# 03: Sequence of Real Numbers

A (seq) of real no.. is a ( ) from the set N of natural no.. to set R of real no..

$\boxed{f : N \to R}$ → (( ) from ntral no.. to real no)

* If $a : N \to R$ is a (seq), its terms are denoted by $a_n$ or $a(n)$ → $n^{th}$ term of (seq).

$\{a_n\}$ or $\{a_n : n \in N\}$ or $(a_n)$ can be denoted.

* It is often convenient to specify $a_1$ & formula for obtaining $a_{n+1}$ $(n \geq 1)$ when $a_n$ is known. This method → Inductive / Recursive definitions.

eg :-

1) consider the seq → 1, 2, 3, 4 ... n ... of N no..,

$a_1 = 1$ $a_2 = 2$ $a_3 = 3$ ... $a_n = n$.

→ $a : N \to R$ : $a(n) = n$.
(: → such that.)

by Inductive / R method →
here $a_1 = 1$ ; $a_{n+1} = a_n + 1$ $(n \geq 1)$

(or) $a_1 = 1$ , $a_{n+1} = a_n + a_1$ $(n \geq 1)$
This type of seq → Ind / R. method.

2) 2, 4, 6, 8, 10 ... 2n. (even integers)
( ) → $a : N \to R$ : $a(n) = 2n$
by I/R,
$a_1 = 2$ $a_{n+1} = a_n + 2$ $(n \geq 1)$. (add 2)
(or) $a_1 = 2$ $a_{n+1} = a_n + a_1$

3) seq → 1, 1, 1, .... ($a_1 = a_2 = a_3 = 1$ ...)
( ) → $a : N \to R$ : $a(n) = 1$ for all $n \in N$
This seq → constant seq.

4) seq → 2, 1, 4, 3, 6, 5, 8, 7 ....
$a_1 = 2$ . $a_2 = 1$ , $a_3 = 4$.
here $a_1, a_3, a_5, a_7$ → even no
$a_2, a_4, a_6, a_8$ → odd no
( ) → $a : N \to R$ : $a(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$

5) statemnts $a_1 = 1$ & $a_n = n \cdot a_{n-1}$ define the seq 1, 2, 6, 24 ... n! of factorials.
with $a_1 = 1$, we hve $a_2 = 2 \cdot a_1 = 2 = 2!$,
$a_3 = 3 \cdot a_2 = 6 = 3!$,
$a_4 = 4 \cdot a_3 = 24 = 4!$ ....

6) seq → $a_1 = 1$, $a_2 = 1$ & $a_{n+1} = a_{n-1} + a_n$ $n \geq$ (by Ind. def)

⇒ <u>fibonacci seq.</u>

eg → $a_1 = 1$, $a_2 = 1$ $a_3 = ?$ $a_4 = ?$ $a_5 = ?$ .....

$a_3 = a_1 + a_2$

$= 1 + 1 = 2$

$a_4 = a_2 + a_3 = 1 + 2 = 3$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

∴ fib. seq → 1, 1, 2, 3, 5, 8, $\underset{5+8}{13}$, 21 .....

⇒ <u>Limits of seq :-</u>

Consider the seq, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$.

$$\boxed{\begin{aligned} a_n &= \frac{1}{n} \\ b_n &= \frac{(-1)^{n+1}}{n} \\ c_n &= \sqrt{n}. \end{aligned}}$$

$a_n = \left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots\right\}$

give 1, 2, 3 ... to n.

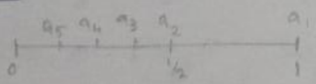
$b_n = \left\{\frac{(-1)^{n+1}}{n}\right\} = \left\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5} \cdots\right\}$

$c_n = \{\sqrt{n}\} = \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5} \cdots\}$

$d_n = \{n^2\} = \{1, 4, 9, 16 \ldots\}$

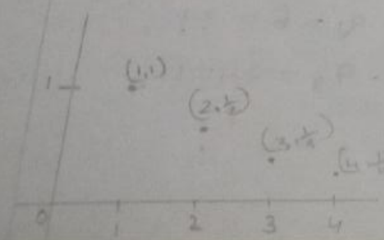
* <u>Graphs →</u>

1) <u>$a_n$</u>,

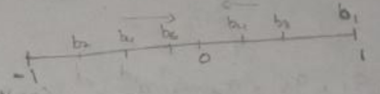



$a_1 = 1$ $a_2 = \frac{1}{2}$ ....

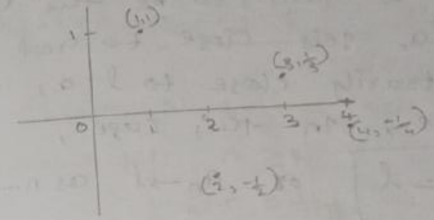
($a_n$ approaches to zero)



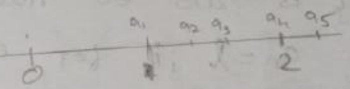

2) <u>$b_n$</u>,

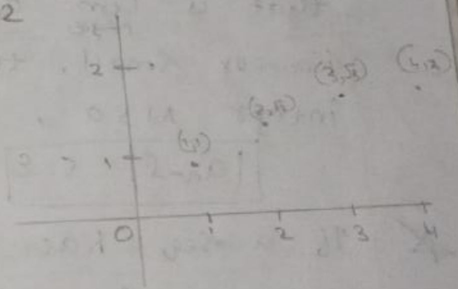



($b_n$ approaches to 0)




3) <u>$c_n$</u>,




From the graph seq $\{a_n\}$, $\{b_n\}$ & $\{c_n\}$ it is clr that seq do not behave in same way as n ↑se.

<u>The terms $a_n = \frac{1}{n}$ ↓se steadily & get arbitrarily close to 0 as n ↑se.</u>

$a_n = \frac{1}{n}$ $\quad b_n = (-1)^{n+1}\left(\frac{1}{n}\right)$ $\quad c_n = n^{1/2}$

* Def →

The seq $a_1, a_2, a_3 \cdots a_n$ converges to the no. $l$ as $l$ is said to be the limit of seq, if $a_n$ gets close to and remains arbitrarily close to $l$ as $n$ becomes large, In this case,

$$\boxed{\lim_{n\to\infty} a_n = l} \text{ or } a_n \to l \text{ as } n \to \infty \quad ①$$

(E-s ...)

* E-N Def :-

A seq $\{a_n\}$ is said to have a limit $l$ that is $\lim_{n\to\infty} a_n = l$, if for every $\varepsilon > 0$ however small, there corresponds an integer $N > 0$,

$$\boxed{|a_n - l| < \varepsilon} \text{ for all } n \geq N. \quad ②$$

★ If a seq has a limit → seq → cvg

If a seq has no limit → seq → dvg.

Q

1) Using E-N def, P.T $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

A) By def, 0 will be limit of seq $\{a_n\}$, where $a_n = \frac{1}{n}$ if for any given $\varepsilon > 0$, we are able to find a N no. 'N' : $|a_n - 0| < \varepsilon$ for all $n \geq N$.

by ① $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$ $\quad \because a_n = \frac{1}{n}$

↳ $\lim_{n\to\infty} a_n = l$ $\quad l = 0$.

$N = ?$

$$|a_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n}.$$

hence $|a_n - 0| < \varepsilon$ if $\frac{1}{n} < \varepsilon$

(i.e) if $n \geq \frac{1}{\varepsilon}$

choose N be a +ve integr $> \frac{1}{\varepsilon}$

we get, $n \geq N$

$$\Rightarrow n \geq \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon$$

$$\Rightarrow |a_n - 0| < \varepsilon.$$

hence by def $\lim_{n\to\infty} a_n = 0$.

$\lim_{n\to\infty} a_n = l$

2) P.T $\lim_{n\to\infty} S_n = \frac{2}{3}$, if $S_n = \frac{2n+1}{3n+5}$ for all $n \in N$.

A) $\left|S_n - \frac{2}{3}\right| < \varepsilon$ for all $n \geq N$.

$$= \left|\frac{2n+1}{3n+5} - \frac{2}{3}\right| = \left|\frac{3(2n+1) - 2(3n+5)}{3(3n+5)}\right|$$

$$= \left|\frac{-7}{3(3n+5)}\right| = \frac{7}{3(3n+5)}$$

hence $\left|S_n - \frac{2}{3}\right| < \varepsilon$ if $\frac{7}{3(3n+5)} < \varepsilon$

by reciprocal,

(i.e) $\frac{3(3n+5)}{7} > \varepsilon$

(i.e) $\frac{9n+15}{7} > \varepsilon$ → $\quad 9n+15 > \frac{7}{\varepsilon}$

(ie) if $9n > \frac{7}{\varepsilon} - 15$ $\quad \frac{15}{9} = \frac{5}{3}$

(ie) if $n > \frac{7}{9\varepsilon} - \frac{5}{3}$

Choose $N$ to be an +ve integr,

$$N > \left(\frac{7}{9\varepsilon} - \frac{5}{3}\right)$$

Then $n \geq N \Rightarrow n > \frac{7}{9\varepsilon} - \frac{5}{3}$

$$\Rightarrow \left|S_n - \frac{2}{3}\right| < \varepsilon$$

hence, $\lim_{n\to\infty} S_n = \frac{2}{3}$

3) If $a_n = \frac{1}{10} + \frac{1}{10^2} + \dots + \frac{1}{10^n}$,
how large must $n$ be for $\frac{1}{9} - a_n$ to be less than $10^{-6}$?

A) Here, $a_n$ is sum of $1^{st}$ $n$ terms of geometric seq, with $1^{st}$ term $a = a_1 = \frac{1}{10}$ & common ratio $\frac{1}{10}$.

$$\therefore a_n = \frac{a(1-r^n)}{1-r} = \frac{\frac{1}{10}\left[1-\left(\frac{1}{10}\right)^n\right]}{1-\frac{1}{10}}$$

$$= \frac{1}{10}\left[1 - \frac{1}{10^n}\right] \times \frac{10}{9}$$

$$= \frac{1}{9} - \frac{1}{9} \cdot \frac{1}{10^n}.$$

we get, $\frac{1}{9} - a_n = \frac{1}{9} \cdot \frac{1}{10^n} < \frac{1}{10^n}$

for all $n = 1, 2 \dots$

$\therefore$, $\frac{1}{9} - a_n$ to be less than $10^{-6}$, $n$ must be grtr than or equal to 6.

$\Rightarrow$ <u>properties of limits of sequence</u> :-

1) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

2) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

3) $\lim_{n\to\infty} (c\, a_n) = c \lim_{n\to\infty} a_n$

4) If $\lim_{n\to\infty} b_n \neq 0$ and $b_n \neq 0$ for all $n$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

5) If $f$ is contin at $\lim_{n\to\infty} a_n$,

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

6) $\lim_{n\to\infty} c = c$

7) $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

8) If $\lim_{x\to\infty} f(x) = l$, then $\lim_{n\to\infty} f(n) = l$.

9) If $|r| < 1 \rightarrow \lim_{n\to\infty} r^n = 0$.

If $|r| > 1$ or $r = -1 \rightarrow \lim_{n\to\infty} r^n$ doesnot exist.

Q 1) $\lim_{n\to\infty} \left(\frac{n-1}{n}\right)$

A) $\lim_{n\to\infty} \left(\frac{n}{n} - \frac{1}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)$

$= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n}$

$\frac{1}{\infty} = 0$ $\quad = 1 - 0$

$= \underline{\underline{1}}$

2) $\lim_{n\to\infty} \left(\frac{(3n+1)(n-2)}{n(n+3)}\right)$.

take n outside on both numrtr & deno.

$= \lim_{n\to\infty} \frac{n(3 + 1/n)\ n(1 - 2/n)}{n^2(1 + 3/n)}$

$= \lim_{n\to\infty} \frac{\cancel{n^2}(3 + \frac{1}{n})(1 - 2/n)}{\cancel{n^2}(1 + 3/n)}$

$= \lim_{n\to\infty} \frac{(3 + 1/n)(1 - 2/n)}{(1 + 3/n)}$ (give limit)

$= \frac{(3 + 1/\infty)(1 - 2\cdot 1/\infty)}{(1 + 3 \cdot 1/\infty)} = \frac{3 \cdot 1}{1} = \underline{\underline{3}}$

(we need to get $\frac{1}{n} \to \frac{1}{\infty} = 0$)

3) $\lim_{n\to\infty} \sin\left(\frac{\pi n}{2n+1}\right)$

([illegible])

A) By ~~[illegible]~~.

$\lim_{n\to\infty} \frac{\pi n}{2n+1} \begin{matrix}\div n \\ \div n\end{matrix}$

$\lim_{n\to\infty} \frac{(\pi)}{(2 + 1/n)} = \lim_{n\to\infty} \frac{\pi}{2 + \frac{1}{n}}$ $\quad \frac{1}{\infty} = 0$

$= \frac{\pi}{2}$

$\lim_{n\to\infty} \sin\left(\frac{\pi n}{2n+1}\right) = \lim_{n\to\infty} f\left(\frac{\pi n}{2n+1}\right)$ (by pro-5)

$= f\left(\lim_{n\to\infty} \frac{\pi n}{2n+1}\right)$

$= f\left(\frac{\pi}{2}\right) \Rightarrow \sin\frac{\pi}{2} = \underline{\underline{1}}$

4) $\lim_{n\to\infty} \left[\frac{3n^2 - 2n + 1}{n(n+1)} - \frac{n(n+2)}{(n+1)(n+3)}\right]^2$.

A) $\lim_{n\to\infty} \left[\frac{3n^2 - 2n + 1}{n(n+1)} - \frac{n(n+2)}{(n+1)(n+3)}\right]$ (give limit to both)

$\lim_{n\to\infty} \frac{3n^2 - 2n + 1 \div n^2}{n(n+1) \div n^2} - \lim_{n\to\infty} \frac{n(n+2) \div n^2}{(n+1)(n+3) \div n^2}$

$\div$ by $n^2$ in numr & deno.

$\lim_{n\to\infty} \frac{\cancel{n^2}(3 - 2/n + 1/n^2)}{\cancel{n^2}(1 + 1/n)} - \lim_{n\to\infty} \frac{\frac{n(n+2)}{n^2}}{\frac{(n+1)(n+3)}{n^2}}$

$\lim_{n\to\infty} \frac{3 - 2/n + 1/n^2}{1 + 1/n} - \lim_{n\to\infty} \frac{1 + 2/n}{(1 + \frac{1}{n})(1 + 3/n)}$

give limit,

$= \frac{3}{1} - \frac{1}{1} = 3-1 = 2$

here $f(x) = x^2$ which is contin (·) on R

(4) $\lim_{n\to\infty} \left[ \frac{3n^2-2n+1}{n(n+1)} - \frac{n(n+2)}{(n+1)(n+3)} \right]^2$

$= \lim_{n\to\infty} f\left[ \frac{3n^2-2n+1}{n(n+1)} - \frac{n(n+2)}{(n+1)(n+3)} \right]$

$= f\left( \lim_{n\to\infty} \left( \frac{3n^2-2n+1}{n(n+1)} - \frac{n(n+2)}{(n+1)(n+2)} \right] \right.$.

$= f(2)$

$f(x) = f(x)^2$

$\therefore f(2) = f(2^2) = 4$

4) P.T $\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r>1 \text{ — (1)} \\ 1 & \text{if } r=1 \text{ — (2)} \\ 0 & \text{if } 0 \le r < 1 \text{ — (3)} \end{cases}$

A) —(1)

Let $r > 1$.

Then $r = 1+s$ , $s > 0$   $s \to$ real no,

Then by binomial expansion,

$r^n = (1+s)^n = 1 + ns +$ positive terms with higher powers of s.

$\therefore r^n > 1+ns$ & $ns \to \infty$ as $n \to \infty$

hence $\lim_{n\to\infty} r^n = \infty$

—(2) when $r=1$, then $r^n = 1$ for all values of n.

So $\lim_{n\to\infty} r^n = 1$

—(3) when $0 \le r < 1$. $\begin{cases} \to r=0 \\ \to 0<r<1 \end{cases}$

If $r=0$, then $r^n = 0$,

So $\lim_{n\to\infty} r^n = 0$.

let $0<r<1$, then $\frac{1}{r} > 1$   $\frac{1}{r} = \rho$ (row)

$\rho = \frac{1}{r} > 1$

(If $r>1$, $\lim_{n\to\infty} r^n = \infty$)

So $\lim_{n\to\infty} \rho^n = \infty$

$\rho = \frac{1}{r} \Rightarrow r = \frac{1}{\rho}$

$\therefore \lim_{n\to\infty} r^n = \lim_{n\to\infty} \frac{1}{\rho^n}$

$= \frac{1}{\lim_{n\to\infty} \rho^n} = \frac{1}{\infty} = 0$

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} \infty & , r>1 \\ 1 & , r=1 \\ 0 & , 0 \le r < 1 \end{cases}}$$

5) $\lim_{n\to\infty} 3^n$

$r^n = 3^n$.

$3 > 1$,

by limits of powers we get,

$\lim_{n\to\infty} r^n = \infty$, if $r > 1$

$\therefore \lim_{n\to\infty} 3^n = \infty$

6) $\lim_{n\to\infty} e^{-n}$

(by quo) we know that $e > 1$   $e \to 2.71$ so $e > 1$.

So $\frac{1}{e} < 1$

by limits of powers,

$$\lim_{n\to\infty} e^{-n} = \lim_{n\to\infty}\left(\frac{1}{e^n}\right) = \lim_{n\to\infty}\left(\frac{1}{e}\right)^n = 0$$

$\because \lim_{n\to\infty} r^n = 0$ if $0 \le r < 1$

7) $\lim_{n\to\infty}\left[e+\left(\frac{2}{3}\right)^n\right]^4.$

$$\lim_{n\to\infty}\left[e+\left(\frac{2}{3}\right)^n\right] = \lim_{n\to\infty} e + \lim_{n\to\infty}\left(\frac{2}{3}\right)^n$$

constant

$$= e + 0 = e$$

$\left[\because \frac{2}{3} \text{ is } < 1. \; ; \; r^n = \frac{2}{3} < 1 \Rightarrow 0\right]$

let $f(x) = x^4 \to$ contin (?).

$$\lim_{n\to\infty}\left[e+\left(\frac{2}{3}\right)^n\right]^4 = \lim_{n\to\infty} f\left(e+\left(\frac{2}{3}\right)^n\right)$$
$$= f\left[\lim_{n\to\infty}\left(e+\left(\frac{2}{3}\right)^n\right)\right]$$
$$= f(e)$$
$$= e^4$$

⇒ Comparison Test :-

**Theorem** → If $\lim_{n\to\infty} a_n = 0$ and $|b_n| \le |a_n|$,

then $\lim_{n\to\infty} b_n = 0$

$\lim_{n\to\infty} a_n = 0$ when $|a_n| = 0$

$\lim_{n\to\infty} b_n = 0$ when $|b_n| = 0$

proof The seq $\{a_n\}$ and $\{b_n\}$ are such that $|b_n| \le |a_n|$ & $\lim_{n\to\infty} a_n = 0$.

Then we hve to prove that $\lim_{n\to\infty} b_n = 0$.

Choose any $\varepsilon > 0$, Since $\lim_{n\to\infty} a_n = 0$ corresponding to this $\varepsilon$ there exist a +ve integer $N$ such that,

$$|a_n - 0| = |a_n| < \varepsilon \quad \text{for all } n \ge N.$$

Then, since $|b_n| \le |a_n|$ for all $n$,

$$|b_n| \le |a_n| < \varepsilon \quad \text{for all } n \ge N.$$

Hence by $\varepsilon - N$ def. of limits of seq, we gt, $\lim_{n\to\infty} b_n = 0$

Q

1) $\lim_{n\to\infty} \frac{(\sin n)^2}{n+2}$ by C. test.

A) $\boxed{|\sin n| \le 1}$ for any $n$.

$$\left|\frac{(\sin n)^2}{n+2}\right| \le \frac{1}{n+2} < \frac{1}{n}$$

$\because \lim_{n\to\infty} \frac{1}{n} = 0$, by C. test,

$$\lim_{n\to\infty} \frac{(\sin n)^2}{n+2} = 0.$$

2) $\lim_{n\to\infty} \frac{(-1)^n + n}{n}$

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$ and $|(-1)^n| = 1$

$\lim_{n\to\infty} \frac{1}{n} = 0$

by c. test,
$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

want to prove → $\lim_{n\to\infty} \frac{(-1)^n + n}{n} = \lim_{n\to\infty} \frac{(-1)^n}{n} + \lim_{n\to\infty} \frac{n}{n}$

$= 0 + 1 = 1$

## ⇒ Newton's method of finding roots :-

$\boxed{f(x) = 0}$.

The solution of $f(x) = 0$ is called zeroes / roots of $f$.

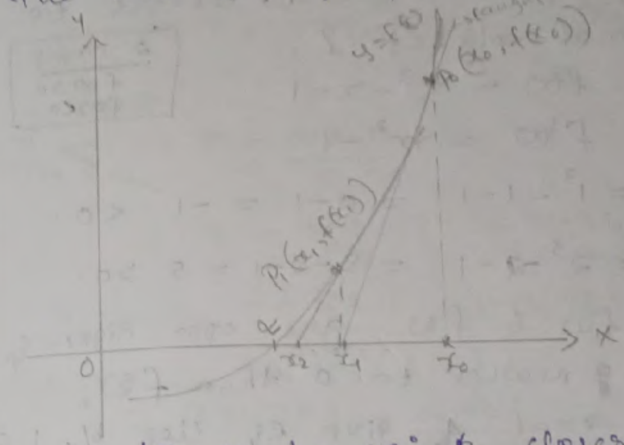
* Newton's method :- To find the root of eq $f(x) = 0$, where $f$ is diff - ( ) such that '$f$' is contin, start with a guess $x_0$, which is reasonably close to a root. Then produce the seq $x_0, x_1, x_2 \dots$ by iterative formula.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

If $\lim_{n\to\infty} x_n = \bar{x}$, then $f(\bar{x}) = 0$

* Geometric Interpretation :-

Suppose the graph of $y = f(x)$ crosses the x-axis at $\alpha$, then $x = \alpha$ is the root of eq $f(x) = 0$.




$\alpha$ = roots
$x_1 = ?$
$x_2 = ?$

$x_1 \to (x_1, 0)$
$x_2 \to (x_2, 0)$

let $x_0$ be point closer to root $\alpha$, then eq of taugnt at $P_0(x_0, f(x_0))$ is.

$$\boxed{y - y_0 = f'(x_0)(x - x_0) \longrightarrow \text{taugnt eq.}}$$

$(x_0, f(x_0))$, $x_1 = ?$

$(x_1, 0)$
$x = x_1$
$y = 0$

$(x_1, 0)$ $y - f(x_0) = f'(x_0)(x_1 - x_0)$

$-f(x_0) = f'(x_0)(x_1 - x_0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_2 = ?$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

here $x_1$ written in terms of $x_0$ & $x_2$ written in terms of $x_1$, so Generally we can write,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Q

1) Find a real root of eq $x^3 - x - 1 = 0$ using (N)'s method, correct to 4 decimal places ?

A) let $f(x) = x^3 - x - 1$

$f'(x) = 3x^2 - 1$

$\boxed{\begin{array}{c}\text{2 cases} \\ f(x) > 0 \\ f(x) < 0\end{array}}$

$f(x)$ is nearest to 0

(take any no) 1, 2

$f(1) = 1^3 - 1 - 1 = 0 - 1 = -1 < 0.$

$f(2) = 2^3 - 2 - 1 = 8 - 2 - 1 = 5 > 0.$

Since $f(1)$ & $f(2)$ are oppo. signs & $f(1)$ is nearer to 0 than $f(2)$, a real root of given eq lies b/w 1 & 2.

here $f(1) < 0$ & $f(2) > 0$,

choose $x_0 = 1$

by (N)'s method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

~~$x_1 = ?$~~
~~$x_2 = ?$~~
~~$x_3 = ?$~~
~~$x_4 = ?$~~
~~$x_5 = ?$~~

~~when~~ ~~$n = 0$~~.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$$

$f'(x) = 3x^2 - 1$

$f'(1) = 3 - 1 = 2$

$$= 1 - \frac{(-1)}{2}$$

$$x_1 = 1 + \frac{1}{2} = 1.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} \quad \text{--- (1)}$$

$f(x) = f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875$

$f'(1.5) = 3 \times (1.5)^2 - 1 = 5.75$

in eq (1)

$$x_2 = 1.5 - \frac{0.875}{5.75}$$

$= 1.5 - 0.15217 = 1.34783$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.34783 - \frac{f(1.34783)}{f'(1.34783)}$$

$$= 1.34783 - \frac{0.1007}{4.44994} = 1.34783 - 0.0226$$

$= 1.3252$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.3252 - \frac{f(1.3252)}{f'(1.3252)}$$

$$= 1.3252 - \frac{0.00206}{4.2687} = 1.32472$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.32472 - \frac{0.0000087}{4.26465}$$

$x_5 = 1.324718$

hence the root of given eq lying b/w 1 & 2 correct to 4 decimal places is 1.3247 //

2) find a real root of eq $xe^x - 2 = 0$ using (N)'s method.?

A) $f(x) = xe^x - 2$.

$f'(x) = xe^x + e^x$.

we can take consecutive no 0 i.e 1,

$f(0) = 0.e^0 - 2 = -2 < 0$ $(e \to 2.718281)$

$f(1) = 1e^1 - 2 = e - 2 = 0.71828 > 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_0 = 1$ closer to 0.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{0.71828}{2 \times 2.71828}$$

$(2.718281 - 2)$

$$= 0.86788$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.86788 - 0.0151$$
$$= 0.85278$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.85278 - 0.0001745$$
$$= 0.85261$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.85261 - 0.0000045$$
$$= 0.85261$$

hence root of given eq, correct to 4 decimal places to 0.85261

⇒ Numerical Integration :-

the definite $\int^b f(x)\,dx$ is defined in the calculus as a limit of wht are → Riemann sum

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$F(x)$ is any anti deri... of $f(x)$

This is → fundamental . T of calculus

→ Riamann Sums :-

Suppose we are given $f(x)$ on $[a,b]$ & ÷ $[a,b]$ into sub interval

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

Then the definite $\int$, $\int_a^b f(x)\,dx$ is approximated by,

$$\boxed{\int_a^b f(x)\,dx \doteq \sum_{i=1}^{n} f(c_i)\,\Delta x_i}$$ → find area.

$c_i$ → lies in $[x_{i-1}, x_i]$

$$\Delta x_i = \frac{b-a}{n} \qquad x_i = a + \frac{i(b-a)}{n}$$

$c_i = x_{i-1}$ or $x_i$

$$\int_a^b f(x)\,dx = \frac{b-a}{n}\left[f(x_0) + f(x_1) + f(x_2) + \dots f(x_{n-1})\right]$$
$$\doteq \frac{b-a}{n}\left[f(x_1) + f(x_2) + \dots f(x_n)\right].$$

Q)

1) find approx value of $\int$, $\int_{-1}^{1}(x^2+1)\,dx$. using R. sum with $n = 10$ (i.e ÷ $[-1,1]$ into 10 sub intervals of equal length)

Compare with actual value ?

1) $[-1,1]$ $a=-1$
$b=1$

partition by $n=10$

($-1$ and 10 non to 1 numbers)

$-0.8, -0.6, 0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, 1$ — $c_i$

$x_0=-1$ (starting no)

area $\Rightarrow \sum_{i=1}^{n} f(c_i)\,\Delta x_i$ $\qquad \Delta x = \frac{b-a}{n}$

$f(x) = x^2+1$

$\therefore f(c_i) = c_i^2+1$

| $c_i$ | $-0.8$ | $-0.6$ | $-0.4$ | $-0.2$ | $0$ | $0.2$ | $0.4$ | $0.6$ | $0.8$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(c_i)$ | 1.64 | 1.36 | 1.16 | 1.04 | 1 | 1.04 | 1.16 | 1.36 | 1.64 |

$$\int_a^b f(x)\,dx = \frac{b-a}{n}\left[f(x_0)+f(x_1)+f(x_2)+\cdots f(x_n)\right]$$

$$= \frac{1-(-1)}{10}\left[1.64+1.36+1.16+1.04+1+1.04+1.16+1.36+1.64+2\right]$$

$$\left(\frac{2}{10}=\frac{1}{5}\right) = \frac{1}{5}\left[13.4\right] = 0.2\times 13.4$$

$$= 2.68$$

$$\int x^2+1\,dx.$$

$$=\left[\frac{x^3}{3}+x\right]_{-1}^{1} = \frac{1}{3}+1-\frac{(-1)^3}{3}+(-1)$$

$$= \frac{1}{3}+1+\frac{1}{3}-1$$

$$= \frac{4}{3}+\frac{4}{3} = \frac{8}{3} = 2.667$$

Approxi —

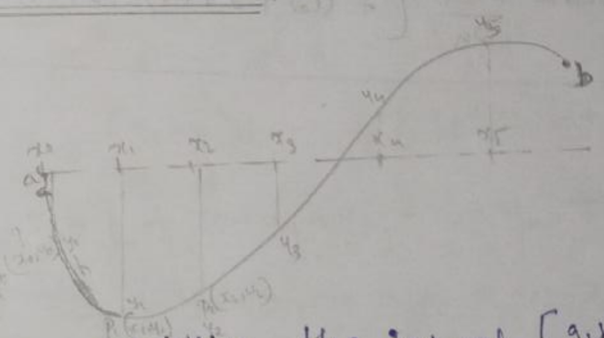
$2.68 - 2.667 = 0.013$

2) using R. Sum evaluate $\int_0^{\pi/2} \cos x\,dx$, taking 10 equally spaced points $x_0=0$, $x_1=\pi/20$, $x_2 = 2\pi/20$ ... $x_{10} = 10\pi/20$ — $c_i = x_i$

compare the ans with actual value.

3) using R. Sums with $n=4$ to estimate $\int_0^2 \frac{1}{1+x}\,dx$. compare the estimate with the exact value of $\int$.

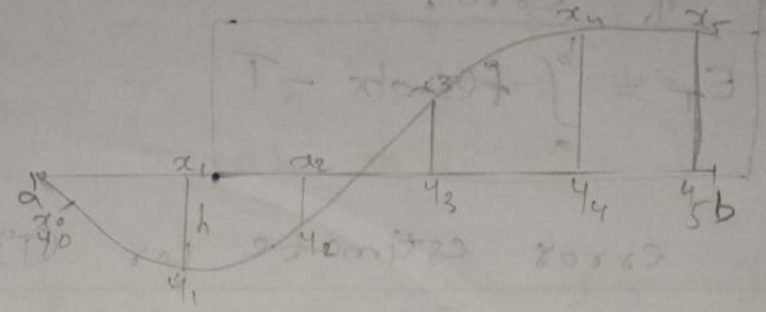
$\Rightarrow$ Trapezoidal Rule :-



Let us partition the intrval $[a,b]$. into $n$ sub intervals of length $\Delta x = h = \frac{b-a}{n}$

let $a = x_0, x_1 \ldots x_n = b$, be the points of partition & let $y_0, y_1 \ldots y_n$ be the corresponding values of $y$.

## Trapezoidal Rule :- (T)




Let us partition $[a, b]$ into $n$ subintervals of length $\boxed{\Delta x = h = \frac{b-a}{n}}$,

Let $a = x_0, x_1, x_2 \ldots x_n = b$ be the points of partition, let $y_0, y_1, y_2 \ldots y_n$ be $y$ values. $y_i = f(x_i)$, for $i = 0, 1, 2, \ldots n$, then $(x_0, y_0)$, $(x_1, y_1)$ $(x_2\ y_2)$ are points on the curve corresponding to this partition.

Let $T$ be the area,

$$T = \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \ldots$$

$$= h\left[\frac{1}{2}y_0 + y_1 + y_2 + \ldots y_{n-1} + \frac{1}{2}y_n\right.$$

$$T = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \ldots 2y_{n-1} + y_n\right]$$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) \ldots + f(n)\right]$$

$$= \frac{h}{2} = \frac{\frac{b-a}{n}}{2} = \frac{b-a}{2n}.$$

$$\frac{h}{2} = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + f(x_2) \cdots f(x_n)\right].$$

* Controlling the error in (T) Approximation

Mag of error

(error) → $\boxed{E_T = \int_a^b f(x)\,dx - T}$

* The error estimate for (T) Rule :-

If $f''$ is contin~ i.e $M_2$ is any upper bound for the values of $|f''|$ on $[a, b]$, then,

error estimate → $\boxed{|E_T| \le \frac{b-a}{12} h^2 M_2}$.

$h = \frac{b-a}{n}$

Q i) Use (T) rule with $n=4$ estimate

$$\int_0^2 \frac{1}{1+x}\,dx.$$

compare estimate with exact value of integral. Also find an upper bound for error in above approxi~

A)

step size → $h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$

here $n=4$. So divide interval $[0, 2]$ into $4$ → 0, 0.5, 1, 1.5, 2.

x values
$$\begin{cases} x_0 = a = 0 \\ x_1 = x_0 + h = 0 + 0.5 = 0.5 \\ x_2 = x_1 + h = 0.5 + 0.5 = 1 \\ x_3 = x_2 + h = 1 + 0.5 = 1.5 \\ x_4 = x_3 + h = 2 \end{cases}$$

intervals → $x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

(4 intervals) = $n=4$

y values

$$f(x) = \frac{1}{1+x}.$$

decimal ✗ so, ×2

$y_0 = f(0) = \frac{1}{1+x_0} = \frac{1}{1+0} = 1$

$y_1 = f(1) = \frac{1}{1+x_1} = \frac{1}{1+0.5} = \frac{1}{1.5}\frac{\times 2}{\times 2} = \frac{2}{3}$

$y_2 = \frac{1}{1+x_2} = \frac{1}{2}$

$y_3 = \frac{1}{1+x_3} = \frac{1}{2.5}\frac{\times 2}{\times 2} = \frac{2}{5}$

$y_4 = \frac{1}{1+x_4} = \frac{1}{3}$

by T. rule,

$$T = \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4.\right]$$

$$= \frac{1}{2\times 2}\left[1 + 2\cdot\frac{2}{3} + \cancel{2}\cdot\frac{1}{\cancel{2}} + 2\cdot\frac{2}{5} + \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[\frac{1\times 15}{1\times 15} + \frac{4\times 5}{3\times 5} + \frac{1\times 15}{1\times 15} + \frac{4\times 3}{5\times 3} + \frac{1\times 5}{3\times 5}\right]$$

$$= \frac{1}{4}\left[\frac{15 + \cancel{4}20 + 15 + 12 + 5}{15}\right]$$

$$= \frac{1}{4}\left[\frac{67}{15}\right] = \frac{67}{60} \approx 1.1167$$

exact value of $\int$,

$\int \frac{1}{x} = \ln x$

$$\int_0^2 \frac{1}{1+x} \cdot dx = \left[\ln(1+x)\right]_0^2$$
$$= \left[\ln(1+2) - \ln(1+0)\right]$$
$$= \ln 3 - 0 = \ln 3$$
$$= 1.0986$$

Then error,

$$= 1.1167 - 1.0986 = 0.0181$$

upper bound ?.

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

$f(x) = \frac{1}{1+x}$

$f'(x) = \frac{-1}{(1+x)^2}$

$\frac{-1}{x^2} = x^{-2} = -2x^{-2-1} = 2x^{-3}$

$f''(x) = \frac{2}{(1+x)^3}$

$\frac{-1}{(1+x)^2} = (1+x)^{-2} = -2(1+x)^{-2-1} = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$

$M_2$ = upper bound values of $|f''|$ on $[0,2]$,

$$M_2 = \frac{2}{(1+0)^3} = 2$$ $\longrightarrow$ take 0 as least value from $[0,2]$ to $x$. 0 is upper un.

error estimation,

$$|E_T| \leq \frac{b-a}{12} h^2 M_2$$
$$= \frac{2-0}{12} \cdot (0.5)^2 \cdot 2.$$
$$|E_T| = \frac{2}{12} \cdot \cancel{0.25}\,0.5 = \frac{1}{12} = 0.083$$

value b/w $[0,2]$.

2) find upper bound for error,

$$\int_0^{\pi} x \sin x \cdot dx. \quad , n=10$$

using (T) rule.

A) $n = 10$, $[0,\pi]$

$$h = \frac{\pi - 0}{10} = \frac{\pi}{10}$$

$x_0 = a = 0$.

$x_1 = \cancel{x_0 + h = a +}\ \frac{\pi}{10}$

$x_2 = \pi/20$, $x_3 = \pi/30$, $x_4 = \pi/40$

.... $x_{10} = \pi/\cancel{100}$.

upper bound ? $M_2 = ?$.

$f(x) = \cancel{\int} x \cdot \sin x \ dx$. (p. rule)

$f'(x) = x \cdot \cos x + \sin x$

$f''(x) = x \cdot -\sin x + \cos x + \cos x$

$= -x\sin x - 2\cos x = 2\cos x - x\sin x$

$M_2 \cdot \text{ms} |f''x| = |2\cos x - x\sin x|$

$\leq 2\,|\cos x| + |x|\,|\sin x|$

(by $\Delta$ Inequality) $\rightarrow$ $M_2 \leq 2\cdot 1 + \pi \cdot 1 = \underline{\underline{2+\pi}}$ $[0,\pi]$

$2\cos x - x\sin x$

when we rule 0,

$2\cos 0 - \frac{0\cdot\sin 0}{0} = x$

$\sin 0 = 0$
$\sin\pi = 0$
$\therefore \sin\frac{\pi}{2} = 1$

So, $\cos \to 0$, i.e. $\sin \to \pi/2$

$\pi/2 \to [0,\pi]$
$\sin\pi/2 = 1$

$\Rightarrow 2\,|\cos 0| + |\pi|\cdot|\sin \pi/2|$

$\Rightarrow 2\times 1 + \pi\cdot 1 = \underline{\underline{2+\pi}}$

$\boxed{\begin{matrix}\cos 0 \\ \sin\pi/2\end{matrix}}$ $\to$ upper bound always

$M_2 = 2+\pi$

error estimation,

$|E_T| \leq \frac{b-a}{12} h^2 M_2$

$= \frac{\pi-0}{12}\cdot\left(\frac{\pi}{10}\right)^2\cdot(2+\pi)$

$= \frac{\pi}{12}\cdot\frac{\pi}{10}\cdot\frac{\pi}{10}\cdot(2+\pi)$

$= \frac{\pi^3(2+\pi)}{1200} = \frac{30.959144(5.14)}{1200}$

$= \underline{\underline{0.133}}$

3) find min no. of rule intervals needed to approximate the $\int$ given below using T. rule with an error mag $< 10^{-4}$.

$$\int_0^3 \sqrt{1+x}\,dx.$$

A) $M_2$ ?

$f(x) = \sqrt{1+x}$

$f'(x) = \frac{1}{2\sqrt{1+x}}$

$f''(x) = \frac{-1}{4(1+x)^{3/2}}$

$|f''(x)| = \frac{-1}{4(1+x)^{3/2}}$

$\frac{1}{2\sqrt{x}} = \frac{1}{2x^{1/2}}$

$= \frac{1}{2}\frac{d}{dx}(x^{-1/2})$

$= \frac{1}{2}\cdot\frac{1}{2}x^{-\frac{1}{2}-1}$

$= \frac{1}{4}x^{-3/2}$

$= -\frac{1}{4x^{3/2}}$

$[0,3]$

$x=0$ $M_2 = \frac{-1}{4(1+0)^{3/2}}$

$= \underline{\underline{\frac{1}{4}}}$

~~when $x=3$~~

$h = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}.$

$|E_T| \leq \frac{b-a}{12} h^2 M_2 = \frac{3-0}{12}\cdot\left(\frac{3}{n}\right)^2\cdot\frac{1}{4}$

$= \frac{3}{12}\cdot\frac{9}{n^2}\cdot\frac{1}{4} = \frac{27\div 3}{48\div 3} \Rightarrow \underline{\frac{9}{16n^2}}$

in Qs

$\frac{9}{16n^2} < 10^{-4}$ (i.e)

$n^2 > \frac{9\times 10^4}{16}$

$n > \sqrt{\frac{9\times 10^4}{16}}$ $|n| > \frac{300}{4} = \underline{\underline{75}}$

hence to get an approxi.. units an error of mag less than $10^{-4}$ the no. of subintervals should be atleast 76.

⇒ Simpson's Rule :- (S)

~~mag of error~~

$$\int_a^b f(x)\,dx \approx S = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 2y_{n-2} + 4y_{n-1} + y_n\right].$$

where y's are values of f at partition points $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h \dots x_{n-1} = a+(n-1)h$, $x_n = b$, n is even & $h = \frac{b-a}{n}$

* Simpson's 1 by 3rd rule (parabolic rule)

$$\boxed{\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]}$$

* Error Estimate for (S) rule:-

$$\boxed{|E_S| \le \frac{b-a}{180} h^4 M_4}$$

If $f^{(4)}$ is contins & $M_4$ is an upper bound for values of $|f^{(4)}|$ on $[a,b]$.

Q 1) Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ using S $\frac{1}{3}$ rd rule. (n [illegible] n=6)

A) here $n = 6$.

$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$ $\qquad y = f(x) = \frac{1}{1+x^2}$

(n=6)

$x_0 = a = 0$ $\qquad y_0 = \frac{1}{1+x_0^2} = \frac{1}{1+0} = 1$

$x_1 = x_0 + h = 1$ $\qquad y_1 = \frac{1}{1+1^2} = \frac{1}{2}$

$x_2 = 2$ $\qquad y_2 = \frac{1}{1+2^2} = \frac{1}{5}$

$x_3 = 3$ $\qquad y_3 = \frac{1}{1+3^3} = \frac{1}{10}$

$x_4 = 4$ $\qquad y_4 = \frac{1}{17}$

$x_5 = 5$ $\qquad y_5 = \frac{1}{26}$

$x_6 = 6$ $\qquad y_6 = \frac{1}{37}$

by S rule,

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right].$$

$$= \frac{1}{3}\left[1 + \cancel{4}^{2}\cdot\frac{1}{\cancel{2}} + 2\cdot\frac{1}{5} \cdot 4\cdot\frac{1}{10} + 2\cdot\frac{1}{17} + 4\cdot\frac{1}{26} + \frac{1}{37}\right].$$

$$= \frac{1}{3}\left[1 + 2 + \frac{2}{5} + \frac{4}{10} + \frac{2}{17} + \frac{4}{26} + \frac{1}{37}\right]$$

$$= \frac{1}{3}\left[3 + \frac{2\times 2}{5\times 2} + \frac{4}{10} + \frac{2}{17} + \frac{4}{26} + \frac{1}{37}\right]$$

$= \frac{1}{63}\left[\frac{3}{7} + \frac{8}{10} - \frac{180}{442} - \frac{1}{37}\right] = 1.366173$

$\cancel{= \frac{1}{3}\left[\frac{38}{10}\right.}$ (or)

$= \frac{h}{3}\left(y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right)$

$= \frac{1}{3}\left[1 + \frac{1}{37} + 4\left(\frac{1}{2} + \frac{1}{10} + \frac{1}{26}\right) + 2\left(\frac{1}{5} + \frac{1}{17}\right)\right]$

$= 1.366173$

$\int_0^6 \frac{1}{1+x^2}\,dx = \left[\ln|1+x^2|\right]_0^6 = \ln|37| - \ln 1$
$= \ln 37 = 3.6109$

error,

$= 1.366173 - 3.6109$

$= 2.24$ $[0,6]$

2) using S.rule with $n=4$

$\int_0^2 \frac{1}{1+x}\,dx.$

Compare estimate with exact value of $\int$. Also find upper bound for error?

A) $h = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$ $y = \frac{1}{1+x}.$

$x_0 = a = 0$ $y_0 = 1$

$x_1 = x_0 + h = 0 + \frac{1}{2} = \frac{1}{2}$ $y_1 = \frac{1}{1.5}\frac{\times 2}{\times 2} = \frac{2}{3}$

$x_2 = 1$ $y_2 = \frac{1}{2}$

$x_3 = 1.5$ $y_3 = \frac{1}{1+1.5} = \frac{1}{2.5}\frac{\times 2}{\times 2} = \frac{2}{5}$

$x_4 = 2.$ $y_4 = \frac{1}{3}$

$\int_0^2 \frac{1}{1+x}\,dx = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$

$= \frac{1}{6}\left[1 + 4\cdot\frac{2}{3} + 2\cdot\frac{1}{2} + 4\cdot\frac{2}{5} + \frac{1}{3}\right]$

$= \frac{1}{6}\left[\frac{1\times 15}{1\times 15} + \frac{8\times 5}{3\times 5} + \frac{1\times 15}{1\times 15} + \frac{8\times 3}{5\times 3} + \frac{1\times 5}{3\times 5}\right]$

$= \frac{1}{6}\left[\frac{15 + 40 + 15 + 24 + 5}{15}\right]$

$= \frac{1}{6}\left[\frac{99}{15}\right] = 1.1$

$\int \frac{1}{x} = \ln x.$

exact value,

$\int_0^2 \frac{1}{1+x}\,dx = \left[\ln|1+x|\right]_0^2 = \ln 3 - \ln 1$
$= \ln 3 = 1.0986$

error is approximately

$= \underline{0.0014}$ $\quad (1.4 \cdot 10^{-3})$

$M_4$ = upper bound for values of $|f^{(4)}|$ on $[0,2]$,

$f(x) = \frac{1}{1+x}$ $\quad f'(x) = \frac{-1}{(1+x)^2}$

$f''(x) = \frac{-1}{(1+x)^2} = -1\,(1+x)^{-2}$

$\cancel{= -1\,(1+x)^{-2} = -1\,(1+x)^{-3}}$

$= -1 \cdot -2\,(1+x)^{-2-1}$

$f''(x) = -1 \cdot -2\,(1+x)^{-3} = \frac{2}{(1+x)^3}$

$f'''(x) = \frac{2}{(1+x)^3} = 2 \cdot (1+x)^{-3}$

$= 2 \cdot -3\,(1+x)^{-3-1}$

$= -6\,(1+x)^{-4} = -\frac{6}{(1+x)^4}$

$f^{(4)}(x) = \frac{-6}{(1+x)^4} = -6 \cdot (1+x)^{-4}$

$= -6 \cdot -4 \cdot (1+x)^{-4-1}$

$= 24\,(1+x)^{-5} = \frac{24}{(1+x)^5}$

$[0,2]$ $=0$ take $f^{(4)}(0)$

$M_4 = \frac{24}{(1+0)^5} = \underline{24}$

$|E_S| = \frac{b-a}{180} h^4 M_4 = \frac{2-0}{180} \cdot \left(\frac{1}{2}\right)^4 \cdot 24$

$= \frac{2}{180} \cdot 1.5 = \frac{3}{180} = \underline{0.0167}$

3) use S. rule with $n=6$ to Estimate lowing $\int$ correct to 4 places of decimals. $\rightarrow \int_0^1 \sqrt{1+x^3}\,dx$.

A) $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} = \underline{0.16667}$

$x_0 = 0$ $\quad y_0 = \sqrt{1+0^3} = 1$

$x_1 = 0.16667$ $\quad y_1 = \sqrt{1+(0.16667)^3} = 1.00231$

$x_2 = 0.33334$ $\quad y_2 = 1.01835$

$x_3 = 0.50001$ $\quad y_3 = 1.06066$

$x_4 = 0.66668$ $\quad y_4 = 1.13856$

$x_5 = 0.83335$ $\quad y_5 = 1.25648$

$x_6 = 1.0$ $\quad y_6 = \sqrt{1+1^3} = 1.41421$

by S. rule,

$$S = \int_0^1 \sqrt{1+x^3}\,dx = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\right].$$

$$= \frac{0.16667}{3}\left[0 + 4 \cdot 1.00231 + 2 \cdot 1.01835 + 4 \cdot 1.06066 + 2 \cdot 1.13856 + 4 \cdot 1.25648 + 1.41421\right]$$

$S = \underline{1.1115}$ (4 decimal places)

4) Suppose you are given following table -

$f(0)=0.846$ $\quad f(0.4)=1.121$ $\quad f(0.8)=2.321$

$f(0.1)=0.928$ $\quad f(0.5)=1.221$ $\quad f(0.9)=3.101$

$f(0.2)=0.882$ $\quad f(0.6)=1.661$ $\quad f(1.0)=3.010$

$f(0.3)=0.953$ $\quad f(0.7)=2.101$

Evaluate $\int_0^1 f(x)\,dx$ by S. rule ?

A) $n = 10$

~~$x_i$ 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8~~

| $x_i$ | $y_i$ |
|---|---|
| 0 | 0.846 |
| 0.1 | 0.928 |
| 0.2 | 0.882 |
| 0.3 | 0.953 |
| 0.4 | 1.121 |
| 0.5 | 1.221 |
| 0.6 | 1.661 |
| 0.7 | 2.101 |
| 0.8 | 2.321 |
| 0.9 | 3.101 |
| 1.0 | 3.010 |

$h = \frac{b-a}{n} = \frac{1-0}{11} = \frac{1}{11} = 0.1$

$h = \Delta x_i = 0.1$

$\int_0^1 f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + \dots 2y_8 + 4y_9 + y_{10}\right]$

$\frac{0.1}{3} = \frac{1}{30}$

$= \frac{1}{30}\left[49.042\right] = 1.6347$

5) find min no. of subintervals needed to approximate the $\int$ given below using S. rule with an error of mag $< 10^{-4}$!

$\int_0^3 \sqrt{1+x}\,dx$

A) $h = \frac{b-a}{n} = \frac{3}{n}$

$f(x) = \sqrt{1+x}$

$f'(x) = \frac{1}{2\sqrt{1+x}}$

$f''(x) = \frac{1}{2\sqrt{1+x}}$

$= \frac{1}{2}\frac{d}{dx}\left(\frac{1}{\sqrt{1+x}}\right)$

$= \frac{1}{2}\cdot\frac{d}{dx}(1+x)^{-1/2}$

$= \frac{1}{2}\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}-1}$

$-\frac{3}{2} - \frac{1}{1} = \frac{-3-2}{2} = \frac{-5}{2}$

$= \frac{1}{4}(1+x)^{-3/2}$

$= \frac{-1}{4(1+x)^{3/2}}$

$f'''(x) = \frac{-1}{4(1+x)^{3/2}}$

$= -\frac{1}{4}\frac{d}{dx}\left(\frac{1}{(1+x)^{3/2}}\right)$

$= \frac{-1}{4}\cdot(1+x)^{-3/2} = \frac{-1}{4}\cdot\frac{-3}{2}\cdot(1+x)^{\frac{-3}{2}-1}$

$= \frac{3}{8}(1+x)^{-5/2} = \frac{3}{8(1+x)^{5/2}}$

$f^{(4)}(x) = \frac{3}{8(1+x)^{5/2}} = -\frac{15}{16(1+x)^{7/2}}$

$M_4$ = upper bound for values of $|f^{(4)}|$ on $[0,3]$

$$= \frac{15}{16(1+0)^{7/2}} = \frac{15}{16}$$

absolute error,

$$\therefore |E_S| = \frac{b-a}{180} h^4 M_4 = \frac{3}{180} \cdot \left(\frac{3}{n}\right)^4 \cdot \frac{15}{16}$$

$$= \frac{3}{180} \cdot \frac{81}{n^4} \cdot \frac{15}{16}$$

$$= \frac{3^4}{2^6 n^4}$$

Since A. error is $< 10^{-4}$, we get

$$\frac{3^4}{2^6 n^4} < 10^{-4} \quad \text{(i.e)} \quad n^4 > \frac{3^4 \times 10^4}{2^6}$$

$$\text{(i.e)} \; |n| > \frac{30}{2\sqrt{2}} > 10.6217$$

Since in S. rule no. of subintervals should be even, min no. of subintervals needed to approximate the gvn interval, using S. rule with an error of mag $< 10^{-4}$ is 12.

6) How small must we take $\Delta x$ in S. rule to evaluate the definite $\int$ gvn below to within $10^{-6}$.

$$\int_2^4 e^{-x^2} dx.$$

Also find smallest value of $\Delta x$ in T. rule to evaluate the above $\int$ with an error of mag $< 10^{-6}$?

A) $f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$, $f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2}$
$= 2(2x^2-1)e^{-x^2}$

$f'''(x) = 8xe^{-x^2} - 4x(2x^2-1)e^{-x^2} = (12x - 8x^3)e^{-x^2}$

$f^{(4)}(x) = [(12-24x^2) - 2x(12x-8x^3)]e^{-x^2}$
$= 4(4x^2 - 12x^2 + 3)e^{-x^2}$.

on $[2,4]$, $4x^4 - 12x^2 + 3$ is $\uparrow$ing since $1^{st}$ deri... of $4x^4 - 12x^2 + 3$ is $> 0$ on $[2,4]$. & $e^{-x^2}$ is $\downarrow$ing on $[2,4]$.

$$M_4 = 4(4x^2 - 12x^2 + 3)\Big|_{x=4} \cdot e^{-x^2}\Big|_{x=2}$$

$$= 4(4 \cdot 4^4 - 12 \cdot 4^2 + 3)e^{-4} \approx 61.174$$

$$h = \Delta x = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n}$$

absolute error,

$$|E_S| \leq \frac{b-a}{180} h^4 M_4 = \frac{2}{180} \cdot \left(\frac{2}{n}\right)^4 \cdot 61.174$$

$$= \frac{10.875}{n^4}$$

Since error's of absolute value is $< 10^{-6}$

$$\frac{10.875}{n^4} < 10^{-6} \quad \text{(i.e)} \quad n^4 > 10.875 \times 10^6$$

$$|n| > \sqrt[4]{10.875 \times 10^6} > 57.426$$

hence to get an approximation with an error of mag $< 10^{-6}$ no. of subintervals should be $> 57$.,

using S. rule with an error of mag $< 10^{-6}$ is 58

In this case length of subinterval is $\Delta x = h = \frac{2}{n} = \frac{2}{58} = 0.0345$

$\Delta x$ should be $< 0.0345$

Since $f'''(x) = 4x(3-2x^2)e^{-x^2} < 0$ on $[2,4]$

$f''(x)$ is decreasing,

Also $f''(x) > 0$ on $[2,4]$ i.e so $|f''(x)| =$

$$f''(x) \leq f''(2) = 14e^{-4}$$

$$M_2 = 14e^{-4} \approx 0.256$$

$$h = \frac{b-a}{n} = \frac{2}{n}.$$

$$|E_T| \leq \frac{b-a}{12} h^2 M_2 = \frac{2}{12}\left(\frac{2}{n}\right)^2 \cdot 0.256$$

$$= \frac{0.171}{n^2}$$

Since the error's of absolute value $< 10^{-6}$,

$$\frac{0.171}{n^2} < 10^{-6} \quad \text{(i.e)} \quad n^2 > 0.171 \times 10^6$$

$$\text{(i.e)} \quad |n| > 413.5$$

To get an approximation with an error of mag $< 10^{-4}$ the no. of subintervals should be atleast 414,

here length of subintervals is

$$\Delta x = h = \frac{2}{n} = \frac{2}{414} = 0.0048$$

$\Delta x$ should be less than 0.0048