \\ 5/\sqrt{26} & 1/\sqrt{26} \end{pmatrix}$.

Then formulas of transformation of coordinates are the following:

$$x_1 = \frac{1}{\sqrt{26}}x'_1 - \frac{5}{\sqrt{26}}x'_2,\ x_2 = \frac{5}{\sqrt{26}}x'_1 + \frac{1}{\sqrt{26}}x'_2.$$

Thus, $f = 27\left(\dfrac{1}{\sqrt{26}}x'_1 - \dfrac{5}{\sqrt{26}}x'_2\right)^2 - 10\left(\dfrac{1}{\sqrt{26}}x'_1 - \dfrac{5}{\sqrt{26}}x'_2\right)\left(\dfrac{5}{\sqrt{26}}x'_1 + \dfrac{1}{\sqrt{26}}x'_2\right) +$

$$+3\left(\frac{5}{\sqrt{26}}x'_1 + \frac{1}{\sqrt{26}}x'_2\right)^2 = 2x_1'^2 + 28x_2'^2.$$

This result can be obtained at once since $f = \lambda_1 x_1'^2 + \lambda_2 x_2'^2$.

Example. Reduce the quadratic form $f = 6x_1^2 + 3x_2^2 + 3x_3^2 + 4x_1x_2 + 4x_1x_3 - 8x_2x_3$ to a canonical type.

Solution: Here $a_{11} = 6, a_{22} = 3, a_{33} = 3, a_{12} = 2, a_{13} = 2, a_{23} = -4.$ Compose the characteristic equation:

$$\begin{vmatrix} 6-\lambda & 2 & 2 \\ 2 & 3-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

Solving this equation we obtain the characteristic numbers $\lambda_1 = \lambda_2 = 7, \lambda_3 = -2.$

Find eigenvectors. If $\lambda = 7$ we obtain the following system of equations: $\begin{cases} -\xi_1 + 2\xi_2 + 2\xi_3 = 0 \\ 2\xi_1 - 4\xi_2 - 4\xi_3 = 0 \\ 2\xi_1 - 4\xi_2 - 4\xi_3 = 0 \end{cases}$.

It is reduced to one equation $\xi_1 = 2\xi_2 + 2\xi_3$.

The solution of this system can be written as: $\xi_1 = 2a + 2b,\ \xi_2 = a,\ \xi_3 = b.$

Thus, we obtain the family of eigenvectors $u = 2(a+b)e_1 + ae_2 + be_3$ depending on two parameters *a* and *b*.

If $\lambda = -2$ we obtain the following system of equations: $\begin{cases} 8\xi_1 + 2\xi_2 + 2\xi_3 = 0 \\ 2\xi_1 + 5\xi_2 - 4\xi_3 = 0 \\ 2\xi_1 - 4\xi_2 + 5\xi_3 = 0 \end{cases}$.

Solving for example the last two equations we have $\xi_1 / 9 = \xi_2 /(-18) = \xi_3 /(-18),$ or

$$\xi_1 = -\xi_2 / 2 = -\xi_3 / 2;\ \xi_1 = c,\ \xi_2 = -2c,\ \xi_3 = -2c.$$

Thus, we obtain the one-parameter family of eigenvectors $v = c(e_1 - 2e_2 - 2e_3)$.

Let's select two orthogonal vectors from the family of eigenvectors $u = 2(a+b)e_1 + ae_2 + be_3$. Assuming for example *a* = 0, *b* = 1, we obtain an eigenvector $u_1 = 2e_1 + e_3$. Choose parameters *a* and *b* in order to hold the equality (*u*, *u*1) = 0. Then we obtain the equation $2 \cdot 2(a+b) + b = 0,$ i.e. 4*a* + 5*b* = 0. Now we can take *a* = 5, *b* = – 4. Thus, we find another vector of the considered family: $u_2 = 2e_1 + 5e_2 - 4e_3$.

Thus, we obtain three pairwise orthogonal vectors: $u_1 = 2e_1 + e_3,\ u_2 = 2e_1 + 5e_2 - 4e_3,$ $v = e_1 - 2e_2 - 2e_3$. The eigenvectors *u*1 and *u*2 correspond to the characteristic number λ = 7, and the eigenvector *v* corresponds to the characteristic number λ = – 2 (for *c* = 1).

Normalizing these vectors we obtain a new orthonormal basis and the transition matrix to a new basis is the following: $B = \begin{pmatrix} 2/\sqrt{5} & 2/(3\sqrt{5}) & 1/3 \\ 0 & \sqrt{5}/3 & -2/3 \\ 1/\sqrt{5} & -4/(3\sqrt{5}) & -2/3 \end{pmatrix}$.

Applying the formulas of transformation of coordinates

$$x_1 = \frac{2}{\sqrt{5}} x_1' + \frac{2}{3\sqrt{5}} x_2' + \frac{1}{3} x_3',\ x_2 = \frac{\sqrt{5}}{3} x_2' - \frac{2}{3} x_3',\ x_3 = \frac{1}{\sqrt{5}} x_1' - \frac{4}{3\sqrt{5}} x_2' - \frac{2}{3} x_3'$$

to our quadratic form we obtain $f = 7x_1'^2 + 7x_2'^2 - 2x_3'^2$.

Law of inertia of quadratic forms: Both the number of positive canonical coefficients and the number of negative canonical coefficients are constant and don't depend on way of reducing a quadratic form to a canonical type.

Definite quadratic forms (quadratic forms of fixed sign)

A quadratic form $Q(x_1, x_2, ..., x_n)$ is called *positive definite* (*negative definite*) if for all values $x_1, x_2, ..., x_n$ the condition $Q(x_1, x_2, ..., x_n) \geq 0$ ($Q(x_1, x_2, ..., x_n) \leq 0$) holds and $Q(x_1, x_2, ..., x_n) = 0$ only for $x_1 = x_2 = ... = x_n = 0$.

For example, $Q(x_1, x_2) = 2x_1^2 + 3x_2^2$ is positive definite; $Q(x_1, x_2) = -x_1^2 - 2x_2^2$ is negative definite. Positive definite and negative definite quadratic forms are called *definite*.

A quadratic form $Q(x_1, x_2, ..., x_n)$ is called *quasi-definite* (either *non-negative* or *non-positive*) if it takes either only non-negative values or non-positive values, but it takes 0 not only for $x_1 = x_2 = ... = x_n = 0$.

For example, $Q(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$ is non-negative since $Q(x_1, x_2) \geq 0$ for all x_1, x_2, but $Q(x_1, x_2) = 0$ not only for $x_1 = x_2 = 0$; so $Q(1, 1) = 0$.

A quadratic form is called *alternating* if it takes both positive and negative values.

For example, $Q(x_1, x_2) = -x_1^2 + 2x_2^2$ is alternating since it takes both positive and negative values: $Q(1,\ 0) = -1 < 0,\ Q(0,\ 1) = 2 > 0.$

Criterion of Sylvester for definiteness of a quadratic form

The determinants

$$\Delta_1 = a_{11},\ \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \ldots,\ \Delta_k = \begin{vmatrix} a_{11} & \ldots & a_{1k} \\ \ldots & \ldots & \ldots \\ a_{k1} & \ldots & a_{kk} \end{vmatrix}, \ldots,\ \Delta_n = \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

are called *angular minors* of a matrix $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}.$

Theorem (Sylvester criterion).

1. A quadratic form is positive definite if and only if all the angular minors of its matrix are positive.
2. A quadratic form is negative definite if and only if the signs of angular minors alternate as follows: $\Delta_1 < 0,\ \Delta_2 > 0,\ \Delta_3 < 0, \ldots.$

Example. Determine whether the quadratic form $Q(x_1, x_2, x_3) = x_2^2 - 4x_1x_2 + 2x_2x_3$ is definite.

Solution: Compose the matrix of the quadratic form $A = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and calculate its angular minors $\Delta_1 = 0,\ \Delta_2 = -4,\ \Delta_3 = 0.$ By the Sylvester criterion the present quadratic form is neither positive definite nor negative definite, i.e. is not definite.

Reducing an equation of curve of the second order to a canonic type

Let an equation of a curve of the second order in a rectangular system of coordinates Oxy be given:
$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2b_1x + 2b_2y + c = 0$ (1)
It is required by turning and parallel transfer of axes of coordinates to pass to such a rectangular system of coordinates in which the equation of the curve has a canonic type.
Consider the quadratic form connected with the equation (1): $a_{11}x^2 + 2a_{12}xy + a_{22}y^2$.

Its matrix is $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$. Reduce the quadratic form to a canonic type $\lambda_1(x')^2 + \lambda_2(y')^2$ by an orthogonal transformation of variables: $\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} x' \\ y' \end{pmatrix}$ (2)

Recall that λ_1, λ_2 are eigen-values of the matrix A, and the columns of the matrix P are orthogonal normalized eigen-vectors (columns) of the matrix A. The matrix P by properties of orthogonal matrices has the following type: $P = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, i.e. P is the matrix of operator of turning on angle φ in the space V_2 of vectors on plane. At such a turning the rectangular system Oxy with the coordinate vectors $\vec{i}, \vec{j}$ (a basis in the space V_2) transits to a rectangular system $Ox'y'$ with the coordinate vectors $\vec{i}\,', \vec{j}\,'$ (another basis in the space V_2), and $(\vec{i}\,'\ \ \vec{j}\,') = (\vec{i}\ \ \vec{j})P.$ Using the formulas (2), express the linear terms $2b_1x + 2b_2y$ of the equation (1) by the coordinates x', y'. In result in the system $Ox'y'$ the equation of the curve takes the following type: $\lambda_1(x')^2 + \lambda_2(y')^2 + 2b_1'x' + 2b_2'y' + c = 0$, i.e. there is a mixed term (with the product $x'y'$) in the equation. Further, extracting complete squares on both variables by parallel transfer of axes of coordinates of the system $Ox'y'$ pass to the system $Ox''y''$ in which the equation of the curve has a canonic type.

An equation of a surface of the second order can be reduced to a canonic type by analogy.

__Example__. Reduce the equation of the curve of second order $11x^2-20xy-4y^2-20x-8y+1=0$ to a canonic type by turning the axes of coordinates of the system Oxy and the consequent parallel transfer.

Solution: Reduce the quadratic form $11x^2-20xy-4y^2$ by an orthogonal transformation to a canonic type. Compose the matrix of the quadratic form: $A=\begin{pmatrix} 11 & -10 \\ -10 & -4 \end{pmatrix}$ and write the characteristic equation: $\begin{vmatrix} 11-\lambda & -10 \\ -10 & -4-\lambda \end{vmatrix}=\lambda^2-7\lambda-144=0$. It has the roots: $\lambda_1=-9, \lambda_2=16$. Further find mutually orthogonal normalized eigen-vectors (columns) F_1 and F_2 of the matrix A: if $\lambda_1=-9$ then $F_1=\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$; if $\lambda_2=16$ then $F_2=\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$. Consequently, the required orthogonal transformation has the matrix $P=\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$. The matrix P is the matrix of operator of turning on angle φ such that $\cos\varphi=1/\sqrt{5}$, $\sin\varphi=2/\sqrt{5}$. Turning the axes of coordinates of the system Oxy on angle $\varphi=\arccos(1/\sqrt{5})$ (by anticlockwise way), we obtain the rectangular system $Ox'y'$. At this orthogonal transformation the quadratic form transits to the form

$$\lambda_1(x')^2+\lambda_2(y')^2=-9(x')^2+16(y')^2$$

Write in new coordinates the linear terms of the equation: $-20x-8y=-\frac{36}{\sqrt{5}}x'+\frac{32}{\sqrt{5}}y'$. In the system of coordinates $Ox'y'$ the equation of the curve takes the following type:

$$-9(x')^2+16(y')^2-\frac{36}{\sqrt{5}}x'+\frac{32}{\sqrt{5}}y'+1=0$$

Extracting complete squares on both variables, we obtain $-9\left(x'+\frac{2}{\sqrt{5}}\right)^2+16\left(y'+\frac{1}{\sqrt{5}}\right)^2+5=0$.

Assuming $x''=x'+2/\sqrt{5}$, $y''=y'+1/\sqrt{5}$, i.e. doing parallel transfer of axes of coordinates so that the origin of coordinates transits to the point $O'(-2/\sqrt{5}, -1/\sqrt{5})$, come to a canonic type:

$$\frac{(x'')^2}{5/9}-\frac{(y'')^2}{5/16}=1$$

It is a canonic equation of hyperbola in the system of coordinates $O'x''y''$.

Glossary

canonic type – канонический вид; **quadratic form of fixed sign** – знакоопределенная форма
positive definite form – положительно определенная форма
negative definite form – отрицательно определенная форма
quasi-definite form – квазизнакоопределенная форма
alternating form – знакопеременная форма; **angular minor** – угловой минор

Exercises for Seminar 11

11.1. Is a transformation $Ax=-\xi_1e_1+\xi_2e_2+\xi_3e_3+\xi_4e_4$ orthogonal (where $x=\xi_1e_1+\xi_2e_2+\xi_3e_3+\xi_4e_4$ is an arbitrary vector and e_1,e_2,e_3,e_4 is an orthonormal basis)?

11.2. Is a transformation which is turning any vector lying in plane *xOy* on a fixed angle α orthogonal?

11.3. Reduce the quadratic form $f = 2x_1^2 + 8x_1x_2 + 8x_2^2$ to a canonical type.

11.4. Reduce the quadratic form $f = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ to a canonical type.

11.5. Reduce the quadratic form $f = 17x^2 + 12xy + 8y^2$ to a canonical type.

11.6. Reduce the quadratic form $f = 6x^2 + 2\sqrt{5}xy + 2y^2$ to a canonical type.

11.7. Determine whether the following quadratic form is definite:

$$Q(x_1, x_2, x_3) = 4x_1^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + 2x_3^2$$

11.8. Determine whether the following quadratic form is positive definite:

$$Q(x_1, x_2, x_3) = 17x_1^2 - 16x_1x_2 + 8x_1x_3 + 17x_2^2 - 8x_2x_3 + 11x_3^2$$

11.9. Find all values of the parameter a for which the following quadratic form is positive definite: $Q(x_1, x_2, x_3) = x_1^2 - 2x_1x_2 - 2x_1x_3 + 4x_2^2 + 2x_2x_3 + ax_3^2$

11.10. Find all values of the parameter b for which the following quadratic form is definite:

$$Q(x_1, x_2, x_3) = 4x_1^2 + 2x_1x_2 + 3x_2^2 - 2x_1x_3 - 2x_2x_3 + bx_3^2$$

11.11. Reduce the equation of the curve of second order $6xy + 8y^2 - 12x - 26y + 11 = 0$ to a canonic type by turning of axes of coordinates of the system Oxy and the consequent parallel transfer. Find the angle of turning and the coordinates of new origin of coordinates (the point O') in the system of coordinates $Ox'y'$ obtained in result of turning the axes of coordinates of Oxy. Find the type of the curve.

Exercises for Homework 11

11.12. What values λ is a transformation A defined by the equality $Ax = \lambda x$ orthogonal for?

11.13. Let $e_1, e_2, e_3, e_4, e_5, e_6$ be an orthonormal basis. Prove that A is an orthogonal transformation if

$$Ae_1 = e_1,\ Ae_2 = -e_2,\ Ae_3 = e_3\cos\alpha + e_4\sin\alpha,\ Ae_4 = -e_3\sin\alpha + e_4\cos\alpha,\ Ae_5 = e_5\cos\beta + e_6\sin\beta,$$
$$Ae_6 = -e_5\sin\beta + e_6\cos\beta.$$

11.14. Reduce the quadratic form $f = 4xy + 3y^2$ to a canonical type.

11.15. Reduce the quadratic form $f = 5x^2 + 4\sqrt{6}xy + 7y^2$ to a canonical type.

11.16. Reduce the quadratic form $f = 3x_2^2 + 3x_3^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3$ to a canonical type.

11.17. Reduce the quadratic form $f = x_2^2 - x_3^2 + 4x_1x_2 - 4x_1x_3$ to a canonical type.

11.18. Determine whether the following quadratic form is definite:

$$Q(x_1, x_2, x_3) = 4x_1^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + x_3^2$$

11.19. Determine whether the following quadratic form is positive definite:

$$Q(x_1, x_2, x_3) = 2x_1^2 + x_2^2 - 4x_1x_2 - 4x_2x_3$$

11.20. Find all values of the parameter b for which the following quadratic form is definite: $Q(x_1, x_2, x_3) = -2x_1^2 - 6x_1x_2 + 6x_1x_3 - 5x_2^2 + 10x_2x_3 + bx_3^2$

11.21. Find all values of the parameter a for which the following quadratic form is positive definite: $Q(x_1, x_2, x_3) = 4x_1^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3 + ax_3^2$.

11.22. Reduce the equation of the curve of second order $9x^2 + 24xy + 16y^2 - 40x + 30y = 0$ to a canonic type by turning the axes of coordinates of the system Oxy and the consequent parallel transfer. Find the angle of turning and the coordinates of new origin of coordinates (the point O') in the system of coordinates $Ox'y'$ obtained in result of turning the axes of coordinates of Oxy. Find the type of the curve.