LECTURE 9

Invariant subspaces, eigenvectors and eigenvalues of a linear transformation

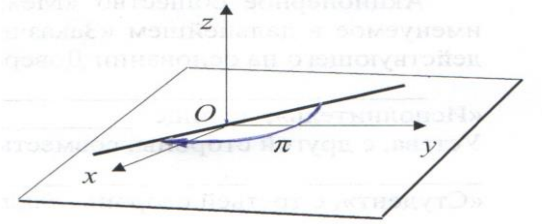
A subspace V_1 of a linear space V is called *invariant* with respect to a linear transformation A if for every element x of V_1 its image Ax also belongs to V_1.

Examples: 1) The subspace consisting of one zero-element 0 is an invariant subspace with respect to any linear transformation.

2) A linear space V itself is invariant with respect to any linear transformation acting in this space.

Zero-subspace and V are called *trivial* invariant subspaces of a linear transformation.

3) The set of radius-vectors of the points of some line on plane Oxy passing through the origin of coordinates is an invariant subspace of the operator of turning these radius vectors on angle around the axis Oz.



4) For the operator of differentiation in the linear space of functions $f(t)$ having on (α, β) the derivative of any order the linear hull of elements $\{e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t}\}$ where $\lambda_1, \lambda_2, ..., \lambda_n$ are pairwise distinct constants is an n-dimensional invariant subspace.

Consider now the conditions for which there exists a one-dimensional invariant subspace of a linear transformation.

A non-zero vector $x \in V$ is called an *eigenvector* of a linear transformation A if there is such a number λ that the equality $Ax = \lambda x$ holds. The number λ is called a *characteristic number* (*eigenvalue*) of the linear transformation A corresponding to the vector x.

Remark on importance of eigenvectors. Assume that for some linear transformation A acting in an n-dimensional linear space V n linearly independent eigenvectors $\{f_1, f_2, ..., f_n\}$ have been found. It means that the following equalities hold: $Af_1 = \lambda_1 f_1, Af_2 = \lambda_2 f_2, ..., Af_n = \lambda_n f_n$. Taking these elements as a basis we can conclude that the matrix of a linear transformation A in this basis will have the following diagonal form: $A_f = \begin{pmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{pmatrix}$ for which studying the properties of this transformation is essentially simplified.

<u>*Exercise*</u>. Show that if a linear transformation A has an eigenvector f with the corresponding eigenvalue λ then the element f will be also an eigenvector of the linear transformation $A^2 = A \cdot A$ with the eigenvalue λ^2.

Solution: By the hypothesis $Af = \lambda f$. Then by linearity of the transformation A we have

$$A^2 f = A(Af) = A(\lambda f) = \lambda^2 f .$$

Choose in V some basis $\{e_1, e_2, ..., e_n\}$ and let $f = \sum_{i=1}^{n} \xi_i e_i$. Let a linear transformation A in the basis $e_1, e_2, \dots, e_n$ have the matrix $A_e = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$.

The equality $Af = \lambda f$ in the coordinate form has the following:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} = \lambda \begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix}, \text{ i.e. } \begin{cases} a_{11}\xi_1 + a_{12}\xi_2 + \dots + a_{1n}\xi_n = \lambda\xi_1 \\ a_{21}\xi_1 + a_{22}\xi_2 + \dots + a_{2n}\xi_n = \lambda\xi_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\xi_1 + a_{n2}\xi_2 + \dots + a_{nn}\xi_n = \lambda\xi_n \end{cases} \text{ or}$$

$$\begin{cases} (a_{11} - \lambda)\xi_1 + a_{12}\xi_2 + \dots + a_{1n}\xi_n = 0 \\ a_{21}\xi_1 + (a_{22} - \lambda)\xi_2 + \dots + a_{2n}\xi_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\xi_1 + a_{n2}\xi_2 + \dots + (a_{nn} - \lambda)\xi_n = 0 \end{cases} \quad (*)$$

Since an eigenvector must be non-zero by definition, we are interested only by non-trivial solutions of the system (*). The necessary condition for existence of non-trivial solutions is the equality of the determinant of the basic matrix of the system (*) to zero:

$$\det(A - \lambda E) = 0 \qquad \text{or} \qquad \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0.$$

The equation $\det(A - \lambda E) = 0$ is called a *characteristic equation*, and $\det(A - \lambda E)$ – *characteristic polynomial* of the linear transformation A acting in V.

Example. Find characteristic numbers and eigenvectors of a linear transformation given in some basis by the matrix A:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution. Compose the characteristic equation:

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$(2 – \lambda)^2(1 – \lambda) – (1 – \lambda) = 0$

$(1 – \lambda)\ [(2 – \lambda)^2 – 1] = 0$

$(1 – \lambda)^2\ (3 – \lambda) = 0$

Thus, $\lambda_{1,\,2} = 1$, $\lambda_3 = 3$ are characteristic numbers of the linear transformation.

If $\lambda = 1$ then to find the coordinates of an eigenvector we obtain the following system of equations: $\begin{cases} \xi_1 - \xi_2 + \xi_3 = 0 \\ -\xi_1 + \xi_2 - \xi_3 = 0 \end{cases}$

From the equations we have $\xi_3 = \xi_2 - \xi_1$. Let $\xi_1 = c_1$ and $\xi_2 = c_2$ where c_1, c_2 are arbitrary non-zero numbers. Then $\xi_3 = c_2 – c_1$ and consequently the eigenvalue $\lambda = 1$ is corresponded the family of eigenvectors $u = c_1 e_1 + c_2 e_2 + (c_2 - c_1) e_3$.

If $\lambda = 3$ then to find the coordinates of an eigenvector we obtain the following system of equations:

$$\begin{cases} -\xi_1 - \xi_2 + \xi_3 = 0 \\ -\xi_1 - \xi_2 - \xi_3 = 0 \\ \qquad\qquad -2\xi_3 = 0 \end{cases}$$

From the third equation we have $\xi_3 = 0$. Then substituting this value in the first equation (or the second equation) we obtain $\xi_1 = – \xi_2$. Let $\xi_1 = c_3$ where c_3 is an arbitrary non-zero number. Then $\xi_2 = – c_3$ and consequently the eigenvalue $\lambda = 3$ is corresponded the family of eigenvectors $v = c_3\ (e_1 – e_2)$.

Thus, substituting all possible numeric values for c_1, c_2 and c_3 in the equalities $u = c_1e_1 + c_2e_2 + (c_2 - c_1)e_3$ and $v = c_3 (e_1 - e_2)$ we will obtain all possible eigenvectors of the linear transformation A.

Theorem 1. The set V_λ containing zero-element and all the eigenvectors of a linear transformation A corresponding to a characteristic number λ is an invariant subspace of the linear transformation A.

Proof. Let $Af_1 = \lambda f_1$ and $Af_2 = \lambda f_2$. Then for any numbers α and β we have

$A(\alpha f_1 + \beta f_2) = \alpha Af_1 + \beta Af_2 = \alpha\lambda f_1 + \beta\lambda f_2 = \lambda(\alpha f_1 + \beta f_2)$. □

The subspace V_λ is called an *eigen-subspace* of a linear transformation corresponding to a characteristic number λ.

Theorem 2. If $\lambda_1, \lambda_2, ..., \lambda_m$ are distinct characteristic numbers of a linear transformation A then the corresponding to them eigenvectors $f_1, f_2, ..., f_m$ are linearly independent.

Proof. One eigenvector is linearly independent as a non-zero vector.

Let $f_1, f_2, ..., f_m$ be linearly independent eigenvectors corresponding to distinct eigenvalues. Show that in this case $m + 1$ eigenvectors $f_1, f_2, ..., f_m, f_{m+1}$ corresponding to distinct eigenvalues will be linearly independent. Assume the contrary:

$$\kappa_1 f_1 + \kappa_2 f_2 + ... + \kappa_m f_m + \kappa_{m+1} f_{m+1} = 0 \quad (1)$$

and without loss of generality we can assume that the number $\kappa_{m+1} \neq 0$.

Apply to both parts of (1) the linear transformation A:

$A(\kappa_1 f_1 + \kappa_2 f_2 + ... + \kappa_m f_m + \kappa_{m+1} f_{m+1}) = \kappa_1\lambda_1 f_1 + \kappa_2\lambda_2 f_2 + ... + \kappa_m\lambda_m f_m + \kappa_{m+1}\lambda_{m+1} f_{m+1} = 0$

On other hand, multiplying both parts of (1) on λ_{m+1} and subtracting the result from the last equality, we obtain $\kappa_1(\lambda_1 - \lambda_{m+1}) f_1 + \kappa_2(\lambda_2 - \lambda_{m+1}) f_2 + ... + \kappa_m(\lambda_m - \lambda_{m+1}) f_m = 0$.

Since all the eigenvalues are distinct, and the vectors $f_1, f_2, ..., f_m$ are linearly independent, then $\kappa_1 = \kappa_2 = ... = \kappa_m = 0$. And then by (1) we have $\kappa_{m+1} = 0$ contradicting our assumption. Thus, by principle of mathematical induction a linear independence of elements $f_1, f_2, ..., f_m$ implies a linear independence of elements $f_1, f_2, ..., f_m, f_{m+1}$. □

Corollary 3. A linear transformation acting in a linear space V of dimension n cannot have more than n distinct characteristic numbers.

Theorem 4. If a linear transformation acting in a linear space V of dimension n has n distinct characteristic numbers $\lambda_1, \lambda_2, ..., \lambda_n$ then the corresponding to them eigenvectors $f_1, f_2, ..., f_n$ form a basis of the space V.

Theorem 5. In a complex n-dimensional linear space V every linear transformation has at least one eigenvector.

Proof: Since the characteristic equation is algebraic equation of the n-th degree regarding to λ, such equation has at least one complex root. □

In case of a real linear space Theorem 5 doesn't hold in general. For example, the linear transformation being the operator of turning the plane Oxy around the origin of coordinates on angle $\varphi \neq k\pi$ has no eigenvectors. Indeed, the characteristic equation for this transformation has the following form:

$$\begin{vmatrix} \cos\varphi - \lambda & -\sin\varphi \\ \sin\varphi & \cos\varphi - \lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 2\lambda\cos\varphi + 1 = 0$$

i.e. $\lambda = \cos\varphi \pm i\sin\varphi$. Then we have that the characteristic equation has no real solutions for $\varphi \neq k\pi$.

Theorem 6. In a real n-dimensional linear space V every linear transformation has either at least one eigenvector or a 2-dimensional invariant subspace.

Proof. If the characteristic equation has a real root, we can find an eigenvector. Let the characteristic equation have a complex root $\lambda = a + bi$ and let $f = u + wi$ be a corresponding to λ eigenvector where $u, w \in V$. Show that u and w are linearly independent. Assume the

contrary: $u = \kappa w$. Then from $Af = \lambda f$ we have $A((\kappa + i)w) = \lambda(\kappa + i)w$, or $Aw = \lambda w$, i.e. λ is real contradicting the assumption on non-reality of the eigenvalue.
Further we have $A(u + wi) = (a + bi)(u + wi)$ or $Au + (Aw)i = (au - bw) + (bu + aw)i$ and from the equality of real and imaginary parts we obtain: $\begin{cases} Au = au - bw \\ Aw = bu + aw \end{cases}$.
The last system means that the transformation A has a 2-dimensional invariant subspace coinciding with the linear hull of elements u and w, since
$A(\xi u + \eta w) = \xi Au + \eta Aw = \xi(au - bw) + \eta(bu + aw) = (\xi a + \eta b)u + (\eta a - \xi b)w$. □
Theorem 7. If a matrix A of a linear transformation A is symmetric then all the roots of the characteristic equation $|A - \lambda E| = 0$ are real numbers.

Example. A linear transformation A in a three-dimensional linear space V with the basis e_1, e_2, e_3 is defined as follows: $f_1 = Ae_1 = 1{,}5e_1 + 0{,}5e_2 + 0{,}5e_3$, $f_2 = Ae_2 = 0{,}5e_1 + e_2$, $f_3 = Ae_3 = 0{,}5e_1 + e_3$. Find all subspaces of the space V that are invariant with respect to A.
Solution: First of all indicate trivial invariant subspaces: zero-space and V itself. Further find eigenvalues and eigenvectors of the transformation A.

$$\begin{vmatrix} 1{,}5 - \lambda & 0{,}5 & 0{,}5 \\ 0{,}5 & 1 - \lambda & 0 \\ 0{,}5 & 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)(\lambda^2 - 2{,}5\lambda + 1).$$

Its roots $\lambda_1 = 0{,}5;\ \lambda_2 = 1,\ \lambda_3 = 2$ are characteristic numbers of the transformation A.
If λ = 0,5 then to find the coordinates of an eigenvector we obtain the following system of equations:

$$\begin{cases} x_1 + 0{,}5x_2 + 0{,}5x_3 = 0 \\ 0{,}5x_1 + 0{,}5x_2 = 0 \\ 0{,}5x_1 + 0{,}5x_3 = 0 \end{cases}$$

There are three unknowns in this system, and the rank of the matrix is equal to 2. Therefore the dimension of the subspace of solutions is equal to 1. Solving the system we find a vector of fundamental system of solutions u_1 = (1; – 1; – 1). Then a one-dimensional invariant with respect to A eigen-subspace corresponding the eigenvalue λ_1 is the linear hull $L(u_1)$.
If λ = 1 then to find the coordinates of an eigenvector we obtain the following system of equations:

$$\begin{cases} 0{,}5x_1 + 0{,}5x_2 + 0{,}5x_3 = 0 \\ 0{,}5x_1 = 0 \\ 0{,}5x_1 = 0 \end{cases}$$

Analogously, the dimension of the subspace of solution is equal to 1. Solving the system we find a vector of fundamental system of solutions u_2 = (0; 1; – 1). Then a one-dimensional invariant with respect to A eigen-subspace corresponding the eigenvalue λ_2 is the linear hull $L(u_2)$.
At last if λ = 2 then to find the coordinates of an eigenvector we obtain the following system of equations:

$$\begin{cases} -0{,}5x_1 + 0{,}5x_2 + 0{,}5x_3 = 0 \\ 0{,}5x_1 - x_2 = 0 \\ 0{,}5x_1 - x_3 = 0 \end{cases}$$

The rank of the matrix is also equal to 2. Therefore an invariant with respect to A subspace corresponding to the characteristic number λ_3 has dimension 1 and represents the linear hull $L(u_3)$ where u_3 = (2; 1; 1) is an eigenvector corresponding to λ_3.
Further, the linear hulls $L(u_1, u_2)$, $L(u_1, u_3)$, $L(u_2, u_3)$ are two-dimensional invariant with respect to A subspaces. In fact, if we consider for example $L(u_1, u_2)$ then we have
$A(\alpha u_1 + \beta u_2) = \alpha Au_1 + \beta Au_2 = \alpha\lambda_1 u_1 + \beta\lambda_2 u_2 = 0{,}5\alpha u_1 + \beta u_2 \in L(u_1, u_2)$.
There is no other invariant with respect to A subspaces.

Glossary

similarity – подобие; **homogeneous** – однородный
eigenvector – собственный вектор; **eigenvalue** – собственное значение
eigen-subspace – собственное подпространство

Exercises for Seminar 9

9.1. Find characteristic numbers and eigenvectors of a linear transformation A defined by the equations $x' = 5x + 4y$, $y' = 8x + 9y$.

9.2. Let a linear transformation with the matrix $A = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$ be given. Find characteristic numbers and eigenvectors of this transformation.

9.3. Let a linear transformation with the matrix $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ be given. Find characteristic numbers and eigenvectors of this transformation.

9.4. Find characteristic numbers and eigenvectors of the following matrix:
$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

9.5. Find characteristic numbers and eigenvectors of a linear transformation with the matrix $A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}$.
Prove that the linear hull $L(e_1 + 2e_2, e_2 + e_3 + 2e_4)$ is an invariant with respect to A subspace.

9.6. Find characteristic numbers and eigenvectors of a linear transformation with the matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

9.7. Find eigenvalues and eigenvectors of the operator of differentiation D as a linear transformation of the space of all polynomials of degree $\le n$.

Exercises for Homework 9

9.8. Find characteristic numbers and eigenvectors of a linear transformation with the matrix $A = \begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix}$.

9.9. Find characteristic numbers and eigenvectors of a linear transformation with the matrix $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

9.10. Determine characteristic numbers and eigenvectors of a linear transformation with the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

9.11. Find characteristic numbers and eigenvectors of a linear transformation with the matrix $A=\begin{pmatrix}\alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha\end{pmatrix}$.

9.12. Find all subspaces that are invariant with respect to a linear transformation A with the matrix $A=\begin{pmatrix}1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3\end{pmatrix}$.

9.13. Find eigenvalues and eigenvectors of the operator of differentiation D as a linear transformation of the linear hull of the system of functions $\{e^{\lambda_1 t}, e^{\lambda_2 t}, ..., e^{\lambda_n t}\}$ where $\lambda_1, \lambda_2, ..., \lambda_n$ are pairwise distinct numbers.