Chapter = 05
Analytic function.

⇒ **( ) of a complex variable =**

Let 'S' be a set of c. nos. A ( ) 'f' defined on S is a rule that assigns to each set in S a c. no 'w', the no. $w \to$ the value of f at z & denoted by f(z) i.e.

$$\boxed{w = f(z)}$$

Here z varies in S & is → a c. variable. The set 'S' → Domain of definition of f. The set of c. nos which f(z) assumes as z varies in S → the range of f(z).

1) find domain of def^n of the follow ( ).

$f(z) = \dfrac{z}{z+\bar{z}}$

A) f(z) is well defined, $z+\bar{z} \neq 0$
$\Rightarrow 2\,Re\,z \neq 0$
$\Rightarrow Re\,z \neq 0$.

Domain of $f(z) = \{z : Re\,z \neq 0\}$.

$z = x+iy$, $\bar{z} = x-iy$, let $Re\,z = x$
$z+\bar{z} = x+iy+x-iy = 2x$
$z+\bar{z} = 2\,Re\,z$
$Re\,z \neq 0$ ⇒ domain = $\{z : Re\,z \neq 0\}$

2) $\dfrac{1}{z}$

$z \neq 0$

Domain of $f(z) = \{z : z \neq (0,0)\}$.
$= \mathbb{C} - \{0,0\}$.

* Real & img parts of c. ( ) f at $z = x+iy$
i.e. $f(z) = f(x+iy)$

let $u(x,y) \to$ real, $v(x,y) \to$ img.

$z = x+iy$
$f(z) = f(x+iy)$
$= u+iv$
$= u(x,y) + iv(x,y)$
$f(z) = Re\,f(z) + i\,Im\,f(z)$

$u + iy$ (real, img)

3) find Real & img parts of follow c. ( ) –
$f(z) = z^2$

A) $z = x+iy$
$z^2 = (x+iy)^2 = x^2 + 2xiy - y^2$
$Re\,f(z) = u(x,y) = x^2 - y^2$
$Im\,f(z) = v(x,y) = 2xy$.

4) $f(z) = z + \dfrac{1}{z} \Rightarrow x+iy + \dfrac{1}{x+iy}$

$\Rightarrow x+iy + \dfrac{1}{x+iy} \times \dfrac{x-iy}{x-iy} = x+iy + \dfrac{x-iy}{x^2-(iy)^2}$

$(x+iy)(x-iy) = x^2-(iy)^2$

$\Rightarrow x+iy + \dfrac{x-iy}{x^2+y^2} \Rightarrow \dfrac{(x+iy)(x^2+y^2)+x-iy}{x^2+y^2} \Rightarrow \dfrac{x^3+xy^2+x^2yi+y^3i+x-iy}{x^2+y^2}$

$\Rightarrow \dfrac{x^3+xy^2+x+i(x^2y+y^3-y)}{x^2+y^2} = \dfrac{x^3+xy^2+x}{x^2+y^2} \Rightarrow Re\,f(z)$, $\dfrac{x^2y+y^3-y}{x^2+y^2} \Rightarrow Im\,f(z)$

5) Let $f(z) = 2x - y^2 + i(xy^3 - 2x^2 + 1)$, evaluate f(z) at follow points.
(a) 2i.

A) $Re\,f(z) = 2x - y^2 \to u(x,y)$
$Im\,f(z) = xy^3 - 2x^2 + 1 \to iv(x,y)$

a) 2i, ~~let z = 2i~~ real part = 0, im part = 2.

~~u(x,y) = 0~~
~~v(x,y) = 2~~

$z = x+iy$, $x = 0$, $y = 2$.

$u(x,y) = 2x - y^2$
$u(0,2) = 2\times0 - 2^2 = -4$.
$v(x,y) = xy^3 - 2x^2 + 1$
$v(0,2) = 0 - 0 + 1 = 1$ //

$\therefore f(2i) = u(0,2) + v(0,2)$
$= \underline{-4 + 1i}$

2) $2-i$.

$z = x+iy$
$x = 2 \quad , y = -1$

$u(x,y) = 2x - y^2$
$u(2,-1) = 2\times2 - (-1)^2 = 4-1 = \underline{3}$
$v(x,y) = xy^3 - 2x^2 + 1$
$v(2,-1) = 2\times(-1)^3 - 2\times2^2 + 1 = -2-8+1 = \underline{-9}$
$f(z) = \underline{3 - i9}$

3) $5+3i$

$x = 5 \qquad y = 3$
$u(5,3) = 2\times5 - 3^2 = 10 - 9 = 1$
$v(5,3) = 5\times3^3 - 2\times5^2 + 1 = 135 - 50 + 1 = 86$
$f(z) = \underline{1 + i86}$

4) find img of line $Re(z) = 1$ under the mapping $w = f(z) = z^2$

A) $z = x+iy \quad \rightarrow Re z = x \quad , im z = y$
$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$.
$Re f(z) = u(x,y) = x^2 - y^2$ —— ①
$im f(z) = v(x,y) = 2xy$ —— ②
To find img of line $Re(z) = 1$
$\therefore Re z = x = 1$
Sub $x = 1$ in ① & ②.
$u(x,y) = 1 - y^2$
$v(x,y) = 2y \quad , v = 2y$
$y = v/2$.
$\therefore u(x,y) = 1 - y^2$
$= 1 - (v/2)^2 = 1 - \frac{v^2}{4}$ //

=> Limits of C. ( ) :-

A( ), F(z) defined in a neighbourhood of z_0 (except z_0 itself) is said to hue the limit 'w_0' as ~~said~~ 'z' approaches z_0,

$$\boxed{\lim_{z\to z_0} f(z) = w_0}$$

z_0 — f(z)

z_0 ... points → f(z)

if for every $\varepsilon > 0$ (epsalon), there exist a +ve real no $\delta > 0$ (delta) such that,
$|f(z) - w_0| < \varepsilon$ when evr $|z - z_0| < \delta$

* Therom =
Suppose that $\lim_{z\to z_0} f(z) = w_1$ & $\lim_{z\to z_0} g(z) = w_2$
then
$$\boxed{\begin{array}{l} \lim_{z\to z_0} (f(z) + g(z)) = w_1 + w_2. \\ \lim_{z\to z_0} (f(z) \cdot g(z)) = w_1 \cdot w_2 \\ \text{if } w_2 \neq 0, \lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_1}{w_2}. \end{array}}$$

1) let $f(z) = \begin{cases} \frac{Re z}{|z|} & , z \neq 0 \\ 0 & , z = 0. \end{cases}$

S.T $\lim_{z\to 0} f(z)$ doesn't exist

A) $z = x+iy$
$Re z = x$
$|z| = \sqrt{x^2+y^2}$
$\therefore f(z) = \frac{Re z}{|z|}$

$f(z) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

(এখানে limit exist ...)

$\lim\limits_{\substack{y \to 0 \\ x=0}} f(z) = \lim\limits_{\substack{y\to 0 \\ x=0}} \frac{(x+iy)}{\sqrt{x^2+y^2}}$ ~~$\lim\limits_{y\to 0} f(0+iy)$~~ along img axis

$\therefore \lim\limits_{y\to 0} \frac{x}{\sqrt{x^2+y^2}} = \frac{0}{\sqrt{0^2+y^2}} = 0$

$(z\to 0 \Rightarrow y\to 0, x=0)$

along $x=0$ & $y\to 0$.

$\lim\limits_{\substack{x\to 0 \\ y=0}} f(z) = \lim\limits_{\substack{x\to 0 \\ y=0}} \frac{(x+iy)}{\sqrt{x^2+y^2}}$ ~~$= \lim\limits_{x\to 0} f(x+i0)$~~ along real axis

$z\to 0 \Rightarrow x\to 0, y=0$

$= \lim\limits_{x\to 0} \frac{x}{\sqrt{x^2+0}} = \lim\limits_{x\to 0} \frac{x}{|x|}$ doesnot exist

$\boxed{\sqrt{x^2} = |x|}$

$\therefore \lim\limits_{z\to 0} f(z) = \lim\limits_{z\to 0} \frac{Re\, z}{|z|}$ doesnot exist

① & ② we get 2 diff. values. So limit doesn't exist.

2) Given that the follow limit exist

$\lim\limits_{z\to 1-i} \frac{5z^2+2z+2}{z+1}$

$\Rightarrow \frac{5\times(1-i)^2 - 2(1-i)+2}{(1-i)+1}$

$= \frac{5(1-2i+i^2)-2(1-i)+2}{1-i+1}$

$= \frac{5-10i+5i^2-2-2i+2}{2-i}$

$= \frac{\cancel{5}-10i-\cancel{5}-\cancel{2}-2i+\cancel{2}}{2-i}$

$= \frac{-10i-2i}{2-i}$

$= \frac{-8i}{2-i}$

$= \frac{5(1-2i+i^2)-\cancel{2}-2i+\cancel{2}}{(1-i)+1}$

$= \frac{5(\cancel{1}-2i-\cancel{1})-2i}{(1-i)+1}$

$= \frac{5(-2i)-2i}{(1-i)+1}$

$= \frac{-10i-2i}{(1-i)+1}$

$= \frac{-8i}{(1-i)+1} = \frac{-8i}{1-i+1}$

$= \frac{-8i}{2-i}$

$\frac{-8i}{2-i} \times \frac{2+i}{2+i} = \frac{-8i}{2-i} \times \frac{2+i}{2+i}$

$= \frac{-8i(2+i)}{(2-i)(2+i)} = \frac{-8i(2+i)}{4+\cancel{2i}-\cancel{2i}-i^2}$

$= \frac{-16i+8}{4+1} = \frac{8-16i}{5}$

$\Rightarrow$ **Continuity of C.no** =

A C.F 'f' is said to be continu at a point z_0 of its domain if corresponding to any $\epsilon = 0$, however there exist δ, such that,

$|f(z)-f(z_0)| < \epsilon$, $|z-z_0| < \delta$

A C.F f is said to be continu at a point z_0 of its domain if all 3 of the follow conditions are satisfied.

1) $\lim\limits_{z\to z_0} f(z)$ exists

2) $f(z_0)$ exists

3) $\lim\limits_{z\to z_0} f(z) = f(z_0)$

$\Rightarrow$ **Differentiability** =

let f be a C.F whose domain definition contain a neighbourhood of point 'z_0' then f is said to be,

$f'(z_0) = \lim\limits_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ exist.

Constant Rule :-

$\frac{d}{dz}(c) = 0$

$\frac{d}{dz} c f(z) = c \cdot f'(z)$

* Sum =

$$\frac{d}{dz}(f(z)+g(z)) = f'(z) + g'(z)$$

* Product =

$$\frac{d}{dz}(f(z)\cdot g(z)) = f(z)g'(z) + g(z)f'(z)$$

* Quatient =

$$\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{g(z)\cdot f'(z) - f(z)g'(z)}{(g(z))^2}$$

* Chain rule =

$$\frac{d}{dz}f(g(z)) = f'(g(z))\cdot g'(z)$$

* power rule

$$\frac{d}{dz}z^n = nz^{n-1}.$$

1) S.T $f(z) = \bar{z}$ is nowhere differentiable

[limit exist [illegible] diff~ [illegible] otherwise Not]

A) $z = x+iy$

$\bar{z} = x-iy = f(z)$

$\Delta z = \Delta x + i\Delta y$, $\Delta\bar{z} = \Delta x - i\Delta y$

$$\boxed{\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}} = \frac{\overline{z+\Delta z}-\bar{z}}{\Delta z} = \frac{\cancel{\bar{z}}+\Delta\bar{z}-\cancel{\bar{z}}}{\Delta z}$$

$f(z) = \bar{z}$ — a

$f(z+\Delta z) = \overline{z+\Delta z} = \bar{z}+\Delta\bar{z}$

$$= \frac{\Delta\bar{z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

along real axis

$$\lim_{\substack{\Delta x\to 0\\ \Delta y=0}}\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x} = 1$$

$\Delta x\to 0$, $\Delta y = 0$

$y\to 0$, $x=0$

along img axis

$$\lim_{\substack{\Delta y\to 0\\ \Delta x=0}}\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y\to 0}\frac{-i\Delta y}{i\Delta y} = -1$$

limit value is not same ∴ not exist.

2) S.T $f(z) = |z|^2$ is diff~ only at $z=0$.

3) using the rules of diff~ find the deri~ of follo~ ().

$f(z) = (2z+1)(z^2-4z+8i)$

A) p.rule $\frac{d}{dz}\left[(2z+1)(z^2-4z+8i)\right]$

$= (2z+1)\frac{d}{dz}(z^2-4z+8i) + (z^2-4z+8i)\frac{d}{dz}(2z+1).$

$= (2z+1)(2z-4) + (z^2-4z+8i)\times 2$

$= 4z^2 - 8z + 2z - 4 + 2z^2 - 8z + 16i$

$= 6z^2 - 14z - 4 + 16i$

4) $f(z) = \frac{5z^2 - z}{z^3+1}$

⇒ Analytical () =

A () f(z) is said to be analytic at z_0 if its deri~ $f'(z_0)$ exist not only at z_0, bt at every point z in some neighbourhood of z_0.

A () f(z) is said to be analytic in a domain d if f(z) is anytic at all points of d.

1) find $\lim_{z\to i}(4z^3 - 5z^2 + 4z + 1 - 5i)$

$i^3 = -i$

A) $4\times i^3 - 5\times i^3 + 4i + 1 - 5i$

$-\cancel{4i} + 5 + \cancel{4i} + 1 - 5i$

$5 + 1 - 5i = 6 - 5i$

2) S.T $f(z) = |z|^2$ is diff only at $z=0$,

A) $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

at $z=0$ &
at $z \neq 0$
If limit exist → diff

at $z \neq 0$

$f(z) = |z|^2$

$z = x+iy$

$\Delta z = \Delta x + i\Delta y$

$|z| = \sqrt{x^2+y^2}$

$|z|^2 = x^2+y^2$.

$f(z+\Delta z) = |z+\Delta z|^2 = |(x+iy) + (\Delta x + i\Delta y)|^2$

$= |\underbrace{(x+\Delta x)}_{x} + i\underbrace{(y+\Delta y)}_{y}|^2$. $\quad |z|^2 = x^2+y^2$.

$= (x+\Delta x)^2 + (y+\Delta y)^2$

$= x^2 + 2x\Delta x + \Delta x^2 + y^2 + 2y\Delta y + \Delta y^2$.

$= x^2+y^2 + 2x\Delta x + 2y\Delta y + \Delta x^2 + \Delta y^2$.

$f(z) = |z|^2 = |x+iy|^2 = x^2+y^2$.

$f(z+\Delta z) - f(z) = \cancel{x^2+y^2} + 2x\Delta x + 2y\Delta y + \Delta x^2 + \Delta y^2 - \cancel{(x^2+y^2)}$

$= 2x\Delta x + 2y\Delta y + \Delta x^2 + \Delta y^2$.

$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{2x\Delta x + 2y\Delta y + \Delta x^2 + \Delta y^2}{\Delta z}$

$(z = \Delta x + i\Delta y)$

$= \lim_{\Delta z \to 0} \frac{2x\Delta x + 2y\Delta y + \Delta x^2 + \Delta y^2}{\Delta x + i\Delta y}$

along real axis $= \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{2x\Delta x + 2y\Delta y + \Delta x^2 + \Delta y^2}{\Delta x + i\Delta y}$

$= \lim_{\Delta x \to 0} \frac{2x\Delta x + 2y \cdot 0 + \Delta x^2 + 0}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$.

$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}}$

$= \lim_{\Delta x \to 0} 2x + \Delta x = 2x$

along imag axis $\lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{2x\Delta x + 2y\Delta y + \Delta x^2 + \Delta y^2}{\Delta x + i\Delta y}$

$= \lim_{\Delta y \to 0} \frac{2x \times 0 + 2y\Delta y + 0 + \Delta y^2}{0 + i\Delta y}$

$= \lim_{\Delta y \to 0} \frac{2y\Delta y + \Delta y^2}{i\Delta y} = \lim_{\Delta y \to 0} \frac{\cancel{\Delta y}(2y + \Delta y)}{i\cancel{\Delta y}}$

$= \lim_{\Delta y \to 0} \frac{2y + \Delta y}{i} = -2yi \qquad \frac{1}{i} = -i$

does not exist

at $z=0$,

$f(z) = |z|^2 = z \cdot \bar{z}$

$\boxed{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$

(at a point → limit ...)

at $z_0 = 0$.

$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$

$\lim_{z \to 0} \frac{z\bar{z} - 0}{z} = \lim_{z \to 0} \frac{\cancel{z}\bar{z}}{\cancel{z}}$

$= \lim_{z \to 0} \bar{z}$

$\lim_{\substack{(x+iy) \to 0 \\ x \to 0 \\ y \to 0}} (x - iy) = 0$ limit exist

3) S.T $f(z) = x + 4iy$ is nowhere diff.

A) $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$.

Ans $f(z) = x+4iy$

$f(z+\Delta z) = x+\Delta x + 4i(y+\Delta y)$

$= x + \Delta x + 4iy + 4i\Delta y$.

$f(z+\Delta z) - f(z) = x+\Delta x + 4iy + 4i\Delta y - (x+4iy)$

$= \cancel{x} + \Delta x + \cancel{4iy} + 4i\Delta y \cancel{-x} \cancel{-4iy}$

$= \Delta x + 4i\Delta y$

$\lim\limits_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim\limits_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}$.

along real axis,

$\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta x}{\Delta x} = 1$

along img axis,

$\lim\limits_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{4i\Delta y}{i\Delta y} = 4$

$\therefore$ doesnot exist.

4) Differentiate $3z^4 - 5z^3 + 2z$

A) let $f(z) = 3z^4 - 5z^3 + 2z$.

$\frac{d}{dz} f(z) = \frac{d}{dz}(3z^4 - 5z^3 + 2z)$

$f'(z) = 12z^3 - 15z^2 + 2$

5) Differentiate $\frac{z^2}{4z+1}$

A) by quotient rule,

$= \frac{(4z+1) \times 2z - (4+0)z^2}{(4z+1)^2}$

$= \frac{8z^2 + 2z - 4z^2}{(4z+1)^2} = \frac{4z^2 + 2z}{(4z+1)^2}$

6) Diff.. $(z^2 - 4i)^3$

A) $\frac{d}{dz} z^n = nz^{n-1}$.

$(z^2-4i)^3 \Rightarrow 3(z^2-4i)^2 \cdot \frac{d}{dz}(z^2-4i)$

$\Rightarrow 3(z^2-4i)^2 \cdot 2z = 6z(z^2-4i)^2$

$\Rightarrow$ Cauchy Riemann eq :- (CR eq)

Theorum = (necessory conditions)

Suppose that $f(z) = u(x,y) + i\,v(x,y)$ & that $f'(z)$ exist at a point $z_0 = x_0 + iy_0$, then the 1st order partial deri.. of u & v must exist at (x_0, y_0). & they must satisfy C. Riemann eq.

$$\boxed{\begin{array}{l} u_x = v_y, \quad u_y = -v_x \quad \text{(i.e)} \\ \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad \text{at } z_0 = (x_0, y_0) \\ f'(z_0) = u_x + iv_x = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{array}}$$

$u_x = v_y$ & $u_y = -v_x$

where these partial deri.. are to be evaluated at (x_0, y_0).

7) S.T $f(z) = Re\,z$ is no where diff..

(diff & analytic [illegible] CR eq [illegible])

A) $z = x+iy$

$Re\,z = x = u(x,y)$ — $f(z) = Re\,z$

$v(x,y) = 0$.

$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ $u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = 0$

$\therefore u_x \neq v_y$.

not diff

$f(z) = Re\,z = x+0$

$f(z) = u(x,y) + i\,v(x,y)$

$u = x$ & $v = 0$

$\therefore u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = 0$.

by CR eq

$u_x = v_y$ $\quad u_y = -v_x$.

$u_x \neq v_y$ $\quad u_y = v_x$.

2) S.T for the ( ) $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

Even though partial deriv of the component ( ) exist & satisfy Cauchy Riemann eq. at $z=0$, $f(z)$ is not diff at $z=0$.

A)

$z = x+iy$

$\bar{z} = x-iy$

$$f(z) = \frac{(\bar{z})^2}{z} = \frac{(x-iy)^2}{x+iy} \times \frac{\overline{x+iy}}{\overline{x+iy}}$$

$$= \frac{(x-iy)^2 (x-iy)}{(x+iy)(x-iy)} \longrightarrow (a+b)(a-b) = a^2-b^2.$$

$$= \frac{(x-iy)^3}{x^2+y^2} \longrightarrow (a-b)^3 = a^3 - 3a^2b + 3ab^2 + b^3$$

$$= \frac{x^3 - 3x^2 iy + 3x(iy)^2 + (iy)^3}{x^2+y^2}$$

$$= \frac{x^3 - 3x^2 iy - 3xy^2 + iy^3}{x^2+y^2}$$

$$= \frac{x^3 - 3xy^2}{x^2+y^2} + i\,\frac{y^3 - 3x^2y}{x^2+y^2}$$

$$u(x,y) = \begin{cases} \dfrac{x^3-3xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = 0. \end{cases}$$

$$v(x,y) = \begin{cases} \dfrac{y^3-3x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = 0 \end{cases}$$

$$\therefore u_x = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{(\Delta x)^3/\Delta x^2}{\Delta x} = 1$$

$$u_y = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y^2/\Delta y} = 0$$

$$v_x = \lim_{\Delta x \to 0} \frac{\frac{0-0}{0}}{\Delta x} = 0$$

$$v_y = \lim_{\Delta y \to 0} \frac{\Delta y^3/\Delta y^2}{\Delta y} = 1$$

$u_x = v_y \quad u_y = -v_x$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\frac{\bar{z}^2}{z} - 0}{z - 0}$$

$$\lim_{\substack{x \to 0 \\ y = 0}} \frac{(x-iy)^2}{(x+iy)^2} = 1$$

$$\lim_{y \to 0} \frac{(x-iy)^2}{(x+iy)^2} = -1$$

$\Longrightarrow$ not diff.

$\Rightarrow$ Criterion for analyticity :-

Suppose the real valued ( ) $u(x,y)$ & $v(x,y)$ are contin... & hve contin 1st order partial deriv in a domain d, if the 1st order partial deriv of u & v satisfy the C.R eq. at all points of D. Then the C. ( ) $f(z) = u(x,y) + i\,v(x,y)$ is analytic in d & it's deriv is gvn by,

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Rmrk,

If the real valued ( ) $u(x,y)$ & $v(x,y)$ are contin & hve contin 1st order partial deriv in a neighbourhood of z & if the 1st order partial deriv u & v satisfy the CR eq at the point z then C.( ) $f(z) = u(x,y) + i\,v(x,y)$ is diffn at z & its derivn gvn by.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

⇒ Harmonic ( ) :- / ~~la~~places eq

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

Therom: [A source of H·( )]

If a ( ) $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component ( ) u & v are harmonic in D.

D.P.T $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$, both satisfying Laplaces eq. but $u+iv$ is not an analytic ( ) of z.

A) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

$u_{xx} + u_{yy} = 0$

$u_x = 2x$ $\quad u_y = -2y$.

$u_{xx} = 2$ $\quad u_{yy} = -2$.

$v_{xx} + v_{yy} = 0$

$\cancel{v_x = \frac{(x^2+y^2) \times 0 - -y \times (2x + \ldots)}{(x^2+y^2)^2}}$

$\cancel{x^2 + y^2 + 2xy}$

$\cancel{\frac{(x^2+y^2) + 2xy + 2y^2}{(x^2+y^2)^2}} = \frac{y(2x)}{(x^2+y^2)^2}$

$\cancel{v_x \Rightarrow \frac{2xy}{(x^2+y^2)^2}}$

Proof.

$$V_x = \left(\frac{-y}{x^2+y^2}\right)_x = \frac{(x^2+y^2)\cdot \times 0 - -y\,(2x)}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$V_y = \left(\frac{-y}{x^2+y^2}\right)_y = \frac{(x^2+y^2)\times -1 - -y\,(2y)}{(x^2+y^2)^2}$$

$$= \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$V_{xx} = \left(\frac{2xy}{(x^2+y^2)^2}\right)_x = \frac{(x^2+y^2)^2 \times 2y - 2xy \times 2(x^2+y^2)\times 2x}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^{\cancel{2}}\,2y - 8x^2y\,\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{4}}}$$

$$= \frac{(x^2+y^2)2y - 8x^2y}{(x^2+y^2)^3} = \frac{2x^2y + 2y^3 - 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$V_{yy} = \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)_y = \frac{(x^2+y^2)^{\cancel{2}} \times 2y - (y^2-x^2)\cdot 2\,\cancel{(x^2+y^2)}\cdot 2y}{(x^2+y^2)^{\cancel{4}}}$$

$$= \frac{2x^2y + 2y^3 - 2y^2 + 2x^2 \times 2y}{(x^2+y^2)^3}$$

$$= \frac{2x^2y + 2y^3 - 4y^3 + 4x^2y}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\therefore\; V_{xx} + V_{yy} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3} + \frac{6x^2y - 2y^3}{(x^2+y^2)^3} = 0$$

=> **Harmonic Conjugate** =

A () $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is H. conjugate of u.

i) Let $u(x,y) = 4xy - x^3 + 3xy^2$, S.T u is a H. () ? find its H. conjugate & hence the most general analytical () $f(z)$ with u as real component.

A) $u(x,y) = 4xy - x^3 + 3xy^2$

$u_x = 4y - 3x^2 + 3y^2$

$u_{xx} = 0 - 6x = \underline{-6x}$

$u_y = 4x - 6yx$

$u_{yy} = \underline{6x}$

$u_{xx} + u_{yy} = -6x + 6x = 0.$

∴ it is analytic

$$\boxed{\begin{aligned} v_y(x,y) &= u_x(x,y) \\ v_x(x,y) &= -u_y(x,y) \end{aligned}}$$

$\Rightarrow 4y - 3x^2 + 3y^2$ — ①

$\Rightarrow -4x - 6xy$ — ②

$\int$ ① w.r.t y, $\Rightarrow \frac{4y^2}{2} - 3x^2y + \frac{3y^3}{3}$

$v(x,y) = 2y^2 - 3x^2y + y^3 + \phi(x)$ — ③ $\Rightarrow 2y^2 - 3x^2y + y^3 + \phi(x)$.

diff ③ w.r.t x, $0 - 6xy + 0 + \phi'(x)$.

$v_x(x,y) = -6xy + \phi'(x)$ — ④ $\Longrightarrow -6xy + \phi'(x)$

② & ④

$-4x - \cancel{6xy} = -\cancel{6xy} + \phi'x$

$\phi'(x) = -4x$.

$\phi'(x) = -4x$

$\phi(x) = -4\frac{x^2}{2}$

$= -2x^2 + k$ //

∴ $\phi(x) = -2x^2 + k$.

∴ $F(z) = u(x,y) + iv(x,y)$

$= 4xy - x^3 + 3x^2y + i(2y^2 - 3x^2y + y^3 + \phi(x))$.

=> Elementary () :-

→ **Exponential ()** =

for any c.no. $z = x+iy$ (x,y) belongs to R, the exponential () of z denoted by e^z is defined by,

$e^z = e^{x+iy}$

$$\boxed{e^z = e^x(\cos y + i\sin y)}$$

a) e^z should reduce to e^x when $z = x + i0$ when img $z = 0$.

b) e^z should be an entire (). (i.e) analytic for all z.

c) $\frac{d}{dz} e^z = e^z$

→ **Algebraic (pro) of e^z** =

* $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

* $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

* $(e^z)^n = e^{zn}$, $n = 0, \pm1, \pm2 \ldots$

→ **Periodicity of ex. ()** =

$\cos n\pi = (-1)^n$

$= (-1)^2 = 1$

$\sin n\pi = 0.$

$e^{z+2\pi i} = e^z \cdot e^{2\pi i}$

$= e^z \cdot (\cos 2\pi + i\sin 2\pi)$

$\underline{= e^z}$

$e^z(1 + i \times 0)$

$\underline{= e^z}$

→ Polar form of a c.no =

$z = r(\cos\theta + i\sin\theta)$ $\quad (e^{i\theta} = \cos\theta + i\sin\theta)$

$z = re^{i\theta}$ $\quad |z| = \sqrt{x^2+y^2}$

Rmrk

* $|e^z| = |e^x \cos y + i\sin y| = |e^x| \, |(\cos y + i\sin y)| = e^x \cdot \sqrt{\cos^2 y + \sin^2 y} = e^x$

* $\arg(e^z) = \arg\left(e^x(\cos y + \sin y)\right) = y + 2n\pi$

argmnt (e^z)

1) write e^z in the form of $a+ib$ in

$z = 2 - \left(\frac{\pi}{3}\right)i$.

A)

$e^z = e^{x+iy}$ $\quad x = 2, \; y = -\pi/3$ $\quad e^{x+iy} = e^x(\cos y + i\sin y)$

$e^{2-(\frac{\pi}{3}i)} = e^{2+(-\pi/3)i}$

$= e^2\left(\cos(-\pi/3) + i\sin(-\pi/3)\right)$

$= e^2\left(\frac{1}{2} + i - \sqrt{3}/2\right)$

$= \frac{e^2}{2} - \frac{ie^2\sqrt{3}}{2}$

2) e^{z^2}

$e^{z^2} = e^{(x+iy)^2} = e^{x^2+2xiy-y^2}$

$= e^{x^2-y^2+2xiy}$

$= e^{x^2-y^2}\left[\cos 2xy + i\sin 2xy\right]$

$\cancel{e^{2-(\frac{\pi}{3})i} = e^{(2+(-\pi/3)i)^2} = e^{(2^2-2\times2\times-\frac{\pi}{3}+(\frac{\pi}{3})^2)} = e^{4-4\pi/3+''}} = e$

3) find whole values of z such that $e^z = 3+4i$

$x=3$, $y=4$

A)

$z = x+iy$

$e^x = |e^z| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$x = \log_e 5 = 1.6094$

$y = \arg(e^z) \pm 2n\pi$

$= \arg(3+4i) \pm 2n\pi$

$= \tan^{-1}(3/4) \pm 2n\pi$

$= 0.9262 \pm 2n\pi$

$z = x+iy$

$\Rightarrow 1.6094 + (0.9262 \pm 2n\pi)$

$\cancel{e^z = 3+4i}$

$\ln e^z = \ln(3+4i)$

$z = \ln(3+4i)$

$|z| = \sqrt{3^2+4^2} = 5$

$\theta = \tan^{-1} y/x$

$= \tan^{-1} 4/3$

4) write $|\exp(2z+i)|$ & $|\exp(iz^2)|$. & T

$|\exp(2z+i) + \exp(iz^2)| \le e^{2x} + e^{-2xy}$.

* Log of a non zero complex variable $z = re^{i\theta}$ $(-\pi < \theta \le \pi)$ is defined by

$\ln z = \log_e r + i\theta$ $\quad n = 0, \pm1, \pm2 ...$

$= \log_e |z| + i(\arg z \pm 2n\pi)$

* Principle value = p.value of $\log z$ is the value obtained from the above def. when $n=0$ & is denoted by 'Ln z'

$\operatorname{Ln} z = \log_e r + i\theta$

$= \log_e |z| + i \arg z$

$\boxed{\ln z = \operatorname{Ln} z + i2n\pi}$

~~ln 1 = Ln + i2nπ~~

$\boxed{\operatorname{Ln} z = \log_e |z| + i \operatorname{Arg} z}$

$\ln 1 \Rightarrow 1 = 1(\cos 0 + i\sin 0)$

$= 1e^{i0}$

$\therefore \ln 1 = \log|1| + i \arg(1 + 2n\pi)$

$= \underbrace{\log_e 1}_{0} + i(0 + 2n\pi) \quad = \underline{\underline{2n\pi i}}$

$(\log 1 = 0)$

$\Rightarrow$ <u>Algebraic (pro) of logarithm</u> =

* For any 2 non zero c. no, z_1 & z_2,

$\ln(z_1 z_2) = \ln z_1 + \ln z_2$

* $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$

* $\ln z^n = n \log z$

1) S.T $Ln(-ei) = 1 - \frac{\pi}{2} i$

A) $Ln(z) \rightarrow z = -ei$

$x = 0, \; y = -ei.$

$|z| = |-ei| = \sqrt{0 + (-e)^2} = e$

$\arg z = \arg(-ei) = \theta = \tan^{-1} y/x.$

$= \tan^{-1} \infty = \pi/2$

$(x, y) = (0, -ei) \rightarrow 4^{th}$ Q.

$\therefore \theta = -\pi/2$

$Ln(-ei) = \log(-ei) + i \arg(-ei)$

$= \log_e e + i(-\pi/2)$

$= 1 - \pi/2 i$

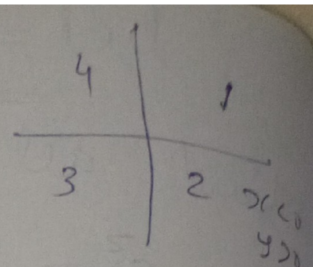
Hw

2) $Ln(1-i) = \frac{1}{2} \log_e 2 - \pi/4 i$

3) S.T $Ln z$ is contius on the -ve real axis ?.

A) $z = x+iy$ (i.e $x_0 \leq 0$.

$z_0 = x_0+iy_0$



$\frac{-\pi}{2} < \text{Arg } z < \pi$

$\lim_{z\to z_0} \text{Arg } z = \lim_{x\to x_0} \text{Arg}(x+iy)$

$= \pi$. (1^{st} Q)., So $\theta = 0$.

$\lim_{y\to y_0} \text{Arg}(x+iy) = -\pi$. ($4^{th}$ Q)

$\lim_{z\to z_0} \text{Ln } z = \lim_{z\to z_0} \log_e |z_i| + i \text{ Arg } z_i$.

doesn't exist

principle value of $z^{\alpha} = e^{\alpha \text{Ln} z}$

3) S.T $n = 0, \pm 1, \pm 2 \ldots$ $(1+i)^i = \exp\left[-\frac{\pi}{4} + 2n\pi\right] \exp\left[\frac{i}{2}\log\right.$

A) $1+i = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$.

$\arg(1+i) = \pi/4 + 2n\pi$

$\ln(1+i) = \log_e|1+i| + i \arg(1+i)$

$= \log_e \sqrt{2} + i(\pi/4 + 2n\pi)$

$= \log_e (2)^{1/2} + i(\pi/4 + 2n\pi)$

$= \frac{1}{2}\log 2 + i(\pi/4 + 2n\pi)$

$(1+i)^i = e^{i(\ln(1+i))}$

$= e^{i\left(\frac{1}{2}\log 2 + i(\pi/4 + 2n\pi)\right)}$