Script for the equations of motion of a T orsobot with rimless wheel

Constants:
L: Spoke length
l: distance of torso CoM from the wheel center
I_w: Moment of inertia of the wheel along the out of plane axis (y), about wheel CoM
I_t: Moment of inertia of the torso along the out of plane axis (y), about torso CoM
M: mass of the wheel
m: mass of the torso

State variables:
θ: angle of wheel with respect to slope normal
ϕ: angle of the torso with respect to slope normal
$\dot{\theta}$: angular rate of the wheel
$\dot{\phi}$: angular rate of the torso

State vectors

$$q = \begin{bmatrix} \theta \\ \phi \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} q_3 \\ q_4 \\ f_3(q, u) \\ f_4(q, u) \end{bmatrix}$$

$\dot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q}, \boldsymbol{u})$

<u>Stance Dynamics</u>
During stance, equations of motion are derived using Newton-Euler equations.

$q := Vector([q1, q2, q3, q4])$

$$q := \begin{bmatrix} q1 \\ q2 \\ q3 \\ q4 \end{bmatrix} \tag{1}$$

$R := m \cdot l \cdot L \cdot \cos(q1 - q2)$

$$R := m\, l\, L \cos(q1 - q2) \tag{2}$$

$S := m \cdot l^2 + I_t$

$$S := l^2\, m + I_t \tag{3}$$

$V := simplify\left(m \cdot L^2 + M \cdot L^2 + I_w\right)$

$$V := I_w + (m + M)\, L^2 \tag{4}$$

$f1 := m \cdot l \cdot L \cdot q3^2 \cdot \sin(q1 - q2) + m \cdot g \cdot l \cdot \sin(\text{alpha} + q2)$

$$f1 := m\, l\, L\, q3^2 \sin(q1 - q2) + m\, g\, l \sin(\alpha + q2) \tag{5}$$

$f2 := m \cdot l \cdot L \cdot q4^2 \cdot \sin(q2 - q1) + (m + M) \cdot g \cdot L \cdot \sin(\text{alpha} + q1)$

$$f2 := -m\, l\, L\, q4^2 \sin(q1 - q2) + (m + M)\, g\, L \sin(\alpha + q1) \tag{6}$$

$N := Matrix([[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, R, S], [0, 0, V, R]])$

$$N := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m\, l\, L \cos(q1 - q2) & l^2\, m + I_t \\ 0 & 0 & I_w + (m + M)\, L^2 & m\, l\, L \cos(q1 - q2) \end{bmatrix} \tag{7}$$

$f := Vector([q3, q4, f1, f2])$

$$f := \begin{bmatrix} q3 \\ q4 \\ m\, l\, L\, q3^2 \sin(q1 - q2) + m\, g\, l \sin(\alpha + q2) \\ -m\, l\, L\, q4^2 \sin(q1 - q2) + (m + M)\, g\, L \sin(\alpha + q1) \end{bmatrix} \tag{8}$$

$H := Vector([0, 0, -1, 1])$

$$H := \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \tag{9}$$

$q\_dot := LinearAlgebra[MatrixInverse](N) . (f + H \cdot T)$

$$q\_dot := \tag{10}$$

$$
\begin{bmatrix}
\cdots \\
\cdots \\
\dfrac{m\, l\, L \cos(q1 - q2)\, \left(-T + m\, l\, L\, q3^2 \sin(q1 - q2) + m\, g\, l \sin(\cdots\right.}{m^2\, l^2\, L^2 \cos(q1 - q2)^2 - L^2\, M\, l^2\, m - L^2\, l^2\, m^2 - I_t\, L^2\, M - I_t\, L^2\, m - \cdots} \cdots \\
\dfrac{\left(-L^2\, M - L^2\, m - I_w\right)\, \left(-T + m\, l\, L\, q3^2 \sin(q1 - q2) + m\, g\, l \sin(\cdots\right.}{m^2\, l^2\, L^2 \cos(q1 - q2)^2 - L^2\, M\, l^2\, m - L^2\, l^2\, m^2 - I_t\, L^2\, M - I_t\, L^2\, m - \cdots} \cdots
\end{bmatrix}
$$

$q\_dot := simplify(q\_dot)$

$q\_dot :=$ **(11)**

$$
\begin{bmatrix}
\cdots \\
\cdots \\
\dfrac{-m\, l\, L \cos(q1 - q2)\, \left(-T + m\, l\, L\, q3^2 \sin(q1 - q2) + m\, g\, l \sin(\cdots\right.}{-m^2\, l^2\, L^2 \cos(q1 - \cdots} \cdots \\
\dfrac{m\, \left(-T + m\, l\, L\, q4^2 \sin(q1 - q2) - (m + M)\, g\, L \sin(\alpha + q1)\right)\, L\, l \cos\cdots}{-m^2\, l^2\, L^2 \cos(q1 - \cdots} \cdots
\end{bmatrix}
$$

$A := VectorCalculus[Jacobian](q\_dot, convert(q, list))$

$A :=$ **(12)**

$$
\begin{bmatrix}
\cdots \\
\cdots \\
\dfrac{m\, l\, L \sin(q1 - q2)\, \cdots}{\cdots} \cdots \\
\dfrac{m\, \left(m\, l\, L\, q4^2 \cos(q1 - q2) - (m + M)\, \cdots\right.}{\cdots} \cdots
\end{bmatrix}
$$

$A := eval(A, [q1 = 0, q2 = 0, q3 = 0, q4 = 0, T = 0])$

$A :=$ **(13)**

$$\begin{bmatrix} 0 & \cdots \\ 0 & \cdots \\ \frac{\left(l^2 m + I_t\right) (m+M)\, g\, L \cos(\alpha)}{-L^2 l^2 m^2 + \left(l^2 m + I_t\right) \left(I_w + (m+M) L^2\right)} & -\frac{m^2\, l}{-L^2 l^2 m^2 + \left(l^2\, n\right.} \cdots \\ -\frac{m\, (m+M)\, g\, L^2 \cos(\alpha)\, l}{-L^2 l^2 m^2 + \left(l^2 m + I_t\right) \left(I_w + (m+M) L^2\right)} & \frac{m\, g\, l \cos(\alpha)}{-L^2 l^2 m^2 + \left(l^2\, m\right.} \cdots \end{bmatrix}$$

Stance Dynamics

During collision, momentum is preserved about the collision point.

Before collision, the velocity of the center of mass of the wheel

$vg1 := Vector([L \cdot q3 \cdot \cos(q1), 0, -L \cdot q3 \cdot \sin(q1)])$

$$vg1 := \begin{bmatrix} L\, q3 \cos(q1) \\ 0 \\ -L\, q3 \sin(q1) \end{bmatrix} \tag{14}$$

$vg2 := Vector([L \cdot q3 \cdot \cos(q1) + l \cdot q4 \cdot \cos(\mathrm{q2}), 0, -(L \cdot q3 \cdot \sin(q1) + l \cdot q4 \cdot \sin(q2))])$

$$vg2 := \begin{bmatrix} L\, q3 \cos(q1) + l\, q4 \cos(q2) \\ 0 \\ -L\, q3 \sin(q1) - l\, q4 \sin(q2) \end{bmatrix} \tag{15}$$

$rg1 := Vector([-L \cdot \sin(q1), 0, L \cdot \cos(q1)])$

$$rg1 := \begin{bmatrix} -L \sin(q1) \\ 0 \\ L \cos(q1) \end{bmatrix} \tag{16}$$

$rg2 := Vector([(-L \cdot \sin(q1) + l \cdot \sin(q2)), 0, (L \cdot \cos(q1) + l \cdot \cos(q2))])$

$$rg2 := \begin{bmatrix} -L \sin(q1) + l \sin(q2) \\ 0 \\ L \cos(q1) + l \cos(q2) \end{bmatrix} \tag{17}$$

$mom\_1 := simplify(LinearAlgebra[CrossProduct](M \cdot rg1, vg1))$

$$mom\_1 := \begin{bmatrix} 0 \\ M\,q3\,L^2 \cos(2\,q1) \\ 0 \end{bmatrix} \quad \mathbf{(18)}$$

$mom\_2 := (LinearAlgebra[CrossProduct](m \cdot rg2, vg2))$

$$mom\_2 := \quad \mathbf{(19)}$$

$$\begin{bmatrix} 0 & \cdots \\ -m\,(-L\sin(q1) + l\sin(q2))\,(-L\,q3\sin(q1) - l\,q4\sin(q2)) + m\,(L\,\mathrm{cc} & \cdots \\ 0 & \cdots \end{bmatrix}$$

$RHS := \left(Vector(\left[0,\ I_t \cdot q4, 0\right]) + Vector(\left[0, I_w \cdot q3, 0\right]) + mom\_1 + mom\_2\right)$

$$RHS := \quad \mathbf{(20)}$$

$$\begin{bmatrix} \cdots \\ I_t\,q4 + I_w\,q3 + M\,q3\,L^2\cos(2\,q1) - m\,(-L\sin(q1) + l\sin(q2))\,(-L \cdots \\ \cdots \end{bmatrix}$$

The vector $\boldsymbol{q}$ denotes the state right before collision (it could also be represented by $\boldsymbol{q}^-$)
The vector $\boldsymbol{q}^+$ represents the state right after collision

$\boldsymbol{R}(\boldsymbol{q}) \cdot \boldsymbol{q}^+ = \boldsymbol{s}(\boldsymbol{q}^-)$

$F := (m + M) \cdot L \cdot \sin(q1)$

$$F := (m + M)\,L\sin(q1) \quad \mathbf{(21)}$$

$G := -m \cdot l \cdot \sin(q2)$

$$G := -m\,l\sin(q2) \quad \mathbf{(22)}$$

$s1 := -F \cdot q3 + G \cdot q4$

$$s1 := -(m + M)\,L\sin(q1)\,q3 - m\,l\sin(q2)\,q4 \quad \mathbf{(23)}$$

$Q := I_w + M \cdot L^2 + m \cdot L^2 + m \cdot l \cdot L \cdot (\cos(q1 + q2))$

$$Q := I_w + L^2\,M + L^2\,m + m\,l\,L\cos(q1 + q2) \quad \mathbf{(24)}$$

$P := I_t + m \cdot l^2 + m \cdot l \cdot L \cdot (\cos(q1 + q2))$

$$P := I_t + l^2\,m + m\,l\,L\cos(q1 + q2) \quad \mathbf{(25)}$$

$s2 := \left(I_w + \mathrm{L}^2 \cdot (m + M) \cdot \cos(2 \cdot q1) + m \cdot l \cdot L \cdot (\cos(q1 - q2))\right) \cdot q3 + P \cdot q4$

$$s2 := \left(I_w + L^2\,(m + M)\cos(2\,q1) + m\,l\,L\cos(q1 - q2)\right)q3 + \left(I_t + l^2\,m + m\,l\,L\cos(q1 + q2)\right)q4 \quad \mathbf{(26)}$$

$R := Matrix([[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, F, G], [0, 0, Q, P]])$

$$R := \quad \mathbf{(27)}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & (m+M)\,L\sin(q1) & -m\,l\sin(q2) & \cdots \\ 0 & 0 & I_w + L^2 M + L^2 m + m\,l\,L\cos(q1+q2) & I_t + l^2 m + m\,l\,L\cos(q1 + & \cdots \end{array}\right]$$

$s := Vector([-q1, q2, s1, s2])$

$s :=$ **(28)**

$$\left[\begin{array}{cc} -q1 & \cdots \\ q2 & \cdots \\ -(m+M)\,L\sin(q1)\,q3 - m\,l\sin(q2)\,q4 & \cdots \\ \left(I_w + L^2\,(m+M)\cos(2\,q1) + m\,l\,L\cos(q1-q2)\right)q3 + \left(I_t + l^2 m + m\,l\,L\,c\right. & \cdots \end{array}\right]$$

$q\_after := LinearAlgebra[MatrixInverse](R).\,(s)$

$q\_after :=$ **(29)**

$$\left[\begin{array}{cc} & \cdots \\ & \cdots \\ \overline{L^2 M\sin(q1)\cos(q1+q2)\,l\,m + L^2} & \cdots \\ \overline{L^2 M\sin(q1)\cos(q1+q2)\,l\,m + L^2} & \cdots \end{array}\right]$$

$q\_after := simplify(q\_after)$

$q\_after :=$ **(30)**

$$\left[\begin{array}{r} \cdots \\ \cdots \\ \cdots \\ \dfrac{L^2\, l\, m\, q3\, (m + M)\, \sin(-q2 + 2\, q1) + l\, L^2\, m\, (q3 + q4)}{} \cdots \end{array}\right]$$