04 : VECTOR INTEGRATION.

⇒ Line integrals :-

we integrate the integrand $f(x)$ from $x=a$ along x-axis to $x=b$.
In the line ∫ the interval $[a,b]$ is replaced by a curve C in plane/ space described by a parametric eq. & integrand is a () $G(x,y,z)$ defined & bounded on this curve.
The resulting ∫ → line integral / a curvilinear ∫ / contour ∫.

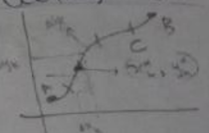
Suppose, C is a curve in the xy-plane parametrized by ~~x=f(t)~~
$x=f(t)$, $y=g(t)$; $a \le t \le b$ & A & B are points $(f(a), g(a))$ & $(f(b), g(b))$.
we call C the path of ∫-tion, A its initial point & B is terminal point.

* C is smooth curve, if f' & g' are continuous in closed interval $[a,b]$ & not simultaneously 0 in open interval (a,b).

* C is piecewise smooth, if it consist of a finite no. of smooth curves $C_1, C_2 \ldots C_n$ joined end to end, (i.e) $C = C_1 \cup C_2 \cup \ldots \cup C_n$.

* C is a closed curve if $A=B$
* C is a simple closed curve if $A=B$ & the curve does not cross itself.

This same terminology carries over in a natural manner to curves in space.



→ Line ∫ in the plane :-

1) Let $z = G(x,y)$ be defined in some region that contains the smooth curve C defined by $x=f(t)$, $y=g(t)$, $a \le t \le b$.
2) Divide C into n subarcs of length Δs_k according to the partition.
$$a = t_0 < t_1 < t_2 \ldots < t_n = b.$$
3) Let $\|P\|$ be the norm of partition / length of longest subarc.
4) Choose a sample point (x_k^*, y_k^*) on each subarc.
5) form the sums,
$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta x_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta y_k,$$
$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta s_k.$$

Then the defn of line ∫ in the plane given below —

def → Let G be a ( ) of 2 variables x & y defined on a region of the plane containing a smooth curve C

a) line $\int$ of G along C from A to B with respect to x is,

$$\boxed{\int_C G(x,y)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta x_k}$$

b) line $\int$ of G along C from A to B with respect to y,

$$\boxed{\int_C G(x,y)\,dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta y_k}$$

c) line $\int$ of G along C from A to B with respect to arc length,

$$\boxed{\int_C G(x,y)\,dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\,\Delta s_k}$$

=> Method of Evaluation - curve defined parametrically :-

$f'(t)\,dt$ or $g'(t)\,dt$ or $\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$

The exp., $ds = \sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$ → differential of arc length.

$\int$-tion is carried out with respect to variable in usual manner :

$$\int_C G(x,y)\,dx = \int_a^b G[f(t), g(t)]\, f'(t)\,dt$$

$$\int_C G(x,y)\,dy = \int_a^b G[f(t), g(t)]\, g'(t)\,dt$$

$$\int_C G(x,y)\,ds = \int_a^b G[f(t), g(t)]\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$$

y+x=0 → Implicit

y+x → Explicit

=> Methods of Evaluation - curve defined by an Explicit ( ) :-

If the curve 'C' is defined by an explicit ( ) $y = f(x)$, $a \le t \le b$, x as a paramtr,

$$dy = f'(x)\,dx \quad \& \quad ds = \sqrt{1+[f'(x)]^2}\,dx.$$

line $\int$ becomes,

$$\int_C G(x,y)\,dx = \int_a^b G(x, f(x))\,dx.$$

$$\int_C G(x,y)\,dy = \int G(x, f(x))\, f'(x)\,dx.$$

$$\int_C G(x,y)\,ds = \int G(x, f(x))\sqrt{1+(f'(x))^2}\,dx.$$

* line $\int$ along a piecewise-smooth curve C is defined as sum of $\int$ over various smooth curves whose union comprises C.

$$\int_C G(x,y)\,ds = \int_{C_1} G(x,y)\,ds + \int_{C_2} G(x,y)\,ds + \int_{C_3} G(x,y)\,ds.$$

Rmrk

* In many appli..., line $\int$ appear as a sum

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

* Common to write this sum as 1 $\int$ without parentheses,

$$\int_C P(x,y)\,dx + Q(x,y)\,dy \quad \text{as} \quad \int_C P\,dx + Q\,dy.$$

* Line $\int$ along a ~~closed~~ closed curve C is often,

$$\oint_C P\,dx + Q\,dy. \qquad \oint \to \text{closed } \int$$

Q 1) Let C denote the quarter-circle defined by $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le \frac{\pi}{2}$, Evaluate,

a) $\int_C xy^2\,dx$ (b) $\int_C xy^2\,dy$ (c) $\int_C xy^2\,ds$. (ds → arc length)

A. $x = 4\cos t$ $y = 4\sin t$

$G(x,y) = xy^2$

$f(t) = x = 4\cos t$ $a = 0$

$g(t) = y = 4\sin t$ $b = \pi/2$.

dx ?

$dx = f'(t)\,dt$

$= -4\sin t\,dt$.

$dy = g'(t)\,dt$

$= 4\cos t\,dt$. ds ?

$$ds = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

$$= \sqrt{(-4\sin t)^2 + (4\cos t)^2} = \sqrt{16\sin^2 t + 16\cos^2 t}$$

$$\sqrt{16(\sin^2 t + \cos^2 t)} = \sqrt{16} = \underline{\underline{4\,dt}}.$$

a) $\int_C xy^2\,\underline{dx} = ?$

$$\boxed{\int_C G(x,y)\,\underline{dx} = \int_a^b G[f(t), g(t)]\,\underline{f'(t)}\,dt}$$

$\int xy^2$
$x \to f(t) \to 4\cos t$
$y^2 \to (g(t))^2 \to (4\sin t)^2$

$$= \int_0^{\pi/2} 4\cos t \cdot (4\sin t)^2 \cdot (-4\sin t)\,dt$$

$$= -256 \int_0^{\pi/2} \cos t \cdot \sin^3 t\,dt$$

$$= -256 \int_0^{\pi/2} \sin^3 t \cdot \cos t\,dt.$$

Substitution method

Let $u = \sin t$

$du = \cos t\,dt$

limit

$t = 0$, $u = 0$ ($\sin 0 = 0$) $\begin{pmatrix} u = 0 \\ u = 1 \end{pmatrix}$

$t = \pi/2$, $u = \sin(\pi/2) = 1$

$$= -256 \int_0^1 u^3\,du. = -256\left[\frac{u^4}{4}\right]_0^1$$

$$= -256 \cdot \frac{1}{4} \cdot 0 = \frac{-256}{4} = \underline{\underline{-64}}$$

b) $\int_C xy^2\,dy = ?$

$$\boxed{\int_C G(x,y)\,\underline{dy} = \int_a^b G[f(t), g(t)] \cdot \underline{g'(t)}\,dt}$$

$$= \int_0^{\pi/2} 4\cos t \cdot (4\sin t)^2 \cdot 4\cos t\,dt$$

$$= 256 \int_0^{\pi/2} \sin^2 t \cos^2 t\,dt.$$

$$= 256 \int_0^{\pi/2} (\cos t \sin t)^2\,dt.$$

$= 64 \int_0^{\pi/2} (2\sin t \cos t)^2 \, dt.$

$2\sin t \cos t = \sin 2t$

$= 64 \int_0^{\pi/2} (\sin 2t)^2 \, dt.$

$= 64 \int_0^{\pi/2} \sin^2 2t \, dt$

$\boxed{\sin^2 t = \frac{1-\cos 2t}{2}}$

$= 64 \int_0^{\pi/2} \frac{1-\cos 4t}{2} \, dt.$

$= \frac{64}{2} \int_0^{\pi/2} (1-\cos 4t) \, dt.$

diff → chain rule → x

$\int \to \div$

$\frac{d}{dt}(-\cos t) = \sin t.$

$= 32 \left[t - \frac{\sin 4t}{4} \right]_0^{\pi/2}$

$= 32 \left[\frac{\pi}{2} - \frac{\sin 4 \cdot \pi/2}{4} - 0 \right].$

$= 32 \left[\frac{\pi}{2} - \frac{\sin 2\pi}{4} \right]$

$\boxed{\sin \pi = 0 \quad ; \sin 2\pi = 0}$

$= 32 \left[\frac{\pi}{2} \right] = \underline{16\pi}$

c) $\int_C xy^2 \, ds =$

$\boxed{\int_C G(x,y) \, ds = \int_a^b G[f(t), g(t)] \cdot \sqrt{(f'(t))^2 + (g'(t))^2} \, dt}$

$= \int_0^{\pi/2} 4\cos t \cdot (4\sin t)^2 \cdot \sqrt{(-4\sin t)^2 + (4\cos t)^2}$

$= \int_0^{\pi/2} 4\cos t \; 4^2 \sin^2 t \cdot \sqrt{16(\sin^2 t + \cos^2 t)}$

$= \int_0^{\pi/2} 4\cos t \cdot 16\sin^2 t \cdot 4 \, dt.$

$= 256 \int_0^{\pi/2} \sin^2 t \cdot \cos t \, dt. \quad \text{—(1)}$

Sub. method

$u = \sin t.$

$du = \cos t \, dt.$

limit: $t = 0, \; u = 0$; $t = \frac{\pi}{2}, \; u = 1$

(1) $= 256 \int_0^1 u^2 \cdot du.$

$= 256 \left[\frac{u^3}{3} \right]_0^1 du. = 256 \cdot \frac{1}{3} = \underline{\frac{256}{3}}$

2) Let C denote the line segment $y = 2x+1$; $-1 \le x \le 0$ & $G(x,y) = 3x^2 + 6y^2$ find line $\int$ -

a) $\int_C G(x,y)\,dx$ (b) $\int_C G(x,y)\,dy$ (c) $\int_C G(x,y)\,ds.$

y = explicit (C)

A) $y = 2x+1. = f(x)$

$\boxed{dy = f'(x)\,dx} = 2\,dx.$

$ds = \sqrt{1+(f'(x))^2}\,dx = \sqrt{1+2^2}\,dx = \underline{\sqrt{5}\,dx}$

a) $\boxed{\int_C G(x,y)\,dx = \int_a^b G(x, f(x))\,dx}$ (1 variable)

$= \int_{-1}^0 3x^2 + 6(2x+1)^2 \, dx.$

$(a+b)^2 = a^2+2ab+b^2$ $= 4x^2+4x+1$

$3x^2 + 6y^2 \Rightarrow 3x^2 + 6(4x^2+4x+1)$

$= 3x^2 + 24x^2 + 24x + 6$

$= 27x^2 + 24x + 6$

$= \int_{-1}^0 27x^2 + 24x + 6 \, dx.$

$= \left[27 \frac{x^3}{3} + 24 \frac{x^2}{2} + 6x \right]_{-1}^0$

$= \left[9x^3 + 12x^2 + 6x \right]_{-1}^0$

$-[3-6] = -[-3] = 3$

$= [0 - (-9+12-6)] = 3$

b) $\int_c G(x,y)\,dy = \int_a^b G(x, f(x))\, f'(x)\,dx$.

$= \int_{-1}^{0} [3x^2 + 6(2x+1)^2]\, 2\,dx$.

$= 2\int_{-1}^{0} 27x^2 + 24x + 6\,dx$.

$= 2\left[9x^3 + 12x^2 + 6x\right]_{-1}^{0}$ $\quad 2[3] = 6$

$= 2\left(0 - (-9 + 12 - 6)\right) = 6$

c) $\int_c G(x,y)\,ds = \int_a^b G(x, f(x)) \cdot \sqrt{1 + (f'(x))^2}\,dx$

$= \int_{-1}^{0} \left[3x^2 + 6(2x+1)^2\right] \sqrt{5}\,dx$.

$= \sqrt{5} \int_{-1}^{0} 27x^2 + 24x + 6\,dx$.

$= \sqrt{5}\left[9x^3 + 12x^2 + 6x\right]_{-1}^{0}$

$= \sqrt{5}\left[0 - (-9 + 12 - 6)\right] = 3\sqrt{5}$

3) Evaluate $\int_c xy\,dx + x^2\,dy$.

c is gvn by $y = x^3$, $-1 \le x \le 2$.

A) [$y =$ ... explicit ...]

$\therefore y = x^3 = f(x)$

$dy = 3x^2\,dx$,

$\int_c xy\,dx + x^2\,dy = \int_{-1}^{2} x(x^3)\,dx + x^2(3x^2\,dx)$

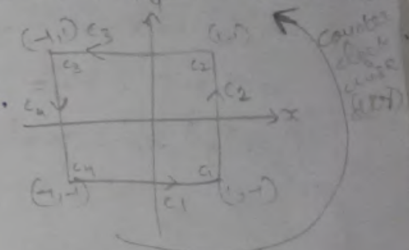
$= \int_{-1}^{2} [x^4 + 3x^4]\,dx = \int_{-1}^{2} 4x^4\,dx$

$= 4\int_{-1}^{2} x^4\,dx = 4\left[\frac{x^5}{5}\right]_{-1}^{2} = 4\left[\frac{32}{5} - \frac{-1}{5}\right] = \frac{132}{5}$

4) Evaluate $\oint_c x^2 y^3\,dx - xy^2\,dy$.

c is the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ & $(-1,1)$ in the counter clockwise (dir)?

A) Let c_1, c_2, c_3 & c_4 be the sides of square c, joining $(-1,-1)$ to $(1,-1)$, $(1,-1)$ to $(1,1)$, $(1,1)$ to $(-1,1)$ & $(-1,1)$ to $(-1,-1)$



c is composed of smooth curves c_1, c_2, c_3, c_4.

(para) eq —

$c_1 = x = t$, $y = -1$, $-1 \le t \le 1$

$c_2 = x = 1$, $y = t$, $-1 \le t \le 1$

$c_3 = x = -t$, $y = 1$, $-1 \le t \le 1$

$c_4 = x = -1$, $y = -t$, $-1 \le t \le 1$

I) On c_1: $x = t$, $y = -1$ & $-1 \le t \le 1$. Then

$dx = dt$, $dy = 0$

$\int_{c_1} x^2 y^3\,dx - xy^2\,dy = \int_{-1}^{1} t^2(-1)^3\,dt - t(-1)^2 \cdot 0$

$= -\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$

(ii) On C_2, $x=1$, $y=t$, $-1 \le t \le 1$. Then
$dx = 0$, $dy = dt$.

$$\int_{C_2} x^2y^3\,dx - xy^2\,dy = \int_{-1}^{1} 1\cdot t^3\cdot 0 - 1\cdot t^2\,dt$$

$$= -\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$$

(iii) On C_3, $x=-t$, $y=1$ & $-1 \le t \le 1$. Then
$dx = -dt$, $dy = 0$

$$\int_{C_3} x^2y^3\,dx - xy^2\,dy = \int_{-1}^{1} (-t)^2(1)^3(-dt) - (-t)\cdot 1\cdot 0$$

$$= -\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$$

(iv) On C_4, $x=-1$, $y=-t$, $-1 \le t \le 1$. Then
$dx = 0$, $dy = -dt$

$$\int_{C_4} x^2y^3\,dx - xy^2\,dy = \int_{-1}^{1} (-1)^2(-t)^3\cdot 0 - (-1)(-t)^2(-dt)$$

$$= -\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{-2}{3}$$

$$\therefore \oint_C x^2y^3\,dx - xy^2\,dy = \frac{-2}{3} + \left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right)$$

$$= \frac{-8}{3}$$

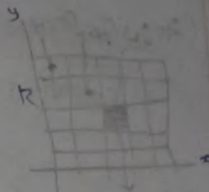
Rmrk

$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$$

Line $\int$ is independent of C.

⇒ Double integrals :-

- Let $z = f(x,y)$ be defined in a closed & bounded region 'R' of 2 space.



- By means of a grid of vertical & horizontal lines ‖ to co-ordinates axis form a partition $P(R)$. n rectangular sub-regions r_k of areas ΔA_k, that lie entirely in R.
- Let $\|P\|$ be the norm of the partition or the length of longest diagonal of r_k.
- Choose a sample point (x_k^*, y_k^*) in each of sub-region r_k.
- Form the sum, $\boxed{\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k}$

$$\boxed{\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k}$$

- Let f be a ( ) of 2 variables defined on a closed region 'R' of 2 space, then the $\iint$ of f over R is given by.

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k.$$

$$\boxed{\text{Area, } A = \iint_R dA}$$

$$\boxed{\text{Volume},\ v = \iint_R f(x,y)\,dA}$$

Properties →

1) constant multiple rule :-

$$\iint_R c\, f(x,y)\,dA = c\iint_R f(x,y)\,dA.$$

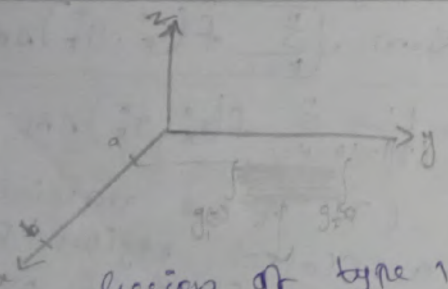
2) Sum & difference rule :-

$$\iint_R \left[f(x,y) \pm g(x,y)\right] dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA.$$

3) Additivity rule :-

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

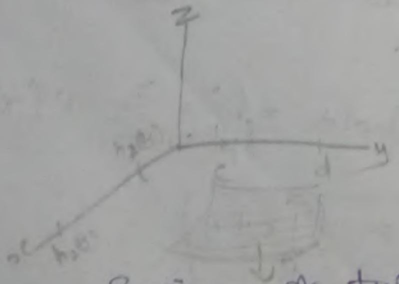
⇒ Evaluation of double $\iint$:-



Region of type 1

$R_1 :\ a \le x \le b$

$g_1(x) \le y \le g_2(x).$



$R_2 :\ c \le y \le d$

$h_1(y) \le x \le h_2(y).$

Region of type II

→ Fubini's Theorem :-

1) $\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx,$ for type 1

2) $\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$ for type II.

Q/1) If $f(x,y) = x^2y - 2xy$, $R: 0 \le x \le 3$, $-2 \le y \le 0$, then Evaluate $\iint_R f(x,y)\,dA$?

A) $\iint_R f(x,y)\,dA = \int_0^3 \int_{-2}^0 x^2y - 2xy\ dy\,dx$

$= \int_0^3 x^2\frac{y^2}{2} - \frac{2xy^2}{2}\ dx.$

$= \int_0^3 \left[x^2\frac{y^2}{2} - xy^2\right]_{-2}^0 dx.$

$= \int_0^3 \left[0 - \left(x^2\cdot 2 - x\cdot 4\right)\right] dx.$

$= \int_0^3 -(2x^2 - 4x)\ dx.$ (arrange)

$= \int_0^3 (-2x^2 + 4x)\ dx.$

$= \int_0^3 4x - 2x^2 \cdot dx.$

$= \left[4\cdot\frac{x^2}{2} - 2\cdot\frac{x^3}{3}\right]_0^3$

$= \left[2x^2 - 2\frac{x^3}{3}\right]_0^3$

$= 2\cdot 9 - 2\cdot\frac{27}{3} = 18 - 18$

$= 0$ //

2) $\int_0^1 \left[\int_0^2 xy\,(x-y)\,dx\right] dy.$

A) $= \int_0^1 \left[\int_0^2 x^2y - xy^2\,dx\right] dy.$

$= \int_0^1 \left[\frac{x^3}{3}y - \frac{x^2}{2}y^2\right]_0^2 dy.$

$= \int_0^1 \left[\frac{8}{3}y - 2y^2\right] dy.$

$= \left[\frac{8}{3}\frac{y^2}{2} - 2\frac{y^3}{3}\right]_0^1$

$= \left[\frac{8}{3}\cdot\frac{1}{2} - 2\cdot\frac{1}{3}\right] = \frac{4}{3} - \frac{2}{3}$

$= \frac{4-2}{3} = \frac{2}{3}$

⇒ **Line ∫-s in space :-**

line ∫ of a CD $G(x,y,z)$ of 3 variables along a curve c in space are defined in a way similar to line ∫ in plane.

Let c be a smooth curve in 3space & (para) eq be $x=f(t)$, $y=(t)$ $z=h(t)$ $a \le t \le b$. Then line ∫ -

☆ $\int_c G(x,y,z)\,dx = \int_a^b G(f(t),g(t),h(t))\,f'(t)\,dt.$

☆ $\int_c G(x,y,z)\,dy = \int_a^b G(f(t),g(t),h(t))\,g'(t)\,dt$

☆ $\int_c G(x,y,z)\,dz = \int_a^b G(f(t),g(t),h(t))\,h'(t)\,dt.$

☆ $\int_c G(x,y,z)\,ds = \int_a^b G(f(t),g(t),h(t))\cdot\sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\,dt$

$x=f(t)$
$y=g(t)$
$z=h(t)$

Q 1) Let c denote curve defined by $x = \frac{1}{3}t^3$, $y=t^2$, $z=2t$, $0\le t\le 1$. evaluate

a) $\int_c 4xyz\,dx$ (b) $\int_c 4xyz\,dy$

(c) $\int_c 4xyz\,dz$ (d) $\int_c 4xyz\,ds.$

A) $G(x,y) = 4xyz$ $g(t)=t^2=y$ $h(t)=2t=z$

$f(t) = \frac{1}{3}t^3 = x$

$a=0,\ b=1$

$dx = f'(t)\,dt$ ~~$= t^2\,dt$~~ $\frac{1}{3}\cdot 3t^2 = t^2\,dt$

$dy = g'(t)\,dt = 2t\,dt.$

$dz = h'(t)\,dt = 2\,dt.$

$ds = \sqrt{(f'(t))^2+(g'(t))^2+(h'(t))^2} = \sqrt{t^4+4t^2+4}\,dt$

$= \sqrt{(t^2+2)^2} = (t^2+2)\,dt$

$(a+b)^2$ expansion

(a) $\int_c 4xyz\,dx = \int_c \underset{(3\ \text{variable})}{G(x,y,z)}\,dx = \int_a^b G(f(t),g(t),h(t))\,f'(t)\,dt$

$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2\cdot 2t\cdot t^2\,dt$

$= \int_0^1 \frac{8t^8}{3}\,dt = \frac{8}{3}\int_0^1 t^8\,dt.$

$= \frac{8}{3}\left[\frac{t^9}{9}\right]_0^1 = \frac{8}{3}\cdot\frac{1}{9} = \frac{8}{27}$

b) $\int_C 4xyz\, dy = \int_a^b G(f(t), g(t), h(t))\, g'(t)\, dt$

$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t \cdot \cancel{x}\, dt$

$= \frac{16}{3}\int_0^1 t^7\, dt$

$= \frac{16}{3}\left[\frac{t^8}{8}\right]_0^1 = \frac{16}{3}\cdot\frac{1}{8} = \frac{16}{24} = \frac{2}{3}$

c) $\int_C 4xyz\, dz = \int_a^b G(f(t), g(t), h(t))\, h'(t)\, dt$

$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t \cdot 2\, dt$

$= \frac{16}{3}\int_0^1 t^6\, dt = \frac{16}{3}\left[\frac{t^7}{7}\right]_0^1$

$= \frac{16}{3}\cdot\frac{1}{7} = \frac{16}{21}$

d) $\int_C 4xyz\, ds = \int_a^b G(f(t), g(t), h(t))\, ds$

$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t \cdot (t^2+2)\, dt$

$= \frac{8}{3}\int_0^1 t^6 (t^2+2)\, dt$

$= \frac{8}{3}\int_0^1 t^8 + 2t^6\, dt$

$= \frac{8}{3}\left[\frac{t^9}{9} + \frac{2t^7}{7}\right]_0^1 = \frac{8}{3}\left[\frac{1}{9}+\frac{2}{7}\right]$

$= \frac{200}{189}$

2) $\int_C y\, dx + z\, dy + x\, dz$ $C_1 \to (0,0,0)\ (2,3,4)$, $C_2 \to (2,3,4)\ (6,8,5)$

'C' consists of line segment from $(0,0,0)$ to $(2,3,4)$ & from $(2,3,4)$ to $(6,8,5)$

A) Let $C_1 \to (0,0,0)$ to $(2,3,4)$. Then C is (subtract same component) || to vector.

$(2-0)i + (3-0)j + (4-0)k$ (i.e) $2i+3j+4k$

& passes through $(0,0,0)$, hence (para) eq is $x = 0+2t$, $y = 0+3t$; $z = 0+4t$

$(0+2)t \quad (0+3)t \quad (0+4)t$

(i.e) $x = 2t$; $y = 3t$, $z = 4t$.

$(0,0,0)$ when $t=0$, & passes through $(2,3,4)$ when $t=1$.

(para) eq of C_1,

$x = 2t \quad y = 3t \quad z = 4t \quad 0 \le t \le 1$

* Let $C_2 \to (2,3,4)$ to $(6,8,5)$. Then C_2 is || vector

$(6-2)i + (8-3)j + (5-4)k$ (i.e) $4i+5j+k$

& passes through $(2,3,4)$. (para) eq,

$x = 2+4t$, $y = 3+5t$, $z = 4+t$

$(2+4)t \quad (3+5)t \quad (4+t)$

above lines passes through $(2,3,4)$ when $t=0$. & passes through $(6,8,5)$ when $t=1$

(para) eq,

$x = 2+4t \quad y = 3+5t \quad z = 4+t \quad 0 \le t \le 1$

* On C_1, $x = 2t \quad y = 3t \quad z = 4t \quad 0 \le t \le 1$

$dx = 2 \quad dy = 3 \quad dz = 4$

Then $dx = 2\,dt$, $dy = 3\,dt$.
$dz = 4\,dt$

$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 \overset{6}{3t \cdot 2\,dt} + \overset{12}{4t \cdot 3\,dt} + \underset{=26}{2t \cdot 4\,dt}$

$= \int_0^1 26t\,dt = \left(\frac{13}{26}\frac{t^2}{2}\right)_0 = [13t^2]_0^1 = \underline{\underline{13}}$

* On C_2, $x = 2+4t$, $y = 3+5t$, $z = 4+t$, $0 \le t \le 1$

$dx = 4\,dt$, $dy = 5\,dt$ $dz = dt$.

$\int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 (3+5t) \cdot 4\,dt + (4+t) \cdot 5\,dt + (2+4t)\,dt$

$= \int_0^1 (34 + 29t)\,dt$

$= \left[34t + \frac{29}{2}t^2\right]_0^1 = \underline{\underline{\frac{97}{2}}}$

Since C is composed of smooth curves C_1 & C_2, the $\int$ over C = sum of $\int$ over C_1 & C_2

$\therefore \int_C y\,dx + z\,dy + x\,dz = \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz$

$= 13 + \frac{97}{2} = \frac{26 + 97}{2}$

$= \underline{\underline{\frac{123}{2}}}$

Remark → we can use the concept of a (v) (?) of several variable to write a general line $\int$ in a compact fashion

eg → $f(x,y) = P(x,y)i + Q(x,y)j$ is defined along the curve $C : x = f(t)$, $y = g(t)$, $a \le t \le b$

& suppose $r(t) = f(t)i + g(t)j$ is position (v) of points on C. Then $r'(t)$,

$\frac{dr}{dt} = f'(t)i + g'(t)j$

$= \frac{dx}{dt} i + \frac{dy}{dt} j$

to define , $dr = \frac{dr}{dt}\,dt = dx\,i + dy\,j$

Since $F(x,y) \cdot dr = P(x,y)\,dx + Q(x,y)\,dy$,

$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C F \cdot dr$

≈ for a line $\int$ on space curve,

$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$

$= \int_C F \cdot dr$

where $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$

& $dr = dx\,i + dy\,j + dz\,k$.

Q) If $F = 3xy\,i + y^2 j$, evaluate $\int_C F \cdot dr$

C is curve in xy-plane, $y = 2x^2$ from $(0,0)$ to $(1,2)$.

A) $y = 2x^2$

let $x = t$ then $y = 2t^2$

$dx = dt$ $dy = 4t\,dt$

$\boxed{dr = dx\,i + dy\,j}$ $\quad F = 3xy\,i - y^2\,j$

$0 \le t \le 1$

$\therefore \int_C F \cdot dr = \int_0^1 (3xy\,i - y^2\,j) \cdot (dx\,i + dy\,j)$

$x = t$
$y = 2t^2$
$dx = dt$
$dy = 4t\,dt$

$= \int_0^1 (3xy \cdot dx) - (y^2 \cdot dy)$

$= \int_0^1 3 \cdot t \cdot 2t^2\,dx - (2t^2)^2\,dy$

$= \int_0^1 3 \cdot t \cdot 2t^2 \cdot dt - 4t^4 \cdot 4t\,dt$

$= \int_0^1 6t^3 - 16t^5\,dt$

$= \left[6\frac{t^4}{4} - 16\frac{t^6}{6}\right]_0^1$

$= 6 \cdot \frac{1}{4} - 16\frac{1}{6} - 0$

$= \frac{6}{4} - \frac{16}{6} = \frac{3\times3}{2\times3} - \frac{8\times2}{3\times2}$

$= \frac{9-16}{6} = -\frac{7}{6}$

2) $F = (3x^2 + 6y)\,i - 14yz\,j + 20xz^2\,k$.
evaluate $\int_C F \cdot dr$ from $(0,0,0)$ to $(1,1,1)$
along the path $x = t$, $y = t^2$, $z = t^3$

A) $dr = dx\,i + dy\,j + dz\,k$; $\quad 0 \le t \le 1$

$\therefore \int_C F \cdot dr = \int_0^1 \left[(3x^2 + 6y)\,i - 14yz\,j + 20xz^2\,k\right] \cdot (dx\,i + dy\,j + dz\,k)$

$= \int_0^1 (3x^2 + 6y)\,dx - (14yz\,dy) + 20xz^2\,dz$

$= \int_0^1 (3t^2 + 6t^2)\,dt - (14t^2 \cdot t^3 \cdot 2t)\,dt + 20t(t^3)^2 \cdot 3t^2\,dt$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$

$= \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1$

$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$

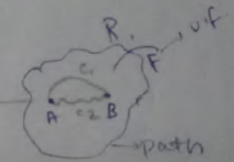
$= (3 - 4 + 6) - 0 = 5$

$\Rightarrow$ path independence and conservative vector field :-

$\boxed{\int F \cdot dr.}$

Let F be a field defined on an open region R in space & suppose that for any 2 points A & B in R, the value of the line $\int$ from A to B is same for every path in R from A to B. Then the $\int F \cdot dr$ is path independent in R.

path is =



$\int_A^B F \cdot dr$

(i.e.) $\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$

⇒ Theorem :-

Fundamental Theorem of line $\int$:-

[Relation b/w path independence & conservative v.f.]

Suppose C is a path in an open region R of xy plane & is defined by $r(t) = x(t)i + y(t)j$, $a \le t \le b$.

If $F(x,y) = p(x,y)i + q(x,y)j$ is a conservative v.f in R & ϕ is a potential ( ) for F, then

$$\int_C F\cdot dr = \int_{C\,(C\cdot V\cdot F)} \nabla\phi \cdot dr = \phi(B) - \phi(A)$$

where $A = (x(a), y(a))$ & $B = (x(b), y(b))$

* **Conservative v.f :-** (or gradient v.f)

F is said to be conservative, if F can be written as the gradient of a scalar ( ) ϕ.

$F(x,y) = yi + xj$

$$\boxed{F = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j}$$

$$= yi + xj = F(x,y).$$

Pf: Let us prove the theorem for a smooth curve C,

since F is conservative & ϕ is a potential ( ) for F,

$$\boxed{F = \nabla\phi} = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j \quad \text{— (1)}$$

path of $\int$ tion defined by,

$r(t) = x(t)i + y(t)j$

$\int F\cdot dr \longrightarrow$

$$\frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j \quad \text{— (2)} \quad a \le t \le b$$

$$\int_C F\cdot dr = \phi(B) - \phi(A) \quad \text{→ To prove}$$

Along the curve, ϕ is ~~defined by~~ differentiable ( ) of t & by chain rule,

$$\cancel{\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dt}} \quad \phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j$$

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\cdot\frac{dy}{dt}$$

(split)

$$= \left[\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j\right]\cdot\left[\frac{dx}{dt} i + \frac{dy}{dt} j\right]$$

(1) (2)

$$= \nabla\phi \cdot \frac{dr}{dt}$$

$$\frac{d\phi}{dt} = F\cdot\frac{dr}{dt} \quad \text{— (3)}$$

wants to prove,

$$\int_C F\cdot dr = \int_a^b F\cdot\frac{dr}{dt}\, dt.$$

$$\overset{(3)}{=} \int_a^b \frac{d\phi}{dt}\, dt.$$

$\therefore \phi[x(t), y(t)]_a^b$ $\quad$ $\cancel{\int \frac{d\phi}{dt}\, dt} = \phi$.

definite $\int$ නිසාවෙන් limit ආදේශකළ.

$\Rightarrow \phi[x(b), y(b)] - \phi[x(a), y(a)]$

$$\boxed{\int_C F \cdot dr = \phi(B) - \phi(A)}$$

Q) Sketch the region of ∫tion & evaluate (scalar ∫tion →)

$$\int_0^1 \int_0^{1-x} (x^2+y^2)\, dy\, dx.$$

A) 

A) $= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx.$

$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3} - 0 \right] dx.$ $\quad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$= \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3} \right] dx.$

$= \int_0^1 \left[x^2 - x^3 + \frac{(1 - 3x + 3x^2 - x^3)}{3} \right] dx.$

$= \int_0^1 \left[\frac{3x^2 - 3x^3 + 1 - 3x + 3x^2 - x^3}{3} \right] dx.$

$= \frac{1}{3}\int_0^1 1 - 4x^3 + 6x^2 - 3x \, dx.$

$= \frac{1}{3} \left[x - \cancel{4} \cdot \frac{x^4}{\cancel{4}} + \cancel{6}^{2} \cdot \frac{x^3}{\cancel{3}} - 3 \cdot \frac{x^2}{2} \right]_0^1$

$= \frac{1}{3}\left[1 - 1 + 2 - \frac{3}{2} \right] = \cancel{\frac{2}{1} + \frac{3}{2} = \frac{4+3}{2} = \frac{8}{2} = 4}$ $\quad \cancel{= 2 - \frac{3}{2} = \frac{4-3}{2} = \frac{1}{2}}$

$= \frac{1}{3}\left[\frac{1}{2}\right] = \frac{1}{6}$

graph

$0 \le x \le 1$

$0 \le y \le 1-x$

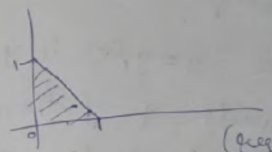
$y = 1 - x.$

$x + y = 1.$

if $x = 0$, $y = 1$

if $y = 0$, $x = 1$

$(1, 1)$



(region)

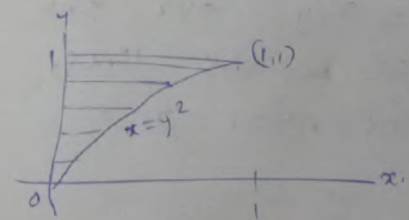
Q) $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy.$

$= \int_0^1 \left[3y^{3} \frac{e^{xy}}{y} \right]_0^{y^2} dy$

$= \int_0^1 \left[3y^2 e^{y^3} - 3y^2 \right] dy$ $\quad \boxed{\int 3y^2 e^{y^3} dy = e^{y^3}}$

$= \left[e^{y^3} - y^3 \right]_0^1 = e - 2$

$0 \le y \le 1 \qquad 0 \le x \le y^2$



To evaluate $\iint_R f(x,y)\,dA$, $\int$-ing with respect to x & y.

1) Sketch the region of $\int$-tion & label the bounding curves.

2) Imagine a vertical line 'l' cutting through 'r' in (dir) of increasing y. mark the y values, where 'l' enters & leaves the region 'r'.

Let these values be $y = g_1(x)$ & $y = g_2(x)$. this are the y limits of $\int$-tion.

3) Choose x-limits that includes all vertical lines through r.

4) Evaluate $\iint$ using the formula:

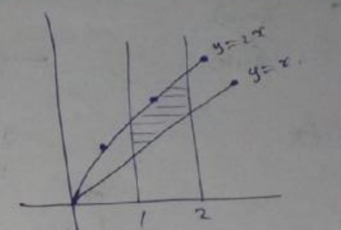
$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

5) To evaluate the $\iint$ as an iterated with the order of $\int$-tion reversed, use horizontal lines instead of vertical lines, then required $\int$ is,

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

6) $\int$-ate $f(x,y) = x/y$ over the regions bounded by the lines,

$y = x$, $y = 2x$, $x = 1$, $x = 2$



$x = 1,\ x = 2 \Rightarrow \int_1^2$

$y = x,\ y = 2x \Rightarrow \int_x^{2x}$

$$= \int_1^2 \int_x^{2x} \frac{x}{y}\,dy\,dx$$

$$= \int_1^2 \left[x \cdot \ln y\right]_x^{2x} dx \Rightarrow \int_1^2 \left[x \cdot \ln 2x - x \cdot \ln x\right] dx.$$

$\boxed{\ln A - \ln B = \ln (A/B)}$

$$= \int_1^2 x\left[\ln 2x - \ln x\right] dx.$$

$$= \int_1^2 x\left[\ln\left(\frac{2x}{x}\right)\right] dx = \int_1^2 x \cdot \ln 2 \cdot dx$$

$$= \ln 2 \int_1^2 x \cdot dx = \ln 2 \cdot \left[\frac{x^2}{2}\right]_1^2$$

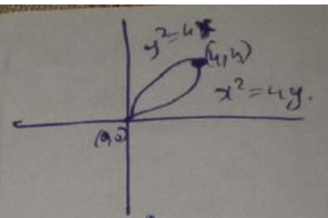
$$= \frac{\ln 2}{2}\left[4 - 1\right] = \frac{\ln 2}{2}\left[3\right]$$

$$= \frac{3}{2} \ln 2$$

Q/1) Evaluate $\iint_R y\,dy\,dx$, where R is region bounded by parabola's

$y^2 = 4x$, $x^2 = 4y$

if $\begin{pmatrix} x=0 \\ x=4 \end{pmatrix} \begin{pmatrix} y=0 \\ y=4 \end{pmatrix}$ $\begin{matrix}(0,0)\\(4,4)\end{matrix}$



$$= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y \, dy \, dx.$$

<u>y limit</u>

$y^2 = 4x$

$y = \sqrt{4x} = 2\sqrt{x}.$

$\int_{x^2/4}^{2\sqrt{x}} \Rightarrow$ $x^2 = 4y$

$y = x^2/4$

$$= \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx.$$

$$= \int_0^4 \left[\frac{(2\sqrt{x})^2}{2} - \frac{(x^2/4)^2}{2}\right] dx. \quad \cancel{= \int_0^4 \left[\frac{4x - x^2}{2}\right] dx.}$$

$\left(\frac{x^2}{4}\right)^2 = \frac{x^4}{16}$

$$= \frac{1}{2} \int_0^4 \left[4x - \frac{x^4}{16}\right] dx.$$

$$= \frac{1}{2} \int_0^4 \left[\frac{64x - x^4}{16}\right] dx.$$

$$= \frac{1}{32} \int_0^4 (64x - x^4)\, dx = \frac{1}{32}\left[64\frac{x^2}{2} - \frac{x^5}{5}\right] dx$$

$$= \left[\frac{64}{32} \cdot \frac{x^2}{2} - \frac{x^5}{5 \times 32}\right]_0^4 = \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}} - \frac{x^5}{160}\right]_0^4$$

$$= \left[x^2 - \frac{x^5}{160}\right]_0^4 = 16 - \frac{1024}{160}$$

$$= \frac{2560 - 1024}{160} = \frac{\cancel{1536}^{\cancel{768}\ 384\ 192\ \cancel{96}\ 48}}{\cancel{160}_{80\ 40\ 20\ 10\ 5}}$$

$$= \frac{48}{5}$$