2) <u>Rectangular distri.....</u> :- (uniform distri...) (<u>continous</u>)

* A very simple distri... for a contin... R.V is the uniform distri....
* It is perticularly useful in theoretical statistics, bcz it is convenient to deal with mathematically.

<u>def</u> →

~~A~~ contin... R.V 'x' is said to hve rectangular distri..., if its pdf givn <u>by</u>,

$$\boxed{f(x) = \frac{1}{b-a}}, \quad a \le x \le b$$
$$= 0, \quad \text{elsewhere.}$$

Rmrk →

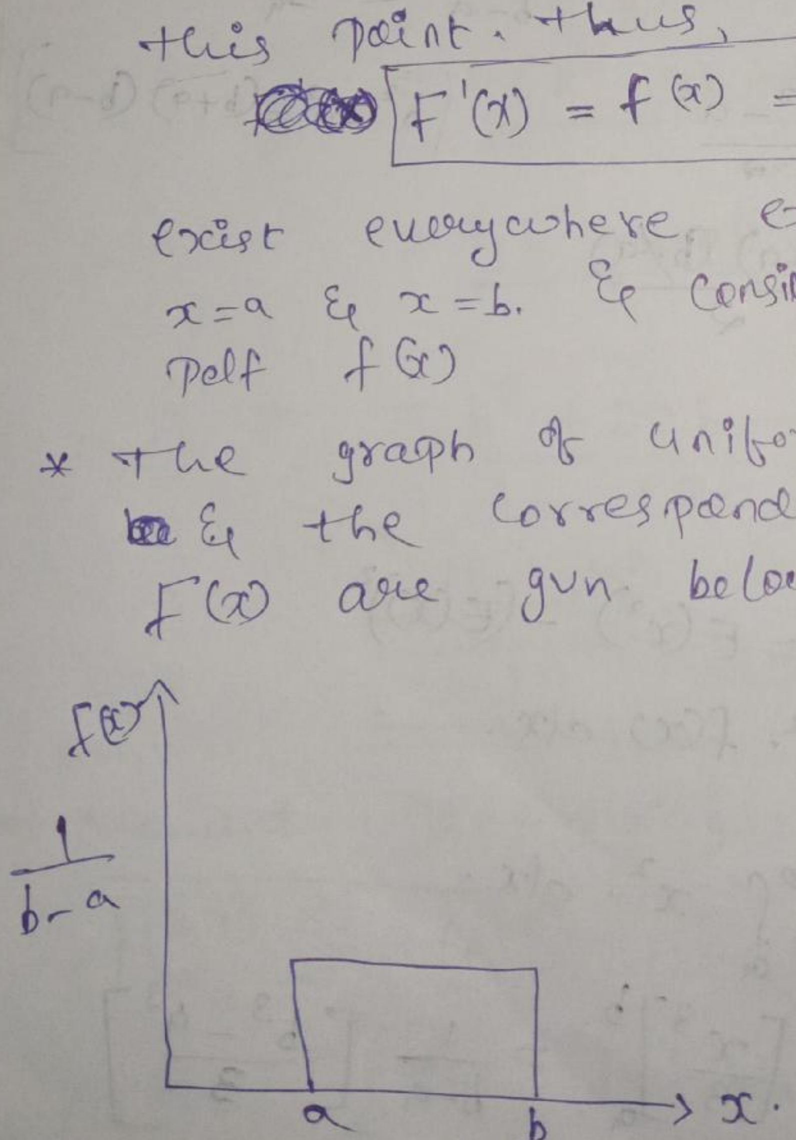
* a and b be $(a<b)$ are 2 paramtr of uniform distri... on a, b.
uniform distri... → rectangular distri...
* The distri... describes a since the curve, $y = f(x)$, discribes a ▭ over the x-axis and b/w the ordinates at $x = a$ & $x = b$.
* The distri... () $F(x) = \begin{cases} 0, & \text{if } -\infty < x < a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & b < x < \infty. \end{cases}$

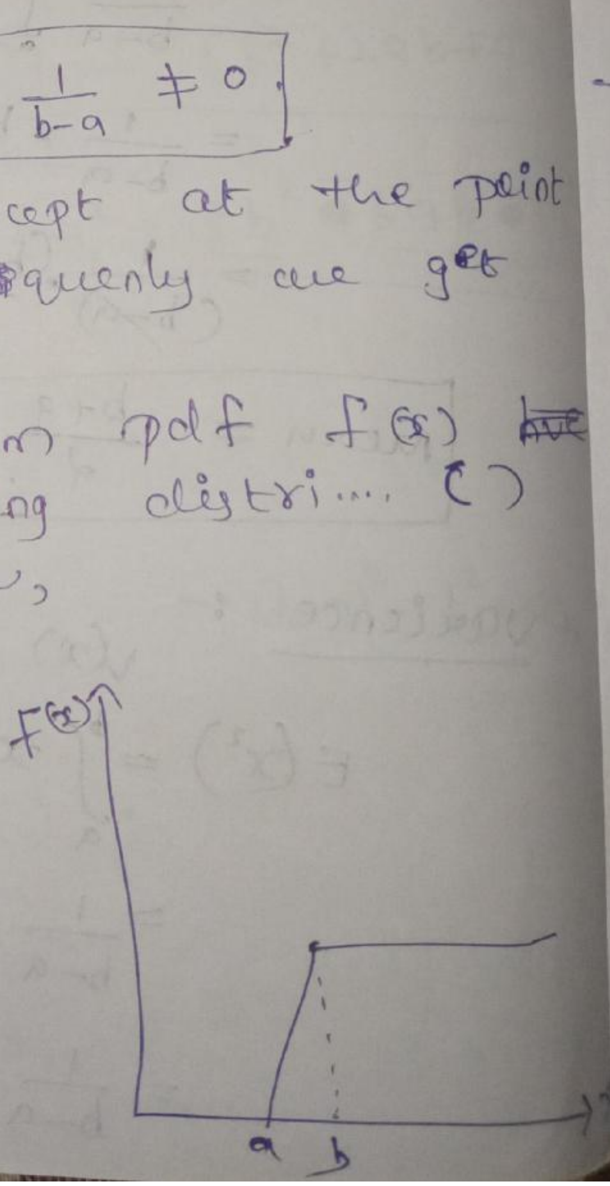
Since $F(x)$ is not contin... at $x = a$ & $x = b$, it is not differentiable at this point. thus,

$$\boxed{F'(x) = f(x) = \frac{1}{b-a} \neq 0}$$

exist everywhere except at the point $x = a$ & $x = b$. & consequenly we get pdf $f(x)$

* The graph of uniform pdf $f(x)$ & the corresponding distri.... () $F(x)$ are gvn below,





* for a ▭-lar / uniform variate 'x' in $(-a, a)$ pdf is gvn by,

$$f(x) = \begin{cases} \frac{1}{2a} , & -a < x < a \\ 0 , & \text{elsewhere} \end{cases}$$

~~* moments :-~~

* <u>Mean</u> :- $E(x)$

$$= \int_a^b x \cdot f(x) \, dx .$$

$$= \int_a^b x \cdot \frac{1}{b-a} \, dx .$$

$$= \frac{1}{b-a} \int_a^b x \cdot dx = \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a} \times \frac{b^2 - a^2}{2}$$

$\boxed{b^2 - a^2 = (b+a)(b-a)}$

$$= \frac{1}{\cancel{(b-a)}} \times \frac{(b+a)\cancel{(b-a)}}{2}$$

$$\boxed{\text{Mean} = \frac{b+a}{2}.}$$

* <u>Varience</u> :- $V(x) = E(x^2) - (E(x))^2$

$$E(x^2) = \int_a^b x^2 \cdot f(x) \, dx .$$

$$= \frac{1}{b-a} \int_a^b x^2 \cdot dx .$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a} \left[\frac{b^3 - a^3}{3}\right]$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$b^3 - a^3 = (b-a)\cdot(b^2+ab+a^2)$

$$= \frac{\cancel{(b-a)}(b^2-ab+a^2)}{3\cancel{(b-a)}}$$

$$E(x^2) = \frac{b^2 - ab + a^2}{3}$$

$$\therefore V(x) = \frac{b^2-ab-a^2}{3} - \left[\frac{b+a}{2}\right]^2$$

$$= \frac{b^2-ab-a^2}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{b^2-ab-a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4(b^2-ab-a^2) - 3(b^2+2ab+a^2)}{12}$$

$$= \frac{4b^2 - 4ab - 4a^2 - 3b^2 - 6ab - 3a^2}{12}$$

$$= \frac{b^2 + 2ab - a^2}{12}$$

$(b-a)^2 = b^2 + 2ab - a^2$

$$= \frac{(b-a)^2}{12}$$

⇒ <u>Moment generating ( )</u> :-

$$M_x(t) = E(e^{tx})$$

$$= \int_a^b e^{tx} \cdot f(x) \, dx .$$

$\Gamma_{n+1} = n! = n(n-1)! = n\Gamma_n$

$\Gamma_{n+1} = n\Gamma_n$ — (a)

Putting $m=1$ & $p=\frac{1}{2}$

$\cancel{f(x) = \frac{m^p}{\Gamma p} \cdot e^{-mx} \cdot x^{p-1}}$

— ① $\int_0^\infty e^{-mx} \cdot x^{p-1}\, dx = \frac{\Gamma p}{m^p}$

$m=1$, $p=1/2$ $\Rightarrow \int_0^\infty e^{-x} x^{p-1}\, dx = \Gamma_{1/2} = \sqrt{\pi}$

$\Rightarrow$ moments :-

(i) Mean →

$E(x) = \int x \cdot f(x)\, dx$

$= \int_0^\infty x \frac{m^p}{\Gamma p} e^{-mx} \cdot x^{p-1}\, dx$

$= \frac{m^p}{\Gamma p} \int_0^\infty x e^{-mx} \cdot x^{p-1}\, dx$

$x \cdot x^{p-1} = x^{p+1-1} \Rightarrow x^{(p+1)-1}$

$= \frac{m^p}{\Gamma p} \int_0^\infty e^{-mx} x^{(p+1)-1}\, dx$

$\int_0^\infty e^{-mx} x^{p-1}\, dx = \frac{\Gamma p}{m^p}$

$\int_0^\infty e^{-mx} x^{(p+1)-1}\, dx = \frac{\Gamma_{p+1}}{m^{p+1}}$

$= \frac{m^p}{\Gamma p} \cdot \frac{\Gamma_{p+1}}{m^{p+1}}$

— (a) like, $\Gamma_{n+1} = n\Gamma_n$

$\therefore \Gamma_{p+1} = p\Gamma p$

(b)

$= \frac{m^p}{\Gamma p} \cdot \frac{p\Gamma p}{m^{p+1}} = \frac{\cancel{m^p}}{\cancel{\Gamma p}} \cdot \frac{p\cancel{\Gamma p}}{\cancel{m^p} \cdot m}$

$= \frac{p}{m}$ //

$= \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx$

$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{1}{b-a}\left[\frac{e^{tb} - e^{at}}{t}\right]$

$\boxed{M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}}$, $t \neq 0$

$\Rightarrow$ Gamma distri... :-

A contin... R.V 'x' is said to hue a gamma distri... , its its pdf is gvn by,

$\boxed{f(x) = \frac{m^p}{\Gamma p} \cdot e^{-mx} \cdot x^{p-1}}$, $x > 0$

gamma p

$= 0$, elsewhere.

where $m > 0$, $p > 0$ ⟶ paramtr of G. distri.

Γp ⟶ Gamma p.

Note
Being a pdf we know that $\int f(x)\, dx = 1$.

$= \int_0^\infty \frac{m^p}{\Gamma p} \cdot e^{-mx} \cdot x^{p-1}\, dx = 1$

$= \frac{m^p}{\Gamma p} \int_0^\infty e^{-mx} \cdot x^{p-1}\, dx = 1$

$\boxed{\Rightarrow \int_0^\infty e^{-mx} \cdot x^{p-1}\, dx = \frac{\Gamma p}{m^p}}$ — ①

If $p = n$, a +ve integr $\Gamma n = (n-1)!$

$\boxed{\text{if } p = n, \text{ then } \Gamma n = (n-1)!}$

using ∫tion by parts we can S.T

$$= \int_0^\infty e^{tx} \cdot \frac{m^p}{\Gamma p} \cdot e^{-mx} \cdot x^{p-1}\, dx.$$

$$= \frac{m^p}{\Gamma p} \int_0^\infty \underline{e^{tx}} \cdot \underline{e^{-mx}} \cdot x^{p-1}\, dx.$$

$$= \frac{m^p}{\Gamma p} \int_0^\infty e^{-(m-t)x} \cdot x^{p-1}\, dx$$

$$= \frac{m^p}{\cancel{\Gamma p}} \cdot \frac{\cancel{\Gamma p}}{(m-t)^p}$$

$$= \frac{m^p}{(m-t)^p} = \left(\frac{m}{m-t}\right)^p$$

$$= \left(\frac{\cancel{m}}{\cancel{m}(1-\frac{t}{m})}\right)^p$$

$$= \left(\frac{1}{1-\frac{t}{m}}\right)^p = \frac{1}{(1-\frac{t}{m})^p}$$

$$= \left(1-\frac{t}{m}\right)^{-p}$$

$$e^{tx} \cdot e^{-mx} = e^{tx-mx} = e^{-mx+tx} = e^{-x(m-t)} = e^{-(m-t)x}$$

$$\int_0^\infty e^{-mx} x^{p-1} \cdot dx = \frac{\Gamma p}{m^p}$$

$$\therefore \int_0^\infty e^{-(m-t)x} x^{p-1}\, dx = \frac{\Gamma p}{(m-t)^p}$$

⇒ Exponential distri.... :-

If a contin... r.v x has a pdf

$$\boxed{f(x) = \theta e^{-\theta x}} \quad x > 0,\ \theta > 0$$

* Mean :- $\boxed{E(x) = \frac{1}{\theta}}$

$$E(x) = \int_0^\infty x \cdot f(x)\, dx = \int_0^\infty x \cdot \theta e^{-\theta x} \cdot dx.$$

$x' = x^{2-1}$

$$= \int_0^\infty x^{2-1} \cdot \theta e^{-\theta x} = \theta \int_0^\infty \underbrace{x^{2-1} \cdot e^{-\theta x}} \cdot dx.$$

(2) Varience →

$$V(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_0^\infty x^2 \cdot \frac{m^p}{\Gamma p} \cdot e^{-mx} \cdot x^{p-1}\, dx.$$

$$= \frac{m^p}{\Gamma p} \int_0^\infty \underline{x^2} \cdot e^{-mx} \cdot \underline{x^{p-1}}\, dx.$$

$$= \frac{m^p}{\Gamma p} \int_0^\infty e^{-mx} \cdot x^{(p+2)-1}\, dx.$$

$$\Rightarrow \frac{m^p}{\Gamma p} \times \frac{\Gamma p+2}{m^{p+2}}$$

$$\Rightarrow \frac{m^p}{\Gamma p} \cdot \frac{(p+1)\, p\, \Gamma p}{m^{p+2}}$$

$$\Rightarrow \frac{\cancel{m^p}}{\cancel{\Gamma p}} \cdot \frac{(p+1)\, p\, \cancel{\Gamma p}}{\cancel{m^p} \cdot m^2}$$

$$E(x^2) \Rightarrow \frac{(p+1)\, p}{m^2}$$

$x^2 \cdot x^{p-1}$ $x^{(p+2)-1}$

$2+p-1 \Rightarrow 2$

(5)

$$\int_0^\infty e^{-mx} x^{(p+1)-1}\, dx = \frac{\Gamma p+1}{m^{p+1}}$$

$$\therefore \int_0^\infty e^{-mx} \cdot x^{(p+2)-1}\, dx = \frac{\Gamma p+2}{m^{p+2}}$$

$$\Gamma n+1 = n\Gamma n$$

$$\therefore \Gamma p+2 = (p+1)\, \Gamma p+1 = (p+1) \cdot p\, \Gamma p$$

$$V(x) = \frac{(p+1)\, p}{m^2} - \left(\frac{p}{m}\right)^2$$

$$= \frac{(p+1)p}{m^2} - \frac{p^2}{m^2} = \frac{\cancel{p^2} + p - \cancel{p^2}}{m^2}$$

$$\boxed{V(x) = \frac{p}{m^2}}$$

(3) Moment generating ( ) :-

$$M_x(t) = E(e^{tx})$$

$$= \int_a^b e^{tx} \cdot f(x)\, dx.$$

* **Mgf** :– $\boxed{M_x(t) = \left[1 - \frac{t}{\theta}\right]^{-1}}$

$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot f(x)\, dx.$

$= \int_0^\infty e^{tx} \cdot \theta e^{-\theta x}\, dx$

$= \theta \int_0^\infty e^{tx} \cdot e^{-\theta x}\, dx = \theta \int_0^\infty e^{tx - \theta x}\, dx.$

$= \theta \int_0^\infty e^{-(\theta - t)x}\, dx.$

$= \theta \cdot \left[\frac{e^{-(\theta - t)x}}{-(\theta - t)} \right]_0^\infty \quad (x \to \infty, 0)$

$e^{\infty} = \infty$
$e^{-\infty} = 0$

$= \cancel{\theta \left[\frac{e^{-\infty}}{-\theta - t} - 0 \right]} = \cancel{\theta}$

$= \theta \left[\frac{e^{-\infty}}{-\theta - t} - \frac{e^{0}}{-\theta - t} \right] = \theta \left[\frac{e^{-\infty} - e^{0}}{-(\theta - t)} \right]$

$\theta \left[\frac{0 - 1}{-(\theta - t)} \right] = \theta \left[\frac{+1}{+(\theta - t)} \right]$

$= \theta \cdot \frac{1}{\theta - t} = \frac{\theta}{\theta - t}$ (split not possible)

$= \left[\frac{\theta - t}{\theta} \right]^{-1} = \left[\frac{\theta}{\theta} - \frac{t}{\theta} \right]^{-1}$

$M_x(t) = \left[1 - \frac{t}{\theta} \right]^{-1}$

* Exp. distri. sometimes is the –ve Expo. distri.

$\boxed{\int_0^\infty e^{-mx} \cdot x^{p-1}\, dx = \frac{\sqrt{p}}{m^p}}$

$m = \theta$
$p = 2$

$\therefore \Rightarrow \theta \cdot \frac{\sqrt{2}}{\theta^2} = \frac{\sqrt{2}}{\theta}$

$\sqrt{n+1} = n(n-1)!$

$\boxed{\sqrt{n} = (n-1)\sqrt{n-1}}$

$\therefore \Rightarrow \sqrt{2} = (2-1)\sqrt{2-1} = 1\sqrt{1} = 1$

$\therefore \Rightarrow \frac{1}{\theta}$

$\boxed{\text{mean} = \frac{1}{\theta}}$

* Variance :– $\boxed{V(x) = \frac{1}{\theta^2}}$

$V(x) = E(x^2) - (E(x))^2$

$E(x^2) = \int_0^\infty x^2 \cdot f(x)\, dx = \int_0^\infty x^2 \cdot \theta e^{-\theta x}\, dx.$

$\boxed{x^2 = x^{3-1}}$

$= \theta \int_0^\infty x^{3-1} \cdot e^{-\theta x}\, dx$

$\int_0^\infty e^{-mx} \cdot x^{p-1}\, dx = \frac{\sqrt{p}}{m^p}$

$= \theta \cdot \frac{\sqrt{3}}{\theta^{3}} = \frac{\sqrt{3}}{\theta^2}$

$\sqrt{n} = (n-1)\sqrt{n-1}$
$\sqrt{3} = (3-1)\sqrt{3-1}$
$= 2\sqrt{2} = 2$

$E(x^2) \Rightarrow \frac{2}{\theta^2}$

$V(x) = E(x^2) - (E(x))^2$

$= \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{2-1}{\theta^2} = \frac{1}{\theta^2}$

1) Mean :-

$$E(x) = \int_0^1 x \cdot f(x)\, dx = \int_0^\infty x \cdot \frac{1}{\beta(m,n)} x^{m-1} (1-x)^{n-1} dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^1 \cdot x^{m-1} (1-x)^{n-1} dx.$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^{(m+1)-1} (1-x)^{n-1} dx.$$

$$\cancel{\frac{\Gamma p}{m^p} = \Gamma(m+1)}$$

$$= \frac{\beta(m+1,n)}{\beta(m,n)} \quad (1)$$

$$\boxed{\beta(m+r,n) = \frac{\Gamma(m+r)\,\Gamma n}{\Gamma(m+r+n)}} \quad 2(a)$$

$$= \frac{\dfrac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+1+n)}}{\dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}}$$

$\Gamma(n+1) = n\Gamma n$

$\Gamma(m+1) = m\Gamma m$

$\Gamma(m+1+n) = (m+n)\Gamma(m+n)$

$$= \frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+1+n)} \cdot \frac{\Gamma(m+n)}{\Gamma m\,\Gamma n}$$

$$= \frac{m\Gamma m\,\Gamma n}{(m+n)\,\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma m\,\Gamma n} = \frac{m}{m+n}$$

$$\boxed{E(x) = \frac{m}{m+n}}.$$

Q) If X is a r.v with contin... distri... (?) F the f(x) has a uniform distri... on [0,1]?

A) Let f(x) be the pdf of x.

Let $y = F(x)$.

$$\therefore f(x) = \frac{dF(x)}{dx} = \frac{dy}{dx}$$

Let g(y) be the pdf of y.

$$g(y) = f(x) \cdot \left|\frac{dx}{dy}\right|$$

$$= f(x) \cdot \frac{1}{\frac{dy}{dx}} = \frac{F(x)}{F(x)} = 1$$

$0 < y < 1$.

$\therefore$ y → a uniform distri... in [0,1].

=> Beta Distribution :- [1st kind]

If a r.v 'x' has pdf given by

$$\boxed{f(x) = \frac{1}{\beta(m,n)} x^{m-1} (1-x)^n} \quad 0 < x < 1$$

$m > 0$ & $n > 0$ then x is said to hue beta distri...,

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \cdot dx} \to \beta(\,)$$

(1)

$\boxed{\beta(m,n) \text{ or } \beta_1(m,n)}$

$$= \frac{(m+1)m\,\Gamma(m+n)}{\Gamma(m+2+n)} = \frac{(m+1)m\,\Gamma(m+n)}{(m+n+1)\,\Gamma(m+n+1)}$$

$$E(x^2) = \frac{(m+1)m\,\cancel{\Gamma(m+n)}}{(m+n+1)(m+n)\,\cancel{\Gamma(m+n)}} = \frac{(m+1)m}{(m+n+1)(m+n)}$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{(m+1)m}{(m+n+1)(m+n)} - \left(\frac{m}{m+n}\right)^2$$

$$= \frac{m}{m+n}\left[\frac{(m+1)}{m+n+1} - \frac{m}{m+n}\right]$$

$$= \frac{m}{m+n}\left[\frac{(m+1)(m+n) - m(m+n+1)}{(m+n+1)(m+n)}\right]$$

$$= \frac{m}{m+n}\left[\frac{\cancel{m^2} + \cancel{mn} + \cancel{m} + n - \cancel{m^2} - \cancel{mn} - \cancel{m}}{(m+n+1)\,(m+n)}\right]$$

$$= \frac{m}{m+n}\left[\frac{n}{(m+n+1)(m+n)}\right]$$

$$V(x) = \frac{mn}{(m+n+1)(m+n)^2}$$

2) Varience :-

$$V(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_0^1 x^2 \cdot \frac{1}{\beta(m,n)} x^{m-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{\beta(m,n)} \int_0^1 x^2 \cdot x^{m-1}(1-x)^{n-1}\,dx.$$

$$= \frac{1}{\beta(m,n)} \underbrace{\int_0^1 x^{m+2-1}(1-x)^{n-1}\,dx}_{\beta(?)}.$$

(1)

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

$$\therefore \beta(m+2,n) = \int_0^1 x^{m+2-1}(1-x)^{n-1}\,dx.$$

$$\Rightarrow \frac{1}{\beta(m,n)} \cdot \beta(m+2,n)$$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \quad (b)$$

$$(a)\ \beta(m+2,n) = \frac{\Gamma(m+2)\,\Gamma n}{\Gamma(m+2+n)}$$

$$\Rightarrow \frac{\beta(m+2,n)}{\beta(m,n)} = \frac{\dfrac{\Gamma(m+2)\,\Gamma n}{\Gamma(m+2+n)}}{\dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}} \quad \begin{matrix}(a)\\ (b)\end{matrix}$$

$$= \frac{\Gamma(m+2)\,\Gamma n}{\Gamma(m+2+n)} \cdot \frac{\Gamma(m+n)}{\Gamma m\,\Gamma n}$$

$$\Gamma(m+2) = (m+1)\,\Gamma(m+1) = (m+1)\,m\,\Gamma m$$

$$= \frac{(m+1)\,m\,\cancel{\Gamma m}\,\cancel{\Gamma n}\,\Gamma(m+n)}{\Gamma(m+2+n)\,\cancel{\Gamma m}\,\cancel{\Gamma n}}$$

⇒ Beta Distribution of 2nd kind :-

Let 'x' be a continous r.v assuming values from 0 to ∞, if the pdf of x is gvn by;

$$\boxed{f(x) = \frac{1}{\beta(m,n)} \frac{x^{m-1}}{(1+x)^{m+n}}} \quad , m,n>0 ,$$
$$0<x<1$$

then x is said to follow a Beta distri... of 2nd kind & is denoted by $\beta_2(m,n)$,

Note →

* β distri... of 2nd kind can be transfomed to β distri... of 1st kind by the transformation,

$$\boxed{1+x = \frac{1}{y}} \quad (i.e)$$

if $x \to \beta_2(m,n)$ then y is defined above follow $\beta_1(m,n)$

* As above we can show that,

$$\boxed{\begin{aligned} \text{Mean} = E(x) &= \frac{m}{n-1} \\ \text{Varience} &= \frac{m(m+n-1)}{(n-1)^2(n-2)} \end{aligned}}$$

~~2) Rectangular distribution :~~

→ Moments of normal distri... =.

1) Mean :-

$$\boxed{E(x) = \mu}$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$(x = (x-\mu)+\mu)$

$$= \int_{-\infty}^{\infty} \underbrace{[(x-\mu)+\mu]}_{} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \mu \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu \underbrace{\int_{-\infty}^{\infty} f(x)\cdot dx}_{1}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} (x-\mu)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{①} + \mu .$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \times 0 + \mu .$$

$$\boxed{E(x) = \mu}$$

① $\int_{-\infty}^{\infty} (x-\mu)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \to$ odd (·).

$\Rightarrow 0$

$f(-x) = -f(x)$. by the (pro) $\cdot \int_{-\infty}^{\infty}$ odd (·) $= 0$

⇒ varience :- $\boxed{V(x) = \sigma^2}$

$$V(x) = E\,(x - E(x))^2$$

$$V(x) = E\,(x-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)\cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} , dx$$

$-\left(\frac{x-\mu}{\sigma}\right)^2 \frac{1}{2}$ ‖ $-\frac{1}{2}z^2$ — ©

by Rule method

$$z = \frac{x-\mu}{\sigma} \Rightarrow x-\mu = z\sigma$$

$\therefore \; x-\mu = z\sigma$ — ⓐ

diff. on both side,

$$d(x-\mu) = d(z\sigma)$$

$$dx - 0 = dz\,\sigma . \Rightarrow dx = dz\,\sigma \text{ — ⓑ}$$

$$\therefore = \int_{-\infty}^{\infty} (\sigma z)^2 \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\, \sigma\, dz$$

$(\sigma z)^2 = \sigma^2 z^2$

$$= \int_{-\infty}^{\infty} (\sigma z)^2 \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, \cancel{\sigma}\, dz .$$

$$= \cancel{\frac{\sigma z^2}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} dz}$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2\, e^{-z^2/2}\, \cancel{\sigma}\, dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\, e^{-z^2/2}\, dz$$

② both side are =.

$$\int_{-\infty}^{\infty} \Rightarrow 2\int_{0}^{\infty}$$

Rule method

let $u = \frac{z^2}{2}$. — ©

dz?. $\therefore\; z^2 = 2u \Rightarrow z = \sqrt{2u}$

$d(z^2) = d(2u)$

$\cancel{2}z\,dz = \cancel{2}\,du$

$z\,dz = du$

$dz = \frac{du}{z} \Rightarrow \frac{du}{\sqrt{2u}}$ — Ⓕ

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2\, e^{-z^2/2}\, dz$$

becomes $\Rightarrow \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} (2u)\, e^{-u}\, \frac{du}{\sqrt{2u}}$

$$= \frac{2\sigma^2}{\sqrt{2}\cdot\sqrt{\pi}} \int_{0}^{\infty} 2u\, e^{-u}\, \frac{du}{\sqrt{2}\cdot\sqrt{u}}$$

$\sqrt{2} \times \sqrt{2} = 2$

$$= \frac{2\sigma^2 \times \cancel{}}{\cancel{2}\sqrt{\pi}} \int_0^\infty u\, e^{-u} \frac{du}{u^{1/2}}$$

$\frac{u}{u^{1/2}} \Rightarrow u^{1-1/2} \Rightarrow u^{1/2}$

$\frac{x^a}{x^b} = x^{a-b}$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty u^{1/2} e^{-u}\, du$$

$u^{1/2} \Rightarrow u^{\frac{3}{2}-1}$

$\Rightarrow u^{\frac{3-2}{2}}$

$\Rightarrow u^{1/2}$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty u^{\frac{3}{2}-1} e^{-u}\, du.$$

$$\Rightarrow \frac{2\sigma^2}{\sqrt{\pi}} \underbrace{\int_0^\infty e^{-u} u^{\frac{3}{2}-1}\, du}$$

$\int_0^\infty e^{-mp} \cdot x^{p-1} = \frac{\Gamma p}{m^p}$, $m = 1$, $p = 3/2$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{\Gamma 3/2}{1^{3/2}}$$

$\Gamma n = (n-1)$

$\Gamma 3/2 = \frac{3}{2} - 1 = \frac{1}{2}$

$\Gamma n+1 = n \Gamma n.$

$= \left(\frac{3}{2}-1\right) \Gamma \frac{3}{2}-1$

$= \frac{1}{2} \Gamma 1/2$

$$= \frac{\cancel{2}\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{\cancel{2}} \Gamma 1/2$$

$$= \frac{\sigma^2}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}}$$

$\boxed{\Gamma 1/2 = \sqrt{\pi}}$

$\boxed{V(x) = \sigma^2}$

$SD(x) = \sqrt{V(x)}$

$= \sqrt{\sigma^2}$

$\boxed{SD(x) = \sigma}$

$\Rightarrow$ odd order moments about mean μ

$\boxed{\mu_{2r+1} = 0}$ for $r = 1, 2, 3 \ldots$

$2r \to$ even
$2r+1 \to$ odd
$2r-1 \to$ odd.

$$\mu_{2r+1} = E(x-\mu)^{2r+1}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r+1} \cdot f(x)\, dx.$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r+1} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx.$$

Euler method

Put $z = \frac{x-\mu}{\sigma}$.

$\sigma z = x - \mu$

$d(\sigma z) = d(x-\mu)$

$\sigma dz = dx$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2r+1} e^{-1/2 z^2} \sigma dz$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^{2r+1} \cdot z^{2r+1} \cdot e^{-1/2 z^2} \cancel{\sigma} dz.$$

$$= \frac{\sigma^{2r+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2r+1} \cdot e^{-1/2 z^2} dz$$

$z^{2r+1} \to$ odd ()
$(z^{r} \to$ even ()$)$

$$= \frac{\sigma^{2r+1}}{\sqrt{2\pi}} \times 0$$

$[f(-x) = -f(x)$ by the prop $\int_{-\infty}^{\infty}$ odd $() = 0]$

$\boxed{\mu_{r+1} = 0}$

→ Even order central moment :-

when $x \to N(\mu, \sigma)$

$$\boxed{\mu_{2r} = 1 \cdot 3 \cdot 5 \cdots (2r-1) \cdot \sigma^{2r}.}$$

$r=1$

$\mu_2 = 2r-1 = \sigma^2$

$r=2$

$\mu_4 = 2r-1 = 2\times2-1 = 3$

$\Rightarrow 1 \cdot 3\,\sigma^2 = 3\sigma^2$

$$\mu_{2r} = E(x-\mu)^{2r}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2r} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2r} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2r} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Put, $z = \frac{x-\mu}{\sigma}$

$x-\mu = \sigma z$

$dx = \sigma dz$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2r} \cdot e^{-\frac{1}{2}z^2} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^{2r} \cdot z^{2r} \cdot e^{-z^2/2} dz$$

$$= \frac{\sigma^{2r}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2r} \cdot e^{-z^2/2} dz.$$

fun ()

$$\int_{-\infty}^{\infty} \Rightarrow 2\int_0^{\infty}$$

$$= \frac{2\sigma^{2r}}{\sqrt{2\pi}} \int_0^{\infty} z^{2r} \cdot e^{-z^2/2} dz.$$

Put $u = \frac{z^2}{2}$

$z^2 = 2u \Rightarrow z = \sqrt{2u}$

$\cancel{2}z\,dz = \cancel{2}du$

$z\,dz = du$

$dz = \frac{du}{z}$

$z = \sqrt{2u}. \quad = \frac{du}{\sqrt{2u}}.$

$$= \frac{2\sigma^{2r}}{\sqrt{2\pi}} \int_0^{\infty} (2u)^r \cdot e^{-u} \cdot \frac{du}{\sqrt{2u}}$$

$$= \frac{2\sigma^{2r}}{\sqrt{2}\cdot\sqrt{\pi}} \int_0^{\infty} 2^r \cdot u^r \cdot e^{-u} \cdot \frac{du}{\sqrt{2}\cdot\sqrt{u}}.$$

$$= \frac{\cancel{2}\sigma^{2r}}{\cancel{\sqrt{2}\sqrt{2}}\sqrt{\pi}} 2^r \int_0^{\infty} u^r \cdot e^{-u} \cdot \frac{du}{u^{1/2}}$$

$\frac{u^r}{u^{1/2}} = u^{r-1/2}$

$$= \frac{\sigma^{2r} 2^r}{\sqrt{\pi}} \int_0^{\infty} u^{r-1/2} \cdot e^{-u} du$$

x^{p-1} → gamma form-er ongsho

$u^{r-1/2} \Rightarrow u^{r+\frac{1}{2}-1}$

$$= \frac{\sigma^{2r} 2^r}{\sqrt{\pi}} \int_0^{\infty} u^{(r+\frac{1}{2})-1} \cdot e^{-u} du.$$

$$\left(\int_0^{\infty} e^{-mx} \cdot x^{p-1} = \frac{\Gamma p}{m^p}\right)$$

$p = r+1/2$

$m = 1$

$$= \frac{\sigma^{2r} 2^r}{\sqrt{\pi}} \cdot \frac{\Gamma(r+1/2)}{1^{r+1/2}}$$

$\Gamma n = (n-1)\Gamma(n-1)$

$\Gamma(r+\frac{1}{2}) = (r+\frac{1}{2}-1)\Gamma(r+\frac{1}{2}-1)$

$= (r-1/2)\Gamma(r-\frac{1}{2})$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \cdot (r-\tfrac{1}{2})(r-\tfrac{1}{2}-1)\Gamma(r-\tfrac{1}{2}-1)$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \cdot (r-\tfrac{1}{2})(r-\tfrac{3}{2})\Gamma(r-3/2)$$

$$= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} (r-\tfrac{1}{2})(r-3/2) \cdots \underbrace{\tfrac{1}{2}\Gamma(1/2)}_{\text{will end}}$$

$$= \frac{2^r \sigma^{2r}}{\cancel{\sqrt{\pi}}} \left(\frac{2r-1}{2}\right)\left(\frac{2r-3}{2}\right) \cdots \frac{1}{2}\cancel{\sqrt{\pi}}$$

$$= \frac{\cancel{2^r}\,\sigma^{2r}(2r-1)(2r-3)\cdots 1}{\cancel{2^r}}$$

$$= 1\cdot 3\cdot 5 \cdots (2r-3)(2r-1)\sigma^{2r}$$

⇒ Recurrance relation for even order central moments :-

(even) $\mu_{2r} = 1\cdot 3\cdot 5 \cdots (2r-1)\sigma^{2r}$.

μ_{2r} (even) μ_{2r+1} (odd) μ_{2r+2} (even)

(even) $\mu_{2r+2} = 1\cdot 3\cdot 5 \cdots (2r-1)(2r+1)\sigma^{2r+2}$

$$\frac{\mu_{2r+2}}{\mu_{2r}} = \frac{1\cdot 3\cdot 5\cdots(2r-1)(2r+1)\sigma^{2r+2}}{1\cdot 3\cdot 5\cdots(2r-1)\sigma^{2r}}$$

$$= \frac{2r+1\cdot \cancel{\sigma^{2r}}\cdot\sigma^2}{\cancel{\sigma^{2r}}} \qquad [\sigma^{2r+2} = \sigma^{2r}\cdot\sigma^2]$$

$$\frac{\mu_{2r+2}}{\mu_{2r}} = (2r+1)\cdot\sigma^2 \longrightarrow$$

$$\boxed{\mu_{2r+2} = (2r+1)\,\sigma^2\,\mu_{2r}}$$

⇒ Moment generating ( ) :- (normal mgf)/ (row mgf)

$$\boxed{M_x(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}}$$

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}\cdot\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}\cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$z = \frac{x-\mu}{\sigma}$

$z\sigma = x-\mu$

$x = z\sigma + \mu$

$dx = \sigma\, dz$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t(\sigma z+\mu)}\cdot e^{-\frac{1}{2}z^2}\,\cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t\sigma z + t\mu}\cdot e^{-\frac{1}{2}z^2}\, dz \qquad \boxed{a^{m+n} = a^m\cdot a^n}$$

(split)

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t\sigma z}\cdot e^{t\mu}\, e^{-z^2/2}\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t\sigma z}\cdot e^{-z^2/2}\, dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t\sigma z - z^2/2}\, dz$$

$$\sigma t z - \frac{z^2}{2} - \frac{1}{2}t^2\sigma^2 + \frac{1}{2}t^2\sigma^2$$

$$= \frac{-1}{2}\left[-2\sigma t z + z^2 + t^2\sigma^2\right] + \frac{1}{2}t^2\sigma^2$$

$$= -\frac{1}{2}\left[-2\sigma t z + z^2 + (t\sigma)^2\right] + \frac{1}{2}t^2\sigma^2$$

$$= \frac{-1}{2}(z-t\sigma)^2 + \frac{1}{2}t^2\sigma^2$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2 + \frac{1}{2}t^2\sigma^2}\, dz.$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\, e^{\frac{1}{2}t^2\sigma^2}\, dz$$

$$= \frac{e^{t\mu}\, e^{\frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\, dz$$

$$= \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2}\, dz.$$

$$= \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, du.$$

$u = z - t\sigma$

$du = dz$

$\int_{-\infty}^{\infty} \to$ ②

$$= 2\, \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-u^2/2}\, du.$$

Let $v = \frac{u^2}{2}$

$2v = u^2$

$u = \sqrt{2v}$

$\cancel{2}\, dv = \cancel{2}\, u\, du$

$dv = u\, du$

$du = \frac{dv}{u}$

$du = \frac{dv}{\sqrt{2v}}$

$$= 2\, \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2}}{\sqrt{2\pi}} \int_0^{\infty} e^{-v} \frac{dv}{\sqrt{2v}}$$

$$= \cancel{2}\, \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2}}{\sqrt{\cancel{2}}\sqrt{\pi}} \int_0^{\infty} e^{-v} \frac{dv}{\sqrt{\cancel{2}}\sqrt{v}}$$

$$= \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \int_0^{\infty} e^{-v} \frac{dv}{v^{1/2}}$$

$$= \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \int_0^{\infty} v^{-1/2} e^{-v}\, dv.$$

$-\frac{1}{2} = \frac{1}{2} - 1$

$$= \frac{e^{t\mu + 1/2\, t^2\sigma^2}}{\sqrt{\pi}} \int_0^{\infty} v^{\frac{1}{2}-1} e^{-v}\, dv.$$

$$\left[\int e^{-mx} x^{p-1}\, dx = \frac{\Gamma p}{m^p}\right]$$

$p = \frac{1}{2}$, $m = 1$

$$= \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \cdot \frac{\Gamma 1/2}{1}$$

$$\boxed{\Gamma 1/2 = \sqrt{\pi}}$$

$$= \frac{e^{t\mu + \frac{1}{2}t^2\sigma^2} \cdot \cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}}$$

$$M_x(t) = e^{\mu t + 1/2\, t^2\sigma^2}$$

$\Rightarrow$ central mgf :-

$$M_x(t) = E\left(e^{t(x-\mu)}\right)$$
$$= E\left(e^{tx - t\mu}\right)$$
$$= E\left(e^{tx} \cdot e^{-t\mu}\right)$$
$$= e^{-t\mu} \cdot E\left(e^{tx}\right)$$
$$= e^{-t\mu} \cdot e^{\mu t + 1/2\, t^2\sigma^2}$$
$$= e^{-t\mu + \mu t + 1/2\, t^2\sigma^2}$$
$$\boxed{M_x(t) = e^{\frac{1}{2}t^2\sigma^2}}$$

$\Rightarrow$ Mean deviation of normal distri.....:-

$E(x) = \mu$

$V(x) = \sigma^2$

$SD = \sigma$

$$\boxed{MD = E\,|x - E(x)|}$$
$$= E\,|x - \mu|$$
$$= \int_{-\infty}^{\infty} |x - \mu| f(x)\, dx.$$
$$= \int_{-\infty}^{\infty} |x - \mu| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

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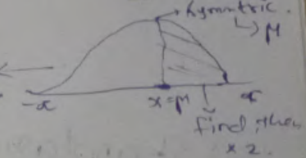

Curve is symmetric $\leftarrow$
$$= 2\int_{\mu}^{\infty} (x-\mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$
$$= \frac{2}{\sigma\sqrt{2\pi}} \int_{\mu}^{\infty} (x-\mu)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx.$$

Put $z = \frac{x-\mu}{\sigma}$

$\sigma z = x - \mu$

$\sigma\, dz = dx.$

= (limit transform) $\int \to$ when $x = \mu$ in z

$\Rightarrow z = 0$

$x = \infty$, $z = \infty$.

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^{\infty} \sigma z\, e^{-\frac{1}{2}z^2} \sigma\, dz$$

$$= \frac{2\sigma}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} z\, e^{-z^2/2}\, dz.$$ put $u = \frac{z^2}{2}$

$2u = z^2$

$\sqrt{2u} = z$

$d(2u) = d(z^2)$

$2\, du = 2z\, dz$

① $du = z\, dz$

$dz = \frac{du}{z}$

when $z=0, u=0$

when $z=\infty$, $u=\infty$

$$= \frac{2\sigma}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} e^{-u}\, du \quad ①$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \left[\frac{e^{-u}}{-1}\right]_0^{\infty}$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \left[\frac{e^{-\infty} - e^{0}}{-1}\right] = \frac{2\sigma}{\sqrt{2\pi}} \left[\frac{e^{-\infty}}{-1} - \frac{e^{0}}{-1}\right]$$

$$= \frac{2\sigma}{\sqrt{2\pi}} [0 + 1] = \frac{2\sigma}{\sqrt{2\pi}} \cdot 1 \qquad 2 = \sqrt{2}\sqrt{2}.$$

$$= \frac{\sqrt{2}\cdot\sqrt{2}\,\sigma}{\sqrt{2}\sqrt{\pi}} = \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}}$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma \doteq 0.79788\sigma \approx \frac{4}{5}\sigma$$

⇒ Standard Normal distri..... :-

(For doing probm, we have to do in S.Nd)

$x \to N(\mu, \sigma)$

(x-ന്നെ z-ലേക്ക് മാറ്റുക)

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$z \to N(0, 1)$

$f(z) = f(x) \left|\frac{dx}{dz}\right|$

$z = \frac{x-\mu}{\sigma}$

$\sigma z = x - \mu.$

$\sigma\, dz = dx.$

$\sigma = \frac{dx}{dz}$

$\therefore f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \cdot \sigma$

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}}$$