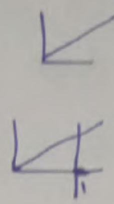
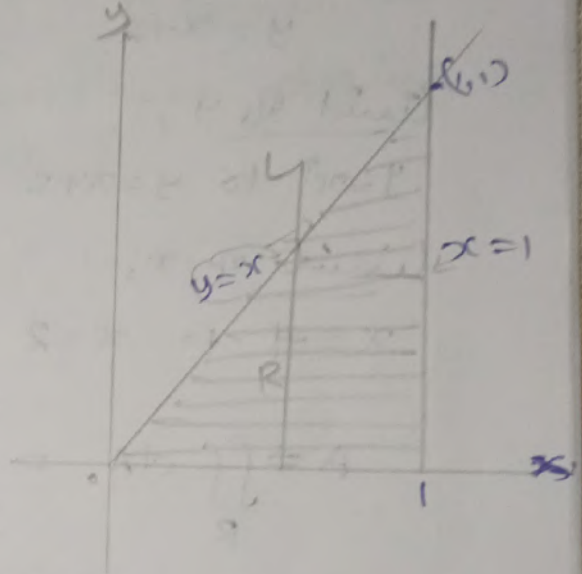
2) Find the vol of prism whose base is the triangle in the xy-plane bounded by x-axis & lines $y=x$ & $x=1$ and whose top lines in the plane $z = f(x,y) = 3-x-y$.

A) $y = x \rightarrow$

$x = 1 \rightarrow$





limit

If $y = 0$, $x = 0$

$x = 1$ $\quad \int_0^1 \int_1^x$

$$V = \iint_R f(x,y)\, dA$$

$$= \iint_R (3-x-y)\, dA$$

$$= \int_0^1 \int_1^x (3-x-y)\, dy\, dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_1^x dx$$

$$= \int_0^1 \left[3x - x^2 - \frac{x^2}{2} - \left(3 - x - \frac{1}{2}\right)\right] dx$$

$$= \int_0^1 3x - \frac{x^2}{2} - 3 + x + \frac{1}{2} - \frac{x^2}{2}$$

$$= \int_0^1 \left[3x - \frac{3x^2}{2}\right] dx$$

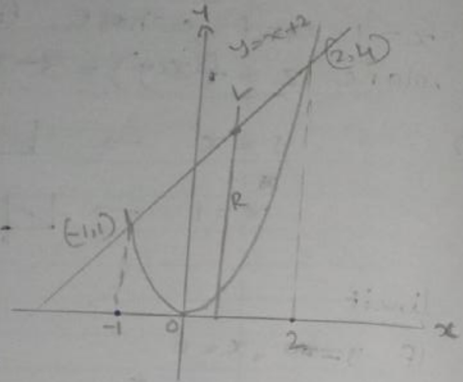
$$= \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1$$

$x^2 - \frac{x^2}{2} = \frac{2x^2 - x^2}{2} = \frac{x^2}{2}$

$3 + \frac{1}{2} = \frac{6+1}{2} = \frac{7}{2}$

3) find area of region enclosed by parabola $y=x^2$ & line $y=x+2$.

A) $A=\iint_R dA$.



$y=x^2$

$y=x+2$

limit of y,

$y=x^2$ to $y=x+2$

limit of x,

$x=-1$ to $x=2$

$$A=\iint_R dA = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2}\left[\int_{x^2}^{x+2} dy\right]dx$$

$$=\int_{-1}^{2}\left[y\right]_{x^2}^{x+2} dx = \int_{-1}^{2}\left[x+2-x^2\right]dx$$

$$=\left[\frac{x^2}{2}+2x+\frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

=> **Laminas with variable Density :-**

* If ρ is a constant density (mass per unit area) then mass of lamina coinciding with a region bounded by the graphs of $y=f(x)$, the x-axis & lines $x=a$ & $x=b$,

$$m=\lim_{\|P\|\to 0}\sum_{k=1}^{n}\rho f(x^*_k)\,\Delta x_k.$$

$$\boxed{m=\int_a^b \rho f(x)\,dx}$$

* If a lamina to a region R has a variable density $\rho(x,y)$, ρ is non-ve & continous on R, then we define its mass m by double $\int$.

(mass). $m=\lim_{\|P\|\to 0}\sum_{k=1}^{n}\rho f(x^*_k, y^*_k)\,\Delta A_k$

$$\boxed{m=\iint_R \rho(x,y)\,dA}$$

* Center of mass of lamina :

with x coordinates, $\boxed{\bar{x}=\frac{M_y}{m}}$

with y ", $\boxed{\bar{y}=\frac{M_x}{m}}$

M_y & M_x are moments,

$$\boxed{\begin{aligned}M_y &= \iint_R x\rho(x,y)\,dA\\ M_x &= \iint_R y\rho(x,y)\,dA\end{aligned}}$$

$\int$-s M_x & M_y also → 1^{st} moments of a lamina about x & y-axes.

2^{nd} moments of a lamina | moments of inertia about x & y-axes, → (I_x), (I_y)

$$\boxed{\begin{aligned} I_x &= \iint_R y^2 \rho(x,y)\, dA \\ I_y &= \iint_R x^2 \rho(x,y)\, dA \end{aligned}}$$

* Kinetic energy :-

$k = \frac{1}{2} m v^2 = \frac{1}{2} m (r\omega)^2$ $v = r\omega$ (circle)

$= \frac{1}{2} (m r^2) \omega^2$ $I = m r^2$. moment of inertia

$$\boxed{k = \frac{1}{2} I \omega^2}$$

1) A lamina has the shape of the region in 1st quadrant that is bounded by graphs of $y = \sin x$, $y = \cos x$, b/w $x = 0$ & $x = \pi/4$. Find the center of mass if density is $\rho(x,y) = y$?

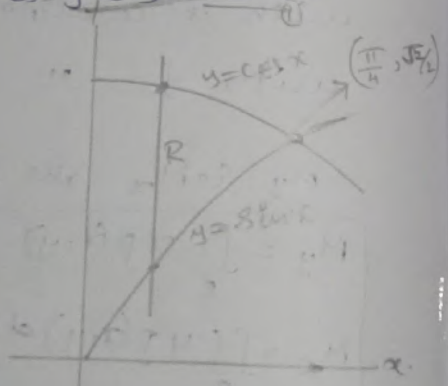
A) $y = \sin x$ ~~$\sin x = 0$~~

$x = 0$

$y = \cos x$.

$x = \pi/4$.

$\int_0^{\pi/4} \int_{\sin x}^{\cos x}$



$\boxed{m = \iint_R \rho(x,y)\, dA}$ → ~~center of mass~~.

$= \int_0^{\pi/4} \int_{\sin x}^{\cos x} y \cdot dy\, dx = \int_0^{\pi/4} \left[\frac{y^2}{2}\right]_{\sin x}^{\cos x}$

$= \frac{1}{2} \int_0^{\pi/4} [\cos^2 x - \sin^2 x]\, dx$.

$= \frac{1}{2} \int_0^{\pi/4} \cos 2x\, dx$. $\boxed{\cos^2 x - \sin^2 x = \cos 2x}$

$= \frac{1}{2} \left[\frac{\sin 2x}{2}\right]_0^{\pi/4} \Rightarrow \frac{1}{2} \cdot \frac{1}{2} [\sin 2\, \pi/4 - 0]$

$= \frac{1}{4} [\sin \frac{\pi}{2}] = \frac{1}{4} \cdot 1 = \frac{1}{4}$

M_x, M_y ?

$$\boxed{M_x = \iint_R y\, \rho(x,y)\, dA}$$ → ① ⇒ $y \times y\, dy dx$

$= \int_0^{\pi/4} \int_{\sin x}^{\cos x} y^2\, dy\, dx$.

$= \int_0^{\pi/4} \left[\frac{y^3}{3}\right]_{\sin x}^{\cos x} dx. = \frac{1}{3} \int_0^{\pi/4} [y^3]_{\sin x}^{\cos x} dx$.

$= \frac{1}{3} \int_0^{\pi/4} [\cos^3 x - \sin^3 x]\, dx$. $\cos^2 x \to 1 - \sin^2 x$ $\sin^2 x \to 1 - \cos^2 x$

$\cos^3 x$ $\cos^2 x$ ... $\cos^2 x = 1 - \sin^2 x$ $\sin x$

$= \frac{1}{3} \int_0^{\pi/4} [\cos x(1 - \sin^2 x) - \sin x(1 - \cos^2 x)]\, dx$.

$(\cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x)$

$= \frac{1}{3} \int_0^{\pi/4} \cos x - \sin^2 x \cos x - \sin x + \sin x \cos^2 x$.

$= \frac{1}{3} \left[\int_0^{\pi/4} \cos x\, dx - \int \sin^2 x \cos x\, dx - \int \sin x\, dx + \int \sin x \cos^2 x\, dx.\right]$

$= \frac{1}{3} \left\{ [\sin x]_0^{\pi/4} - \frac{1}{3} [\sin^3 x]_0^{\pi/4} - [-\cos x]_0^{\pi/4} + \frac{1}{3} [-\cos^3 x]_0^{\pi/4} \right\}$.

$\boxed{\begin{aligned} &\cos x \to \sin x \\ &\sin^2 x \cos x \to \frac{\sin^3 x}{3} \\ &\sin x \to -\cos x. \\ &\sin x \cos^2 x \to -\frac{\cos^3 x}{3} \end{aligned}}$

$$= \frac{1}{3}\left\{\left[\frac{1}{\sqrt2}-0\right]-\frac{1}{3}\left[\frac{1}{2\sqrt2}-0\right]-\left[1-\frac{1}{\sqrt2}\right]+\frac{1}{3}\left[1-\frac{1}{2\sqrt2}\right]\right\}$$

$$= \frac{1}{3}\left\{\frac{1}{\sqrt2}-\frac{1}{6\sqrt2}-1+\frac{1}{\sqrt2}+\frac{1}{3}-\frac{1}{6\sqrt2}\right\}. \text{ (cm)}$$

$$= \frac{6-1-6\sqrt2+6+2\sqrt2-1}{18\sqrt2} = \frac{10-4\sqrt2}{18\sqrt2}$$

$$\therefore M_x = \frac{5\sqrt2-4}{18}$$

$$\boxed{M_y = \iint_R x\,\rho(x,y)\,dA} = \int_0^{\pi/4}\int_{\sin x}^{\cos x} xy\,dy\,dx.$$

$\rho = \rho(x,y) = y$

$$= \int_0^{\pi/4}\left[x\cdot\frac{y^2}{2}\right]_{\sin x}^{\cos x} dx = \frac{1}{2}\int_0^{\pi/4}\left[x\,y^2\right]_{\sin x}^{\cos x} dx.$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[x\cdot(\cos^2 x-\sin^2 x)\right] dx.$$

$\boxed{\cos^2 x-\sin^2 x = \cos 2x}$

$$= \frac{1}{2}\int_0^{\pi/4}\left[x\cdot\cos 2x\right] dx.$$

(I.L.A.T.E. rule) $\rightarrow$ $$= \frac{1}{2}\left[x\cdot\frac{\sin 2x}{2}\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\sin 2x\,dx.$$

integrating by parts.

$$= \frac{1}{2}\left\{\frac{\pi}{8}-\left[-\frac{1}{4}\cos 2x\right]_0^{\pi/4}\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi}{8}-\frac{1}{4}\right\} = \frac{\pi-2}{16} \Rightarrow \frac{1}{2}\left[\frac{4\pi-8}{32}\right] \Rightarrow \frac{4\pi-8}{64}\,\frac{\div 4}{\div 4} = \frac{\pi-2}{16}$$

hence coordinates of center of mass,

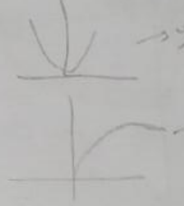
$$\bar{x} = \frac{M_y}{m} = \frac{(\pi-2)/16}{1/4} = \frac{\pi-2}{4} \approx 0.29$$

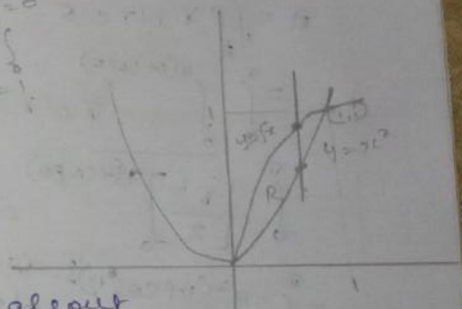
$$\bar{y} = \frac{M_x}{m} = \frac{(5\sqrt2-4)/18}{1/4} = \frac{10\sqrt2-8}{9} \approx 0.68$$

coordinates $(0.29, 0.68)$

2) find moment of inertia about x-axis of lamina that has slope of region bounded by graphs of $y=x^2$ & $y=\sqrt{x}$ & density $\rho(x,y)=x^2$?

A).





moment of inertia about x-axis. $\rightarrow$

$$I_x = \iint_R y^2\rho(x,y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} y^2x^2\,dy\,dx.$$

$$= \int_0^1 x^2\left[\frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} dx = \frac{1}{3}\int_0^1 x^2\left[x^{3/2}-x^6\right]dx.$$

$y^3 \rightarrow (\sqrt{x})^3 \rightarrow (x^{1/2})^3 = x^{3/2}$

$y^3 \rightarrow (x^2)^3 \Rightarrow x^6$

$x^2 \cdot x^{3/2} = x^{\frac{4+3}{2}} = x^{7/2}$

$$= \frac{1}{3}\int_0^1\left[x^{7/2}-x^8\right]dx.$$

$$= \frac{1}{3}\left[\frac{x^{9/2}}{9/2}-\frac{x^9}{9}\right] = \frac{1}{3}\left[\frac{2}{9}-\frac{1}{9}\right] = \frac{1}{27}$$

⇒ **Area in polar coordinates :-**

The area of a closed bounded R in the polar coordinates,

$$\boxed{A = \iint_R r \, dr \, d\theta}$$

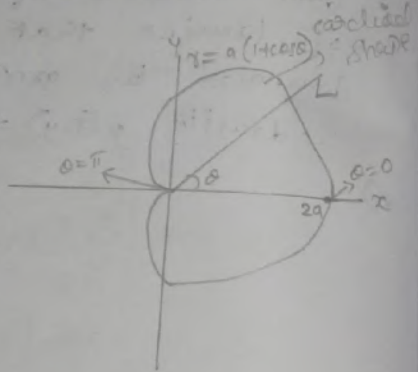
→ polar coordinates

want to evaluate $\iint_R f(r,\theta) \, dA$.

$$\boxed{\iint_R f(r,\theta) \, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{g_2(\theta)} f(r,\theta) \, r \, dr \, d\theta.}$$

Q/ 1) find area enclosed by cardioid

$r = a(1+\cos\theta)$



A) $x = \pi \to 0$.

$$A = \iint_R r \, dr \, d\theta.$$

$$= \int_0^{\pi} \int_0^{a(1+\cos\theta)} r \, dr \, d\theta.$$

$$= \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta.$$

$$= \int_0^{\pi} \frac{a^2(1+\cos\theta)^2}{2} \, d\theta. \quad (a+b)^2. \quad = \int_0^{\pi} \frac{a^2 + (\cos^2\theta + 2\cos\theta + 1)}{2} \, d\theta.$$

$$= a^2 \int_0^{\pi} \frac{\cos^2\theta + 2\cos\theta + 1}{2} \, d\theta = a^2 \int_0^{\pi} \frac{\cos^2\theta}{2} + \frac{\cancel{2}\cos\theta}{\cancel{2}} + \frac{1}{2}$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= a^2 \int_0^{\pi} \left[\frac{1+\cos 2\theta}{2\times 2} + \cos\theta + \frac{1}{2}\right] d\theta = a^2 \int_0^{\pi} \frac{1+\cos 2\theta}{4} + \cos\theta + \frac{1}{2}$$

$$= a^2 \left[\frac{3\theta}{4} + \frac{\sin 2\theta}{8} + \sin\theta\right]_0^{\pi} = a^2\left[\frac{3\pi}{4} + 0 - 0\right]$$

$$= \frac{3\pi a^2}{4}$$

∴ Area enclosed by cardioid =

$$2 \times R = \cancel{2} \cdot \frac{3\pi a^2}{\cancel{4}_2} = \frac{3\pi a^2}{2}$$

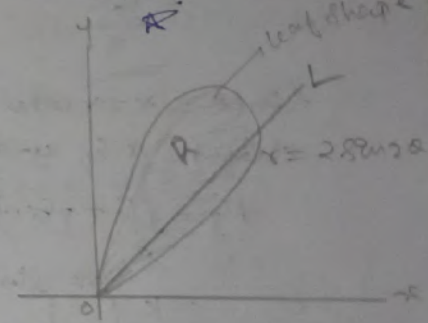
2) find center of mass of lamina that corresponds to the region bounded by 1 leaf of rose. $r = 2\sin 2\theta$ in the 1st quadrant if the density at a point P in the lamina is directly proportional to the distance from pole?

A) y limits

$0 \to 2\sin 2\theta$

x limits

$0 \to \frac{\pi}{2}$.



$$m = \iint_R \rho(x,y) \, dA = \iint_R k|r| \, dA$$

$$= k \int_0^{\pi/2} \int_0^{2\sin 2\theta} r \cdot r \, dr \cdot d\theta.$$

$$= k \int_0^{\pi/2} \int_0^{2\sin 2\theta} r^2 \cdot dr \, d\theta = k \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\sin 2\theta} d\theta.$$

$$= \frac{k}{3} \int_0^{\pi/2} 8\sin^3 2\theta \cdot d\theta = \frac{8k}{3} \int_0^{\pi/2} (1-\cos^2 2\theta)\sin 2\theta \, d\theta.$$

let $\cos 2\theta = u$.

$-2\sin 2\theta \, d\theta = du$.

$\sin 2\theta \, d\theta = -du/2$

Limits
$\theta = 0 \quad , u = 1$
$\theta = \pi/2 \quad , u = \cos\pi = -1$.

$\therefore \; m = \frac{8k}{3} \int_1^{-1} (1-u^2)\left(-\frac{du}{2}\right)$

$= \frac{4k}{3} \int_1^{-1} (1-u^2)\, du$.

$= \frac{4k}{3}\left[u - \frac{u^3}{3}\right]_1^{-1} = \frac{4k}{3}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$

$= \frac{16k}{9}$.

Since $x = r\cos\theta$ & $y = r\sin\theta$, M_x & M_y about x & y-axes ,

$M_y = \iint_R x\,\rho(x,y)\, dA = \iint_R r\cos\theta \cdot k|r|\, dA$.

$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^2\cos\theta \cdot r\, dr \cdot d\theta$

$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^3 \cos\theta\, dr\, d\theta$

$= k\int_0^{\pi/2} \cos\theta \left[\frac{r^4}{4}\right]_0^{2\sin 2\theta} d\theta$.

$= 4k\int_0^{\pi/2} \sin^4 2\theta \cos\theta\, d\theta$.

$= 4k\int_0^{\pi/2} (2\sin\theta\cos\theta)^4 \cos\theta\, d\theta$.

$= 64k\int_0^{\pi/2} \sin^4\theta \cos^5\theta\, d\theta$.

$= 64k\int_0^{\pi/2} \sin^4\theta\,(1-\sin^2\theta)^2 \cos\theta\, d\theta$.

$= 64k\int_0^{\pi/2} [\sin^4\theta - 2\sin^6\theta + \sin^8\theta]\cos\theta\, d\theta$

Sub. $\sin\theta = u$.
$\cos\theta\, d\theta = du$
$\sin 0 = 0$ } limit
$\sin\pi/2 = 1$

$= 64k\int_0^1 [u^4 - 2u^6 + u^8]\, du$

$= 64k\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right]$

$= 64k\left[\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right] = \frac{512}{315}k$.

$M_x = \iint_R y\,\rho(x,y)\, dA = \iint_R r\sin\theta \cdot k|r|\, dA$

$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^2 \sin\theta \cdot r \cdot dr\, d\theta$.

$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^3 \sin\theta\, dr\, d\theta$.

$= k\int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\sin 2\theta} \sin\theta \cdot d\theta$.

$= 4k\int_0^{\pi/2} \sin^4 2\theta \sin\theta\, d\theta$.

$= 4k\int_0^{\pi/2} (2\sin\theta\cos\theta)^4 \sin\theta\, d\theta$

$= 64k\int_0^{\pi/2} \sin^5\theta \cos^4\theta\, d\theta$.

$= 64k\int_0^{\pi/2} \cos^4\theta\,(1-\cos^2\theta)^2 \sin\theta\, d\theta$.

$= 64k\int_0^{\pi/2} [\cos^4\theta - 2\cos^6\theta + \cos^8\theta]\sin\theta\, d\theta$

$\cos\theta = u$.
$-\sin d\theta = du$
$\sin d\theta = -du$

$= 64k\int_0^1 (u^4 - 2u^6 + u^8)\,(du)$

$= 64k\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right]_0^1$

$= 64k\left[\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right] = \frac{512}{315}k$

Centres of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{512k/315}{16k/9}$$
$$= \frac{32}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{512k/315}{16k/9} = \frac{32}{35}$$

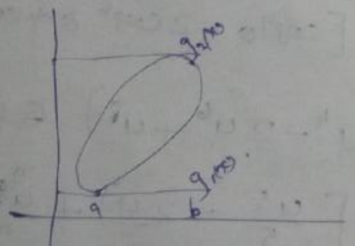
Coordinates $\left(\frac{32}{35}, \frac{32}{35}\right)$

⇒ Green's Theorem in the Plane:-

Suppose that 'C' is a piecewise smooth simple closed curve ($\oint$) bounding a simply connected region 'R', if $P, Q, \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ are contin- on R.

then,

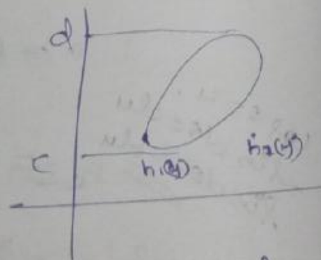
$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$



Type I

$a \le x \le b$

$g_1(x) \le y \le g_2(x)$



Type II

$c \le y \le d$

$h_1(y) \le x \le h_2(y)$

$$\iint_R \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$

$\boxed{\int \frac{\partial P}{\partial y}\,dy \quad P(x,y)}$ $\boxed{\text{Type I}}$

$$= \int_a^b \left[P(x,y)\right]_{g_1(x)}^{g_2(x)} dx$$
$$= \int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx.$$
$$= -\int_a^b P(x, g_1(x))\,dx + \int_a^b P(x, g_2(x))\,dx$$
$$= -\oint_C P(x, g(x))\,dx = -\oint P(x,y)\,dx \quad \text{--- (1)}$$

$\boxed{\text{Type II}}$

$$\iint_R \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy$$
$$= \int_c^d \left[Q(x,y)\right]_{h_1(y)}^{h_2(y)} dy$$
$$= \int_c^d \left[Q(h_2(y), y) - Q(h_1(y), y)\right] dy$$
$$= -\int_c^d Q(h_1(y), y) + Q(h_2(y), y)\,dy$$
$$= \oint_C Q(x,y)\,dy. \quad \text{--- (2)}$$

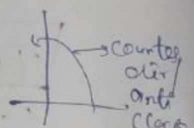
from (1) & (2)

$$\Rightarrow \iint_R \frac{\partial P}{\partial y}\,dA - \iint_R \frac{\partial Q}{\partial x}\,dA$$
$$= -\oint P(x,y)\,dx - \oint Q(x,y)\,dy$$
$$\Rightarrow \cancel{\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA = \iint_R \left(Q(x,y) - P(x,y)\right) dA}$$

$$\iint_R \frac{\partial P}{\partial y}\, dA - \iint_R \frac{\partial Q}{\partial x}\, dA = \int P\, dx + \int Q\, dy$$

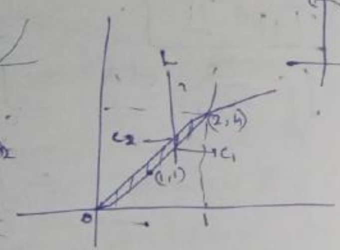
$$\int P\, dx + \int Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Q 1) Verify Green's Theorem for line $\int$ $\oint_C (xy\, dx + x^2 dy)$ where C is the curve enclosed in the region bounded by the parabola & line $y = x^2$, $y = x$ in counter (clis)



A)

(c_1) $y = x^2$
$x = 0$, $y = 0$
$x = 1$, $y = 1$
$x = 2$, $y = 4$



(c_2) $y = x$
$x = 0$ $y = 0$
$x = 1$ $y = 1$
$x = 2$ $y = 2$

$c = c_1 \cup c_2$

(para) eq

on c_1 let $x = t$, $dx = dt$ (with respect to t)

$y = t^2$
$dy = 2t\, dt$ → $y = t^2$, $\frac{dy}{dt} = 2t$, $dy = 2t\, dt$

$0 \le t \le 1$

Green's T.
$0 \to 1$
$1 \to 0$ (oppo).

on c_2 $\oint_c P\, dx + Q\, dy$ $1 \le t \le 0$.

$y = t$

$\therefore x = t$ $y = t$
$dx = dt$ $dy = dt$.

$P = xy$
$Q = x^2$.
$\Rightarrow \int_{c_1 \cup c_2} xy\, dx + x^2 dy$

$C = C_1 \cup C_2$.

~~$= \int_{c_1} xy\, dx + \int_{c_2} x^2 dy$ $+ \int_{c_2} xy\, dx + x^2 dy$~~

~~$= \int_0^1 xy\, dx + \int_1^0 x^2 dy + \int_1^0 xy\, dx + x^2 dy$~~

~~$= \int_0^1 t \cdot t^2 dt + \int_0^1 t^2 \cdot 2t\, dt + \int_1^0 t \cdot t^2 dt +$~~

~~$= \int_0^1 t^3 dt - \int_0^1 2t^3 dt$~~

$\int_1^0 \Rightarrow 0 \ldots$
$\& \; -\int_0^1$

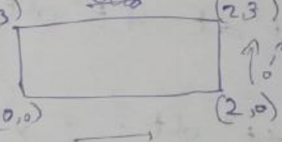
~~$= \left[\frac{t^4}{4}\right]_0^1 - 2\left[\frac{t^4}{2}\right]_0^1$~~

~~$= \frac{1}{4} - 0 - \left(0 - \frac{1}{2}\right) = \frac{1}{4} \left(\frac{-1}{2}\right)$~~

~~$= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$~~

2) Using Green's theorem, evaluate $\oint x^2 e^y\, dx + y^2 e^x\, dy$ by counter clock wise around the bounded in C, where C → ▭ with vertices (0,0) (2,0) (2,3) (0,3)

$0 \le x \le 2$
$0 \le y \le 3$

A) 

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int P\, dx + Q\, dy$$

$P = x^2 e^y$
$Q = y^2 e^x$

$\Rightarrow \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$\frac{\partial Q}{\partial x} = y^2 e^x$ $\Rightarrow \iint_R$

$\Rightarrow \int_0^2 \int_0^3 (y^2 e^x - x^2 e^y)\, dy\, dx$

$\frac{\partial P}{\partial y} = x^2 e^y$

$\Rightarrow \int_0^2 \left[\frac{y^3}{3} e^x - x^2 e^y\right]_0^3 dx.$

$$= \int_0^2 \left[\frac{27}{3} e^x - x^2 e^3 + x^2\right] dx.$$

$$= \int_0^2 9e^x - x^2 e^3 dx. = \left(9e^x - \frac{x^3 e^3}{3} + \frac{x^3}{3}\right)_0^2$$

$$= 9e^2 - \frac{8e^3}{3} - \frac{8}{3} + \frac{9}{1}$$

$$= 9e^2 - \frac{8e^3}{3} - \frac{19}{3}$$

① → continous.

$$\oint_{C_1} p\,dx + q\,dy \qquad 1 \le t \le 0$$

$$P = xy \qquad C = C_1 \cup C_2.$$
$$Q = x^2.$$

$$= \int_{C_1 \cup C_2} xy\,dx + x^2 dy.$$

$$= \int_{C_1} xy\,dx + x^2 dy + \int_{C_2} xy\,dx + x^2 dy$$

Rule

$$= \int_0^1 t \cdot t^2 dt + t^2 \cdot 2t\,dt + \int_1^0 t \cdot t\,dt + t^2 dt$$

$$= \int_0^1 t^3 dt + 2t^3 dt + \int_1^0 t^2 dt + t^2 dt$$

$$= \int_0^1 t^3 + 2t^3 dt + \int_1^0 t^2 + t^2 dt.$$

$$= \int_0^1 3t^3 dt + \int_1^0 2t^2 dt.$$

$$\cancel{= 3\left[\frac{t^4}{4}\right]_0^1 + 2\left[\frac{t^3}{3}\right]_1^0 = \frac{3}{4} + \frac{2}{3}}$$

$$= \int_0^1 3t^3 dt - \int_0^1 2t^2 dt.$$

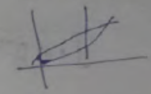
$$\left(\frac{2t^3}{3}\right)_1^0 \quad \left(\frac{3t^4}{4}\right)_0^1 \quad \left(0 - \frac{2}{3}\right) \quad \left(-\frac{2}{3}\right)$$

$$= \frac{1}{12}$$

$$= \left[3\frac{t^4}{4} - 2\frac{t^3}{3}\right]_0^1$$

$$= \frac{3}{4 \times 3} - \frac{2}{3 \times 4} = \frac{9 - 8}{12} = \frac{1}{12}$$

$$y = x^2, \quad y = x$$
$$x^2 \le y \le x.$$



$$P = xy \quad Q = x^2$$

$$x: 0 \to 1$$
$$y: x^2 \to x.$$

$$\frac{\partial Q}{\partial x} = 2x \qquad \frac{\partial P}{\partial y} = x.$$

$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_0^1 \int_{x^2}^x (2x - x)\,dy\,dx.$$

$$= \int_0^1 \int_{x^2}^x x\,dy\,dx = \int_0^1 \left[xy\right]_{x^2}^x dx.$$

$$\cancel{= \int_0^1 \frac{x^3}{2} - \frac{x^4}{2} dx} = \int_0^1 x \cdot x - x \cdot x^2 dx.$$

$$= \int_0^1 x^2 - x^3 dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

LHS = RHS

3) $\oint_C y^2 dx + x^2 dy$

C is the boundary of Δ bounded

$x = 0$, $x + y = 1$, $y = 0$ in counter clockwise

(dis?)

4) $y = 2x + 1$, $-1 \le x \le 0$

$G(x,y) = 3x^2 + 6y^2$ $\quad \int_C G(x,y)\,ds$?.

ii) $\int_C G(x,y)\, ds = \int_a^b G(x, f(x)) \sqrt{1+(f'(x))^2}\, dx$ —①

$f(x) = 2x+1$ $(f'(x))^2 = 4$

$f'(x) = 2$

$1+(f'(x))^2 = 1+4 = 5$

$\sqrt{1+(f'(x))^2} = \sqrt{5}$

$G(x, f(x)) = G(x, 2x+1)$

$= 3x^2 + 6(2x+1)^2 \leftarrow G(x,y) = 3x^2 + 6y^2$

$= 3x^2 + 6[(2x)^2 + 2\cdot 2x \cdot 1 + 1^2]$

$= 3x^2 + 6[4x^2 + 4x + 1]$

$= 3x^2 + 24x^2 + 24x + 6.$

$= 27x^2 + 24x + 6$

—①

$= \int_{-1}^{0} (27x^2 + 24x + 6)\sqrt{5}\, dx.$

$= \sqrt{5}\int_{-1}^{0} (27x^2 + 24x + 6)\, dx.$

$= \sqrt{5}\left[\cancel{27}^{9}\frac{x^3}{\cancel{3}} + \cancel{24}^{12}\frac{x^2}{\cancel{2}} + 6x\right]_{-1}^{0}$

$= \sqrt{5}\left[9x^3 + 12x^2 + 6x\right]_{-1}^{0}$

$= \sqrt{5}\left[0 - \left[(9(-1)^3 + 12(-1)^2 + 6\times(-1)\right]\right]$

$= \sqrt{5}\left[9 - 12 + 6\right] = \sqrt{5}\,[3]$

$= 3\sqrt{5}$

5) $x = \left(\frac{1}{3}\right)t^3$, $y = t^2$, $z = 2t$ $0 \le t \le 1$

$\int_C 4xyz\, dy$?

$dx = f'(t)\,dt = t^2$

$dy = g'(t)\,dt = 2t\, dt$

$dz = h'(t)\,dt = 2\,dt$

ii) $f(t) = \frac{1}{3}t^3$ $h(t) = 2t$

$g(t) = t^2$

$f'(t) = \frac{1}{3}\cdot 3t^2 = t^2$

$h'(t) = 2$

$g'(t) = 2t$

$\int_C 4xyz\, dy = \int_C G(x,y,z)\, dx = \int_a^b G(f(t), g(t), h(t))\, g'(t)\, dt.$

$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t \cdot 2t\, dt$

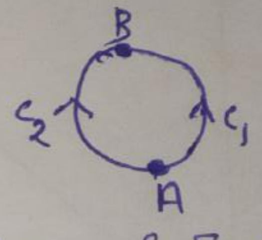
$= \frac{16}{3}\int_0^1 t^7\, dt = 16\left[\frac{t^8}{8}\right]_0^1$

$= 2/3$

$= 2/3$

⇒ Integrals around closed paths :-

In an open connected region 'R' $\int_C F \cdot dr$ is independent of path, if and only if $\int_C F \cdot dr = 0$ for every closed path C in R.



$$\int_C F \cdot dr = \int_{C_1} F \cdot dr - \int_{C_2} F \cdot dr = 0$$

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

If/ Suppose that.
$\int_C F \cdot dr$ is independent of path of $\int$-tion C is R. Let C be any ~~$\oint$ R~~ closed curve R, let A & B be a points in C. we get 2 paths from ~~A to~~ A to B say C_1

& c_2 along c. Such that I say: c has the same orientation. then by the (Pro) of line $\int$.

$\int_c F\cdot dr = \int_{c_1} F\cdot dr - \int_{c_2} F\cdot dr$.

$\int_{c_1} F\cdot dr = \int_{c_2} F\cdot dr$.

$\int_c F\cdot dr = \int_{c_1} F\cdot dr - \int_{c_2} F\cdot dr = 0$.

Let us assume that the $\int$ around any closed curve c in R is 0. Give any 2 points → A & B, & c_1 & c_2 from A to B in R, we see that c_2 with orientation reversed & c_1 together,

(Th 137)

$c = c_1 \cup (c_2)$.

$$0 = \int_c F\cdot dr = \int_{c_1 \cup (c_2)} F\cdot dr = \int_{c_1} F\cdot dr + \int_{-c_2} F\cdot dr$$

$$= \int_c F\cdot dr - \int_{c_2} F\cdot dr.$$

$$\int_{c_1} F\cdot dr = \int_{c_2} F\cdot dr$$

If $\int_c F\cdot dr = 0$ for every closed path c in R line $\int$, $\int_c F\cdot dr$ is independent of path.

Q) $\int_c 4xyz\, ds$. $x = (1/3)t^3$ $y = t^2$ $z = 2t$

$0 \le t \le 1$

A) $\int_c 4xyz\, ds = \int (f(t), g(t), h(t)) \sqrt{(f'(t))^2 (g'(t))^2 (h'(t))^2}$

$$= \int_0^1 4\cdot(t^3/3)\, t^2 \cdot 2t \cdot (t^2+2)\, dt.$$

$$= \int_0^1 \frac{4t^3}{3}\, 2t \cdot t^2 (t^2+2)\, dt.$$

$$= \frac{8}{3} \int_0^1 t^3 \cdot t^1 \cdot t^2 (t^2+2)\, dt$$

$$= \frac{8}{3} \int_0^1 t^6 (t^2+2)\, dt.$$

$$= \frac{8}{3} \int_0^1 t^8 + 2t^6\, dt.$$

$$= \frac{8}{3} \left[\frac{t^9}{9} + 2\frac{t^7}{7} \right]_0^1$$

$$= \frac{8}{3} \left[\frac{1}{9} \cancel{\times} + \frac{2}{7} \right] = \frac{8}{3} \left[\frac{25}{63} \right]$$

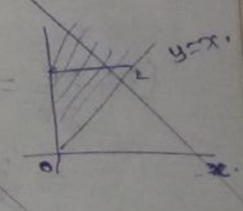
$$= \frac{200}{189}$$

Q) Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$.

R: $x \le y \le \infty$, $0 \le x \le \infty$.

A) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$.

$= \int_0^\infty \left[\int_x^\infty \right.$



⇒ Test for conservative field :-

let $f = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$ whose component ( ) hue continu... 1st partial derivation, then f is conservative if and only if,

(1) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (2) $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ (3) $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

A) 1

$f = Pi + Qj + Rk = \nabla \phi$.

$= \underset{P}{\frac{\partial \phi}{\partial x}}\, i + \underset{Q}{\frac{\partial \phi}{\partial y}}\, j + \underset{R}{\frac{\partial \phi}{\partial z}}\, k.$

$\frac{\partial P}{\partial y} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ proved

(2) $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

$\Rightarrow \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial R}{\partial y}.$

$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ //.

(3) $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right)$

$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

* Corollary-1

Suppose $f(x,y) = P(x,y)\,i + Q(x,y)\,j$ be a field whose compound ( ) hue contin... 1st partial derivatives, then F is conservative if and only if,

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.}$$

⇒ Physical eg of line $\int$:-

Total workdone $\longrightarrow w = \int_c F \cdot dr$

Q) find total work done in moving a particle in a force gvn by,
$F = 3xy\,i - 5z\,j + 10x\,k$ along the curve.
$x = t^2+1$ $y = 2t^2$ $z = t^3$ from $t = 1$ to $t = 2$.

A) $F = 3xy\,i - 5z\,j + 10x\,k$ $1 < t < 2$

$dr = dx\,i + dy\,j + dz\,k.$

$x = t^2+1$ $y = 2t^2$ $z = t^3$

$dx = 2t\,dt$ $dy = 4t\,dt.$ $dz = 3t^2\,dt$

$w = \int_c F \cdot dr. = \int_c (3xy\,i - 5z\,j + 10x\,k) \cdot (dx\,i + dy\,j + dz\,k)$

$= (3xy \cdot dx)\,i - (5z \cdot dy)\,j + (10x \cdot dz)\,k.$

$= 3xy\,dx - 5z\,dy + 10x\,dz.$

Sub

$= \int_1^2 3 \cdot (t^2+1)\, 2t^2 \cdot 2t\,dt - 5\,t^3 \cdot 4t\,dt + 10(t^2+1)\, 3t^2\,dt.$

$= \int_1^2 3t^2 + 3 \cdot 2t^2 \cdot 2t\,dt - 20\,t^4\,dt + 10t^2 + 30t^2\,dt.$

$12t^5 + 12t^3$

$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$

$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$

$= (28+64+48+80) - (2+2+3+10)$

$= 303$ units of work

* **Corollary - 2 →**

$F = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$ is conservative if and only if curl $F = 0$

Pf

$$\text{curl } F = \nabla \times F$$

$$= \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \cdot \times (Pi + Qj + Rk)$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= i\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] - j\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + k\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right].$$

$$= \left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] i + \left[\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right] j + \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] k.$$

* **Rmrk →**

F is conservative, we usually want to find a potential (?) for F. This requires solving the eq $\nabla\phi = F$.

$$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = Pi + Qj + Rk.$$

$$\frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q, \quad \frac{\partial \phi}{\partial z} = R.$$

Q.1) S.T $F = [e^x \cos y + yz]\,i + [xz - e^x \sin y]\,j + (xy+z)\,k$

A) $P = e^x \cos y + yz$.

$Q = xz - e^x \sin y$

$R = (xy + z)$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(e^x \cos y + yz)$$

$$= -e^x \sin y + z = \frac{\partial Q}{\partial x}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(xz - e^x \sin y) = z - e^x \cos y.$$

$$\frac{\partial R}{\partial x} = y = \frac{\partial P}{\partial z}$$

Therefore by the component test for conservative field, F is conservative i.e. hence there is a (?) ϕ such that $\nabla\phi = F$.

$$\nabla\phi = F \Rightarrow \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = Pi + Qj + Rk$$

~~$Pi + Qj + Rk$.~~

$P = e^{x}\cos y + yz$

$$\Rightarrow \frac{\partial \phi}{\partial x} = P, \quad \frac{\partial \phi}{\partial y} = Q, \quad \frac{\partial \phi}{\partial z} = R.$$

$$\frac{\partial \phi}{\partial x} = e^x \cdot \cos y + yz \quad \text{— (1)}$$

$$\frac{\partial \phi}{\partial y} = xz - e^x \sin y \quad \text{— (2)}$$

$$\frac{\partial \phi}{\partial z} = xy + z \quad \text{— (3)}$$

$\int$-ing (1) with respect to x, y & z constant.

$$\phi(x,y,z) = e^x \cos y + xyz + g(y,z) \quad \text{— (4)}$$

the arbitrary constant of function $g(y,z)$ is free from x.

differentiating ④ with respect to y,

$$\frac{\partial\phi}{\partial y} = -e^x \sin y + xz + \frac{\partial g}{\partial y} \quad — ⑤$$

Comparing ② & ⑤ $\qquad \frac{\partial\phi}{\partial y} = xz - e^x \sin y$

we get $\frac{\partial g}{\partial y} = 0$

hence g is a () of z,

let $g(y,z) = h(z)$

$$\phi(x,y,z) = e^x \cos y + xyz + h \quad — ⑥$$

differentiating ⑥ w.r.to z

$$\frac{\partial\phi}{\partial z} = xy + \frac{dh}{dz}$$

Comparing ③ & ⑦

we get $\frac{dh}{dz} = x \qquad h(z) = (z^2/2) + c$

then from —⑥

$$\phi(x,y,z) = e^x \cos y + xyz + (z^2/2) + c$$

2) S.T field $F = (z+y)i + zj + (y+x)k$ is not conservative.

A) $P = (z+y)$, $Q = z$, $R = y + x$.

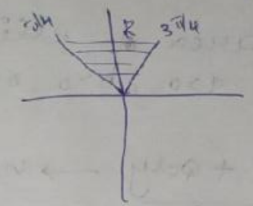
$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(z) = 0$$

$$\therefore \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

F is not conservative

3) find the workdone by the force, $F = (-16y + \sin x^2)i + (4e^y + 3x^2)j$ acting along simple closed curve C, boundary of the region enclosed by ○, $x^2+y^2=1$ & lines $y = -x$ & $y = x$ above x axis in counter clockwise (dir)?

A) workdone, $\boxed{W = \oint_C F \cdot dr}$. force × displacement.



$F = (-16y + \sin x^2)i + (4e^y + 3x^2)j$ (P, Q)

$$\frac{\partial Q}{\partial x} = 6x$$

$$\frac{\partial P}{\partial y} = -16$$

$$\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dy = \int P\,dx + Q\,dy$$

$$= \iint_R 6x + 16\, dA.$$

eq ○ on area

$x^2 + y^2 = 1$

$x = r\cos\theta \quad y = r\sin\theta$

$(r\cos\theta)^2 + (r\sin\theta)^2 = 1$

$r^2\cos^2\theta + r^2\sin^2\theta = 1$

$r^2(\cos^2\theta + \sin^2\theta) = 1$

$r^2 = 1$

$r = 1$

$0 \le r \le 1 \qquad \pi/4 \le \theta \le 3\pi/4$

$$W = \iint_R 6x + 16\, dA$$

$$= \int_{\pi/4}^{3\pi/4}\int_0^1 (6r\cos\theta + 16)\, r\,dr\,d\theta \qquad \iint 6r^2\cos\theta + 16r$$

$dA = r\,dr\,d\theta$

$$= \int_{\pi/4}^{3\pi/4} \left[\frac{{}^2\cancel{6}\, r^3}{\cancel{3}} \cos\theta + {}^8\cancel{16}\frac{r^2}{\cancel{2}}\right]_0^1 d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \left[2r^3\cos\theta + 8r^2\right]_0^1 d\theta = \int (2\cos\theta + 8)\, d\theta$$

$$= \left[2\sin\theta + 8\theta\right]_{\pi/4}^{3\pi/4} = \left[2\sin 3\pi/4 + 8\cdot 3\frac{\pi}{4} - \left(2\sin\frac{\pi}{4} + 8\frac{\pi}{4}\right)\right]$$

$= 2\sin 3\pi/4 + 8 \cdot 3\pi/4 - (\sin\frac{\pi}{2} + 2\pi)$

$= 6.45 + 24.45 - (1 + 2.180)$

4) a) Let R be the region bounded by a piecewise smooth simple closed curve using green's .T, .P.T $\frac{1}{2}\oint_C(-ydx + xdy)$ ⓐ

= area of R.

b) using ⓐ, find the area of ellipse $x = a\cos t$, $y = b\sin t$, $a>0$, $b>0$, $0 \le t \le 2\pi$

A) $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \underbrace{\int Pdx + Qdy}_{①} \longrightarrow$ G.T. ⓑ

a) $\frac{1}{2}\oint_C -y\,dx + x\,dy$ is in the form

$\oint Pdx + Qdy.$ $P = -y$, $Q = x$

$\therefore \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x) = 1$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(-y) = -1.$

$\frac{1}{2}\oint_C Pdx + Qdy.$ ——ⓐ

$\therefore \frac{1}{2}\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$

[① ഒന്നാമത്തെ ~ ഉത്തരം
② രണ്ട് ഒന്നി'] ⓐ → ⓑ

$\Rightarrow \frac{1}{2}\iint_R 1 - (-1)\, dA.$

$\Rightarrow \frac{1}{2}\iint_R 2\, dA.$

$= \iint_R dA.$ = area of region R

b) $\frac{1}{2}\oint_C(-ydx + xdy)$ — ellipse eq.,

$x = a\cos t$ $y = b\sin t$.

$dx = -a\sin t\,dt$ $dy = b\cos t\,dt$.

$= \frac{1}{2}\oint -b\sin t\,dt\; a\cos\, dy.$

$= \frac{1}{2}\int_0^{2\pi} -b\sin\cdot -a\sin t\,dt + a\cos t \cdot b\cos t\,dt.$

$= \frac{1}{2}\int_0^{2\pi}(ab\sin^2 t\,dt + ab\cos^2 t)dt.$

$= \frac{1}{2}\int_0^{2\pi} ab(\sin^2 t + \cos^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt.$

$= \frac{1}{2}ab\int_0^{2\pi} 1\,dt. = \frac{1}{2}ab[t]_0^{2\pi}$

$= \frac{1}{2}ab[2\pi - 0] = \frac{1}{\cancel{2}}ab[\cancel{2}\pi]$

$= \pi ab.$

⇒ Extension of Green's .T to regions with holes :-

$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_{R_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA + \iint_{R_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA.$

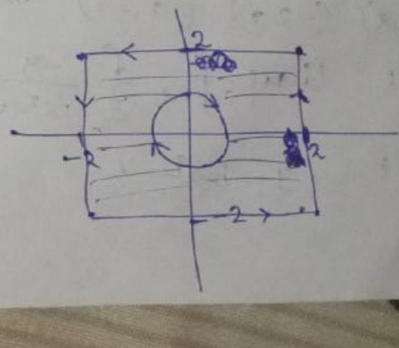
$\Rightarrow \int_C Pdx + Qdy = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy.$

i) Evaluate line $\int$, $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, $C = C_1 \cup C_2$, C_1 is □ with vertices $(2,-2), (2,2), (-2,2), (-2,-2)$ in the counter clock wise (dir), i.e C_2 is the ○ $x^2+y^2=1$ in clock wise (dir).

counter c.w. dir

c.w dir

A) $x^2+y^2 = \frac{1}{r}$



$P = \frac{-y}{x^2+y^2}$ $\quad Q = \frac{x}{x^2+y^2}$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$

(Q. rule) $= \frac{(x^2+y^2)\cdot -1 - -y \cdot 2y}{(x^2+y^2)^2}$

$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$

by G.T.,

$\int \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \iint_R \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}\,dA$

$= \iint_R \frac{-x^2+y^2}{(x^2+y^2)^2} - \frac{-x^2+y^2}{(x^2+y^2)^2}\,dA = 0$

$\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$ — (a)

$\frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$ — (b)

5) let C is □ with vertices $(2,-2)$ $(2,2)$ $(-2,2)$ $(-2,-2)$ in counter clock. w (dir)

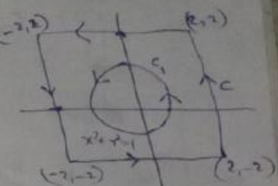
Evaluate line $\int$, $\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

A) $x = a\cos t$, $y = b\sin t$;

(pg-187) $P = \frac{-y}{x^2+y^2}$ $\quad Q = \frac{x}{x^2+y^2}$;

$\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad \frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$

Let C_1 denote $x^2+y^2=1$.



$\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$

$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \oint_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ — ①

$C_1 \rightarrow$ ○, $r = 1$

$\therefore x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

on C_1, $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

$dx = -\sin t\,dt$, $dy = \cos t\,dt$.

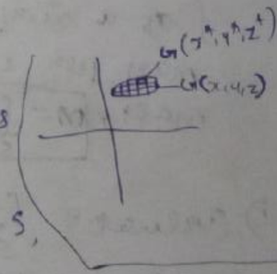
$\therefore \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \oint_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$

$= \int_0^{2\pi} (-\sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)$ $\quad$ ($x^2+y^2=1$) $\cos^2 t + \sin^2 t = 1$

$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$

⇒ Surface $\int$:-

Let G be a ( ) of 3 variables defined over a region of 3 space containing surface S. Then S. $\int$ of G over S



$$\boxed{\iint_S G(x,y,z)\,dS = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*)\,\Delta S_k}$$

* Methods of evaluation of S.∫ :-

(1) If $z = f(x,y)$ (2 variables $\rightarrow x, y$)

$$\iint_S G(x,y,z)\, ds = \iint_R G(x,y,f(x,y)) \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\, dA$$

$$\left(ds = \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\, dA\right)$$

(2) If $y = g(x,z)$

$$\iint_S G(x,y,z)\, ds = \iint_R G(x, g(x,z), z) \sqrt{1+g_x(x,z)^2+g_z(x,z)^2}\, dA$$

(3) If $x = h(y,z)$

$$\iint_S G(x,y,z)\, ds = \iint_R G(h(y,z),y,z) \sqrt{1+h_y(y,z)^2+h_z(y,z)^2}\, dA$$

If $x = h(y,z)$ is the eq of the surface S that project into a region R of (y,z) plane & if G, f, h_y, h_z are continuous throughout a region containing S.

⇒ Mass of a surface :-

Suppose $\delta(x,y,z)$ represents the δ of a surface at any point / mass per unit area, then

mass, $\boxed{M = \iint_S \delta(x,y,z)\, ds}$

density $(\delta) = \dfrac{\text{mass}}{\text{vol}}$

mass $= \int \delta \times \text{vol}$

$= \int \delta(x,y,z)\, ds$.

1) Evaluate $\iint_S xz\, ds$, where S is the part of the plane $x+y+z=1$ that lies in the 1st octant ?

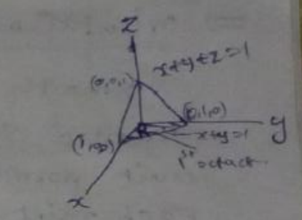
A) draw plane

$x+y+z = 1$

$x=0,\ y=0,\ z=1$ $(0,0,1)$

$x=0,\ z=0,\ y=1$ $(0,1,0)$

$x=1,\ y=0,\ z=0$ $(1,0,0)$



R is the region bounded by lines $x=0, y=0, x+y=1$

$x+y=1$

$y = 1-x$

$0 \le y \le 1-x$, $0 \le x \le 1$

$x+y+z=1$

$z = 1-x-y$

$z = f(x,y) = 1-x-y$. $G(x,y) = xz$

$f_x(x,y) = -1$

$f_y(x,y) = -1$

$\therefore$ eq $\rightarrow \iint_S G(x,y)\, ds = \iint_S G(x,y,z)$

$= \iint_R G(x,y,f(x,y)) \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\, dA$

$f(x,y) = 1-x-y$

$$= \int_0^1 \int_0^{1-x} \underbrace{(1-x-y)x}_{xz} \sqrt{1+(-1)^2+(-1)^2}\, dy\, dx.$$

$$= \iint x - x^2 - yx \sqrt{3}\, dy\, dx.$$

$$= \sqrt{3} \int_0^1 \left[xy - x^2 - \frac{y^2}{2}x\right]_0^{1-x} dx.$$

$$= \sqrt{3} \int_0^1 \left[x(1-x) - x^2(1-x) - \frac{1}{2}x(1-x)^2\right] dx$$

$$= \sqrt{3} \int_0^1 \left[\frac{x}{2} - x^2 + \frac{1}{2}x^3\right] dx$$

$$= \sqrt{3} \left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1$$

$$= \sqrt{3} \left[\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right] = \frac{\sqrt{3}}{24}$$

⇒ **orientation of a surface** :-

A smooth surface 's' is orientable / is an oriented surface if there exist a continuous unit normal vector () 'n' defined at each point (x,y,z) on the surface.

The vector field $n(x,y,z)$ → the orientation of 's'.

Since a unit normal to the surface 's' at (x,y,z) can be either $n(x,y,z)$ /(or) $-n(x,y,z)$, an orientable surface has 2 orientation.

* A surface 's' is defined by $z = f(x,y)$ has an upward orientation, when the unit normals directed upward (ie) have +ve k components & has downward orientation when the unit normals are directed downward (ie) have -ve k components.

$$g(x,y,z) = 0$$

(unit normal) → $\hat{n} = \dfrac{\nabla g}{\|\nabla g\|}$ $\quad \nabla g$.

~~$\nabla g = \hat{n} \times \|\nabla g\|$~~

If a smooth surface 's' is defined by $g(x,y,z) = 0$ then a unit normal,

$$\hat{n} = \frac{\nabla g}{\|\nabla g\|}$$

(gradient of g) $\nabla g = \dfrac{\partial g}{\partial x} i + \dfrac{\partial g}{\partial y} j + \dfrac{\partial g}{\partial z} k.$

* If 's' is defined by $z = f(x,y)$, then one can use, $\boxed{\begin{aligned} g(x,y,z) &= z - f(x,y) = 0 \\ g(x,y,z) &= f(x,y) - z = 0 \end{aligned}}$ ① (or)

depending on the orientation of S.

eg →

Q) consider the sphere of radius $a > 0$ defined by $x^2+y^2+z^2 = a^2$ by ①.

A) $g(x,y,z) = x^2+y^2+z^2-a^2$.

$$\nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k$$

$$= 2xi + 2yj + 2zk$$

$$\|\nabla g\| = \sqrt{(2x)^2+(2y)^2+(2z)^2} = \sqrt{4x^2+4y^2+4z^2}$$

$$= \sqrt{4\underbrace{(x^2+y^2+z^2)}_{a^2 \to a}} = \sqrt{4a^2} = 2a$$

$$\therefore n = \frac{\nabla g}{\|\nabla g\|} = \frac{2xi+2yj+2zk}{2a} = \frac{\cancel{2}xi}{\cancel{2}a} + \frac{\cancel{2}yj}{\cancel{2}a} + \frac{\cancel{2}zk}{\cancel{2}a}$$

$$n = \frac{xi+yi+zk}{a} = \frac{x}{a}i + \frac{y}{a}j + \frac{z}{a}k$$

and

$$-n = -\frac{x}{a}i - \frac{y}{a}j - \frac{z}{a}k$$

⇒ **flux** =

The total vol of a fluid passing through S per unit time — flux of 'f' through 's'

& is gvn by, $\iint_S f \cdot n \, ds$

Suppose S is the orientable piecewise smooth closed surface bounded by the paraboloid

S_1, $z = x^2+y^2$ &

S_2, $z = 1$ then flux,

$$\boxed{\iint_{S_1} f \cdot n\, ds + \iint_{S_2} f \cdot n\, ds = \iint_S f \cdot n\, ds}$$

where S_1 oriented downward

where S_2 " upward.

1) Suppose that a curved lamina 'S' with constant density $\rho(x,y,z) = k$ is the portion of the paraboloidal $z = x^2+y^2$ below the plane $z = 1$, find mass of lamina?

A) $z = x^2+y^2$ we know that $z = f(x,y)$.

$\therefore f(x,y) = x^2+y^2$.

$$\text{mass} = \iint_S \rho(x,y,z)\, ds. \qquad z = f(x,y)$$

$$= \iint \rho(x,y,z) \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}$$

$f(x,y) = x^2+y^2$

$f_x = 2x$

$f_y = 2y$.

$$= \iint_R k \sqrt{1 + (2x)^2 + (2y)^2}\, dA.$$

$$= \iint_R k \sqrt{1 + 4x^2 + 4y^2}\, dA$$

$\int$ limit

$z = x^2+y^2$, $z = 1$

$x^2+y^2 = 1 \longrightarrow r = 1$

$0 \le r \le 1$ $\qquad 0 \le \theta \le 2\pi$ $\qquad$ (dA & r means $r\,dr\,d\theta$)

$x = r\cos\theta$

$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^1 k \sqrt{1 + 4(r\cos\theta)^2 + 4(r\sin\theta)^2}\; r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 k \sqrt{1 + 4r^2\cos^2\theta + 4r^2\sin^2\theta}\; r\,dr\,d\theta.$$

$$\cancel{\int_0^{2\pi}\int_0^1 k \sqrt{r + 4r^3\cos^2\theta + 4r^3\sin^2\theta}}$$

$$k\int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2(\cos^2\theta + \sin^2\theta)}\; r\,dr\,d\theta$$

$$k\int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2} \cdot r\,dr\,d\theta.$$

Rule method

put $1 + 4r^2 = u$

$r = 0$, $r = 1$

$\therefore u = 0$, $r = 5$

$8r\,dr = du$

$r\,dr = \frac{du}{8}$

$$k\int_0^{2\pi}\int_1^5 \sqrt{u}\, \frac{du}{8} \cdot d\theta.$$

$$\frac{k}{8}\int_0^{2\pi}\int_1^5 \sqrt{u}\, du\, d\theta$$

$u^{1/2} = \frac{u^{3/2}}{3/2} = \frac{2}{3}u^{3/2}$

$$\frac{k}{8}\int_0^{2\pi} \left[\frac{2}{3}u^{3/2}\right]_1^5 d\theta \; =$$

$$\frac{k}{8_4}\cdot\frac{2}{3}\int_0^{2\pi} \left[u^{3/2}\right]_1^5 d\theta$$

$$\frac{k}{12}\int_0^{2\pi} 5^{3/2} - 1\; d\theta = \frac{k}{12}\int_0^{2\pi} (5\sqrt{5} - 1)\, d\theta.$$

$$= \frac{k}{12}(5\sqrt{5}-1)\int_0^{2\pi} d\theta = \frac{k}{12}(5\sqrt{5}-1)(\theta)_0^{2\pi}$$

$$= \frac{k}{\cancel{12}_6}(5\sqrt{5}-1)\,\cancel{2}\pi \qquad = \frac{k}{6}(5\sqrt{5}-1)\pi$$

$$= \frac{k\pi}{6}(5\sqrt{5}-1)$$

2) find mass of surface of paraboloid $z = 1 + x^2 + y^2$ in the 1^{st} octact for $1 \le z \le 5$, if the density at a point 'P' on the surface is directly proportional to its distance from the xy plane ?

⇒ Vector form of Green's Theorem :-

$f(x,y) = P(x,y)i + Q(x,y)j$

$F \cdot dr = (P(x,y)i + Q(x,y)j) \cdot (dx\,i + dy\,j)$

$= P(x,y)\,dx + Q(x,y)\,dy$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$

$$= i\left|\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}Q\right| - j\left|\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}P\right| + k\left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)$$

$$= \underbrace{-\frac{\partial Q}{\partial z}i + \frac{\partial P}{\partial z}j}_{z\ [illegible]\ 0} + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\boxed{\oint_C F \cdot dr = \oint_C F \cdot T\, ds = \iint_R \text{Curl} F \cdot k\, dA} \text{ — eq.}$$

→ Stoke's Theorem :-

let 'S' be a piecewise smooth orientable surface bounded by a piecewise smooth simple closed curve 'C'.

$F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$ be a v.field for which P,Q,R are continuous & have continuous 1^{st} partial derivative in a region of space containing f. if C is traversed in the +ve (dir) then

$$\boxed{\oint_C F \cdot dr = \oint_C F \cdot T\, ds = \iint_S \text{curl} F \cdot n\, ds}$$

where n is a unit normal to S in the (dir) of the orientation of S.

1) let S be the part of cylinder $z = 1 - x^2$ for $0 \le x \le 1$, $-2 \le y \le 2$, varify Stock's therem for the v.field $f = xyi + yzj + xzk$. Assume S is oriented upward.

A) make 4 curves.

$c_1 \to (1,-2,0)$ to $(1,2,0)$ y limit change.

$c_2 \to (1,2,0)$ to $(0,2,1)$ x limit "

$c_3 \to (0,2,1)$ to $(0,-2,1)$ y limit "

$c_4 \to (0,-2,1)$ to $(1,-2,0)$ x limit "

$$\oint_C F \cdot dr = \oint_C F \cdot T\, ds = \iint_S \text{curl} f \cdot n\, ds.$$

$f = xyi + yzj + xzk$

$$\text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & xz \end{vmatrix}$$

$$= i\left|\frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(yz)\right| - j\left|\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial z}(xy)\right|$$

$+ k \left| \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy) \right|$

$\iint_R \text{curl } F \cdot n \, ds$

$\text{curl } F = -y\, i - z\, j - x\, k.$

$g(x,y,z) = z + 1 - x^2 = 0$ $\quad z = 1 - x^2 = 0$

$= z + 1 - x^2 = 0$

(unit normal vector) $n = \frac{\nabla g}{\|\nabla g\|}$

$\nabla g = \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k.$

$= 2x\, i + 0 j + 1 k \quad = 2x\, i + 1k.$

$\|\nabla g\| = \sqrt{(2x)^2 + (k)^2} = \sqrt{4x^2 + 1^2} = 2x + 1$

$\therefore n = \frac{2x i + k}{\sqrt{4x^2+1}} = \frac{2x}{\sqrt{4x^2+1}} i + \frac{k}{\sqrt{4x^2+1}} j.$

$\text{curl } F \cdot n = (-y\, i - \underset{0}{z\, j} - x\, k) \cdot \left(\frac{2x}{\sqrt{4x^2+1}} i + \frac{k}{\sqrt{4x^2+1}} j \right)$

$= \frac{-2xy}{\sqrt{4x^2+1}} + \frac{-x}{\sqrt{4x^2+1}}$

$\iint_S \text{curl } F \cdot n \, ds = ?$ $\quad$ we know $z = f(x,y)$

$f(x,y) = 1 - x^2$

$f_x = 2x \quad f_y = 0$

$ds \to (x,y)$

$\iint_R \frac{-2xy}{\sqrt{4x^2+1}} + \frac{-x}{\sqrt{4x^2+1}} \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2} \, dA.$

$\int_0^1 \int_{-2}^{2} \frac{-2xy}{\sqrt{4x^2+1}} + \frac{-x}{\sqrt{4x^2+1}} \sqrt{1+4x^2} \, dA = \iint \frac{-2xy + -x}{\sqrt{4x^2+1}} \sqrt{1+4x^2}$

$= \int_0^1 \int_{-2}^{2} -2xy - x \, dy\, dx.$

$= \int_0^1 \left[-xy^2 - xy\right]_{-2}^{2} dx = \int_0^1 -4x \, dx$

$= \left[-2x^2\right]_0^1 = -2$