Module = 02

vector fields

* A vector field (v.f) on a domain in the plane / space is a () that assigns a vector to each point on the domain,

$$\boxed{f(x,y,z) = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k}$$

is the field of 3D vector.

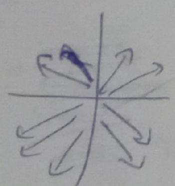
* The real valued () P,Q,R [component ()] of v.f 'f'.
* v.f 'f' is contin & diff..., if component () P,Q,R are contin & diff...
* A field of 2D (v) →

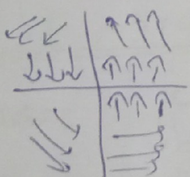
$$\boxed{f(x,y) = P(x,y)\,i + Q(x,y)\,j}$$

v.f

1) $f = xi + yj$

2) $f = -\dfrac{yi + xj}{\sqrt{x^2+y^2}}$





→ Gradient of a v.f :-

$f(x,y)$

$$\nabla f(x,y) = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j.$$

$$f\,\phi(x,y,z) = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k.$$

→ curl Divergence :-

$$\boxed{\text{curl } f = \nabla \times F}$$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$i\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - j\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\boxed{\text{Div } f = \nabla \cdot f}$$

$$\left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(Pi + Qj + Rk)$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

* A v.f 'f' said to be irrotational if curl f = 0
* A v.f 'f' is said to be Solenoidal / incompressible if div f = 0

$$\text{div}(\text{grad } f) = \nabla \cdot (\nabla f)$$

$$= \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \cdot \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\boxed{\operatorname{div}(\operatorname{grad} f) = \nabla^2 f}$$

1) Compute the div & curl of (V).

$f = xyz\, i + 3x^2y\, j + (xz^2 - y^2z)\, k$

at 1, 2, −1.

A) $\boxed{\operatorname{div} f = \nabla \cdot f}$ = grad f.

$\boxed{f = Pi + Qj + Rk.}$

$P = xyz$

$Q = 3x^2y$

$R = xz^2 - y^2z$

$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$= yz + 3x^2 + (2xz - y^2)$.

$\boxed{\operatorname{div}(\operatorname{curl} f) = \nabla \cdot (\nabla \times f)}$

~~curl f = ∇ × f~~

at $(1, 2, -1)$,

$= yz + 3x^2 + (2xz - y^2)$

$= -2 + 3 + (-2-4) = \underline{-5}$

$\boxed{\operatorname{curl} f = \nabla \times f}$

$\boxed{\operatorname{curl}(\operatorname{grad} f) = \nabla \times (\nabla \cdot f)}$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times (Pi + Qj + Rk)$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right] + j\left[\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right] + k\left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right].$$

y → as terms so 0

$= i[(0 - 2yz) - 0] + j(z^2 - 0) - xy + k(6xy - xz)]$

$= i[-2yz] + j[z^2 - xy] + k[6xy - xz]$

at (1,2,−1)

$= [-2 \times 2 \times -1] + [(-1)^2 - 1 \times 2] + [6 \times 1 \times 2 - 1 \times -1]$

$= 4 - 1 + 12 + 2 = 3 + 14 = 17$

$12 - [1 \times -1]$

$12 - [-1]$

$12 + 1 = 13$

$4i + j + 13k$

$[-4 \times -1] + (1 - 2) + [12 + 1]$

$4 - (-1) + 1 \ 1$

$4i + 1j + 13k$

2) If f is a v.f having continuous 2nd partial derivatives then, P.T $\operatorname{div}(\operatorname{curl} \hat{f}) = 0$?

A) $\operatorname{div} f = \nabla \cdot f$

$\boxed{\operatorname{div}(\operatorname{curl} f) = \nabla \cdot (\operatorname{curl} f)}$

$\operatorname{div} f = \nabla \cdot f$.

$\operatorname{div} x = \nabla \cdot x$.

$\operatorname{div}(\operatorname{curl} f) = \nabla (\operatorname{curl} f)$

$\operatorname{curl} f = \nabla \times f$

$\operatorname{div}(\operatorname{curl} f) = \nabla \cdot (\nabla \times f)$

$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$.

$F = Pi + Qi + Rk$.

$$\bar{\nabla} \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= i\left| \frac{\partial}{\partial y} R - \frac{\partial}{\partial z} Q \right| - j\left| \frac{\partial}{\partial x} R - \frac{\partial}{\partial z} P \right| + k\left| \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right|.$$

$\therefore$ $\text{div}(\text{curl} f) = \bar{\nabla} \cdot (\nabla \times F)$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left[i\left(\frac{\partial}{\partial y} R - \frac{\partial Q}{\partial z}\right) - j\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + k\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$$

$$= \left[\cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial x \partial z}}\right] - \left[\cancel{\frac{\partial^2 R}{\partial y \partial x}} - \cancel{\frac{\partial^2 P}{\partial y \partial z}}\right] + \left[\cancel{\frac{\partial^2 Q}{\partial z \partial x}} - \cancel{\frac{\partial^2 P}{\partial z \partial y}}\right] = 0$$

3) P.T for any scalar () f(x, y, z) whose 2nd partial derivative are contin $\underline{\text{Curl}(\text{grad} f) = 0}$

A) $\text{curl}(\text{grad} f) = 0$.

$\text{grad} f = \bar{\nabla} \cdot f$.

$$\boxed{\text{curl}(\text{grad} f) = \bar{\nabla} \times (\bar{\nabla} \cdot f)}$$

$$\bar{\nabla} = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$\bar{\nabla} \cdot f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$$

$$\bar{\nabla} \times (\nabla \cdot f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left| \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \cdot \frac{\partial f}{\partial y} \right| - j\left| \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \cdot \frac{\partial f}{\partial x} \right| - k\left| \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial x} \right|$$

$$= i\left| \cancel{\frac{\partial^2 f}{\partial y \partial z}} - \cancel{\frac{\partial^2 f}{\partial z \partial y}} \right| - j\left| \cancel{\frac{\partial^2 f}{\partial x \partial z}} - \cancel{\frac{\partial^2 f}{\partial z \partial x}} \right| + k\left| \cancel{\frac{\partial^2 f}{\partial x \partial y}} - \cancel{\frac{\partial^2 f}{\partial y \partial x}} \right| = 0.$$

4) P.T the V.f $3y^4z^2 i + 4x^3z^3 j - 3x^2y^2 k$ is solenoidal ?

A) $\boxed{\text{Solenoidal} \rightarrow \text{div} f = 0}$ $= \bar{\nabla} \cdot f = 0$

$f = Pi + Qj + Rk$.

$P = 3y^4z^2$, $Q = 4x^3z^3$, $R = -3x^2y$.

$$\bar{\nabla} \cdot f = \frac{\partial P}{\partial x} i + \frac{\partial P}{\partial y} j + \frac{\partial P}{\partial z} k$$

$$= 0i + 0j + 0k. = 0$$

$\therefore$ $\text{div} f = 0$ //

5) find the value of a if $F = (axy - z^2)i + (x^2+2yz)j + (y^2 - axz)k$ is irrotational?

A) curl f = 0 ? $= \bar{\nabla} \times F = 0$.

$\bar{\nabla} = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$.

$F = Pi + Qi + Ri$.

$P = axy - z^2$.

$Q = x^2 + 2yz$.

$R = y^2 - axz$

$\boxed{\text{curl } f = \bar{\nabla} \times q.}$

$$\text{curl } f \Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^2 & x^2+2yz & y^2 - axz \end{vmatrix}$$

$$= i\left|\frac{\partial}{\partial y}(y^2 - axz) - \frac{\partial}{\partial z}(x^2+2yz)\right| - j\left|\frac{\partial}{\partial x}(y^2 - axz) - \frac{\partial}{\partial z}(axy - z^2)\right| + k\left|\frac{\partial}{\partial x}(x^2+2yz) - \frac{\partial}{\partial y}(axy - z^2)\right|.$$

$$= i\,\underbrace{|2y - 2y|}_{0} - j|az - 2z| + k|2x - ax|$$

$= (az - 2z)j + (2x - ax)k$.

$\Rightarrow$ curl f = 0 $\Rightarrow (az - 2z)j + (2x - ax)k = 0$

$z(a-2)j + x(2-x) = 0$.

$a - 2 = 0 \qquad 2 - x = 0$.

$a = 2, \qquad x = 2$

6) If a is a constant (v) & $r = xi + yj + zk$ S.T curl $(a \times r) = 2a$.

A) let $a = a_1 i + a_2 j + a_3 k$

$\therefore$, $$a \times r = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$= (a_2 z - a_3 y)i + (a_3 x - a_1 z)j + (a_1 y - a_2 x)k$.

curl$(a \times r) = \bar{\nabla} \times (a \times r)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(a_1 y - a_2 x) - \frac{\partial}{\partial z}(a_3 x - a_1 z)\right] i + \left[\frac{\partial}{\partial z}(a_2 z - a_3 y) - \frac{\partial}{\partial x}(a_1 y - a_2 x)\right] j + \left[\frac{\partial}{\partial x}(a_3 x - a_1 z) - \frac{\partial}{\partial y}(a_2 z - a_3 y)\right] k.$$

$= 2a_1 i + 2a_2 j + 2a_3 k = 2a$ //