INTEGRATION.

⇒ Sigma Notation for finite sums :-

The finite sum $a_1 + a_2 + \dots a_n$ is denoted by $\sum_{k=1}^{n} a_k$. where a_i's are the terms of the sum.

$a_1 \to 1^{st}$ term

$a_2 \to 2^{nd}$ term

$a_k \to k^{th}$ term

$a_n \to n^{th}$ term i.e last term.

eg = 1) $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$= 1 + 4 + 9 + 16 + 25$

$= \underline{\underline{55}}$

2) $\sum_{k=2}^{4} \frac{k}{k+1} = \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1}$

$= \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

$= \frac{8+9}{12} + \frac{4}{5}$

$= \frac{17}{12} + \frac{4}{5}$

$17 \times 5 = 85$

$= \frac{85 + 48}{60} = \cancel{132} = \frac{133}{60}$

(odd) $(-1)^3 = -1$

$(-1)^4 = 1$ (even)

3) $\sum_{n=1}^{3} (-1)^n \cdot n(n+1) = (-1)^1 \cdot 1(1+1) + (-1)^2 \cdot 2(2+1) + (-1)^3 \cdot 3(3+1)$

$= -1 \cdot 1(2) + (1) \cdot 2(3) + (-1) \cdot 3(4)$

$= -1 \cdot 2 + 1 \cdot 6 + (-1) \cdot 12$

$= -2 + 6 - 12$

$= 6 - 14 = -8$ $\quad -2 - 12 = -14$

11/11

1) Evaluate the following sum :-

① $\sum_{i=1}^{4} i^2+1$ ② $\sum_{k=1}^{2} \frac{6k}{k+1}$

① $\sum_{i=1}^{4} i^2+1$

$= (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)$

$= 2+5+10+17 = \underline{\underline{34}}$

② $\sum_{k=1}^{2} \frac{6k}{k+1}$

$= \frac{6\times1}{1+1} + \frac{6\times2}{2+1} = \frac{6}{2} + \frac{12}{3} = 3+4 = \underline{\underline{7}}$

⇒ Properties of summation :-

1) $\sum_{k=1}^{n} (a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$. (Sum rule)

2) $\sum_{k=1}^{n} (a_k-b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$. (Difference rule)

3) $\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k$ (Constant multiple rule)

4) $\sum_{k=1}^{n} c = c\times n$ (constant value rule)

5) If $m \le n$ and $n \le p$ then,

$\sum_{i=m}^{p} a_i = \sum_{i=m}^{n} a_i + \sum_{i=n}^{p} a_i$ $\quad m-p$ [illegible] $n-p$ minus [illegible]

6) If $a_i = c$, for all i with $m \le i \le n$ then,

$\sum_{i=m}^{n} a_i = c(n-m+1)$

7) If $a_i \le b_i$ for all i with $m \le i \le n$,

$a_i \le b_i = \sum_{i=m}^{p} a_i \le \sum_{i=m}^{p} b_i$

* Problems :-

1) $\sum_{j=3}^{102} (j-2) = \sum_{j=3}^{102} j - \sum_{j=3}^{102} 2$

③ rule → $= \left[\sum_{j=1}^{102} j - \sum_{j=1}^{2} j\right] - 2(102-3+1)$ According to ⑥ rule

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$= \frac{102(102+1)}{2} - \frac{2(2+1)}{2} - 2(102-3+1)$

$= \frac{10506}{2} - 3 - 2(100)$

$= 5253 - 3 - 200 = \underline{\underline{5050}}$

$\boxed{\text{Remark}}$

$1+2+3\cdots\cdots+n = \sum_{k=1}^{n} k = n\frac{(n+1)}{2}$

$1^2+2^2+3^2\cdots+n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$1^3+2^3+3^3\cdots+n^3 = \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

1) $\sum_{k=1}^{7} k(2k+1)$ $\cancel{= 1(2+1)+2(4+1)+3}$ $\sum_{k=1}^{7}(2k^2+k)$

$= \sum_{k=1}^{7} 2k^2 + \sum_{k=1}^{7} k$

$= 2\sum_{k=1}^{7} k^2 + \sum_{k=1}^{7} k$

$= 2\left[\frac{7(8)(2\times7+1)}{6}\right] + \frac{7(8)}{2}$

$= \frac{56 \cdot 15}{6} + \frac{56}{2}$

$= 140 + 28$

$= \cancel{168}\ 308$

2) $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

$= \frac{1}{225} \sum_{k=1}^{5} k^3 + \left(\sum_{k=1}^{5} k\right)^3$

$= \frac{1}{225} \left(\frac{5(5+1)}{2}\right)^2 + \left(\frac{5(5+1)}{2}\right)^3$

$= \frac{1}{225} \cdot \left(\frac{30}{2}\right)^2 + \left(\frac{30}{2}\right)^3$

$= \frac{1}{225} \cdot 225 + 15^3$

$= \frac{1}{225} \cdot \cancel{240}\ 225 + 3375$

$= \frac{3600}{225} = 3375 + 1 = 3376$

3) $\sum_{i=1}^{4} (i^2+1) = \sum_{i=1}^{4} i^2 + \sum_{i=1}^{4} 1$

$= \frac{4(5)\cdot 8+1}{6} + \frac{4(4+1)}{2} 1(4-1+1)$

$= \frac{20 \cdot 9}{6} + \frac{20}{2} 1(4)$

$= \frac{180}{6} + \cancel{10}\ 4$

$= 30 + 4\ \cancel{40}$

$= 34$

4) $\sum_{i=1}^{5} i(i-1) = \sum_{i=1}^{5} i^2 - \sum_{i=1}^{5} i$

$= \frac{5(6)\cdot 10+1}{6} - \frac{5(6)}{2}$

$= \frac{30 \cdot 11}{6} - \frac{30}{2}$

$= 55 - 15 = 40$

5) $\sum_{i=1}^{45} i$ (6) $\sum_{j=4}^{80} j-3$ (7) $\sum_{i=1}^{99} ((i+1)^2 - i^2)$

8) $\sum_{k=1}^{10} k^3$ (9) $\sum_{k=2}^{6} 2^k$ (10) $\sum_{i=-2}^{2} i^3$

5.A) $\sum_{i=1}^{45} i = \frac{45(45+1)}{2} = \frac{45(46)}{2} = \frac{2070}{2} = 1035$

6.A) $\sum_{j=4}^{80} j-3 = \sum_{j=4}^{80} j - \sum_{j=4}^{80} 3$

$= \frac{80(81)}{2} - 3(80-4+1)$

$= \frac{6480}{2} - 3(77)$

$= 3240 - 237 = 3003$

8.A) $\sum_{k=1}^{10} k^3 = \left(\frac{10(10+1)}{2}\right)^2 = \left(\frac{10(11)}{2}\right)^2 = \left(\frac{110}{2}\right)^2 = (55)^2$

$= 3025$

9.A) $\sum_{k=2}^{6} 2^k = 2^2 + 2^3 + 2^4 + 2^5 + 2^6$

$= 4 + 8 + 16 + 32 + 64 = 124$

10.A) $\sum_{i=-2}^{2} i^3 = (-2)^3 + (-1)^3 + (0)^3 + (1)^3 + (2)^3$

$= -8 - 1 + 0 + 1 + 2 = 0$

7. i) $\sum_{i=1}^{99} ((i+1)^2 - i^2)$

$(a+b)^2 = a^2 + 2ab + b^2$
$= i^2 + 2 \times i \times 1 + 1^2$
$= i^2 + 2i + 1$

$\sum_{i=1}^{99} (i^2+2i+1) - i^2$.

$\sum_{i=1}^{99} i^2 + \sum_{i=1}^{99} 2i + \sum_{i=1}^{99} 1 - \sum_{i=1}^{99} i^2$. (c×n)

$= \frac{99(100)\,199}{6} + 2\sum_{i=1}^{99} i + 99 \times 1 - \frac{9900 \cdot 199}{6}$

$= \frac{9900 \cdot 199}{6} + 2 \times \frac{99(100)}{2} + 99 - \frac{1970100}{6}$

$= \frac{1970100}{6} + 2 \times 4950 + 99 - 328350$

$= 328350 + 9900 + 99 - 328350.$

$= 338349 - 328350$

$= 9999$

ii) $\sum_{i=1}^{4} (i^2+1) = \sum_{i=1}^{4} i^2 + \sum_{i=1}^{4} 1$

$= \frac{4(5) \cdot 9}{6} + 4 \times 1$

$= \frac{180}{6} + 4$

$= 30 + 4 = 34$

* Express the following sum in sigma notation.

a) $\frac{1}{1} + \frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25}$ [numerator → odd is $2k+1$ or $2k-1$, Even = $2k$.]

$\sum_{k=0}^{4} \frac{2k+1}{(k+1)^2}$ or $\frac{2k-1}{k^2}$

$\frac{1}{1} = \frac{2k+1}{(k+1)^2} = \frac{2 \times 0 + 1}{(0+1)^2} = \frac{1}{1}$ (k should be continuous 1, 2, 3, 4 ...)

$\frac{3}{4} = \frac{2 \times 1 + 1}{(1+1)^2} = \frac{3}{4}$

$\frac{5}{9} = \frac{2 \times 2 + 1}{(2+1)^2} = \frac{5}{9}$

$\frac{7}{16} = \frac{2 \times 3 + 1}{(3+1)^2} = \frac{7}{16}$

$\frac{9}{25} = \frac{2 \times 4 + 1}{(4+1)^2} = \frac{9}{25}$

(or) we can use $\frac{2k-1}{k^2}$ eq also by this form.

b) $1+2+3+4+5+6 \rightarrow \sum_{k=1}^{6} k$.

c) $2+4+6+8+10 \rightarrow$ (even → 2k)

$\sum_{k=1}^{5} 2k$

$1 \times 2 = 2$
$2 \times 2 = 4$
$2 \times 3 = 6$
$2 \times 4 = 8$
$2 \times 5 = 10$

d) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

$\sum_{k=1}^{4} \frac{1}{2^k}$

$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$

e) $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

$\sum_{k=1}^{5} (-1)^{k+1} \frac{1}{k}$

$(-1)^{odd} = -1$
$(-1)^{even} = 1$

alter → (-1)

$(-1)^{1+1} \frac{1}{k}$

$(-1)^2 \frac{1}{1}$ = even $1 \cdot \frac{1}{1} \rightarrow (1)$.

$(-1)^{2+1} \frac{1}{k} = (-1)^3 \frac{1}{2}$ $-1 \cdot \frac{1}{2} \rightarrow (-\frac{1}{2})$

→ Step Function :-

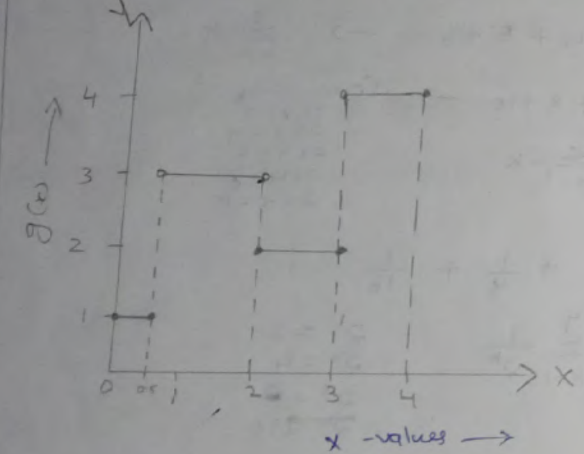
A ( ) 'g' on [a,b] is a step ( ) when [a,b] can be broken upto intervals of width Δx_i, on each of which g = a constant 'k_i'

If each k_i is +ve / 0 then the area under the graph of 'g' is $\boxed{\sum_{i=1}^{n} k_i \Delta x_i}$

1) Draw a graph of step () defined on $[1,4]$ by $g(x) = \begin{cases} 1 & \text{if } 0 \le x \le 0.5 \\ 3 & \text{if } 0.5 < x < 2 \\ 2 & \text{if } 2 \le x \le 3 \\ 4 & \text{if } 3 < x \le 4 \end{cases}$ Δx_i: 0.5, 1.5, 1, 1

Also find the area of region under graph.

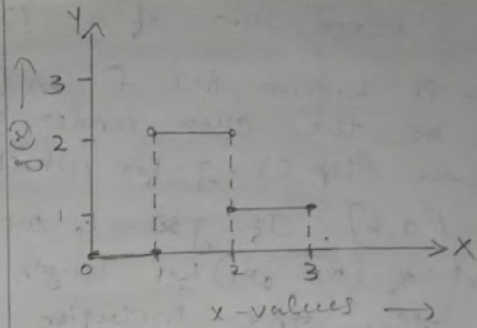
A)



$$\text{Area} = \sum_{i=1}^{4} k_i \Delta x_i$$
$$= (1 \times 0.5) + (3 \times 1.5) + (2 \times 1) + (4 \times 1)$$
$$= 0.5 + 4.5 + 2 + 4 = 5 + 6 = \underline{\underline{11}}$$

2) $g(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 2, & 1 < x < 2 \\ 1, & 2 \le x \le 3 \end{cases}$ (k_i; Δx_i: 1, 1, 1)

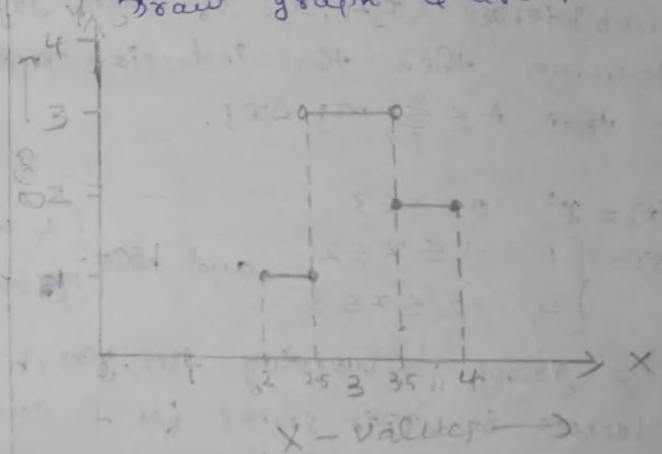
Draw graph & find area.



$$\text{Area} = \sum_{i=0}^{3} k_i \Delta x_i$$
$$= (0 \times 1) + (2 \times 1) + (1 \times 1) = 0 + 2 + 1 = \underline{\underline{3}}$$

3) $g(x) = \begin{cases} 1, & 2 \le x \le 2.5 \\ 3, & 2.5 \le x \le 3.5 \\ 2, & 3.5 \le x \le 4 \end{cases}$ 0.5, 1, 0.5

Draw graph & area.



$$\text{Area} = \sum_{i=2}^{4} k_i \Delta x_i$$
$$= (1 \times 0.5) + (3 \times 1) + (2 \times 0.5)$$
$$= 0.5 + 3 + 1 = \underline{4.5}$$

→ Lower sum & upper sum of a ( ) :-

* Lower Sum - A L.sum for 'f' on $[a,b]$ is defined to be the area under graph of a non-ve step ( ) . g for which $g(x) \le f(x)$ on $[a,b]$. If $g(x) = k_i$ on the ith sub interval of (x_{i-1}, x_i) for length Δx_i for $i = 1,2,\dots n$ then the inclusion property of areas says that $\sum_{i=1}^{n} k_i \Delta x_i \le A$

* upper sum for f on $[a,b]$ is defined to be the area under the graph of a non-negative step ( ) h for which $f(x) \le h(x)$ on $[a,b]$. If $h(x) = m_j$ on the jth sub interval (x_{j-1}, x_j) of length Δx_j, for $j = 1,2\dots n$ then the inclusion property says that $A \le \sum_{i=1}^{n} m_j \Delta x_j$.

1) Let $f(x) = x^2 \quad 0 \le x \le 3$

$$g(x) = \begin{cases} 1 & 1 \le x \le 2 \\ 4 & 2 \le x \le 3 \end{cases} \quad \text{and } h(x) = \begin{cases} 4 & 1 \le x < 2 \\ 9 & 2 \le x \le 3 \end{cases}$$

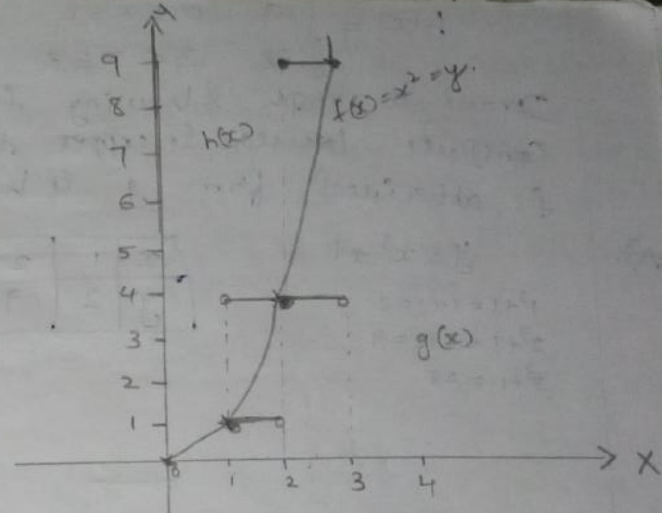
Draw a graph showing $f(x), g(x), h(x)$ wht lower & upper sums for f can be obtained from g & h.

A) $f(x) = x^2 \quad 0 \le x \le 3$

$0 \to x^2 = 0^2 = 0$
$1^2 = 1$
$2^2 = 4$
$3^2 = 9$

$y = x^2$

x	0	1	2	3
y	0	1	4	9



lower sum $= g(x) = k_i \Delta x_i$
$= (1 \times 1) + (4 \times 1) = \underline{\underline{5}}$

upper sum $= h(x) = k_i \Delta x_i$
$= (4 \times 1) + (9 \times 1) = 4 + 9 = \underline{\underline{13}}$

— $g(x) = f(x) = x^2$ for all $1 \le x \le 3$ & hence the graph step ( ) g lies below that of f.
∴ area under 'g' is lower sum & is given by 5.

— $f(x) \le h(x)$ for all $1 \le x \le 3$ & hence the graph of step ( ) h lies above that of f.
∴ area under 'h' is upper sum & is given by 13.

2) Let $f(x) = x^3 + 1$ for $1 \le x \le 3$

$$g(x) = \begin{cases} 2 & 1 \le x \le 1.5 \\ 4 & 1.5 \le x < 2 \\ 9 & 2 \le x \le 3 \end{cases}$$

0.5
0.5
1

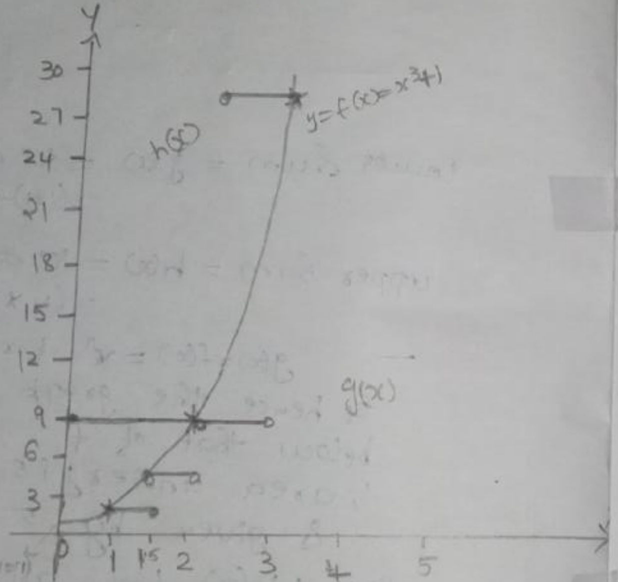
$$h(x) = \begin{cases} 9 & 0 \le x \le 2 \\ 28 & 2 < x \le 3 \end{cases}$$

Draw a graph showing $f(x), g(x), h(x)$. Compute lower & upper sum for f obtained from g & h.

A) $y = x^3 + 1$

x	1	2	3
y	2	9	28

$1^3+1 = 1+1 = 2$
$2^3+1 = 8+1 = 9$
$3^3+1 = 28$



Since $g(x) \le f(x)$

lower sum $= g(x) = \sum k_i \Delta x_i$
$= (2 \times 0.5) + (4 \times 0.5) + (9 \times 1)$
$= 1 + 2 + 9 = \underline{\underline{12}}$

Since $h(x) \ge f(x)$

upper sum $= h(x) = \sum k_i \Delta x_i$
$= (9 \times 2) + (28 \times 1)$
$= 18 + 28 = \underline{\underline{46}}$

⇒ **Fundamental theorem of calculus :-**

Suppose that the (') 'F' is differentiable everywhere on $[a, b]$ & that F' is intergrable on $[a, b]$ then $\boxed{\int_a^b F'(x) \cdot dx = F(b) - F(a)}$

In other words, if 'f' is integrable on $[a, b]$ & has antiderivative F. then

$\int_a^b f(x) \cdot dx = F(b) - F(a)$.

$\int x^n dx = \frac{x^{n+1}}{n+1}$
$\Rightarrow f(x) = x$
$F = \int f = \int x = \frac{x^2}{2}$

1) using (F) theorm of calculus combute $\int_0^1 x \cdot dx$

A) $\int_0^1 x \cdot dx$.

$f(x) = x$.

$F(x) = \int f(x) \cdot dx = \cancel{\int \frac{x^{1+1}}{1+1} = \int \frac{x^2}{2}}$

$= \int x \cdot dx = \frac{x^{1+1}}{1+1} = \underline{\underline{\frac{x^2}{2}}}$

$b = 1$
$a = 0$

$\int_a^b F'(x) \cdot dx = F(b) - F(a)$.

$\int x \cdot dx \underline{\underline{\frac{x^2}{2}}} = F(1) - F(0)$

$= \frac{1^2}{2} - \frac{0^2}{2}$

$= \frac{1}{2} - 0 = \underline{\underline{\frac{1}{2}}}$

2) using F.T of calculus combute $\int_a^b x^2 \cdot dx$.

$f(x) = x^2$ $\left(\frac{x^{n+1}}{n+1}\right)$

$F(x) = \int f(x) \cdot dx$

$= \int x^2 \cdot dx = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}$ $\left(\frac{x^3}{3} = \frac{b^3}{3} - \frac{a^3}{3}\right)$

$\therefore$ by F.T of calculus

$\int_a^b x^2 \cdot dx = F(b) - F(a)$

$= \frac{b^3}{3} - \frac{a^3}{3} = \frac{b^3 - a^3}{3}$

3) $\int_4^6 3x \cdot dx$.

$f(x) = 3x$

$F(x) = \int fx \cdot dx \cdot = \int_4^6 3x \cdot dx$

$= 3\int_4^6 x \cdot dx$.

$= 3 \int \frac{x^2}{2}$

$F(x) = \frac{3x^2}{2}$

$\therefore$ by F.T of calculus,

$\int_4^6 3x \cdot dx = F(b) - F(a)$ $\cancel{3 \times 6^2}$

$= F(6) - F(4)$

$= \frac{3 \times 6^2}{2} - \frac{3 \times 4^2}{2}$

$= \frac{3 \times 36}{2} - \frac{3 \times 16}{2}$

$= \frac{108}{2} - \frac{48}{2}$

$= 54 - 24 = 30$

4) $\int_1^8 (1 + \sqrt{x})\, dx$. $\sqrt{x} = x^{1/2}$

$f(x) = (1 + \sqrt{x})$

$F(x) = \int f(x) \cdot dx$ $\int 1 \cdot dx = x$

$= \int (1 + \sqrt{x})\, dx$. $\int 2 \cdot dx = 2x$

$= \int (1 + x^{1/2})\, dx$. $\int x^{1/2} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$

$= x + \frac{x^{3/2}}{3/2}$ $= \frac{x^{3/2}}{3/2}$

$= x + \frac{2}{3} x^{3/2}$

$\therefore$ by F.T of calculus,

$\int_1^8 (1 + \sqrt{x})\, dx = F(b) - F(a)$

$= F(8) - F(1)$

$\boxed{8^{3/2}} = 8^{2 \cdot 3/2}$ $= \left(8 + \frac{2}{3} 8^{3/2}\right) - \left(1 + \frac{2}{3} 1^{3/2}\right)$

$= 2\sqrt{2}$ $\sqrt{2} \times \sqrt{2} \times \sqrt{2} = 2\sqrt{2}$

$= (2\sqrt{2})^3$ $= 8 + \frac{2}{3} 16\sqrt{2} - 1 + \frac{2}{3}$

$= 8 \times 2\sqrt{2}$

$= 16\sqrt{2}$ $= 7 + \frac{2}{3}(16\sqrt{2} - 1)$

$\sqrt{8}$ $8 = 2 \times 2 \times 2$ $= 2\sqrt{2}$

5) $\int_1^3 x^3\, dx$.

6) $\int_2^3 x^2 \cdot dx$

$F(x) = \int x^2 \cdot dx = \frac{x^3}{3}$.

by F.T of calculus.

$\int_2^3 x^2 \cdot dx = F(3) - F(2)$

$= \frac{3^3}{3} - \frac{2^3}{2} = \frac{27}{3} - \frac{8}{3}$

$= \frac{27 - 8}{3} = \frac{19}{3}$ $\cancel{= 9 - 2.6} = 6.4$

5. A) $F(x) = \int x^3 \cdot dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4}$

by F.T of calculus,

$\int_1^3 x^3 \cdot dx = F(b) - F(a) = F(3) - F(1)$

$= \frac{3^4}{4} - \frac{1^4}{4}$ ~~$= 20.25 - 0.25 = 20$~~ $= \frac{26}{4}$ //

⇒ Fundamental Integration Method :-

To integrate the ( ) $f(x)$ on $[a, b]$ find an antiderivative $F(x)$ for $f(x)$ then evaluate 'F' at a and b. & substract the results.

$\int_a^b f(x) \cdot dx = F(b) - F(a)$.

→ Notation for the F. Theorem =

$\boxed{F(x) \Big|_a^b} \Rightarrow F(b) - F(a)$.

Q

1) find $(x^3+5)\Big|_2^3$

A) $F(x)\Big|_a^b \Rightarrow F(b) - F(a)$.

$(x^3+5)\Big|_2^3 \Rightarrow F(3) - F(2)$

$= 3^3+5 - 2^3+5$

$= 27+5 - 8+5$

$= 32 - 13 = 19$

2) find $\frac{x^2}{2}\Big|_2^3$

$f(x)\Big|_a^b = F(b) - F(a)$

$\frac{x^2}{2}\Big|_2^3 = F(3) - F(2)$

$= \frac{3^2}{2} - \frac{2^2}{2}$

$= \frac{9}{2} - \frac{4}{2}$ ~~$= 4.5 - 2 = 2.5$~~

$= \frac{9-4}{2} = \frac{5}{2}$

3) find $\int_2^6 (x^2+1)\, dx$ .. [by F.T of (c)] $\int \frac{1}{x^6} = \int x^{-6}$ $= \int \frac{x^{-6+1}}{-6+1}$

A) $\int_2^6 (x^2+1)\, dx$.

$f(x) = x^2+1$

$F(x) = \int f(x) = \int x^2+1 \cdot dx$.

$= \frac{x^3}{3} + x$

$\int_2^6 (x^2+1)\, dx = F(b) - F(a)$

$= F(6) - F(2)$

$= \frac{6^3}{3} + 6 - \frac{2^3}{3} + 2$

$= \frac{216}{3} + \frac{6 \times 3}{1 \times 3} - \frac{8}{3} + \frac{2 \times 3}{1 \times 3}$

$= \frac{216+18}{3} - \frac{8+6}{3}$

$= \frac{234-14}{3} = \frac{220}{3}$ //

(a) $\int_2^6 (x^2+1)\,dx = \left(\frac{x^3}{3}+x\right)_2^6$

$= \left(\frac{6^3}{3}+6\right) - \left(\frac{2^3}{3}+2\right)$

$= \frac{220}{3}$

4) Evaluate $\int_1^2 \frac{1}{x^4}\,dx$.

$\int \frac{1}{x^4} = \frac{x^{-4+1}}{-4+1} = \frac{x^{-3}}{-3}$

A) $\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx$

$= \left(\frac{x^{-4+1}}{-4+1}\right)_1^2$

$= \left(\frac{x^{-3}}{-3}\right)_1^2$

$= \left(\frac{-1}{3x^3}\right)_1^2 = \frac{-1}{3\times 2^3} - \frac{-1}{3\times 1^3}$

$= -\frac{1}{3\times 8} + \frac{1}{3}$

$= \frac{-1}{24} + \frac{1}{3}$

$= \frac{-3+24}{72} = \frac{21\div 3}{72\div 3}$

$= \frac{7}{24}$

5) Find $\int_0^4 t^2 + 3t^{7/2}\,.dt$.

A) $\int_0^4 t^2+3t^{7/2}\,dx = \int \frac{t^3}{3} + 3\int t^{7/2}$

$\int \frac{t^{7/2+1}}{\frac{7}{2}+1}$

$= \left(\frac{t^3}{3} + 3\times\frac{t^{\frac{7}{2}+1}}{\frac{7}{2}+1}\right)_0^4$

$= \left(\frac{t^3}{3} + 3\frac{t^{\frac{9}{2}}}{9/2}\right)_0^4$

$= \left(\frac{t^3}{3} + \cancel{3}\times\frac{2}{\cancel{9}_3}\,t^{\frac{9}{2}}\right)_0^4$

$= \left(\frac{t^3}{3} + \frac{2t^{9/2}}{3}\right)_0^4$

Substitude upper & lower limit,

$= \left(\frac{4^3}{3} + \frac{2\times 4^{\frac{9}{2}}}{3}\right) + 0$

$4^{9/2} = 4^{9\times 1/2} = (\sqrt{4})^9$

$\sqrt{4} = \frac{2\,|4}{2} = 2$

$\Rightarrow (2)^9 = 512$

$= \frac{64}{3} + \frac{2\times 512}{3}$

$= \frac{64+1024}{3}$

$= \frac{1088}{3}$

6) $\int_1^2 \frac{(s+5)^2}{s^4}\,ds$.

$a^2+2ab+b^2$

$= s^2 + 2\times s\times 5 + 5^2$

$= s^2 + 10s + 25$

$= \int_1^2 \frac{s^2+10s+25}{s^4}.$

$= \frac{s^3}{3} + 10\int s\, t$

$\frac{1}{s^4} = \frac{s^{-4+1}}{-4+1} = \frac{s^{-3}}{-3}$

$= \frac{-1}{3s^3}$

(give s^4 for all)

$$= \int_1^2 \left(\frac{s^2}{s^{4+2}} + \frac{10s}{s^{4+3}} + \frac{25}{s^4} \right) ds$$

$$= \int_1^2 \left(\frac{1}{s^2} + \frac{10}{s^3} + \frac{25}{s^4} \right) ds \qquad \frac{1}{s^2} = s^{-2}$$

$$= \int_1^2 \left(s^{-2} + 10 s^{-3} + 25 s^{-4} \right) ds.$$

$$= \left(\frac{s^{-2+1}}{-2+1} + \frac{10 s^{-3+1}}{-3+1} + 25 \frac{s^{-4+1}}{-4+1} \right)_1^2$$

$$= \left(\frac{s^{-1}}{-1} + \frac{10 s^{-2}}{-2} + 25 \frac{s^{-3}}{-3} \right)_1^2$$

$$= \left(-\frac{1}{s^1} - \frac{5}{s^2} - \frac{25}{3s^3} \right)_1^2$$

$$= \left(\frac{-1 \times 12}{2 \times 12} - \frac{5 \times 6}{4 \times 6} - \frac{25}{24} \right) - \left(\frac{-1 \times 3}{1 \times 3} - \frac{5 \times 3}{1 \times 3} - \frac{25}{3} \right)$$

take lcm.

$$= \left(\frac{-12 - 30 - 25}{24} \right) - \left(\frac{-3 - 15 - 25}{3} \right)$$

$$= \left(\frac{-67}{24} \right) - \left(\frac{-43}{3} \right)$$

$$= -\frac{67}{24} + \frac{43}{3} = \frac{-201 + 1032}{72} = \frac{831 \div 3}{72 \div 3}$$

$$= \frac{277}{24}$$

7) $\int_1^3 x^3 \, dx.$

$$= \int_1^3 \left(\frac{x^{3+1}}{3+1} \right) = \left(\frac{x^4}{4} \right)_1^3 = \frac{3^4}{4} - \frac{1^4}{4}$$

$$= \frac{27 - 1}{4} = \frac{26}{4}$$

→ Signed Area :-

The signed area of a region is the area of the portion above the x axis minus the area of the portion below the x axis.

For the region b/w x axis & the graph of the step ( ) 'g', the signed area is $\boxed{\sum_{i=1}^{n} k_i \Delta x_i}$

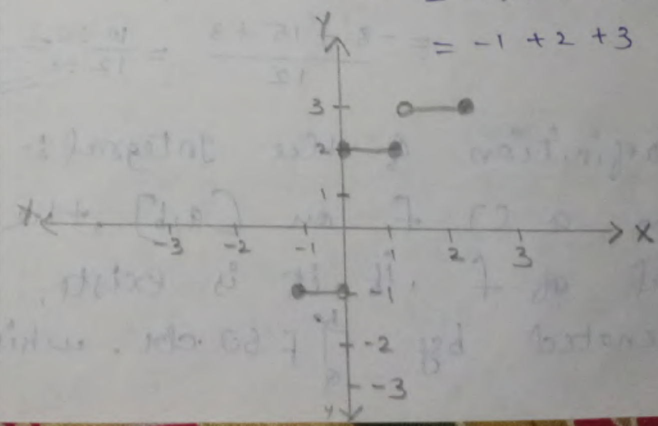
i) Draw the graph of step ( ) g on [-1, 2], defined by $g(x) = \begin{cases} -1 & , -1 \le x < 0, \\ 2 & , 0 \le x \le 1, \\ 3 & , 1 < x \le 2. \end{cases}$ (in graph)

Compute the signed area of the region b/w the graph & x axis.

A) Signed area $= \sum_{i=1}^{3} k_i \Delta x_i$

$$= -1 \times 1 + 2 \times 1 + 3 \times 1$$

$$= -1 + 2 + 3 = 5 - 1 = 4$$

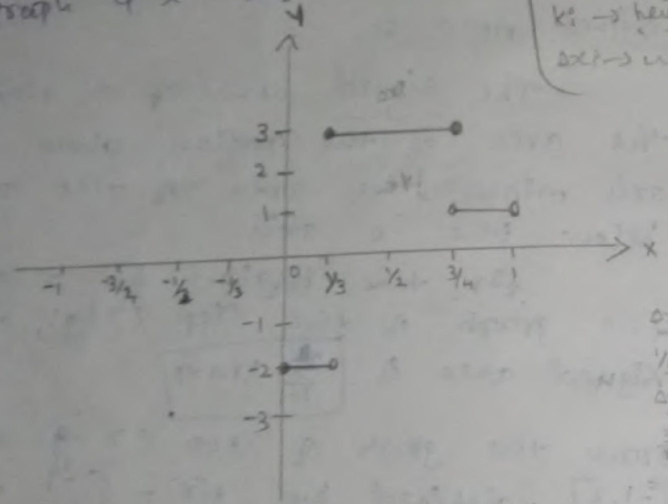


2) Draw a graph of step () 'g' on $[0,1]$ defined by $g(x) = \begin{cases} -2 & \text{if } 0 \le x < 1/3 \\ 3 & \text{if } 1/3 \le x \le 3/4 \\ 1 & \text{if } 3/4 < x < 1 \end{cases}$

compute signed area of the region b/s its graph & x axis.

$K_i \to$ height
$\Delta x_i \to$ width

A)



$\Delta x_1 = 1/3 - 0 = 1/3$

$\Delta x_2 = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$

$\Delta x_3 = 1 - \frac{3}{4} = \frac{4-3}{4} = \frac{1}{4}$

Signed area $= \sum_{i=1}^{3} k_i \Delta x_i$

$= \frac{-2}{1} \times \frac{1}{3} + 3 \times \frac{5}{12} + 1 \times \frac{1}{4}$

$\cancel{= \frac{-6 \times 1}{3}}$

$= \frac{-2 \times 4}{3 \times 4} + \frac{15}{12} + \frac{1 \times 3}{4 \times 3}$

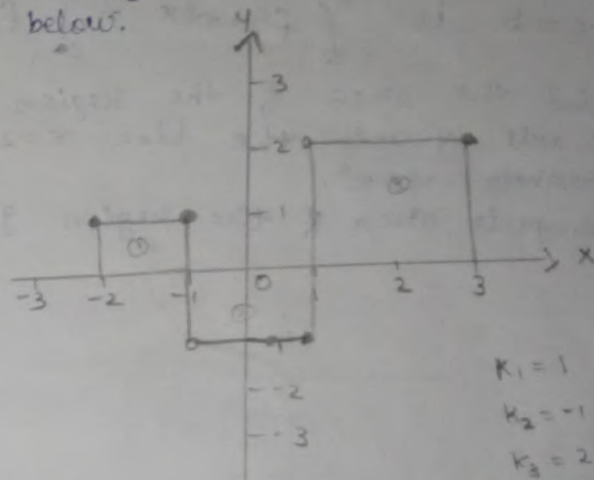
$= \frac{-8 + 15 + 3}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$

⇒ The Definition of the Integral :-

Given a () f on $[a,b]$, the integral of f, if it is exists, is the no. denoted by $\int_a^b f(x) \cdot dx$. which

Separate the upper & lower sums. This no. is the signed area of the region b/s the graph of 'f' & the x axis

7) Compute $\int_{-2}^{3} f(x) \cdot dx$ for the () f given below.



$K_1 = 1$ $\Delta x_1 = -1-(-2) = 1$

$K_2 = -1$ $\Delta x_2 = 1-(-1) = 2$

$K_3 = 2$ $\Delta x_3 = 2$

A) $\int_{-2}^{3} f(x) \cdot dx = \sum_{i=1}^{3} k_i \Delta x_i$

$= 1 \times 1 + -1 \times 2 + 2 \times 2$

$= 1 + -2 + 4 = 3$

⇒ Area under a graph :-

* If $f(x) \ge 0$ for all x in $[a,b]$ the area under the graph of 'f' b/s $x = a$ & $x = b$ is $\int_a^b f(x) \cdot dx$.

* If 'f' is -ve at some points of $[a,b]$ then, $\int_a^b f(x) \cdot dx$ is the signed area of the region b/s the graph of 'f', the x axis & the lines $x = a$ & $x = b$.

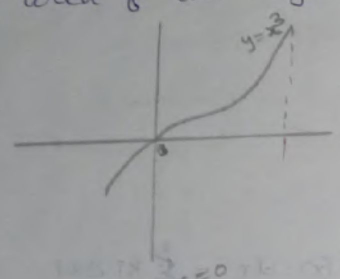
* If $f(x) \ge 0$ for all x in $[a,c]$ &

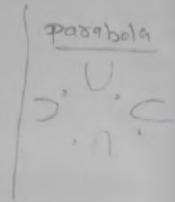
if $f(x) \le 0$ for all x in $[c,b]$ for some $c \in (a,b)$, then the area of the region b/w the graph of f, the x-axis and the lines $x=a$ & $x=b$ is $\int_a^c f(x)\,dx - \int_c^b f(x)\,dx$.

Q

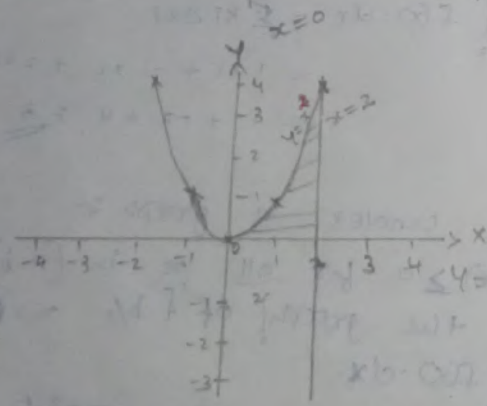
1) find the area of the region bounded by x axis, y axis, the line $x=2$ & the parabola $y=x^2$.

2) Compute area of the region given below,





(1)



a)

$y=x^2$

x	0	1	2	-1	-2
y	0	1	4	1	4

(to get parabola: take 5 points)

Eq for $x = y=0$

Eq for $y = x=0$

Area of the region $= \int_0^2 (x^2-0)\,dx$

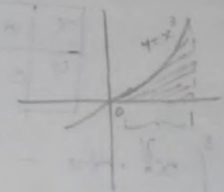
$= \int_0^2 x^2\,dx$

$= \frac{x^{2+1}}{2+1}\,dx$

$= \left(\frac{x^3}{3}\right)_0^2$

$= \left(\frac{2^3}{3} - 0\right) = \frac{8}{3}$

(2)



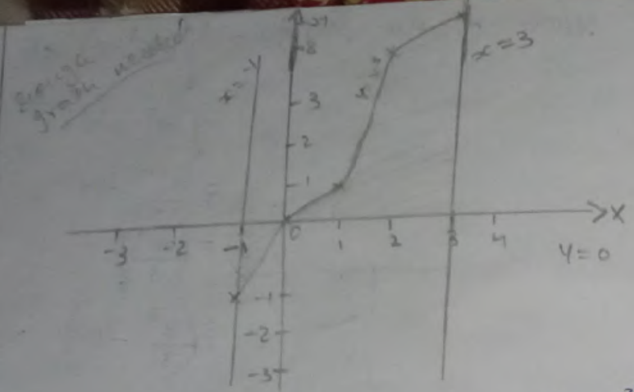
Area of the graph $= \int_0^1 (x^3-0)\,dx$

$= \int_0^1 x^3\,dx$

$= \left(\frac{x^4}{4}\right)_0^1 \quad = \left(\frac{1^4}{4} - 0\right)$

$= \frac{1}{4}$

2) find the area of the region bounded by the curve $y=x^3$ x-axis & vertical line $x=-1$ & $x=3$



$y = x^3$

x	0	1	2	3	-1
y	0	1	8	27	-1

big region   small region

Area of region $= \int_0^3 x^3 \cdot dx - \int_{-1}^{0} x^3 \cdot dx$

$= \left(\frac{x^4}{4}\right)_0^3 - \left(\frac{x^4}{4}\right)_{-1}^{0}$

$= \left(\frac{3^4}{4} - 0\right) - \left(0 - \frac{(-1)^4}{4}\right)$

$= \left(\frac{81}{4}\right) - \left(\frac{-1}{4}\right)$

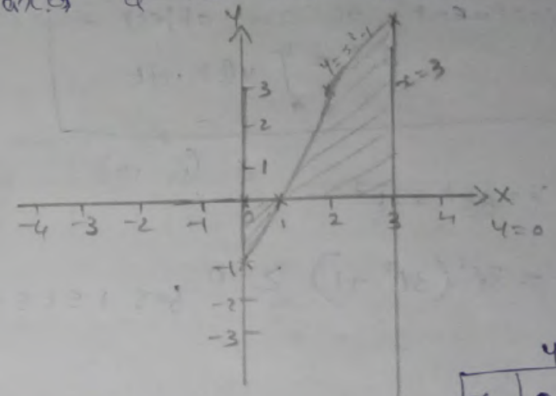
$= \frac{81}{4} + \frac{1}{4} = \frac{81+1}{4}$

$= \frac{82}{4} \frac{\div 2}{\div 2}$

$= \frac{41}{2}$ unit^2

(unit □)

3) find the area of the region bounded by curve $y = x^2 - 1$, x axis, y-axis & the line $x = 3$



$y = x^2 - 1$

x	0	1	2	3
y	-1	0	3	8

Area of region,

$= \int_1^3 (x^2-1)\, dx - \int_0^1 (x^2-1)\, dx.$

$= \left(\frac{x^3}{3} - x\right)_1^3 - \left(\frac{x^3}{3} - x\right)_0^1$

$= \left[\left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)\right] - \left[\left(\frac{1^3}{3} - 1\right) - 0\right]$

$= \left(\frac{27}{3} - \frac{3}{1}\right) - \left(\frac{1}{3} - \frac{1}{1}\right) - \left(\frac{1}{3} - \frac{1}{1}\right) - 0$

$= \left[\frac{27-9}{3} - \frac{1-3}{3}\right] - \left[\frac{1-3}{3} - 0\right]$

$= \frac{18+2}{3} + \frac{2}{3} = \frac{20+2}{3} = \frac{22}{3}$

4) An object moving in a straight line has velocity $v = 6t^4 + 3t^2$ at time t how far the object travel b/w $t=1$ & $t=10$

$$\boxed{\text{displacement of an object} = \Delta d = \int_a^b v(t)\cdot dt}$$

A) $(1, 10)$

$v = 6t^4 + 3t^2$ (make factors)

graph → ↑ +ve

graph → ↓ -ve

$v = 3t^2(3t^2+1) \geq 0$ for $1 \leq t \leq 10$

eg = $3\times1^2 \, (3\times1^2+1)$ +ve +ve.

Thus () is +ve.

$\Delta d = \int_a^b v(t)\cdot dt$.

$= \int_1^{10} (6t^4 + 3t^2)\, dt$.

$= \left(6\frac{t^5}{5} + 3\frac{t^3}{3}\right)_1^{10}$

$= \left[\frac{6\times10^5}{5} + \frac{\cancel{3}\times10^3}{\cancel{3}}\right] - \left[\frac{6\times1^5}{5} + \frac{\cancel{3}\times1^3}{\cancel{3}}\right]$

$= \left[\frac{600000}{5} + \cancel{\frac{3000}{3}}\, 1000\right] - \left[\frac{6}{5} + \frac{1}{1}\right]$

$= \frac{\overset{600000}{\cancel{120000}} + \cancel{1}5000}{5} - \frac{6+5}{5}$

$= \frac{605000 - 11}{5} = \frac{604989}{5} = 120997.8$

5) velocity of an object on x-axis is $v = 4t - 2t^2$, if the object is at $x = 1$ at $t = 0$. where is it at $t = 4$? how far has it travelled?

$f(0)=0$ · $x=1$ $f(0)=1$ — $f(4)=?$ $x=?$

$t=0$ ———— $t=4$

A) $v = 4t - 2t^2$

$\cancel{v = 2t(2 - t)}$

$\Delta d = f(4) - f(0)$

$\Delta d + f(0) = f(4)$

$f(t) = f(0) = 1$

$f(t) = f(4) = ?$

$(f(4)$ consider as object → ?)

$\Delta d + 1 = f(4)$. — ①

$\Delta d = \int_a^b v(t)\cdot dt$.

$= \int_0^4 (4t - 2t^2)\, dt$.

$= \left(\overset{2}{\cancel{4}}\frac{t^2}{\cancel{2}} - \frac{2t^3}{3}\right)_0^4$

$= \left(2t^2 - \frac{2t^3}{3}\right)_0^4$

$= \left(2\times4^2 - \frac{2\times4^3}{3}\right) - 0$.

$= \frac{32}{1} - \frac{128}{3}$

$= \frac{96-128}{3} = \frac{-32}{3}$

— ①

∴ $f(4) = \Delta d + 1$

$= \frac{-32}{3} + \frac{1}{1} = \frac{-32+3}{3} = \frac{-29}{3}$

How far it has travelled ?

$(0,4)$

$v = 4t - 2t^2$

$= 2t(2-t) < 0 \quad 2 \to 4 \quad (-ve)$

$2t(2-t) > 0 \quad 0 \to 2 \quad (+ve)$

$\therefore$ distance from $t=0$ to $t=4$

$= \int_0^2 (4t - 2t^2)\,dt - \int_2^4 (4t-2t^2)\,dt$

$= \left(\frac{4t^2}{2} - 2\frac{t^3}{3}\right)_0^2 - \left(\frac{4t^2}{2} - 2\frac{t^3}{3}\right)_2^4$

$= \left(2\times2^2 - \frac{2\times2^3}{3}\right) - 0 - \left[\left(2\times4^2 - \frac{2\times4^3}{3}\right) - \left(2\times2^2 - \frac{2\times2^3}{3}\right)\right]$

$= \frac{8}{1} - \frac{16}{3} - \left[\left(\frac{32}{1} - \frac{128}{3}\right) - \left(\frac{8}{1} - \frac{16}{3}\right)\right]$

$= \frac{24-16}{3} - \left[\left(\frac{96-128}{3} - \frac{24-16}{3}\right)\right]$

$= \frac{8}{3} - \left[-\frac{32}{3} - \frac{8}{3}\right]$

$= \frac{8}{3} + \frac{40}{3} = \frac{48}{3} = 16$

$\Rightarrow$ Definite & Indefinite Integral :-

$\int f(x)\,dx$ — Indefinite

$\int_a^b f(x)\,dx$ — Definite

$F(x) = \int f(x)\,dx$ → anti derivative

when an anti derivative $f(x)$ is ( ) $F(x)$ such that

$F'(x) = f(x)$

In this case we write $F(x) = \int f(x)\,dx$.

( ) on right hand on above eq → the indefinite integral of f. i.e f → the integrand.

An expression of the form $\int_a^b f(x)\,dx$ with the end points specify → Definite integral.

Q Check the formula –

1) $\int 3x^8\,dx = \frac{x^9}{3} + c$

$\frac{d}{dx}\left(\frac{x^9}{3} + c\right)$

$\frac{d}{dx}\left(\frac{x^9}{3}\right) + \frac{d}{dx}(c)$. $\quad \frac{d}{dx}(c) = 0 \quad \frac{d}{dx}(x^n) = nx^{n-1}$

$= \frac{1}{3}\frac{d}{dx}(x^9) + 0$

$= \frac{1}{3} \cdot 9x^8 = 3x^8$

$\therefore \int 3x^8\,dx = \frac{x^9}{3} + c$.

3) $\int x(1+x)^6\,dx = \frac{1}{56}(7x-1)(1+x)^7 + C$

3) find $\int_0^2 x(1+x)^6\,dx$.

2.A) $\frac{d}{dx}\left(\frac{1}{56}(7x-1)(1+x)^7 + C\right)$

$= \frac{1}{56}\frac{d}{dx}\left[(7x-1)(1+x)^7\right]$ $\quad \frac{d}{dx}(1^{st} \times 2^{nd}) = 1^{st}\frac{d}{dx}(2^{nd}) + 2^{nd}\frac{d}{dx}(1^{st})$

$= \frac{1}{56}\left[(7x-1)\frac{d}{dx}(1+x)^7 + (1+x)^7\frac{d}{dx}(7x-1)\right]$ $\quad \frac{d}{dx}(x) = 1$, $\frac{d}{dx}(1) = 0$

$= \frac{1}{56}\left[(7x-1)\,7(1+x)^6 \times (0+1) + (1+x)^7(7-0)\right]$ $\quad n x^{n-1}$

$= \frac{1}{56}\left[7(7x-1)(1+x)^6 + (1+x)^7 \times 7\right]$ $\quad (1+x)^6$ common

$= \frac{1}{56}(1+x)^6\left[7(7x-1) + 7(1+x)\right]$

$= \frac{1}{56}(1+x)^6\left[49x \cancel{-7} \cancel{+7} + 7x\right]$

$= \frac{1}{\cancel{56}}(1+x)^6\left[\cancel{56}x\right]$

$= x(1+x)^6$

3.A) $\cancel{\int x(1+x)^6}$

$\int_0^2 x(1+x)^6 \cdot dx = \left[\frac{1}{56}(7x-1)(1+x)^7\right]_0^2$

$= \frac{1}{56}(7\times2-1)(1+2)^7 - \left(\frac{1}{56}(0-1)(1+0)^7\right)$

$= \frac{1}{56}\left(13 \times 3^7 - \frac{1}{56}(-1)\times1\right)$

$= \frac{1}{56}\left(13 \times 2187 + \frac{1}{56}\right)$

$= \frac{1}{56}\left(28431 + \frac{1}{56}\right)$

$= \frac{28431}{56} + \frac{1}{56} = \frac{28431+1}{56}$

$= \frac{28432}{56} = 507.71$

⇒ **Properties of Definite Integrals :-**

1) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. (Sum rule)

2) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$. $\quad c$ = constnt (constnt multiple rule)

3) if $a < b < c$, then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.

4) if $f(x) = c$ is constnt, then $\int_a^b f(x)\,dx = c(b-a)$

5) if $f(x) \le g(x)$, for all x satisfying $a \le x \le b$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Q

1) Assume that f, g have antiderivative prove that $\int_a^b (f(x) + g(x))\cdot dx = \int_a^b f(x)\cdot dx + \int_a^b g(x)$.

A) Let f be the antiderivative of f.
G be the " of g.

$F(x) = \int f(x)$ or $F'(x) = f(x)$

$G(x) = \int g(x)$

$(F+G)$ be the antiderivative $f+g$.

$F(x) + G(x) = \int f(x) + g(x)$.

$$\int_a^b (f(x) + g(x))\,dx = [F(x) + G(x)]_a^b$$
$$= [F(b) + G(b)] - [F(a) + G(a)]$$
$$= F(b) + G(b) - F(a) - G(a)$$
$$= F(b) - F(a) + G(b) - G(a)$$

$$\int_a^b (f(x) + g(x)) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2) Prove if $f(x) \leq g(x)$ for all x satisfying $a \leq x \leq b$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

A) Let F be the antiderivative of f.
G be the " of g

$F(x) = \int f(x)$ or $F'(x) = f(x)$ ✓

$G'(x) = g(x)$.

$(F-G) \to (F-G)'$

$f(x) \leq g(x)$

$f(x) - g(x) = 0$.

$(F-G)'(x) = F'(x) - G'(x)$
$= f(x) - g(x) \leq 0$.

$(F-G)'(x) \leq 0$

Since antiderivative of a ( ) is -ve, the ( ) use.

by giving limits, $(F(b) - F(a)) - (G(b) - G(a)) \leq 0$

$F(b) - F(a) \leq G(b) - G(a)$

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Note :-

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ [To chang a-b to b-a]

2) $\int_a^a f(x)\,dx = 0$

3) $\int_a^b f'(x)\,dx = f(b) - f(a)$, for all a, b.

=> Fundamental theorum of calculus :-

If f is continuous on $[a, x]$, then,

$$\frac{d}{dx}\int_a^x f(s)\,ds = f(x).$$

($\frac{d}{dx}$ & $\int$ cancel)

1) Varify the formula $\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$ for $f(x) = x$.

A) $f(x) = x$

$\therefore f(s) = s$

(1st $\int$) $\int_a^x f(s)\,ds = \int_a^x s\,ds$

$$= \left(\frac{s^2}{2}\right)_a^x = \frac{x^2}{2} - \frac{a^2}{2}.$$

(2nd $\frac{d}{dx}$)

$$\frac{d}{dx}\int_a^x f(s)\,ds = \frac{d}{dx}\left(\frac{x^2}{2} - \frac{a^2}{2}\right)$$
$$= \frac{d}{dx}\left(\frac{x^2}{2}\right) - \frac{d}{dx}\left(\frac{a^2}{2}\right)$$

$= \frac{1}{2} \frac{d}{dx}(x^2) - \frac{1}{2} \frac{d}{dx}(a^2)$ $\qquad \frac{x^2}{2} = \frac{1}{2}(x^2)$

$a \to$ constant $\to 0$.

$= \frac{1}{\cancel{2}} \times \cancel{2}x - 0.$

$= x \Rightarrow f(x)$

2) let $F(x) = \int_2^x \frac{1}{1+s^2+s^3} \cdot ds$ find $F'(3)$

A) $F(x) = \int_2^x \frac{1}{1+s^2+s^3} \cdot ds.$

$\therefore F'(x) = \frac{1}{1+s^2+s^3} \cdot ds.$

$F'(3) = ?$

$F'(3) = \frac{1}{1+3^2+3^3} = \frac{1}{1+9+27} = \frac{1}{37}$

* Usual version :- [fundametal T. of (C)]

$\int_a^b F'(x) \cdot dx = F(b) - F(a)$

Integrating (deri) of F gives the change in F.

* Alternative version :-

$\frac{d}{dx} \int_a^x F(s) \cdot ds = f(x).$

differenciating the $\int$ of F with respect to the upper limit give F.