Mathematical Modelling of Traffic Flow at Bottlenecks

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June 10, 2008

Abstract

This master's thesis gives a brief overview of mathematical modelling of traffic flow from different perspectives. Focus lies on the fluid flow analogy first derived by Lighthill, Whitham and Richards (LWR) in the 1950's which treats a traffic stream as a compressible one-dimensional fluid.

The LWR theory is applied to bottleneck queueing situations, more specifically to junctions with one outgoing and two incoming roads. Problems are analysed through two different models, utilising two different sets of flow merge rules. The models are compared and contrasted through steady-state solutions and more general queueing situations.

Benefits and drawbacks of the respective models are discussed and some further modifications and extensions are suggested.

Contents

1 Introduction

1.1 Background

To express the space-time $(x - t)$ interrelation of macroscopic traffic quantities, i.e. the relationship between vehicle flux or flow, f, density or concentration of vehicles, ρ, and speed, v, in the modelling of traffic flow one may use continuum approximations. This technique was first adopted by M.J. Lighthill, G.B. Whitham and P.I. Richards (LWR) in the 1950's ([22, 29]). In it, the traffic stream is treated as a one-dimensional compressible fluid ruled by two main postulates:

- traffic flow is conserved
- there is a one-to-one relationship between speed and density which leads to $f = f(\rho) = v(\rho)\rho$

Translated into mathematics, this resulted in the model

$$\rho_t + f(\rho)_x = 0$$

which is a non-linear scalar conservation law.

In this approach flow (or speed) is a function of density but this really only holds at equilibrium. Equilibrium, however, can seldom be observed in practice why it is hard to achieve a satisfactory flow-density relationship. One thus often has to refer to this idealised relationship in theory only.

Solutions of the continuum model sometimes lead to the generation of shock waves which are discontinuities in the density of traffic. Physically, this means that cars for some reason change speed abruptly, in theory without the chance to decelerate or accelerate at all. For someone in the traffic stream this has the consequence of queuebuilding. Not all discontinuities result in a shock wave though, as we will see in 2.2, there is also the possibility of a rarefaction wave corresponding to a traffic stream in which cars are speeding up and increasing the spacing between them.

The main benefit of continuum modelling is that compressibility is built into the state equations through the flow-density relationship defined in the above postulates. This means that when traffic streams enter areas of higher density, compression characteristics can be observed and in the same manner, when traffic streams enter areas of lower density, a diffusion can be seen. The result is a realistic model that does not have to rely on empirically determined dispersion models. These models thus offer good insight into the formation and dissipation of traffic congestion.

1.2 Aim and Purpose

This master's thesis strives to give a brief overview of the development of mathematical modelling of traffic flow – through time as well as from different perspectives – looking into the use and development of different analysis methods. Focus lies on queueing at bottlenecks, illustrated by the behaviour of traffic flow at a motorway slipway or at a junction with two incoming and one outgoing road. This is compared and contrasted through the "eyes" of two different mathematical models.

The master's thesis is aimed at anyone with an interest in applied mathematics and mathematical modelling. To avail oneself of the mathematical theory and the numerical implementation of the same, some prior knowledge of partial differential equations is recommended. However, anyone with a base in engineering should be able to understand and read this paper with benefit.

1.3 Paper Outline

This paper takes off in chapter 2 describing the mathematical background needed for understanding of general macroscopic mathematical modelling of traffic flow and the analytical construction of solutions of traffic flow problems. Then follows an overview of different modelling perspectives and their significance and dominance in traffic flow modelling research. Chapter 3 gives a presentation of two different ways to model traffic in general and bottleneck queueing in particular. This is succeeded, in chapter 4, by a comparison between the two perspectives and an analysis of their respective benefits and disadvantages togehter with some illustrations of the different results they yield when applied to a specific situation. To wrap up, a few other approaches to modelling of bottleneck queueing are presented and briefly discussed. The paper is brought to a close with a general discussion in chapter 5.

2 Mathematical Background

2.1 The Scalar Conservation Law

As mentioned in the introduction, the mathematical modelling of traffic flow often rests on a fluid flow analogy, treating the traffic stream as a one dimensional compressible fluid from which it can be deducted that traffic flow is conserved.

Generally, a conservation law states that *the change of the total amount of some physical entity in a region of space is equal to the inward net flux across the boundary of that region (provided there are no sources and sinks).*

To convert this into mathematics, consider traffic flow along a one-way road. Let the road run along the x-axis and let $\rho(x,t)$ denote the density of cars, i.e. the number of cars per unit length of road, at time t and point x. Now let (x_1, x_2) be an arbitrary interval of the x-axis. In integral form the conservation law can then be written as

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = f|_{x=x_1} - f|_{x=x_2} \tag{1}$$

where the first term represents the increase of cars per unit time and the second and third term represent the flux in per unit time and flux out per unit time respectively.

Now assuming that $\rho(x,t) \in C^1$, the left hand side of (1) can be written as $\int_{x_1}^{x_2} \frac{\partial \rho}{\partial t}\,dx$. Further, rewriting the right hand side of (1) as $-\int_{x_1}^{x_2} \frac{\partial f}{\partial x}\,dx$ one arrives at

$$\int_{x_1}^{x_1} (\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x})\,dx = 0 \tag{2}$$

This holds for every interval (x_1, x_2), why it follows that

$$\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x} = 0 \quad \Longleftrightarrow \quad \rho_t + f_x = 0 \tag{3}$$

This is the conservation law in form of a partial differential equation.

As we saw in 1.1, it is natural to assume that f is a function of ρ why it – in the modelling of traffic flow – takes the form

$$f = f(\rho(x,t)) \tag{4}$$

where f is assumed smooth. Further, it is reasonable – based on studies performed by Greenshields ([16]) already in 1935 but still considered accurate today – to expect that the speed of a car depends solely on the concentration of cars in its direct vicinity at (x,t), that is, $v = v(\rho)$. Thus one can use the simple function $v(\rho) = v_0(1 - \frac{\rho}{\rho_{\max}})$ to describe the speed-density relationship. Here v_0 is the "free" speed of the car – usually set to the maximum allowed speed of the road – and $\rho_{\max}$ is the maximal concentration of vehicles – corresponding to them being more or less bumper to bumper. It follows that

$$f(\rho) = v(\rho)\rho = v_0(1 - \frac{\rho}{\rho_{\max}})\rho, \quad 0 \le \rho \le \rho_{\max} \tag{5}$$

see figure 1.

For future reference we now define σ as the value of ρ for which f is maximised, i.e.

$$\sigma = \frac{\rho_{\max}}{2}$$

Flow corresponding to ρ values that are smaller than σ is hereafter referred to as light traffic and correspondingly, flow as a result of ρ values larger than σ, is referred to as heavy traffic.

With the help of (4),(3) can then be written as

$$\rho_t + f(\rho)_x = 0 \Longleftrightarrow \rho_t + f'(\rho)\rho_x = 0 \tag{6}$$

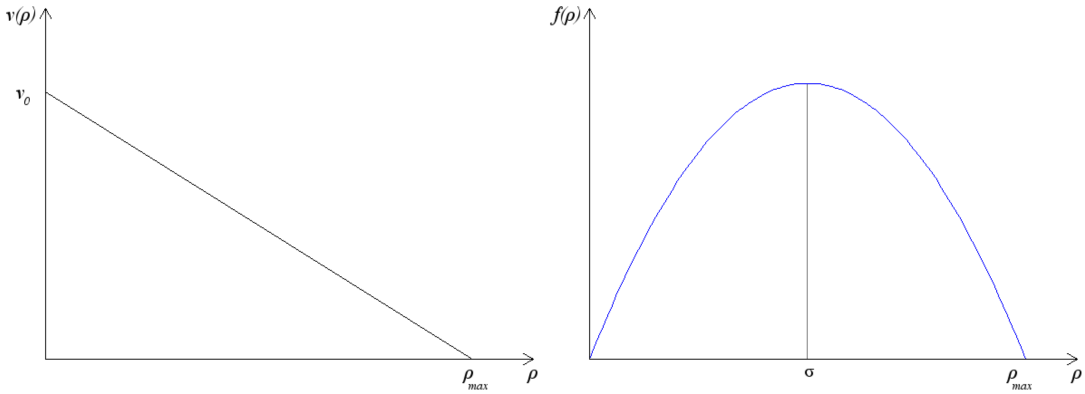

Figure 1: Speed and traffic flux as functions of density

2.2 Characteristics

In order for (6) to be soluble, an initial density distribution needs to be given which leads the problem

$$\begin{aligned} \rho_t + f(\rho)_x = 0 \quad &, \quad x \in \mathbb{R}, t > 0 \\ \rho(x, 0) = \rho_0(x) \quad &, \quad x \in \mathbb{R} \end{aligned}$$

Letting $x = x(t)$ be a level curve in the $x - t$ plane;

$$\rho(x(t), t) = \text{constant} = P_0$$

differentiating with respect to t and using $\rho_t = -f'(\rho)\rho_x$ one gets

$$0 = \rho_x x'(t) + \rho_t = \rho_x(x'(t) - f'(P_0))$$

Conventionally it holds that $x'(t) = f'(P_0)$, so the level curve is a straight line in the $x - t$ plane and is known as a *characteristic*. $f'(P_0)$ is known as the *signal speed* since it is the speed with which a wavefront will propagate.

To get an idea of what a solution to this non-linear – and thus rather mentally challenging – problem might look like, one can construct a geometrical solution for given initial data. The method is to, through each point x_0 on the x-axis, draw a straight line with speed $f'(\rho_0(x_0))$ in the $x - t$ plane. Along this line the solution has the value $\rho_0(x_0)$. Analytically, the solution connecting ρ, x and t can be implicitly expressed in the following manner:

$$\begin{cases} x &= f'(\rho_0(x_0))t + x_0 \\ \rho &= \rho_0(x_0) \end{cases}$$

2.3 Discontinuities and the Jump and Entropy Conditions

Despite having initial data $\rho_0(x) \in C^1$ one may find, as characteristics are drawn, that a continuous solution cannot be defined after a certain point in time; characteristics with different concentrations intersect.

A C^1 solution is acquirable if, from the implicit solution expression, one can solve for x_0 in the first equation and then substitute this into the second one. The implicit function theorem says that this can be done if

$$\frac{dx}{dx_0} = f''(\rho(x_0))\rho_0'(x_0)t + 1 \neq 0 \tag{7}$$

This holds for small positive t, so if the function $\rho_0(x)$ is smooth then there exists a smooth solution $\rho(x,t)$ for the aforementioned t's.

In order to continue to carry the solution out after the discontinuity occurs, the solution concept needs to be generalised. By multiplying (6) with a test function φ and integrating by parts one gets the following condition:

$$\int_0^\infty \int_{-\infty}^\infty (\rho\varphi_t + f(\rho)\varphi_x)\, dxdt + \int_{-\infty}^\infty \rho(x,0)\varphi(x,0)\, dx = 0, \forall \varphi \in C_0^1 \tag{8}$$

A function ρ that satisfies (8) is then said to be a *weak* solution of (6).

From the conservation law it can also be determined how a discontinuity moves. To derive this condition, let ρ be a piecewise C^1 solution of the conservation law, with a discontinuity curve $x = x(t) \in C^1$ in the $x - t$ plane. Further, let (a, b) be an interval running parallel with the x-axis so that the $x(t)$ intersects the interval at a time t. Now, let $\rho^\pm = \rho(x(t) \pm 0, t)$ be the values of the solution to the left and right of the of the discontinuity curve respectively. The conservation law then gives

$$\begin{aligned} f(\rho(a,t)) - f(\rho(b,t)) &= \frac{d}{dt}\int_a^b \rho\, dx = \frac{d}{dt}\Big(\int_a^{x(t)} \rho\, dx + \int_{x(t)}^b \rho\, dx\Big) = \\ &= \int_a^{x(t)} \rho_t\, dx + \rho^- x'(t) + \int_{x(t)}^b \rho_t\, dx - \rho^+ x'(t) = [\rho_t = -f_x] = \\ &= f(\rho(a,t)) - f(\rho(b,t)) + f(\rho^+) - f(\rho^-) - (\rho^+ - \rho^-)x'(t) \end{aligned}$$

Solving for x' one finds that the speed of the discontinuity satisfies

$$x'(t) = \frac{f(\rho^+) - f(\rho^-)}{\rho^+ - \rho^-} \overset{\text{def.}}{=} s \tag{9}$$

This last expression, (9), is known as the *Rankine-Hugoniot condition* or, more generally, the *jump condition* and simply says that the speed of the discontinuity is equal to the slope of the straight line through the points $(\rho^-, f(\rho^-))$ and $(\rho^+, f(\rho^+))$ on the graph of f.

With $\rho(x,t)$ being piecewise smooth and satisfying the inital data $\rho(x,0) = \rho_0(x)$, it can be shown that $\rho(x,t)$ is a weak solution of (8) iff

- the conservation law is satisfied at points where $\rho \in C^1$,
- (9) is satisfied at discontinuities.

By introducing weak solutions a problem arises in the form of it being possible to obtain many different solutions for the same initial data. In order to chose a unique and physically relevant one, yet another condition needs to be imposed; the *entropy condition*. This can be obtained by examining what happens when diffusion or viscosity is introduced. Invoking Fick's law and replacing f with $f(\rho) - \varepsilon\rho_x$ in the derivation of (4), one obtains the viscous equation

$$\rho_t + f(\rho)_x = \varepsilon\rho_{xx} \tag{10}$$

If ε is small one gets approximately the same solutions as those to (6) but the discontinuitues – the *shocks* – are now smoothed.

Consider now a solution of (6) with left and right limits ρ^- and ρ^+ respectively, and consisting of a single discontinuity with speed s. Such a shock is said to be allowed if it satisfies the *viscous profile condition*: For given constants ρ^+, ρ^- and s satisfying (9), there exists a viscous profile or travelling-wave solution

$$\rho(x,t) = v(\xi), \qquad \xi = \frac{x - st}{\varepsilon} \tag{11}$$

of (10) with $v(\xi) \to \rho^\pm$ as $\xi \to \pm\infty$. Thus, $\varepsilon \searrow 0$ implies that the viscous profile converges to the wanted shock with speed s.

Futher, it holds that the viscous profile $v(\xi)$ satisfies the ODE [1]

$$v' = f(v) - f(\rho^-) - s(v - \rho^-) \equiv \psi(v) \tag{12}$$

Regardless of whether $\rho^- > \rho^+$ or it is the other way around, it can be shown that

$$s < \frac{f(v) - f(\rho^-)}{v - \rho^-}, \quad \text{for all } v \text{ between } \rho^- \text{ and } \rho^+$$

is a necessary condition for an admissable shock. In the case of modelling of traffic flow with f as described in this section (i.e. f being strictly concave), the entropy condition reduces to

$$f'(\rho^-) > s > f'(\rho^+)$$

This follows from 13 through the use of the definition of differentiation.

[1]for details on this the reader is referred to [11]

2.4 The Riemann Problem

The conservation law in combination with piecewise constant data having a single discontinuity is known as the Riemann problem. To illustrate, consider for example the traffic flow function as defined by (5) with piecewise constant initial data

$$\rho(x,0) = \begin{cases} \rho_l, & x < 0 \\ \rho_r, & x > 0 \end{cases}$$

where ρ_l of course corresponds to data to the left of an arbitrary point of interest and ρ_r corresponds to data to the right of said point. The solution can now take two different appearances depending on the relation between ρ_l and ρ_r;

I. $\rho_l < \rho_r$

In this case there is a unique weak solution satisfying the entropy condition,

$$\rho(x,t) = \begin{cases} \rho_l, & x < st \\ \rho_r, & x > st \end{cases}$$

We have a shock that acts as a barrier between the two sections with constant data and the characteristics from either side go *into* the shock. A general graphic illustration of this can be seen in Figure 2.

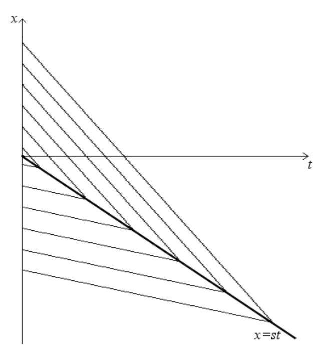

Figure 2: Shock wave solution to Riemann problem with $\rho_l > \rho_r$

II. $\rho_l > \rho_r$

With these conditions the solution will have the general appearance seen in Figure 3. By taking the jump that results in the shock at different places, the shock will change speed and direction which leads to the size of the fan like waves varying, again, see Figure 3. Since the jump quite naturally can be undertaken at infinitely many places this gives that we in fact have infinitely

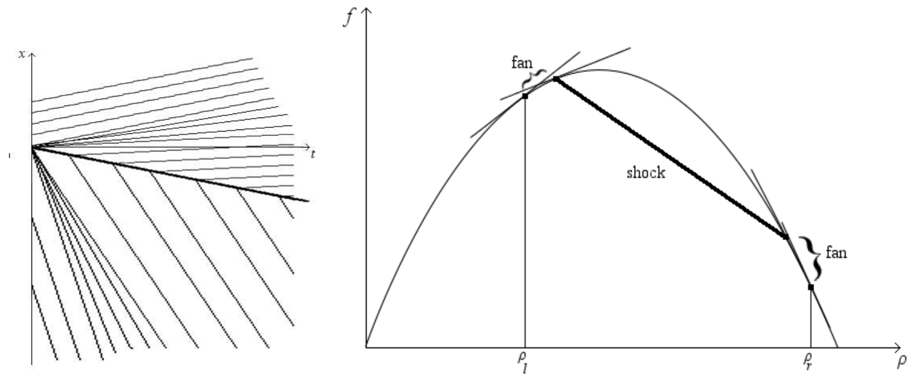

Figure 3: Shock wave solution to Riemann problem with $\rho_r < \rho_l$

many solutions in this case. In comparison to the solution in Figure 2 we see that the characteristics here travel *away* from the shock.

As mentioned there are infinitely many solutions to choose from in this case and as it turns out, the one we just saw is not the physically relevant one due to its violation of the entropy condition. Respecting the entropy condition we arrive at the solution

$$\rho(x,t) = \begin{cases} \rho_l, & x < f'(\rho_l)t \\ \frac{x}{t}, & f'(\rho_l)t \leq x \leq f'(\rho_r)t \\ \rho_r, & x > f'(\rho_r)t \end{cases}$$

This is known as a rarefaction or acceleration wave and has the appearance seen in Figure 4.

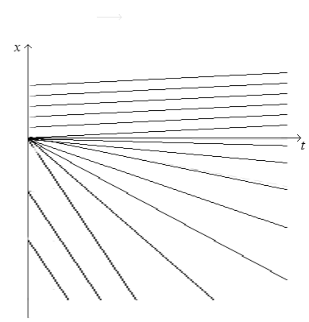

Figure 4: Rarefaction wave solution to Riemann problem with $\rho_r < \rho_l$

3 Different Modelling Perspectives

Traffic flow theory entered the scientific mind in the 1930's with studies performed by B.D. Greenshields [16] and W.F. Adams [1]. These early attempts at traffic modelling made use of probability theory to describe traffic flow and tried to find connections between speed and traffic density. This was followed by the development of different approaches to modelling in the 1950's introducing car-following, wave and queueing theory approaches which resulted – among other things – in the much dominant work by M.J. Lighthill and G.B. Whitham, [22], see discussions in [2, 5, 19, 27, 28, 29].

When attempting to model traffic flow there are three main perspectives to choose from: *micro-*, *meso-* and *macroscopic*. As one might suspect, *microscopic* models look at a single vehicle in a chain of others and how it responds to the traffic around it whereas *macroscopic* models look at all vehicles as a stream, treating them as a single unit. Mesoscopic models, also known as *kinetic* models, are somewhere in the border country between the other two, trying to marry the best elements of both worlds. In ongoing research macroscopic outlooks are clearly dominating, followed by the microscopic ones whereas mesoscopic models are rarely seen at all.

3.1 Microscopic Models: Follow-the-leader

The foundation of the follow-the-leader (FTL) theory rests upon the assumption that drivers react in a specific way to a stimulus from the vehicle(s) directly ahead (or, in some cases, behind) them and that nothing else interferes with this. Thus the basic equation of the FTL theory can be described in psychological terms:

$$\text{response} = \text{sensitivity} \times \text{stimulus}$$

where the response is taken to be the acceleration of the following vehicle (i.e. the vehicle one is observing, in (13) denoted j). Empirically, it has been acertained that a high correlation is present between the response of a driver and the relative speed of his vehicle and the one he is following. In the basic equation, this relative speed is taken to be the stimulus. To determine the sensitivity factor a lot of suggestions have been made. Most commonly, it is chosen as a constant or as inversely proportional to the spacing between consecutive vehicles but this is really a question of stability and steady-state flow.

Expressed in mathematical terms, the FTL-model can be described in the following manner

$$\ddot{x}_j(t+T) = \lambda(\dot{x}_{j-1}(t) - \dot{x}_j(t)) \tag{13}$$

where x_j is the position of the j^{th} vehicle (cf. Figure 5), T is the time lag, i.e. the reaction time of the driver, and λ is the sensitivity.

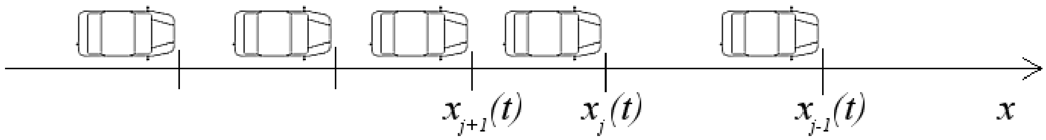

Figure 5: Follow-the-leader model

If traffic moves at constant speed (13) is always satisfied; there is no unique steady-state equation. Transitions from one steady state to another, after a change of velocity of the leading car, can be achieved by a process governed by (13), namely by integrating (13) on both sides. Doing this disposes of T and generates a relationship between the spacing and speed of vehicles in a steady-state stream, i.e. it provides a nice and logical transition from micro- to macroscopic modelling.

To illustrate, let us first make the definition

$$x_{j-1}(t) - x_j(t) \overset{\text{def.}}{=} h_j(t)$$

$h_j(t)$ is thus the relative spacing between the leading and the following vehicle. Integrating (13) with λ equal to a constant it now follows that

$$\int_{-\infty}^{\infty} \ddot{x}_j(t+T)\,dt = \lambda \int_{-\infty}^{\infty} (\dot{x}_{j-1}(t) - \dot{x}_j(t))\,dt \tag{14}$$

$$\Big[\dot{x}_j(t+T)\Big]_{-\infty}^{\infty} = \lambda\Big[x_{j-1}(t) - x_j(t)\Big]_{-\infty}^{\infty}$$

$$v^{\text{finish}} - v^{\text{start}} = \lambda(h_j^{\text{finish}} - h_j^{\text{start}}) \tag{15}$$

Clearly, the $(h_j^{\text{finish}} - h_j^{\text{start}})$ part of the right hand side of (15) has the unit length. Inverting $h_j(t)$ we thus get the unit $\frac{1}{\text{length}}$ which, as it so happens, is the same unit as that for density ρ. Hence, we can rewrite (15) as

$$v^{\text{finish}} - v^{\text{start}} = \lambda(\frac{1}{\rho_j^{\text{finish}}} - \frac{1}{\rho_j^{\text{start}}}) \tag{16}$$

Now, assuming that when we start our analysis our road is jam packed, we have $\rho_j^{\text{start}} = \rho_{\max}$ and $v^{\text{start}} = 0$. Generalising and setting $v := v^{\text{finish}}$ and $\rho := \rho^{\text{finish}}$, Eq. (16) becomes

$$v = \lambda(\frac{1}{\rho} - \frac{1}{\rho_{\max}})$$

Assuming λ is no longer a constant but of the form $\lambda = \frac{c}{h_j^x}, x \in \mathbb{Z}$ in accordance with earlier discussions, different $v - \rho$ relationships can now be determined. Going back to Eq. (14) we see that choosing $x = 1 \Rightarrow \lambda = \frac{c}{h_j}$ it holds that

$$v = c \int_{-\infty}^{\infty} \frac{\dot{h}_j(t)}{h_j(t)} \, dt = c[\ln(h_j(t))]_{-\infty}^{\infty} = c \ln \frac{h_j(\text{finish})}{h_j(\text{start})} = c \ln \frac{\rho_{\max}}{\rho}$$

where c is some constant. v and the resulting f can be seen in Figure 6, where for simplicity c has been set to 1.

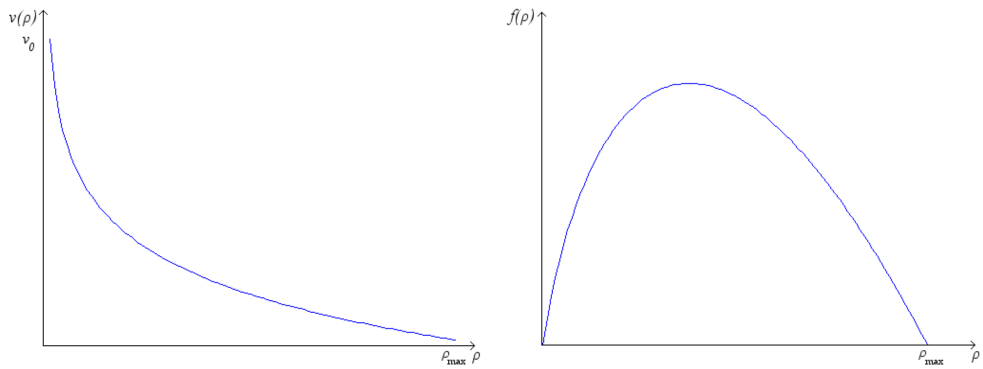

Figure 6: Speed and traffic flux as a function of density, $\lambda = \frac{c}{h_j}$

Letting $\lambda = \frac{c}{h_j^2}$ we get

$$v = c \int_{-\infty}^{\infty} \frac{\dot{h}_j(t)}{h_j^2(t)} \, dt = c[-\frac{1}{h_j(t)}]_{-\infty}^{\infty} = c\,(\rho_{\max} - \rho) = c\,\rho_{\max}(1 - \frac{\rho}{\rho_{\max}})$$

Comparing this latter result to the speed-density relationship defined in Eq. (5) we see that they are identical presuming we let $c = \frac{v_0}{\rho_{\max}}$. From these calculations it is clear how micro- and macroscopic modelling are connected.

As can be expected, the FTL theory can be made more elaborate, that is, it can be made to account for more factors. (13) can be extended to let a vehicle respond to more vehicles than just the one right in front of it. If one wants to, for instance, include also the vehicle right in front of the currently leading vehicle one gets the following:

$$\ddot{x}_{j+1}(t + T) = \lambda_1(\dot{x}_j(t) - \dot{x}_{j+1}(t)) + \lambda_2(\dot{x}_{j-1}(t) - \dot{x}_{j+1}(t)) \qquad (17)$$

This can be generalised to hold for yet more vehicles.

3.2 Mesoscopic Models

Mesoscopic models describe the traffic entities at a high level of detail but their behaviour and interactions at a lower level of the same. Roughly speaking, mesoscopic models can be divided into three main categories[2]:

- **Headway distribution models:** These models neither considers nor tracks single vehicles separately but describe the distribution of headways of the individual vehicles.
- **Cluster models:** Vehicles sharing a certain property are grouped together and then followed through the network as an entity. In this type of model all roads are given their specific speed-density relationship.
- **Gas-kinetic continuum models:** These models describe the dynamics of velocity distributions, implicitely bridging the gap between microscopic driver behaviour and the aggregated macroscopic modelling approach.

3.3 Macroscopic Models

Macroscopic models accumulate all individual vehicles and describe them as flows. The main approach, pioneered by B.D. Greenshields in 1935 ([16]), is to find a speed-density relationship. As stated in the introduction this speed-density relation is then generally put into the scalar conservation law derived by Lighthill, Whitham and Richards. From this, modifications can be made to make the model more accurate for a specfic situation or the equation can be kept as it is for a more general analysis. Different adaptations to the classic LWR theory and the results thereof can be found in [8, 18, 33].

4 Queueing at Bottlenecks

As stated in the introduction, the main part of this paper deals with analysis of queueing at bottlenecks, here in the shape of motorway merges and simple junctions with two incoming and one outgoing road. We are thus dealing with traffic behaviour in situations like the one illustrated in Figure 7. We will carry out the analysis through two different modelling approaches, one for which – besides analytical results – some numerical simulations are carried out in MatLab[3] by use of Godunov's method[4]. Following this is a brief presentation and discussion of alternative ways to model bottleneck queueing.

4.1 Main Roads With Slipways Considered as Source Terms

For this model we consider a main road on which the conservation law (1) holds. To this we add a slipway, feeding yet more cars into an already existing

[2][31], p.438. In actuality, models are in [31] divided into four groups but the last one is titled simulation and can thus hardly be considered an actual theory

[3]concrete details on how this is done can be found in the code in appendix B

[4]details can be found in appendix A

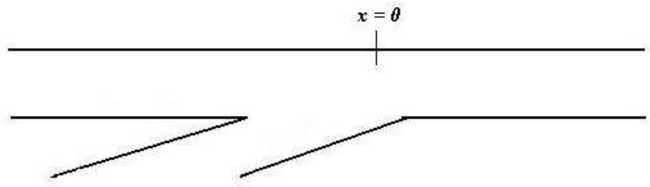

Figure 7: Motorway merge point

traffic stream. This slipway can then be treated as a source term $s(t), s > 0$.

It then holds weakly that

$$\rho_t + f(\rho)_x = s(t)\delta(x) \tag{18}$$

$$\Longleftrightarrow$$

$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \neq 0 \\ f(\rho_-) + s = f(\rho_+), & x = 0 \end{cases}$$

Observing that

$$\delta(x) = H'(x) = -(1 - H(x))'$$

where $H(x)$ is Heaviside's step function, (18) can be written as

$$\rho_t + (f(\rho) - s(t)H(x))_x = 0$$

$$\Longrightarrow \rho_t + (f(\rho) + s(t)(1 - H(x)))_x = 0, \;\; x \in \mathbb{R} \tag{19}$$

Letting the road run along the x-axis and placing the merge point at $x = 0$ it then, for the fluxes, holds – according to [12] – that

$$\begin{cases} f(\rho) + s \equiv g(\rho), & x < 0; \\ f(\rho), & x > 0. \end{cases}$$

see Figure 8.

For this situation we thus have different flow curves for $x < 0$ and $x > 0$ and when traffic passes the merge point we must – while making sure to conserve the flow in accordance with the above specified flow condition for the point $x = 0$ – jump between the respective flow curves depicted in Figure 9. Details on how this jump is undertaken can be found in 4.1.1.

To simplify the analysis of the model and make sure it works properly, $s(t)$ is first set to a constant so that it can easily be compared to analytically constructed solutions. Later on, as accuracy is confirmed, s is allowed to change over time in various ways. Further, the slipway is given a complete right of way meaning that any car coming up to the merge point on the

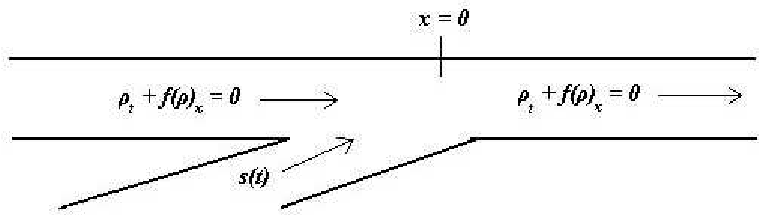

Figure 8: Flux conditions before and after a merge point

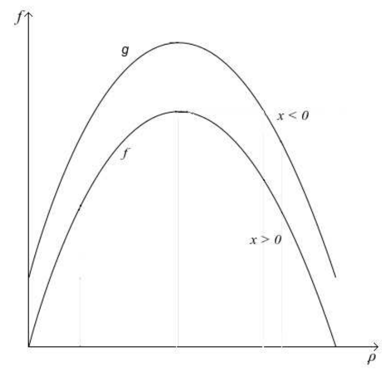

Figure 9: Flow curves corresponding to functions f and g as defined in 4.1

"main" road has to give way to cars from the slipway. In practice this is ensured by never letting the total flow of the "main" road plus the incoming flow from the slipway exceed the maximum capacity of the part of the main road that is stream of the merge point. In the MatLab program found in appendix B this is guaranteed through the validity checks performed for each source term option.

4.1.1 Flux Help Functions

As we have seen, we have different flux conditions on either side of the merge point. This means that, when constructing the solution we must – when passing the merge point – jump between the curves illustrated in Figure 9. To make sure that the flow is conserved this has to be done horizontally. Obviously though, there are infinitely many locations on which this jump can take place. In order to choose a correct and physically relevant solution, it is therefore useful to introduce so called "help functions" which uniquely determine the solution of interest. These functions can be defined in the following matter, as they are in [12]:

Given initial and final traffic densities, ρ_- and ρ_+ respectively, and flux functions f and g (see 4.1), define

$$\hat{f}(\rho;\rho_+) = \begin{cases} \min\limits_{v\in[\rho,\rho_+]} f(v), & \rho \le \rho_+ \\ \max\limits_{v\in[\rho_+,\rho]} f(v), & \rho > \rho_+ \end{cases}$$

$$\check{g}(\rho;\rho_-) = \left\{ \begin{array}{ll} \max\limits_{v\in[\rho,\rho_-]} g(v), & \rho \le \rho_- \\ \min\limits_{v\in[\rho_-,\rho]} g(v), & \rho > \rho_- \end{array} \right\} = \hat{g}(\rho_-;\rho)$$

$\hat{f}(\cdot;\rho_+)$ is here a non-decreasing function consisting of increasing parts separated by plateaus where the function is constant. Correspondingly, $\check{g}(\cdot;\rho_-)$ is a non-increasing function consisting of decreasing parts separated by plateaus. $\hat{f}$ and $\check{g}$ are both continuous. The correct "place" from where to jump from one flux curve to another is the point (there will be only one) where the curves $\hat{f}$ and $\check{g}$ intersect[5]. The accuracy of this jumping point can be verified by viscous profile analysis, see [13].

To illustrate, consider the Riemann problem depicted in the left part of Figure 10. At $t = 0$ we have initial densities ρ_- and ρ_+ and we see that the help functions $\hat{f}$ and $\check{g}$ intersect at exactly one point corresponding to the density ρ^-. This is thus the relevant density for $t > 0$. Going from the upper curve to the lower one at this very point will ensure a correct and unique solution.

To construct the characteristic plot and get a picture of what the solution looks like one proceeds as follows: Draw the tangents to the flux curve at the points ρ_- and ρ_+. These tangents correspond exactly to the characteristics for the solution at $t < 0$ and $t > 0$ respectively so one simply transfers them from the flow curve diagram into the $x-t$-plane. We are now left with a big gap which – for the picture of the solution to be complete – of course must be filled while making sure that the flux is conserved as we pass the point $t = 0$. The first step in filling this gap is to transfer the tangents from the upper flux curve at $\rho = \rho^-$ into the $x - t$ plot, letting the characteristics go out from the t-axis. As can be clearly seen, the density ρ^- obtained with the help functions, is smaller than the starting density ρ_- which, in accordance with the theory presented in 2.4, means that we get a rarefaction wave filling the empty space created as we go from ρ_- to ρ^-. To conserve the flux we must now jump from the upper to the lower curve, going from density ρ^-

[5]should the two graphs not intersect this of course corresponds to a problem that for some reason is not soluble

on the upper curve to σ on the lower curve. By doing so we connect the two respective quarter plane problems (one for $x < 0$ and one for $x > 0$). The remaining gap in the upper quarter plane is then filled with a rarefaction wave as we go from σ to ρ_+. In the end we arrive at the solution seen to the right in Figure 10.

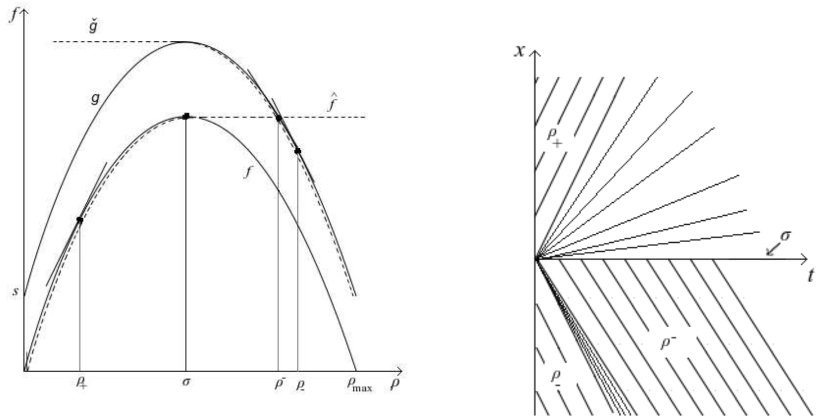

Figure 10: Selecting a point to jump between the flow curves using the functions described in 4.1.1, and corresponding characteristic plot for the complete solution

4.1.2 Steady-State Solutions and Simulation Results

Letting s be constant and assuming that plenty of time has passed since we started observing our bottleneck, we get the steady-state solutions in Figure 11. Thus there are two qualitatively different steady-state solutions, one for light traffic and one for heavy. From here, we make a step in s which – analytically – results in the characteristic plots seen in Figures 12–15. It is important to note here that it does not matter whether s is the result of light or heavy traffic on the slipway at this point since the flow is the same either way.

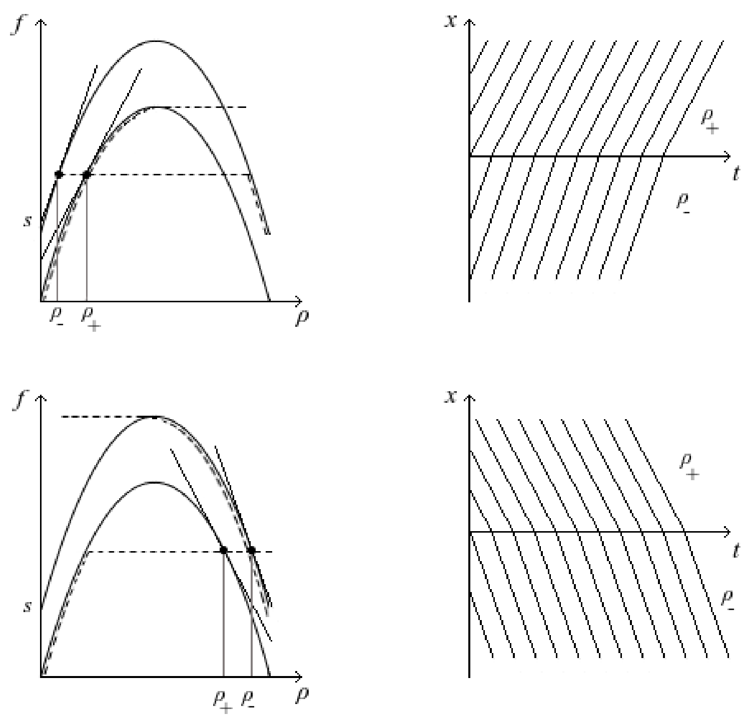

Figure 11: Steady-state solutions used as a starting point for a source term in the shape of a step function

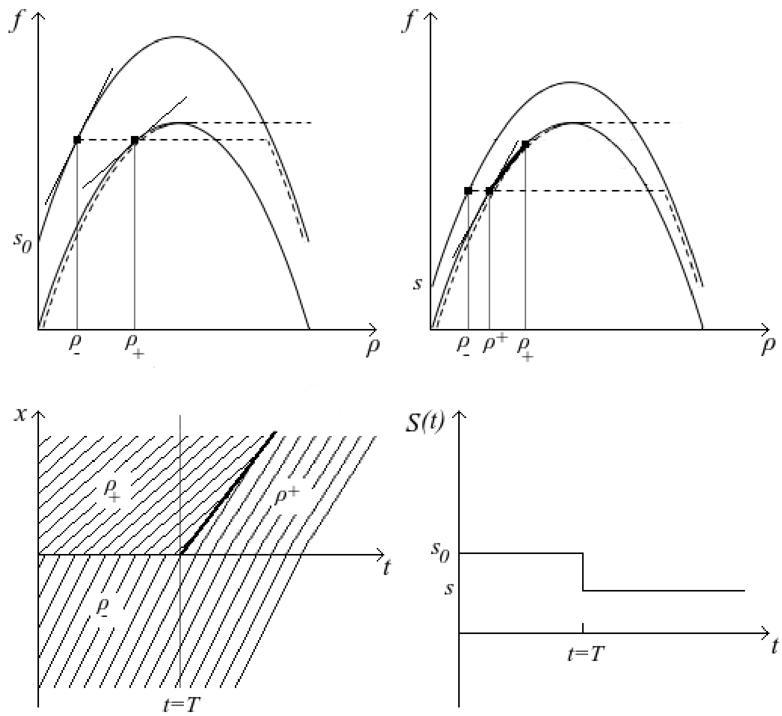

Figure 12: Step down from a steady-state solution with ρ_- and ρ_+ both on the light side

The jump spot determined by the flux functions corresponds to a density lower than the one present downstream of the merge point. There will thus be a build up of traffic and accordingly, a shock can be seen here.

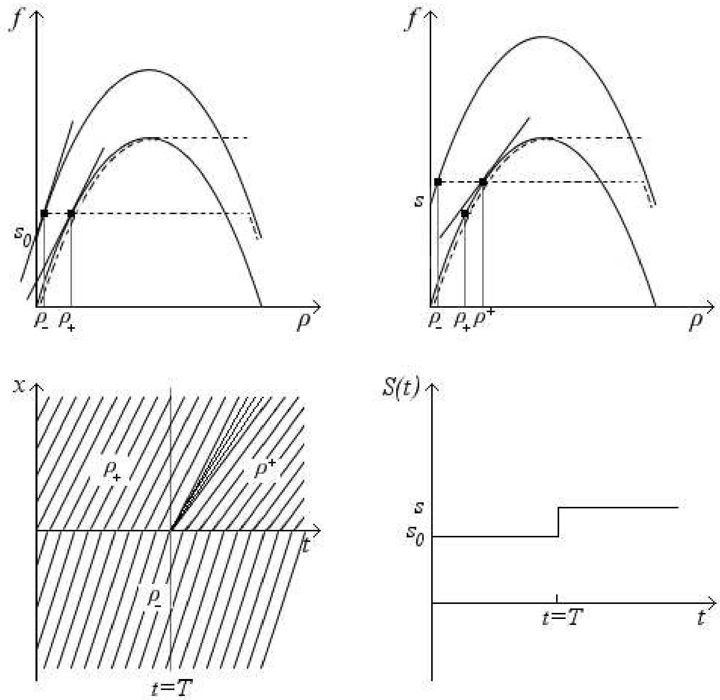

Figure 13: Step up from a steady-state solution with ρ_- and ρ_+ both on the light side

This represents the opposite of the situation in Figure 12. The total incoming traffic density after the change in s is here greater than the density downstream of the merge point. We will thus experience a drop in density which results in a rarefaction wave travelling downstream of the merge point. The bigger the difference in ρ^+ and ρ_+ the bigger the rarefaction wave with the limit being the step size bringing s up to the maximum capacity of the outgoing road.

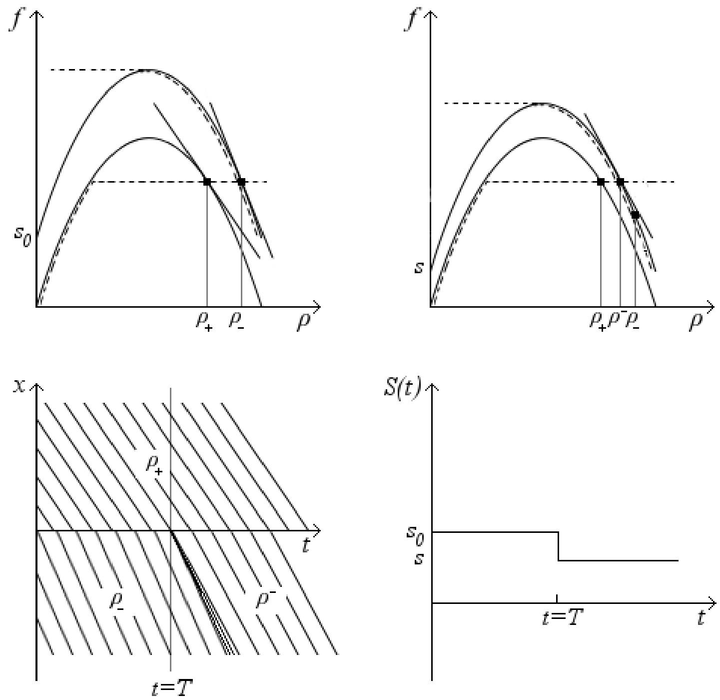

Figure 14: Step down from a steady-state solution with ρ_- and ρ_+ both on the heavy side

Letting s take a step down corresponds to the incoming traffic dropping. There is thus more room for the cars on the main road to go through the merge which results in the dispersion of cars on the main road upstream of the merge point. Correspondingly we have a rarefaction wave travelling upstream.

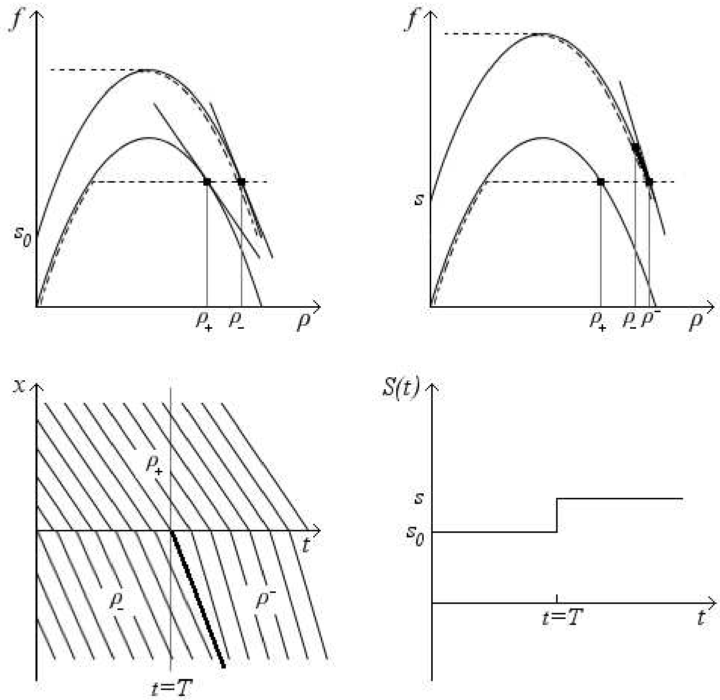

Figure 15: Step up from a steady-state solution with ρ_- and ρ_+ both on the heavy side

Again, we here have the opposite situation to that depicted in Figure 14. An increase in s means that more cars from the main road will have to wait and slow down to get through the merge point with the result of build up congestion.

As can be expected, we generally, when going from lighter to heavier traffic see a result in the shape of congestion building up in the form of queues. In the same manner, going from heavier to lighter traffic results in cars being able to speed up and thus increase the space between them.

At this point the reader might wonder if there is not ever a time when traffic goes from being light to being heavy. Unsurprisingly, there is. This happens when the total flux coming into the junction exceeds the capacity of the outgoing road. In this case the help functions determine a jumping point in accordance with Figure 16.

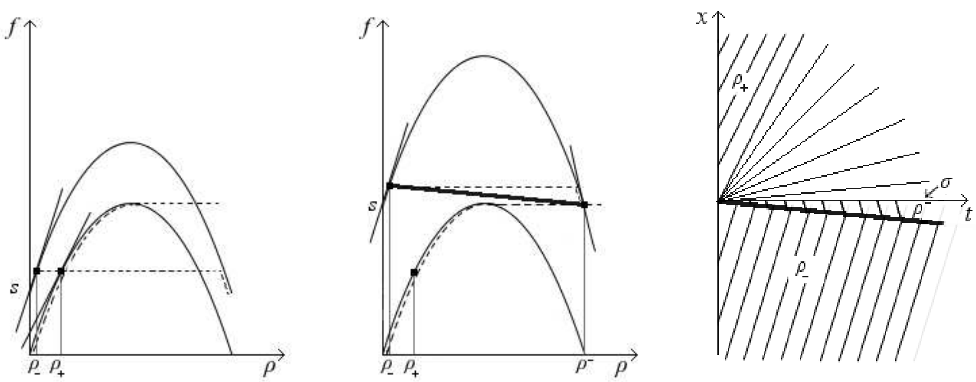

Figure 16: Step up from a steady-state solution with ρ_- and ρ_+ both on the "light" side. The step is here so large that it causes the traffic to go from light to heavy

The opposite scenario, a sudden change in s leading to traffic going from being heavy to being light is however not a possibility. Starting from a steady-state solution, a ρ_- value corresponding to heavy initial traffic will always cause the flux help functions to intersect at a ρ value larger than σ. Accordingly, traffic in this case remains heavy.

To confirm the analytically determined characteristic plots, simulations were carried out in MatLab using the program found in appendix B, see Figures 17–18.

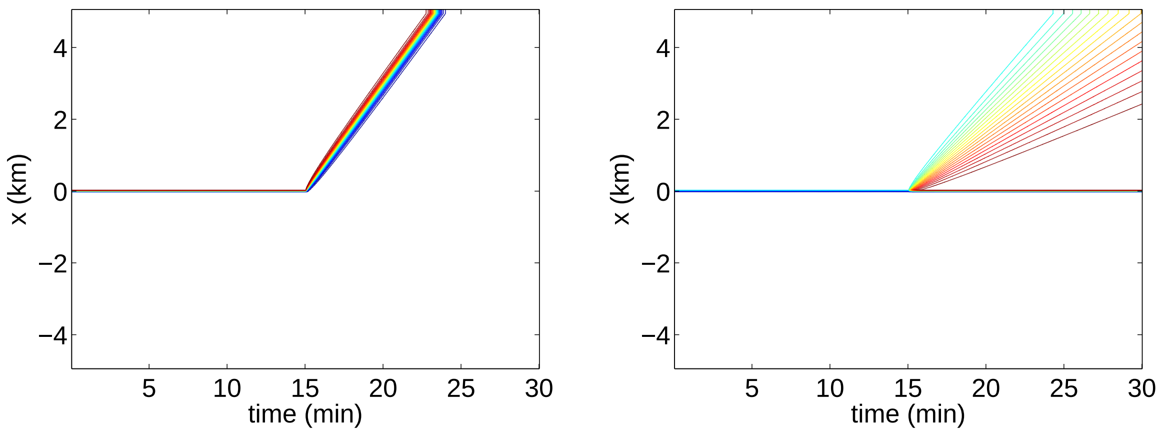

Figure 17: Step down and up respectively from steady-state solution with ρ_- and ρ_+ both on the "light" side

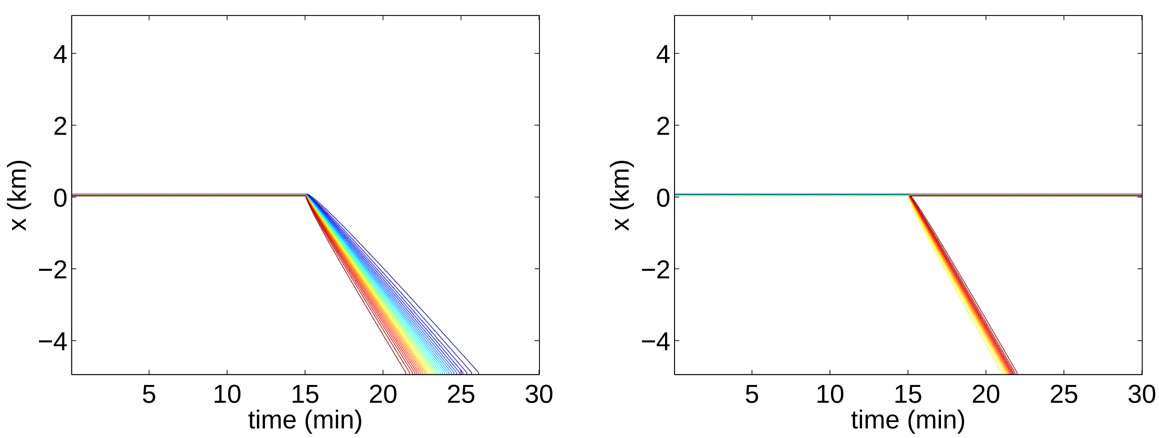

Figure 18: Step down and up respectively from steady-state solution with ρ_- and ρ_+ both on the "heavy" side

It can be analytically confirmed that the numerically and analytically produced plots in fact do match. A minor inconsistency can be detected for shocks at a low number of numerical run throughs. This is however merely the result of Godunov's method only being first order accurate which causes shocks to get smeared out.

To get some more interesting figures, source terms more interesting than just constants and step functions have been tried, see Figures 19–26.

The sinusoidal source term plots correspond to traffic coming into the main stream in waves. This could be representative of a situation where traffic lights control the source term flow causing the stream to fluctuate in a repetetive manner. As for the formation of rarefaction and shock waves, the results remain consistent with earlier discussions.

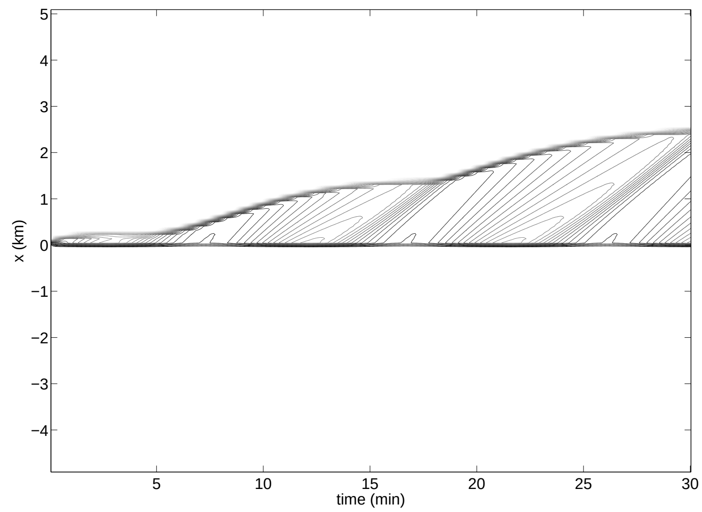

Figure 19: Sinusoidal source term, $\rho_- < \rho_+$

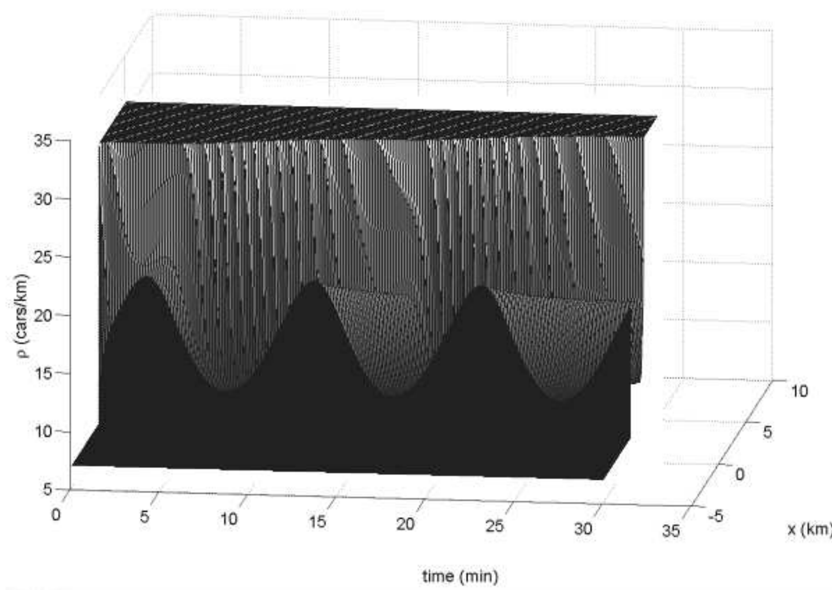

Figure 20: Sinusoidal source term, $\rho_- < \rho_+$

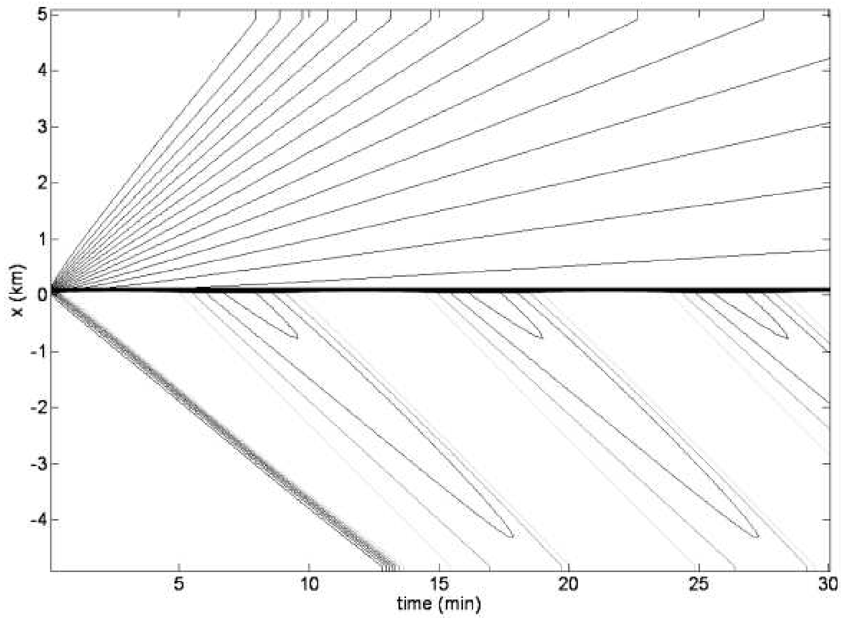

Figure 21: Sinusoidal source term, $\rho_- > \rho_+$

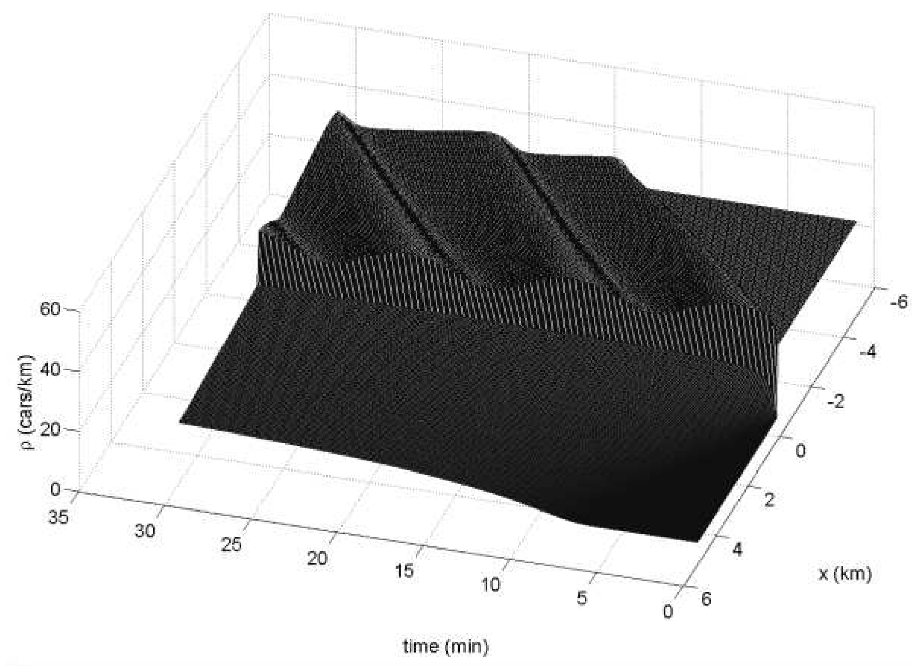

Figure 22: Sinusoidal source term, $\rho_- > \rho_+$

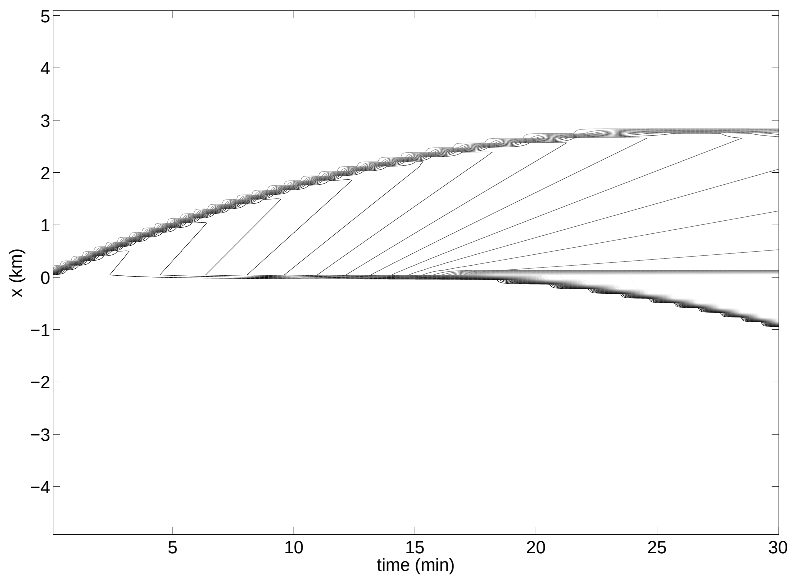

Figure 23: Constantly increasing source term, $\rho_- < \rho_+$

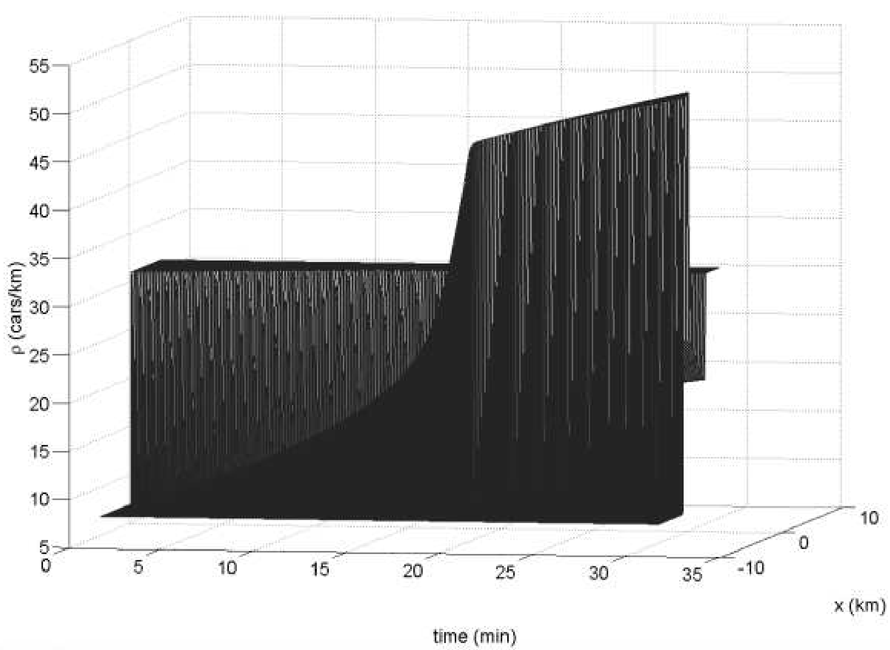

Figure 24: Constantly increasing source term, $\rho_- < \rho_+$

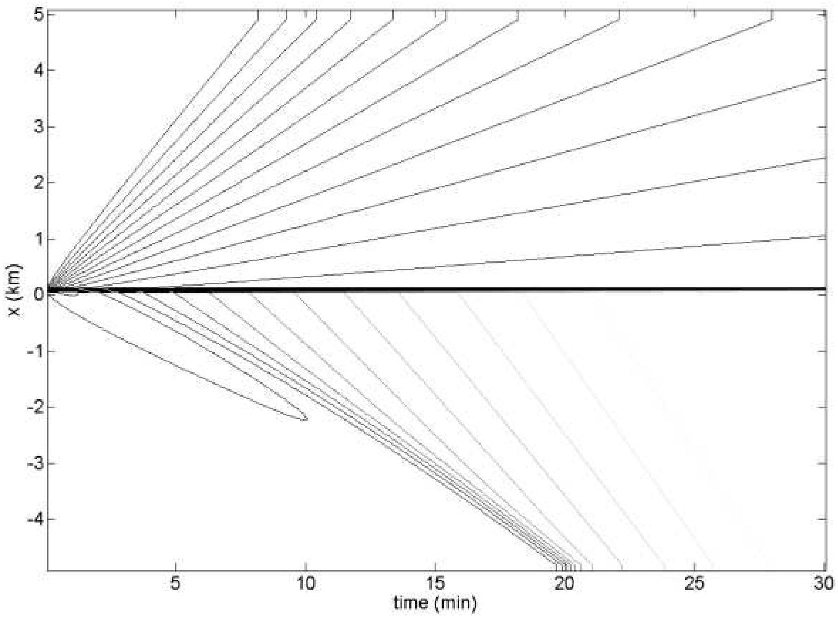

Figure 25: Constantly increasing source term, $\rho_- > \rho_+$

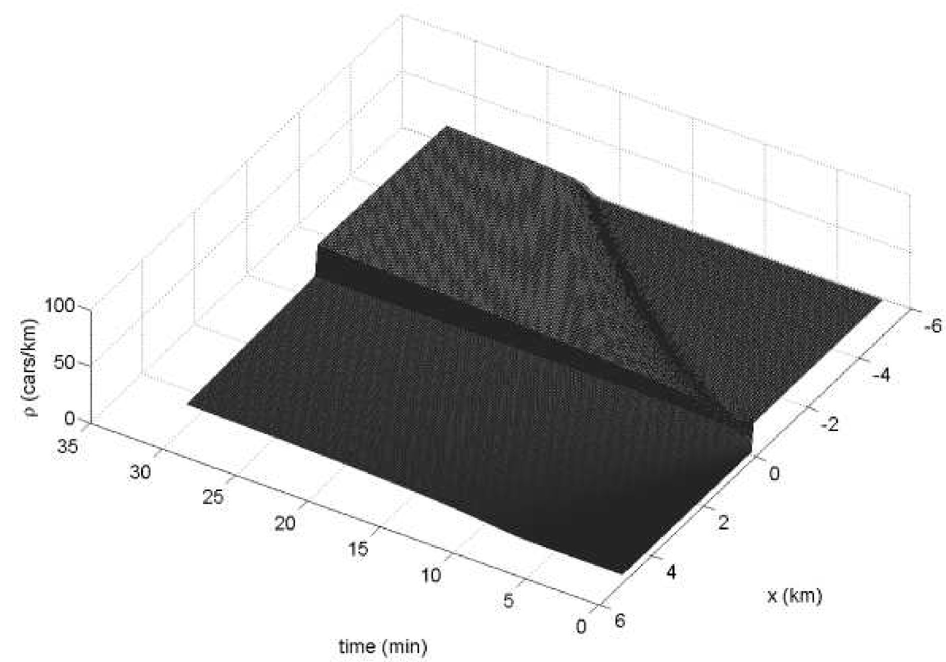

Figure 26: Constantly increasing source term, $\rho_- > \rho_+$

The constantly increasing source terms could be used to represent a situation where, say, a big event ends upon which everyone is trying to get home at the same time, using the only way on to the main road. In this case shocks arise regardless of whether we are going from heavy to light traffic or the other way around. As can be expected, queues are far more dominant when we go from light to heavy traffic. In fact, we here, as time goes by, actually end up with shocks travelling backwards as well as forwards, with the former starting to build up as the flow on the main road closes in on maximum capacity. Going from incoming traffic that is heavier than the outgoing traffic results only in a backwards travelling shock, being the result of too many cars wanting to enter the main road at the same time. Once the pressure drops though, cars can accelerate freely which is reflected in the plot by rarefaction waves.

4.2 The Coclite/Piccoli/Garavello Way

Many modifications have been made to the LWR theory over the years in order for the model to fit a specific situation. Since it can be used to analyse the bottleneck situation that we are interested in, and doing so while treating the slipway in the same way as the other roads, we choose here to look at the theory developed by G. Coclite, B. Piccoli and M. Garavello in [10] which has been further extended by Piccoli et al. in [6].

The paper [10] is based on two main assumptions:

(A) drivers behave in a specific way which in this case means that traffic from incoming roads are distributed onto outgoing roads according to fixed coefficients

(B) with respect to (**A**) drivers seek to maximise fluxes[6]

Consider a junction J with n incoming and m outgoing roads, cf. Figure 27.

The cofficients mentioned in **(A)** are determined in accordance with the following:

Fix a matrix

$$A = \{\alpha_{ji}\}_{j=n+1,\dots,n+m,\, i=1,\dots,n} \in \mathbb{R}^{m\times n}$$

such that

$$0 < \alpha_{ji} < 1, \quad \sum_{j=n+1}^{n+m} \alpha_{ji} = 1 \tag{20}$$

for each $i = 1, \dots, n$ and $j = n+1, \dots, n+m$ and where α_{ji} is the percentage of drivers arriving from the i^{th} incoming road that take the j^{th} outgoing

[6] in [6] this is specified to "drivers behave in order to maximize the flux through junctions"

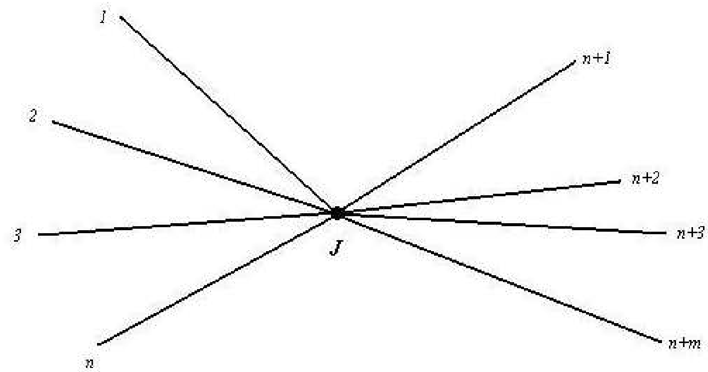

Figure 27: Junction with n incoming roads and m outgoing roads

road. These two assumptions represent a situation where drivers have a final destination and thus choose their outgoing road in accordance with this but maximise the flux whenever possible.

To see how these assumptions can be turned into a functioning mathematical theory quite a few definitions are introduced in [10]. To begin with, consider a network of roads modelled by a finite collection of intervals $I_i = [a_i, b_i] \subset \mathbb{R}, i = 1, \dots, N$ on which the conservation law (6) is analysed. For the junction J it then – as a total – holds that

$$\sum_{i=1}^{n} f(\rho_i(t, b_i)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, a_j)) \tag{21}$$

where $\rho_i, i = 1, \dots, n$ are the car densities of the incoming roads and $\rho_j, j = n+1, \dots, n+m$ are the outgoing ones. For simplicity, let these densities lie between zero and one, that is, let

$$(t, x) \in \mathbb{R}_+ \times I_i \mapsto \rho_i(t, x) \in [0, 1],\ \ i = 1, \dots, n$$

be the incoming traffic densities and

$$(t, x) \in \mathbb{R}_+ \times I_j \mapsto \rho_j(t, x) \in [0, 1],\ \ j = n+1, \dots, n+m$$

the outgoing ones.

We need – for each road I_i – the function ρ_i to be a weak entropic solution, i.e. for $\varphi : [0, \infty[\ \times\ I_i \to \mathbb{R}$ smooth with compact support on $]0, \infty[\ \times\ ]a_i, b_i[$

$$\int_0^\infty \int_{a_i}^{b_i} \left(\rho_i \frac{\partial \varphi}{\partial t} + f(\rho_i) \frac{\partial \varphi}{\partial x} \right) dx\, dt = 0$$

Now, assume that every junction J on the network is given by a finite number of incoming and outgoing roads, $(i_1, \dots, i_n)$ and $(j_1, \dots, j_m)$ respectively. Then the complete model (for the network) is given by a couple $(\mathcal{I}, \mathcal{J})$ where $\mathcal{I} = \{I_i : i = 1, \dots, N\}$ is the set of roads and $\mathcal{J}$ the set of junctions. Fixing a junction with incoming and outgoing roads $I_1, \dots, I_n$ and $I_{n+1}, \dots, I_{n+m}$ respectively, the weak solution at J is a collection of functions $\rho_l : [0, \infty[\times I_l \to \mathbb{R}, l = 1, \dots, n+m$ such that

$$\sum_{l=1}^{n+m} \int_0^\infty \int_{a_l}^{b_l} \left(\rho_l \frac{\partial \varphi_l}{\partial t} + f(\rho_l) \frac{\partial \varphi_l}{\partial x} \right) dx\, dt = 0$$

for each smooth $\varphi_1, \dots \varphi_{n+m}$ with compact support in $]0, \infty[\times]a_l, b_l]$ for $l = 1, \dots, n$ (for incoming roads) and $]0, \infty[\times [a_l, b_l[$ for $l = n+1, \dots, n+m$ (for outgoing roads) that are also smooth across the junction. That is, the φ's satisfy

$$\varphi_i(\cdot, b_i) = \varphi_j(\cdot, a_j), \quad \frac{\partial \varphi_i}{\partial x}(\cdot, b_i) = \frac{\partial \varphi_j}{\partial x}(\cdot, a_j), \quad i = 1, \dots, n,\ j = n+1, \dots, n+m$$

Now, let $\rho = (\rho_1, \dots, \rho_{n+m})$ be such that $\rho_i(t, \cdot)$ is of bounded variation for all positive t. Then ρ is an admissible weak solution of (6), related to the matrix A, satisfying (20) at the junction J iff the following holds:

(i) ρ *is a weak solution at the junction;*
(ii) $f(\rho_j(\cdot, a_j+)) = \sum_{i=1}^n \alpha_{ji} f(\rho_i(\cdot, b_i-))$ *for each* $j = n+1, \dots, n+m$*;*
(iii) $\sum_{i=1}^n f(\rho_i(\cdot, b_i-))$ *is maximum subject to (ii).*

(ii) is here a sort of adaptation of Kirchhoff's circuit law, "what goes in must come out", and *(iii)* refers to the aforementioned will of drivers to maximise fluxes through junctions.

For the flux (or flow), f, the following is imposed in [10]:

$(\mathcal{F})$ $f : [0, 1] \to \mathbb{R}$ is smooth, strictly concave and satisfies $f(0) = f(1) = 0$, Further, there exists a unique maximum $\sigma \in]0, 1[$ such that $f'(\sigma) = 0$, i.e. σ is a strict maximum.

The solution at junction J is obtained from solving the Riemann problem for said junction. How this is done will be demonstrated next.

Define a mapping as follows:

Let $\tau : [0, 1] \to [0, 1]$ *be the map satisfying*

1. $f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0, 1]$
2. $\tau(\rho) \neq \rho$ for every initial density $\rho \in [0, 1] \setminus \{\sigma\}$

This is a well defined map for which it holds that

$$0 \leq \rho \leq \sigma \Longleftrightarrow \sigma \leq \tau(\rho) \leq 1, \quad \sigma \leq \rho \leq 1 \Longleftrightarrow 0 \leq \tau(\rho) \leq \sigma$$

All these definitions and notations considered, the authors of [10] then present their main result:

Let f be a real valued function defined on $[0,1]$ satisfying $(\mathcal{F})$ at the junction J. For every $\rho_{1,0}, \dots, \rho_{n+m,0} \in [0,1]$ there then exists an admissible weak solution $\rho = (\rho_1, \dots, \rho_{n+m})$ (under *(i)-(iii)*) to the conservation law (6) at the junction J such that

$$\rho_1(0,\cdot) \equiv \rho_{1,0}, \dots, \rho_{n+m}(0,\cdot) \equiv \rho_{n+m,0}$$

Further, there exists a unique $(n+m)$-tuple $(\hat{\rho}_1, \dots, \hat{\rho}_{n+m}) \in [0,1]^{n+m}$ such that

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1] & \text{if } 0 \leq \rho_{i,0} \leq \sigma \\ [\sigma, 1] & \text{if } \sigma \leq \rho_{i,0} \leq 1 \end{cases}$$

and

$$\hat{\rho}_j \in \begin{cases} [0, \sigma] & \text{if } 0 \leq \rho_{j,0} \leq \sigma \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[& \text{if } \sigma \leq \rho_{j,0} \leq 1 \end{cases}$$

for $i = 1, \dots, n$ and $j = n+1, \dots, n+m$ respectively. By drawing a simple picture of the admissible regions of values for $\hat{\rho}_{i,j}$ it is clear that they exactly match the regions obtained using the help functions from 4.1.1.

From this, the correct value of ρ, resulting in a shock or a rarefaction wave, can always be determined for fixed i and j as defined above. For details on how this solution is picked the reader is referred to [10].

The results presented in [10] however only apply to situations where there the number of incoming roads is smaller than or equal to the number of outgoing roads, that is, when $n \leq m$. In order for it to be relevant also for the queueing situation depicted in Figure 7, (i.e. when $n = 2$ and $m = 1$ and, more generally, for any situation where $n > m$), one further condition must be imposed, as is done in [6]; fix a *right of way* parameter $q \in]0, 1[$ where q represents the percentage of cars coming from the incoming road labelled $i = 1$, and designate the rule

(C) Let C be the quantity of vehicles that actually can enter the outgoing road. Then qC cars through the junction come from the incoming road labelled $i = 1$ and $(1-q)C$ come from the road $i = 2$.

There are however a few problems with this rule (**C**). One of them is when the sum of the incoming flows is smaller than the maximum allowed outgoing flow. In this case it just does not make sense to use (**C**) since it would in fact make the flux through the junction smaller than it maximally could be. To find out whether (**C**) really should be used in this context, Piccoli has been contacted upon which it has been clarified that (**C**) in fact only should come into play when the maximum sum of the incoming flows exceeds the capacity of the outgoing road.

With this, not all problems are resolved though, it is still possible for rule (**C**) to violate rule (**B**), consider for instance the situation in Figure 28. In

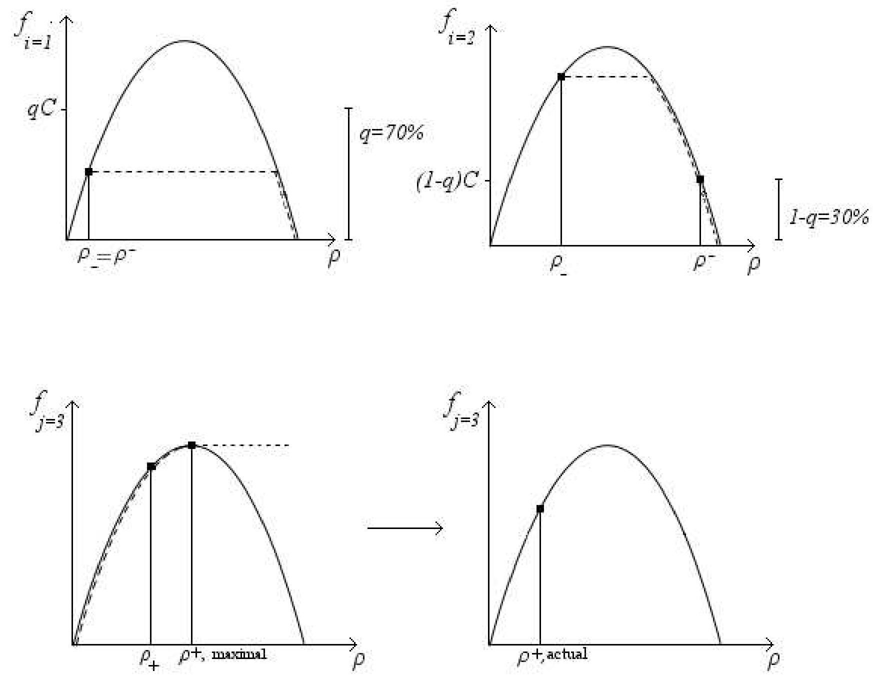

Figure 28: Maximum outgoing flow not obtained due to value of q

this case, the maximum outgoing flow should, according to the help function $\hat{f}$ be equal to $f(\sigma) = C$. Clearly, the maximised sum of the fluxes of the two incoming roads is larger than σ why this is in fact possible. However, due to the very same reason that σ is a possible flow through the junction, we must also invoke rule (**C**). Assume, for instance, that $q = 0.7$. This means that we are supposed to take a flow corresponding to 70% of C from road one and add that to 30% of C taken from road two to make up the flux going through the junction. Taking these latter 30 % is not a problem but it is obvious that that $0.7C$ is well above the maximally available flow from road one. Taking the allowed maximum from road one and adding that to the $(1-q)C$ from road two we get a flow through the junction that is nowhere near as large as C. As a result the flux through the junction does not get maximised at all. Since the entire model presented in [10] rests upon the flux maximisation this is of course a considerable problem.

In response to this, Piccoli says that there is in fact no exact way to satisfy rule (**C**) but that one should choose the solution that is “more close to satisfy rule (**C**)”. He further refers to [3] and [9] where indeed some further explanation is offered.

The following, new, definitions are made in [3]:

$$\gamma_i^{\max} = \begin{cases} f(\rho_{i,0}), & \rho_{i,0} \in [0, \sigma] \\ f(\sigma), & \rho_{i,0} \in]\sigma, 1] \end{cases}$$

$$\gamma_j^{\max} = \begin{cases} f(\sigma), & \rho_{j,0} \in [0, \sigma] \\ f(\rho_{j,0}), & \rho_{j,0} \in]\sigma, 1] \end{cases}$$

for $i = 1, 2$ and $j = 3$.

In terms of the help functions this corresponds to

$$\gamma_i^{\max} = \max_{0 \le v \le 1} \check{f}(v; \rho_{i,0})$$

$$\gamma_j^{\max} = \max_{0 \le v \le 1} \hat{f}(v; \rho_{j,0})$$

These new quantities $\gamma_i^{\max}$ and$\gamma_j^{\max}$ represent the maximum flux that can be obtained by a single wave solution on each piece of road.

Further, to maximise the number of cars through J the authors make the definition

$$\Gamma = \min\{\Gamma_{\text{in}}^{\max}, \gamma_j^{\max}\}$$

where $\Gamma_{\text{in}}^{\max} = \gamma_1^{\max} + \gamma_2^{\max}$. The desired – and maximally obtainable – flux through the junction is thus equal to Γ.

To solve the Riemann problem we must determine the fluxes $\hat{\gamma}_i = f(\hat{\rho}_i)$ in accordance with the earlier presented main result. Two scenarios must be taken under consideration:

I. $\Gamma_{\text{in}}^{\max} = \Gamma$

II. $\Gamma_{\text{in}}^{\max} > \Gamma$

In case (**I**) one sets $\hat{\gamma}_i = \gamma_i^{\max}$ and a solution can be constructed through the analysis of three quarter plane problems. In case (**II**) however, define, in the space (γ_1, γ_2), the lines

$$r_q : \gamma_2 = \frac{1-q}{q}\gamma_1$$

$$r_\Gamma : \gamma_1 + \gamma_2 = \Gamma$$

One might find the definition of r_q a touch confusing but as $\gamma_1 = qC$ it holds that $\gamma_2 = (1-q)C$ why it makes perfect sense. It is obvious that the final fluxes must be in the region

$$\Omega \stackrel{\text{def.}}{=} \{(\gamma_1, \gamma_2) : 0 \le \gamma_i \le \gamma_i^{\max}, i = 1, 2\}$$

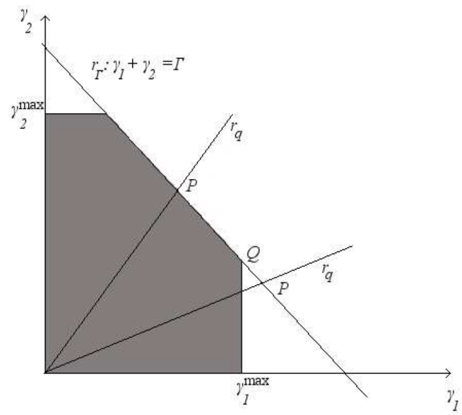

Figure 29: Admissible region for final fluxes and points P and Q depending on r_q

as seen in Figure 29. Now, let P be the point of intersection between the two lines. We then get two possibilities; $P \in \Omega$ and $P \notin \Omega$. For the case where P is in Ω we simply set $(\hat{\gamma_1}, \hat{\gamma_2}) = P$. When P is not in Ω, we set $(\hat{\gamma_1}, \hat{\gamma_2}) = Q$ where Q is the projection of P onto the set Ω along the line r_Γ. This latter case – $P \notin \Omega$ – must then be considered to correspond to Piccolis urge to choose the solution that most closely satisfies (**C**).

With all this considered, solution uniqeness can finally be obtained.

Returning now to the example seen in Figure 28 we get the admissible region and point $P(\notin \Omega)$ depicted in Figure 30. Drawn to scale, this corresponds to the fluxes and characteristic quarter plane plots seen in Figures 31 and 32. As expected, the determination of P and subsequently Q, keeps γ_1 at $\gamma_1^{\max}$ and reduces γ_2 just enough to ensure that the flux through the junction is equal to $f(\sigma)$. Due to the admissible set of ρ values this renders a shock on the ingoing road labelled two, but changes nothing on the road labelled one. Correspondingly, with the flux through the junction being equal to $f(\sigma)$ we experience a rarefaction wave on the outgoing road as the traffic density drops to ρ_+.

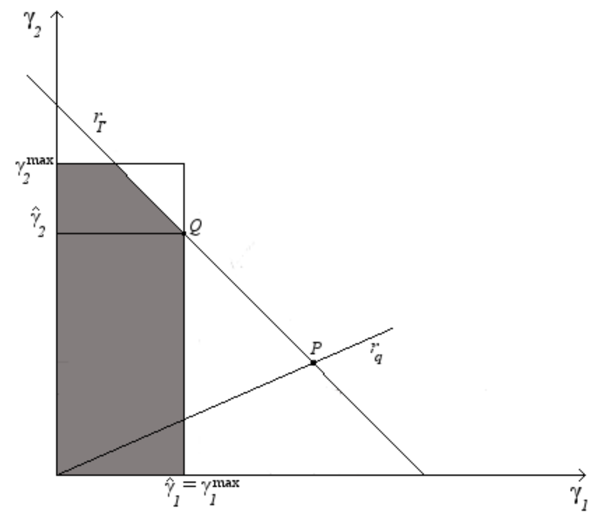

Figure 30: Final fluxes determined with the help of the lines r_q and $r_\Gamma, q = 0.7$

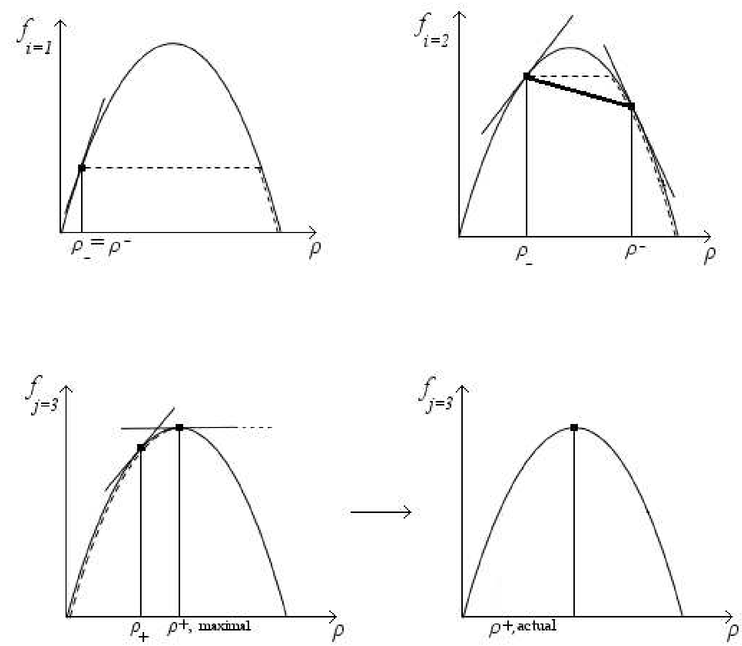

Figure 31: Final fluxes as a result of P and Q from Figure 30

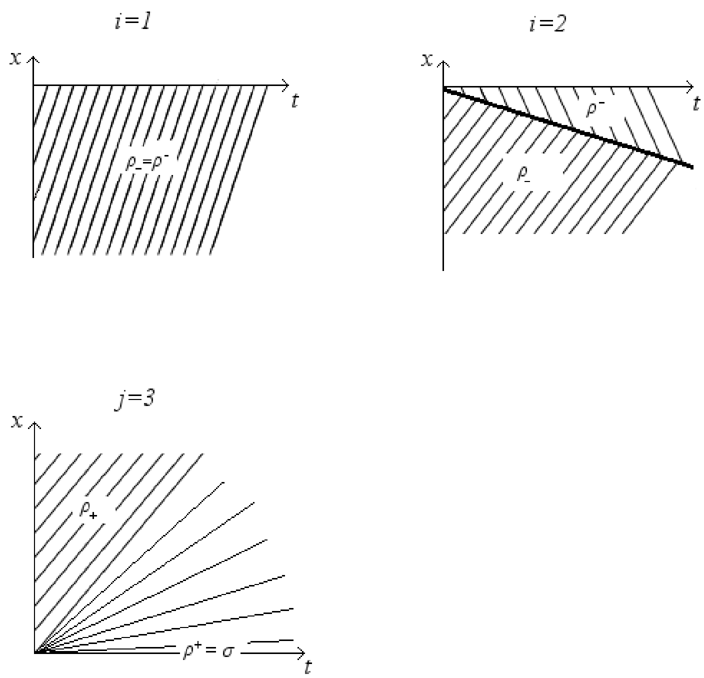

Figure 32: Characteristic plots corresponding to the fluxes in Figure 31

Should we instead have $q = 0.3$ we get the admissible region and point $P(\in \Omega)$ seen in Figure 33 with corresponding final fluxes and characteristic plots in Figures 34 and 35. In this case, both incoming fluxes are reduced

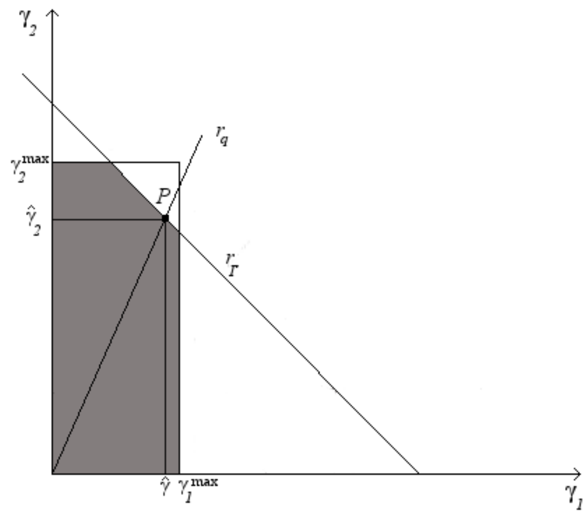

Figure 33: Final fluxes determined with the help of the lines r_q and $r_\Gamma, q = 0.3$

as a result of P which renders shocks on both roads. Naturally, the result for the outgoing road remains consistent with the result obtained for $q = 0.7$.

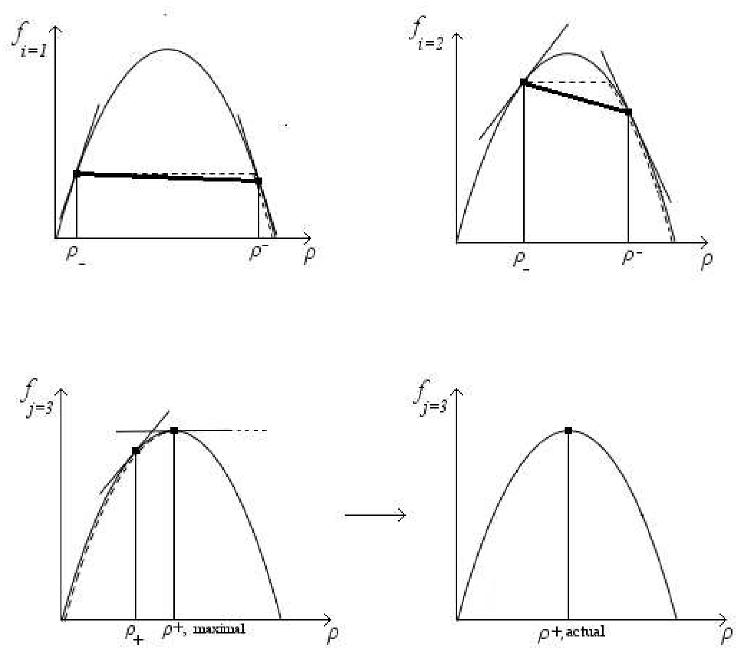

Figure 34: Final fluxes as a result of P

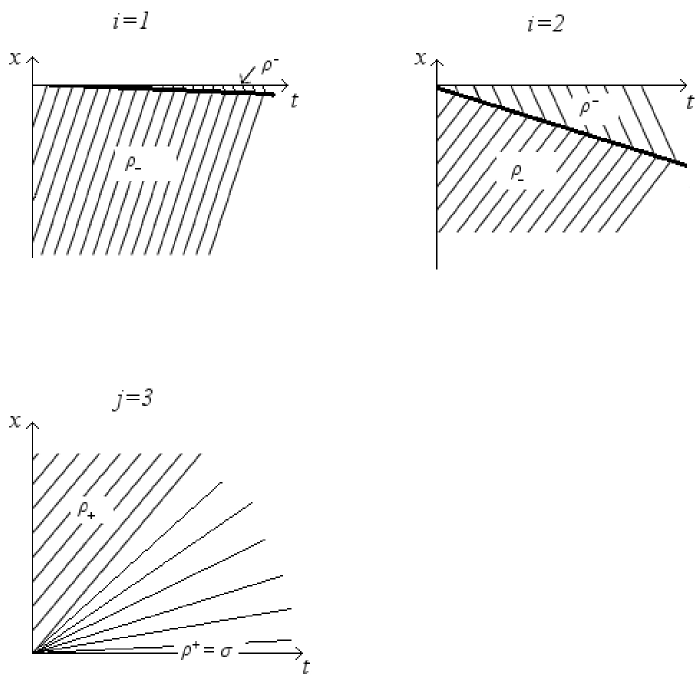

Figure 35: Characteristic plots corresponding to the fluxes in Figure 34

Solving the same problem using the model from 4.1 we get the resulting flows and characteristic plots seen in Figures 36 and 37. The flow on road two is here, again, considered as a constant source term s and stays fixed throughout the analysis. Since s lies fixed, the entire flux reduction required

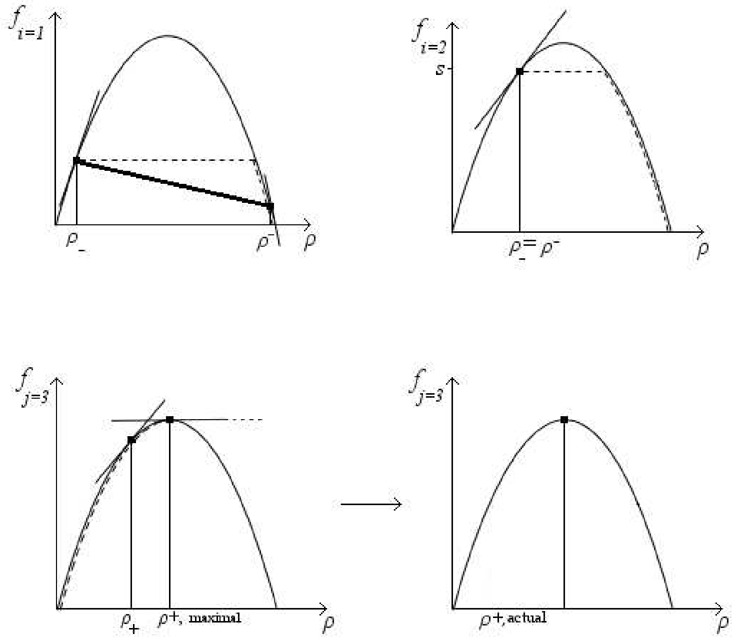

Figure 36: Final fluxes as a result of P

to get the incoming flux down to σ must be accounted for by road one. Qualitatively, we get the same result for this road as we did when using the Coclite/Piccoli/Garavello model with $q = 0.7$ but with the shock going down the opposite road. For the outgoing road the densities σ and ρ_+ are exactly the same as before why the solution does not change.

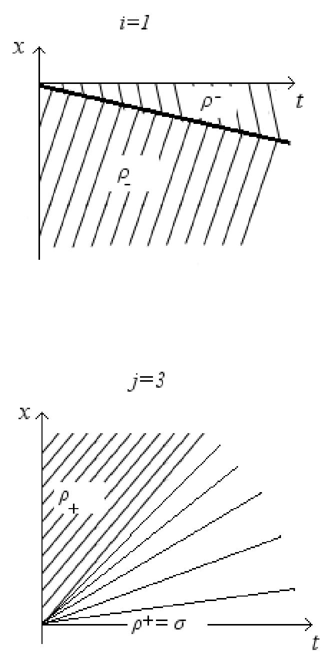

Figure 37: Characteristic plots corresponding to the fluxes seen in Figure 36

4.3 Comparison Between Analysed Models

Comparing the two models presented in 4.1 and 4.2, there are of course a number of questions to answer regarding the validity and realism of them both.

For the former model, needless to say, giving one road complete right of way over the other in this sense is possibly not very common on heavily trafficked roads. That said, it corresponds to a right of way sign being placed at the merge point, forcing one traffic stream to make way for another, something that is entirely possible to implement in reality. It is however improbable that maximum capacity would never be reached; complete motorway stops are far too common for anyone to believe that. However, it stands to reason that on fairly low trafficked roads the 4.1 model could hold quite well. Due to its relative simplicity, it is also illustrative of the study of strictly qualitative results of flow merges under certain conditions.

Of course, the model can be used to describe other situations than just motorway merge point behaviour, it also serves a description of situations where – for some reason or other – a road's surface condition suddenly changes leading to a drop in maximum free speed and thus capacity, see figure 38. The theory can also be used to describe a situation where a road goes down in number of lanes which corresponds to the flow curves changing in accordance with Figure 39.

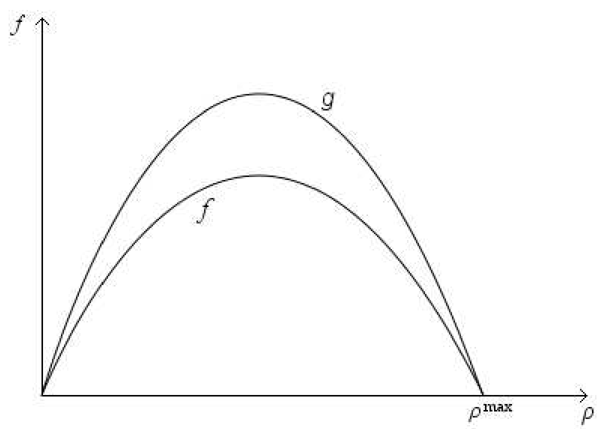

Figure 38: Changing flux curve as a result of a drop in the free speed

For the Coclite/Piccoli/Garavello model, the first question that comes to mind is whether it is actually reasonable to assume that drivers indeed do behave as to maximise the traffic flow. Presuming that they do though, this model of course offers a welcome and needed complement to the one presented in 4.1 since it can handle far more scenarios.

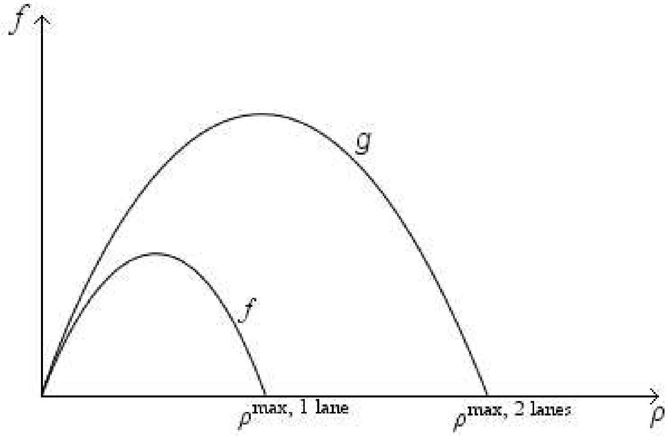

Figure 39: Changing flux curve as a result of a road going down from two lanes to one

To illustrate, consider the situation seen in Figure 40 where, for clarity, the flow curves are drawn to scale. This corresponds exactly to the situation in Figure 38, the outgoing road here has a lower free speed than the two incoming roads. Here, we have a maximised possible incoming flow that exceeds the capacity of the outgoing road. If we were to try to use the model from 4.1 here, we would not be able to resolve the situation. This is due to the fact that s – corresponding to the flow of the second incoming road – is higher than the capacity of the outgoing road and cannot be changed. Applying the technique presented in [3] though, we get the gamma region and subsequent point P (q has been set to 0.7) seen in Figure 41 with corresponding characteristic plot in Figure 42. The problem is solved wihtout complication while making sure to maximise the flux through the junction.

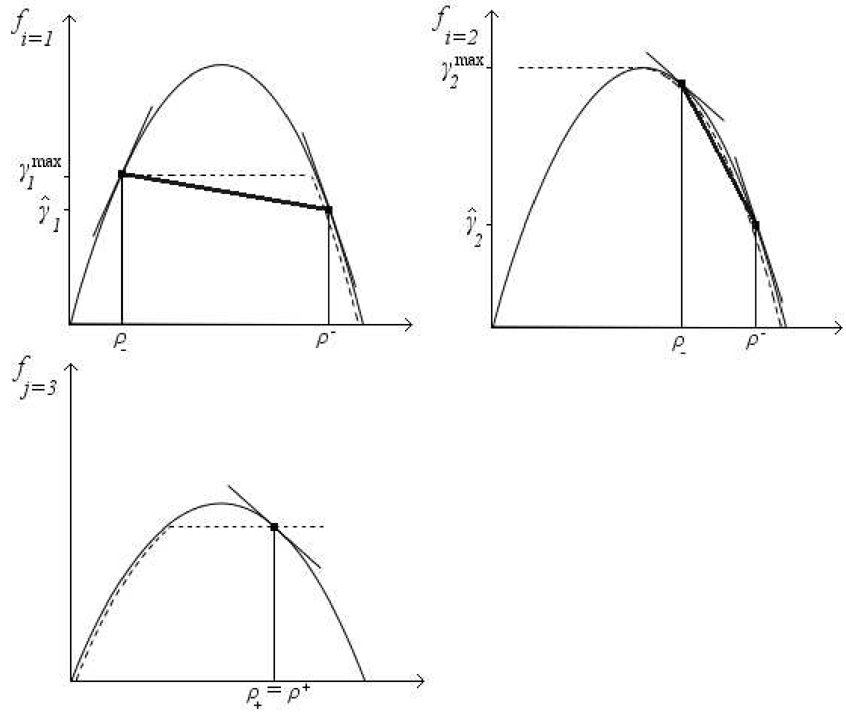

Figure 40: Situation that cannot be resolved with the theory presented in 4.1 but is treatable with the theory from [3]

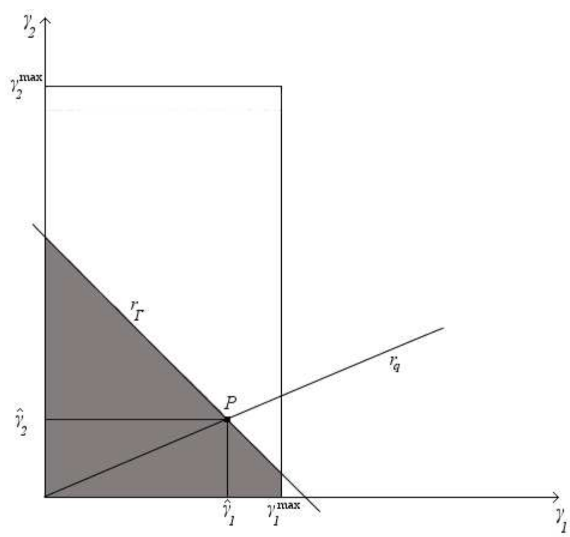

Figure 41: Gamma region for the situation depicted in Figure 40, $q = 0.7$

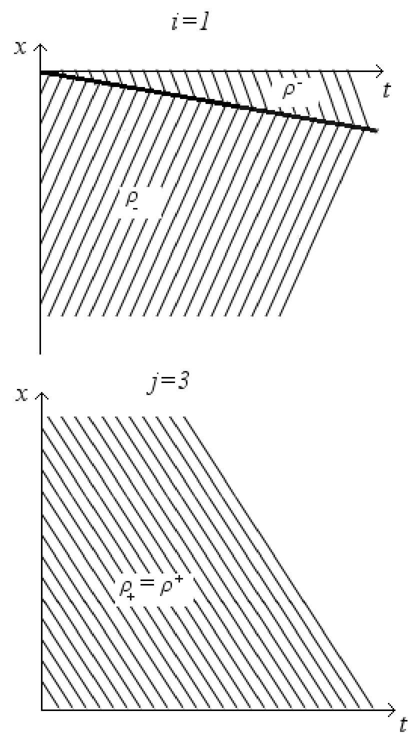

Figure 42: Characteristic plot corresponding to Figures 40 and 41

The maximisation assumption is not the only matter of concern here though. There is also the question of the parameter q as an idea. My first objection is the fact that it is defined as a *right of way* parameter. Saying that a certain percentage of traffic on the outgoing road should come from the first road and so on is not really giving a right of way in either direction. It is simply stating a desired traffic distribution without really considering reality. The idea of q was originally developed in [3] which deals with telecommunication networks. These are supposedly easier to manipulate why it makes sense there. For traffic flow though, it is reasonable that one cannot interfere in the same way. Some sort of distribution parameter is of course necessary for the queueing problem to be soluble but I feel that it could be implemented with greater care.

Another thing worth looking at in terms of model comparison is how the steady-state solutions look and if they are the same for both models. For the model in 4.1 we know that the steady-state solutions look like in Figure 11. Finding steady-state solutions for the Coclite/Piccoli/Garavello model is a lot harder.

If ρ_+ for the outgoing road is representative of light traffic the maximum allowed flow through the junction is equal to $f(\sigma)$. In this case two scenarios are possible; either the maximised sum of the incoming fluxes is smaller than or equal to $f(\sigma)$ or it is larger than $f(\sigma)$.

In the first case a steady-state solution is a possibility only if the sum of the incoming fluxes happens to equal $f(\rho_+)$. In the second case a steady-state solution is not possible since the incoming fluxes will have to be reduced in order to get through the junction. When ρ_+ for the outgoing road happens to equal σ a steady-state solution is a possibility only when the incoming fluxes equal $f(\sigma)$ or when q happens to be set so that the incoming fluxes get "locked" in their current positions. Any other value of q will cause the incoming fluxes to change.

If ρ_+ for the outgoing road is representative of heavy traffic we again have two scenarios depending on the maximised sum of the incoming fluxes.

If the maximised incoming fluxes are smaller than $f(\rho_+)$ for the outgoing road, no steady-state solution exists since the density on the outgoing road invariably will drop. If the maximised incoming flux happens to equal the outgoing flux $f(\rho_+)$, the incoming fluxes – together with $f(\rho_+)$ for the outgoing road – make up a steady-state solution. Finally, if the maximised incoming flux is larger than the outgoing $f(\rho_+)$ a steady-state solution again is obtainable only if q manages to "lock" the incoming fluxes in their current positions.

To compare the models the values of ρ that resulted in steady state solutions for the first model have been inserted in to the Coclite/Piccoli/Garavelli model to see what happens there.

For the situation seen in the upper part of Figure 11 the solution remains in steady state as the maximised sum of the incoming fluxes is well below the maximised allowed flux out of the juncion and it happens to equal ρ_+ for the outgoing road, rules (**A**) and (**B**) are satisfied without question. Entering the values giving the solution seen in the lower part of Figure 11 though, the maximised incoming flux exceeds both ρ_+ and the capacity of the outgoing road. We must thus invoke rule (**C**). If, when doing so, we set $q = 0.5$, we actually manage to make this a steady-state solution because the construction of the gamma region in this specific case locks the incoming fluxes in their positions. Any other value of q changes this balance though, why the solution moves out of steady state for the source term model.

5 Alternative Modelling Approaches and Discussion

A fairly different way of modelling traffic flow has been presented by Newell in [26]. He starts out with the traditional LWR equation but chooses – instead of solving the equation analytically – to construct a graphical solution. In order to do this he looks at the *cumulative flow, A,* of vehicles to pass some location x at time t starting from the passage of some reference vehicle. By then drawing the graph of A at different locations the vertical distance between the A-curves at a specific time t can then be interpreted as the "queue", i.e. the number of vehicles between these different locations[7] at this specific time. From the smoothed A-curve the flow and density can be deduced from differentiating A once with respect to t and x respectively. From this it is clear that shocks are found when there is a discontinuity in one (or both) of these derivatives.

According to Newell, the main advantage of analysing traffic flow this way is related to shock conditions. For reasons that will not be repeated here but that can be found in [26], a shock path will emerge automatically as the solution is constructed and in order to locate the shock at a specific time $t*$ or to find out at what time a shock is at certain position $x*$ one then simply has to construct the graph of $A(x, t*)$ vs. x or $A(x*, t)$ vs. t respectively and locate any discontinuity in f or ρ. The actual path of the shock need not be followed.

From this rather straight-forward way of analysing traffic behaviour – although it is not quite as simple as Newell lets on – he then moves on to introduce a moving coordinate system in order to evaluate delays. In doing so he is able to subtract the "free trip time" from the total trip time. At this point however, a whole new cumulative count and therefore also new flow and density relations have to be determined which rather complicate the situation. As if this was not a lot to handle already, he then goes on to introduce "departure" curves D, which are meant to replace A when the

[7]since the number of vehicles of course is integer valued these curves really are step functions but they are smoothed as they are drawn

size of A exceeds the capacity of the road. These are rather complicated to construct though and it is only very briefly explained how one should undertake this construction. For the small piece of information they give, i.e. the number of vehicles who would like to pass a point x at time t minus the ones that actually can do this, it all – in my opinion – turns out to be a rather cost ineffective procedure. Needless to say all these extra conditions and coordinate systems also make the geometrical solution not only more and more difficult to construct but also to – at last – decypher and interpret.

Other attempts at queueing modelling – that are significantly different from what has already been discussed – can be found in Heidemann ([17]) and Brilon ([7]). The former uses various statistical distributions to predict and describe traffic stream behaviour and the latter looks at a more generalised situation with multiple crossings including a larger number of roads. These methods and situations however fall outside the frame of this paper why a closer look at them will not be offered here.

Obviously, there are severe flaws to the continuum-flow approach of mathematical modelling of traffic flow; speed-density relationships are oversimplified, drivers are assumed to behave in a specific manner no matter the density of the traffic stream they are a part of, no real care to sudden external circumstances are taken etc.

Another thing to consider is the fact that the continuum-flow models are only first order accurate, that is, the scalar comservation law only incorporates first order derivatives. Obviously, a higher accuracy is desirable and a number of higher order models have been presented over the years, several which give results deemed to be more true to reality (see for instance [8, 15, 20, 32, 33]). However, it is generally also considered that these higher order models are far too complex and hard to handle in relation to the – generally – small improvement in results that they actaully yield. Thus it is generally considered sufficient to work only with first order models which is also the reason why higher order models have not been discussed further in this paper.

The models that have been described in chapter 4 are of course not flawless. The limitations of the model presented in 4.1 have already been touched upon. It is obvious that the Coclite/Piccoli/Garavello model [10] is supreme to this former model in terms of the larger number of scenarios it can handle. That does not mean that it is perfect though, I maintain that the right of way – as it is implemented in 4.1 – is more realisitic than its Coclite/Piccoli/Garavello correspondant q.

One way to modify the model from [10] and [3] could be to introduce yet another rule (one to replace or expand (**C**)), preferably one that states the "importance" of the branches of traffic flows that are about to merge. It is far from certain that rule (**C**) – while guaranteeing flux maximisation – actually gives the *desired* maximisation. If q is set completely voluntarily then we do not have a problem since we will just modify it in accordance with our desires. If q happens to be the result of poorly placed traffic lights for instance, then another maximising solution might be preferable. This could be obtained through good implementation of an “importance” rule.

Another way to modify the Coclite/Piccoli/Garavello model could be to introduce a “reduction” rule to replace rule (**C**) in the case of the maximised incoming flow exceeding the capacity of the road going out of the junction. What one could do then, is to reduce each of the incoming flows with exactly as much as is needed for the total incoming flow to be equal to the maximum admissible flow of the outgoing road. This will ensure uniqeness of the solution and a maximisation through the junction. In mathematical terms this can be stated in the following manner:

If $f(\rho_{1-}) + f(\rho_{2-}) > C$ where C is the maximum flux allowed through the junction, then find $r \in (0, 1)$

$$r = \frac{C}{f(\rho_{1-}) + f(\rho_{2-})}$$

For the fluxes $f(\rho_1^-)$ and $f(\rho_2^+)$ it then holds

$$\begin{aligned} f(\rho_1^-) &= r\,f(\rho_{1-}) \\ f(\rho_2^-) &= r\,f(\rho_{2-}) \end{aligned}$$

Again, to illustrate, consider the situation in Figure 40. Using the reduction rule, the result is as seen in Figures 43 and 44. Qualitatively we thus get the same result but the concentrations $\rho_i^-, i = 1, 2$, are not the same.

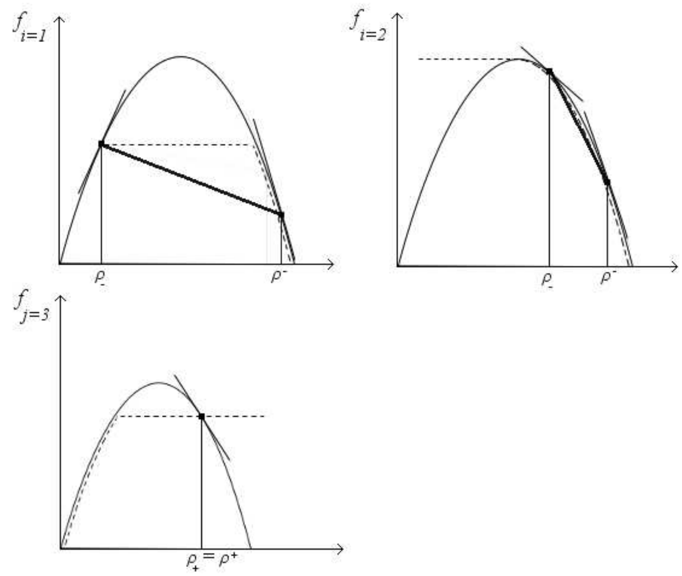

Figure 43: Final fluxes for the problem seen in Figure 40 using the reduction rule

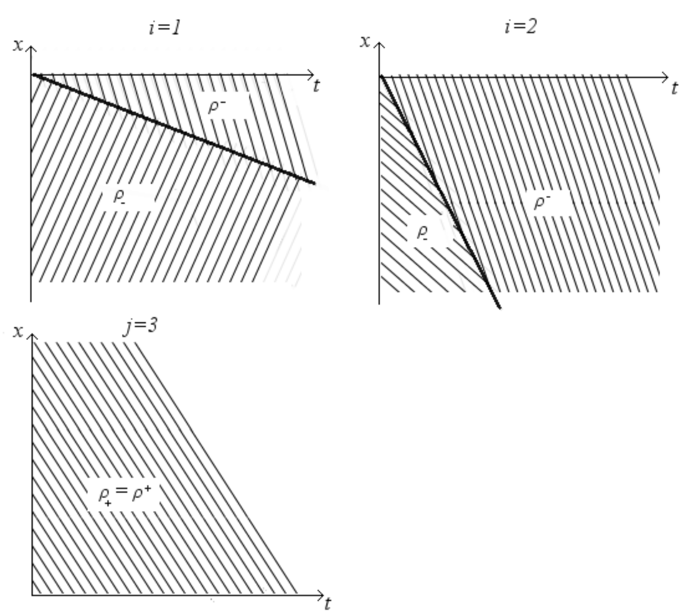

Figure 44: Characteristic plot corresponding to the fluxes from Figure 44 but with use of the reduction rule

A Godunov's Method

The idea behind Godunov's method is to follow characteristics forward in time instead of backwards, as is tradionally the case. For Δx and $\Delta t > 0$, define, in the plane $\mathbb{R}\times\mathbb{R}_+$ a grid $\{(i\Delta x, j\Delta t) : i, j \in \mathbb{Z}, j \geq 0$. Further, let

$$a = \max_{u \in M} |f'(u)|, \quad b = \max_{u \in M} |g'(u)|$$

where the interval M depends om the initial data and on f, g and s. In this case f and g are the functions defined in relation to Figure 8. Due to the identical shape of the flow curves used in the model from 4.1, a is of course equal to b here which is reflected in the code in appendix B.

Considering analytical solutions of parallel Riemann problems originating from piecewise constant initial data, the Godunov scheme can be derived under the CFL condition[8]

$$\lambda \equiv \frac{\Delta t}{\Delta x} < \frac{1}{2}\min(\frac{1}{a}, \frac{1}{b}) \overset{\text{(here)}}{=} \frac{1}{2a}$$

Using U_i^j as the approximate solution at grid point (i, j) and treating the incoming traffic flow from the slip way as a source term S^j, the scheme can be written as

$$U_i^{j+1} = U_i^j + \lambda(f(u^j_{i-\frac{1}{2}}) - f(u^j_{i+\frac{1}{2}})), \quad i > 0$$

$$U_0^{j+1} = U_0^j + \lambda(f(u^j_{-\frac{1}{2}}) - f(u^j_{\frac{1}{2}}) + S^j), \quad i = 0$$

$$U_i^{j+1} = U_i^j + \lambda(f(u^j_{i-\frac{1}{2}}) - f(u^j_{i+\frac{1}{2}})), \quad i < 0$$

where

$$U_i^0 = \frac{1}{\Delta x}\int_{(i-\frac{1}{2})\,\Delta x}^{(i-\frac{1}{2})\,\Delta x} u_0(x)\,dx \quad , \; S^j = \frac{1}{\Delta t}\int_{j\,\Delta t}^{(j+1)\,\Delta t} s(t)dt$$

Thus the scheme simply states that the "new" flow in a cell is equal to the old flow plus the change in flux over said cell. The fluxes on the straight lines in the t-direction beteen the grid points are given by

$$f(u^j_{i-\frac{1}{2}}) = \begin{cases} \displaystyle\min_{v \in [U^j_{i-1}, U^j_i]} f(v) & \text{if} \quad U^j_{i-1} \leq U^j_i, \\ \displaystyle\max_{v \in [U^j_i, U^j_{i-1}]} f(v) & \text{if} \quad U^j_{i-1} > U^j_i, \end{cases}$$

in correspondance with whether going from one concentration to another results – respectively – in a shock or a rarefaction wave.

[8]this is a stability condition regulating the gridsize. In the case of Godunov's method it ensures that these parallel solutions do not intersect

B MatLab Code

```
function U = godunov(Nx,rol,ror,k,romax,v0,T,a,b)

% For an odd number of points Nx, calculate changes in traffic density on road with merge
% point located at x = 0.

% INPUT VARS:rol = initial density of cars upstream of merge point
%            rol = initial density of cars downstream of merge point
%            k = source term factor (see resp.  source terms for usage)
%            romax = maximum capacity of road
%            v0 = free speed (km/h)
%            T = time interval of interest (minutes)
%            a = starting point
%            b = endpoint (km)


% INITIAL VALIDATION CHECK; MUST NOT EXCEED CAPACITY OF ROAD
rom = romax/2;
if ror > romax || rol > romax
   'invalid input'
end

% INITIATE PARAMETERS
fprimmax = v0;
lambda = 0.49/fprimmax; % CFL-condition for stability
dx = (b-a)/Nx; % step size, x-direction
T = T/60; % rescale from minutes to hours
dt = lambda*dx; % step size, t-direction
Nt = T/dt; % number of steps to be taken in time
tvector = (1:Nt+1)*dt; % create time and space vectors
H = (b-a)/2;
xvector = (1:Nx+1)*(dx)-H;
n = Nx/2;

% SOURCE TERMS

% SOURCE TERM NO 1
% NN = ceil(Nt/2);
% % % check validity for this particular source term, s is assumed to change with size 50
% if k + flode2(v0,romax,rol) > max(flode2(v0,romax,rom))
     || (50+k + flode2(v0,romax,rol)) > max(flode2(v0,romax,rom))
%  'invalid input'
% end
% % s1 = ones(NN,1)*k; % create step function with step of size k
% s2 = ones(NN,1)*k-200;
% s = [s1;s2];

% SOURCE TERM NO 2
% s = k + 0.5*k*sin(0.2*k*tvector); % create sinuslike input wave
% % check validity for this particular source term
% for i = 1:length(s)
%  if s(i) > flode2(v0,romax,rom) || s(i) < 0
%     'invalid output'
%  end
% end

% % SOURCE TERM NO 3
% for I = 1:Nt % create constantly increasing source term
%  s(I) = I*1.15;
% end
% % check validity for this particular source term
```

```
% for i = 1:length(s)
%  if s(i) > flode2(v0,romax,rom) || s(i) < 0
%     'invalid output'
% end
% end

% NUMERICAL SCHEME

U1 = ones(n-0.5,1)*rol; % create vectors with initial car densities
Ux = rol;
U2 = ones(n-0.5,1)*ror;
U = [U1;Ux;U2];
for j = 1:Nt
   U(1,j) = U(2,j);
   U(Nx,j) = U(Nx-1,j);
   for i = 1:Nx-1
      if U(i,j) <= U(i+1,j)
         F(i,j) = min(flode2(v0,romax,U(i,j)),flode2(v0,romax,U(i+1,j)));
      else    % U(i,j) > U(i+1,j)
          if U(i,j) < rom && U(i+1,j) < rom
             F(i,j) = max(flode2(v0,romax,U(i,j)),flode2(v0,romax,U(i+1,j)));
          elseif U(i,j) > rom && U(i+1,j) > rom
             F(i,j) = max(flode2(v0,romax,U(i,j)),flode2(v0,romax,U(i+1,j)));
          elseif U(i,j) > rom && U(i+1,j) < rom
             F(i,j) = flode2(v0,romax,rom);
          end
      end
   end
   for i = 1:Nx-2
      G(i,j) = (F(i,j)-F(i+1,j))*lambda;
   end
   for i = 2:Nx-1
      if i == n+0.5
          U(i,j+1) = U(i,j)+lambda*(F(i-1,j)-F(i,j)+s(j));
      else
          U(i,j+1) = U(i,j)+lambda*(F(i-1,j)-F(i,j));
      end
   end
end

U(Nx,end) = U(Nx-1,end);
U(1,end) = U(2,end);
U(Nx+1,:)  = U(Nx,:);

% GRAPHICS

[X,Y] = meshgrid(tvector*60,xvector);
figure(1)
contour(X,Y,U,30)
xlabel('time (min)')
ylabel('x (km)')

figure(2)
surf(X,Y,U)
xlabel('time (min)')
ylabel('x (km)')
zlabel('concentration (cars/km)')
```

References

[1] W.F. Adams, *"Road Traffic Considered as a Random Series", J. Inst. Civil Engineers* **4**, pp. 121-130, 1936

[2] R. Akcelik, *"Relating Flow, Density, Speed and Travel Time Models for Uniterrupted and Interrupted Traffic", Traffic Engineering and Control* **37(9)**, pp. 511-516, 1996

[3] C. D'Apice, R. Manzo, B. Piccoli, *"Packet flow on telecommunication networks", SIAM J. Math. Anal.* **Vol. 38, No. 3**, pp. 717-740, 2006

[4] W.D. Ashton, *"The theory of Road Traffic Flow", Methmen & Company Ltd.*, 1966

[5] D. Branston, *"Models of Single Lane Time Headway Distributions", Transportation Science*, **10**, pp. 125-148, 1976

[6] G. Bretti, R. Natalini, B. Piccoli, *"Numerical algorithms for simulations of a traffic model on road networks", Journal of Comp. amd Appl. Math.*, **210**, pp. 71-77, 2007

[7] W. Brilon, W. Ning, *"Capacity at unsignalized two-stage priority intersections", Transp. Research Pt. A*, **33**, pp. 275-289, 1999

[8] R. Bürger, K.H. Karlsen, *"On a diffusively corrected kinematic.wave traffic flow model with changing road surface conditions", Math. Models and Methods in Appl. Sciences*, **Vol. 13, No. 12**, pp. 1767-1799, 2003

[9] Y. Chitour, B. Piccoli, *"Traffic circles and timing of traffic lights for cars flow", Disc. and Cont. Dyn. Systems,* ***Ser. B 5***, pp. 599-630, 2005

[10] G.M. Coclite, M. Garavello, B. Piccoli, *"Traffic Flow on a Road Network", SIAM J. Math. Anal.*, **Vol. 36**, pp. 1862-1886, 2005

[11] S. Diehl, *"Introduction to the Scalar Non-Linear Conservation Law", Dept. of Mathematics, Lund Institute of Technology*, lecture notes

[12] S. Diehl, *"On scalar conservation laws with point source and discontinous flux function", SIAM J. Math. Anal.*, **Vol. 26, No. 6**, pp. 1425-1451, 1995

[13] S. Diehl, *"Scalar conservation laws with discontinous flux function: I. The viscous profile condition", Comm. Math. Phys.* **176**, pp. 23-44, 1996

[14] A.L. Garcia, *"Numerical Methods for Physics", Addison Wesley*, 1999

[15] N. Gartner, C.J. Messer, A.K. Rathi, (Ed.) *"Traffic Flow Theory, A State-of-the-Art Report", http://www.tfhrc.gov/its/tft/tft.htm*, 1995

[16] B.D. Greenshields,*"A Study in Highway Capacity", Highway Research Board Proceedings* **Vol. 14**, p. 458, 1935

[17] D. Heidemann, H. Wegmann, *"Queuing at Unsignalized Intersections", Transpn. Res.-B*, **Vol. 31, No. 3**, pp. 239-263, 1997

[18] D. Helbing, A. Hennecke, V. Shvetsov, M. Treiber, *"Micro- and Macro-Simulation of Freeway Traffic", Math. and Comp. Modelling,* **35**, pp. 517-547, 2002

[19] R. Herman (Ed.), *"Theory of Traffic Flow", Elsevier Scientific Publishers*, 1961

[20] A. Klar, R.D. Kühne, R. Wegener, *"Mathematical models for vehicular traffic", Surv. Math. Ind,* **6**, pp. 215-239, 1996

[21] R. J. LeVeque *"Numerical Methods for Conservation Laws", BirkhÃďuser Verlag*, 1990

[22] M.J. Lighthill, G.B. Whitham, *"On kinematic waves: II. A theory of traffic flow on long crowded roads", Proc. R. S.* **A229**, pp. 317-345, 1955

[23] S. Mochon, *"An analysis of the traffic on highways with changing surface conditions", Math. Modelling* **9**, pp. 1-11, 1987

[24] C.S. Morawetz, *"Lectures on nonlinear waves and shocks", Springer-Verlag*, 1981

[25] G.F. Newell, *"A moving bottleneck", Transpn. Res. B,* **Vol. 32, No. 8**, pp. 531-537, 1998

[26] G.F. Newell, *"A simplified theory if kinematic waves in highway traffic, Part I: General theory, Part II: Queueing at freeway bottlenecks, Part III: Multi-destimation flows", Transpn. Res.*, **27B**, pp. 281-313, 1993

[27] G.F. Newell, *"Mathematical Models for Freely Flowing Highway Traffic", Operations Research* **3**, pp. 176-186, 1955

[28] I. Prigogine, F.C. Andrews, *"A Boltzmann-like Approach for Traffic Flow", Operations Research*, **4**, pp. 789-797, 1960

[29] P.I. Richards, *"Shock Waves on the Highway", Operations Research*, **4**, pp. 42-51, 1956

[30] A. Sparr, G. Sparr, *"Kontinuerliga system", Studentlitteratur*, 1999,2000

[31] T. van Woensel, N. Vandaele, *"Modeling traffic flows with queueing models: a review", Asia-Pacific Journal of Operational Research*, **Vol. 24, No. 4**, pp. 435-461, 2007

[32] H.M. Zhang, *"A theory of nonequilibrium traffic flow", Transpn. Res.-B*, **Vol. 32, No. 7**, pp. 485-498, 1998

[33] H.M. Zhang, *"New Perspectives on Continuum Traffic Flow Models", Networks and Spatial Economics*, **1**, pp. 9-33, 2001