L'Hopital's Rule.

let f & g be differentiable on open interval containing x_0
assume that,

$$f(x_0) = g(x_0) = 0$$

If $g'(x_0) \neq 0$ then,

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}}$$

($\frac{0}{0}$ or $\frac{\alpha}{\alpha}$, use this rule) $\frac{d}{dx}$ of numtr. in limit

Note

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x_0)}{g'(x_0)} = l$$

Q
1) find $\lim_{x \to 1} \left(\frac{x^3-1}{x-1}\right)$

A) $\lim_{x \to 1} \left(\frac{x^3-1}{x-1}\right) = \frac{1^3-1}{1-1} = \frac{0}{0}$ form

by (L) rule,

$$\lim_{x \to 1} \left(\frac{x^3-1}{x-1}\right) = \lim_{x \to 1} \frac{3x^2-0}{1-0}$$

$$= \lim_{x \to 1} 3x^2 \quad \text{(give 1 to x)}$$

$$= 3 \times 1^2 = 3$$

$\frac{d}{dx}(c) = 0$

$\frac{d}{dx}(x) = 1$

(or)

$(a^3-b^3) = (a-b)(a^2+ab+b^2)$

$(x^3-1) = (x^3-1^3) = (x-1)(x^2+x+1)$

give limit —

$$\lim_{x \to 1} \left(\frac{x^3-1}{x-1}\right) = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}}$$

$$= \lim_{x \to 1} x^2+x+1$$

$$= 1+1+1 = 3$$

2) $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

$\frac{d}{dx}(\sin x) = \cos x$.

$\frac{d}{dx}(\cos x) = -\sin x$

$\sin 0 = 0$

$\cos 0 = 1$.

A) $\lim_{x\to 0} \frac{\cos 0 - 1}{\sin 0}$

$= \frac{1-1}{0} = 0/0$ form

$\tan x = \frac{\sin x}{\cos x}$

$\downarrow$

$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$

by (L),

$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \frac{-\sin x - 0}{\cos x}$ $(\Rightarrow \lim_{x\to 0} \tan 0 = 0)$

$= \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$

3) $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

A) $\lim_{x\to 0} \frac{\sin 0 - 0}{0} = \frac{0}{0}$ form

by (L),

$\lim_{x\to 0} \frac{\sin x - x}{x^3} \rightarrow \frac{d}{dx}$

$\lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \frac{\cos 0 - 1}{3\times 0} = \frac{1-1}{0} = \frac{0}{0}$

$\lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \frac{-\sin x - 0}{6x} = \frac{-\sin 0 - 0}{6\times 0}$

$= \frac{0-0}{0} = \frac{0}{0}$

$\lim_{x\to 0} \frac{-\sin x}{6x} = \frac{-\cos x}{6} = \frac{-\cos 0}{6} = \frac{-1}{6}$

4) $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

$\frac{d}{dx}(\tan x) = \sec^2 x$

A) $\lim_{x\to 0} \frac{\cos 0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

by (L),

$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x - 1}{2x}$

$= \lim_{x\to 0} \frac{-\sin 0 - 0}{2\times 0}$

$= \frac{0-0}{0} = \frac{0}{0}$ form

$\lim_{x\to 0} \frac{-\sin x}{2x}$

$= \frac{-\cos x}{2} = \frac{-\cos 0}{2} = \frac{-1}{2}$

Reciprocal

$\sec x = \frac{1}{\cos x}$

$\cot x = \frac{1}{\tan x}$

5) $\lim_{x\to 0} \frac{\sin x - x}{\tan x - x}$

A) $\lim_{x\to 0} \frac{\sin 0 - 0}{\tan 0 - 0} = \frac{0-0}{0-0} = \frac{0}{0}$

by (L),

$\lim_{x\to 0} \frac{\sin x - x}{\tan x - x} = \lim_{x\to 0} \frac{\cos x - 1}{\sec^2 x - 1}$

$(\sec^2 0) = (\sec 0)^2$

$= \frac{1}{\cos 0} = \frac{1}{1}$

$= 1$

$= \frac{\cos 0 - 1}{\sec^2 0 - 1} = \frac{1-1}{1-1} = \frac{0}{0}$

$\lim_{x\to 0} \frac{\cos x - 1}{\sec^2 x - 1} = \lim_{x\to 0} \frac{-\sin x - 0}{2\sec x(\sec x \tan x)}$

$\lim_{x\to0} \frac{-\sin x}{2\sec^2 x \tan x}$ (Rearrange only)

$\frac{d}{dx}x^2 = 2x.$

$\frac{d}{dx}(\sec^2 x) = 2\sec x\,(\sec x \tan x)$

(by p.rule)

$\lim_{x\to0} \frac{-\cos x}{2(\sec^2 x\cdot\sec^2 x + \tan x\cdot 2\sec x\cdot\sec x\tan x)}$

$= \frac{-\cos 0}{2(\sec^2 0\cdot\sec^2 0+\tan 0\cdot 2\sec 0\cdot\sec 0\tan 0)}$

$= \frac{-1}{2(1^2\cdot1^2+0)} = \frac{-1}{2}$

6) find $\lim_{x\to\infty} \frac{\ln x}{x^p}$, where $p>0$.

A) $\lim_{x\to\infty} \frac{\ln x}{x^p} = \frac{\ln\infty}{\infty^p} = \frac{\infty}{\infty}$

by (L),

$\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}}$

$= \lim_{x\to\infty}\left(\frac{1}{x}\cdot\frac{1}{px^{p-1}}\right)$

$= \lim_{x\to\infty} \frac{1}{x\cdot px^{p-1}}$

$= \frac{1}{\infty\cdot p\infty^{p-1}} = \frac{1}{\infty} = 0$

(∞ → large value)

$e^0 = 1$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}(1/x) = \frac{-1}{x^2}$

$\frac{a/b}{c/d} = \frac{a}{b}\times\frac{d}{c}$

$\frac{1}{\infty} = 0$

$\ln\infty = \infty$

$\ln 0 = -\infty$

$\ln 1 = 0$

$1/0 = \infty$

7) find $\lim_{x\to0^+} x\ln x$ (ng)

A) $\lim_{x\to0^+} x\ln x = 0\cdot\ln 0$

$= 0\cdot-\infty = -\infty$ (indeterminate form)

(we can't stop here. So we arrange it)

by (L),

$\lim_{x\to0^+} x\ln x = \lim_{x\to0^+}\frac{\ln x}{1/x}$

$= \frac{\ln 0}{1/0} = \frac{-\infty}{\infty}.$

$\lim_{x\to0^+}\frac{\ln x}{1/x} = \lim_{x\to0^+}\frac{1/x}{-1/x^2}$ $\left(\frac{a/b}{c/d}\right)$

$= \lim_{x\to0^+} \frac{-1}{x}\cdot\frac{x^2}{1}$ (we can give to numtr)

$= \lim_{x\to0^+}(-x)$

$= -0 = 0$

Note

In determinate forms $(-\infty)$ of the type $0^0, 1^\infty$ can be handled by using Log.

1) find $\lim_{x\to0^+} x^x$.

A) $\lim_{x\to0^+} x^x = 0^0$

let $y = x^x$.

$\ln y = \ln x^x$

$= x\ln x.$

$\ln x^a = a\ln x.$

$\ln x^x = x\ln x$

$\lim_{x\to0} x\ln x = 0$

give exponential,

$$e^{\ln y} = e^{x\ln x}$$

(e [illegible] ln [illegible] which cancel)

$$y = e^{x\ln x}$$

$$\lim_{x\to 0} x^x = \lim_{x\to 0} y = \lim_{x\to 0^+} e^{x\ln x}$$

$$= e^{\lim_{x\to 0} x\ln x} \quad {=0}$$

(e [illegible] exponential [illegible] give limit to power)

$$= e^0 = 1$$

2) $\lim_{x\to 1} x^{\frac{1}{1-x}}$.

1) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

$$= 1^{\frac{1}{1-1}} = 1^\infty$$

let $y = x^{\frac{1}{1-x}}$

$$\ln y = \ln x^{\frac{1}{1-x}}$$

$$\ln y = \frac{1}{1-x}\ln x$$

$$e^{\ln y} = e^{\frac{1}{1-x}\ln x}$$

$$y = e^{\frac{1}{1-x}\ln x}$$

∴ $\lim_{x\to 1} y = \lim_{x\to 1} e^{\frac{1}{1-x}\ln x}$

$$= e^{\lim_{x\to 1}\frac{1}{1-x}\ln x} = e^{\lim_{x\to 1}\frac{\ln x}{1-x}}$$

$$= e^{\frac{\ln 1}{1-1}} = e^{\frac{0}{0}}$$

by (L)

$$e^{\lim_{x\to 1}\frac{\ln x}{1-x}} = e^{\lim_{x\to 1}\frac{1/x}{0-1}} = e^{\lim_{x\to 1}\frac{1/x}{-1}}$$

$$= e^{-1/x}$$

$$= e^{-1/1} = e^{-1}$$

1/6.

3) $\lim_{x\to 0}\left(\frac{1}{x\sin x} - \frac{1}{x^2}\right)$.

1) $\lim_{x\to 0}\left(\frac{1}{x\sin x} - \frac{1}{x^2}\right)$

$$= \frac{1}{0\cdot\sin 0} - \frac{1}{0^2} = \infty - \infty = \infty$$

So cross multiply,

$$\lim_{x\to 0}\frac{1}{x\sin x}\times\frac{1}{x^2} = \lim_{x\to 0}\frac{x^2 - x\sin x}{x^3\sin x}$$

$$= \lim_{x\to 0}\frac{\cancel{x}(x-\sin x)}{x^{\cancel{3}2}\sin x}$$

$$= \frac{x-\sin x}{x^2\sin x} = \frac{0-\sin 0}{0^2\cdot\sin 0} = \frac{0}{0}$$

$x^2 \cdot \sin x$

$x^2 \cdot \cos x + \sin x \cdot 2x$.

by (L),

$$\lim_{x\to 0} \frac{x - \sin x}{x^2 \cdot \sin x} \text{ (p. rule)}$$

$$\lim_{x\to 0} \frac{x - \sin x}{x^2 \cdot \cos x + \sin x \cdot 2x}$$

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2 \cdot \cos x + \sin x \cdot 2x}$$

$$= \frac{1 - \cos 0}{0 \cdot \cos 0 + \sin 0 \cdot 2\times 0} = \frac{1-1}{0} = \frac{0}{0}$$

by (L),

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2 \cdot \cos x + \sin x \cdot 2x} \text{ (p. rule)}$$

$$\lim_{x\to 0} \frac{0 - (-\sin x)}{x^2 \cdot -\sin x + \cos x \cdot 2x + \sin x \cdot 2 + 2x \cdot \cos x} = \frac{0}{0}$$

by L Hopi

rearrange

$$\lim_{x\to 0} \frac{\sin x}{-x^2 \sin x + 2x \cos x + 2\sin x + 2x \cos x}$$

$$\lim_{x\to 0} \frac{\sin x}{\underbrace{-x^2}_{1^{st}} \cdot \underbrace{\sin x}_{2^{nd}} + \underbrace{4x}_{1^{st}} \underbrace{\cos x}_{2^{nd}} + 2\sin x} \text{ (p. rule)}$$

$$\lim_{x\to 0} \frac{\cos x}{-x^2 \cdot \cos x + \sin x \cdot -2x + 4x \cdot \sin x + \cos x \, 4 + 2\cos x} = \frac{1}{6}$$

$$= \frac{}{4\cdot 1 + 2\cdot 1}$$

$$= 1/6$$