LECTURE 5

The set of real numbers. The set of complex numbers. Actions over complex numbers.

1. Rational numbers

Natural numbers $N = \{1; 2; 3; \ldots\}$ have appeared in connection with necessity of calculation of subjects, i.e. with necessity to answer the following question: «How many elements does a given finite set contain?».

However the following situation is possible: one of shelves of a bookcase is free from books, i.e. the set of books on this shelf is an empty set. In this case the number zero enables to answer the question: «How many elements does the given set contain?».

If we attach the number 0 to N we receive *the set of non-negative integers* $\{0; 1; 2; 3; \ldots\}$. Non-negative integers have appeared insufficiently for solving problems put by practice. So, in order to characterize the temperature of air above and below zero opposite numbers are required. For example, the temperature of air in six degrees of heat and six degrees of frost characterize accordingly + 6°C and – 6°C. The numbers 6 and – 6 are called *opposite*: – 6 is opposite to 6, and 6 is opposite to – 6.

Natural numbers, zero, and also the numbers that are opposite to natural, make Z – *the set of integers*: $\{0; 1; 2; 3; \ldots\} \cup \{-1; -2; -3; \ldots\}$.

Measuring quantities has led to necessity of expanding the set of integers by introducing the fractional numbers. For example, the academic hour (lesson period) proceeds 45 minutes or ¾ hour, and break – 5 minutes, or 1/12 hours.

The integers and fractional numbers make Q – *the set of rational numbers*. Any rational number can be written down as fraction m/n, where $m \in Z$ and $n \in$ N. An arbitrary rational number can be written down by different fractions, for example, $\frac{2}{3} = \frac{4}{6} = \frac{10}{15}$. Among the fractions representing the given rational number, always there is an irreducible fraction, for example, for our case this is $\frac{2}{3}$.

Let a rational number m/n be given. Dividing the numerator on the denominator, we receive finite or infinite decimal fraction. For example, $\frac{1}{2} = 0{,}5$; $\frac{1}{3} = 0{,}3333\ldots$.

Thus, any rational number is presented as an infinite decimal fraction: $a_0, a_1a_2a_3 \ldots$ where a_0 – an integer, and each of $a_1, a_2, a_3, \ldots$ is one of digits 0, 1, 2, …, 9.

2. Real numbers

The set of all infinite decimal fractions is called *the set of real numbers* and is denoted by R. The set Q of all rational numbers is a subset of the set R. The real numbers which are not rational are called *irrational*.

Theorem. There is no rational number of which the square is equal to 2.

Proof. Assume the contrary: there is a rational number of which the square is equal to 2, and it is presented by an irreducible fraction m/n.

Then we have: $\left(\frac{m}{n}\right)^2 = 2$, or $m^2 = 2n^2$ (1)

Thus, m^2 is an even number. And then the number m is also even. In fact, if $m = 2k + 1$ (i.e. m would be odd) then $m^2 = (4k^2 + 4k) + 1$ is odd since $(4k^2 + 4k)$ is even. But if m is even then $m = 2k$ and by formula (1) we have $n^2 = 2k^2$, i.e. the number n^2 is even, and consequently the number n is also even. Thus, our assumption implies that both numbers m and n are even contradicting the hypothesis on irreducibility of the fraction m/n.

Consequently, the number $\sqrt{2}$ is irrational.

The number π which is the ratio of the length of a circle to its diameter ($\pi = 3,\ 1415926\ldots$) is an example of irrational number.

The *module* of a real number x is such a non-negative number denoting by $|x|$:

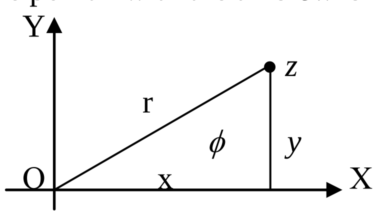
$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

For example, $|-5| = 5$, $|3| = 3$.

Let's consider a horizontal straight line on plane. Choose on it positive direction (from left to right) which we will denote by an arrow, then we choose the origin of reference – a point O and another arbitrary point A. The line OA is called *a numeric line*.
Every real number x can be depicted by a point M on the line OA if we construct the segment OM of length $|x|$ by putting it to the right side from the point O in the case $x > 0$ and to left from the point O in the case $x < 0$. The zero is the image of the point O. Therefore each real number is answered with quite certain point of the numeric line.
And the converse is true: each point of the numeric line is answered with the certain real number (the image of which is this point). Often the set R of real numbers is named a numeric line, and the real numbers – points of this line.

3. Complex numbers
The set of real numbers is not enough for solving any algebraic equation. Indeed, the following equations have no solutions on the set of real numbers: $x^2 + 1 = 0$, $x^2 + x + 1 = 0$ and etc. To overcome the specified difficulty we enter the set of complex numbers which includes the set of real numbers as a subset. Since we wish that the equation $x^2 + 1 = 0$ have a solution in the set of complex numbers it is necessary to enter some new number and think it as a solution of the equation. We denote such new number by symbol i and we call it by *imaginary unit*. Thus, $i^2 + 1 = 0$, или $i^2 = -1$.
Complex numbers are expressions of the following form $a + bi$ (a and b are real numbers, i is the imaginary unit).
Two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ are equal $\Leftrightarrow$ $a_1 = a_2$ and $b_1 = b_2$.
The sum of numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is called the number $a_1 + a_2 + (b_1 + b_2)\, i$.
The product of numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is called the number $a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1)\, i$.
The set of all complex numbers is denoted by C. The set of real numbers R is a subset of the set of complex numbers, i.e. $R \subset C$. A real number a is the *real part* of a complex number $a + bi$. A real number b is the *imaginary part* of the complex number $a + bi$.
Numbers $a + bi$ and $a - bi$, i.e. numbers differing only the sign of the imaginary part, are called *conjugate* complex numbers.
The module of a complex number $z = a + bi$ is denoted by $|z|$ and is determined by the formula $|z| = \sqrt{a^2 + b^2}$.

Let's consider a plane with the rectangular system of coordinates Oxy. The point of a plane $z\ (x, y)$ can be put to each complex number $z = x + yi$ in correspondence, and this correspondence is one-to-one. A plane on which such correspondence is realized is called *a complex plane*. The *real* numbers $z = x + 0i = x$ are located on the axis Ox; and therefore it is called *the real axis*. *Pure imaginary* numbers $z = 0 + yi = yi$ are located on the axis Oy; it is called *the imaginary axis*.
Observe that $r = |z|$ represents the distance between the point z and the origin of coordinates. Every point z is corresponded the radius-vector $\overrightarrow{Oz}$ of the point z. The angle formed by the radius-vector of the point z with the axis Ox is called the *argument* $\phi = Arg\, z$ of the point. Here $-\infty < Arg\, z < +\infty$.



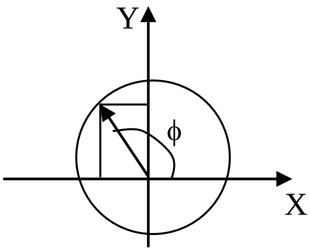
Every solution ϕ of the system of equations $\cos\phi = \dfrac{x}{\sqrt{x^2 + y^2}}$, $\sin\phi = \dfrac{y}{\sqrt{x^2 + y^2}}$ $(*)$ is called the argument of a complex number $z = x + yi \neq 0$. All the arguments of a number z are differed on whole

multiples 2π and are denoted by one symbol *Arg z*. The value of *Arg z* satisfying the condition $0 \le Arg\, z < 2\pi$ is called the *principal value* of the argument and is denoted by *arg z*.
The formulas (*) imply that for every complex number z the following equality is true:

$$z = |z|\,(\cos\phi + i\sin\phi)$$

It is called the *trigonometric form* of the number z.
And the form of a complex number $z = x + yi$ is called *algebraic*.
Example 1. Represent the complex number $z = -2 + 2i\sqrt{3}$ in trigonometric form.

Solution. $|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$, $\cos\phi = -\dfrac{1}{2}$, $\sin\phi = \dfrac{\sqrt{3}}{2}$



The principal value of the argument is equal to $arg\, z = 2\pi/3$,
Consequently, the required trigonometric form has the following form: $z = 4(\cos 2\pi/3 + i\sin 2\pi/3)$.

Theorem 1. The module of the product of complex numbers is equal to the product of the modules of these numbers, and the argument of the product is equal to the sum of the arguments of the multipliers, i.e.

$$|z_1 \cdot z_2| = r_1 \cdot r_2 = |z_1| \cdot |z_2|, \quad Arg\, z_1 \cdot z_2 = \phi_1 + \phi_2 = Arg\, z_1 + Arg\, z_2.$$

Proof.

$$z_1 = r_1(\cos\phi_1 + i\sin\phi_1), \quad z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$$

$$z_1 \cdot z_2 = r_1 \cdot r_2[(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) + i(\sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2)] =$$

$$= r_1 \cdot r_2[\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)].$$

Corollary. The module of an integer positive power of a complex number is equal to this power of the module of the number, and the argument of the power is equal to the argument of the number multiplied on the power exponent, i.e. $|z^n| = |z|^n$, $Arg\, z^n = n\, Arg\, z$ (n is an integer positive number):

$$z^n = [r(\cos\phi + i\sin\phi)]^n = r^n(\cos n\phi + i\sin n\phi) \text{ – Moivre formula}$$

Theorem 2. The module of the quotient of two complex numbers is equal to the quotient of the modules of these numbers, and the argument of the quotient is equal to the difference of the arguments of the dividend and the divisor, i.e

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}, \quad Arg\,\frac{z_1}{z_2} = \phi_1 - \phi_2 = Arg\, z_1 - Arg\, z_2.$$

Extracting the root from a complex number

Let $\sqrt[n]{z} = \rho(\cos\psi + i\sin\psi)$, where $z = r(\cos\phi + i\sin\phi)$.
Then by Corollary we have: $z = [\rho(\cos\psi + i\sin\psi)]^n = \rho^n(\cos n\psi + i\sin n\psi)$.
Then $\rho^n = r$, $n\psi = \phi + 2\pi k$ ($k = 0, \pm 1, \pm 2, \ldots$).

Consequently, $\rho = \sqrt[n]{r} = \sqrt[n]{|z|}$, $\psi = Arg\sqrt[n]{z} = \dfrac{\phi + 2\pi k}{n}$.

Here for number k we can take only values $k = 0, 1, 2, \ldots, n-1$ since at all other values k we obtain recurrences of already found values of the root.
Consequently, we have finally:

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\phi + 2\pi k}{n} + i\sin\frac{\phi + 2\pi k}{n}\right), \; k = 0,1,2,\ldots,n-1.$$

Thus, the root of the n–th power from any complex number $z \ne 0$ has exactly n different values.
Example. Find $w = \sqrt[3]{-1+i}$.

Solution: .

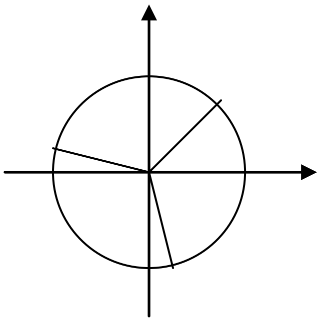
$$r=\sqrt{(-1)^2+1^2}=\sqrt{2};\ \cos\phi=-\frac{1}{\sqrt{2}},\ \sin\phi=\frac{1}{\sqrt{2}}\Rightarrow\phi=\frac{3\pi}{4}\Rightarrow -1+i=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right).$$

$$w=\sqrt[3]{\sqrt{2}}\left(\cos\frac{\frac{3\pi}{4}+2\pi k}{3}+i\sin\frac{\frac{3\pi}{4}+2\pi k}{3}\right)=\sqrt[6]{2}\left[\cos\left(\frac{\pi}{4}+\frac{2\pi k}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2\pi k}{3}\right)\right],\ k=0,1,2.$$

$$w_0=\sqrt[6]{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$w_1=\sqrt[6]{2}\left(\cos\frac{11}{12}\pi+i\sin\frac{11}{12}\pi\right)$$

$$w_2=\sqrt[6]{2}\left(\cos\frac{19}{12}\pi+i\sin\frac{19}{12}\pi\right).$$



The points w_0, w_1, w_2 represent equidistant each other points located on the circumference of raduis $\sqrt[6]{2}$.

4. Field. A set F of elements for which two algebraic operations: addition and multiplication have been determined (such that the sum $a + b \in F$ and the product $ab \in F$ for any two elements $a, b \in F$) is called a *field* if the following conditions hold:

1. $a + b = b + a$ for all $a, b \in F$ (the addition is commutative).
2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in F$ (the addition is associative).
3. There is zero-element $0 \in F$ such that $a + 0 = a$ for every $a \in F$.
4. For every $a \in F$ there is such an element (*opposite* for a) $-a$ that $a + (-a) = 0$.
5. $ab = ba$ for all $a, b \in F$ (the multiplication is commutative).
6. $(ab)c = a(bc)$ for all $a, b, c \in F$ (the multiplication is associative).
7. There is unit $1 \in F$ such that $a \cdot 1 = a$ for every $a \in F$.
8. For every non-zero element $a \in F$ there is such an element (*inverse* for a) a^{-1} that $aa^{-1} = 1$.
9. $(a + b)c = ac + bc$ for all $a, b, c \in F$ (the multiplication is distributive regarding to the addition).

Examples of fields: Q (the field of rational numbers), R (the field of real numbers) and C (the field of complex numbers).

Glossary

irreducible – несократимый; **numerator** – числитель; **denominator** – знаменатель
the origin of reference – начало отсчета; **imaginary unit** – мнимая единица
conjugate – сопряженный; **principal value** – главное значение
power – степень; **exponent** – показатель (степени)
quotient – частное; **dividend** – делимое; **divisor** – делитель
recurrence – повторение; **equidistant** – равноотстоящий
circumference – окружность; **field** – поле

Exercises for Seminar 5

5.1. Perform the following actions:
a) $(3 + 4i) + (2 - 5i)$; b) $(7 - 4i) - (9 - 2i)$; c) $(2 + 3i)(3 - 2i)$; d) $(1 - 2i)(3 - 4i)$;
e) $(3 - 2i)^2$; f) $\dfrac{1+i}{1-i}$.

5.2. Solve the equations:
a) $x^2 + 25 = 0$; b) $x^2 - 2x + 5 = 0$.

5.3. Write the following complex numbers in trigonometric form:
a) $z = 3$; b) $z = -2i$; c) $z=1+i\sqrt{3}$; d) $z=-\sqrt{2}+i\sqrt{2}$

5.4. Calculate:
a) $(1 + i)^{10}$; b) $(1-i\sqrt{3})^6$; c) $\left(1+\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^4$; d) $(\sqrt{3}+i)^3$

Direction for (c): Use the following trigonometric formulas:

$$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2};\ \sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2};\ \cos^2\frac{x}{2}=\frac{1+\cos x}{2}$$

5.5. Find all the values $z=\sqrt[6]{1}$ and depict them as radius-vectors by constructing a circle of radius 1.

5.6. Find:

a) $\sqrt[3]{1}$; b) $\sqrt[3]{i}$; c) $\sqrt[6]{-1}$; d) $\sqrt[3]{-2+2i}$

Exercises for Homework 5

5.7. Perform the following actions:

a) $(2 + 7i) + (5 – 6i)$; b) $(4 – 7i) – (3 – 5i)$; c) $(3 – 4i)(5 – 3i)$; d) $(7 + 6i)(3 + 2i)$;

e) $(1 + i)^3$; f) $\frac{2i}{1+i}$.

5.8. Solve the equations:

a) $x^6 + 64 = 0$; b) $x^2 + 4x + 13 = 0$.

5.9. Write the following complex numbers in trigonometric form:

a) $z = 5$; b) $z = – i$; c) $z=-1+i\sqrt{3}$; d) $z=-\sqrt{3}-i$

5.10. Calculate:

a) $(1 – i)^6$; b) $(2+i\sqrt{12})^5$; c) $\left(1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6$; d) $(-1+i)^4$

5.11. Find all the values $z=\sqrt[4]{-1}$ and depict them as radius-vectors by constructing a circle of radius 1.

5.12. Find:

a) $\sqrt[5]{1}$; b) $\sqrt{i}$; c) $\sqrt[3]{-1+i}$; d) $\sqrt[4]{-8+8i\sqrt{3}}$