02: CALCULUS OF MULTIVARIABLE FUNCTIONS.

$w = f(x,y,z) = 3$ variable ( ) $\qquad z = f(x,y) \to 2$ variable ( )

Q 1) Sketch the graph & some of the level curves of the ( ) $\to f(x,y) = x^2 + y^2$.

A) $f(x,y)$ is a well defined real no. for all ordered pairs (x,y) of real no. Hence domain of f consists of entire xy-plane. Since $x^2+y^2 \geq 0$ for all (x,y) in xy plane, range of f is the set of all non -ve no.

The graph of f is the circular paraboloid $z = x^2+y^2$, ~~as shown in the~~

$x^2+y^2=r^2$

The level curve $z=4$ is the set of points in xy plane,

Suppose, $r=0 \to x^2+y^2=0 \quad z=0$

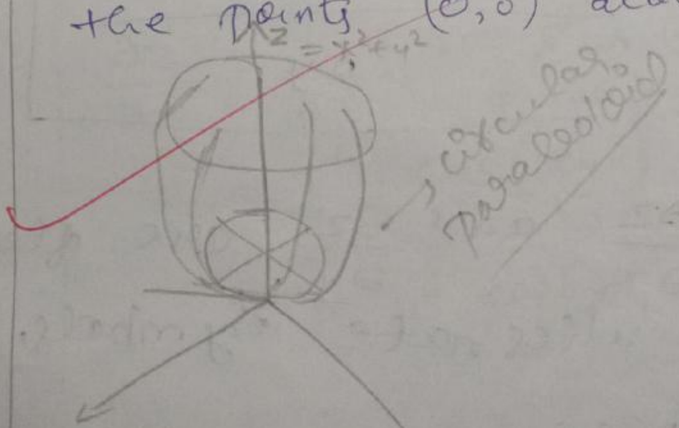
$r=1 \to x^2+y^2=1 \quad z \to z=1$

$r=2 \to x^2+y^2=4 \quad z=4$

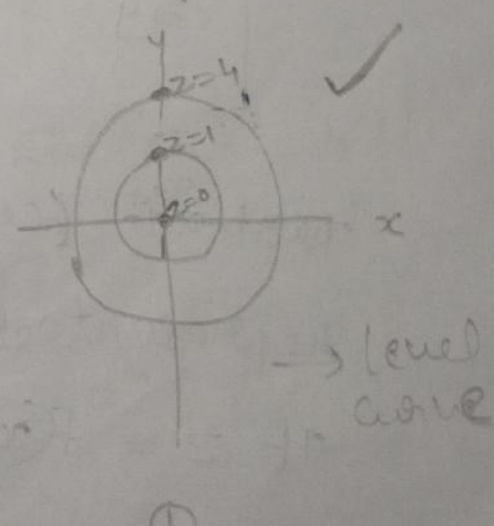
$z = f(x,y) = x^2+y^2 = 4$, (i.e) $x^2+y^2=4$.

which is a circle with center at the origin & radius 2.

≈ the level curve $z=1$ is a circle with center at the origin & radius 1 & level curve $z=0$ is a ○ at origin, the points $(0,0)$ alone.



(a)



(b)

2) Describe level surfaces of ()

$$f(x,y,z) = \frac{x^2+y^2}{z}$$

A) for $c \neq 0$. level surfaces,

$$\frac{x^2+y^2}{z} = c \quad , (i.e) \; x^2+y^2 = cz$$

which are paraboloids in 3-space.

⇒ **Partial Derivatives :-**

→ **Partial Derivatives of $z = f(x,y)$:-**

(Deri) of a () of 1 variable

$y = f(x)$,

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

If $z = f(x,y)$ then partial (deri) of z with respect to x is,

$$\boxed{\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}}$$

& with respect to y is

$$\boxed{\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}}$$

partial (deri) $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are often represented by alternate symbols, If $z = f(x,y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x \qquad \text{\&}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y$$

Definition of partial (deri) of z with respect x → calculate $\frac{\partial z}{\partial x}$, we have to diff z with respect to x in usual way treating y as constant. Same as $\frac{\partial z}{\partial y}$.

Q)

1) If $z = 3x^3y + 4xy^2 - 2x + 4y - 5$, $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$?

A) Diff z with respect to x, y as constant,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(3x^3y + 4xy^2 - 2x + 4y - 5\right)$$

$$= 9x^2y + 4y^2 - 2$$

(x not terms as constant)

with respect to y,

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(3x^3y + 4xy^2 - 2x + 4y - 5\right)$$

$$= 3x^3 + 8xy - 4$$

(y not terms as constant)

2) $z = x^4 \sin(xy^3)$, $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$?

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^4 \cdot \sin(xy^3)\right) \quad \text{(P. rule)}$$

$$= x^4 \cdot \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3$$

$$= x^4y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^4 \cdot \sin(xy^3)\right) = x^4 \left[\frac{\partial}{\partial y} \sin(xy^3)\right]$$

with respect to y only, and x constant

$= x^4 \cos(xy^3) \cdot 3xy^2$

$= 3x^5y^2 \cos(xy^3)$

$\boxed{\tan^{-1}(x) = \frac{1}{1+x^2}}$

3) $f(x,y) = x \tan^{-1}(xy)$, $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$?

A) $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x \cdot \tan^{-1}(xy)\right) \rightarrow$ p. rule

$= x \cdot \frac{1}{1+(xy)^2} \cdot y + \tan^{-1}(xy)$

$= \frac{xy}{1+x^2y^2} + \tan^{-1}(xy)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x \cdot \tan^{-1}(xy)\right) = x \cdot \left[\frac{\partial}{\partial y} \tan^{-1}(xy)\right]$

$= x \cdot \frac{1}{1+(xy)^2} \cdot x = \frac{x^2}{1+x^2y^2}$

$\Rightarrow$ partial deri. of () of more than 2 variables

Q/ 1) $F(x,y,t) = -4e^{3\pi t} \cos 4x \sin 6y$.

find partial deri. of F with respect to x, y & t

(-4) → constant → take 2nd

A) x

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{-3\pi t} \cos 4x \sin 6y\right)$ $e^{-3\pi t} \rightarrow e^{-3\pi t}$

$\cos 4x \rightarrow -\sin 4x \cdot 4$

$\sin 6y \rightarrow \sin 6y$

$= -4e^{-3\pi t} \sin 4x \sin 6y$

y

$\frac{\partial}{\partial y}\left(e^{-3\pi t} \cos 4x \sin 6y\right) \rightarrow e^{-3\pi t} \rightarrow e^{-3\pi t}$

$\cos 4x \rightarrow \cos 4x$

$\sin 6y \rightarrow \cos 6y \cdot 6$

$= 6e^{-3\pi t} \cos 4x \cos 6y$

at

$\frac{\partial}{\partial t}\left(e^{-3\pi t} \cos 4x \sin 6y\right) = -3\pi e^{-3\pi t} \cos 4x \sin 6y$

$\frac{\partial}{\partial t}(e^{-3\pi t}) \rightarrow e^{-3\pi t} \cdot \frac{\partial}{\partial t}(-3\pi t) = e^{-3\pi t} \cdot -3\pi$

2) $f(x,y,z) = \ln(x+2y+3z)$, f_x, f_y & f_z ?

$\frac{d}{dx} \ln x = \frac{1}{x}$

A) $f_x = \frac{\partial}{\partial x}\left(\ln(x+2y+3z)\right)$

$= \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial x}(x+2y+3z)$

$= \frac{1}{x+2y+3z}$

$f_y = \frac{\partial}{\partial y}\left(\ln(x+2y+3z)\right)$

$= \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial y}(x+2y+3z) = \frac{2}{x+2y+3z}$

$f_z = \frac{\partial}{\partial z}\left(\ln(x+2y+3z)\right)$

$= \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial z}(x+2y+3z) = \frac{3}{x+2y+3z}$

3) If resistors of R_1, R_2 & R_3 ohms are connected in || to make an R-ohm resistor, the value of R can be found from the eq —

$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.

find value of $\frac{\partial R}{\partial R_2}$ when $R_1 = 30$ $R_2 = 45$ $R_3 = 90$ ohms

A) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

diff. on both sides with respect to R_2

$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$

$-\frac{1}{R^2} \frac{\partial R}{\partial R_2} = 0 + -\frac{1}{R_2^2} + 0$

$$= -\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 + -\frac{1}{R_2^2} + 0$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2} = \left(\frac{R}{R_2}\right)^2$$

when $R_1 = 30$ $R_2 = 45$ $R_3 = 90$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{1}{15}$$

So $R = 15$ & $\boxed{\frac{\partial R}{\partial R_2} = \left(\frac{R}{R_2}\right)^2} = \left(\frac{15}{45}\right)^2 = \frac{1}{9}$

⇒ partial deri.. of higher orders :-

second order partial deri... of $f(x,y)$ are –

$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}$ or $f_{xx} = (f_x)_x$.

diff... f partially, twice with respect to x,

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ or $f_{yy} = (f_y)_y$

diff... f partially, twice with respect to y,

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ or $f_{xy} = (f_x)_y$

here f is differentiated partially with respect to both x & y (mixed 2nd order partial deri...) –

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ or $f_{yx} = (f_y)_x$.

According to Euler's theorem (mixed deri... theorem) → the mixed 2nd order partial deri... of $f(x,y)$ are =

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}}$$

whenever f, f_x, f_y, f_{xy} & f_{yx} are contin...

$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)$ or $f_{yyx} = (f_{yy})_x$

$\frac{\partial^4 f}{\partial x^2 \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y^2}\right)$ or $f_{yyxx} = (f_{yyx})_x$

Q

1) compute 2nd order partial deri... of ( ) $g(x,y) = x^2y + \cos y + y\sin x$.

A) diff... $g(x,y)$ partially → x & y.

$\frac{\partial g}{\partial x} = 2xy + y + \cos x$

$\frac{\partial g}{\partial y} = x^2 - \sin y + \sin x$.

diff... $\partial g/\partial x$ ... → x & y,

$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = 2y - y\sin x$

$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = 2x + \cos x$.

diff... $\partial g/\partial y$ .... → x & y.

$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = 2x + \cos x$

$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = -\cos y$

2) verify that $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$ when

$w = x^y + \sin(xy)$.

$\boxed{\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)}$?

A) diff. w partially → x & y.

$\frac{\partial w}{\partial x} = y x^{y-1} + y\cos(xy)$

$\frac{\partial w}{\partial y} = x^y \ln x + x\cos(xy)$

diff. $\frac{\partial w}{\partial x}$, → y,

$\boxed{\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)} = \frac{\partial}{\partial y}\left(\underset{(\text{P rule})}{y \cdot x^{y-1}} + \underset{(\text{P rule})}{y \cdot \cos(xy)}\right)$

$= x^{y-1} + y x^{y-1}\ln x + \cos(xy) - xy\sin(xy)$ — ①

diff. $\frac{\partial w}{\partial y}$, → x.

$\boxed{\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)} = \frac{\partial}{\partial x}\left[\underset{(\text{P rule})}{x^y \ln x} + \underset{(\text{"})}{x \cdot \cos(xy)}\right]$

$= x^y \cdot \frac{1}{x} + y x^{y-1}\ln x + \cos(xy) - xy\sin(xy)$

$= x^{y-1} + y x^{y-1}\ln x + \cos(xy) - xy\sin(xy)$ — ②

① = ②

$\ln(x^{x+y}+y^{x+y}) = \ln(x+y)$

∴ $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$

$\ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{1/2}$
$= \ln(x+y)$

3) $z = \ln\sqrt{x^2+y^2}$, P.T $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

A) $z = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$ (last pg)

diff z → x & y

$\frac{\partial z}{\partial x} = \frac{x}{x^2+y^2}$ & $\frac{\partial z}{\partial y} = \frac{y}{x^2+y^2}$

diff. $\frac{\partial z}{\partial x}$ → x.

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$

$(x^2+y^2) \to 2x$

$= \frac{(x^2+y^2) \cdot 1 - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

diff. $\frac{\partial z}{\partial y}$ → y

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$= \frac{(x^2+y^2) \cdot 1 - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2-y^2)^2}$

hence, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$

4) S.T $w = 5\cos(3x+3ct) + e^{x+ct}$, where c → constant, satisfies the wave eq–

$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$.

A) diff. $w = 5\cos(3x+3ct) + e^{x+ct}$, → x & t.

$\frac{\partial w}{\partial x} = -15\sin(3x+3ct) + e^{x+ct}$

$\frac{\partial w}{\partial t} = -15c\sin(3x+3ct) + ce^{x+ct}$

diff. $\partial w/\partial x$ → x.

$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left[-15\sin(3x+3ct) + e^{x+ct}\right]$

$= -45\cos(3x+3ct) + e^{x+ct}$.

diff— $\partial w/\partial t$ — w.r.t.

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial t}\left[-15c \sin(3x+3ct) + ce^{x+ct}\right.$$

$$= -45c^2 \cos(3x+3ct) + c^2 e^{x+ct}$$

$$= c^2\left[-45 \cos(3x+3ct) + e^{x+ct}\right]$$

$$= c^2 \frac{\partial^2 w}{\partial x^2}$$

⇒ **chain rule :-**

let $w = f(x)$ be a diff… ( ) of x & let $x = x(t)$ be a diff… ( ) of t, then we know that w is a diff… ( ) of t & its deri… with respect to t can be evaluated using the formula.

$$\boxed{\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}}$$

This is chain rule for ( ) of a single variable. For a composite ( ) of 2 variables $z = f(u,v)$, $u = g(x,y)$ & $v = h(x,y)$.

Chain rule for ( ) of 2 variables →

* **Theorem :-**

If $z = f(u,v)$ is diff… & $u = g(x,y)$ & $v = h(x,y)$ have contin… 1st partial deri…, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}, \quad \text{—— (2)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

* Sple case → If $z = f(u,v)$ is diff… & $u = g(t)$ & $v = h(t)$ are diff… ( ) of a single variable t,

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}} \quad \text{—— (1)}$$

* **Generalizations** → results given in (2) & (3) immediately generalize to any no. of variables.

If $z = f(u_1, u_2, \ldots, u_n)$ is diff… & each of the variables $u_1, u_2, \ldots u_n$ are ( ) of $x_1, x_2, \ldots x_k$ which have contin… partial deri… then for $i = 1, 2, \ldots k$.

(1) $$\frac{\partial z}{\partial x_i} = \frac{\partial z}{\partial u_1} \frac{\partial u_1}{\partial x_i} + \frac{\partial z}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial z}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_i} \quad \text{—— (4)}$$

partial

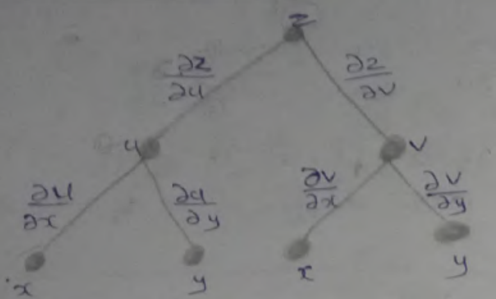
If $z = f(u_1, u_2 \ldots u_n)$ is diff… & each of variables $u_1, u_2 \ldots u_n$ are diff… ( ) of a single variable t,

(2) $$\frac{dz}{dt} = \frac{\partial z}{\partial u_1} \cdot \frac{du_1}{dt} + \frac{\partial z}{\partial u_2} \cdot \frac{du_2}{dt} + \cdots + \frac{\partial z}{\partial u_n} \cdot \frac{du_n}{dt} \quad \text{—— (5)}$$

fully diff

⇒ Tree Diagram :-

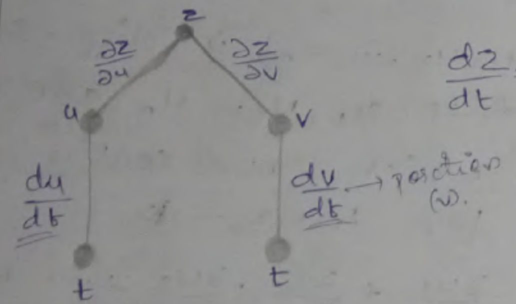
$z = f(u, v)$
$u = g(x, y)$
$v = h(x, y)$



(a)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$



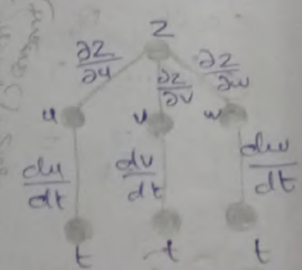
$\frac{dv}{dt} \rightarrow$ practice (w).

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}$$

(b)

Q 1) If $z = u^2 v^3 w^4$ & $u = t^2$, $v = 5t - 8$, $w = t^3 + t$, using tree diagram, find $\frac{dz}{dt}$.

A) here z is f of u, v, w & u, v are f of a single variable.



$$\frac{\partial z}{\partial u} = 2uv^3w^4 \text{ (only u)}$$

$$\frac{\partial z}{\partial v} = 3u^2v^2w^4 \text{ (only v)}$$

$$\frac{\partial z}{\partial w} = 4u^2v^3w^3 \text{ (only w)}$$

$\frac{du}{dt} = 2t$, $\frac{dv}{dt} = 5$, $\frac{dw}{dt} = 3t^2 + 1$

from tree diagram,

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dt}$$

(c)

$$= 2uv^3w^4 \cdot 2t + 3u^2v^2w^4 \cdot 5 + 4u^2v^3w^3 (3t^2+1)$$

$$= uv^2w^3 [4tvw + 15uw + 4uv(3t^2+1)]$$

$$= uv^2w^3 [4t(5t-8)(t^3+t) + 15t^2(t^3+t) + 4t^2(5t-8)(3t^2+1)]$$

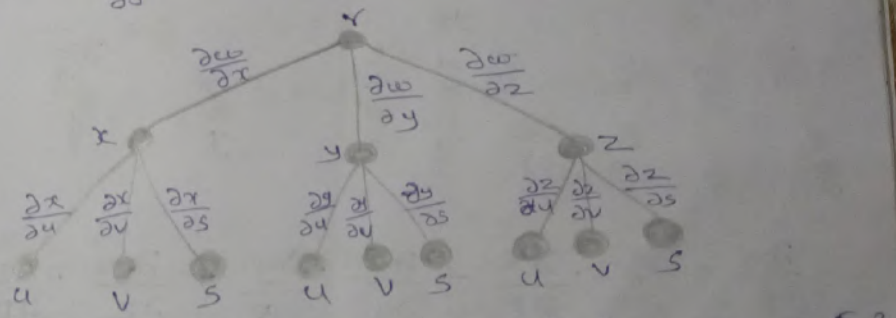
$$= uv^2w^3t^2 [95t^3 - 128t^2 + 55t - 64]$$

$$= t^4(5t-8)^2(t^3+t)^3 [95t^3 - 128t^2 + 55t - 64]$$

$$= t^7(5t-8)^2(t^2+1)^3 [95t^3 - 128t^2 + 55t - 64]$$

2) If $r = x^2 + y^5 z^3$ & $x = uve^{2s}$, $y = u^2 - v^2 s$, $z = \sin(uvs^2)$ using tree diagram find $\frac{\partial r}{\partial s}$.

A)



$\frac{\partial r}{\partial x} = 2x$, $\frac{\partial r}{\partial y} = 5y^4z^3$, $\frac{\partial r}{\partial z} = 3y^5z^2$

$\frac{\partial x}{\partial s} = 2uve^{2s}$, $\frac{\partial y}{\partial s} = -v^2$, $\frac{\partial z}{\partial s} = 2uvs \cos(uvs^2)$

from tree diagram,

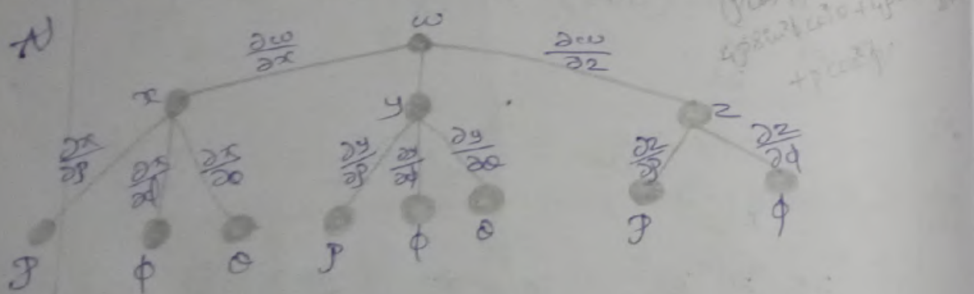
$$\frac{\partial r}{\partial s} = \frac{\partial r}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial r}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial r}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= 2x\left(2uve^{2s}\right) + 5y^4z^3(-v^2) + 3y^5z^2\left[2uvs\cos(uvs^2)\right]$$

$$= 4u^2v^2e^{4s} - 5v^2(u^2-v^2s)^4\sin^3(uvs^2) + 6uvs(u^2-v^2s)\sin^2(uvs^2)\cos(uvs^2).$$

3) Suppose that $\omega = 4x^2+4y^2+z^2$ & $x = \rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$, using tree diagram, find appropriate forms of chain rule for the partial deri. $\frac{\partial\omega}{\partial\rho}$, $\frac{\partial\omega}{\partial\phi}$ & $\frac{\partial\omega}{\partial\theta}$ & evaluate them.

A)



$$\frac{\partial\omega}{\partial\rho} = \frac{\partial\omega}{\partial x}\cdot\frac{\partial x}{\partial\rho} + \frac{\partial\omega}{\partial y}\cdot\frac{\partial y}{\partial\rho} + \frac{\partial\omega}{\partial z}\cdot\frac{\partial z}{\partial\rho}$$

$$= 8x\cdot\sin\phi\cos\theta + 8y\cdot\sin\phi\sin\theta + 2z\cdot\cos\phi.$$

$$= 8\rho\sin^2\phi\cos^2\theta + 8\rho\sin^2\phi\sin^2\theta + 2\rho\cos^2\phi$$

$$= 8\rho\sin^2\phi + 2\rho\cos^2\phi.$$

$$= 6\rho\sin^2\phi + 2\rho\left[\sin^2\phi+\cos^2\phi\right]$$

$$= 6\rho\sin^2\phi + 2\rho = 2\rho\left(3\sin^2\phi+1\right)$$

$$\frac{\partial\omega}{\partial\phi} = \frac{\partial\omega}{\partial x}\cdot\frac{\partial x}{\partial\phi} + \frac{\partial\omega}{\partial y}\cdot\frac{\partial y}{\partial\phi} + \frac{\partial\omega}{\partial z}\cdot\frac{\partial z}{\partial\phi}$$

$$= 8x\cdot\rho\cos\phi\cos\theta + 8y\cdot\rho\cos\phi\sin\theta + 2z\cdot(-\rho\sin\phi)$$

(give x, y, z values)

$$= 8\rho^2\cdot\sin\phi\cos\phi\cos^2\theta + 8\rho^2\sin\phi\cos\phi\sin^2\theta - 2\rho^2\cos\phi\sin\phi.$$

$$= 8\rho^2\sin\phi\cos\phi\left(\cos^2\theta+\sin^2\theta\right) - 2\rho^2\cos\phi\sin\phi$$

$$= 6\rho^2\sin\phi\cos\phi$$

$$\frac{\partial\omega}{\partial\theta} = \frac{\partial\omega}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial\omega}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$= 8x(-\rho\sin\phi\sin\theta) + 8y\cdot\rho\sin\phi\cos\theta.$$

$$= -8\rho^2\sin^2\phi\cos\theta\sin\theta + 8\rho^2\sin^2\phi\sin\theta\cos\theta = 0$$

⇒ The Gradient of ( ) :-

A new vector associated with a ( ) of 2/3 variables based on partial diff...

when the vector differential operator,

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$ or

$\vec{\nabla} \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$\hat{\nabla}$

∇ is applied to a diff... (1) $z = f(x,y)$ or $w = F(x,y,z)$,

2(1). $\boxed{\nabla f(x,y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j}$ — gradient

$\nabla F(x,y,z) = \frac{\partial F}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} k.$

3(1) → are the gradients of (1).

(Symbol ∇ → delta)

Q/1) find gradient of follow (1) →

a) $f(x,y) = xe^y + \cos(xy)$. at $(2,0)$

$\frac{\partial f}{\partial x} = e^y - y\sin(xy)$

$\frac{\partial f}{\partial y} = xe^y - x\sin(xy)$

$\nabla f(x,y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j.$

$= [e^y - y\sin(xy)]i + [xe^y - x\sin(xy)]j$

$\nabla f(2,0) = \left[\left(\frac{\partial f}{\partial x}\right)i + \left(\frac{\partial f}{\partial y}\right)j\right]_{(2,0)}$

$= [e^0 - 0\sin 0]i + [2\cdot e^0 - 2\cdot\sin 0]j$

$= i + 2j$

(b) $F(x,y,z) = xy^2 + 3x^2 - z^3$ at $(2,-1,4)$

A) $\frac{\partial F}{\partial x} = y^2 + 6x \quad \frac{\partial f}{\partial y} = 2xy \quad \frac{\partial f}{\partial z} = -3z^2$

$\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial F}{\partial z} k.$

$= (y^2+6x)i + 2xyj - 3z^2k.$

$\therefore \nabla F(2,-1,4) = \left[\frac{\partial F}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial F}{\partial z} k\right]_{(2,-1,4)}$

$= [(-1)^2 + 6\cdot 2]i + 2\cdot 2\cdot(-1)j - 3\cdot 4^2 k.$

$= 13i - 4j - 48k.$

⇒ Directional Derivative :-

def → The (D.D) of $z = f(x,y)$ in the (dir) of a unit vector $u = \cos\theta i + \sin\theta j$ is,

$$D_u f(x,y) = \lim_{h\to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h}$$

above def.. is a generalization of the partial deri... since $\theta = 0$ implies $u = i$ &

$$D_i f(x,y) = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial z}{\partial x}$$

ie $\theta = \frac{\pi}{2}$ implies $u = j$ &

$$D_j f(x,y) = \lim_{h\to 0}\frac{f(x, y+h) - f(x,y)}{h} = \frac{\partial z}{\partial y}$$

* Theorem :- If $z = f(x,y)$ is a diffⁿ () of x & y & $u = \cos\theta\, i + \sin\theta\, j$, then
$$D_u f(x,y) = \nabla f(x,y) \cdot u$$

A) let x, y are → fixed
$$D_u f(x,y) = \nabla f(x,y) \cdot u.$$
$$g(t) = f(x + t\cos\theta,\ y + t\sin\theta)$$
is a () of single variable t,
since (x,y) is a diffⁿ () g is also diffⁿ
$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$$
$$= \lim_{h\to 0} \frac{f(x + h\cos\theta,\ y + \sin\theta\, h) - f(x,y)}{h}$$

diffⁿ $g(t)$ using chain rule,
$$g'(t) = f_1(x + t\cos\theta,\ y + t\sin\theta) \cdot \frac{d}{dt}(x + \cos\theta\, t)$$
$$f_2(x + t\cos\theta,\ y + \sin\theta\, t) \cdot \frac{d}{dt}(x + t\sin\theta)$$
$$= f_1(x + t\cos\theta,\ y + t\sin\theta)\cos\theta + f_2(x + t\cos\theta,\ y + t\sin\theta)\sin\theta.$$

here the subscripts 1 & 2 refer to partial deriⁿ of $f(x + t\cos\theta,\ y + t\sin\theta)$ with respect to $x + t\cos\theta$ & $y + t\sin\theta$ when $t = 0$, we note that $x + t\cos\theta$ & $y + t\sin\theta$ are simply x & y &

hence from above eq,
$$g'(0) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$

by defⁿ of D.D,
$$D_u f(x,y) = \lim_{h\to 0} \frac{f(x + h\cos\theta,\ y + \sin\theta\, h) - f(x,y)}{h}$$
$$= g'(0) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$
$$= [f_x(x,y)\, i + f_y(x,y)\, j] \cdot [\cos\theta\, i + \sin\theta\, j]$$
$$= \nabla f(x,y) \cdot u$$

2) find D.D of $f(x,y) = \sqrt{xy}$ at $(1,4)$ in the (dirⁿ) of unit vector that makes an angle $\pi/3$ with the +ve x axis.

A) $f_x(x,y) = \dfrac{y}{2\sqrt{xy}}$ $\quad f_y(x,y) = \dfrac{x}{2\sqrt{xy}}$

$f_x(1,4) = \dfrac{4}{2\sqrt{4}} = \dfrac{4}{4} = 1$ $\quad f_y(1,4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{2\cdot 2} = \dfrac{1}{4}$

$$\nabla f(1,4) = f_x(1,4)\, i + f_y(1,4)\, j = i + \frac{1}{4} j.$$

unit vector u that makes an angle of $\pi/3$ →
$$u = \cos(\pi/3)\, i + \sin(\pi/3)\, j$$
$$= \left(\frac{1}{2}\right) i + \left(\sqrt{3}/2\right) j.$$

∴ D.D = $D_u f(1,4)$ of f at $(1,4)$ in (dirⁿ) of unit (v) u,
$$D_u f(1,4) = \nabla f(1,4) \cdot u = \left[i + \frac{1}{4} j\right] \cdot \left[\frac{1}{2} i + \frac{1}{2}\sqrt{3}\, j\right]$$

$= 1 \times \frac{1}{2} + \frac{1}{4} \times \frac{\sqrt{3}}{2} = \frac{4+\sqrt{3}}{8}$

3) D.D. of $f(x,y) = xe^y + \cos(xy)$ in (dis) of (v) $i+j$ at the point $(2,0)$

*) $f_x(x,y) = e^y - y \sin(xy)$

$f_x(2,0) = 1 - 0 = 1$

$f_y(x,y) = xe^y - x \sin(xy)$

$f_y(2,0) = 2 - 0 = 2$

Then $\nabla f(2,0) = f_x(2,0)\, i + f_y(2,0)\, j$

$= i + 2j$

unit (v),

$u = \frac{i+j}{\sqrt{1^2+1^2}} = \frac{i+j}{\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

∴ D.D of f,

$D_u f(2,0) = \nabla f(2,0) \cdot u$

$= (i+2j) \cdot \left(\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j\right)$

$= \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

4) Consider the plane that is ⊥ to the xy-plane & passes through the points $P(2,1)$ & $Q(3,2)$. What is the slope of the tangent line to the curve of intersection of this plane with the surface $f(x,y) = 4x^2 + y^2$ at $(2,1,17)$ in the (dis) of Q?

*) Slope of the tangent line to curve of intersection of this plane with the surface $f(x,y) = 4x^2 + y^2$ at $(2,1,17)$ in (dis) of Q is = the (dis) deri... of f at $(2,1)$ in (dis) of $\overline{PQ}$.

The partial deri... of f,

$f_x(x,y) = 8x$

$f_x(2,1) = 16$ $\to 8x \to x=2 \to 8\times 2 = 16$

$f_y(x,y) = 2y$

$f_y(2,1) = 2$ $\to 2y \to y=1 \to 2\times 1 = 2$

$\nabla f(2,1) = f_x(2,1)\, i + f_y(2,1)\, j = 16i + 2j$

$\overline{PQ} = (3-2)i + (2-1)j = i + j$

unit (v) → $\overline{PQ}$ is,

$u = \frac{i+j}{\sqrt{1^2+1^2}} = \frac{i+j}{\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

∴ required slope,

$D_u f(2,1) = \nabla f(2,1) \cdot u = (16i + 2j) \cdot \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

$= 16 \times \frac{1}{\sqrt{2}} + 2 \times \frac{1}{\sqrt{2}} = 8\sqrt{2} + \sqrt{2}$

$= 9\sqrt{2}$

⇒ Directional deri... of (f) of 3 variables :-

def ⇒ The directional deri... of $w = f(x,y,z)$ in the (dis) of unit (v) u is,

$$D_u F(x,y,z) = \lim_{h \to 0} \frac{F(x + h\cos\alpha,\ y + h\cos\beta,\ z + h\cos\gamma) - F(x,y,z)}{h}$$

α, β & γ → (dir) angles of unit (v) u measured relative to the x-y & z axes.

as in the case of 2 variables (),

$D_u F(x,y,z) = \bar{\nabla} F(x,y,z) \cdot u$

Q/

1) find D.D of $F(x,y,z) = e^{x+y+3z}$ at the point $(-2,2,-1)$ in (dir) of $a = 20i - 4j + 5k$.

A) $F_x(x,y,z) = e^{x+y+3z}$ $F_x(-2,2,-1) = e^{-3}$

$F_y(x,y,z) = e^{x+y+3z}$ $F_y(-2,2,-1) = e^{-3}$

$F_z(x,y,z) = 3e^{x+y+3z}$ $F_z(-2,2,-1) = 3e^{-3}$.

$\therefore \bar{\nabla} F(-2,2,-1) = F_x(-2,2,-1) + F_y(-2,2,-1) + F_z(-2,2,-1)$

$= e^{-3} i + e^{-3} j + 3e^{-3} k$.

unit (v) u in (dir) $a = 20i - 4j + 5k$,

$$u = \frac{20i - 4j + 5k}{\sqrt{20^2 + 4^2 + 5^2}} = \frac{20}{21} i + \frac{4}{21} j + \frac{5}{21} k.$$

$\therefore$ D.D is,

$D_u F(-2,2,-1) = \bar{\nabla} F(-2,2,-1) \cdot u$

$= [e^{-3} i + e^{-3} j + 3e^{-3} k] \cdot \left[\frac{20}{21} i + \frac{4}{21} j + \frac{5}{21} k\right]$.

$= \frac{20}{21} e^{-3} - \frac{4}{21} e^{-3} + \frac{15}{21} e^{-3} = \frac{31}{21} e^{-3}$

2) D.D of $F(x,y,z) = x^3 y^2 z^5 - 2xz + yz + 3x$ at $P(-1,-2,1)$ in (dir) of -ve z-axis.

A) $F_x(x,y,z) = 3x^2 y^2 z^5 - 2z + 3$.

$F_y(x,y,z) = 2x^3 y z^5 + z$

$F_z(x,y,z) = 5x^3 y^2 z^4 - 2x + y$.

$\therefore F_x(-1,-2,1) = 13$,

$F_y(-1,-2,1) = 5$

$F_z(-1,-2,1) = -20$

$\therefore \bar{\nabla} F(-1,-2,1) = F_x(-1,-2,1) + F_y(-1,-2,1) + F_z(-1,-2,1)$

$= 13i + 5j - 20k$.

$\therefore$ D.D,

$D_u F(-1,-2,1) = \bar{\nabla} F(-1,-2,1) \cdot u$

$= [13i + 5j - 20k] \cdot -k = 20$

⇒ (pro) of gradient & directional deri..:-

$D_u f = \bar{\nabla} f \cdot u = \|\bar{\nabla} f\| \, \|u\| \cos\theta = \|\bar{\nabla} f\| \cos\theta$.

θ is angle b/w $\bar{\nabla} f$ & u.

(pro) →

1) Max value of $\cos\theta$ is 1 & it is attained when $\theta = 0$. Hence the () f rises most rapidly, → $D_u f$ is max when u is in the (dir) of $\bar{\nabla} f$.

$$\boxed{D_u f = \|\bar{\nabla} f\| \cos 0 = \|\bar{\nabla} f\|}$$

for () of 2 variables, geometrically this means that at (x,y), the surface $z = f(x,y)$ has its max. slope in (dis) of gradient i.e. max slope is $\|\nabla f\|$ at (x,y).

2) min value of $\cos\theta$ is -1. i.e. it is attained when $\theta = \pi$.
hence () f uses most rapidly → $D_u f$ is min when u is in the opposite (dis) of ∇f. → (dis) of $-\nabla f$.

$$\boxed{D_u f = \|\nabla f\| \cos\pi = -\|\nabla f\|}$$

for () of 2 variables, geometrically, the this means that at (x,y), the surface $z = f(x,y)$ has its min slope in the (dis) that is opposite to gradient i.e. the min slope is $-\|\nabla f\|$ at (x,y).

3) when $\theta = \pi/2$, $\cos\theta = 0$ i.e. hence $D_u f = 0$
hence () f has no change → $D_u f$ is 0 when u is orthogonal to the (dis) of ∇f.
Thus at each point p in its domain, the (dis) orthogonal to the (dis) of ∇f, is a (dis) of 0 change in f.

Q/1) find (dis) in which $f(x,y) = x^2 + xy + y^2$ rises most rapidly i.e. uses most rapidly at (1,1). what are the (dis) of zero change in f at (1,1)?

A) $f_x(x,y) = 2x+y$ $\qquad f_y(x,y) = x+2y$
$f_x(1,1) = 2+1 = 3$ $\qquad f_y(1,1) = 1+2 = 3$
$\therefore \nabla f(1,1) = f_x(1,1)\,i + f_y(1,1)\,j = 3i + 3j$

by (pro) of D.D, the () f rises most rapidly in the (dis) of ∇f at (1,1).
hence () rises most rapidly in (dis) $3i+3j$ i.e. unit (v) in this (dis),

$$u = \frac{3i+3j}{\|3i+3j\|} = \frac{3i+3j}{\sqrt{3^2+3^2}} = \frac{1}{\sqrt{2}}\,i + \frac{1}{\sqrt{2}}\,j.$$

() uses most rapidly in (dis) of $-\nabla f$ at (1,1), hence the unit (v) in this (dis),

$$-u = -\frac{1}{\sqrt{2}}\,i - \frac{1}{\sqrt{2}}\,j.$$

Again by (pro) of D.D, the (dir) of 0 change in f at (1,1) are orthogonal to ∇f at (1,1). hence if $ai+bj$ is a (dis) of 0 change then

$$\nabla f \cdot (ai+bj) = 0.$$

$\nabla f \cdot (ai+bj) = 0 \Rightarrow (3i+3j)\cdot(ai+bj) = 0.$
$\Rightarrow 3a + 3b = 0.$

Choosing $a = -1$ i.e. $b = 1$, $3a+3b = 0$

& so $-i+j$ is a vector orthogonal to ∇f. Hence a unit vector orthogonal to ∇f is,

$$n = \frac{-i+j}{\sqrt{(-1)^2+1^2}} = -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j.$$

hence (dis) of zero change are,

$$n = -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \quad \&$$

$$-n = \frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j$$

2) Let $f(x,y) = 4x^3y$, find max value of a (dis) deriv at $(-1,1)$ & find the unit (v) in which the max value occur?

A) we know, D.D, $D_uf(-1,1)$ is max when u is in the (dis) of $\nabla f(-1,1)$ & max value is $\|\nabla f(-1,1)\|$.

by def,

$$\nabla f(x,y) = f_x(x,y)i + f_y(x,y)j$$
$$= 12x^2y\,i + 4x^3j$$

$$\nabla f(-1,1) = 12i - 4j$$

hence max value of D.D at $(-1,1)$,

$$\|\nabla f(-1,1)\| = \sqrt{12^2+(-4)^2} = \sqrt{144+16} = 4\sqrt{10}$$

This max value occurs in (dis) of $\nabla f(-1,1)$. The unit vector in this (dis),

$$u = \frac{\nabla f(-1,1)}{\|\nabla f(-1,1)\|} = \frac{12i-4j}{4\sqrt{10}} = \frac{3}{\sqrt{10}}i - \frac{1}{\sqrt{10}}j$$

3) find a unit (v) in (dis) in which $f(x,y,z) = x^3z^2 + y^3z + z - 1$ Ises most rapidly at $(1,1,-1)$ & find the rate of change of f at $(1,1,-1)$ in that (dis).

A) $f_x(x,y,z) = 3x^2z^2$, $f_y(x,y,z) = 3y^2z$.

$f_x(1,1,-1) = 3$, $f_y(1,1,-1) = -3$

$f_z(x,y,z) = 2x^3z + y^3 + 1$,

$f_z(1,1,-1) = -2+1+1 = 0$

$\therefore \nabla f(1,1,-1) = f_x(1,1,-1)i + f_y(1,1,-1)j + f_z(1,1,-1)k = 3i - 3j$

by (pro) of gradient of a (f)f, the (f)f Ises most rapidly in (dis) of $-\nabla f$ at $(1,1,-1) \Rightarrow -(3i-3j) = -3i+3j$ & unit (v) (dis),

$$u = \frac{-3i+3j}{\|-3i+3j\|} = \frac{-3i+3j}{\sqrt{(-3)^2+3^2}} = -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j.$$

rate of change of f in (dis). f Ises most rapidly at $(1,1,-1)$ is (dis) deriv of f at $(1,1,-1)$ in (dis) of $-\nabla f$. & $= -\|\nabla f\|$ at $(1,1,-1)$, hence rate of change of f in (dis) in which f ↓ses most rapidly at (1,1,-1),

$D_u f(1,1,-1) = -\|\nabla f(1,1,-1)\| = -\|-3i+3j\|$

$= -\sqrt{(-3)^2+3^2} = -3\sqrt{2}$

4) a) find gradient of $h(x,y,z) = \cos(xy) + e^{yz} + \ln zx$ at point $(1,0,\frac{1}{2})$ & hence find the (dis) deriv of h at $(1,0,1/2)$ in (dis) of (v) $i+2j+2k$.

(b) In wht (dis) does h change most rapidly at p & wht are the rates of change in these (dis)?

A) (a) $h_x(1,0,1/2) = \left[-y\sin xy + \frac{1}{x}\right]_{(1,0,1/2)} = 1.$

$h_y(1,0,1/2) = \left[-x\sin xy + ze^{yz}\right]_{(1,0,1/2)} = 1/2$

$h_z(1,0,1/2) = \left[ye^{yz} + \frac{1}{z}\right]_{(1,0,1/2)} = 2$

hence gradient of h at $(1,0,1/2)$,

$\nabla h(1,0,1/2) = h_x(1,0,1/2)\,i + h_y(1,0,1/2)\,j + h_z(1,0,1/2)\,k$

$= i + (1/2)j + 2k.$

unit (v) u,

$u = \frac{i+2j+2k}{\|i+2j+2k\|} = \frac{i+2j+2k}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k.$

∴ D.D,

$D_u h(1,0,1/2) = \nabla h(1,0,1/2)\cdot u.$

$= (i+\frac{1}{2}j+2k)\cdot(\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)$

$= \frac{1}{3}+\frac{1}{3}+\frac{4}{3} = \frac{6}{3} = 2$

(b) At $(1,0,1/2)$ the (). ↑ses most rapidly in the (dis) of $\nabla h = i+(1/2)j+2k$ & ↓ses most rapidly in (dis) of $-\nabla h = -i-(\frac{1}{2})j-2k$. The rate of change in these (dis).

$\|\nabla h\| = \sqrt{1^2+(\frac{1}{2})^2+2^2} = \frac{\sqrt{21}}{2}$ & $-\|\nabla h\| = -\sqrt{21}/2$

5) Temperature in ▭ box is approximated by $T(x,y,z) = xyz(1-x)(2-y)(3-z)$, $0\le x\le 1, 0\le y\le 2, 0\le z\le 3$.

If a mosquito is located at $(\frac{1}{2},1,1)$ in which (dis) should it fly to cool off as rapidly as possible?

A) $T_x(1/2,1,1) = \left[yz(1-x)(2-y)(3-z) - xyz(2-y)(3-z)\right]_{(1/2,1,1)}$

~~$T_y(1/2,1,1)$~~ $= \left[yz(2-y)(3-z)(1-2x)\right]_{(1/2,1,1)} = 0$

$T_y(1/2,1,1) = \left[xz(1-x)(2-y)(3-z) - xyz(1-x)(3-z)\right]_{(1/2,1,1)}$

$= \left[xz(1-x)(3-z)(2-2y)\right]_{(1/2,1,1)} = 0$

$$T_z(1/2,1,1) = \Big[xy(1-x)(2-y)(3-z) - xyz\,(1-x)(2-y)\Big]_{(1/2,1,1)}$$
$$= \Big[xy(1-x)(2-y)(3-2z)\Big]_{(1/2,1,1)} = \frac{1}{4}$$

hence gradient of T at $(\frac{1}{2},1,1)$,

$$\nabla T(1/2,1,1) = T_x(1/2,1,1)\,i + T_y(1/2,1,1)\,j + T_z(1/2,1,1)\,k = \frac{1}{4}k$$

hence to cool of most rapidly, the mosquito should fly in the (dir) of $-\nabla T = -\frac{1}{4}k$, $\rightarrow$ it should fly towards the floor of the box, where the temperature $T(x,y,z) = 0$

$\Rightarrow$ Tangent planes and Normal lines :-

$\rightarrow$ Tangent planes & normal lines to level surfaces
$F(x,y,z) = c$

let $p(x_0,y_0,z_0)$ be a point on the graph of $F(x,y,z) = c$, where ∇F is not equal to 0

The tangent plane at p is that plane through p that is normal to ∇F evaluated at p.

Thus if $Q(x,y,z)$ & $p(x_0,y_0,z_0)$ are points on the tangent plane & r and r_0 are their vectors, then a (v) eq of tangent plane is $\nabla F(x_0,y_0,z_0)\cdot(r-r_0) = 0$

results $\rightarrow$

* Eq of a tangent plane :-

Let $p(x_0,y_0,z_0)$ be a point on the graph of $F(x,y,z) = c$, where ∇F is not 0, then tangent line eq $\rightarrow$

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0}$$

[partial diff with respect to x]

The line through the point p $\parallel$ to normal (v) ∇F is $\perp$ to the above tangent plane $\rightarrow$ normal line / normal to the surface $S: F(x,y,z) = c$ at $p(x_0,y_0,z_0)$ can be expressed parametrically as

$$x = x_0 + F_x(x_0,y_0,z_0)\,t,$$
$$y = y_0 + F_y(x_0,y_0,z_0)\,t,$$
$$z = z_0 + F_z(x_0,y_0,z_0)\,t.$$

$x = x_0 + F_x(x_0,y_0,z_0)t$
$y = y_0 + F_y(x_0,y_0,z_0)t$
$z = z_0 + F_z(x_0,y_0,z)t$

when expressed as symmetric eq the normal line to a surface, $F(x,y,z) = c$ at $p(x_0,y_0,z_0)$ is,

symmetric eq

$$\boxed{\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}}$$

Q/ 1) consider the surface $xz - yz^3 + yz^2 = 2$

a) find an eq of the tangent ~~line~~ plane to the surface at the point $(2,-1,1)$

$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)$

b) Find parametric eq & symmetric eq of the line that is normal to the surface at point $(2,-1,1)$.

A) $F_x(2,-1,1) = \frac{\partial}{\partial x}\left[xz - yz^3 + yz^2\right]\Big|_{(2,-1,1)}$

(only look x $\frac{d}{dx}(xz)=z$)

$= [z]\big|_{(2,-1,1)} = 1$

$F_y(2,-1,1) = \frac{\partial}{\partial y}\left[xz - yz^3 + yz^2\right]\Big|_{(2,-1,1)}$

$= \left[-z^3 + z^2\right]\Big|_{(2,-1,1)} = 0$

$-(1)^3+1^2=0$

$F_z(2,-1,1) = \frac{\partial}{\partial z}\left[xz - yz^3 + yz^2\right]\Big|_{(2,-1,1)}$

$2-3\times-1\times1+2\times-1\times1$
$-\cancel{2}+3-\cancel{2}=3$

$= \left[x - 3yz^2 + 2yz\right] = 3$

$\therefore \nabla F(2,-1,1) = F_x(2,-1,1)\,i + F_y(2,-1,1)\,j + F_z(2,-1,1)\,k.$

$= i + 3k \neq 0$

a) eq of tangent plane to the surface $F(x,y,z) = 2$, at $P(2,-1,1)$,

$F_x(2,-1,1)(x-2) + F_y(2,-1,1)(y-(-1)) + F_z(2,-1,1)(z-1) = 0.$

$\Rightarrow 1(x-2) + 0(y+1) + 3(z-1) = 0$

$x-2+3z-3$
$x+3z=5$
$x+3z=5$

$\Rightarrow x + 3z = 5$

b) Since $\nabla F(2,-1,1) = i + 3k$ is a normal (v) to surface $F(x,y,z) = 2$ at $P(2,-1,1)$, the parametric eq of line that is normal to level surface at $P(2,-1,1)$,

$x = 2 + F_x(2,-1,1)\,t,$

$y = -1 + F_y(2,-1,1)\,t,$

$z = 1 + F_z(2,-1,1)\,t.$

$\Rightarrow x = 2 + t, \quad y = -1 + 0t, \quad z = 1 + 3t$

$\Rightarrow x = 2, \quad y = -1, \quad z = 1 + 3t$

symmetric eq of line at $P(2,-1,1)$ is

$$\frac{x-2}{1} = \frac{z-1}{3}, \quad y = -1$$

$\Rightarrow$ Tangent planes & Normal line to surface, $z = f(x,y)$

$f(x,y) = 0 \rightarrow$ implicit ()
$f(x,y) = \text{no.} \rightarrow$ explicit ()

For a surface gvn explicitly by a diff. () $z = f(x,y) \rightarrow$

$F(x,y,z) = f(x,y) - z$ or $F(x,y,z) = z - f(x,y)$.

Thus a point (x_0, y_0, z_0) is on the graph of $z = f(x,y)$ if and only if it is also on level surface $F(x,y,z) = 0$

This follows from $F(x_0,y_0,z_0) = f(x_0,y_0) - z_0 = 0$

Tangent plane & normal lines to the surface $z = f(x,y)$ at (x_0, y_0, z_0) is the same tangent planes & normal lines to the level surface $F(x,y,z) = 0$

$z = f(x,y)$

where $\boxed{F(x,y,z) = f(x,y) = z}$

Q/ i) find an eq for tangent plane & parametric eq for normal line to a surface $z = \ln(x^2+y^2)$ at $P(1,0,0)$.

A) Let $F(x,y,z) = z - \ln(x^2+y^2)$ is the 0 level surface, $z = \ln(x^2+y^2)$ is the 0 level surface of (n).

$z = f(x,y) \Rightarrow F(x,y,z) = z - f(x,y)$

$z = \ln(x^2+y^2) = F(x,y,z) = z - \ln(x^2+y^2)$

to avoid we $\ln(x^2+y^2) - z$

$F_x(1,0,0) = \frac{\partial}{\partial x}\left[\ln(x^2+y^2) - z\right]\Big|_{(1,0,0)}$

$= \left[\frac{2x}{x^2+y^2}\right] = 2 \rightarrow \frac{2\times1}{1+0} = \frac{2}{1} = 2$

$F_y(1,0,0) = \frac{\partial}{\partial y}\left[\ln(x^2+y^2) - z\right]_{(1,0,0)}$

$= \left[\frac{2y}{x^2+y^2}\right] = 0 \rightarrow \frac{2\times0}{1+0} = 0$

$F_z(1,0,0) = \frac{\partial}{\partial z}\left[\ln(x^2+y^2) - z\right] \quad 0 - 1 = -1$

$= [-1]_{(1,0,0)} = -1.$

eq of tangent plane to level surface $F(x,y,z) = 0 \rightarrow$ to the surface $z = \ln(x^2+y^2)$ at $P(1,0,0)$,

$F_x(1,0,0)(x-1) + F_y(1,0,0)(y-0) + F_z(1,0,0)(z-0) = 0$

$\Rightarrow 2(x-1) + 0(y-0) + (-1)(z-0) = 0$

$2x - 2 - 1(z-0) \Rightarrow 2x - 2 - z - 0 = 0$

$\Rightarrow 2x - z - 2 = 0$

$2x - z - 2 = 0$

parametric eq to normal line to level surface $F(x,y,z) = 0 \rightarrow$ to the surface $z = \ln(x^2+y^2)$ at $P(1,0,0)$ is,

$x = 1 + F_x(1,0,0)\,t,$

$y = 0 + F_y(1,0,0)\,t,$

$z = 0 + F_z(1,0,0)\,t.$

$x = 1+2t, \quad y = 0, \quad z = -t$

2) find an eq for tangent plane & parametric eq for normal line to a surface $z = x^2y$ at $P(2,1,4)$

A) Let $F(x,y,z) = z - x^2y$.

surface, $z = x^2y$.

$F_x(2,1,4) = \frac{\partial}{\partial x}\left[x^2y - z\right]\Big|_{(2,1,4)} = [2xy] = 4$

$F_y(2,1,4) = \frac{\partial}{\partial y}\left[x^2y - z\right]\Big|_{(2,1,4)} = [x^2] = 4$

$F_z(2,1,4) = \frac{\partial}{\partial z}\left[x^2y - z\right]\Big|_{(2,1,4)} = [-1] = -1$

eq of tangent plane, $F(x,y,z) = 0 \rightarrow$ to the surface $z = x^2y$ at $P(2,1,4)$,

$F_x(2,1,4)(x-2) + F_y(2,1,4)(y-1) + F_z(2,1,4)(z-4) = 0.$

$\Rightarrow 4(x-2) + 4(y-1) + (-1)(z-4) = 0$

$\Rightarrow 4x + 4y - z - 8 = 0$

parametric eq $\rightarrow$

$F(x,y,z) = 0 \rightarrow$ surface $z = x^2y$ at $P(2,1,4)$

$x = 2 + F_x(2,1,4)\,t$

$y = 1 + F_y(2,1,4)\,t$

$z = 4 + F_z(2,1,4)\,t.$

$\Rightarrow x = 2+4t, \quad y = 1+4t, \quad z = 4-t$ //

(3). P.T $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$.

$z = \ln\sqrt{x^2+y^2} \Rightarrow \ln(x^2+y^2)^{1/2} \Rightarrow \ln\left(x^{2\cdot 1/2} + y^{2\cdot 1/2}\right)$
$= \ln(x+y)$

$\frac{\partial^2 z}{\partial x^2}$ $\frac{\partial}{\partial x}\ln(x+y) = \frac{1}{x+y}\frac{\partial}{\partial x}(x+y)$

$= \frac{1}{x+y}\cdot(1+0) = \frac{1}{x+y}$.

$\frac{\partial^2 z}{\partial x^2}\left(\frac{1}{x+y}\right) = \frac{-1}{(x+y)^2}$

$\frac{\partial}{\partial y}\left\{\ln(x^2+y^2)^{1/2} = \ln(x+y) = \frac{1}{x+y}\right.$

$\frac{\partial^2 z}{\partial y^2}\left(\frac{1}{x+y}\right) = \frac{-1}{(x+y)^2}$.

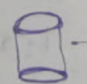
$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{-1}{(x+y)^2} - \left(-\frac{1}{(x+y)^2}\right)$

$= \frac{-1}{(x+y)^2} + \frac{1}{(x+y)^2} = 0$

31/8/22

⟹ Geometric interpretation of gradient :-

For a () of 2 variables
of. at 0 at point P is
orthogonal ~~for a () of 2 variables~~
to the level curve at P.



Let $f(x,y) = C$
Param eq →

$x = g(t) \quad y = h(t)$
$x_0 = g(t_0) \quad y_0 = h(t_0)$

deri— of $f(g(t), h(t)) = c$.
$$\boxed{\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = \frac{d}{dt}(c) = 0.}$$

$$\boxed{\text{gradient, } \bar{\nabla} = \frac{\partial f}{\partial x}\,i + \frac{\partial f}{\partial y}\,j} \Rightarrow \bar{\nabla} f(x,y).$$

$\frac{dx}{dt}\,i + \frac{dy}{dt}\,j = r'(t)$

$$\boxed{\bar{\nabla} f(x,y)\cdot r'(t) = 0}$$

≈ for a () $f(x,y,z)$ of 3 variables
$\bar{\nabla} f$ at a point P is normal to the level surface at P.
→ orthogonal.
$[\bar{\nabla} f(x,y)\cdot r'(t) = 0]$