01: INTRODUTION TO DIFFERENTIAL EQUATION.

=> <u>ordinary & partial diff... eq</u> =

A D.eq involving a single independent variable & hence only ordinary deri... → <u>ordinary D.eq</u> (ODE)

eg → ① $(y^2+x)\dfrac{d^2y}{dx^2} + 2y\left(\dfrac{dy}{dx}\right)^2 = 7.$ → dependnt → x-nn depent ആശ്രിതമായി y.

② $y'' + (8x+3)y' + e^y \sin x = 0$

③ $y\left(\dfrac{dy}{dt}\right)^2 + 2t\dfrac{dy}{dt} - y = 0$

* A D.eq involving more than 1 independnt variable, & hence partial deri... → <u>partial diff... eq</u> (PDE)

eg → ① $t^2\dfrac{\partial^2 u}{\partial t^2} - x\left(\dfrac{\partial u}{\partial x}\right)^2 - \sin t\dfrac{\partial u}{\partial t} = 0$ u → dependnt, t, x → independnt

② $u_{xx} + u_{yy} + u_{zz} = 0$ → $\dfrac{\partial^2 u}{\partial x^2} = u_{xx}$

u → depndnt
x, y, z → indi...

* <u>Rmrk</u>

1) <u>Leibniz notation</u> = $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}$

2) <u>prime notation</u> = $y', y'', y''', y^{(4)}, y^{(5)}$; (4) → 4th deri...

nth deri... of f = $f^{(n)}$

=> <u>order & degree of D.eq</u> =

* The order of a D.eq is the order of highest deri... occuring in the eq.

* The degree of a D.eq is the degree of the highest deri... which occurs in it.

eg for order →

① $(y^2+x)\dfrac{d^2y}{dx^2} + 2y\left(\dfrac{dy}{dx}\right)^2 = 7$ (2nd deri., 1st deri.)

here order = 2.

Degree = 3 → ($\dfrac{d^2y}{dx^2}$ of (m) times 3rd)

② $(y'')^3 + 5xy' - 5xy = 8$

O = 2

D = 3

③ $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ O = 2, D = 1 (power → 1)

④ $\dfrac{\partial^2 z}{\partial x^2} + \left(\dfrac{\partial^2 z}{\partial y^2}\right)^2 = 0$ O = 2, D = 2

+ Rmrk

we can express n^{th} order ODE in 1 independent variable by the general form,

$$\boxed{F(x, y, y', y'', \dots, y^{(n)}) = 0} \quad ①$$

1st order ODE → $y' + x = 0$

2nd " → $y'' + x = 0$

F → real valued ( ) with n+2 variables.

($x, y, y', \dots, y^n$ = n+2)

* The D.eq $\dfrac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$

f → real valued contin ( ) of n+1 variables. $x, y, y', \dots, y^{(n-1)}$ is referred to as normal form of eq — ①

⇒ Linear & Non-linear D.eq =

$x' + 3 = 1$ → linear D.F.

$y' + 3y + x = 0$ → L.DF.

$(y'')^3 + 3y' = 0$ → non-L.

$x + 3 = 1$ → linear

$x^2 + 3 = 1$, $x^3 + 3 = 1$ → non-linear

* The ODE of order n $F(x, y, y', \dots, y^{(n)}) = 0$ is said to be linear if F is linear ( ) of variables $y, y', y'', \dots, y^{(n)}$.

* Any linear ODE of degree n can be return as,

$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = \phi(x)$.

$a_n(x)y^n$, $a_2(x)y''$

$a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ & $\phi(x)$ are ( )s of independent variable x.

* 2 imp sple cases are linear 1st order & linear 2nd order ODE'S,

1st o ← $a_1(x)\dfrac{dy}{dx} + a_0(x)y = \phi(x)$ — ① &

2nd o ← $a_2(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_0(x)y = \phi(x)$ — ②

format

* eg of linear D.eq →

1) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$. by — ②

2) $\dfrac{d^4y}{dt^4} + t^2\dfrac{d^3y}{dt^3} + 5t^3\dfrac{d^2y}{dt^2} + 6\sin t\, y = te^t$

* eq for non-linear D.eq →

1) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y^2 = 0$ y – where y^2.

2) $\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right)^2 + 6y = 0$. [in general eq no whole square]

3) $\dfrac{d^2y}{dx^2} + 5y\dfrac{dy}{dx} + 6y = 0$

⇒ Solution of a D.eq =

Any () ϕ defined on an interval 'I' & possessing atleast n deri... that are contin... on I, which when substituted into an n^{th} order ODE reduces the eq to an identity, it is said to be a soln of the eq

Q
1) S.T the () ϕ is defined by $\phi(t) = 2\sin t + 3\cos t$ is a soln of the follo... D.eq for all real t. by,
$y'' + y = 0$.

$x-2=0$
$x=0-2$
$x=2$
(or)
$x^2-2=0$
$2-2=0$

A) gvn $\phi(t) = 2\sin t + 3\cos t$

$y = \phi(t)$ $\quad y = \phi(t) = 2\sin t + 3\cos t$ —①
$y' = \phi'(t) = 2\cos t - 3\sin t$
$y'' = \phi''(t) = -2\sin t - 3\cos t$ —②

② + ①
$\cancel{y''+y=0}$
$y''+y = -2\sin t - 3\cos t + 2\sin t + 3\cos t = 0$

hence $\phi(t)$ is a soln of gvn D.eq.

2) Verify that the () $y = xe^x$ is a soln of the D.eq by $y'' - 2y' + y = 0$

A) $y = x \cdot e^x$
(product) $y' = x \cdot e^x + e^x = xe^x + e^x$
$y'' = x \cdot e^x + e^x + e^x$ or $xe^x + 2e^x$

$y'' - 2y' + y = xe^x + 2e^x - 2(xe^x + e^x) + xe^x$
$= xe^x + 2e^x - 2xe^x - 2e^x + xe^x$
$= x^{ex} - 2xe^x + xe^x$
$= 2xe^x - 2xe^x = 0$

hence y is the soln gvn D.eq

Rmrk

* A soln of a D.eq that is identically 0 on an interval 'I' is said to be a trivial soln

3) Determine the value of r for which the D.eq $y''' - 3y'' + 2y' = 0$ has the soln of the form $y = e^{rx}$?

A) Since y satisfies the gvn D.eq
$y = e^{rx}$
$y' = re^{rx}$
$y'' = r^2 e^{rx}$
$y''' = r^3 e^{rx}$

$y''' - 3y'' + 2y' = 0 \Rightarrow r^3 e^{rx} - 3(r^2 e^{rx}) + 2(re^{rx}) = 0$
$= r^3 e^{rx} - 3r^2 e^{rx} + 2re^{rx} = 0$
$= e^{rx}[r^3 - 3r^2 + 2r] = 0$
① ②

Q soln ...

① 0 ... / ② ... then only we get 0 as answer but $e^x \neq 0$ (always), so

$\Rightarrow r^3 - 3r^2 + 2r = 0$
$\Rightarrow r[r^2 - 3r + 2] = 0$ (qua eq) $\rightarrow \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
$\Rightarrow r(r-2)(r-1) = 0$
$r=0$ or $r-2=0$ or $r-1=0$
$r=0 \quad r=2 \quad r=1$
∴ r values 0, 1, 2

4) find the value of m so that $y=x^m$ is a solⁿ of $x^2y''-7xy'+15y=0$.

A)
$y=x^m$
$y'=m\,x^{(m-1)}$
$y''=(m-1)m\,x^{((m-1)-1)}=(m^2-m)x^{(m-2)}$

$x^2y''-7xy'+15y \Rightarrow x^2\left[(m^2-m)x^{(m-2)}\right]-7x\left[mx^{(m-1)}\right]+15x^m=0.$

$=x^{((m-2)+2)}(m^2-m)-7x^{((m-1)+1)}m+15x^m=0$

$=x^m(m^2-m)-7x^m m+15x^m$

$=x^m\left[m^2-m-7m+15\right]=0$

$=x^m\left[m^2-8m+15\right]=0 \rightarrow \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=x^m\left[(x-3)(x-5)\right]=0$

$\therefore m=3,5$ $\quad y_1=x^3,\ y_2=x^5 \Big\}\ y=x^m$.

⇒ **Explicit & implicit solⁿ :-**

A solⁿ in which the dependent variable is expressed in terms of the independent variable & constants is said to be Explicit solⁿ (i.e.) $\boxed{y=\phi x}$

A relation $\boxed{G(x,y)=0}$ is said to be an implicit solⁿ of an ODE on an interval 'I' provided there exist at least 1 () ϕ that satisfies the relation as well as the D.eq on 'I'.

eg → consider a relation $x^2+y^2=1$

diff w.r.t x, $\qquad \frac{d}{dx}(1)=0.$

$2x+2y\frac{dy}{dx}=0.$ $\quad$ on $-1<x<1$

$\therefore$ $x^2+y^2=1$ is an implicit solⁿ of $2x+2y\frac{dy}{dx}=0.$

y-কে x-এর () [illegible] express [illegible] → ex.

$x^2+y^2=1 \Rightarrow y^2=1-x^2$
$y=\pm\sqrt{1-x^2}.$

let (i) $y=\sqrt{1-x^2}=\phi_1(x)$ &
$y=-\sqrt{1-x^2}=\phi_2(x)$
satisfies the relation $x^2+(\phi_1(x))^2=1$ & $x^2+(\phi_2(x))^2=1$

& are ex. solⁿ defined on $(-1,1)$.

Q 1) (i) Verify that $-2x^2y+y^2=1$ is an implicit $G(x,y)=0$ solⁿ of $2xy\,dx+(x^2-y)\,dy=0$. (ii) find ex. solⁿ.

A) $-2x^2y+y^2=1$
diff w.r.t x, $\quad -2(x^2\cdot y)+y^2=1$

(i) ~~$-4xy+2y\frac{dy}{dx}=0$~~ $\quad =-2\left[x^2\cdot\frac{dy}{dx}+y\cdot 2x\right]+2y\frac{dy}{dx}=0$

$=-2x^2\frac{dy}{dx}-4xy+2y\frac{dy}{dx}=0$

$=2\left[-x^2\frac{dy}{dx}-2xy+y\frac{dy}{dx}\right]=0$

$=-x^2\frac{dy}{dx}-2xy+y\frac{dy}{dx}=0$

$=-2xy+\frac{dy}{dx}\left[-x^2+y\right]=0$

$=-\left[2xy+\frac{dy}{dx}(x^2-y)\right]=0$

$= 2xy + \frac{dy}{dx}(x^2 - y) = 0$

$\times\, dx$ through out,

$= 2xy\,dx + (x^2 - y)\,dy = 0$

$\therefore -2x^2y + y^2 = 1$ is an imp. Sol^n of given D. eq.

② ex Sol^n $\longrightarrow y = (\quad)$

To find ex. Sol^n consider $-2x^2y + y^2 = 1$

$y^2 - 2x^2y - 1 = 0$ (qua. eq)

$a = 1 \quad b = -2x^2 \quad c = -1$

$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2x^2 \pm \sqrt{(-2x^2)^2 - 4 \times 1 \times -1}}{2 \times 1}$

$= \frac{2x^2 \pm \sqrt{4x^4 + 4}}{2} = \frac{2x^2 \pm \sqrt{4(x^4+1)}}{2}$

$= \frac{\cancel{2}x^2 \pm \cancel{2}\sqrt{x^4+1}}{\cancel{2}} = x^2 \pm \sqrt{x^4+1}$

$\therefore y = x^2 + \sqrt{x^4+1} = \phi_1(x)$ &

$y = x^2 - \sqrt{x^4+1} = \phi_2(x)$

$y = (\quad)$ — sex. ()

$\phi_1(x)$ & $\phi_2(x)$ are the ex Sol^n of the relation $-2x^2y + y^2 = 1$

⇒ General & Particular Sol^n =

Consider the eq $y' = \cos x$

$\therefore \int y' = \int \cos x \Rightarrow y = \sin x + c$ //

c-ന് ഓരോ value കൊടുത്തിട്ട് നോക്കണം.
∴ general eq.

$\therefore \quad y = \sin x + c \longrightarrow$ General Sol^n.

If $y = \sin x + 1$, $y = \sin x + 0.5$, $y = \sin x \longrightarrow$ P. eq

* A Sol^n of a D. eq that is free of arbitory paramtrs $\longrightarrow$ particular Sol^n & obtained by giving a specific values to the paramtrs.

* If every Sol^n of an n^{th} order ODE on an interval 'I' can be obtained from an n-paramtr family $G(x, y, c_1, c_2 \cdots c_n) = 0$ by appropriate choices of the paramtr c_i, $i = 1, 2, 3 \cdots n$, we say that the family is the general Sol^n of the ODE.

⇒ System of D. eq =

A System of ODE is 2/more eq involving the deri.. of 2/more unknown ( )s of a single independnt variable.

eg → Consider the System

$\frac{dx}{dt} = f(t, x, y)$

$\frac{dy}{dt} = g(t, x, y)$.

dependnt .v → x, y

i. n. v → t.

independt (ഒന്നേ ഉള്ളൂ)

1) Varify that the pair of ( )s $x = e^{-2t} + 3e^{6t}$, $y = -e^{-2t} + 5e^{6t}$ is a Sol^n of the D. eq $\frac{dx}{dt} = x + 3y$ & $\frac{dy}{dt} = 5x + 3y$?

A) ~~To prove $\frac{dx}{dt} = x + 3y = e^{-2t} + 3e^{6t}$ & $\frac{dy}{dt} =$~~

$\frac{dx}{dt} = x + 3y$.

$\frac{dx}{dt} = \frac{d}{dt}\left(e^{-2t} + 3e^{6t}\right)$

$= e^{-2t} \times -2 + 3e^{6t} \times 6$

$= -2e^{-2t} + 18e^{6t}$ — ①

(want to get $x+3y$.)

eq ① ൽ x & y -ന്റെ terms ൽ ആക്കുക.

$\Rightarrow x+3y$

eg $5b = 3b+2b$

$x = \dots$, $y = \dots$ ②

$1 - 3e^{\prime\prime}$

$x+3y = e^{-2t} + 3e^{6t} + 3\left(-e^{-2t} + 5e^{6t}\right)$

$= 1e^{-2t} + 3e^{6t} - 3e^{-2t} + 15e^{6t}$.

$= -2e^{-2t} + 18e^{6t}$ — ②

∴ from ① & ②

$\frac{dx}{dt} = x + 3y$

$\frac{dy}{dt} = 5x + 3y$.

$\frac{dy}{dt} = \frac{d}{dt}\left(-e^{-2t} + 5e^{6t}\right)$

$= -e^{-2t} \times -2 + 5e^{6t} \times 6$

$= 2e^{-2t} + 30e^{6t}$ — ③

$5x+3y = 5\left(e^{-2t} + 3e^{6t}\right) + 3\left(-e^{-2t} + 5e^{6t}\right)$

$= 5e^{-2t} + 15e^{6t} - 3e^{-2t} + 15e^{6t}$

$= 2e^{-2t} + 30e^{6t}$ — ④

③ & ④ ∴ $\frac{dy}{dt} = 5x + 3y$

The pair of eq is a sol^n

2) Verify that each of given ( ) is the sol^n of the D.eq.

a) $y'' - y = 0$; $y_1(t) = e^t$, $y_2(t) = \cosh t$.

$y'' - y = 0$ ന്റെ sol^n, $y_1(t)$ & $y_2(t)$ ആണോ എന്ന് നോക്കുക. $=0$ ആണോ എന്ന് check ചെയ്യുക. If yes → sol^n.

A) consider $y_1 = e^t$

$y_1' = e^t$

$y_1'' = e^t$

$y''-y=0$, ∴ $y_1'' - y = e^t - e^t = 0$ → sol^n.

$y_2 = \cosh t$

$y_2' = \sinh t$

$y_2'' = \cosh t$

$y_2'' - y_2 = \cosh t - \cosh t = 0$ → sol^n

3) b) $2t^2y'' + 3ty' - y = 0$; $y_1(t) = t/3$, $y_2(t) = t^{1/2}$.

A) $y_1 = t/3$

$y_1' = 1/3 \rightarrow \frac{1}{3} \times t \rightarrow \frac{1}{3} \times 1 = 1/3$.

$y_1'' = 0$

$2t^2y_1'' + 3ty_1' - y_1 = 2t^3 \times 0 + 3t \times 1/3 - t/3$

$= t - \frac{t}{3} = \frac{3t-t}{3} = \frac{2t}{3} \neq 0$

not a sol^n

$\frac{1}{2} - 1 = \frac{1-2}{2} = -\frac{1}{2}$

$y_2 = t^{1/2}$

$y_2' = \frac{1}{2}t^{\frac{1}{2}-1} = \frac{1}{2}t = \frac{1}{2}t^{-1/2}$

$-\frac{1}{2} - 1 = \frac{-1-2}{2} = \frac{-3}{2}$

$y_2'' = \frac{1}{2} \times -\frac{1}{2}t^{-\frac{1}{2}-1} = \frac{-1}{4}t^{-3/2}$

$2t^2y_2'' + 3ty_2' - y_2 = \cancel{2}t^2\left(-\frac{1}{\cancel{4}_2}t^{-3/2}\right) + 3t\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2}$

$= -\frac{1}{2}$

$$= \frac{-1}{2} \times t^{2-3/2} + \frac{3}{2} t^{1-1/2} - t^{1/2}$$

$$t^2 \cdot t^{-3/2}$$

$$2 + {-3/2}$$

$$\frac{2}{1} - \frac{3}{2} = \frac{-3-4}{2}$$

$$= \frac{-2}{2} = \frac{-1}{2}$$

$$= \frac{-1}{2} t^{-1/2} + \frac{3}{2} t^{1/2} - t^{1/2}$$

$$= \frac{-1}{2} t^{-1/2} + \cancel{\frac{3}{2}} t^{1/2} \left[\frac{3}{2} - 1\right]$$

$$\frac{3}{2} - \frac{1}{1} = \frac{3-2}{2} = +\frac{1}{2}$$

$$= \frac{-1}{2} t^{-1/2} + \frac{1}{2} t^{1/2} = \frac{1}{2}\left[-t^{\cancel{-\frac{1}{2}} + \cancel{\frac{1}{2}}}\right]$$

$$= \frac{1}{2} \times -1 = -1/2 \neq 0 \quad \text{not a sol}^n$$

~~3) find the~~.