04: Complex Numbers

Q1) write the following no. in the form $a+ib$

$$\frac{1}{(1+i)(1-2i)(1+3i)}$$

A) $(1+i)(1-2i)(1+3i)$

$(1+i)(1-2i) = 1-2i+i+2 = \underline{3-i}$

$(3-i)(1+3i) = 3+9i+i+3 = \underline{6+8i}$

$\frac{1}{6+8i} \times \frac{\overline{6+8i}}{\overline{6+8i}}$ (x & ÷ by $\overline{6+8i}$)

$\overline{6+8i} = 6-8i$

$= \frac{1 \times 6-8i}{(6+8i)(6-8i)} \longrightarrow z \cdot \bar{z} = |z|^2$

$= \frac{6-8i}{|6-8i|^2} = \frac{6-8i}{6^2+8^2} = \frac{6-8i}{36+64} = \frac{6-8i}{100}$

$= \frac{6}{100} - \frac{8}{100}i$

$= \frac{3}{50} - \frac{2}{25}i$

$\frac{6}{100} = \frac{3}{50}$

$\frac{8}{100} = \frac{4}{50} = \frac{2}{25}$

2) $\frac{(4+5i)+2i^3}{(2+i)^2}$

$2i^3 = 2i^2 \cdot i$

$(i^2) = -1$.

A) $\frac{(4+5i)+2i^2 \cdot i}{2^2+2\times 2i+i^2} = \frac{(4+5i)+-2 \cdot i}{4+4i+i^2}$

$= \frac{4+5i-2i}{4+4i+i^2} = \frac{4+3i}{4+4i-1} = \frac{4+3i}{3+4i}$

$\frac{4+3i}{3+4i} \times \frac{\overline{3+4i}}{\overline{3+4i}} = \frac{(4+3i)(3-4i)}{|3+4i|^2} = \frac{12-16i+9i+12}{9+16}$

$(z \cdot \bar{z} = |z|^2)$

$= \frac{24+7i}{25} = \frac{24}{25} + \frac{7}{25}i$

2) $Im(\bar{z}^2 + z^2)$ Im → imaginary.

A) $z = x + iy$ $\qquad i^2 = -1$

$\bar{z} = x - iy$

$z^2 = (x+iy)^2 = x^2 + 2xiy - y^2$

$\bar{z}^2 = (x-iy)^2 = x^2 - 2xiy - y^2$

$\bar{z}^2 + z^2 = x^2 + \cancel{2xiy} - y^2 + x^2 - \cancel{2xiy} - y^2$

$= 2x^2 - 2y^2$

$= 2(x^2 - y^2)$

$(x+iy)^2$
$= x^2 + 2xiy + (iy)^2$
$= x^2 + 2xiy - y^2$

$(x-iy)^2 = x^2 - 2xiy + (iy)^2$
$= x^2 - 2xiy - y^2$

3) $|z + 5\bar{z}|$

$z = x + iy \qquad \bar{z} = x - iy.$

$5\bar{z} = 5(x - iy) = 5x - 5iy$

$z + 5\bar{z} = x + iy + 5x - 5iy$

$= 6x - 4iy$

$|z + 5\bar{z}| = |6x - 4iy|$ $\qquad |z| = \sqrt{x^2 + y^2}$

$= \sqrt{6x^2 + 4y^2}$

$= \sqrt{36x^2 + 16y^2}$

$= \sqrt{4(9x^2 + 4y^2)}$

4) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$
interpret the result geometrically.

A) $|z_1 + z_2|^2 + |z_1 - z_2|^2 -$ $\qquad |z|^2 = z \cdot \bar{z}$

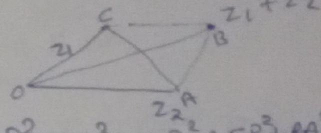
$(z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2})$ $\qquad \overline{z_1 - z_2} = \bar{z_1} - \bar{z_2}$

$= (z_1 + z_2)(\bar{z_1} + \bar{z_2}) + (z_1 - z_2)(\bar{z_1} - \bar{z_2})$

$= z_1\bar{z_1} + \cancel{z_1\bar{z_2}} + \cancel{z_2\bar{z_1}} + z_2\bar{z_2} + z_1\bar{z_1} - \cancel{z_1\bar{z_2}} -$
$\cancel{z_2\bar{z_1}} + z_2\bar{z_2}$

$= 2z_1\bar{z_1} + 2z_2\bar{z_2}$ $\qquad (z_1 \cdot \bar{z_1} = |z_1|^2)$

$= 2|z_1|^2 + 2|z_2|^2$



$OB^2 + AC^2 = OC^2 + CB^2 + BA^2 + OA^2$

5) find complex no. satisfying the eq

$z + 2\bar{z} = \frac{2-i}{1+3i}$

A) $z = x + iy \quad , \bar{z} = x - iy$

$\frac{2-i}{1+3i} = \frac{2-i}{1+3i} \times \frac{\overline{1+3i}}{\overline{1+3i}} = \frac{2-i(1-3i)}{1+3i(1-3i)}$

$= \frac{2-6i-i+3i^2}{1-3i+3i-9i^2} = \frac{2-6i-i+3\times-1}{1-9\times-1}$

$= \frac{2-7i-3}{10} = \frac{-1-7i}{10} = \frac{-1}{10} - \frac{7}{10}i$

$z + 2\bar{z} = (x+iy) + 2(x-iy) = (x+iy) + 2x - 2iy$

$= x + iy + 2x - 2iy = 3x - iy.$

$z + 2\bar{z} = \frac{2-i}{1+3i} \Rightarrow 3x - iy = \frac{-1}{10} - \frac{7}{10}i$

$3x = -\frac{1}{10} \quad , -iy = -\frac{7}{10}i$

$-y = -7/10.$

equating real & img parts,

$x = \frac{-1}{30}$ and $y = \frac{7}{10}$

$z = x + iy \Rightarrow -\frac{1}{30} + i\frac{7}{10}$ //

⇒ polar form of a complex no :-

c.no can also be expressed in terms of polar (co). r, θ. in complex plane defined by $x = r\cos\theta$, $y = r\sin\theta$

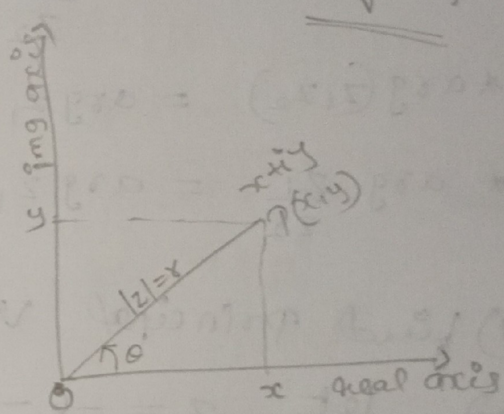
$$\boxed{\begin{aligned} z &= x + iy \\ z &= r(\cos\theta + i\sin\theta) \end{aligned}}$$ ⟶ polar form of c.no.

r → absolute value / modulus of z, & is denoted by $|z|$.

$$|z| = \sqrt{x^2 + y^2} = r.$$

$x^2 + y^2 = r^2$
$r = \sqrt{x^2 + y^2}$

The directed angle θ measured from +axis to OP → argument / Amplitude of z & is denoted by arg z or amp z.



$$\boxed{\theta = \arg z = \tan^{-1}\left|\frac{y}{x}\right|} \quad z \neq 0.$$

* For z=0 this θ is undefined.
For gvn z ≠ 0 it is determined only up to integr multiple of 2π.
* The value of θ that lies in interval $-\pi < \theta \leq \pi$ → principal value of argument of z(≠0) & denoted by Arg z. Thus by definition $\theta = \text{Arg } z$ satisfies, $-\pi < \text{Arg } z \leq \pi$

eg = $\text{Arg}(i) = \frac{\pi}{2}$.

✱ Note
* If (x,y) belongs to 1st quadrant, Arg z = θ.

* If (x,y) belongs to 2nd q, $\operatorname{Arg} z = \pi - \theta$.
* " " to 3rd q, $\operatorname{Arg} z = -(\pi - \theta)$
* " " to 4th q, $\operatorname{Arg} z = -\theta$. ($\theta \to \infty$ also)

Result

let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ & $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

* product $\Rightarrow z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$
$= r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$
$z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

* Division $\Rightarrow \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

* $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

* $\arg \dfrac{z_1}{z_2} = \arg z_1 - \arg z_2$

1) find principal value of $-6-6i$

A) let $z = -6-6i$

we know, $z = x+yi$

$\therefore x = -6, \ y = -6$

principal value, θ.

$\theta = \arg z = \tan^{-1}\left|\dfrac{y}{x}\right|$

$= \tan^{-1}\left|\dfrac{-6}{-6}\right| = \tan^{-1}\left|\dfrac{6}{6}\right| = \tan^{-1} 1$

$= \dfrac{\pi}{4}$

$(x,y) = (-6,-6)$

$\therefore$ 3rd quadrant

II $(-x,y)$	I (x,y)
III $(-x,-y)$	IV $(x,-y)$

$\arg z = -(\pi - \theta)$

$= -\left(\pi - \dfrac{\pi}{4}\right) = -\left(\dfrac{4\pi - \pi}{4}\right)$

Principal value $\Rightarrow = -3\pi/4$ //

2) principal value of $2+2i$

A) $z = 2+2i$, $x = 2$, $y = 2$.

$\theta = \tan^{-1}\left|\dfrac{y}{x}\right| = \tan^{-1}\left|\dfrac{2}{2}\right| = \tan^{-1} 1 = \dfrac{\pi}{4}$

quadrant is I

(x,y)
$(2,2) \to$ I $\quad \arg z = \theta.$
$= \dfrac{\pi}{4}$

3) Express $1-\sqrt{3}i$ in polar form.

A) $z = x+iy$

$\boxed{z = r(\cos\theta + i\sin\theta)}$.

let $z = 1-\sqrt{3}i$, $x = 1$, $y = -\sqrt{3}$

$r = |z| = \sqrt{x^2+y^2} = \sqrt{1+(\sqrt{3})} = \sqrt{1+3} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left|\dfrac{y}{x}\right| = \tan^{-1}\left|\dfrac{-\sqrt{3}}{1}\right| = \tan^{-1}(\sqrt{3})$

$= \pi/3$

$(x,y) = (1, -\sqrt{3}) \to$ 4th quadrant

$\therefore \theta = \arg z = -\theta = -\pi/3$

polar form

$z = 2\left(\cos\left(\dfrac{-\pi}{3}\right) + i\sin\left(\dfrac{-\pi}{3}\right)\right)$

4) $-2+2i\sqrt{3}$ to polar form.

A) $z = -2+2\sqrt{3}i$ $\quad x = -2$, $y = 2\sqrt{3}$

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

$\theta = \tan^{-1}\left|\dfrac{y}{x}\right| = \tan^{-1}\left|\dfrac{2\sqrt{3}}{2}\right| = \tan^{-1}\sqrt{3}$
$= \pi/3$

$(x,y) \rightarrow (-2, 2\sqrt{3}) \longrightarrow 2^{nd}$ q

$\theta = \arg z = \pi - \theta$

$= \pi - \frac{\pi}{3} = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}$

polar form $\Rightarrow z = 4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

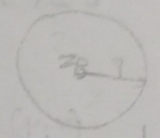
$\Rightarrow$ Sets in the Complex plane :-

* circles =

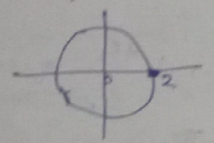
The distance b/w 2 points z & z_0 is $|z - z_0|$. Hence a circle C of radius ρ with centre at a point z_0 can be represented in the form,

$$\boxed{|z - z_0| = \rho}$$

$z_0 \rightarrow$ centre
$\rho \rightarrow$ radius



eg $\rightarrow |z - 0| = 2$.



* Neighborhood = (N)

It consist of all points of a disc / a ○ region including the centre z_0, but excluding the points on boundary circle.

* (N) with radius ε is usually $\longrightarrow$ ε- neighborhood.

* Interior point =

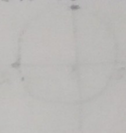
Let S be non-empty subset of complex plane. A point z_0 is said to be an interior point of set S, if there exists a (N) with centre z_0, that contains only points of S.

* Exterior point =

Let S be non-empty subset of complex plane. A point z_0 is said to be exterior point of set S, if there exists a neighborhood with centre z_0, which contains no points of S.

* Boundary point =

If z_0 is neither an interior, nor an exterior point.



(a) * every point b/w 2 concentric circles $|z| = 2$ & $|z| = 4$ $\rightarrow$ interior of S.

(b) * every point within the circle $|z| = 2$ & outside the ○ $|z| = 4$ $\rightarrow$ exterior

* every point on ○ $|z| = 2$ & $|z| = 4$ $\rightarrow$ boundary of S

* open set =

If every point of a set is an interior point $\rightarrow$ o.set.

* Connected set =

If any pair of points z_1 & z_2 in an open set S can be connected by a polygonal line that lies entirely in the set, then the open set S is said to be connected.

* **Domain** = An open connected set.

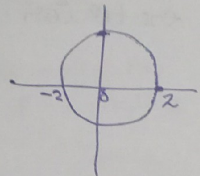
* **Region** = It is a domain in the complex plane with all, some (none of its boundary points.
Since an open connected set does not contain any boundary points, it is automatically a region.

* **Closed** = A region containing all its boundary points is said to be closed.

eg → let $S = \{ z : 2 < |z| < 4 \}$

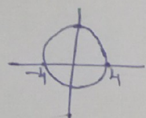
$|z| < 2$

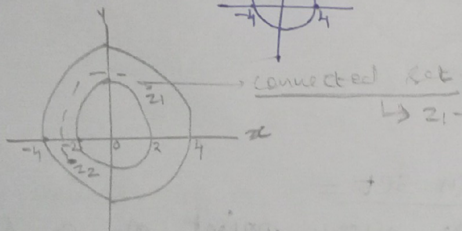
$|z - 0| < 2$ → r.
(0 → centre)



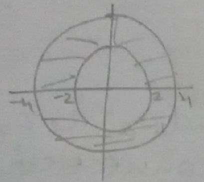
$|z| < 4$

$|z - 0| < 4$





$S = \{ z : 2 < |z| < 4 \}$

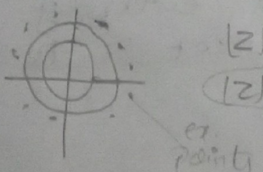


$|z| = 2$

$|z| = 4$

(a) ⇒ Interior point

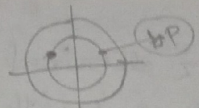
[illegible] I.P
[illegible] E.P



$|z| = 2$

$|z| = 4$

(b) ⇒ Exterior point



(c) ⇒ Boundary points

⇒ **Powers & Roots of a c. no :-**

→ powers of a c. no =

Let $z = r(\cos\theta + i\sin\theta)$ then,

$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$

$z^3 = r^3(\cos 3\theta + i \sin 3\theta$

- - - - - - -

$$\boxed{z^n = r^n(\cos n\theta + i \sin n\theta)} \quad n = 0, \pm1, \pm2 \ldots$$

Note

If $r = 1$, then $z^n = (\cos n\theta + i \sin n\theta)$ $n = 0, \pm1, \pm2 \ldots$

→ **DeMoiver's formula**

→ **n^{th} roots of a nonzero c. no =**

Let $z = \omega^n$, $n = 1, 2 \ldots$ then to each value of ω there coo responds one value of z, the correspond precisely n distinct value of ω, Each of these values → n^{th} root of z.

$\boxed{\omega = \sqrt[n]{z}}$ → n^{th} root of z.

The n^{th} root of a nonzero c. no - $z = r(\cos\theta + i \sin\theta)$ is gun by,

$$\omega_k = r^{\frac{1}{n}} \left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right], \quad k = 0, 1, 2 \ldots n-1.$$

* **principal n^{th} root of z =**

The root ω_0 of a c.no z obtained by using the principal argument of z with $k=0$ is sometimes $\rightarrow$ p. n^{th} root of z.

i) find 3 cube roots of $8i$.

A) want to find $\rightarrow \sqrt[3]{8i}$ or $(8i)^{1/3}$

let $z=(8i)^{1/3}$

$z^3 = ((8i)^3)^{1/3} = 8i$

$z = x+iy$

$x=0$, $y=8i$.

to polar

$z = r(\cos\theta + i\sin\theta)$

$r=|z| = \sqrt{x^2+y^2} = \sqrt{0^2+8^2} = \sqrt{64} = 8$

$\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{8}{0}\right|$

$= \tan^{-1}\infty = \pi/2$

$(x,y) \rightarrow (0,8) \rightarrow$ I quadrnt

$\therefore \theta = \pi/2$

$z = r(\cos\theta + i\sin\theta)$

$= 8\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ ~~$= 8(0+i\cdot 1)$~~

cos & sin are periodic $2k\pi$ $= 8\left(\cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\frac{\pi}{2}+2k\pi\right)\right)$

$z = (8i)^{1/3}$

$\therefore = 8\left[\cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\pi/2+2k\pi\right)\right]^{1/3}$

$= 8^{1/3}\left[\cos\left(\frac{\pi}{2}+2k\pi\right) + i\sin\left(\frac{\pi}{2}+2k\pi\right)\right)^{1/3}$.

$= 8^{1/3}\left[\cos\left(\frac{\pi}{2\times3}+\frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{2\times3}+\frac{2k\pi}{3}\right)\right]$

$= 2\left[\cos\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)\right]$

we know, $(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta \rightarrow$ demorgan law

$\therefore (\cos\theta + i\sin\theta)^{1/n}$. $\frac{\cos\theta}{n} + \frac{\sin\theta}{n}$.

$(8)^{1/3} = (\sqrt{8})^3$

here $n=3 \rightarrow$ 3 cube. a.

$\therefore k=0,1,2$.

$k=0$,

$\omega_0 = 2\left(\cos\left(\frac{\pi}{6}+\frac{2\times0\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{2\times0\pi}{3}\right)\right)$

$= 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} + i\cancel{2}\cdot\frac{1}{\cancel{2}}$

$= \sqrt{3}+i$

$k=1$,

$\omega_1 = 2\left(\cos\left(\frac{\pi}{6}+\frac{2\times1\times\pi}{3}\right) + i\sin\left(\frac{\pi}{6}+\frac{2\times1\times\pi}{3}\right)\right)$

$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\sqrt{3}+i$

$k=2$

$\omega_2 = 2(0+i\times-1) = -2i$

$\therefore (8i)^{1/3} = -2i, \sqrt{3}+i, -\sqrt{3}+i$

2) find 2 squar root of $\sqrt{3}+i$.

A) $z = \sqrt{3}+i$ $x=\sqrt{3}$, $y=1$.

$z = r(\cos\theta + i\sin\theta)$

$r = \sqrt{x^2+y^2} = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$

$\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\sqrt{3} = \pi/6$

$(x,y) = (\sqrt{3},1) \longrightarrow \underline{\text{I}}\ q, \quad \theta = \pi/6.$

$\therefore\ z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2\left(\cos\left(\frac{\pi}{6} + 2k\pi\right) + i\sin\left(\frac{\pi}{6} + 2k\pi\right)\right)$

$n=2$
$k=0,1$

$(\sqrt{3}+i)^{1/2} = 2\left[\cos\left(\frac{\pi}{6}+2k\pi\right) + i\sin\left(\frac{\pi}{6}+2k\pi\right)\right]^{1/2}$

$= 2^{1/2}\left[\cos\left(\frac{\pi}{6}+2k\pi\right) + i\sin\left(\frac{\pi}{6}+2k\pi\right)\right]^{1/2}$

$n=3.$ $= 2^{1/2}\left[\cos\left(\frac{\pi}{6\times 2} + \frac{2k\pi}{2}\right) + i\sin\left(\frac{\pi}{6\times 2} + \frac{2k\pi}{2}\right)\right]$

$= 2^{1/2}\left[\cos\frac{\pi}{12} + \frac{2k\pi}{2} + i\sin\frac{\pi}{12} + \frac{2k\pi}{2}\right]$

$\underline{k=0},$

$\omega_0 = 2^{1/2}\left[\cos\frac{\pi}{12} + \frac{2\times 0\pi}{2} + i\sin\frac{\pi}{12} + \frac{2\times 0\times\pi}{2}\right]$

$= 2^{1/2}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$

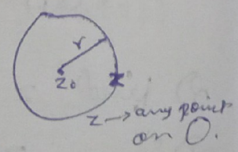
$\underline{k=1}$

$\omega_1 = 2^{1/2}\left[\cos\frac{\pi}{12} + \frac{2\pi}{2} + i\sin\frac{\pi}{12} + \frac{2\pi}{2}\right]$

$= 2^{1/2}\left[\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right]$

3) find the eq of O with centre z_0 & radius r

A) $|z - z_0| = r$



$|z - z_0|^2 = r^2.$

$(z - z_0)(\overline{z - z_0}) = r^2 \qquad |z|^2 = z\cdot\bar{z}$

$(z - z_0)(\bar{z} - \bar{z}_0) = r^2$

$z\bar{z} - z_0\bar{z} - z\bar{z}_0 + z_0\bar{z}_0 = r^2.$

$|z|^2 - (z_0\bar{z} + z\bar{z}_0) + |z_0|^2 = r^2.$

$|z|^2 - 2\operatorname{Re}\bar{z}_0 z + |z_0|^2 = r^2$

$(2\operatorname{Re}\bar{z}_0 z = z_0\bar{z} + z\bar{z}_0)$