Module = 3

Triple Integrals:

let $f(x,y,z)$ be a () defined on a closed bounded region D in space. region containing D into □-lar cells by planes ∥ to the coordinate planes.
The cells that lies inside D from 1 to n
let the dimensions of k^{th} cell be $\Delta x_k, \Delta y_k, \Delta z_k$ then its vol, $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$
in each cell we choose a point x_k, y_k, z_k
& form the sum;

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta V_k.$$

$$= \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta x_k \Delta y_k \Delta z_k$$

The above sum ⟶ a Riemann sum.

* ∭ of f over D,

$$\iiint_D f(x,y,z)\, dv = \lim_{n\to\infty} S_n$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta x_k \Delta y_k \Delta z_k.$$

⇒ Pro of ∭ :-
If $f = f(x,y,z)$, $G = G(x,y,z)$ are continu-

1) Constant multiple rule ⟶

$$\iiint_D k \cdot f\, dv = k \iiint_D f \cdot dv$$

2) Sum & diff rule ⟶

$$\iiint_D (f \pm G)\, dv = \iiint_D f \cdot dv \pm \iiint_D G \cdot dv.$$

3) Additivity ⟶

$$\iiint_D f\, dv = \iiint_{D_1} f\, dv + \iiint_{D_2} f\, dv.$$

where the region D is the union of non overlapping regions D_1 & D_2.

→ Evaluation of $\iiint$-s by iterated $\int$-s :-

$\int_x\int_y\int_z f(x,y,z)\,dz\,dA_{(x,y)}$

$\boxed{\iint_R\int_z f(x,y,z)\,dz\,dy\,dx}$

$f_1(x,y) \le z \le f_2(x,y)$

$$\iiint_D f(x,y,z)\,dv = \iint_R \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\,dA$$

* Type I region,

$a \le x \le b \qquad g_1(x) \le y \le g_2(x)$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

* Type II region,

$c \le y \le d, \quad k_1(y) \le z \le k_2(y) \quad h_1(y,z) \le x \le h_2(y,z)$

$$\iiint_D f(x,y,z)\,dv = \int_c^d \int_{k_1(y)}^{k_2(y)} \int_{h_1(y,z)}^{h_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

(we can $\int$ any way. → no rules)

1) Evaluate $\int_0^3\int_0^2\int_0^1 (x+y+z)\,dz\,dx\,dy.$

A) $= \iint \left(xz+yz+\frac{z^2}{2}\right)_0^1 dx\,dy$

$= \int_0^3\int_0^2 \left[x+y+\frac{1}{2}\right] dx\,dy$

$= \int_0^3 \left[\frac{x^2}{2}+yx+\frac{1}{2}x\right]_0^2 dy$

$= \int_0^3 \left[2+2y+1\right] dy = \left[2y+\cancel{2}\frac{y^2}{\cancel{2}}+y\right]_0^3$

$2\times3+3^2+3 = 6+9+3 = 18$

2) $\int_0^1\int_0^{1-x}\int_0^{x+y} e^z\,dz\,dy\,dx.$

A) $\iint \left(e^z\right)_0^{x+y} dy\,dx. = \iint \left(e^{x+y}-e^0\right) dy\,dx.$ $\quad e^{x\cdot 0} = e^0$

$= \int\int_0^{1-x}\left(e^{x+y}-1\right) dy\,dx.$

$= \int \left[\frac{e^{x+y}}{\frac{d}{dy}(x+y)} - y\right]_0^{1-x} dx = \int \left[e^{x+(1-x)} - (1-x) - e^{x} - 0\right] dx.$

$\cancel{= \int e^{x+1-x} - (1-x)\,dx = \int_0^1 e - (1-x)\,dx.}$

$\cancel{= \int_0^1 e - 1 - x\,dx. = \left[e - x - \frac{x^2}{2}\right]_0^1}$

$\cancel{= e - 1 - \frac{1}{2}}$

$\int e^{x+1-x} - (1-x) - e^x\,dx.$

$= \int_0^1 e^1 - 1 - x - e^x\,dx = \left[ex + x + \frac{x^2}{2} - e^x\right]_0^1$

$= e + 1 + \frac{1}{2} - e^1 - e\times0 + 0 + 0 - e^0$

$= \cancel{e} + \cancel{1} + \frac{1}{2} \cancel{e} - \cancel{1} \qquad = \frac{1}{2}$

2) Evaluate $\iiint_D z\,dv$ where D is the region in the 1st octant bounded by the region $y=x,\ y=x-2,\ y=1,\ y=3,$ $z=0,\ z=5.$

$\int_x^{x-2}\int_1^3\int_0^5$

4) Evaluate $\iiint_D \frac{1}{(x+y+z+1)^3} dV$ where V is the tetrahedron bounded by the planes $x=0, y=0, z=0$ & $x+y+z=1$

A) $x+y+z=1$,

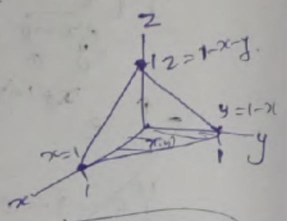
$z = 1-x-y$

limits

if $z=0 \rightarrow x+y=1$ $=z$ $x+y=1 \Rightarrow y=1-x$.

" $y=0 \longrightarrow y = 1-x$. $\longrightarrow$

" $x=0 \longrightarrow x=1$ $\rightarrow 1-0=1$



$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dz\,dy\,dx$$

$\boxed{\int \frac{1}{x^3} = \frac{-1}{2x^2}}$

$\int x^{-3} = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2}$ $\Rightarrow -\frac{1}{2x^2}$

$$= \iint \left(\frac{-1}{2(x+y+z+1)}\right)_0^{1-x-y} dy\,dx.$$

$$= \int_0^1\int_0^{1-x} \left[\frac{-1}{2(\underbrace{x+y+(1-x-y)+1}_{(1+1)^2=2^2})^2} - \frac{-1}{2(x+y+1)^2}\right] dy\,dx.$$

$$= \int\int_0^{1-x} \frac{-1}{2(2^2)} + \frac{1}{2(x+y+1)^2} dy\,dx.$$

$$= \int\int_0^{1-x} \frac{-1}{8} + \frac{1}{2(x+y+1)^2} dy\,dx.$$

$\int \frac{1}{x^2} = \int x^{-2}.$

$= \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1}$

$= \frac{-1}{x}$ (a)

$$= \int \left(\frac{-y}{8} - \underbrace{\frac{1}{2(x+y+1)}}_{(a)}\right)_0^{1-x} dx.$$

$$= \int_0^1 \frac{-(1-x)}{8} - \frac{1}{2(x+1-x+1)} - \frac{-1}{2(x+1)} dx$$

$$= \int_0^1 \frac{-1+x}{8} - \frac{1}{2\cdot 2} + \frac{1}{2(x+1)} dx.$$

$$= \int_0^1 \frac{-1+x}{8} - \frac{1}{4} + \frac{1}{2(x+1)} dx.$$

$$= \int_0^1 \frac{-1}{8} + \frac{x}{8} - \frac{1}{4} + \frac{1}{2(x+1)} dx.$$

$\frac{-1}{8} - \frac{1}{4} = \frac{-4-8}{32} = \frac{-12}{32} = \frac{-6}{16} = \frac{-3}{8}$

$$= \int_0^1 \frac{-3}{8} + \frac{x}{8} + \frac{1}{2(x+1)} dx.$$

$\int \frac{1}{x} = \ln x.$

$$= \left(\frac{-3}{8}x + \frac{x^2}{2\cdot 8} + \frac{1}{2}\ln(x+1)\right)_0^1$$

$$= \frac{-3}{8} + \frac{1}{16} + \frac{1}{2}\ln 2 - \left(\frac{1}{2}\ln 1\right)$$

$$= \frac{-48+8}{8\cdot 16} + \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1$$

$$= \frac{-40}{8\cdot 16} = \frac{-5}{8\cdot 2} + \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1$$

$\ln A - \ln B = \ln\left(\frac{A}{B}\right)$

$$= \frac{-5}{8\cdot 2} + \frac{1}{2}\ln\left(\frac{2}{1}\right)$$

$$= \frac{-5}{16} + \frac{1}{2}\ln 2$$

$\Rightarrow$ Some appli— of $\iiint$:-

1) Volume, $V = \iiint_D dV$.

2) mass, $m = \iiint_D \rho(x,y,z)\, dV$

3) First moments, $m_{xy} = \iiint_D z\rho(x,y,z)\, dV$.

(x,y moment)

$, m_{yz} = \iiint_D x\rho(x,y,z)\, dV$

$m_{xz} = \iiint_D y\rho(x,y,z)\, dV.$

4) Center of mass, $\bar{x} = \frac{m_{yz}}{m}$

$\bar{y} = \frac{M_{xz}}{m}$, $\bar{z} = \frac{M_{xy}}{m}$

6) centroid, if $\rho(x,y,z)$ = constnt, the center of mass $\longrightarrow$ centroid of solid.

7) 2^{nd} moments (moments of inertia) :-

$I_x = \iiint_D (y^2+z^2)\cdot\rho(x,y,z)\,dv$.

$I_y = \iiint_D (x^2+z^2)\cdot\rho(x,y,z)\,dv$

$I_z = \iiint_D (x^2+y^2)\cdot\rho(x,y,z)\,dv$.

8) Radius of gyration :-

$g = \sqrt{I/m}$, I = moments of inertia, m = moments.

Q

1) find the vol of region b/w cylinder $z = y^2$ & xy plane & bounded by planes $x=0,\ x=1,\ y=-1,\ y=1$.

A) vol $= \iiint_D dv$, $z = y^2$; $dv = dz\,dy\,dx$.

$= \int_0^1\int_{-1}^1\int_0^{y^2} dz\,dy\,dx$.

$= \iint (z)_0^{y^2}\,dy\,dx = \iint_{-1}^{1} y^2\,dy\,dx$.

$= \int \left[\frac{y^3}{3}\right]_{-1}^{1} dx = \int_0^1 \left(\frac{1}{3}+\frac{1}{3}\right) dx$.

$= \int_0^1 \frac{2}{3}\,dx = \frac{2}{3}\int_0^1 1\cdot dx = \frac{2}{3}(x)_0^1$

$= \frac{2}{3}$

2) find the total mass of a solid cube in 1^{st} octact bounded by coordinate axis & the plane, $x=1, y=1, z=1$, if the density of cube is $x+y+z+1$

A) $\int_0^1\int_0^1\int_0^1 (x+y+z+1)\,dz\,dy\,dx$.

~~$= \int_0^1\int_0^1 (z)_0^1\,dy\,dx = \int_0^1\int_0^1 dy\,dx$~~

$= \int_0^1\int_0^1 \left(xz+yz+\frac{z^2}{2}+z\right)_0^1 dy\,dx$.

$= \int_0^1\int_0^1 \left(x+y+\frac{1}{2}+1\right) dy\,dx$.

$= \int_0^1 \left(xy+\frac{y^2}{2}+\frac{y}{2}+y\right)_0^1 dx$.

$= \int_0^1 \left(x+\frac{1}{2}+\frac{1}{2}+1\right) dx$

$= \left(\frac{x^2}{2}+\frac{x}{2}+\frac{x}{2}+x\right)_0^1 = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1$

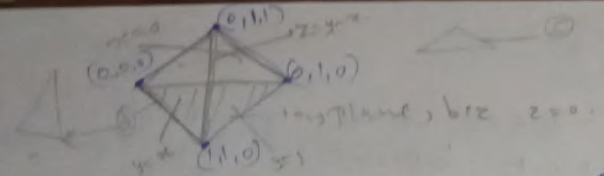
$= \frac{3}{2}+\frac{1}{1} = \frac{3+2}{2} = 5/2$

3) find moment of inertia of a $\square$-lar solid $-\frac{a}{2} \le x \le a/2$, $-\frac{b}{2} \le y \le \frac{b}{2}$, $-\frac{c}{2} \le z \le c/2$ of constnt density $\rho = 1$ about x axis

$I_x = $

4) find the vol of tetrahedran with values $(0,0,0)$ $(1,1,0)$ $(0,1,0)$ & $(0,1,1)$

A)



$z=0$, $(0,0,0)$ $(0,1,0)$ $(1,1,0)$

$y=1$, $(0,1,1)$ $(0,1,0)$ $(1,1,0)$

$x=0$ $(0,0,0)$ $(0,1,0)$ $(0,1,1)$

$z=y-x$ $(0,0,0)$ $(1,1,0)$ $(0,1,1)$

$z=y-x$
$x=0 \to 1$
$y=x \to 1$

~~find the vol of~~

vol.

$$\iiint_D dv = \int_0^1\int_x^1\int_0^{y-x} dz\,dy\,dx$$

$$= \iint [z]_0^{y-x} dy\,dx = \iint_x^1 [y-x]\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2} - xy\right]_x^1 = \int_0^1 \left[\frac{1}{2} - x - \left(\frac{x^2}{2} - x^2\right)\right] dx$$

$$= \left[\frac{1}{2}x - \frac{x^2}{2} - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3} = -\frac{1}{6} + \frac{1}{3}$$

$$= \cancel{\frac{-3+1}{18}} = \cancel{\frac{2}{18}} = \frac{-1+2}{6} = \frac{1}{6}$$

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5) Rewrite the $\int$ as an equalent $\int$ in the (1) $dz\,dx\,dy$ (2) $dy\,dz\,dx$ (3) $dx\,dy\,dz$

A) $\iiint dv = \int_0^1\int_0^y\int_0^{y-x} dz\,dx\,dy$

$\iiint_D dz\,dx\,dy = \iiint_D dy\,dz\,dx = \iiint_D dx\,dy\,dz = \frac{1}{6}$

2) $\iiint_D dv = \int_0^1\int_0^{1-x}\int_{x+z}^1 dy\,dz\,dx$.

$\iiint_D dv = \int_0^1\int_z^1\int_0^{y-z} dx\,dy\,dz$.

⇒ Cylindrical Coordinates :- (cy) (co)

It represent a point in space by ordered $\iiint$-s (r,θ,z) in which

a) (r,θ) are polar (c) of the vertical projection of P on xy plane.

b) z is the □-lar vertical coordinate.

$$\boxed{x = r\cos\theta,\quad y = r\sin\theta,\quad z = z. \qquad \frac{y}{x} = \tan\theta. \qquad x^2+y^2 = r^2} \quad (1)$$

1) Convert cylindrical (co) $(2, \pi/3, 1)$ into □-lar (co) . ?

A) $(2, \frac{\pi}{3}, 1)$ → (x, y, z) $x=2$ $y=\pi/3$ $z=1$

$\sin\theta = \frac{y}{r}$
$\cos\theta = \frac{x}{r}$
$\tan\theta = \frac{y}{x}$

by (1) $x = r\cos\theta = 2$

rectangular (co) → (x, y, z)
cylin (co) → (r, θ, z)

$r = 2$, $\theta = \pi/3$, $z=1$

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$x = 2\cos\frac{\pi}{3}$, $y = 2\sin\frac{\pi}{3}$, $z=1$

$= 2\cdot\frac{1}{2}$, $= 2\frac{\sqrt{3}}{2}$

$x = 1$, $y = \sqrt{3}$, $z=1$

2) convert □-lar (co) $(-1, 1, 2)$ into (cy) (co)

3) find an eq in (cy) (co) of surface with the givn □-lar eq $x^2+y^2 = 9z$

A) Givn □-lar eq represents a paraboloid

putting $x^2+y^2 = r^2$; $r^2 = 9z$

$\therefore$ eq paraboloid in (cy) (co) is

$$r^2 = 9z$$

5) $x^2+y^2 = z^2 \longrightarrow$ cone

$r = z$

A) given eq represent cone.

$$r^2 = z^2$$

6) Express the follow (cy) eqs of surfaces in rectangular eq & identify the surface

a) $\theta = \frac{\pi}{4}$

$\frac{y}{x} = \tan\theta = \tan\frac{\pi}{4} = 1$

$y = x$

rectangular eq of the given surface is $y=x$. it is the vertical plane through,

$(0,0,0)$ $(1,1,0)$ $(1,1,1) \longrightarrow (y=x)$

b) $r^2\cos 2\theta - z^2 = 4, \Rightarrow r^2(\cos^2\theta - \sin^2\theta) - z^2 = 4. = r^2\cos^2\theta - r^2\sin^2\theta - z^2 = 4$ $(\cos 2\theta = \cos^2\theta - \sin^2\theta)$

$r\cos\theta = x$ & $r\sin\theta = y$ $\therefore x^2 - y^2 - z^2 = 4$

$\Rightarrow$ Triple $\int$s in cy (co) :-

let $f(r,\theta,z)$ be a contin ( ) over a region 'D' defined by

$f_1(r,\theta) \leq z \leq f_2(r,\theta)$, $g_1(\theta) \leq r \leq g_2(\theta)$,

$\alpha \leq \theta \leq \beta$

$\iiint$ of f over D,

$$\boxed{\iiint_D f(r,\theta,z)\,dv = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} f(r,\theta,z)\, r\,dz\,dr\,d\theta.}$$

$dv = r\,dz\,dr\,d\theta.$

1) using (cy) (co) evaluate $\iiint x^2\,dv$, where D is the solid that is bounded by the graphs of eqs. $x^2+y^2=9$, $z = 9-x^2-y^2$, $z=0$

A) $x^2+y^2 = r^2$

$r^2 = 9$, $r = 3$ $\qquad 0 \leq r \leq 3$

$z = 9 - x^2 - y^2$

$= 9 - (x^2+y^2)$ $\qquad 0 \leq z \leq 9 - r^2$

$z = 9 - r^2$ $\qquad 0 \leq \theta \leq 2\pi$

$x = r\cos\theta \longrightarrow$ (cy) (co)

$\iiint x^2\,dv$,

$= \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2\cos^2\theta\, r\,dz\,dr\,d\theta$

$= \int\int_0^3 \left[z r^3\cos^2\theta\right]_0^{9-r^2} dr\,d\theta$

$= \int\int_0^3 (9-r^2) r^3\cos^2\theta\,dr\,d\theta.$

$= \int\int_0^3 9r^3 - r^5\cos^2\theta\,dr\,d\theta.$

$= \int \left[\frac{9r^4}{4} - \frac{r^6}{6}\cos^2\theta\right]_0^3 d\theta$

$= \int_0^{2\pi}\left[\frac{9\cdot 81}{4} - \frac{729}{6}\cos^2\theta\right] d\theta.$

$= \int \frac{729}{4} - \frac{729}{6}\cos^2\theta\,d\theta.$ (Lcm take)

$= \int \frac{729\times 3 - 729\times 2}{12}\cos^2\theta\,d\theta.$

$= \int \frac{729(3-2)}{12}\cos^2\theta\,d\theta = \int_0^{2\pi}\frac{729}{12}\cos^2\theta\,d\theta.$

$= \frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{243}{4}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta.$

$$= \frac{243}{4} \int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta.$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{243}{4} \cdot \left[\frac{\theta}{2} + \frac{1}{2}\cdot\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{243}{4}\left[\frac{2\pi}{2} + \frac{1}{4}\cdot \sin 2\pi\cdot 2\right]$$

$(\sin 0 = 0)$

$$= \frac{243}{4}\left[\pi + \underbrace{\frac{\sin 4\pi}{4}}_{0}\right] = \frac{243\pi}{4}$$

Hw

2) $(-1, 1, 2)$; $x=-1,\ y=1,\ z=2$ $\square$ - ~~to~~ (cy) (co).

$x^2+y^2=r^2$

$(-1)^2+1^2 = r^2$

$2 = r^2$

$\therefore r = \pm\sqrt{2}$

$x=-1 \quad r=\sqrt{2}$

$$\boxed{\begin{aligned}\cos\theta &= \frac{x}{r}\\ \sin\theta &= \frac{y}{r}\end{aligned}} \quad \begin{aligned}&= -1/\sqrt{2}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$y=1$, $r=\sqrt{2}$

$\therefore \theta = \pi - \left(\frac{\pi}{4}\right) = 3\pi/4$

when $r=-\sqrt{2}$, $\cos\theta = \frac{1}{\sqrt{2}}$ & $\sin\theta = -\frac{1}{\sqrt{2}}$

& $\theta = -\pi/4$,

(cy) (co),

$\left(\sqrt{2}, \frac{3\pi}{4}, 2\right)$ or $\left(-\sqrt{2}, \frac{\pi}{4}, 2\right)$

Hw

3) $I_x = ?$ $-\frac{a}{2} \le x \le \frac{a}{2}$, $-\frac{b}{2} \le y \le \frac{b}{2}$, $-\frac{c}{2} \le z \le c/2$.

$\rho(x,y,z) = 1$.

$I_x = \iiint_S (y^2+z^2)\,\rho(x,y,z)\,dv.$

~~Let solid consist of 8 symmetric region~~

$$\text{(i) } (\text{zero}\ 2 \to 8) = 8\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} y^2+z^2\,dx\,dy\,dz.$$

$$= 8\int\int\left[(y^2+z^2)x\right]_{-a/2}^{a/2} dy\,dz$$

$\int_0^a \int_{-a}^{a} dx = 2\int_0^a dx$

$$= 4a\int\int (y^2+z^2)\,dy\,dz = 4a\int_0^{c/2}\left(\frac{y^3}{3}+z^2 y\right)_{-b/2}^{b/2} dz$$

$$= 4a\int_0^{c/2}\left(\frac{b^3}{24}+\frac{b}{2}z^2\right)dz = 4a\left(\frac{b^3}{24}z + \frac{bz^3}{6}\right)_0^{c/2}$$

$$= 4a\left(\frac{b^3c}{48} + \frac{bc^3}{48}\right) = \frac{abc}{12}(b^2+c^2)$$

→ **Spherical coordinates :-** (sp) (co)

Represents a point P in space by ordered $\iiint$-s (ρ, ϕ, θ) in which,

a) ρ is the distance from P to origin $(\rho \ge 0)$

b) ϕ is the angle $\overline{OP}$ makes with the +ve z-axis $(0 \le \phi \le \pi)$

c) θ is the angle from cylindrical (co). $(0 \le \theta < 2\pi)$

$\boxed{x^2+y^2+z^2=\rho^2}$

Let P be any point in space & M be its projection in xy plane, Hence, we have following relationship b/w the cartesian, (cy) & (sp) (co) system-

$r = \rho\sin\phi$	$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$
$\theta = \theta$	$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$	$z = z$	$z = \rho\cos\phi$

It is clear that,

$x^2+y^2=r^2$

$\rho^2 = x^2+y^2+z^2 = r^2+z^2$;

(or) $r^2 = \rho^2 - z^2$ $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

$\boxed{dx\,dy\,dz \text{ or } dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$

$\left[\tan\theta = \frac{y}{x} \quad , \quad \cos\phi = \frac{z}{\rho} \right]$ $\boxed{\phi = \cos^{-1}\frac{z}{\rho}}$

Every point in the whole space can be gun (sp) (co) restricted to ranges

$\boxed{\theta = \tan^{-1}\frac{y}{x}}$ $\rho \geq 0 \; , \; 0 \leq \phi \leq \pi$.

Q

1) convert (sp) (co) $(2, 5\frac{\pi}{6}, 0)$ into $\square$-lar (co) & (cy) (co) ?

A) $\rho = 2 \; , \; \phi = 5\frac{\pi}{6} \; , \; \theta = 0$.

(sp) (co)
(ρ, ϕ, θ)
$2 \quad 5\pi/6 \quad 0$.

Then cartesian (co) x, y, z ,

$x = \rho \sin\phi \cos\theta$
$= 2\sin\frac{5\pi}{6}\cos 0$
$= 2\sin\left(\pi - \frac{\pi}{6}\right) = 2\sin\frac{\pi}{6} = 1$

$\left(\frac{5\pi}{6} = \pi - \frac{\pi}{6}\right)$

$\sin\frac{\pi}{2} = 1$
$\sin\frac{\pi}{6} = \frac{1}{2}$

$2 \times \frac{1}{2} = 1$

$y = \rho\sin\phi\sin\theta$
$= 2\sin\frac{5\pi}{6}\sin 0 = 0$

$z = \rho\cos\phi = 2\cos\frac{5\pi}{6} = 2\cos\left(\pi - \frac{\pi}{6}\right)$
$= -2\cos\frac{\pi}{6} = -\sqrt{3}$
$= -2\cdot\frac{\sqrt{3}}{2}$

$\therefore$ cartesian (co) $\longrightarrow \left(1, 0, -\sqrt{3}\right)$

The (cy) (co) (r, θ, z) ,

$r = \rho\sin\phi$
$= 2\sin\frac{5\pi}{6} = 1$

$\theta = 0$

$z = \rho\cos\phi = 2\cos\frac{5\pi}{6} = -\sqrt{3}$

(or)
$r^2 = \rho^2 - z^2 \; , \; r = \sqrt{\rho^2 - z^2}$
$r = \sqrt{2^2 - (-\sqrt{3})^2} = \sqrt{4-3} = \sqrt{1}$
$r = 1$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{1}$
$= \tan^{-1} 0 = 0$

$z = -\sqrt{3}$

$\therefore$ (cy) (co) $\longrightarrow \left(1, 0, -\sqrt{3}\right)$

2) convert (cy) (co) $\left(-1, \frac{-\pi}{3}, -1\right)$ into (sp) (co).
$\quad r \quad \theta \quad z$

A) $r = -1 \; , \; \theta = -\frac{\pi}{3} \; , \; z = -1$

(sp) (co)
ρ ?
ϕ ?
θ ?

Then (sp) (co) ρ, ϕ, θ ,

$\boxed{\rho^2 = r^2 + z^2} = (-1)^2 + (-1)^2 = 2$

$(\rho \geq 0)$

$\therefore \rho = \sqrt{2}$

$\boxed{\cos\phi = \frac{z}{\rho}} = -\frac{1}{\sqrt{2}}$

$\phi = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$
$= -\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = -\pi/4$

$(0 \leq \phi \leq \pi)$

- normally we can write it as $\pi - \pi/4$.

$\therefore \phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$-\frac{\pi}{4} = \pi - \frac{\pi}{4}$
$-\frac{\pi}{3} = 2\pi - \frac{\pi}{3}$

$\theta = -\frac{\pi}{3} = 2\pi - \left(\frac{\pi}{3}\right) = 5\pi/3$.

$\left(\sqrt{2}, \frac{3\pi}{4}, \frac{5\pi}{3}\right)$

3) find cartesian eq for surface $\rho = \cos\phi$, identify the surface ?.

A) from relationship b/w (sp) (co) & cartesian (co) , we hue ,

$\rho = \sqrt{x^2+y^2+z^2} \quad , \quad \cos\phi = \frac{z}{\rho}$.

hence cartesian eq of surface, $\rho = \cos\phi$,

$\sqrt{x^2+y^2+z^2} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

(i.e) $x^2+y^2+z^2 = z$ (i.e) $x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$

which is a sphere of radius $\frac{1}{2}$ & centre $\left(0, 0, \frac{1}{2}\right)$

~~Q) find (SP) (co) eq to (SP) (co) eq put,~~
~~$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$~~

Q) find (SP) (co) eq for cone, $z=\sqrt{x^2+y^2}$

A) To convert above cartesian eq to spherical (co) put,

$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$ $z=\rho\cos\phi$

$$\rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2}$$
$$=\sqrt{\rho^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta)}$$

$\rho\cos\phi = \rho\sin\phi$ $[\cos^2\theta+\sin^2\theta=1]$

$\rho\geq 0$, $\sin\phi\geq 0$.

$\therefore$, $\cos\phi=\sin\phi$ & so $\phi=\frac{\pi}{4}$ $(0\leq\phi\leq\pi)$.

$\therefore$, (SP) (co) eq for cone is $\phi=\frac{\pi}{4}$

$\Rightarrow$ Triple $\int$-s in (SP) (co) :-

vol element in (SP) (co) is the vol of a (SP) wedge defined by the differentials $d\rho$, $d\phi$ & $d\theta$.

The wedge is approximately a ▭ box with 1 side a ○ arc of length $\rho\,d\phi$, another side a ○ arc of length $\rho\sin\phi\,d\theta$ & thickness $d\rho$.

$\therefore$, vol element in (SP) (co),

$$\boxed{dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

& $\iiint$ takes,

$$\boxed{\iiint_D F(\rho,\phi,\theta)\,dV = \iiint_D F(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

Q) find moment of inertia about z-axis of homogeneous solid bounded b/w spheres $x^2+y^2+z^2=a^2$ & $x^2+y^2+z^2=b^2$, $a<b$.

A) Convert cartesian eq to (SP) (co),

$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$.

$(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2+(\rho\cos\phi)^2=a^2$

$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi=a^2$

$\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)+\rho^2\cos^2\phi=a^2$

$\rho^2\sin^2\phi+\rho^2\cos^2\phi=a^2$

$\rho^2=a^2$ $\therefore$ $\underline{\rho=a}$

$\approx$ other eq becomes $\rho^2=b^2$, $\rightarrow$ $\underline{\rho=b}$

hence eq of given spheres in (SP) (co) are $\rho^2=a^2$ & $\rho^2=b^2$ where $a<b$

$a\leq\rho\leq b$, $0\leq\phi\leq\pi$, $0\leq\theta\leq 2\pi$.

$\rho=a$, $\rho=b$

density of solid is a constant say k, Then moment of inertia about z-axis,

$$I_z=\iiint_D k(x^2+y^2)\,dV.$$
$$=\iiint_D k\left[(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2\right]dV$$
$$=\iiint k\rho^2\sin^2\phi\,\underbrace{dV}.$$
$$=\int_0^{2\pi}\int_0^{\pi}\int_a^b k\rho^2\sin^2\phi\,\underline{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$
$$=k\int\int\int_a^b \rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta.$$

$I_z=\iiint_D (x^2+y^2)\underbrace{f(x,y,z)}_{k}dV$

$\iiint_D (x^2+y^2)k\,dV$

$= k\int\int \left[\frac{p^5}{5}\sin^3\phi\right]_a^b d\phi\, d\theta$ $\quad \int_a^b p^4 = \frac{p^5}{5} = \frac{1}{5}(b^5-a^5)$

$= \frac{k}{5}(b^5-a^5)\int_0^{2\pi}\int_0^{\pi}(1-\cos^2\phi)\sin\phi\, d\phi\, d\theta.$ $\quad \int \sin^3\phi\, d\phi = \int \sin\phi \cdot \sin^2\phi\, d\phi = \int \sin\phi\,(1-\cos^2\phi)\, d\phi$

$= \frac{k}{5}(b^5-a^5)\int_0^{2\pi}\int_1^{-1}(1-u^2)(-du)\, d\theta$

$(\sin^2\phi = 1-\cos^2\phi)$ $\quad = \left[-u+\frac{u^3}{3}\right]_1^{-1}$

let $u = \cos\phi$.

$du = -\sin\phi\, d\phi$

$-du = \sin\phi\, d\phi$

$\phi = 0,\ u = 1$

$\phi = \pi,\ u = \cos\pi = -1$

$= \frac{k}{5}(b^5-a^5)\int_0^{2\pi}\left[-u+\frac{1}{3}u^3\right]_1^{-1} d\theta$

$= \int(1-\cos^2\phi)\sin\phi\, d\phi$

$= \int(\sin\phi - \cos^2\phi\sin\phi\, d\phi)$

$\frac{4\times}{5\cdot 3}$ $= \frac{4k}{15}(b^5-a^5)\int_0^{2\pi} 1\cdot d\theta$ $\quad [\theta]_0^{2\pi} = 2\pi$

$= \frac{8k\pi}{15}(b^5-a^5)$

(D.T)

⇒ Divergence theorem :- (or Gauss's D.T)

Let D be a closed & bounded region in 3-space with a piecewise-smooth boundary S that is oriented outward.

Let $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$ be a v. field for which P, Q, R are contin. & have contin. 1st partial deriv. in a region of 3 space containing D,

$$\boxed{\iint_S F\cdot n\, ds = \iiint_D \mathrm{div} F\, dV = \iiint_D \nabla\cdot F\, dV}$$

$n = \frac{\nabla f}{\|\nabla f\|}$

Q

1) Verify D. theorem for field $F = xi+yj+zk$ over sphere $x^2+y^2+z^2 = a^2$. — ①

A) $\iint_S F\cdot n\, ds = \iiint_D \mathrm{div}F\, dv$, let $f(x,y,z) = x^2+y^2+z^2-a^2$

S → surface of sphere $x^2+y^2+z^2 = a^2$

D → region enclosed by S.

$f(x,y,z) = x^2+y^2+z^2 = a^2$ $\quad \to a = \sqrt{x^2+y^2+z^2}$

outward unit normal to S,

ⓑ — $n = \frac{\nabla f}{\|\nabla f\|} = \frac{2xi+2yj+2zk}{\sqrt{(2x)^2+(2y)^2+(2z)^2}} = \frac{2(xi+yj+zk)}{\sqrt{4(x^2+y^2+z^2)}}$ $= a^2$

$\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$

$= \frac{2xi+yj+zk}{\sqrt{4\cdot a^2}}$

$= \frac{2xi+yj+2k}{2a}$

$= \frac{xi+yj+zk}{a}$

$F\cdot n = [xi+yj+zk]\cdot\frac{xi+yj+zk}{a}$

$= \frac{x^2+y^2+z^2}{a} = \frac{a^2}{a} = a$

~~on S, since $x^2+y^2+z^2 = a^2$.~~

~~$\therefore F\cdot n = \frac{a^2}{a} = a$~~

$x^2+y^2+z^2 = a^2$

$\therefore r^2 = a^2$

ⓒ — $r = a$

$\therefore \iint_S F\cdot n\, ds = \iint_S a\, ds = a\iint_S ds.$

$= a\times$ surface area of sphere.

$= a(4\pi a^2) = 4\pi a^3$ — ①

ⓒ

2nd part

~~The Divergence of F,~~

$\mathrm{div}F = \nabla\cdot f = \left[\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right]\cdot(xi+yj+zk)$

$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 1+1+1 = 3$

$\therefore \iiint_D \nabla\cdot f\, dv = \iiint_D 3\, dV = 3\iiint_D dv.$

$= 3\times$ vol of sphere

$r = a$

$= 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$ — ②

from ① & ② $\iint_S F\cdot n\,ds = \iiint_D \operatorname{div} F\,dV$

2) Let D be region bounded by hemisphere $x^2+y^2+(z-1)^2=9$, $1\le z\le 4$ & plane $z=1$. verify D. theorem if $F = xi+yj+(z-1)k$.

A) with center $(0,0,1)$, radius $= 3$, $z=1$ $\boxed{\iint_S F\cdot n\,ds = \iiint_D \nabla F}$

$F = xi+yj+(z-1)k$,

$\operatorname{div} F = \nabla\cdot F$

$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\cdot(xi+yj+zk)$ (pg-253)

$= 1+1+1 = 3$

$\therefore \iiint_D \operatorname{div} f\,dv = \iiint_D 3\,dv = 3\iiint_D dV$.

$= 3\times \text{vol of } D$

$= 3\times\frac{2}{3}\pi 3^3 = 54\pi$ — ①

Surface S $\to S_1$ & S_2.

$S_1 \to$ hemisphere

$S_2 \to$ O , $x^2+y^2=9$.

$\iint_S f\cdot n\,ds = \iint_{S_1} f\cdot n_1\,ds_1 + \iint_{S_2} f\cdot n_2\,ds_2$.

S_1 is hemisphere $x^2+y^2+(z-1)^2=9$ $r=3$

S_1 is a level surface of

$g(x,y,z) = x^2+y^2+(z-1)^2$.

$n_1 = \frac{\nabla g}{\|\nabla g\|} = \frac{2xi+2yj+2(z-1)k}{\sqrt{(2x)^2+(2y)^2+(2(z-1))^2}}$

$= \frac{2(xi+yj+(z-1)k)}{2\sqrt{x^2+y^2+(z-1)^2}} = \frac{xi+yj+(z-1)k}{3}$

$\therefore f\cdot n_1 = (xi+yj+(z-1)k)\cdot\frac{xi+yj+(z-1)k}{3}$

$= \frac{x^2+y^2+(z-1)^2}{3}$

on S_1, $x^2+y^2+(z-1)^2=9$, $f\cdot n_1 = \frac{9}{3} = 3$

$\iint_{S_1} F\cdot n\,ds_1 = \iint_{S_1} 3\,ds_1 = 3\iint_{S_1} ds$,

$= 3\times$ S.A of hemisphere

$= 3\times 2\pi 3^2 = 54\pi$ — ①

$f\cdot n_2 = (xi+yj+(z-1)k)\cdot(-k) = -z+1$

on S_2 , $z=1$. $f\cdot n_2 = 0$.

$\iint_{S_2} f\cdot n\,ds_2 = 0$

$\therefore \iint_S F\cdot n\,ds = \iint_{S_1} F\cdot n_1\,ds_1 + \iint_{S_2} F\cdot n_2\,ds_2$

$= 54\pi$ — ②

from ① & ② . $\iint_S F\cdot n\,ds = \iiint_D \operatorname{div} f\,dv$

3) If $A = 2xy\,i + yz^2 j + xz\,k$, & S is ▭ parallelepiped bounded by $x=0$ $y=0$ $z=0$, $x=2$ $y=1$ $z=3$, evaluate $\iint_S A\cdot n\,ds \to \left(\iint_S f\cdot n\,ds\right)$

A) by gauss's d. theorm,

$\iint_S A\cdot n\,ds = \iiint_D \operatorname{div} A\,dv$ bcz we have values of x,y,z

$\operatorname{div} A = \nabla\cdot A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\cdot(2xy\,i + yz^2 j + xz\,k)$

$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)$

$= 2y + z^2 + x$

Solid D is completely determined by inequalities, $0\le x\le 2$, $0\le y\le 1$, $0\le z\le 3$

$$\therefore \iint_S A \cdot n \, ds = \cancel{\iiint_V \nabla \cdot A \, dV} \iiint_D (2y + z^2 + x) \, dV$$

$$= \int_0^3 \int_0^1 \int_0^2 (2y + z^2 + x) \, dx \, dy \, dz$$

$$= \iint \left[2yx + z^2 x + \frac{x^2}{2} \right]_0^2 dy \, dz$$

$$= \int \int_0^1 \left[4y + 2z^2 + 2 \right] dy \, dz$$

$$= \int \left[2y^2 + 2z^2 y + 2y \right]_0^1 dz$$

$$= \int_0^3 \left[4 + 2z^2 \right] dz$$

$$= \left[4z + \frac{2z^3}{3} \right]_0^3 = 30$$

⇒ Change of variables in $\iint$-s :-

If f, g and F have contin partial deriv & $J(u,v)$ is 0 only at isolated point

$$\boxed{\iint_R F(x,y)\,dA = \iint_S F\left(f(u,v), g(u,v)\right) |J(u,v)|\, dA'}$$

dA' → either du dv or dv du. & the factor $J(u,v)$ → Jacobian (co) transformation is defined by,

$(u,v \to x,y)$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v}$$

Inverse transformation,

$(x,y \to u,v)$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}$$

1) Solve the system $u = x - y$, $v = 2x + y$ for x & y in terms of u & v then find the value of Jacobian $J(u,v)$.

(2) find image under transformation

$u = x - y$ & $v = 2x + y$ of Δ-lar region with vertices $(0,0)$ $(1,1)$ $(1,-2)$ in xy plane. Sketch the transformed region in uv plane ?. (2)

A) $(u,v) \longrightarrow (x,y)$

$① \; u = x - y$ +
$② \; v = 2x + y$

$u + v = 3x.$

$\therefore \frac{1}{3}(u+v) = x$

$$\left.\begin{aligned} u &= x - y \\ u &= \frac{1}{3}(u+v) - y \\ u &= \frac{u}{3} + \frac{v}{3} - y \\ u - \frac{u}{3} &= \frac{v}{3} - y \end{aligned}\right\} ①$$

$\cancel{① \times 2.}$

$\cancel{2u = 2x - 2y} \; — ③$

$\cancel{③ - ②}$

$$\cancel{\begin{aligned} 2u &= 2x - 2y \\ v &= 2x + y \\ \hline 2u - v &= -3y \end{aligned}}$$

$\cancel{-\frac{1}{3}(2u - v) = y}$

$\cancel{\frac{1}{3}(v - 2u) = y}$

$\frac{3u - u}{3} = \frac{v}{3} - y$

$\frac{2u}{3} = \frac{v}{3} - y$

$\frac{2u}{3} - \frac{v}{3} = -y$

$\frac{2u - v}{3} = -y$

$\frac{1}{3}(2u - v) = -y$

$-\frac{1}{3}(2u - v) = y$

$\frac{-2u}{3} + \frac{v}{3} = y$

$\frac{-2u + v}{3} = y$

$\frac{1}{3}(v - 2u) = y$

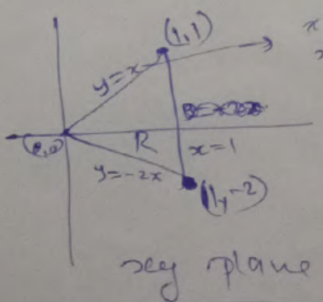
$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$\frac{\partial x}{\partial u} = \frac{1}{3}$ $\quad \frac{\partial x}{\partial v} = \frac{1}{3}$ $\quad \frac{\partial y}{\partial u} = \frac{-2}{3}$ $\quad \frac{\partial y}{\partial v} = \frac{1}{3}.$

$$\begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -2/3 & 1/3 \end{vmatrix} = \frac{1}{3} \cdot \frac{1}{3} - \left(\frac{-2}{3} \times \frac{1}{3}\right)$$

$$= \frac{1}{9} + \frac{2}{9} = \frac{3}{9} = \frac{1}{3}$$

$(0,0)$ $\quad (1,1)$ $\quad (1,-2)$



xy plane

$x = 0, \; y = 0$

$x = 1 \quad y = 1$

$x =$

$x = \frac{1}{3}(u+v)$

$y = \frac{1}{3}(v-2u)$

$\frac{u}{3} + \frac{\cancel{v}}{3} = \frac{\cancel{v}}{3} - \frac{2u}{3}$

$\frac{u}{3} + \frac{2u}{3} = 0$

$3u.$

(1) — $y = x.$

$\frac{1}{3}(u+v) = \frac{1}{3}(v-2u) \Rightarrow \underline{u=0}$

(2) — $x = 1$

$\frac{1}{3}(u+v) = 1.$

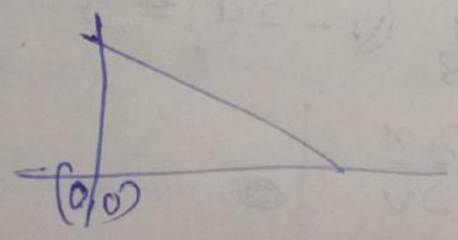
$u+v = 3.$

(3) — $y = -2x$

$\frac{1}{3}(v-2u) = -2\left(\frac{1}{3}(u+v)\right)$

$\underline{v=0}$

$(0,0)$ $(0,3)$ $(3,0)$



$\sqrt{1-x^2}$

$3/2$

$\Rightarrow$ Change in cartisian (co) into polar (co) $\rightarrow$

car. (co) $\rightarrow (x, y)$ polar (co) $\rightarrow (r, \theta)$

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} \Bigg\} \text{ (x,y) ను (r,θ) లోకి మార్చుట.}$$

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = \underline{\underline{r}}$$

$\left.\begin{matrix} dxdy \\ dydx \\ dA \end{matrix}\right\} r\,dr\,d\theta$

Area $= \iint r\,dr\,d\theta$

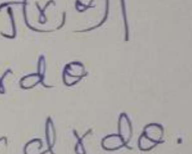
$$\boxed{\begin{gathered} \iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\,|J(r,\theta)|\,dr\,d\theta \\ \iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta. \end{gathered}}$$

1) Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$, where R is the semi ○ region handed by the x. axis & center $y = \sqrt{1-x^2}$

A) $x = r\cos\theta \quad y = r\sin\theta.$

$$\therefore e^{x^2+y^2} = e^{r^2\cos^2\theta + r^2\sin^2\theta} = e^{r^2(\cos^2\theta+\sin^2\theta)}$$

$$= e^{r^2}$$



$x \rightarrow 0 - \pi$

$r \rightarrow 0 - 1$

$$\int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta$$

Let $r^2 = u$

$2r\,dr = du$

$r\,dr = \frac{du}{2}.$

$$= \int_0^{\pi}\left[e^u\,\frac{du}{2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\left[e\cdot\frac{1}{2}\right]_0^1 d\theta. = \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta.$$

$= \frac{1}{2}(e-1)\int_0^{\pi} d\theta = \frac{1}{2}(e-1)[\theta]_0^{\pi}$

$= \frac{1}{2}(e-1)\pi$

2) Evaluate $\int_0^4 \int_{x=y/2}^{x=y/2+1} \frac{2x-y}{2}\,dx\,dy$ by applying transformation $u = \frac{2x-y}{2}$, $v = y/2$

A) $u = \frac{2x-y}{2} \Rightarrow 2u = 2x-y$ — ①

$v = \frac{y}{2} \Rightarrow 2v = y$ — ②

② in ①

$2u = 2x - y$

$2u = 2x - 2v$

$2u + 2v = 2x$

$\frac{2u+2v}{2} = x \Rightarrow \frac{2(u+v)}{2} = x$

$\Rightarrow u+v = x$

$J(x,y) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$y = 2v$
$x = u+v$

$= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$

Limit

$y=0$, $y=4$, $x = \frac{y}{2}$, $x = \frac{y}{2}+1$.

$y=0$, $y=4$ $\quad 2x = y$, $\quad 2x = y+1$

$y = 2x-1$.

$y=0 \Rightarrow 2v=0 \Rightarrow v=0$

$y=4 \Rightarrow 2v=4 \Rightarrow v=2$.

$y = 2x$

$2v = 2(u+v)$

$2v = 2u+2v$

$2u = 2v-2v = 0$

$2u = 0 \Rightarrow u=0$

$2v = 2(u+v) - 2$

$2v = 2u - 2v - 2$

$2u = 2v - 2v + 2$

$2u = 2 \Rightarrow u = 1$

$v \to 0-2$

$u \to 0-1$

$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \iint u\,|J(u,v)|\,du\,dv$

$= \iint u \cdot 2\,du\,dv = 2\int_0^2 \frac{u^2}{2}\,dv = 2\cdot\frac{1}{2}\int_0^2 dv$

$= \int_0^2 dv = (v)_0^2 = 2$

3) $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$.

4) $\iint_R xy\,dA$, R is the region of 1st quadrant of xy plane bounded by the curves $xy=1$, $y=x^2$, $xy=5$, $y=4x^2$

$\Rightarrow$ Substitution of $\iiint$:-

If f, g & h hue contin 1st partial derivatives & ~~Jacobian~~ $J(u,v,w)$ is 0 only at isolated points, if at all then,

$$\boxed{\iiint_D F(x,y,z)\,dV = \iiint_E G(u,v,w)\,|J(u,v,w)|\,dV'}$$

$\sigma(u,v,w) = f\big(f(u,v,w),\ g(u,v,w),\ h(u,v,w)\big)$

$dv' = du\,dv\,dw.$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

1) find the vol of ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

A) $\boxed{\text{Vol} = \iiint dv}$

$u = \frac{x}{a} \quad v = \frac{y}{b} \quad w = \frac{z}{c}$

$au = x \quad bv = y \quad cw = z.$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= a\,|bc| - 0 + 0 = abc$$

transform,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ into $u^2+v^2+w^2 = 1$

$\therefore \text{Vol} = \iiint_D dv.$

$= \iiint_E |J(u,v,w)|\, du\,dv\,dw$

ellipsoid

8-octact.

$= \iiint_E abc\; du\,dv\,dw$ — ①

$u^2+v^2+w^2 = 1$

$\Rightarrow w^2 = 1-u^2-v^2$

$w = \sqrt{1-u^2-v^2}$

$0 \le w \le \sqrt{1-u^2-v^2}$

$\iiint_E dw\,dv\,du = 8 \times \iiint_{E_1} dw\,dv\,du,$

$\iint_R \int_0^{\sqrt{1-u^2-v^2}} dw\,dv\,du. = \iint_R [w]_0^{\sqrt{1-u^2-v^2}}\, dv\,du,$

$= \iint_R \sqrt{1-u^2-v^2}\; dv\,du$

let $u = r\cos\theta$, $v = r\sin\theta$

$\sqrt{1-u^2-v^2} = \sqrt{1-r^2\cos^2\theta - r^2\sin^2\theta}$

$= \sqrt{1-r^2}$

$dv\,du = r\,dr\,d\theta.$

$8\iint_R \sqrt{1-r^2}\; r\,dr\,d\theta.$

$8\int_0^{\pi/2}\int_0^1 \sqrt{1-r^2}\; r\,dr\,d\theta = 8\int\int_0^1 (1-r^2)^{1/2}\, r\,dr\,d\theta.$

$= 8\int_0^{\pi/2} \left((1-r^2)^{3/2} \cdot \frac{2}{3} \cdot \frac{r^2}{2}\right)_0^1 d\theta = 8\int \frac{2}{3}\cdot\frac{1}{2}\, d\theta.$

$= 8\int_0^{\pi/2} \frac{1}{3}\, d\theta = 8 \cdot \frac{1}{3}\, (\theta)_0^{\pi/2} = \frac{8}{3}\cdot\frac{\pi}{2}$

$\iiint_E abc\, du\,dv\,dw = \frac{8\pi}{6} = \frac{4\pi}{3} \times abc.$