chapter = 02

Hyperbolic Functions

$e^x \to$ Exponential ()

(e) → exponential

☆ H.F. are sple combinations of (e) function.

Consider ~~sinh~~ $\boxed{\sinh x = \dfrac{e^x - e^{-x}}{2}}$ → Hyperbolic sin of x = sinh x.

$\boxed{\cosh x = \dfrac{e^x + e^{-x}}{2}}$ → H. cosine of x. = cosh x.

≃ we can define other H.F ⟶

* $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}}$

$\Rightarrow \boxed{\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}$

* $\text{cosech}\, x = \dfrac{1}{\sinh x} = \dfrac{1}{\frac{e^x - e^{-x}}{2}}$

$\boxed{\text{cosech}\, x = \dfrac{2}{e^x - e^{-x}}}$

* $\text{sech}\, x = \dfrac{1}{\cosh x} = \dfrac{1}{\frac{e^x + e^{-x}}{2}}$

$\boxed{\text{sech}\, x = \dfrac{2}{e^x + e^{-x}}}$

* $\coth x = \dfrac{1}{\tanh x} = \dfrac{1}{\frac{e^x - e^{-x}}{e^x + e^{-x}}}$

$\boxed{\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}}$

Remark

sin max ⟶ 1
min → −1
sinh ≠ sin.

* sin x varies b/w −1 & 1.
sinh x varies b/w −∞ to ∞.

* cos x varies b/w −1 & 1.
cosh x varies ~~b/w~~ frm +∞ to 1 to +∞

* we know trignometric () are periodic
eg → $\sin(x + 2\pi) = \sin x$.
But H. () are not periodic.

* $|\sec x|$ is never < 1:
sec x never > 1 & always +ve.

* tan x varies frm −∞ to +∞.
tanh x varies frm −1 to +1

for $x \to -\alpha$ (large & +ve)	for $x \to -\alpha$ (large & -ve)
* $\cosh x \approx \sinh x \simeq \frac{1}{2} e^{x}$.	$\cosh x \approx -\sinh x \simeq \frac{1}{2} e^{-x}$
* $\tanh x \approx \coth x \simeq 1$	$\tanh x \approx \coth x \simeq -1$
* $\operatorname{sech} x \approx \operatorname{cosech} x \simeq 2e^{-x} \approx 0$	$\operatorname{sech} x \approx -\operatorname{cosech} x \simeq 2e^{x} \approx 0$

Remark → (I → Identities)

I → $\boxed{\cosh x + \sinh x = e^{x}}$.

→ $\sinh(-x) = -\sinh x$ → (odd ()) $\sin(-x) = -\sin x$

→ $\cosh(-x) = \cosh x$. $\cos(-x) = \cos x$

↳ (even ())

(odd () even () eg)

$f(-x) = -f(x)$ (odd)

$f(-x) = f(x)$ (evn)

→ Hyperbolic Identities :-

1) $\boxed{\cosh^2 x - \sinh^2 x = 1}$

proof $\cosh x^2 - \sinh^2 x = \left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2$ $(a+b)^2$

$= \frac{e^{2x} + 2e^{0} + e^{-2x}}{4} - \left[\frac{e^{2x} + e^{-2x} - 2e^{x}e^{-x}}{4}\right]$

$(e^{x})^2 = e^{x \times 2} = e^{2x}$ $(e^{-x})^2$ ↓ $e^{x} \times e^{x}$ e^{x^2}

$\frac{(e^{x})^2 + 2 \times e^{x} \times e^{-x} + (e^{-x})^2}{4} - \left[\frac{(e^{x})^2 + 2 \times e^{x} \times e^{-x} - (e^{-x})^2}{4}\right]$

$= \frac{e^{2x} + 2e^{0} + e^{-2x}}{4} - \left[\frac{e^{2x} + e^{-2x} - 2e^{x}e^{-x}}{4}\right]$

$(\cosh x)$ $(\sinh x)$ $e^{0} = 1$

$= \frac{\cancel{e^{2x}} + 2e^{0} + \cancel{e^{-2x}} - \cancel{e^{2x}} - \cancel{e^{-2x}} + 2e^{x}e^{-x}}{4}$

$e^{0} = 1$

$= \frac{2 \times 1 + 2 \times 1}{4} = \frac{4}{4} = 1$

2) $\boxed{\tanh^2 x + \operatorname{sech}^2 x = 1}$

3) $\boxed{\coth^2 x - \operatorname{cosech}^2 x = 1}$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\sinh 2x = 2 \sinh x \cosh x$.

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$$\boxed{\begin{aligned} \cosh\frac{x}{2} &= \sqrt{\frac{\cosh x + 1}{2}} \\ \sinh\frac{x}{2} &= \pm\sqrt{\frac{\cosh x - 1}{2}} \end{aligned}}$$

Rmrk →

$\cosh^2 x - \sinh^2 x = 1$ → the point with coordinates, $x = \cosh x$ & $y = \sinh x$ lies on unit hyperbola.

Q

1) Given $\sinh x = -3/4$. find the other 5 H.F.

A) $\sinh x = \frac{-3}{4}$.

$\operatorname{cosech} x = \frac{1}{-3/4} = \frac{-4}{3}$

$\cosh x = ?$.

$\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - \left(\frac{-3}{4}\right)^2 = 1$

$\cosh^2 x = \frac{1}{1} + \frac{9}{16}$

$\cosh^2 x = \frac{25}{16}$ $\therefore \cosh x = \sqrt{\frac{25}{16}} = \frac{5}{4}$

$\tanh x = \frac{\sinh x}{\cosh x}$

$= \frac{-3/4}{5/4} = \frac{-3}{5}$

$\coth x = \frac{1}{\tanh x}$.

$= \frac{-5}{3}$

$\operatorname{sech} x = \frac{1}{\cosh x}$

$= \frac{4}{5}$

2) S.T $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$?.

A) LHS $\cosh x + \sinh x = e^x$.

$\therefore (\cosh x + \sinh x)^n = (e^x)^n = e^{nx}$ → ①

RHS $\cosh nx = ?$.

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\therefore \cosh nx = \frac{e^{nx} + e^{-nx}}{2}$

$\sinh nx = ?$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\therefore \sinh nx = \frac{e^{nx} - e^{-nx}}{2}$

$= \frac{e^{nx} + \cancel{e^{-nx}} + e^{nx} - \cancel{e^{-nx}}}{2} = \frac{\cancel{2} e^{nx}}{\cancel{2}}$

$(e^{nx} + e^{nx} = 2e^{nx})$

$$= e^{nx} \quad \text{—} ②$$

frm ① & ②.

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$$

→ Derivatives of H. C) :-

we know $\frac{d}{dx}(e^x) = e^x$.

$$\frac{d}{dx}(e^{-x}) = -e^{-x}$$

1) $\therefore \frac{d}{dx}(\sinh x) = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x - (-e^{-x})}{2}$

$$= \frac{e^x + e^{-x}}{2} = \cosh x.$$

2) $\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{e^x + (-e^{-x})}{2}$

$$= \frac{e^x - e^{-x}}{2} = \sinh x$$

3) $\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$. (Q-rule)

$$= \frac{\cosh x \cdot \frac{d}{dx}(\sinh x) - \sinh x \frac{d}{dx}(\cosh x)}{\cosh^2 x}$$

$$= \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x}$$

$$= \frac{\overset{(1)}{\cosh^2 x - \sinh^2 x}}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$= \left(\frac{1}{\cosh x}\right)^2 = \operatorname{sech}^2 x$$

4) $\frac{d}{dx}(\coth x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right)$

$$= \frac{\sinh x \cdot \frac{d}{dx}(\cosh x) - \cosh x \cdot \frac{d}{dx}(\sinh x)}{\sinh^2 x}$$

$$= \frac{\sinh x \cdot \sinh x - \cosh x \cdot \cosh x}{\sinh^2 x}$$

$$= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x}$$

$$= -\frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x}$$

$$= -\frac{1}{\sinh^2 x} = -\operatorname{cosech} x$$

5) $\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) \Rightarrow \frac{d}{dx}(\cosh x)^{-1} = -(\cosh x)^{-2}$

$$= -(\cosh x)^{-2} \cdot \frac{d}{dx}(\cosh x) = -\frac{\sinh x}{\cosh^2 x}.$$

$$= -\frac{\sinh x}{\cosh x} \cdot \frac{1}{\cosh x} = -\operatorname{sech} x \tanh x$$

6) $\frac{d}{dx}(\operatorname{cosech} x) = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) \Leftarrow = \sinh$

$= -(\sinh x)^{-2} \cdot \frac{d}{dx} \sinh x = \frac{-\cosh x}{\sinh^2 x}$

$= \frac{-\cosh x}{\sinh x} \cdot \frac{1}{\sinh x} = -\coth x \operatorname{cosech} x$

Remark →

From deri- formulas of H.( ) derived above. we get the following antidifferentiation formulas ($\int$ formulas) for H.( ) →.

$\int \sinh x = \cosh x + c$

1) $\int \sinh x \; dx = \cosh x + c$ $\int \cosh x = \sinh x + c$

2) $\int \cosh x \; dx = \sinh x + c$

3) $\int \operatorname{sech}^2 x \; dx = \tanh x + c$

4) $\int \operatorname{cosech}^2 x \; dx = -\coth x + c$

5) $\int \operatorname{sech} x \tanh x \; dx = -\operatorname{sech} x + c$

6) $\int \operatorname{cosech} x \cdot \coth x \; dx = -\operatorname{cosech} x + c.$

→ Derivaties :-

1) $\frac{d}{dx}(\sinh x) = \cosh x$

2) $d/dx (\cosh x) = \sinh x$

$\frac{d}{dx}(\sinh x) = \cosh x$
$\sinh x = \int \cosh x$
$\sinh x = \sinh x$

3) $d/dx (\tanh x) = \operatorname{sech}^2 x$

4) $d/dx (\coth x) = -\operatorname{cosech}^2 x.$

5) $d/dx (\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x.$

6) $d/dx (\operatorname{cosech} x) = -\coth x \cdot \operatorname{cosech} x.$

Q/

1) diff $x \sinh x - \cosh x$ (P.S.)

A) Let $y = x \sinh x - \cosh x$.

$\frac{d}{dx}(x \cdot \sinh x)$

$\rightarrow \frac{dy}{dx} = x \cdot \cosh x + \sinh x \times 1 - \sinh x$ $\frac{d}{dx}(\cosh)$

$= x \cosh x.$

2) $\tanh\sqrt{1+x^2} = y$ (explicit c). $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

A) $\frac{dy}{dx} = \frac{d}{dx}\left[\tanh\sqrt{1+x^2}\right]$

$\frac{d}{dx}(\sqrt{1+x^2}) \times \frac{d}{dx}(1+x^2)$

$= \operatorname{sech}^2\left(\sqrt{1+x^2}\right) \times \frac{1}{2\sqrt{1+x^2}} \times 2x$

$= \frac{x}{\sqrt{1+x^2}} \times \operatorname{sech}^2\left(\sqrt{1+x^2}\right)$

3) $\frac{d}{dx} \ln \left| \tanh \frac{x}{2} \right|$

A) $\frac{1}{\tanh(x/2)} \times \operatorname{sech}^2(x/2) \times \frac{1}{2}$

$= \frac{\cosh(x/2)}{\sinh(x/2)} \times \frac{1}{(\cosh(x/2))^2} \times \frac{1}{2}$

$= \frac{1}{2\sinh(x/2)\cdot\cosh(x/2)} = \frac{1}{\sinh 2(x/2)}$

$= \frac{1}{\sinh x} = \operatorname{cosech} x$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\tan x = \frac{\sin x}{\cos x}$ $\therefore \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ $\sec = \frac{1}{\cos}$ $(2\sin x \cdot \cos x = \sin 2x)$

4) compute $\frac{dy}{dx}$ of $\frac{\sinh(x+y)}{xy} = 1$.

A) $\sinh(x+y) = xy$

diff. both side, x, $\frac{d}{dx}(x\cdot y) \rightarrow$ p. rule

$\cosh(x+y) \times \left[1 + \frac{dy}{dx}\right] = x\frac{dy}{dx} + y \times 1.$

$\Rightarrow \cosh(x+y) + \cosh(x+y)\frac{dy}{dx} = x\frac{dy}{dx} + y$

$\Rightarrow \cosh(x+y)\frac{dy}{dx} - x\frac{dy}{dx}$

$= y - \cosh(x+y)$

$\Rightarrow \frac{dy}{dx}\left[\cosh(x+y) - x\right] = y - \cosh(x+y)$

$\therefore \frac{dy}{dx} = \frac{y - \cosh(x+y)}{\cosh(x+y) - x}$

5) Evaluate the $\int$ (a) $\int \tanh 5x \, dx$.

A) $\int \tanh 5x \, dx = \int \frac{\sinh 5x}{\cosh 5x} dx.$

(Sub method)

put $u = \cosh 5x$

$\frac{du}{dx} = \sinh 5x \times 5$

$du = \sinh 5x \cdot 5 \, dx$

$\therefore \int \tanh 5x \, dx = \int \frac{du}{u \times 5} = \frac{1}{5}\int \frac{1}{u} du$

$= \frac{1}{5} \ln |u| + C.$

$= \frac{1}{5} \ln |\cosh 5x| + C.$

b) $\int \frac{\sinh x}{1 + \cosh^2 x} dx.$

A) put $u = \cosh x$ ①

$du = \sinh x \, . dx.$

$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

$\therefore \int \frac{du}{1+u^2} = \tan^{-1} u + C$

$= \tan^{-1}[\cosh x] + C.$

c) $\int_0^{\ln 2} 4e^x \sinh x \, dx = \int_0^{\ln 2} 4 \times e^x \times \left[\frac{e^x - e^{-x}}{2}\right] dx$

$= \int_0^{\ln 2} 2\left[e^{2x} - e^0\right] dx.$

$= 2\int_0^{\ln 2} \left[e^{2x} - 1\right] dx.$

$e^x \cdot e^x = e^{2x}$ $e^x \cdot e^{-x} = e^0 = 1$

$$= 2\left[\frac{e^{2x}}{2} - x\right]_0^{\ln 2}$$

$$= \left[\frac{2e^{2x}}{2} - 2x\right]_0^{\ln 2},$$

$$= \left(e^{2\ln 2} - 2\ln 2\right) - \left(e^{0} - 2\times 0\right).$$

$$= e^{2\ln 2} - 2\ln 2 - 1$$

$$= e^{\ln 2^2} - 2\ln 2 - 1$$

$$= e^{\ln 4} - 2\ln 2 - 1$$

$$= 4 - 2\ln 2 - 1 = 3 - 2\ln 2$$

$$= 3 - \ln 4$$

$$\boxed{\begin{array}{l}\text{pro of } \ln \\ \ln a^b = b\ln a \\ e^{\ln a} = a \\ \ln e^{a} = a\end{array}}$$

6) S.T $y = a\cosh\left(\frac{x}{a}\right)$: satisfies the differential eq,

$$\frac{d^2y}{dx^2} = \frac{W}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Provided $a = \frac{H}{W}$, where H & W are constnt.

A) Given $y = a\cosh\left(\frac{x}{a}\right)$

$$\frac{dy}{dx} = a \times \sinh\left(\frac{x}{a}\right)\times\frac{1}{a}$$

$$= \sinh\frac{x}{a} \quad \text{—(1)}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\sinh\frac{x}{a}\right)^2 \Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\sinh\frac{x}{a}\right)^2}$$

$$\frac{d^2y}{dx^2} = \cosh\frac{x}{a}\times\frac{1}{a} = \frac{1}{a}\cosh\frac{x}{a}$$

$$= \frac{1}{a}\times\sqrt{1+\sinh^2\left(\frac{x}{a}\right)}$$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh^2\left(\frac{x}{a}\right) - \sinh^2\left(\frac{x}{a}\right) = 1$

$\cosh^2\left(\frac{x}{a}\right) = 1 + \sinh^2\left(\frac{x}{a}\right)$

$\cosh\left(\frac{x}{a}\right) = \sqrt{1+\sinh^2\left(\frac{x}{a}\right)}$

$$= \frac{1}{a}\times\sqrt{1+\left(\sinh\frac{x}{a}\right)^2}$$

$$= \frac{1}{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{(from 1.)}$$

$$= \frac{1}{H/W}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad a = \frac{H}{W}.$$

$\left(a = \frac{H}{W}\right)$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{W}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\boxed{\begin{array}{l} y'' = \frac{1}{a}\sqrt{1+(y')^2} \\ \quad \hookrightarrow y = a\cosh\left(\frac{x}{a}\right)\end{array}}$$

Remrk →

Consider the differentional eq $\frac{d^2x}{dt^2} - a^2x = 0$. Hence the soln is given by, $x = x_0\cosh(at) + \frac{V_0}{a}\cdot\sinh(at)$ where $x = x_0$ & $V_0 = \frac{dx}{dt}$ when $t = 0$.

→ Inverse ( ) :

8. The inverse ( ) to a ( ) f is a ( ) g for which $g(y) = x$ when $y = f(x)$ & vice verse.

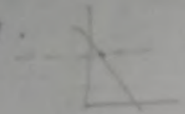
The inverse ( ) of f is denoted by f^{-1}.

eg = let $f(x) = 5x - 7$.

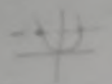
(i.e) $\boxed{y = 5x - 7}$

$y + 7 = 5x$

$x = \frac{y+7}{5}$



→ touching 1 point



→ Touching 2 points

(i.e) $g(y) = \boxed{x = \frac{y+7}{5}}$ is (I) ( ).

To find the formula, for f^{-1}, try to solve the eq $y = f(x)$. for x in terms of y. If the sol^n is unique, let $f(y) = x$. It may be necessary to restrict the domain of f bfre there is an I. ( ).

The graph of f^{-1} is obtained frm that of f by flipping the figine to interchange the horizontal & vertical axes.

eg ① graph → $f(x) = -2x + 4$.

f is well defined objective ( ) on the set of all real no.

for any real no, $y = -2x + 4$

$y - 4 = -2x$

$\frac{-(y-4)}{2} = x$

$x = \frac{-y}{2} + 2$

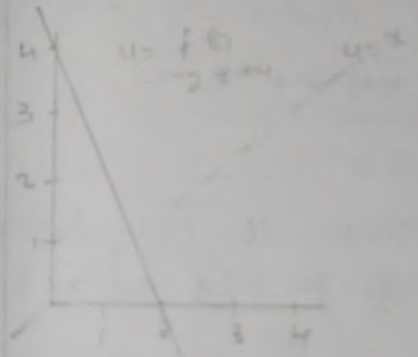
$x = -\frac{1}{2}y + 2$

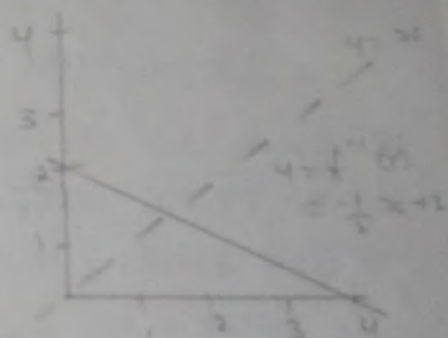
Consider $g(y) = -\frac{1}{2}y + 2$.

∴ if $y = f(x)$, iff i.e only if $x = g(y)$

hence $g = f^{-1}$

(i.e) $f^{-1}(x) = g(x) = \frac{-1}{2}x + 2$





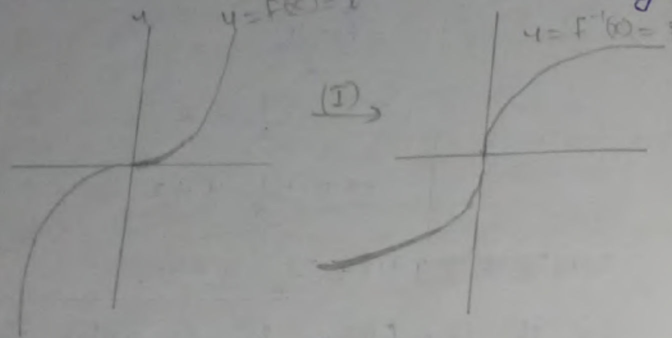
2) let $f(x) = x^3$, f is well defined bijective ( ) ∴ for any real no, $y = x^3$

$x = y^{1/3} = \sqrt[3]{y}$

hence, $g(y) = x$.

$\Rightarrow g = f^{-1}(x)$

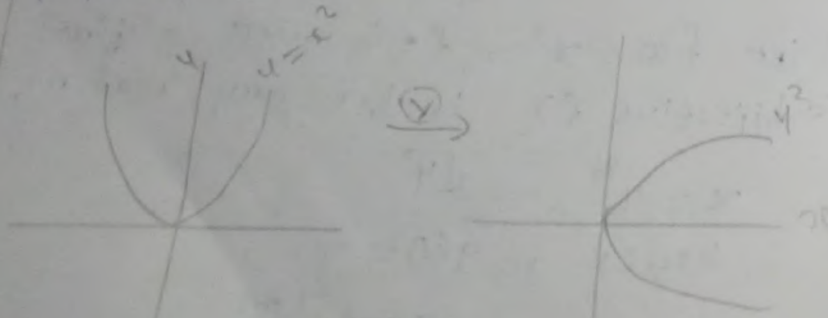
$\therefore$ we can written as $f^{-1}(x) = g(x) = \sqrt[3]{x}$



3) Let $y = f(x) = x^2$
$\Rightarrow x = \pm\sqrt{y}$.

· If $y<0$, the □ root is not defined
· if $y>0$, the choice of +ve / −ve give 2 diff. values of x.

Rmrk →
A ( ) is invertible if each horizontal line meets the graph in at most 1 point.



I. ( ) test :-

Suppose that f is contin. on [a, b] & that f is ↑ing at each point of (a, b), $f'(x) > 0$, for each x in (a, b), then f is invertible on [a, b] & the inverse f^{-1} is defined on the interval $[f(a), f(b)]$. If f is ↓ing rather than ↑ing at each point of (a, b), $f'(x) < 0$, for each x in [a, b], then f is still invertible on [a, b], & the domain of f^{-1} is $[f(b), f(a)]$.

↑ → $f'(x) > 0$ → $[f(a), f(b)]$
↓ → $f'(x) < 0$ → $[f(b), f(a)]$

Q/
1) S.T $f(x) = -x^3 - 2x + 1$ is invertible on [−1, 2]. wht is the domain of (I)?

A) ($f(x) = -x^3 - 2x + 1$ being a polynomial so it is contin. & differentiable every where)

[−1, 1, 2]

$[-1, 2]$ $\quad f'(x) = -3x^2 - 2 = -(3x^2 + 2) = -(3\times1+2) = -(3+2) = -(+) = -ve$

since $3x^2+2$ is +ve.

$\Rightarrow -(3x^2+2) < 0$.

$\Rightarrow f'(x) < 0$ on [−1, 2] $\therefore$ it is ↓ing then by Inverse ( ) test, f is invertible on [−1, 2] & the domain of f^{-1} is $[f(2), f(-1)]$

$f(2) = -11$

$= [-8-4+2, 1+2+1]$

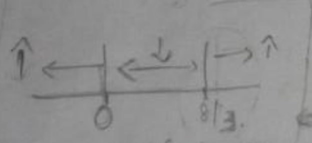
$= [-11, 4]$ $[f(b), f(a)]$

2) let $f(x) = x^3 - 4x^2 + 1$, find an intrval containing 1 on which f is invertible. Denote the (I) by g & find $g(-7)$.

A) $f(x) = x^3 - 4x^2 + 1$ being a polym f is contin & diff everywhere.

$f'(x) = 3x^2 - 8x = x(3x - 8)$

if $f'(x) > 0 \Rightarrow x(3x-8) > 0$



$x > 0$ & $(3x - 8) > 0$

$x > 0$ & $x > 8/3$

$\Rightarrow x > 8/3$

$(-\infty, 0)$ $(0, 8/3)$ $(8/3, \infty)$

$\therefore f'(x) > 0 \Rightarrow x > 8/3$

containing 1

$f'(x) < 0 \Rightarrow x < 0$ or $3x - 8 < 0$

$\Rightarrow x < 0$ or $3x < 8$

$\Rightarrow x < 0$ or $x < 8/3$

$\hookrightarrow x < 0$

$\rightarrow f'(x) < 0 \Rightarrow 0 < x < 8/3$.

Hence the intrval containing 1 is $0 < x < 8/3$ (i.e) $[0, 8/3]$ & f is bsing

Hence f is invertible on $[0, 8/3]$

Let g denote the (I) of f

(i.e) $f^{-1} = g$.

$\therefore$ domain is $[f(8/3), f(0)] = [-229/27, 1]$

clearly, $-7 \in [-229/27, 1]$.

Now we hve to find $g(-7) = f^{-1}(-7)$

Let $f^{-1}(-7) = x$.

$f(x) = x^3 - 4x^2 + 1$

$x \in [0, 8/3]$

$-7 = x^3 - 4x^2 + 1$

$-7 - 1 = x^3 - 4x^2$

$-8 = x^2(x - 4)$

$-8 = 2^2(2 - 4)$

$\Rightarrow x = 2$

$f(2) = -7$

$f^{-1}(-7) = 2$

$g(-7) = 2$

H/w

3) find (I) of each of prbm.

1) $f(x) = 2x + 5$ on $[4, 4]$

$\Rightarrow$ Derivatives of I.( ) :-

* (I) ( ) rule = To diff the I.( ) $g = f^{-1}$ at y, take reciprocal of deriv of given ( ) at $x = f^{-1}(y) = g(y)$.

(i.e) $\boxed{g'(y) = \frac{1}{f'(g(y))}}$

Q. use I-() rule to compute the deri. of $\sqrt{x}$, evaluate the deri. at $x=2$.

→ let $f(x) = x^2$ since f is poly.

$f'(x) = 2x > 0$ when $x > 0$ (incr)

< 0 when $x < 0$ (decr)

(i.e.) $f(x)$ is invertible.

⇒ f is invertible on every interval not containing zero.

let g denote the I() of f on interval $(0, \infty)$, $g = f^{-1}$,

$\Rightarrow g(y) = \sqrt{y}$ where $y = x^2$, $x \in (0, \infty)$

by I-() rule, we get,

$$g'(y) = \frac{1}{f'(g(y))} \quad \text{--- (1)}$$

$$= \frac{1}{2(g(y))} = \frac{1}{2\sqrt{y}}$$

$$\Rightarrow \frac{d}{dy}\sqrt{y} = \frac{1}{2\sqrt{y}}$$

Replacing the variable y by x,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

when $x = 2$,

$$\therefore \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{2}}$$

Remark

using I-() rule to trigonometric identities, we can prove the following diff. formulas →

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ $\quad -1 < x < 1$

$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ $\quad -1 < x < 1$

$d/dx(\tan^{-1}x) = \frac{1}{1+x^2}$ $\quad -\infty < x < \infty$

$d/dx(\cot^{-1}x) = -\frac{1}{1+x^2}$ $\quad -\infty < x < \infty$

$d/dx(\sec^{-1}x) = \frac{1}{\sqrt{x^2(x^2-1)}}$ $\quad -\infty < x < -1,\ 1 < x < \infty$

$d/dx(\operatorname{cosec}^{-1}x) = -\frac{1}{\sqrt{x^2(x^2-1)}}$ $\quad -\infty < x < -1,\ 1 < x < \infty$

$y = x^2$

$y = \sqrt{x}$

→ Inverse hyperbolic ( ) :-

$y = \sinh x$

$\sinh^{-1} y = x$.

For every value of x in the interval $-\infty < x < \infty$, the value of $\boxed{y = \sinh^{-1} x.}$ is the no.. whose H. Singh is x. (i.e) $\sinh y = x$ we call $\sinh^{-1} x$, the I. H. Singh of x.

$y = \sinh^{-1} x$ also written as

$y = \text{arc} \sinh x$.

$\sinh^{-1} = \text{arc sin}$

≈ $y = \cosh^{-1} x$ (or).

$y = \text{arc} \cosh x$. → called the I. H. cosin of x. for every value of x ~~in the interval~~ ~~for~~ $x \geq 1$.

The remaining I. H. ( ) are →

$y = \tanh^{-1} x$.

$y = \coth^{-1} x$

$y = \text{sech}^{-1} x$.

$y = \text{cosech}^{-1} x$.

→ Expression for I. H. ( ) in terms $\ln$:-

1) $\sinh^{-1} x$ in terms of $\ln$ →

$e \xrightarrow{(1)} \ln$.

let $y = \sinh^{-1} x$. (avoid (1))

$\therefore \sinh y = x$.

$\left(\sinh x = \frac{e^x - e^{-x}}{2}\right)$

$\frac{e^y - e^{-y}}{2} = x \rightarrow$

$e^y - e^{-y} = 2x$ or $e^y - e^{-y} - 2x = 0$

$\times$ through out by e^y,

$e^y \times e^y - e^{-y} \times e^y - 2x \times e^y = 0$

$(e^y)^2 - 1 - 2x \cdot e^y = 0$

$(e^y)^2 - 2x(e^y) - 1 = 0$

qua. eq $\rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$e^{-y} \cdot e^y$
$= e^{-y+y}$
$= e^0 = 1$

$e^y = \frac{2x \pm \sqrt{(2x)^2 - 4 \times 1 \times -1}}{2 \times 1}$

$= \frac{2x \pm \sqrt{4x^2 + 4}}{2} = \frac{2x \pm \sqrt{4(x^2+1)}}{2}$

$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.
$\sqrt{4 \cdot (x^2+1)} = \sqrt{4} \cdot \sqrt{x^2+1}$

$= \frac{2x \pm \sqrt{4}\sqrt{x^2+1}}{2}$

$= \frac{\cancel{2}x}{\cancel{2}} \pm \frac{\cancel{2} \cdot \sqrt{x^2+1}}{\cancel{2}} = x \pm \sqrt{x^2+1}$

$e^y = x \pm \sqrt{x^2+1}$

we need y, so remove e by $\ln$.

$\ln e^y = \ln\left(x \pm \sqrt{x^2+1}\right)$

$y = \ln\left(x + \sqrt{x^2+1}\right)$

2) $\cosh^{-1} x$:-

$y = \cosh^{-1} x$.

$\cosh y = x$.

$\frac{e^y + e^{-y}}{2} = x$.

$e^y + e^{-y} = 2x$

$e^y + e^{-y} - 2x = 0$.

$e^y - 2x + e^{-y} = 0$ $\times e^y$

$e^y \cdot e^y - 2x \cdot e^y + e^{-y} \cdot e^y = 0$

$(e^y)^2 - 2x e^y + 1 = 0$.

$e^y = \frac{-(-2x) \pm \sqrt{4x - 4 \times 1 \times 1}}{2}$

$= \frac{2x \pm \sqrt{4x - 4}}{2} = \frac{2x \pm \sqrt{4(x^2-1)}}{2}$

$= \frac{2x \pm \sqrt{4}\sqrt{x^2-1}}{2} = \frac{2x \pm \sqrt{2}\sqrt{x^2-1}}{2}$

$= \frac{\cancel{2}x}{\cancel{2}} \pm \frac{\cancel{2} \cdot \sqrt{x^2-1}}{\cancel{2}} = x \pm \sqrt{x^2-1}$

$e^y = x \pm \sqrt{x^2-1}$. [taking ln on both side]

$\ln e^y = \ln\left(x \pm \sqrt{x^2-1}\right)$

$\boxed{y = \ln\left(x \pm \sqrt{x^2-1}\right)}$

3) $\tanh^{-1} x$:-

$y = \tanh^{-1} x$.

$\tanh y = x$.

$\dfrac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = x$.

$e^{y} - e^{-y} = x(e^{y} + e^{-y})$

$\boxed{(e^{y})^2 = e^{2y}}$

× throughout e^{y}.

$e^{y}.e^{y} - e^{-y}.e^{y} = x(e^{y}.e^{y} + e^{-y}.e^{y})$

$(e^{y})^2 - 1 = x((e^{y})^2 + 1)$

$e^{2y} - 1 = x e^{2y} + x$.

$e^{2y} - xe^{2y} = x + 1$

$e^{2y}(1-x) = x+1$

$e^{2y} = \dfrac{x+1}{1-x} \Rightarrow \dfrac{1+x}{1-x}$.

taking ln on both side,

$\ln e^{2y} = \ln\left(\dfrac{1+x}{1-x}\right)$

$2y = \ln\left(\dfrac{1+x}{1-x}\right)$

$\boxed{y = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)}$

$\approx$

4) $\coth^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{x+1}{x-1}\right)$, $|x| > 1$

5) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$ $0 < x < 1$

6) $\operatorname{cosech}^{-1} x \begin{cases} = \ln\left(\dfrac{1+\sqrt{x^2+1}}{x}\right) & x > 0 \\ = \ln \dfrac{1-\sqrt{x^2+1}}{x} & x < 0 \end{cases}$

2 eq →

⇒ Derivatives I. H. C) :-

1) Deri... of $\sinh^{-1} x = \dfrac{dy}{dx}$?

$y = \sinh^{-1} x$

$\sinh y = x$ — ⓐ

diff — with respect to x on both side. → $\frac{d}{dx} \sinh y \cdot \frac{d}{dx}(y) = \frac{d}{dx}(1)$

$\cosh y \times \dfrac{dy}{dx} = 1$.

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cosh y}$ — ① (want to avoid y) → use identity

Since $\cosh^2 y - \sinh^2 y = 1$.

$\cosh^2 y = 1 + \sinh^2 y$

$\cosh y = \pm\sqrt{1 + \sinh^2 y}$ in — ⓐ we gt. $\cosh y > 0$.

$\cosh y = \sqrt{1 + x^2}$, $\sinh y = x$.

Since ~~sinh~~

Rule in — ①,

$\boxed{\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}}$

2) Deri... of $\cosh^{-1} x$:-

$y = \cosh^{-1} x$

$\cosh y = x$. —— (a)

diff with respect to x on both side

$\sinh y \times \frac{dy}{dx} = 1$.

$\therefore \frac{dy}{dx} = \frac{1}{\sinh y}$ —— (1) (avoid y)

$\cosh^2 y - \sinh^2 y = 1$.

$\sinh^2 y = \cosh^2 y - 1$.

$\sinh y = \pm\sqrt{\cosh^2 y - 1}$

by —(a)

$\sinh y = \pm\sqrt{x^2-1}$ $\quad \cosh y = x$.

Rule in —(1)

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{x^2-1}}$$

$\approx$ we can find deri – of other I. H. ( ).

Deri

* $\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$
* $d/dx \cosh^{-1} x = \pm\frac{1}{\sqrt{x^2-1}}$ $\quad |x| > 1$
* $\frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2}$ $\quad |x| < 1$.
* $\frac{d}{dx} \coth^{-1} x = \frac{1}{1-x^2}$, $|x| > 1$
* $d/dx \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}}$, $0 < x < 1$
* $\frac{d}{dx} \operatorname{cosech}^{-1} x = \frac{-1}{|x|\sqrt{1+x^2}}$ $\quad x \neq 0$

5

$\int$

* $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + c$
  $= \ln\left(x + \sqrt{x^2+1}\right) + c$.
* $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + c$
  $= \ln\left(x + \sqrt{x^2-1}\right) + c$, $|x| > 1$
* $\int \frac{dx}{1-x^2} = \tanh^{-1} x + c$ if $|x| < 1$ $\quad = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$)
  $\coth^{-1} x + c$ if $|x| > 1$
* $\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1} x + c = \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) + c$ $(0<x<1)$
* $\int \frac{dx}{x\sqrt{1+x^2}} = -\operatorname{cosech}^{-1} + c = \ln\left(\frac{1 \mp \sqrt{1+x^2}}{x}\right) + c$ $\quad x > 0$

Q 1) find $\sinh^{-1} 5$ numerically by using ln?

A) $\sinh^{-1} x = \ln\left(x \pm \sqrt{x^2+1}\right) = \ln\left(x + \sqrt{x^2+1}\right)$

$\sinh^{-1} 5 = \ln\left(5 + \sqrt{5^2+1}\right)$

$= \ln\left(5 + \sqrt{26}\right) \approx 2.312$

2) P.T the (1) H.() satisfies follow identities —

1) $\operatorname{sech}^{-1} x = \cosh^{-1}\left(\frac{1}{x}\right)$

2) $\operatorname{cosech}^{-1} x = \sinh^{-1}(1/x)$

3) $\coth^{-1} x = \tanh^{-1}(1/x)$

1) Let $y = \operatorname{sech}^{-1} x$ — ①

$\operatorname{sech} y = x$

we get x, we need $\frac{1}{x}$

$\frac{1}{\operatorname{sech} y} = \frac{1}{x} \Rightarrow \cosh y = \frac{1}{x}$

$\therefore y = \cosh^{-1}(1/x)$ — ②

① = ②

2) $y = \operatorname{cosech}^{-1} x$. — ①

$\operatorname{cosech} y = x$

$\frac{1}{\operatorname{cosech}} = \frac{1}{x} \Rightarrow \sinh y = \frac{1}{x}$

$\Rightarrow y = \sinh^{-1}(1/x)$ — ②

① = ②

3) $y = \coth^{-1} x$.

$\coth y = x$.

$\frac{1}{\coth y} = \frac{1}{x} \Rightarrow \tanh y = \frac{1}{x}$,

$y = \tanh^{-1}(1/x)$

3) Diff — (a) $\sinh^{-1}(3x+\cos x)$

1) ~~want to diff 3x+cos x.~~

$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$

Chain rule

$\therefore \frac{d}{dx}\sinh^{-1}(3x+\cos x) = \frac{1}{\sqrt{1+(3x+\cos x)^2}} \times \frac{d}{dx}(3x+\cos x)$

$\Rightarrow \frac{1}{\sqrt{1+(3x+\cos x)^2}} \cdot [3-\sin x]$ ans

(b) $\cosh^{-1}\sqrt{x^2+1}$

$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$

$\therefore \frac{d}{dx}\cosh^{-1}\left(\sqrt{x^2+1}\right) = \frac{1}{\sqrt{(\sqrt{x^2+1})^2+1}} \times \frac{d}{dx}\left(\sqrt{x^2+1}\right)$

$= \frac{1}{\sqrt{x^2+1+1}} \times \frac{1}{2\sqrt{x^2+1}} \times \frac{d}{dx}(x^2+1)$

$\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\therefore \sqrt{x^2+1} = \frac{1}{2\sqrt{x^2+1}}$

$= \frac{1}{\sqrt{x^2+2}} \times \frac{1}{\cancel{2}\sqrt{x^2+1}} \times \cancel{2}x$

$= \frac{x}{\sqrt{x^2+2}\,\sqrt{x^2+1}}$

Chain rule

$\frac{d}{dx} \to \times$

$\int \to \div$

4) Evaluate (1) $\int \frac{dx}{\sqrt{4x^2-1}}$

we have $\int \to \int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x$

$4x^2 = (2x)^2$

$\frac{1}{2}$ in x

$(2x)^2$.

$\therefore \int \frac{dx}{\sqrt{(2x)^2-1}} = \frac{1}{2}\cosh^{-1}(2x) + C$

5) $\int \frac{dx}{\sqrt{9+4x^2}}$

$= \int \frac{dx}{\sqrt{9(1+\frac{4}{9}x^2)}}$

$(9+4)$
$9(1+\frac{4}{9})$

$\sqrt{ab} = \sqrt{a}\cdot\sqrt{b}$.
$9\cdot(1+\frac{4}{9})$ $\sqrt{9}\cdot\sqrt{1+\frac{4}{9}}$

eg
$\frac{d}{dx}\cos 2x$
$= \sin 2x \times 2$
$\int \cos 2x$
$= \frac{\sin 2x}{2}$

$= \int \frac{dx}{\sqrt{9}\sqrt{1+\frac{4}{9}x^2}} = \int \frac{dx}{3\sqrt{1+\frac{4}{9}x^2}}$

$= \frac{1}{3}\int \frac{dx}{\sqrt{1+\frac{4}{9}x^2}}$

$\frac{4}{9}x^2 \Rightarrow \frac{2}{3}x^2$

$= \frac{1}{3}\int \frac{dx}{\sqrt{1+(\frac{2}{3}x)^2}}$ —①

This method is Substitution method

(x-ന്റെ ... substitution method ...)

put $u = \frac{2}{3}x$.

$\frac{d}{dx}$ on both side for dx

$\frac{d}{dx}(u) = \frac{d}{dx}(\frac{2}{3}x)$

$\frac{du}{dx} = \frac{2}{3}$ $dx = ?$.

$du = \frac{2}{3}dx$.

$\boxed{\frac{3}{2}du = dx}$

Sub in ①

$= \frac{1}{3}\int \frac{dx}{\sqrt{1+(\frac{2}{3}x)^2}} = \frac{1}{3}\int \frac{3/2\, du}{\sqrt{1+u^2}}$

$= \frac{1}{\cancel{3}}\times\frac{\cancel{3}}{2}\int \frac{du}{\sqrt{1+u^2}}$

$= \frac{1}{2}\times \sinh^{-1} u$ (in terms of x) $\int\frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u$.

$\therefore \int \frac{dx}{\sqrt{9+4x^2}} = \frac{1}{2}\sinh^{-1}(\frac{2}{3}x) + c$

⇒ Lengths of a plane curves :-

→ A ( ) is said to be Smooth, if it has a contin- 1st derivative.

★ Def ⇒ Length :- (L)

Suppose that the ( ) f is contin- on [a, b] & deri.... f' exist & is contin (except possibily at finite many points) on [a,b]. Then the length of graph of f on [a,b] is given by,

If (a) is to find y → $\boxed{L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$ → on x axis

(or)

$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$.

Rmrk

If the eq of curve given by,

$x = g(y)$. $c \le y \le d$, then

If (a) is to find x → $\boxed{L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$ → on y axis

$= \int_c^d \sqrt{1+(f'(y))^2}\,dy$

Q/ 1) find L of graph of (?),

$f(x) = \frac{x^4}{8} + \frac{1}{4x^2}$ on $[1, 3]$

about x axis

A) $f(x) = \frac{x^4}{8} + \frac{1}{4x^2}$

$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$ — (1)

$\cancel{f'(x) = \frac{1}{8}\left(\frac{x^5}{5}\right) +}$

$f'(x) = \frac{1}{8} \cdot 4x^3 + \frac{1}{4} \times -2x^{-3}$

$\frac{d}{dx}\left(\frac{1}{x^2}\right) = x^{-2} = -2x^{-2-1} = -2x^{-3}$

$f'(x) = \frac{1}{2}x^3 - \frac{1}{2x^3}$

$(f'(x))^2 = \left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2$

$(a-b)^2$

$= \left(\frac{x^3}{2}\right)^2 - 2\left(\frac{x^3}{2}\right)\left(\frac{1}{2x^3}\right) + \left(\frac{1}{2x^3}\right)^2$

$(f'(x))^2 = \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}$

$1 + (f'(x))^2 = 1 + \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}$ $\quad 1 - \frac{1}{2} = \frac{1}{2}$

(for giving $\int$)

$= \left(\frac{x^3}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2x^3}\right)^2$ → $(a+b)^2$ and expansion

$= \left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2$ → $(a+b)^2$ ... $\int$ ...

$\sqrt{1 + (f'(x))^2} = \sqrt{\left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2} = \frac{x^3}{2} + \frac{1}{2x^3}$

Now — (1)

$L = \int_1^3 \frac{x^3}{2} + \frac{1}{2x^3}\, dx = \int_1^3 \frac{1}{2}\left(\frac{x^4}{4}\right) + \frac{1}{2}x^{-3}\, dx$

$= \left[\frac{1}{2}\left(\frac{x^4}{4}\right) + \frac{1}{2} \times \frac{x^{-2}}{-2}\right]_1^3$

$\frac{1}{2} \times \frac{x^{-2}}{-2} = -\frac{1}{2} \times \frac{1}{2x^2} = -\frac{1}{4x^2}$

$= \left[\frac{x^4}{8} - \frac{1}{4x^2}\right]_1^3$

$= \left(\frac{3^4}{8} - \frac{1}{4 \times 3^2}\right) - \left(\frac{1}{8} - \frac{1}{4}\right)$

$= \frac{81}{8} - \frac{1}{36} - \left(\frac{1}{8} - \frac{1 \times 2}{4 \times 2}\right)$

$= \frac{2916 - 8}{288} - \left(\frac{1 - 2}{8}\right)$

$= \frac{2908}{288} + \frac{1 \times 36}{8 \times 36} = \frac{2908 + 36}{288}$

$= \frac{2944}{288} = \frac{92}{9}$

→ Area of surface of Revolution :-

* Result :-

Suppose that the (?) f is contin on $[a, b]$ & that the deri- f' exist & contin on $[a, b]$,

Then the area of surface generated by revolving the graph of $f(x)$ (≥ 0) on $[a, b]$.

about x-axis,

$$\boxed{S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.}$$

$$S = 2\pi\int_a^b y\sqrt{1+(f'(x))^2}\,dx.$$

* Result 2 :-

The area of surface obtained by revolving the graph of $f(x) \geq 0$ about y axis,

$$S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dx.$$

(about y axis) →

$$\boxed{S = 2\pi\int_a^b x\sqrt{1+(f'(x))^2}\,dx.}$$

Q

1) find the area of surface obtained by revolving the graph $y = x^2$ about y axis for $1 \leq x \leq 2$

A) $S = 2\pi\int_a^b x\sqrt{1+(f'(x))^2}\,dx.$ — (1)

$f(x) = x^2$

$f'(x) = 2x.$

$(2x)^2 = 4x^2$

$1+(f'(x))^2 = 1+4x^2$

∴ Sub in (1) we get,

$$S = 2\pi\int_1^2 x\sqrt{1+4x^2}\,dx.$$

(by sub method, [illegible])

put $u = 1+4x^2$

$\frac{d}{dx}(u) = \frac{d}{dx}(1+4x^2)$

$\frac{du}{dx} = 8x$ — we need $x\,dx$.

$du = 8x\,dx.$

$\frac{du}{8} = x\,dx.$

($1 \leq x \leq 2$ → interms x, but we need in u)

when $x = 1$ $u = 1+4x^2 \Rightarrow 1+4\times1 = 5$

$x = 2$ $u = 1+4\times2^2 \Rightarrow 1+16 = 17$

$x\,dx = \frac{du}{8}$

$$S = 2\pi\int_5^{17}\sqrt{u}\,\frac{du}{8}$$

$$= \frac{\cancel{2}\pi}{\cancel{8}_4}\int_5^{17}(u)^{1/2}\,du = \frac{\pi}{4}\int_5^{17}\frac{u^{3/2}}{3/2}$$

$\sqrt{u} = u^{1/2}$

$\int u^{\frac{1}{2}+1}\big/\left(\tfrac{1}{2}+1\right) = u^{\frac{1+2}{2}}\big/\tfrac{1+2}{2} = \frac{u^{3/2}}{3/2}$

$$= \frac{\pi}{4}\left[\frac{u^{3/2}}{3/2}\right]_5^{17}$$

$$\cancel{\Rightarrow \frac{\pi}{4}\left[\frac{17^{3/2}}{3/2} - \left(\frac{5^{3/2}}{3/2}\right)\right]}$$

$$= \frac{\pi}{\cancel{4}_2}\times\frac{\cancel{2}}{3}\left[u^{3/2}\right]_5^{17}$$

$$= \frac{\pi}{2}\times\frac{1}{3}\left[u^{3/2}\right]_5^{17} = \frac{\pi}{6}\left[u^{3/2}\right]_5^{17}$$

$= \frac{\pi}{6}\left[17^{3/2} - 5^{3/2}\right]$

$= \frac{\pi}{6}\left[\cancel{245.65 - 62.5}\right]$

$= \frac{\pi}{6}\left[\cancel{2384}\right] = \cancel{399\pi}$

$= \frac{\pi}{6}\left[70.09 - 11.18\right]$

$= \frac{\pi}{6}\left[58.909\right] = \pi \cdot 9.818 \rightarrow 3.14 \times 9.818$

$= \underline{30.82}$

2) find length of arc of semi cubical parabola $y^2 = x^3$ extending from origin to $(1,1)$.

A) want to find curve of $y^2 = x^3$ from $(0,0)$ to $(1,1)$,

diff with respect to x on both sides,

$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3)$

$2y\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{2y}$

$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{3x^2}{2y}\right)^2$

$= 1 + \frac{9x^4}{4y^2}$ ($\Rightarrow y^2 = x^3$)

(Integration ചെയ്യുമ്പോൾ y വരരുത്. with respect to x)

$= 1 + \frac{9x^4}{4x^3}$

$= 1 + \frac{9x}{4} = \frac{4 + 9x}{4}$

$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \int_0^1 \sqrt{\frac{4 + 9x}{4}}\, dx$ $\quad \sqrt{4} = 2$

$= \frac{1}{2}\int_0^1 \sqrt{4 + 9x}\, dx$ — ①

using sub- method, $4 + 9x = u$

$u = 4 + 9x$

$dx?\quad \frac{du}{dx} = \frac{d}{dx}(4 + 9x) = 9$

$\frac{du}{dx} = 9$

$\frac{du}{9} = dx$

Limit?

$u = 4 + 9x$

$x = 0$, $u = 4$

$x = 1$, $= 4 + 9 = 13$ $\Rightarrow \int_4^{13}$

eq ① becomes,

$L = \frac{1}{2}\int_4^{13} \sqrt{u} \cdot \frac{du}{9}$ $\quad \frac{1}{2} \times \frac{1}{9} = \frac{1}{18}$

$= \frac{1}{18}\int_4^{13} \sqrt{u} \cdot du$ $\quad \sqrt{u} = u^{1/2}$

$= \frac{1}{18}\int_4^{13} u^{1/2}\, du$.

$$L = \frac{1}{18}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_4^{13} = \frac{1}{18}\left[\frac{u^{3/2}}{3/2}\right]_4^{13}$$

$$= \frac{1}{\cancel{18}_{9}} \times \frac{\cancel{2}}{3}\left[u^{3/2}\right]_4^{13}$$

$$= \frac{1}{27}\left[u^{3/2}\right]_4^{13}$$

$$= \frac{1}{27}\left[13^{3/2} - 4^{3/2}\right]$$

$$L = \frac{1}{27}\left[13\sqrt{13} - 8\right]$$

$4^{3/2} = 8$

$13^{3/2} = (13^{1/2})^3 = (\sqrt{13})^3$
$\sqrt{13} \times \sqrt{13} \times \sqrt{13}$
$= 13\sqrt{13}$

$4^{3/2} = (4^{1/2})^3 = (\sqrt{4})^3$
$= (2)^3 = 8$

3) find area of surface generated by revolving about x axis, the arc of parabola $y^2 = 4ax$ from origin to the point where $x = a$, $a > 0$?

A)

$$S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{—} (1)$$

$y^2 = 4ax$.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$$

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{\cancel{4}^{2}a}{\cancel{2}y}$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{2a}{y}\right)^2$$

$$= 1 + \frac{4a^2}{y^2} = 1 + \frac{\cancel{4}a^{\cancel{2}}}{\cancel{4}ax}$$

$(y^2 = 4ax)$

$$= 1 + \frac{a}{x}$$

$y^2 = 4ax$
$y = \sqrt{4ax}$
$y = 2\sqrt{ax}$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{x+a}{x}$$

in eq (1)

$$S = 2\pi\int_0^a 2\sqrt{ax}\sqrt{\frac{x+a}{x}}\, dx$$

(split $\sqrt{\ }$)

$$= 4\pi\int_0^a \sqrt{a\cancel{x}} \times \frac{\sqrt{x+a}}{\sqrt{\cancel{x}}}$$

$$= 4\pi\int_0^a \sqrt{a}\sqrt{x+a}\, dx$$

$$= 4\pi\sqrt{a}\int_0^a \sqrt{x+a}\, dx$$

$$= 4\pi\sqrt{a}\int_0^a (x+a)^{1/2}\, dx$$

$$= 4\pi\sqrt{a}\left[\frac{(x+a)^{3/2}}{3/2}\right]_0^a$$

$$= 4\pi\sqrt{a}\left[\frac{2}{3}(x+a)^{3/2}\right]_0^a$$

$$= \frac{8\pi\sqrt{a}}{3}\left[(a+a)^{3/2} - (0+a)^{3/2}\right]$$

$$S = \frac{8\pi\sqrt{a}}{3}\left[(2a)^{3/2} - a^{3/2}\right]$$