Module = 03

Vector Spaces (V)

Let 'V' be a set of elements on which 2 operations → vector addition & scalar multiplication, are defined. Then V said to be a vector space if following 10 (pro) are satisfied,

→ **Axioms of vector addition :- (V)**

(vector) 1) If $\vec{x}$ & $\vec{y}$ are in V then $\vec{x}+\vec{y}$ is in V.

2) For all $\vec{x},\vec{y} \in V \Rightarrow \vec{x}+\vec{y} = \vec{y}+\vec{x}$ [commutative law]

3) For all $\vec{x},\vec{y},\vec{z} \in V \Rightarrow \vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$ ↳ (Associative law)

4) There is a unique (V), $\vec{0}$ in V such that $\vec{0}+\vec{x} = \vec{x} = \vec{x}+\vec{0}$

5) For each $\vec{x}$ in V there exist a (V), $-\vec{x}$ such that $\vec{x}+(-\vec{x}) = \vec{0} = (-\vec{x})+\vec{x}$

→ **Axioms of scalar multiplication :- (k)**

6) If k any scalar & $\vec{x} \in V$, then $k\vec{x} \in V$.

7) $k(\vec{x}+\vec{y}) = k\vec{x}+k\vec{y}$.

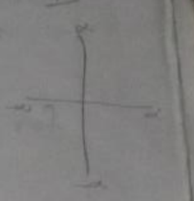
8) $(k_1+k_2)\vec{x} = k_1\vec{x}+k_2\vec{x}$

9) $k_1(k_2\vec{x}) = (k_1k_2)\vec{x}$ (10) $1\vec{x} = \vec{x}$

eg → real no → (-∞,∞) & R^2 → (x, y), (x_1, y_1) ... R^3, polynomials (P_n), set of matrices, continuous valued f(x).

vector addition (real no) ✓ R^n

$R^2 \to (x_1,y_1)\ (x_2,y_2)$



1^{st} (pro) → $(1,3)+(3,7) = (4,10)$

2^{nd} → $(x_1+y_1)(x_2+y_2) = (x_1+x_2, y_1+y_2)$
$= (x_2+x_1, y_2+y_1)$
$= (x_2,y_2)(x_1,y_1)$

3^{rd} → $[(x_1,y_1)+(x_2,y_2)]+(x_3,y_3) = (x_1,y_1)+[(x_2,y_2)+x_3+y_3]$

4^{th} → $\vec{0}$, (0,0)

5^{th} → Inverse. $(10,-8) \to (-10,8) = (0,0)$.
$(x_1,y_1)+(-x_1,-y_1) = (0,0)$

scalar multi

7^{th} → $k \in \mathbb{R}$
$k(x_1,y_1) = (kx_1, ky_1)$

8^{th} → $(2+5)(x_1,y_1) = 7(x_1,y_1)$
$= 7x_1, 7y_1 = ((2+5)x_1, (2+5)y_1)$
$= (2x_1+5x_1), (2y_1+5y_1)$

10^{th} → $1 \times (1,3) = (1,3)$

* Rmrk → The axiom 1 & 4 → closure axioms & we say that a vector space V is closed under vector addition & scalar multi

* Rmrk →

1) Let C[a,b] be set of real valued contin. ( ) defined on closed interval [a,b]. for any $f,g \in C[a,b]$ & $\alpha \in R$. & $x \in [a,b]$, f+g. & αf is defined by, $(f+g)(x) = f(x)+g(x)$. & $(\alpha f)(x) = \alpha f(x)$.

then $C[a,b]$ is a V.S.

2) Let P_n be set of real polynomials of degree less than or equal to n, for some +ve integral value of n.

$$P_n = \{ p(x) = a_0x^n + a_1x^{n-1} + \dots a_{n-1}x + a_n ; \; a_0, a_1, \dots a_n \in R \}.$$

for any $p(x) = a_0x^n + a_1x^{n-1} + \dots a_n$
and
$q(x) = b_0x^n + b_1x^{n-1} + \dots + b_n \in P_n$ & $\alpha \in R$

∴ define $p(x) + q(x) = (a_0+b_0) + (a_1+b_1) \dots (a_n+b_n)$
and
$\alpha\, p(x) = \alpha a_0x^n + \alpha a_1x^{n-1} + \dots \alpha a_n$.

Then P_n is a V.S

3) Let V be set of real no. s.t V is a V.S with addition & scalar multi.,

$$x+y = xy \quad \& \quad kx = x^k \qquad k \in R$$

A) 1) for any $x, y \in V$ & $x+y = xy$ is a +ve real no. ∈ V.
∴ it is closed under v. addition.

2) $x+y = xy = yx = y+x$.
Hence commutative ∀ $x, y \in V$.

3) $x, y, z \in V$,
$x+(y+z) = x+yz = x(yz) = (xy)z$
$= (x+y)z = (x+y)+z$.
Associative

4) Now for any $x \in V \Rightarrow 1+x = 1x = x = x1 = x+1$.
$1 \in V$ & we get 1 is zero vector of V.

5) for any $x \in V$,
define $-x = \frac{1}{x}$ then $x+(-x) = x \times \frac{1}{x} = 1$
The 0 vector. ∴ $\frac{1}{x}$ is -ve vector.

6) for any $x \in V$ & $k \in R$. define $kx = x^k$
if $k \geq 0$, $kx = x^k \in V$
if $k < 0$, $kx = x^{-k} = \frac{1}{x^k} \in V$.
hence it is closed under scalar multi.

7) for any $x, y \in V$ & $k \in R$
$k(x+y) = k(xy) = (xy)^k = x^k y^k$
$= kx + ky$.

8) For any $x \in V$ & $k_1, k_2 \in R$.
$(k_1+k_2)x = x^{(k_1+k_2)}$
$= x^{k_1} \cdot x^{k_2} = (k_1x)(k_2x)$
$= k_1x + k_2x$

9) for any $x \in V$, $k_1, k_2 \in R$
$k_1(k_2x) = k_1 x^{k_2} = (x^{k_2})^{k_1} = x^{k_2k_1}$
$= x^{k_1k_2} = (k_1k_2)x$.

10) for any $x \in V$, $1x = x^1 = x$.
It satisfies all the axioms of a V.S. hence V is a V.S

⇒ **Sub space (S.S) of a vector space (V.S) :-**

A nonempty ~~subset~~ S of vector space V → subspace of V, if 'S' itself be a v.space under the operation of addition & scalar multi defined on 'V'.

* **Rmrk →**

If S is v.space such that $S \subseteq V$ (subset), but the rule of addition / scalar multi / both are not same as for V then S is not a subspace of V.

Theorem :- criteria for a S.S :-

A non empty S.S 'S' of a V.S 'V' is a S.S of 'V', if and only if S closed under ~~the~~ vector addition & scalar multi, (i.e) if S satisfy the following condi,

1) $\vec{x}, \vec{y} \in S \Rightarrow \vec{x}+\vec{y} \in S$ (v.addition) ~~for~~ $\forall\ x, y \in S$

2) $\vec{x} \in S \Rightarrow \alpha\vec{x} \in S$, $\forall\ \vec{x} \in S$ for every scalar α.

* **Rmrk 1 →**

A non empty S.S 'S' of V.S 'V' is a S.S of V if and only if $\alpha\vec{x}+\beta\vec{y} \in S$ $\forall\ x, y \in S$.

and for every scalar α & β.

* **Rmrk 2 →**

Smallest S.S of any V.S is a space of zero vector only, which denote V_0.

$V_0 = \{0\}$ → Trivial vector space.

Q/1) let $S = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0\}$ for some fixed scalar a_1, a_2, a_3, a_4, a_5 S.T S is a S.S of $\mathbb{R}^5$?

A) we know that $\mathbb{R}^5$ is V.S

with v.addi & scalar multi defined by,

for any $\vec{x}, \vec{y} \in \mathbb{R}^5 \Rightarrow (x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)$

$= (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5)$.

$\alpha\vec{x} \Rightarrow (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5)$

here $a_1 0 + a_2 0 + a_3 0 + a_4 0 + a_5 0 = 0$

$\Rightarrow (0, 0, 0, 0, 0) \in S$

$\Rightarrow S$ is non empty.

(i) let $\vec{x}, \vec{y} \in S \Rightarrow a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0$

and

$a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + a_5y_5 = 0$

adding both eq.

$\Rightarrow a_1(x_1+y_1) + a_2(x_2+y_2) + a_3(x_3+y_3) + a_4(x_4+y_4) + a_5(x_5+y_5) = 0$.

$\Rightarrow (x_1+y_1, x_2+y_2, x_3+y_3, x_4+y_4, x_5+y_5) \in$

(2) $\Rightarrow \vec{x} + \vec{y} \in S$

(2) $x \in S$,

$\Rightarrow a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0$.

* with α through out,

$\Rightarrow \alpha a_1 x_1 + \alpha a_2 x_2 + \alpha a_3 x_3 + \alpha a_4 x_4 + \alpha a_5 x_5 =$

$\Rightarrow a_1(\alpha x_1) + a_2(\alpha x_2) + a_3(\alpha x_3) + a_4(\alpha x_4) + a_5(\alpha x_5) = 0$

$\Rightarrow (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \in S$

(3) $\Rightarrow \alpha \vec{x} \in S$

It satisfies all the condin of S.S

$\therefore$ S is S.S of $\mathbb{R}^5$

* Rmrk →

* P_n → space of all polynomials of degree $\leq n$.

* $C[a,b]$ be the space of all contin. () on $[a,b]$

contin.

Linear Combination & linear Span :-

let $S = \{\vec{x}_1, \vec{x}_2, \vec{x}_3 \dots \vec{x}_n\}$ be a S.S of V.S 'V'. Then the vector,

L.C ← $\boxed{\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n}$ — α → scalar constant

for some scalars, $\alpha_1, \alpha_2 \dots \alpha_n$

→ Linear combination. of vectors in S.

$S = \{x_1, x_2, x_3\}$.

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$

α = non

* $(2x_1 + 7x_2 - 3x_3)$ → also L.C.

* $x_1 + 2x_2$ → L.C

$\Rightarrow x_1 + 2x_2 + 0x_3$. can be written

$\boxed{0x_1 + 0x_2 + 0x_3 \downarrow \text{Trivial L.C}}$

(Collection of L.C → Linear span)

($x_1, x_2, x^3 \dots$, power of x is 1, that's y it is Linear)

* If all scalars $\alpha_1, \alpha_2, \alpha_n \dots$ are 0 then L.C → Trivial L.C

* If atleast 1 of α_i's is not 0

→ non trivial L.C.

* let $S = \{\vec{x}_1, \vec{x}_2, \dots \vec{x}_n\}$ be a S.S of V.S 'V'. The set of all vectors, which are L.C of vectors in 'S' → Linear span of S. [Simply span of S].

denoted by span(S).

$\therefore$ span(S) $= \{\alpha_1 x_1 + \alpha_2 x_2 \dots + \alpha_n x_n$: where $\alpha_1, \alpha_2 \dots \alpha_n$ are scalars$\}$.

A Theorem :-

Let $S = \{\vec{x_1}, \vec{x_2} \dots \vec{x_n}\}$ be a s.s of v.s 'V', then span(S) is a S.S of V.

(must satisfy 3 condi of s.s)

Proof $span(S) = \{\alpha_1 x_1 + \alpha_2 x_2 \dots \alpha_n x_n : \alpha_1, \alpha_2 \dots$ are scalar $\}$.

Since $x_1, x_2 \dots x_n \in V$. & V is a v.s which is closed under vector addition & scalar multi.

$\therefore \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \in V$.

hence span(S) is a non empty of V.

Let $x, y \in span(S)$,

$x = \alpha_1 x_1 + \alpha_2 x_2 \dots \alpha_n x_n$ and

~~$y = \beta_1 x_1 + \beta_2 x_2 \dots \beta_n$~~

$y = \beta_1 x_1 + \beta_2 x_2 \dots + \beta_n x_n$

where $\alpha_1, \alpha_2 \dots \alpha_n$ & $\beta_1, \beta_2 \dots \beta_n$ are scalars.

$x+y = (\alpha_1 \beta_1) x_1 + (\alpha_2 \beta_2) x_2 \dots + (\alpha_n \beta_n) x_n \in span(S)$

$\therefore x, y \in span(S) \Rightarrow x+y \in span(S)$.

$\alpha \vec{x} = \alpha (\alpha_1 x_1 + \alpha_2 x_2 \dots + \alpha_n x_n)$.

$= (\alpha \alpha_1) x_1 + (\alpha \alpha_2) x_n \dots + (\alpha \alpha_n) x_n$.

$\in span(S)$.

$x \in span(S) \Rightarrow \alpha x \in span(S)$.

$\therefore$ span(S) is a s.s of V

Example 1 →

consider v.s $\mathbb{R}^2$, & $S = \{(1,0), (0,1)\}$.

$\alpha_1 x_1 + \alpha_2 x_2$

suppose $\alpha = 5, 6$. $\Rightarrow 5(1,0) + 6(0,1) \Rightarrow (5,0)(0,6) = (5,6)$

$\alpha = 7, 3 \Rightarrow 7(1,0) + 3(0,1) = (7,3)$

$\mathbb{R}^3 \Rightarrow S = \{(1,0,0), (0,1,0), (0,0,1)\}$.

Generally for every $(x,y) \in \mathbb{R}^2$ we have

$(x,y) = x(1,0) + y(0,1)$

$\therefore \boxed{span(S) = \mathbb{R}^2}$

Example 2 → (general)

let $S = \{e_1, e_2 \dots e_n\}$. where $e_1 = (1,0, \dots 0)$

$e_2 = (0,1,0,0 \dots)$ ....

$e_n = (0,0,0 \dots 1)$

Then $span(S) = \mathbb{R}^n$

i) write $(1,0)$ as L.C of $(1,1)$ & $(-1,2)$

constnt → a, b.

$(1,0) = a(1,1) + b(-1,2)$

$= (a,a) + (-b, 2b)$

$(1,0) = (a-b, a+2b)$

(i.e) $a - b = 1$ —①   ×2

$a + 2b = 0$ —②.

$2a - 2b = 2$

$a + 2b = 0$

$3a = 2$

$a = 2/3$

$a - b = 1 \Rightarrow a = 1 + b$ — ③

Sub ③ in ②

$a + 2b = 0$.

$1 + b + 2b = 0$

$1 + 3b = 0$

$3b = -1$

$b = \frac{-1}{3}$

$a = 1 + b = 1 + \frac{-1}{3} = 1 - \frac{1}{3}$

$a = \frac{2}{3}$

$\therefore (1,0) = \frac{2}{3}(1,1) + \left(\frac{-1}{3}\right)(1,2)$

★ ⇒ Linear Dependence & Independence :-

Let $S = \{\vec{x_1}, \vec{x_2} \dots \vec{x_n}\}$ be a S.S of V.S V,

Consider a L.C of vectors of the set 'S' equated to $\vec{0}$. (i.e)

$\alpha_1 \vec{x_1} + \alpha_2 \vec{x_2} \dots + \alpha_n \vec{x_n} = \vec{0}$ (all constant)

Now solve for $\alpha_1, \alpha_2 \dots \alpha_n$, if only solution is $\alpha_1 = \alpha_2 = \dots = x_n = 0$

Then the set $S \rightarrow$ a linearly independent set (LI).

If there exist a non trivial solution for $\alpha_1, \alpha_2 \dots \alpha_n$, (i.e) atleast 1 of α_i's is not zero, then the set $S \rightarrow$ linearly dependent set (LD).

Note →

$$\boxed{\begin{aligned} &\alpha_1 x_1 + \alpha_2 x_2 \dots + \alpha_n x_n = (0,0 \dots 0) \\ &\Rightarrow \alpha_1 = \alpha_2 \dots = \alpha_n = 0 \\ &\Rightarrow LI \end{aligned}}$$

Q/1) S.T the set $= \{(2,6,-4)\ (3,9,-6)\}$ is linearly dependent in R^3.

A) Let $a, b \in R$

$\Rightarrow a(2,6,-4) + b(3,9,-6) = \vec{0} = (0,0,0)$ (R^3)

$\Rightarrow (2a, 6a, -4a) + (3b, 9b, -6b) = \vec{0} = (0,0,0)$

$\Rightarrow (2a+3b)\ (6a+9b)\ (-4a+-6b) = (0,0,0)$

$2a + 3b = 0$ — ①

$6a + 9b = 0$ — ②

$-4a - 6b = 0$ — ③

$6a + 9b = 0$

$2a + 3b = 0 \quad \times 3$

$6a + 9b = 0$

~~$6a + 9b = 0$~~

~~$6a + 9b = 0$~~

~~$a = 0$~~

$2a + 3b = 0$.

$2a = -3b$

$a = \frac{-3}{2} b$.

$a \neq 0 \quad b \neq 0 \Rightarrow LD$

$a = 0, b = 0 \Rightarrow L.I.$

$a \neq b \neq 0 \Rightarrow L.D$

Thus there exist infinite non 0 values for a & b

$\therefore$ The set is LD.

★ ⇒ Basis for a V.S :-

Let V be V.S, a S.S 'B' of V → Basis for the V.S V, if B has the following (pro) →

1. B is linearly independent. } 2 condition
2. B spans V, (i.e) span(B) = V.

$B \to$ [illegible] $(n) \to$ Dimension

→ Dimension of a V.S :-

* let B be a basis for a V.S V.
  If B is finite then V.S V ⟶ finite Dimensional & the no. of elements of B ⟶ Dimension of V.
* If B has n vectors then, V ⟶ n-dimensional, return as $\boxed{\dim V = n}$ dimension
* If basis B is infinite then V ⟶ Infinite dimensional.
* If $V = \{0\}$ the zero space, then, $\dim V = 0$.

Q) 1) S.T $S = \{(1,1), (-1,2)\}$ is a basis of $\mathbb{R}^2$.

A) B = S (V) (must satisfy 2 condition)

a, b as constant.

① ∴ $a(1,1) + b(-1,2) = (0,0)$

$\Rightarrow (a,a) + (-b,2b) = (0,0)$

$\Rightarrow (a-b)\ (a+2b) = (0,0)$

$a - b = 0$ —①

$a + 2b = 0$. —②

a, b ?

$a - b = 0$

$a = b$ —③

Sub ③ in ②

$a + 2b = 0$. $\Rightarrow b + 2b = 0$

~~$a + 2x - b = 0$~~

~~$a - 2b = 0$~~

$3b = 0$

$\boxed{b = 0}$ &

$\boxed{a = 0}$

$a = b = 0$

∴ S is linearly independent

② let $(x,y) \in \mathbb{R}^2$ be arbitrary.

$\boxed{\text{span}(S) = \mathbb{R}^2}$

$(x,y) \in \mathbb{R}^2$

$a(1,1) + b(-1,2) =$

we want to get values of a & b if we are getting = span(S) = $\mathbb{R}^2$

$(x,y) = a(1,1) + b(-1,2)$

$(x,y) = (a,a) + (-b,2b)$

$(x,y) = (a-b) + (a+2b)$

$\Rightarrow x = a - b$ —①

$y = a + 2b$ —②

a = ?

b = ?

$x = a - b$

$b = a - x$ —③

Sub ③ in ②.

$y = a + 2b$

$= a + 2(a - x)$

$= a + 2a - 2x$.

$y = 3a - 2x$.

$y + 2x = 3a$.

$\boxed{a = \frac{y+2x}{3}}$ —④

Sub ④ in ③

$b = a - x$.

$= \frac{y+2x}{3} - \frac{x}{1}$.

$= \frac{y+2x-3x}{3} = \frac{y-x}{3}$

$$\boxed{b = \frac{y-x}{3}}$$

If we are taking $(1,2)$ for (x,y).

$a = \frac{y+2x}{3} = \frac{2+2\times1}{3} = \frac{4}{3}$ getting

$b = \frac{y-x}{3} = \frac{2-1}{3} = \frac{1}{3}$ no

$\therefore$ a & b are existing

$(x,y) \in \mathbb{R}^2$

hence $(x,y) = \left(\frac{y+2x}{3}\right)(1,1) + \left(\frac{y-x}{3}\right)(1,2)$, thus every element of $\mathbb{R}^2$ can be written as a linear combination of vectors in S. (i.e)

$\mathrm{span}(S) = \mathbb{R}^2$

hence S is a basis of $\mathbb{R}^2$.

2) S.T $S = \{1, x, x^2\}$ is a basis for the vector space P_2. (polynomial) of all real polynomials of degree ≤ 2.

A) $S = \{1, x, x^2\}$

$a, b, c \in \mathbb{R}^2$.

$\Rightarrow a + bx + cx^2 = 0$.

$a = 0$
$bx = 0 \Rightarrow b = 0$
$cx^2 = 0 \Rightarrow c = 0$.

$a + bx + cx^2 = 0 + 0x + 0x^2$

$\therefore$ S is L.I. $\rightarrow a = b = 0$.

$\mathrm{span}(S) = P_2$?.

Also every poly. of degree ≤ 2 is in the form of $ax^2 + bx + c$ where a, b, c are real no,

$\therefore$ $\mathrm{span}(S) = P_2$.

$\therefore$ hence S is basis of P_2. → S.B

Examples → (standard basis) basis of $\mathbb{R}^2$

1) Consider $S = \{(1,0), (0,1)\}$

$S = \{(0,0,1)\ (1,0,0)\ (0,1,0)\}$ basis of $\mathbb{R}^3$

clearly S is L.I & $\mathrm{span}(S) = \mathbb{R}^2$, here $\boxed{\dim = 2}$

$\therefore$ S form a basis

This B is → Standard B of $\mathbb{R}^2$.

2) $S = \{(1,0,0), (0,0,1), (0,1,0)\}$ standard basis for $\mathbb{R}^3$

$\boxed{\dim = 3}$

$= \dim = n$

3) In general $S = \{e_1, e_2 \ldots e_n\}$.

$e_1 = (1, 0, \ldots 0)$

$e_2 = (0, 1, 0 0 0 \ldots 0)$.

$e_n = (0, 0, \ldots 1)$

is a S.B for $\mathbb{R}^n$ & $\boxed{\dim \mathbb{R}^n = n}$

(v) → vector.

→ The n-space :-

A vector in n-space is any ordered n-tuple,

$\vec{a} = (a_1, a_2, \cdots a_n)$.

of real no., → components of $\vec{a}$.

R^2 elements (1,2) (2,1) ...

R^3 elements (1,1,2) (2,1,1) ...

For any $\vec{a} = (a_1, a_2 \cdots a_n)$ &

$\vec{b} = (b_1, b_2 \cdots b_n)$

then $\vec{a}+\vec{b} = (a_1+b_1, a_2+b_2, \cdots a_n+b_n)$

Also $k\vec{a} = (ka_1, ka_2, \cdots ka_n)$, $k \in \mathbb{R}$.

→ A $\vec{0}$ in $\mathbb{R}^n$ is,

$\vec{0} = (0, 0 \cdots 0)$

length

|| || (norm)

* length / magnitude of $\vec{a}$

$$\boxed{\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 \cdots + a_n^2}}$$

length of a vector → its norm (|| ||)

* zero (v) = $\vec{0}$ = (0, 0, ... 0).

* 16/6/22

Unit (v) is a (v) whose norm is 1

* Normalizing $\vec{a}$ is $\boxed{u = \frac{1}{\|\vec{a}\|} \cdot \vec{a}}$

(normal → perpendicular (v)).

(v)

* orthogonal → 2 non zero (v) $\vec{a}$ & $\vec{b}$ are said to be orthogonal if $\vec{a} \cdot \vec{b} = 0$. (dot product)

(ie) $a_1 b_1 + a_2 b_2 \cdots + a_n b_n = 0$.

(O)

(... ⊥ (v) → orthogonal)

(basis → B)

→ Definition :-

* A basis for a (v) space is said to be an orthogonal basis if the (v) in the basis are mutually orthogonal.

* A basis for a (v) space is said to be orthonormal basis if the (v) in the basis are mutually orthogonal & their norm is 1., (i.e) they are unit (v).

→ Gram-Schmidt orthogonalization process:

(Converting basis to (o) basis → G-S.O process)

1) constructing an (o) basis for R^2 :→

let $B = \{\vec{u_1}, \vec{u_2}\}$ be a given basis then $B' = \{\vec{v_1}, \vec{v_2}\}$ is an (o) basis where.

[$u_1 \to v_1$, $u_2 \to v_2$.] conversion.]

$\boxed{\vec{v_1} = \vec{u_1}}$ &

$$\boxed{\vec{v_2} = \vec{u_2} - \left(\frac{\vec{u_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}}\right) \vec{v_1}}$$

(dot (p))

2) constructing an (o) basis for R^3 :→

let $B = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ be a given basis

Then (o) basis $B' = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ where.

$\boxed{\vec{v_1} = \vec{u_1}}$, $\boxed{\vec{v_2} = \vec{u_2} - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1}$.

$$\boxed{v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{\vec{u_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}}\right) v_2.}$$

Q 1) The set $B = \{(-3,2)\ (1,-1)\}$ is a basis for R^2. Transform B into an (O) basis.

A) let $\vec{u_1} = (-3,2)$ & $\vec{u_2} = (1,-1)$

$v_1, v_2 = ?$

by Gram-Schmidt eq,

$\vec{v_1} = \vec{u_1} = (-3,2)$

$v_2 = \vec{u_2} - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$

$= (1,-1) - \left[\frac{(1,-1)\cdot(-3,2)}{(-3,2)\cdot(-3,2)}\right](-3,2)$

$\left[\begin{matrix} -1\times -3 + -1\times 2 = 3-2 = 1 \\ -3\times -3 + 2\times 2 = 9+4 = 13 \end{matrix}\right]$

$= (1,-1) - \frac{1}{13}(-3,2)$

$= (1,-1) - \frac{-3}{13}, \frac{2}{13}$

$= \left[1 + \frac{3}{13}, -1 - \frac{2}{13}\right] = \left(\frac{-10}{13}, \frac{-15}{13}\right)$

↓ orthognal.

(we get (O) bt we need orthonormal so make (O) on (V) → v_1 & v_2.)

(orthonormal) w_1 & w_2

Nxt we hve to find the orthonormal basis, so normalize the $\vec{v_1}$ & $\vec{v_2}$.

$\vec{w_1} = \frac{\vec{v_1}}{||\vec{v_1}||} = \frac{(-3,2)}{\sqrt{(-3)^2+(2)^2}} = \frac{(-3,2)}{\sqrt{13}}$ //

$\vec{w_2} = \frac{\vec{v_2}}{||\vec{v_2}||} = \frac{\left(-10/13, -15/13\right)}{\sqrt{\left(\frac{-10}{13}\right)^2 + \left(\frac{-15}{13}\right)^2}}$

$\sqrt{\frac{100}{169} + \frac{225}{169}} = \sqrt{\frac{325}{169}} = \frac{5\sqrt{13}}{13}$

$= \frac{13}{5\sqrt{13}} \cdot \left(\frac{-10}{13}, \frac{-15}{13}\right)$

→ System of linear Algebraic eq:-
(power of x ... is 1)
(system → eq → $5x_1 + 9x_2 + 3x_3 = 0$)

General form →
* A system of 'm' linear eq in n-variables / unknowns, has general form,

$$\text{system}\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

($b_1, b_2, \dots b_m$ → no)

(want to find $x_1, x_2 \dots$).

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

(coefficient matrix) (soln) (constant)

A → coefficient of variables
B → constant of system.

* If B is zero then the system → homogenous otherwise non homogenous

(= [illegible] 0 [illegible] → homogenous) eg → homogenous →

$5x_1 + 9x_2 + x_3 = 0$

$x_1 - x_3 = 0$

$4x_1 + 6x_2 - x_3 = 0$.

eg → non homogenous →

$2x_1 + 5x_2 + 6x_3 = 1$ (atleast any one non zero)

$4x_1 + 3x_2 - x_3 = 9$

* A linear system of eq is said to be consistent, if it has atleast 1 solu. & inconsistent, if it has no soln.

* Elementary operations →

eg) $2x_1 + 6x_2 + x_3 = 7$ —— ①

$x_1 + 2x_2 - x_3 = -1$ —— ②

$5x_1 + 7x_2 - 4x_3 = 9$ —— ③

Consider ① & ②

① × 2 $\quad \cancel{2x_1} + 6x_2 + x_3 = 7$

$\cancel{2x_1} + 4x_2 - 2x_3 = -2$

$2x_2 + 3x_3 = 9$ —— ④

$2x_3 + x_3 = 3x_3$

② × 5 $\quad \cancel{5x_1} + 10x_2 - 5x_3 = -5$

$\cancel{5x_1} + 7x_2 - 4x_3 = 9$

$3x_2 - x_3 = -14$ —— ⑤

④ & ⑤

$2x_2 + 3x_3 = 9$

$3x_2 - x_3 = -14$. × 3

$9x_2 - \cancel{3x_3} = -42$

$2x_2 + \cancel{3x_3} = 9$

$11x_2 = -33$

$x_2 = \frac{-33}{11} = -3$

$x_2 = -3$

$3x_2 - x_3 = -14$

$3 \times -3 - x_3 = -14$

$-9 - x_3 = -14$

$x_3 = -14 + 9$

$x_3 = 5$

$x_2 = -3$

$x_3 = 5$

$2x_1 + 6x_2 + x_3 = 7$

$2x_1 + 6 \times -3 + 5 = 7$

$2x_1 - 18 + 5 = 7$

$2x_1 - 13 = 7 \Rightarrow 2x_1 = 7 + 13$

$x_1 = \frac{20}{2} = 10$

$x_1 = 10$

The solⁿ is $x_1 = 10$, $x_2 = -3$ & $x_3 = 5$

→ Augmented metrix → It is a metrix obtained by appending the constnt column into the coefficient of metrix (i.e) $[A|B]$.

(Combaining a & b → A metrix)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

→ Elementary row operations :-

1) multiply a raw by a nonzero constant. $R_i \to kR_i$
2) interchange any 2 rows,
3) Add a nonzero multiple of 1 row to any other row.

→ Elimination method :-

① Gaussian Elimination (using row-echelon form)

② Gauss-Jordan elimination (using reduced row enchelon form)

* Row-echelon form :-

1) 1st nonzero entry in a nonzero is a 1 (... non must be 1)
2) In conclusive nonzeroes, the 1st entry in the lower row appears to the right of the 1 in the higher row.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3) Row consisting of all zeroes are at the bottom of the matrix.

* Reduced Row enchelon form :-

have the same above 3 (pro).

4) * A column containing a 1st entry 1 has zeroes everywhere else.

Q) Solve the system using Gauss elimination method —

$$2x_1 + 6x_2 + x_3 = 7$$
$$x_1 + 2x_2 - x_3 = -1$$
$$5x_1 + 7x_2 - 4x_3 = 9.$$

A) The Augmented matrix is,

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}$$

$R_2 \to R_1$.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$6 - 2\times2$, $6 - 4 = 2$, $1 + 2 = 3$, $7 + 2 = 9$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{array}\right]$$

$R_2 \to \frac{R_2}{2}$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & -3 & 1 & 14 \end{array}\right]$$

$R_3 \to R_3 + 3R_2$

$1 + 3\times\frac{3}{2}$, $1 + \frac{9}{2} = \frac{2+9}{2} = \frac{11}{2}$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 11/2 & 55/2 \end{array}\right]$$

$R_3 \to R_3 \times \frac{2}{11}$

$\frac{55}{2} \times \frac{2}{11} = \frac{55}{11} = 5$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The above metrix is in row echelon form,

$$x_1 + 2x_2 - x_3 = -1$$
$$x_2 + \frac{3}{2}x_3 = \frac{9}{2}$$
$$x_3 = 5$$

Subs values in above eq, we get.

$x_3 = 5$
$x_2 = -3$
$x_1 = 10$

$x_2 + \frac{3}{2}x_3 = \frac{9}{2}$
$x_2 + \frac{3}{2}\times 5 = \frac{9}{2}$
$x_2 + \frac{15}{2} = \frac{9}{2}$
$x_2 = \frac{9}{2} - \frac{15}{2} = \frac{-6}{2}$
$x_2 = -3$

$x_1 + 2x_2 - x_3 = -1$
$x_1 + 2\times -3 - 5 = -1$
$x_1 - 6 - 5 = -1$
$x_1 - 11 = -1$
$x_1 = -1 + 11 = 10$

Q) use Gauss Jordan elimination —

$$x_1 + 3x_2 - 2x_3 = -7$$
$$4x_1 + x_2 + 3x_3 = 5$$
$$2x_1 - 5x_2 + 7x_3 = 19$$

A) $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 1 & 3 \\ 2 & -5 & 7 \end{bmatrix} \begin{bmatrix} -7 \\ 5 \\ 19 \end{bmatrix}$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{array}\right]$ $R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 2R_1$

→ Homogenous system :- $Ax = B,\ B = 0$

It is always consistent since $x_1 = 0,\ x_2 = 0 \ldots x_n = 0$.
The solⁿ consisting of all zeroes → trivial solⁿ otherwise non trivial soln.

→ Theorum :-

Let $Ax = 0$, denote a homogenous system of linear eq →

a) if x_1 is a solⁿ then so is cx_1

b) x_1 & x_2 are solⁿ then so $x_1 + x_2$.

→ Rank of A Metrix =

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ then, $\vec{u_1} = (a_{11}\ a_{12} \cdots a_{1n})$

$\vec{u_2} = (a_{21}\ a_{22} \cdots a_{2n})$

$\vdots$

$\vec{u_m} = (a_{m1}\ a_{m2} \cdots a_{mn})$

→ called row (v) of A

let $\vec{v_1} = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix}$, $\vec{v_2} \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix}$, $\cdots \vec{v_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$

→ column (v) of A.

→ Defⁿ Rank of a metrix :-

∴ A, denoted by Rank(A) is the max no. of linearly Independent row (v) in A.

Q) find rank of $= \begin{bmatrix} 6 & 2 & 3 & 4 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 7 & -2 \end{bmatrix}$

A) $\vec{u_1} = (6\ 2\ 3\ 4)$

$\vec{u_2} = (0\ 5\ -3\ 1)$

$\vec{u_3} = (0\ 0\ 7\ -2)$

linearly dependent / not.

$3(v) \rightarrow \vec{u_1}, \vec{u_2}, \vec{u_3}$ so a, b, c.

$a\vec{u_1} + b\vec{u_2} + c\vec{u_3} = \vec{0}$

$\Rightarrow a(6\ 2\ 3\ 4) + b(0\ 5\ -3\ 1) + c(0\ 0\ 7\ -2) = \vec{0}$

make 3eq

$\Rightarrow 6a + 0b + 0c = 0 \Rightarrow 6a = 0 \quad a = 0$

$\Rightarrow 2a + 5b + 0c = 0 \Rightarrow 2a + 5b = 0$

$5b = -2a \quad a = 0$

$5b = 0$

$\boxed{b = 0}$

$\Rightarrow 3a - 3b + 7c = 0$

$0 - 0 + 7c = 0 \quad 7c = 0$

$c = 0$

$a = b = c = 0$

all the real no. are 0.

$\vec{u_1}, \vec{u_2}, \vec{u_3}$ are linearly independent.

$\therefore$ Rank$(A) = 3$

$a = b = c = 0$

3

→ Rank by Row Reduction :-

Theorem → If a (m) A is row equalent to a row echelon form B, then,

a) The row space of A = row space of B

b) Non zero rows of B from a basis for the row space of A.

c) Rank(A) = No. of nonzero rows of B.

Q) obtain row-echelon form follow. & find its Rank.

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

A) 1st row ok starting 1 ✓

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad R_2 \rightarrow \frac{R_2}{5}.$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - 5R_2.$$

Q) Find rank of $= \begin{bmatrix} 6 & 2 & 3 & 4 \\ 0 & 5 & -3 & 1 \\ 0 & 0 & 7 & -2 \end{bmatrix}$

A) $\vec{u_1} = (6\ 2\ 3\ 4)$
$\vec{u_2} = (0\ 5\ -3\ 1)$
$\vec{u_3} = (0\ 0\ 7\ -2)$

linearly dependent / not.

$3(v) \to \vec{u_1}, \vec{u_2}, \vec{u_3}$ So a, b, c.

$a\vec{u_1} + b\vec{u_2} + c\vec{u_3} = \vec{0}$

$\Rightarrow a(6\ 2\ 3\ 4) + b(0\ 5\ -3\ 1) + c(0\ 0\ 7\ -2)$

make 300

$\Rightarrow 6a + 0b + 0c = 0 \Rightarrow 6a = 0$ $\quad a = 0$

$\Rightarrow 2a + 5b + 0c = 0 \Rightarrow 2a + 5b = 0$
$5b = -2a$ $\quad a = 0$
$5b = 0$
$\boxed{b = 0}$

$\Rightarrow 3a - 3b + 7c = 0$
$0 - 0 + 7c = 0$ $\quad 7c = 0$
$c = 0$

$a = b = c = 0$

all the real no. are 0.

$\vec{u_1}, \vec{u_2}, \vec{u_3}$ are linearly indepent.

$\therefore$ Rank(A) = 3

$a = b = c = 0$

3

→ Rank by Row Reduction :-

Theorem → If a (m) A is row equalent to a row echelon form B, then,

a) The row space of A = row space of B

b) Non zero rows of B form a basis for the row space of A.

c) Rank(A) = No. of nonzero rows of B.

Q) obtain row-echelon form follow. & find its Rank.

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

A) 1st row ok starting 1 ✓

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 5 & -3 \end{bmatrix} \quad R_2 \to \frac{R_2}{5}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 5R_2.$$

non zeroes is 2.

So Rank is 2

→ Theorem → A linear system of eq $Ax=B$ is consistent if & only if the rank of (co) of (m) A is the same as the rank of Argumented (m) of system $[A \mid B]$.

Rmrk, consider system $Ax=B$ with 'm' eq. & n unknowns.

Case 1 : if Rank $(A|B) \neq$ Rank (A) then the systems of eq is consistent (i.e) no soln.

Case 2 : If rank $(A|B)$ = rank $(A) = r$ & $r=n$, then the system is consistent i.e it has a unique soln.

Case 3 : if rank of $[A|B]$ = rank of $(A) = r$ Either the system is consistent & $(n-r)$ of unknown may be chosen assign arbitrary values to $(n-r)$ unknowns.

Q) By using Gaussian elimination method S.T the system
$$x+2y+z=2$$
$$3x+y-2z=1$$
$$4x-3y-z=3$$
$$2x+4y+2z=4$$
is consistent & hence solve?

A) $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}$

$Ax=B$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to \frac{R_2}{-5}$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 + 11R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to \frac{R_3}{6}$$

(A|B) rank →

∴ is in row echlon form.

(A|B)

Rank [A|B] = 3 → (no. of non zeroes).

Also $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Rank (A) = 3

(A) Rank →

hence Rank (A|B) = Rank (A).

System is consistent

hence solve

To find soln, let $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$Ax = B$

mult

* $x + 2y + z = 2$
* $y + z = 1$
* $z = 1$

$y = 0$ → $y + z = 1$, $y + 1 = 1$, $y = 1 - 1$, $y = 0$

$x + 0 + 1 = 2$, $x = 1$

$x + 1 = 2$

$x = 2 - 1$, $x = 1$

$x = 1, z = 1, y = 0$ is a soln

Soln of homogenous system of linear eq. →

$Ax = 0$

$x = 0$ → trivial soln. otherwise nontrivial.

*Theorem →

A homogenous system of m eq in n variables posses non trivial soln if the no. m of eq is < the no. n variables.

Q) Solve,

$x + y - z + t = 0$

$x - y + 2z - t = 0$

$3x + y + t = 0$.

(homogenous system)

3eq ↓ $m = 3$

$\underbrace{x? \; y? \; z? \; t?}_{4 = n}$

find 4 variables → $n = 4$

$Ax = 0$. $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

no z 800

$$A \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

~~Rank (A) = 2 = r~~

$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -3/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{-R_2}{2}$$

$$\text{Rank}(A) = 2 = r$$

$n \neq r$ so $(n-r) = 4-2 = 2$

So take 2 variables $\to a, b$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$① \left\{ \begin{array}{l} x + y - z + t = 0 \\ -2y + 3z - 2t = 0 \end{array} \right.$$

$z = a$ & $t = b$

$$\begin{array}{cccc} 1 & 1 & -1 & 1 \\ 0 & 1 & -3/2 & 1 \\ 0 & 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ x & y & z & t \end{array}$$

$\Rightarrow x$ & y have 1, z & t no 1, so, $z = a$, $t = b$

① become,

$x + y - a + b = 0$ &

$-2y + 3a - 2b = 0$

$-2y = -3a + 2b$

$y = \dfrac{-3a + 2b}{-2}$

$$\boxed{y = \frac{3}{2} a - b}$$

$x = -y + a - b$

$x = -\frac{3}{2} a - b + a - b \Rightarrow \left(\frac{-3}{2} + 1\right) a = -\frac{1}{2} a$

$x = -\frac{1}{2} a$

$y = 3/2\, a - b$

$z = a$

$t = b$