Length of a Space Curve :-

$r(t) = f(t)i + g(t)j + h(t)k$

$r(t)$ - [illegible] length [illegible] $\|r(t)\|$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt.$$

$r'(t) = f'(t)i + g'(t)j + h'(t)k.$

$\cancel{L = \int_a^b \sqrt{\|r'(t)\|}}$

$$\boxed{L = \int_a^b \|r'(t)\| \, dt}$$

Q/ 1) Find the L of the [illegible]

$r(t) = ti + \cosh t\, j$ from 0 to 1

Sol $L = \int_a^b \|r'(t)\| \, dt$

$r(t) = ti + \cosh t\, j$ $\quad$ $\boxed{1 + \sinh^2 t = \cosh^2 t}$

$r'(t) = 1i + \sinh t\, j$

$\therefore L = \int_0^1 \sqrt{1^2 + \sinh^2 t} = \int_0^1 \sqrt{\cosh^2 t}$

$= \int_0^1 \cosh t \Rightarrow [\sinh t]_0^1$

$= \sinh 1 - \sinh 0 = \underline{\sinh 1}$

2) Find arc length b/w $[0, 0, 1]$ i.e $[1, 0, 1]$ of [illegible] helix

$y = \sin 2\pi x \quad z = \cos 2\pi x$

A) 1st find parametric eq

[illegible] $x = t$,

eq becomes → $y = \sin 2\pi t$, $z = \cos 2\pi t$, $x = t$

$[0, 0, 1] \quad [1, 0, 1]$

$r(t) = xi + yj + zk$

$r(t) = ti + \sin 2\pi t\, j + \cos 2\pi t\, k$

$r'(t) = 1i + \cos 2\pi t \cdot 2\pi j - \sin 2\pi t \cdot 2\pi k$

$L = \int_a^b \|r'(t)\| \, dt$

$= \int_0^1 \sqrt{1^2 + (\cos 2\pi t \cdot 2\pi)^2 + (2\pi \cdot \sin 2\pi t)^2}\, dt$

$= \int_0^1 \sqrt{1 + 4\pi^2 \cdot \cos^2 2\pi t + 4\pi^2 \cdot \sin^2 2\pi t}$

$= \int_0^1 \sqrt{1 + 4\pi^2(\underbrace{\cos^2 2\pi t + \sin^2 2\pi t}_{1})}\, dt$ $\quad$ $(\cos^2 x + \sin^2 x = 1)$

$= \int_0^1 \sqrt{1 + \underbrace{4\pi^2}_{\text{const}}}\, dt \quad \int dt = t$

$= \sqrt{1 + 4\pi^2} \int_0^1 dt \Rightarrow \sqrt{1 + 4\pi^2}\, [t]_0^1$

$\Rightarrow \sqrt{1 + 4\pi^2}\,(1 - 0) = \underline{\sqrt{1 + 4\pi^2}}$

3) Find L of one 3rd of helix
$r(t) = \cos t\, i + \sin t\, j + t k$.

$\Rightarrow$ Arc length parameter :-

Arc length function :-

Suppose c is a smooth curve described by,

$r(t) = f(t) i + g(t) j + h(t) k$.

$\boxed{\text{arc length, } s(t) = \int_{t_0}^{t} \| r'(u) \| \, du}$

$s(t) = \int_{t_0}^{t} \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \, dt$.

1) The arc length parametrication of ○, $x^2+y^2=a^2$ with counter clockwise orientation.

A) we know, para. eq of ○ →

$x = a\cos t$ $y = a\sin t$

$r(t) = a\cos t\, i + a \sin t\, j$

$r'(t) = -a\sin t\, i + a\cos t\, j$

Starting → 0
ending → we don't know

$t_0 = 0$ arc length, $s(t) = \int_{t_0}^{t} \| r'(u) \| \, du$.

$t \to u$

$= \int_0^t \sqrt{(-a\sin t)^2 + (a\cos t)^2} \, du$.

$= \int_0^t \sqrt{a^2 \sin^2 u + a^2 \cos^2 u} \, du$.

$= \int_0^t \sqrt{a^2 [\sin^2 u + \cos^2 u]} \, du$.

$= \int_0^t \sqrt{a^2} \, du = \int_0^t a \, du$

$= a \int_0^t du = a [u]_0^t \Rightarrow a(t-0) = \underline{at}$

$\underline{s = at} \rightarrow t = s/a$

arc length parametriazation

$0 \le t \le 2\pi$

$r(t(s)) = a\cos(s/a) i + a \sin(s/a) j$

$s(t) = at$

$\therefore s(0) = 0$

$s(2\pi) = 2\pi a$ interval $\Rightarrow [0, 2a\pi]$.

2) Reparametrise following parameter in terms of length →

a) $r(t) = \frac{t^2}{2} i + \frac{t^3}{3} k$. $0 \le t \le 2$

A) $r'(t) = \frac{1}{2} \cdot 2t\, i + \frac{1}{3} \cdot 3t^2 k$

$r'(t) = t i + t^2 k$

$t_0 = 0$.

$t \to u$.
$r'(u) = u i + u^2 k$

$t \to u$ $s = \int_0^t \| r'(t) \| \, du$.

$= \int_0^t \sqrt{u^2 + u^4} \, du$.

$= \int_0^t \sqrt{u^2(1+u^2)} \, du$

$= \int_0^t u\sqrt{1+u^2}\, du.$

Substitution (a)

put $1+u^2 = v.$

$\frac{d}{du}(1+u^2) = \frac{d}{du}(v)$

$2u\, du = dv$

$u\, du = \frac{dv}{2}$

$v = 1+u^2 = 1+0 = 1$

$v = 1+t^2$

limit

$u = 0 \quad , \quad v = 1$

$u = t \quad , \quad v = 1+t^2$

$\therefore \int_1^{1+t^2} \sqrt{v}\, \frac{dv}{2} = \frac{1}{2}\int_1^{1+t^2} \sqrt{v}\, dv.$

$= \frac{1}{2}\int_1^{1+t^2} v^{1/2}\, dv = \frac{1}{2}\left[\frac{v^{3/2}}{3/2}\right]_1^{1+t^2}$

$= \frac{1}{2} \cdot \frac{2}{3} \cdot \left[v^{3/2}\right]_1^{1+t^2}$

$= \frac{1}{3}\left[v^{3/2}\right]_1^{1+t^2} = \frac{1}{3}\left[(1+t^2)^{3/2} - 1^{3/2}\right]$

$S = \frac{(1+t^2)^{3/2}}{3} - \frac{1}{3}$

$S = \frac{(1+t^2)^{3/2} - 1}{3}$

$3S = (1+t^2)^{3/2} - 1.$

$3S + 1 = (1+t^2)^{3/2}$

(3/2 ~~ m conversion from 2/3 rang)

$(3S+1)^{2/3} = 1+t^2$

$(3S+1)^{2/3} - 1 = t^2$

$\sqrt{(3S+1)^{2/3} - 1} = t.$ (eq)

$\left((3S+1)^{2/3} - 1\right)^{1/2} = t$

$r(t(s)) = \left((3S+1)^{2/3} - 1\right)^{1/2}$

$= \frac{(3S+1)^{2/3} - 1}{2}\, i + \frac{\left[(3S+1)^{2/3} - 1\right]^{3/2}}{3} k.$

Q) $r(t) = \frac{t^2}{2} i + \frac{t^3}{3} k.$

$\frac{\left(\left[(3S+1)^{2/3} - 1\right]^{1/2}\right)^2}{2} i + \frac{\left(\left((3S+1)^{2/3} - 1\right)^{1/2}\right)^3}{3}$

$= \frac{(3S+1)^{2/3} - 1}{2}\, i + \frac{\left[(3S+1)^{2/3} - 1\right]^{3/2}}{3}$

4) $r(t) = (2\cos t)\, i + (2\sin t)\, j, \quad 0 \le t \le 2\pi$

5) find arc length param of helix $r(t) = \cos t\, i + \sin t\, j + t\, k$, then has reference point $r(0) = [1, 0, 0]$ & same orientation has the gun helix.

A) $r'(t) = -\sin t\, i + \cos t\, j + k.$

$= -\sin u\, i + \cos u\, j + k$

$$S = \int_{t_0}^{t} \| r'(u) \| \, du .$$

$$= \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1} \, du .$$

(To find arc length find till t.....)

$$= \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1} \, du .$$

$$= \int_0^t \sqrt{1+1} \Rightarrow \cancel{\left[u \right]_0^t}$$

$$S = \int_0^t \sqrt{2} \Rightarrow S = \sqrt{2}\, t .$$

$$t = S/\sqrt{2} .$$

$$\therefore r(t(S)) = \cos(S/\sqrt{2})\, i + \sin(S/\sqrt{2})\, j + (S/\sqrt{2})\, k .$$

HW 5) L of $r(t) = \cos t\, i + \sin t\, j + t k$.

A) $r'(t) = -\sin t\, i + \cos t\, j + k$.

helix make 1 full turn as t runs from 0 to 2π,

$$L = \int_a^b \| r'(t) \| \, dt$$

$$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} \, dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1} \, dt = \int_0^{2\pi} \sqrt{1+1} \, dt .$$

$$= \int_0^{2\pi} \sqrt{2} \, dt = 2\pi\sqrt{2}$$

4) Arc L of $r(t) = (2\cos t)\, i + (2\sin t)\, j$

$0 \le t \le 2\pi$.

A) velocity $\boxed{v(t) = \dfrac{dr}{dt}} = -2\sin t\, i + 2\cos t\, j$.

$t_0 = 0$,

$$s = s(t) = \int_0^t \| v(t) \| \, dt .$$

$$= \int_0^t \sqrt{(-2\sin t)^2 + (2\cos t)^2} \, dt .$$

$$= \int_0^t 2 \, dt = 2t .$$

$S = 2t$.

$t = S/2$

$\therefore$ S rises from 0 to 4π.

$$u(S) = r(S/2) = 2\cos(S/2) + 2\sin(S/2)\, j$$

$(0 \le S \le 4\pi)$

=> Motion on a curve :-

$\boxed{(\text{dir}) \Rightarrow \dfrac{v(t)}{\| v(t) \|}}$

(v) func → $r(t) = f(t)\, i + g(t)\, j + h(t)\, k$.

$r'(t) = f'(t)\, i + g'(t)\, j + h'(t)\, k$.

velocity → $v(t) = r'(t) = f'(t)\, i + g'(t)\, j + h'(t)\, k$.

acc → $a(t) = r''(t) = f''(t)\, i + g''(t)\, j + h''(t)\, k$.

Speed → $v = \| r'(t) \| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$

$\boxed{\cos\theta = \dfrac{v(t) \cdot a(t)}{\| v(t) \| \| a(t) \|}}$

Q 1) position (v) of a particle in space at time 't' is $r(t) = (t+1)\, i + (t^2 - 1)\, j + 2t\, k$

a) find particle velocity & acc (v)?

b) find particle speed & (dir) of motion at $t = 1$?

Q find angle b/w the velocity & acc (v) at $t=1$?

A) ~~$v(t) = r'(t) =$~~

$r(t) = (t+1)\,i + (t^2-1)\,j + 2t\,k$

a) $v(t) = r'(t) = 1\,i + 2t\,j + 2\,k$

$a(t) = r''(t) = 0i + 2j = 2j$

b) Speed $= \|r'(t)\| = \sqrt{1^2 + (2t)^2 + 2^2}$

$= \sqrt{1 + 4t^2 + 4} = \sqrt{5 + 4t^2}$

Speed at $t=1$

$v \Rightarrow \sqrt{5+4t^2} = \sqrt{5+4\times1} = \sqrt{5+4} = \sqrt{9} = 3$

(dir) of motion →

$\Rightarrow \dfrac{v(t)}{\|v(t)\|}$ at $t=1$

$v(t) = i + 2tj + 2k$

$v(1) = i + 2j + 2k$

$\dfrac{v(1)}{\|v(1)\|} \Rightarrow \dfrac{i + 2j + 2k}{3} = \dfrac{i}{3} + \dfrac{2}{3}j + \dfrac{2}{3}k$

c) $\boxed{\cos\theta = \dfrac{v(1)\; a(1)}{\|v(1)\|\,\|a(1)\|}} = \dfrac{(i+2j+2k)\,2j}{\sqrt{3^2}\cdot\sqrt{2^2}}$

Same component x multiply

$= \dfrac{(i + 2j + 2k)\,2j}{3\cdot 2} = \dfrac{4}{3\cdot 4} \cdot 2 = \dfrac{2}{3}$

$\cos\theta = 2/3$, $\theta = \cos^{-1}(2/3)$

Q) Suppose $r(t) = t^2 i + (t^3-2t)\,j + (t^2-5t)\,k$ is the position (v) of a moving particle at wht points does the particle pass through the (x y) plane? wht are its v & acc at this points?

A) $r(t) = t^2 i + (t^3 - 2t)\,j + (t^2 - 5t)\,k$.

param eq → $r(t) = xi + yj + zk$.

$x = t^2$ $\quad y = t^3 - 2t$ $\quad z = t^2 - 5t$

* ((x y) plane → pass z=0)

$z = 0$

$t^2 - 5t = z$

$t^2 - 5t = 0$

$t(t-5) = 0$

$t = 0$, $t - 5 = 0$

$t = 5$

at $t=0$, $x = y = z = 0$

at $t=5$, $x = t^2 = 25$

$y = t^3 - 2t = 125 - 10 = 115$

$z = 0$

∴ points → $(0,0,0)$ & $(25, 115, 0)$

* $v(t) = r'(t) = 2t\,i + (3t^2 - 2)\,j + (2t-5)\,k$

$a(t) = r''(t) = 2\,i + 6t\,j + 2\,k$

at $t=0$, $v(t)$? $a(t)$?

$v(t) = v(0) = 2\times0\, i + (3\times 0 - 2)\,j + (2\times 0 - 5)\,k$.

$= -2j - 5k$

$a(t) = a(0) = 2i + 2k$