LECTURE 12
Conjugate transformations in a Euclidean space

Consider two distinct orthonormal bases $\{e_1, e_2, ..., e_n\}$ and $\{e'_1, e'_2, ..., e'_n\}$ in a Euclidean space E^n with a transition matrix S from the first basis to the second one. Since in these bases the Gram matrix is identity, we have that the expression $\Gamma_{e'} = S^T \Gamma_e S$ implies $E = S^T S$. Since the transition matrix S is regular, we have $S^{-1} = S^T$. Recall that a matrix Q satisfying to $Q^{-1} = Q^T$ is called *orthogonal*.

Properties of orthogonal matrices:

1) $Q^T Q = QQ^T = E$

2) $\det Q = \pm 1$

3) Orthogonal matrices (and only they) can be transition matrices from one orthonormal basis to another orthonormal basis.

4) Eigen-values of a linear transformation having an orthogonal matrix are equal to 1 by modulus (absolute value).

Check the last assertion: The equality $A_g f_g = \lambda f_g$ implies $f_g^T A_g^T = \lambda f_g^T$. Multiplying these equalities termwise we obtain $f_g^T A_g^T A_g f_g = \lambda^2 f_g^T f_g$. By orthogonality of A_g we have $A_g^T A_g = E$, and therefore $f_g^T f_g = \lambda^2 f_g^T f_g$, i.e. $\lambda^2 = 1$ (since eigen-vectors are non-zero).

A linear transformation A^+ given in a Euclidean space E is called *conjugate* to a linear transformation A if for all $x, y \in E$ the following holds: $(Ax, y) = (x, A^+ y)$.

Example. Consider the Euclidean space formed by infinitely differentiable functions that are equal to zero outside of some finite interval with scalar product $(x, y) = \int\limits_{-\infty}^{+\infty} x(\tau) y(\tau) d\tau$ for the linear transformation $A = \dfrac{d}{d\tau}$ (differentiation) the transformation $A^+ = \dfrac{d}{d\tau}$ is conjugate to A.

Indeed, according to the rule of integrating these intervals by parts we have:

$$(Ax, y) = \int\limits_{-\infty}^{+\infty} \frac{dx(\tau)}{d\tau} y(\tau) d\tau = x(\tau) y(\tau) \Big|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty} x(\tau) \frac{dy(\tau)}{d\tau} d\tau = \int\limits_{-\infty}^{+\infty} x(\tau) \left(-\frac{dy(\tau)}{d\tau} \right) d\tau = (x, A^+ y).$$

Consider now a finite-dimensional Euclidean space E^n with a basis $\{g_1, g_2, ..., g_n\}$ and find a connection of matrices of linear transformations A and A^+ in this basis assuming that there exists a conjugate transformation. Let A_g and $A^+{}_g$ be the matrices of transformations A and A^+ respectively, and elements x and y have the coordinates $(\xi_1, \xi_2, ..., \xi_n)$ and $(\eta_1, \eta_2, ..., \eta_n)$ respectively in the basis. Then the equality $(Ax, y) = (x, A^+ y)$ can be written as follows:

$$\left(A_g \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_n \end{pmatrix} \right)^T \Gamma_g \begin{pmatrix} \eta_1 \\ \eta_2 \\ \cdots \\ \eta_n \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_n \end{pmatrix}^T \Gamma_g A^+{}_g \begin{pmatrix} \eta_1 \\ \eta_2 \\ \cdots \\ \eta_n \end{pmatrix}$$ where Γ_g is the Gram matrix of the chosen basis.

Since $(AB)^T = B^T A^T$ the last equality is written as follows: $\begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_n \end{pmatrix} \left(A_g^T \Gamma_g - \Gamma_g A^+{}_g \right) \begin{pmatrix} \eta_1 \\ \eta_2 \\ \cdots \\ \eta_n \end{pmatrix} = 0$.

Since this equality holds for any x and y, we conclude that $A_g^T \Gamma_g - \Gamma_g A^+{}_g = 0$, i.e.

$A^+{}_g = \Gamma_g^{-1} A_g^T \Gamma_g$ which for an orthonormal basis $\{e_1, e_2, ..., e_n\}$ has the form: $A^+{}_g = A_g^T$.

Lemma 1. If $(x, Ay) = 0$ for all $x, y \in E$ then A is zero transformation.

Proof: Let for all $x, y \in E$ the equality $(x, Ay) = 0$ holds. Then it holds for $x = Ay$. But the equality $(Ay, Ay) = 0$ implies that $Ay = 0$. At last by arbitrariness of element y we have that $A = 0$. □

Theorem 2. Every linear transformation in a Euclidean space E^n has a unique conjugate transformation.

Proof: An existence in E^n of an transformation A^+ that is conjugate to an transformation A follows from possibility to construct a matrix of the form $\Gamma_g^{-1} A_g^T \Gamma_g$ for any linear transformation A.

Show now a uniqueness of A^+. Suppose that A has two conjugate transformations A^+ and $A^\times$. It means that for all $x, y \in E$ the following equalities hold simultaneously:

$$(Ax, y) = (x, A^+ y) \text{ and } (Ax, y) = (x, A^\times y)$$

Subtracting termwise, we obtain $(x, (A^+ - A^\times) y) = 0$. By Lemma 1 we have $A^+ - A^\times = 0$. □

Theorem 3. For any linear transformations A and B acting in E $(AB)^+ = B^+ A^+$.

Proof: For all $x, y \in E$ the following holds:

$$((AB)^+ x, y) = (x, ABy) = (A^+ x, By) = (B^+ A^+ x, y).$$

It means that $(((AB)^+ - B^+ A^+) x, y) = 0$ for all $x, y \in E$ and by Lemma 1 $(AB)^+ - B^+ A^+ = 0$

Theorem 4. $(A^+)^+ = A$.

Proof: For all $x, y \in E$ the following holds: $((A^+)^+ x, y) = (x, A^+ y) = (Ax, y)$. Then $((A - (A^+)^+) x, y) = 0$ for all $x, y \in E$ and consequently by Lemma 1 $A - (A^+)^+ = 0$.

Unitary space

Recall that c*omplex numbers* are expressions of the form *a* + *bi* where *a* and *b* are real numbers, *i* is the imaginary unit, i.e. $i^2 = -1$. The set of all complex numbers is denoted by *C*.

Let $z = a + bi$. Then the complex number $a - bi$ is called *complex conjugate* to z and denoted by $\bar{z}$, i.e. $\overline{a + bi} = a - bi$.

Properties of complex conjugation:

1) $\overline{(\bar{z})} = z$; 2) A complex number z is real iff $\bar{z} = z$;

3) The number $z \cdot \bar{z} = a^2 + b^2$ is always real and nonnegative;

4) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

Consider such a set V of elements *x*, *y*, *z*, …, in which the *sum* $x + y \in V$ for every $x, y \in V$ and the *product* $\lambda x \in V$ for every $x \in V$ and every complex number λ are determined. If an addition of elements of *V* and a multiplication of an element of *V* on a complex number satisfy the axioms of a linear space, then the set *V* is called a *complex linear space.*

Let in a complex linear space U every ordered pair of elements *x* and *y* is put in correspondence a complex number (*x*, *y*) called their *scalar product* so that the following holds:

(1) $(x, y) = \overline{(y, x)}$;

(2) $(\lambda x, y) = \lambda (x, y)$ for every complex number λ;

(3) $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$;

(4) (x, x) is a real nonnegative number, and $(x, x) = 0 \Leftrightarrow x = 0$.

Then we say that U is a *unitary space*.

Remark. The form of the axiom 1 allows avoiding a problem which arises in case of using the Euclidean rule of scalar product for complex vector spaces.

Indeed, if we accept that $(x, y) = (y, x)$ then $(x, \lambda y) = \lambda (x, y)$ and obviously for some non-zero $x \in U$ and $\lambda = i$ we have $(ix, ix) = i \cdot i\,(x, x) = i^2 (x, x) = -(x, x)$. But then either (ix, ix) or (x, x) is not positive, and the axiom 4 doesn't hold.

In a unitary space the following holds: $(x, \lambda y) = \bar{\lambda} (x, y)$ and $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$.

Indeed, $(x, \lambda y) = \overline{(\lambda y, x)} = \bar{\lambda}\, \overline{(y, x)} = \bar{\lambda} (x, y)$;

$(x, y_1 + y_2) = \overline{(y_1 + y_2, x)} = \overline{(y_1, x) + (y_2, x)} = \overline{(y_1, x)} + \overline{(y_2, x)} = (x, y_1) + (x, y_2)$.

Example. 1) The space of n-dimensional columns $x=\begin{pmatrix}\xi_1\\ \xi_2\\ \dots\\ \xi_n\end{pmatrix}$, $y=\begin{pmatrix}\eta_1\\ \eta_2\\ \dots\\ \eta_n\end{pmatrix}$ where ξ_i,η_i, $i=1,\dots,n$ are complex numbers with scalar product defined by the formula $(x,y)=\sum_{i=1}^{n}\xi_i\overline{\eta_i}$ is unitary.

2) The space of continuous on $[\alpha,\beta]$ complex-valued functions with scalar product for elements $x(t)=\alpha(t)+i\beta(t)$ and $y(t)=\delta(t)+i\gamma(t)$ defined by $(x,y)=\int_{\alpha}^{\beta}x(t)\overline{y(t)}dt$ is also unitary.

Let in U^n a basis $\{g_1,g_2,\dots,g_n\}$ be given. The scalar product of elements $x=\sum_{i=1}^{n}\xi_i g_i$ and $y=\sum_{j=1}^{n}\eta_j g_j$ is presented as $(x,y)=\left(\sum_{i=1}^{n}\xi_i g_i,\sum_{j=1}^{n}\eta_j g_j\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i\overline{\eta_j}(g_i,g_j)=\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\xi_i\overline{\eta_j}$, where $\gamma_{ij}=(g_i,g_j),\quad i,j=1,\dots,n$ are components of the matrix Γ_g named the *basis matrix of Gram*. Then the coordinate representation of scalar product can be written as follows:

$$(x,y)=(x)_g^T\,\Gamma_g\,(\overline{y})_g=\begin{pmatrix}\xi_1 & \xi_2 & \dots & \xi_n\end{pmatrix}\begin{pmatrix}(g_1,g_1) & (g_2,g_1) & \dots & (g_n,g_1)\\ (g_1,g_2) & (g_2,g_2) & \dots & (g_n,g_2)\\ \dots & \dots & \dots & \dots\\ (g_1,g_n) & (g_2,g_n) & \dots & (g_n,g_n)\end{pmatrix}\begin{pmatrix}\overline{\eta_1}\\ \overline{\eta_2}\\ \dots\\ \overline{\eta_n}\end{pmatrix}$$

Observe that since $(g_i,g_j)=\overline{(g_j,g_i)}$, the following holds: $\Gamma^T=\overline{\Gamma}$.

A matrix A satisfying the property $A^T=\overline{A}$ is called *Hermitian*. A matrix A satisfying the properties $A^T\overline{A}=E$ and $\overline{A}A^T=E$ is called *unitary*.

The determinant of a unitary matrix is the complex number of which the module is equal to 1. Indeed: $\det\left(A^T\overline{A}\right)=\det A^T\det\overline{A}=\det A\overline{\det A}=\left|\det A\right|^2=\det E=1$.

Example. In quantum mechanics the following matrices (named *Pauli matrices*) are used:

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

It can be proved that each of these matrices is unitary.

Example. Let $A=\begin{pmatrix}0 & -2i\\ 2i & 0\end{pmatrix}$. Then $\overline{A}=\begin{pmatrix}0 & 2i\\ -2i & 0\end{pmatrix}$ and $A^T=\begin{pmatrix}0 & 2i\\ -2i & 0\end{pmatrix}$, i.e. A is Hermitian.

But $\begin{pmatrix}0 & 2i\\ -2i & 0\end{pmatrix}\begin{pmatrix}0 & 2i\\ -2i & 0\end{pmatrix}=\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}=4E$, i.e. A is not unitary.

Linear transformations in a unitary space

A linear transformation A^+ acting in a unitary space U is called *Hermitian conjugate* to a linear transformation A if for all $x,y\in U$ the following holds: $(Ax,y)=(x,A^+y)$.

Theorem 1. For linear transformations A and B acting in a unitary space U the following holds:

$$(AB)^+=B^+A^+ \text{ and } (\lambda A)^+=\overline{\lambda}A^+$$

Proof: Prove the first assertion. We have $(ABx,y)=(Bx,A^+y)=(x,B^+A^+y)$ for all $x,y\in U$, and consequently $(AB)^+=B^+A^+$. Similarly, $(\lambda Ax,y)=\lambda(Ax,y)=\lambda(x,A^+y)=(x,\overline{\lambda}A^+y)$ for all $x,y\in U$ and every complex number λ. □

Theorem 2. The matrix of an transformation A^+ that is Hermitian conjugate to an transformation A in U^n in a basis

$\{g_1,g_2,\dots,g_n\}$ is defined by the following equality: $A_g^+=\Gamma^{-1}\overline{A}_g^T\Gamma$.

Example. Let $\{e_1, e_2\}$ be an orthonormal basis in a unitary space U, $f_1 = e_1 + e_2$, $f_2 = e_1 - ie_2$. A linear transformation A acting in this space has in the basis $\{f_1, f_2\}$ the matrix $A_f = \begin{pmatrix} 2 & 1+i \\ -1-i & 1-i \end{pmatrix}$. Find the matrix A_f^+ of Hermitian conjugate to A transformation A^+ in the orthonormal basis $\{f_1, f_2\}$.

Solution: The matrix A_e^+ of the transformation A^+ in the orthonormal basis $\{e_1, e_2\}$ is connected with the matrix A_e of A by the equality: $A_e^+ = \overline{A}_e^T$. At transition to the basis $\{f_1, f_2\}$ the matrix A_e^+ is transformed by the formula: $A_f^+ = S^{-1}A_e^+S = S^{-1}\overline{A}_e^T S$ where S is the transition matrix from $\{e_1, e_2\}$ to $\{f_1, f_2\}$, i.e. $S = \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix}$. Similarly, the matrix A_f is connected with A_e by the equality $A_f = S^{-1}A_e S$. Then $A_e = SA_f S^{-1}$ and consequently $\overline{A}_e^T = \left(\overline{S^{-1}}\right)^T \overline{A}_f^T \overline{S}^T$. Thus, we have $A_f^+ = S^{-1}\left(\overline{S^{-1}}\right)^T \overline{A}_f^T \overline{S}^T S$. Since $S^{-1} = \dfrac{1}{-1-i}\begin{pmatrix} -i & -1 \\ -1 & 1 \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & -1+i \end{pmatrix}$, we have:

$$A_f^+ = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & -1+i \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & -1-i \end{pmatrix}\begin{pmatrix} 2 & -1+i \\ 1-i & 1+i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 1+i & -1+i \\ 1-i & 2 \end{pmatrix}.$$

Glossary

conjugate – сопряженный; **unitary space** – унитарное пространство
Hermitian matrix – эрмитова матрица

Exercises for Seminar 12

12.1. Let $\{e_1, e_2, e_3\}$ be a basis in a Euclidean space E. Find the Gram matrix Γ for this basis if $e_1 = (1, 0, -1), e_2 = (0, -1, -1), e_3 = (-1, 1, 0)$.

12.2. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ be the matrix of a transformation A in a basis $\{e_1, e_2\}$ with Gram matrix $\Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Find the matrix of the conjugate transformation A^+.

12.3. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ be the matrix of a transformation A in a basis $\{e_1, e_2, e_3\}$ with Gram matrix $\Gamma = \begin{pmatrix} 6 & 4 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix}$. Find the matrix of the conjugate transformation A^+.

12.4. In the space P_2 of polynomials of degree ≤ 2 with standard scalar product $(p, q) = \int_{-1}^{1} p(t)q(t)dt$ for polynomials p and q the transformation A assigns to a polynomial its derivative. Find the matrix of the conjugate transformation A^+: a) in the basis $1, t, t^2$; b) in the basis $1, t, (3t^2 - 1)/2$.

12.5. Is a transformation $Ax = -\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4$ orthogonal (where $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4$ is an arbitrary vector and e_1, e_2, e_3, e_4 is an orthonormal basis)?

12.6. Is a transformation being the turning any vector lying in plane *xOy* on a fixed angle α orthogonal?

12.7. Whether the complex 2-dimensional linear space is unitary if scalar product is defined for vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ as follows: a) $(x, y) = x_1 y_1 + x_2 y_2$; b) $(x, y) = \overline{x}_1 y_1 + \overline{x}_2 y_2$.

12.8. In the complex arithmetic space with standard scalar product find the scalar product of the following pair vectors: a) $(1,i), (i,1)$; b) $(1+2i,-1+2i), (2-i,\, 2+i)$. Find the length of the first vector.

12.9. Find the scalar product of vectors of a unitary space by their coordinates in a basis $\{g_1, g_2, ..., g_n\}$ and the Gram matrix Γ of this basis:

a) $(1,\, i), (2i,\, 1)$, $\Gamma = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$; b) $(2+i,\, 0,\, 1+2i), (2-i,\, 1,\, 2+i)$, $\Gamma = \begin{pmatrix} 1 & i & 0 \\ -i & 2 & -i \\ 0 & i & 2 \end{pmatrix}$.

12.10. By using the procedure of orthogonalization construct an orthonormal basis in the linear hull of the following vectors of the complex arithmetic space with standard scalar product:
a) $(1,\, i), (1,\, 1)$; b) $(1,\, i,\, 1), (i,\, 1,\, 0), (-1,\, 0,\, 1)$.

12.11. A linear transformation A acting in a unitary space U^2 has in an orthonormal basis $\{e_1, e_2\}$ the matrix $A_e = \begin{pmatrix} 2 & 1+i \\ -1-i & 1-i \end{pmatrix}$. Find the matrix of Hermitian conjugate transformation A^+ in the basis $\{f_1, f_2\}$ if $f_1 = e_1 + e_2$, $f_2 = e_1 - ie_2$.

Exercises for Homework 12

12.12. Let $\{e_1, e_2, e_3\}$ be a basis in a Euclidean space E. Find the Gram matrix Γ for this basis if $e_1 = (1,2,3), e_2 = (2,3,1), e_3 = (3,1,2)$.

12.13. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ be the matrix of a transformation A in a basis $\{e_1, e_2\}$ with Gram matrix $\Gamma = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$. Find the matrix of the conjugate transformation A^+.

12.14. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ be the matrix of a transformation A in a basis $\{e_1, e_2, e_3\}$ with Gram matrix $\Gamma = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Find the matrix of the conjugate transformation A^+.

12.15. In the space P_2 of polynomials of degree ≤ 2 with scalar product $(p,q) = \sum_{k=0}^{n} \alpha_k \beta_k$ for polynomials p and q (where α_k and β_k are the coefficients of p and q at the same degrees) the transformation A assigns to a polynomial its derivative. Find the conjugate transformation A^+. Write the matrix of A^+: a) in the basis $1,\, t,\, t^2$; b) in the basis $1,\, t,\, (3t^2 - 1)/2$.

12.16. What values λ is a transformation A defined by the equality $Ax = \lambda x$ orthogonal for?

12.17. Let $e_1, e_2, e_3, e_4, e_5, e_6$ be an orthonormal basis. Prove that A is an orthogonal transformation if

$$Ae_1 = e_1,\ Ae_2 = -e_2,\ Ae_3 = e_3 \cos\alpha + e_4 \sin\alpha,\ Ae_4 = -e_3 \sin\alpha + e_4 \cos\alpha,$$
$$Ae_5 = e_5 \cos\beta + e_6 \sin\beta,\ Ae_6 = -e_5 \sin\beta + e_6 \cos\beta.$$

12.18. Whether the complex 2-dimensional linear space is unitary if scalar product is defined for vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ as follows: a) $(x,y) = \bar{x}_1 \bar{y}_1 + \bar{x}_2 \bar{y}_2$; b) $(x,y) = \bar{x}_1 \bar{y}_1 + x_2 y_2$.

12.19. In the complex arithmetic space with standard scalar product find the scalar product of the following pair vectors: a) $(1,-i), (1,-i)$; b) $(1,\, i,\, 1), (i,\, 1,\, i)$. Find the length of the first vector.

12.20. Find the scalar product of vectors of a unitary space by their coordinates in a basis $\{g_1, g_2, ..., g_n\}$ and the Gram matrix Γ of this basis:

a) $(1+i,\ i),\ (1,\ 1-i),\ \Gamma=\begin{pmatrix} 3 & 1+i \\ 1-i & 1 \end{pmatrix}$; b) $(-1,\ 2+i,\ 1),\ (1,\ -i,\ 1+i),\ \Gamma=\begin{pmatrix} 1 & 1-i & 0 \\ 1+i & 3 & i \\ 0 & -i & 2 \end{pmatrix}$.

12.21. By using the procedure of orthogonalization construct an orthonormal basis in the linear hull of the following vectors of the complex arithmetic space with standard scalar product:
a) $(2-i,\ i),\ (4-i,\ 2-3i)$; b) $(1+i,\ 2+i,\ 1-i),\ (-2,\ 4+i,\ 1-i),\ (1,\ 2+i,\ 2-i)$.

12.22. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis in a unitary space U^3, $f_1=e_1,\ f_2=ie_1+e_2$, $f_3=-ie_1+e_2+e_3$. A linear transformation A acting in this space has in the basis $\{f_1,\ f_2,\ f_3\}$ the following matrix: $A_f=\begin{pmatrix} i & -i & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Find the matrix of Hermitian conjugate transformation A^+ in the basis $\{f_1,\ f_2,\ f_3\}$.