Q2: IMPROPER INTEGRALS

→ The integral $\int_a^b f(x)\,dx$ → I.∫.

① * if $a = -\infty$ or $a = \infty$ or both.

② * $f(x)$ is unbounded at 1/more points of $[a,b]$.

* ∫ corresponding to ① is integrals over unbounded intervals / I.∫ of 1st kind.
* ∫ corresponding to ② → integrals of unbounded () / I.∫ of 2nd kind.
* ∫ with both ① & ② → I.∫ of 3rd kind.

* To distinguish the other integrals from I.∫ the former → proper integrals.

eg →

* $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ → ∫ over unbounded interval.
* $\int_0^{4} \frac{dx}{x-3}$ → " ().
* $\int_0^{\infty} \frac{e^{-x}}{x-2}\,dx$ → unbounded () over u. interval
* $\int_0^{1} \frac{\sin x}{x}\,dx$ → proper ∫ since $\lim_{x\to 0} \frac{\sin x}{x} = 1$

→ Integrals over unbounded Intervals :-

I.∫ of 1st kind → have infinite limits of ∫. i.e integrand is bounded.
here we can have upper / lower limits of ∫ as ∞.

1) Def :- Suppose that for a fixed, f is integrable on $[a,b]$ for all $b > a$,

$$\boxed{\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx}$$

– If limit on RHS is finite → I.∫ conv (non-)
– If limit fails to exist then I.f → Dvg.
$(-\infty,\infty)$

2) If for b, f is ∫ on $[a,b]$ for all $a < b$,

$$\boxed{\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to -\infty} \int_a^b f(x)\,dx}$$

3) If f is ∫ on $[a,b]$ for all $a < b$,

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx}$$

$-\infty \quad 0 \quad \infty$

Q/ i) $\int_1^{\infty} \frac{1}{x^2}\,dx$.

$= \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$.

$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$

$\int \frac{1}{x^2} = \int x^{-2} = \frac{x^{-2+1}}{-2+1}$

$= \frac{x^{-1}}{-1}$

$= -\frac{1}{x}$

$$= \lim_{b\to\alpha}\left[-\frac{1}{b} - \frac{-1}{1}\right]$$

$$= \lim_{b\to\alpha}\left[-\frac{1}{b} + \frac{1}{1}\right] = \lim_{b\to\alpha}\left[-\frac{1}{b}+1\right]$$

$$= \lim_{b\to\alpha}\left[1 + \frac{1}{b}\right] = 1 + 0 = 1$$

2) $\int_1^{\alpha} \frac{\ln x}{x^2}\, dx = \lim_{b\to\alpha}\int_1^{b} \frac{\ln x}{x^2}\, dx$.

$\ln,\ \frac{1}{x^2}$

$$\boxed{\int u\cdot v\, dx = u\cdot\int v\, dx - \int (u'\int v)dx}$$

($\int$ of by parts)

How to choose 1st () & 2nd ()

ILATE

which's order

→ I → inverse

→ L → Log

→ A → Algebric $\frac{1}{x}$, $\frac{1}{x^2}$

→ T → Trignomatry

→ E → exp.

here $\ln x$ → L   & $\frac{1}{x^2}$ → A. So, L → 1st () A → 2nd ()

∴ using $\int$ by parts,

$$= \lim_{b\to\alpha}\left[\ln x\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \frac{1}{x}\left(-\frac{1}{x}\right)dx.$$

$u' \to (\ln x)' \to \frac{1}{x}$

$\int v \to \int \frac{1}{x^2} = -\frac{1}{x}$

$$= \lim_{b\to\alpha}\left[-\frac{\ln b}{b} + \ln 1 - \int_1^b -\frac{1}{x^2}\, dx.\right]$$

$\ln 1 = 0$

$$= \lim_{b\to\alpha}\left[-\frac{\ln b}{b} + \ln 1 + \left(-\frac{1}{x}\right)_1^b\right]$$

$$= \lim_{b\to\alpha}\left[-\frac{\ln b}{b} + 0 + \frac{-1}{b} + 1\right]$$

$$= \lim_{b\to\alpha}\left[1 - \frac{\ln b}{b} - \frac{1}{b}\right]$$

$$= \lim_{b\to\alpha} 1 - \lim_{b\to\alpha}\frac{\ln b}{b} - \lim_{b\to\alpha}\frac{1}{b}$$

$$= 1 - \lim_{b\to\alpha}\frac{\ln b}{b}$$

$\frac{\alpha}{\alpha}$ ⇒ Lopital's rule, → $\frac{d/dx \text{ of numerator}}{d/dx \text{ of deno}}$

$$= 1 - \lim_{b\to\alpha}\frac{1/b}{1} = 1 - 0 = 1$$

cvg

3) Test follo. I.of for cvg.

a) $\int_{-\alpha}^{0} \frac{dx}{(1-3x)^2} = \lim_{a\to-\alpha}\int_a^0 \frac{dx}{(1-3x)^2}$

$1-3x=0$
$x = 1/3$
unbounded

by Sub-method, put $1-3x = u$.

$u = 1-3x$.

$\frac{d}{dx}u = \frac{d}{dx}(1-3x)$

$\frac{du}{dx} = 0-3 = -3$

$\frac{du}{dx} = -3$, $du = -3dx$.

$dx = -\frac{du}{3}$

$$\int\frac{dx}{(1+3x)^2} = \int\frac{1}{(1+9x^2)}\cdot dx.$$

$$= \int\frac{1}{u^2}\cdot\left(-\frac{du}{3}\right) = -\frac{1}{3}\int\frac{du}{u^2} = -\frac{1}{3}\cdot\left(-\frac{1}{u}\right) = \frac{1}{3u}$$

$$\Rightarrow \frac{1}{3(1-3x)}$$

$$\therefore \int_{-\alpha}^{0}\frac{dx}{(1-3x)^2} = \lim_{a\to-\alpha}\int_a^0\frac{dx}{(1-3x)^2}$$

$$= \lim_{a\to-\alpha}\left[\frac{1}{3(1-3x)}\right]_a^0$$

$$= \lim_{a\to-\alpha}\left[\frac{1}{3} - \frac{1}{3(1-3a)}\right]$$

$$= \lim_{a\to-\alpha}\frac{1}{3} - \lim_{a\to\alpha}\frac{1}{3(1-3a)}$$

$= \frac{1}{3} - \frac{1}{3} \lim_{a \to -\infty} \frac{1}{1-3a}$ (0)

$= \frac{1}{3} - 0 = \frac{1}{3}$ finite value ie cvg.

b) $\int_{-\infty}^{0} \cosh x \, dx = \lim_{a \to -\infty} \int_{a}^{0} \cosh x \, dx$.

$= \lim_{a \to -\infty} \left[\sinh x \right]_a^0$

$= \lim_{a \to -\infty} \left[\sinh 0 - \sinh a \right]$

$= \lim_{a \to -\infty} \left[0 - \frac{e^a - e^{-a}}{2} \right]$

$= \infty$

limit is infinite & DVg.

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\sinh 0 = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0$

$\boxed{\begin{matrix} e^{\infty} = \infty \\ e^{-\infty} = 0 \end{matrix}}$

4) S.T $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} dx$ = (split)

$= \int_{-\infty}^{0} \frac{dx}{1+x^2} dx + \int_{0}^{\infty} \frac{dx}{1+x^2} dx$.

$= \lim_{a \to -\infty} \int_a^0 \frac{dx}{1+x^2} dx + \lim_{b \to \infty} \int_0^b \frac{dx}{1+x^2} dx$.

$\int \frac{1}{1+x^2} = \tan^{-1} x$

$= \lim_{a \to -\infty} \left[\tan^{-1} x \right]_a^0 + \lim_{b \to \infty} \left[\tan^{-1} x \right]_0^b$

$= \lim_{a \to -\infty} \left[\tan^{-1} 0 - \tan^{-1} a \right] + \lim_{b \to \infty} \left[\tan^{-1} b - \tan^{-1} 0 \right]$

$= 0 - \lim_{a \to -\infty} \tan^{-1} a + \lim_{b \to \infty} \tan^{-1} b - 0$

$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

finite ie cvg

$\int \frac{1}{1+x^2} = \tan^{-1} x$

$\tan^{-1} 0 = 0$

$\tan \infty = \frac{\pi}{2}$

$\frac{f'(x)}{f(x)} = \ln|f(x)|$

5) S.T $\int_{-\infty}^{\infty} \frac{2x}{1+x^2} dx$ is DVg.

$= \int_{-\infty}^{0} \frac{2x}{1+x^2} dx + \int_{0}^{\infty} \frac{2x}{1+x^2} dx$.

$= \lim_{a \to -\infty} \int_a^0 \frac{2x}{1+x^2} dx + \lim_{b \to \infty} \int_0^b \frac{2x}{1+x^2} dx$.

So, $\frac{f'(x)}{f(x)} = \ln|f(x)|$

$\therefore = \lim_{a \to -\infty} \left[\ln(1+x^2) \right]_a^0 + \lim_{b \to \infty} \left[\ln(1+x^2) \right]_0^b$.

$= \lim_{a \to -\infty} \left[\ln 1 - \ln(1+a^2) \right] + \lim_{b \to \infty} \left[\ln(1+b^2) - \ln 1 \right]$

$\ln \infty$ — doesnot exist

here RHS does not exist, so the given $\int$ is DVg.

* Rmrk :-

$\int_{-\infty}^{\infty} \frac{2x}{1+x^2} dx = \lim_{b \to \infty} \int_{-b}^{b} \frac{2x}{1+x^2} dx$.

$= \lim_{b \to \infty} \left[\ln(1+x^2) \right]_{-b}^{b}$

$= \lim_{b \to \infty} \left[\ln(1+b^2) - \ln(1+b^2) \right]$

$= 0$ // → cvg

(Don't do this method) bcz $\int \frac{2x}{1+x^2}$ is dvg.

This implies that $\int_{-\infty}^{\infty} f(x)\,dx$ need not be equal to $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$.

b)

6) S.T $\int_0^{\infty} e^{-tx}\,dx$, $t \to$ constant, cvg if $t>0$ & Dvg if $t \le 0$. ?

A) $\int_0^{\infty} e^{-tx}\,dx = \lim_{b\to\infty} \int_0^{b} e^{-tx}\,dx$

$= \lim_{b\to\infty} \left[\frac{e^{-tx}}{-t}\right]_0^b$

$= \lim_{b\to\infty} \left[\frac{e^{-tb}-1}{-t}\right]$

$= \lim_{b\to\infty} \left[\frac{1-e^{-tb}}{t}\right]$

$= \frac{1}{t} \lim_{b\to\infty} \left[1-e^{-tb}\right]$

$= \frac{1}{t} \left[\lim_{b\to\infty} 1 - \lim_{b\to\infty} e^{-tb}\right]$

$= \frac{1}{t} \left[1 - \underline{\lim_{b\to\infty} e^{-tb}}\right]$

$\int e^{x} = e^{x}$.

$\int e^{-tx}\,dx = \frac{e^{-tx}}{\frac{d}{dx}(-tx)} = \frac{e^{-tx}}{-t}$.

$\lim_{b\to\infty} e^{-tb} = \begin{cases} \text{if } t>0 \to 0 \\ \text{if } t<0 \to \infty \end{cases}$

$-tb \to -ve$, $b\to\infty$ means $-t\infty \to -\infty \to e^{-\infty} = 0$
when $t>0$

when $t<0 \to -tb$ ~~[illegible]~~ >0 ആയാൽ, $\to b\to\infty$
$e^{\infty} = \infty$

$\therefore \lim_{b\to\infty} e^{-tb} = \begin{cases} 0, & \text{if } t>0 \\ \infty, & \text{if } t<0 \end{cases}$

Hence $\int_0^{\infty} e^{-tx}$ cvg if $t>0$ & dvg if $t<0$.

when $t=0$, $\int_0^{\infty} e^{-0x}\,dx = \int_0^{\infty} dx$

$= \lim_{b\to\infty} \int_0^b dx \Rightarrow \lim_{b\to\infty} [x]_0^b$

$= \lim_{b\to\infty} [b-0] = \lim_{b\to\infty} [b] = \infty$

$\therefore \int_0^{\infty} e^{-tx}\,dx$ cvg if $t>0$ & dvg if $t \le 0$.

7) S.T $\int_a^{\infty} \frac{1}{x^p}\,dx$, $p \to$ constant & $a>0$, cvg if $p>1$ & dvg if $p \le 1$.

eg
$\int_2^{\infty} \frac{1}{x^3} =$
$2>0$, $3>1$ cvg

A) $\int_a^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_a^b \frac{1}{x^p}\,dx$

$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^b$ when $p \ne 1$

$= \lim_{b\to\infty} \left[\frac{b^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right] = \lim_{b\to\infty} \left[\frac{b^{1-p} - a^{1-p}}{1-p}\right]$

$= \frac{1}{1-p} \left[\lim_{b\to\infty} b^{1-p} - a^{1-p}\right]$ $p \ne 1$.

$\lim_{b\to\infty} b^{1-p} = \begin{cases} 0, & \text{if } p>1 \\ \infty, & \text{if } p<1. \end{cases}$

$b^{1-p} \to >1$ ആണെങ്കിൽ $\to b^{-ve} = \frac{1}{b}$

$\lim_{b\to\infty} \frac{1}{b} = 0$

$b^{1-p} \to <1$ ആണെങ്കിൽ $b^{+ve} = \infty$

Hence $\int_a^{\infty} \frac{1}{x^p}\,dx$ cvg if $p>1$ &
dvg if $p<1$

b) when $p=0$, $\int_a^{\infty} \frac{1}{x^p}\,dx = \int_a^{\infty} \frac{1}{x}\,dx$.

$= \lim_{b\to\infty} \int_a^b \frac{1}{x}\,dx = \lim_{b\to\infty} \left[\ln x\right]_a^b$

$= \lim_{b\to\infty} \left[\ln b - \ln a\right] = \infty - \ln a$
$= \infty$

$\therefore \int_a^{\infty} \frac{1}{x^p}\,dx$ cvg if $p>1$ & dvg
if $p \le 0$

* Rmrk :- $a \to 6, \& 7$ are of grt imp since these I.S can be used as comparison integrals in testing the convergence of improper integrals.

2) ⇒ Tests for convergence & Dvg :-

Many I.S cannot be evaluated directly from the def.
Most of tests are for testing cvg of S with unbounded intervals.
The most imp test for cvg of an I.S is comparison test.

→ Comparison Test :-

$\lim_{x\to} f(x) = 84$
convgnt or not

~~1) $|f(x)| \le g(x)$ $x>a$~~
~~2) If $\int_a^{\infty} g(x)\,dx$ is convergnt.~~

Suppose that f & g are ( ) such that
1) $|f(x)| \le g(x)$ for all $x>a$
2) $\int_a^b f(x)\,dx$ & $\int_a^b g(x)\cdot dx$ exist for every $b>a$ then,

* if $\int_a^{\infty} g(x)\,dx$ is convergnt so $\int_a^{\infty} f(x)\,dx$.
* if $\int_a^{\infty} f(x)\,dx$ is divergnt so $\int_a^{\infty} g(x)\,dx$.

Q/ 1) Examine the convergnts for S
(c. or not)

$\int_0^{\infty} \frac{dx}{e^{x+1}}$

A) $\int_0^{\infty} \frac{dx}{e^{x+1}}$

$\int_0^{\infty} \frac{1}{e^{x+1}} \cdot dx$

$e^{x+1} \ge e^{x}.$ } Comparision

$\therefore \frac{1}{e^{x+1}} \le \frac{1}{e^{x}} = \underline{e^{-x}}$.

eg

$3 \ge 2$

$\frac{1}{3} \ge \frac{1}{2}$

$0.33 \ge 0.5$

b) 2) $\int_0^{\infty} e^{-tx} \cdot dx$

$\lim_{b\to\infty} \int_0^b e^{-tx} \cdot dx$.

$\lim_{b\to\infty} \left[\frac{e^{-tx}}{-t}\right]_0^b$

$\lim_{b\to\infty} \left[\frac{e^{-bt}}{-t} - \frac{e^{0}}{-t}\right] = \lim_{b\to\infty} \frac{1}{t}\left[-e^{-bt}+1\right]$

$= \lim_{b\to\infty} \frac{1}{t}\left[1-e^{-bt}\right]$

$t > 0$

$t \le 0$.

3) $\int_0^{\infty} e^{-tx}\, dx$,

$t>0$ convergent.

4) $\int_0^{\infty} e^{-x} \cdot dx$, cgs

$\frac{dx}{e^{x}}$ cgs.

$\therefore \frac{dx}{e^{x+1}}$ cgs.

we know that $\int_0^{\infty} e^{-tx} \cdot dx$ convgs. if t :

hence $\int_0^{\infty} e^{-x}\, dx$ ~~dt~~ cgs.

$\therefore$ by comparison tst it follows that

$\int_0^{\infty} \frac{dx}{e^{x+1}}$ cgs.

4) $\int_2^{\infty} \frac{dx}{\ln x}$

$x \ge 2 \rightarrow \int_2^{\infty}$

$\ln x < x$. $x \uparrow$

$\frac{1}{\ln x} > \frac{1}{x}$, $x \ge 2$

5) $\int_a^{\infty} \frac{1}{x^p}\, dx$.

$\lim_{b\to\infty} \int_a^b \frac{1}{x^p} \cdot dx$

$\lim_{b\to\infty} \int_a^b x^{-p}\, dx$.

$\lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^b$

$\lim_{b\to\infty} \left[\frac{1}{-p+1}\left[b^{-p+1} - a^{-p+1}\right]\right]$, $p \ne 1$

$p > 1$, cgs.

$\int_0^{\infty} e^{-x^2} dx$.

finite → (a,b) (a,b) ....

Infinite → (a,∞) (∞,a) $(b,-\infty)$ $(-\infty, ...$

b) $\boxed{\text{improper } \int \to \lim_{a\to\infty} \int_a^b f(x)\cdot dx}$

↳ $\int_1^{+\infty} \frac{dx}{x^2}$

↳ $\int_{-\infty}^{0} e^{x}\, dx$

↳ $\int_{-\infty}^{\infty} \frac{dx}{x^2}$

I. $\int$

converge = we get a no. 3, 4, ...

Diverge = $-\infty$, ∞.

(a) * $\int_{-3}^{3} \frac{dx}{x^2}$ $(-3, 3)$ ↳ take 0.

$\Rightarrow \frac{1}{0} \cdot dx \to \infty$.

2) * $\int_1^2 \frac{dx}{x-1}$ ⇒ take 1

$\Rightarrow \frac{1}{1-1} \cdot dx$

$\Rightarrow \frac{1}{0} \Rightarrow \infty$.

* $\int_0^{\pi} \tan x \cdot dx \Rightarrow (0-\pi) \to \pi/2$.

$\Rightarrow \frac{\sin \pi/2}{\cos \pi/2} = \frac{1}{0} = \infty$

$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$

$\boxed{e^0 = 1,\ e^{-\infty} = 0,\ e^{\infty} = \infty}$

1) $\int_1^{\infty} \frac{dx}{x^3} \Rightarrow \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\cdot dx$

$\Rightarrow \lim_{b\to\infty} \int_1^b x^{-3}\, dx$.

$= \lim_{b\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^b$

$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$

$= \lim_{b\to\infty} \left[-\frac{1}{b} + \frac{1}{1}\right]$

$= \lim_{b\to\infty} \left[-\frac{1}{b} + 1\right]$

$= -\frac{1}{\infty} + 1 = 0 + 1 = 1$

$\frac{1}{\infty} = 0$

Cnvrg

2) $\int_1^{+\infty} \frac{dx}{x} \Rightarrow \lim_{b\to\infty} \int_1^b \frac{1}{x}\cdot dx$.

$\Rightarrow \lim_{b\to\infty} [\ln x]_1^b$

$\Rightarrow \lim_{b\to\infty} [\ln b - \ln 1]$

$\Rightarrow \lim_{b\to\infty} [\ln b] \Rightarrow \ln \infty = \infty$.

$\boxed{\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2},\ \int \frac{1}{x} = \ln x,\ \ln 1 = 0}$

Dvrg

b)

3) $\int_{-\infty}^{0} e^{4x}\,dx \Rightarrow \lim_{a\to-\infty} \int_{a}^{0} e^{4x}\,dx$

$\Rightarrow \lim_{a\to-\infty} \left[\frac{e^{4x}}{4}\right]_a^0 \Rightarrow \lim_{a\to-\infty}\left[\frac{e^{4\times 0}}{4} - \frac{e^{4a}}{4}\right]$

$\Rightarrow \lim_{a\to-\infty}\left[\frac{1}{4} - \frac{e^{4a}}{4}\right] \Rightarrow \frac{1}{4} - \frac{e^{4\times-\infty}}{4}$

$\Rightarrow \frac{1}{4} - 0 = \frac{1}{4}$

4) $\int_{-\infty}^{2} \frac{dx}{x^2+4}\,dx \Rightarrow \lim_{a\to-\infty}\int_a^2 \frac{dx}{x^2+4}$

$\Rightarrow \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_a^2$

$\Rightarrow \lim_{a\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right)\right]$

$\Rightarrow \lim_{a\to-\infty}\left[\frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right)\right]$

$\Rightarrow \frac{\pi}{8} - \frac{1}{2}\tan^{-1}(-\infty)$ avoid -ve

$\Rightarrow \frac{\pi}{8} + \frac{1}{2}\tan^{-1}(\infty)$

$\Rightarrow \pi/8 + \frac{1}{2}\cdot\pi/2 \Rightarrow \frac{\pi}{8} + \frac{\pi\times 2}{4\times 2} = \frac{\pi+2\pi}{8} = \frac{3\pi}{8}$

$\int \frac{1}{x^2+4} = \frac{1}{a}\tan^{-1}$

here $a =$

$\tan^{-1} 1 =$

$\tan^{-1}\infty =$

$a/2 = -$

*) $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{+\infty}\frac{dx}{1+x^2}$

$\int_{-\infty}^{0}\frac{dx}{1+x^2} = ?\ldots$

$\int_{0}^{+\infty}\frac{dx}{1+x^2} = ?\ldots$

finally + both

→ Comparison Test :-

1) $\int_0^{\infty} e^{-tx}\,dx$ → $t>0$ → cvg

→ $t \le 0$ → Dvg

2) $\int_a^{\infty}\frac{1}{x^p}\,dx$ → p constant & $a>0$

→ $p>1$ → cvg

→ $p\le 1$ → Dvg.

1) $\int_0^{\infty}\frac{dx}{e^x+1} =$

$e^x+1 \ge e^x$

So, $\frac{1}{e^x+1} \le \frac{1}{e^x} = e^{-x} = 1$

$\frac{1}{e^x} = e^{-x}$

$e^{-x} = 1$

we know, $\int_0^{\infty} e^{-tx}\,dx$ cvg if $t>0$

Hence $\int_0^{\infty} e^{-x}\,dx$ cvg / q...

2) $\int_2^{\infty}\frac{dx}{\ln x}$ [we know, for all $x \ge 2$ $\ln x < x$]

$\frac{1}{\ln x} > \frac{1}{x}$

by (2) $\int_2^{\infty}\frac{dx}{x^p}\,dx$ dvg if $p\le 1$

$p=1$ $x^p \to \frac{1}{x} \to$ dvg

3) $\int_0^{\infty} e^{-x^2}\,dx \overset{(split)}{\Rightarrow} \int_0^1 e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx.$

$\int_0^1 e^{-x^2}\,dx \to$ proper $\int$.

b. for every $x \geq 1$, $x \leq x^2$

$x \leq x^2 \Rightarrow e^{x} \leq e^{x^2}$

$\Rightarrow \frac{1}{e^{x}} \geq \frac{1}{e^{x^2}} = e$

$\Rightarrow e^{-x^2} \leq e^{-x} = 1$

we know, $\int_1^{\infty} e^{-tx}\,dx$ cvg if $t > 0$

cvg

1) Examine the convergents of follo... $\int$.

a) $\int_0^{\infty} \frac{dx}{\sqrt{1+x^8}}$, by the (pro) of definite $\int$ we hve,

$$\int_0^{\infty} \frac{dx}{\sqrt{1+x^8}} \overset{(split)}{\to} = \int_0^1 \frac{dx}{\sqrt{1+x^8}} + \int_1^{\infty} \frac{dx}{\sqrt{1+x^8}}$$

(by definite $\int$)

$\frac{1}{\sqrt{1+x^8}}$ is contin on $[0,1]$

1^{st} $\int$ of RHS a proper $\int \to \int_0^1 \frac{dx}{\sqrt{1+x^8}}$

we need only tst the convergnts of the 2^{nd} $\int \to \int_1^{\infty} \frac{dx}{\sqrt{1+x^8}}$

for every value of $x \geq 1$

$$\frac{1}{\sqrt{1+x^8}} < \frac{1}{\sqrt{x^8}} = \frac{1}{x^{8/2}} = \frac{1}{x^4}.$$

$\sqrt{x^8} \to x^{8/2} \to x^4$

we know, $\int_1^{\infty} \frac{dx}{x^p}$ convrgs if $p > 1$

hence $\int_1^{\infty} \frac{dx}{x^4}$ convergs.

∴ by comparison tst it follows that

$\int_1^{\infty} \frac{dx}{\sqrt{1+x^8}}$ convrgs.

b) $\int_0^{\infty} \frac{\sin x}{(1+x)^2}\,dx.$

here & (1) → $\sin x$ & $(1+x)^2$

$\boxed{|\sin x| \le 1}$

we know that $|\sin x| \le 1$ for all x.

hence, we get,

$$\frac{|\sin x|}{(1+x)^2} \le \frac{1}{(1+x)^2}, \quad \text{for all } x \ge 0$$

we def. of $\int$ over unbounded intervals we get,

$$\int_0^\infty \frac{dx}{(1+x)^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{(1+x)^2}$$

$$= \lim_{b\to\infty} \left[\frac{-1}{1+x}\right]_0^b$$

$$\Rightarrow \lim_{b\to\infty} \left(\frac{-1}{1+b} - \frac{-1}{1+0}\right)$$

$$= \lim_{b\to\infty} \left[\frac{-1}{1+b} + 1\right]$$

$$= \lim_{b\to\infty} \left[1 - \frac{1}{1+b}\right]$$

$$= 1$$

(12)

Q) $\int_1^\infty \frac{dx}{\sqrt{1+x^2}}$

$x \ge 1$

$x^2 \ge 1$

$$= \int_1^\infty \frac{1}{1+x^2} \ge \frac{1}{\sqrt{x^2+x^2}}$$

$$\ge \frac{1}{\sqrt{2}\,x}$$

by def. $\int$,

$$\int_1^\infty \frac{dx}{\sqrt{2}\,x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \frac{1}{\sqrt{2}} \lim_{b\to\infty} [\ln x]_1^b$$

→ ~~Integrals of unbounded ( ) :-~~

$$= \lim_{b\to\infty} [\ln b]$$

$$\int_1^\infty \frac{dx}{\sqrt{2}x} = \infty$$

limit is infinite

∴ given improper $\int$ is divergent.

→ Integrals of unbounded ( ) :-

Let $f(x)$ be contin. on $(a,b]$ & $f(x)$ has an infinite discontinuity at $x=a$, then we define,

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\cdot dx.$$

$$\boxed{\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b f(x)\,dx.}$$

If the limit on the RHS infinite then we say that the improper $\int$ converges. & limit is the value of $\int$.

If the limit fails to exist the improper $\int$ is said to diverge.

Let $f(x)$ be contin. on $[a,b)$ & $f(x)$ has an infinite discontin. at $x=b$, then we define,

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

$$\boxed{\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx.}$$

$$\boxed{\int_a^b f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^b f(x)\,dx.}$$

$x_0 \to$ any real no.

LHS is convergent if & only if RHS is convergent otherwise it diverges.

If f becomes infinite at an interior 'd' on $[a,b]$, then

$$\boxed{\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx.}$$

. LHS → convgnt if & only if RHS → c.

If the (?) f has a finite no. of infinite discontin.. $d_1, d_2, \ldots d_m$ in $[a,b]$ where $a \le d_1 \le d_2 \le \ldots \le d_m \le b$.

$$\int_a^b f(x)\,dx = \int_a^{d_1} f(x)\,dx + \int_{d_1}^{d_2} f(x)\,dx \ldots + \int_{d_m}^{b} f(x)\,dx.$$

LHS → c then RHS → c.

Q 1) Examine the cnvgs of $\int_0^1 \frac{dx}{x^2}$

here the integrant $\frac{1}{x^2}$ become infinite at $x=0$ (x-m̃ a atisfommes α f(b))

but it is contin. on $[0,1)$

$$\int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0^+} \int_{0+\varepsilon}^1 \frac{dx}{x^2}$$

(remove so $\varepsilon \to 0^+$)

$$= \lim_{\varepsilon \to 0^+} \left[\frac{-1}{x}\right]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{-1}{1} + \frac{1}{\varepsilon}\right]$$

$$= \lim_{\varepsilon \to 0^+} \left[\frac{1}{\varepsilon} - 1\right] = \frac{1}{0} - 1 = \alpha.$$

$= \alpha$

(Dvg)

2) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$\frac{1}{\sqrt{1-x^2}}$ become infinite at $x = 1$

(x-m̃ l atisform weight α)

$$\boxed{\begin{array}{c}\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x \\ \hline \sin^{-1} 0 = 0 \\ \cos^{-1} 0 = 1\end{array}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx.$$

$\left(\begin{array}{l} x=1 \\ \to 1-\varepsilon \end{array}\right)$

$$= \lim_{\varepsilon \to 0^+} \left[\sin^{-1} x\right]_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \left[\sin^{-1}(1-\varepsilon) - \sin^{-1} 0\right]$$

(cvg) (Give limit) $= \sin^{-1} 1 - 0 \Rightarrow \sin^{-1} 1$

$\Rightarrow \sin \frac{\pi}{2} = 1$

3) $\int_0^2 \frac{dx}{2x-x^2}$

4) Investigate the cvg of $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$.

A) ($x-\infty$, assign α range)

$x=1$. $\int_0^3 \frac{1}{(1-1)^{2/3}} = \int_0^3 \frac{1}{0} = \underline{\underline{\alpha}}$

here $x=1$ so split, bcz in $\int$ no (.

$$\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\,dx.$$

(ε, δ) $\lim_{\varepsilon\to 0^+} \int_0^{1-\varepsilon} \frac{1}{(x-1)^{2/3}}\,dx + \lim_{\delta\to 0^+ \,(\delta \ne 1+\delta)} \int_{1+\delta}^{3} \frac{1}{(x-1)^{2/3}}\,dx.$

$\cancel{\lim_{\varepsilon\to 0^+} \int_0^{1-\varepsilon} \frac{(x-1)^{-2/3+1}}{-2/3+1}}$

$\frac{1}{(x-1)^{2/3}} \Rightarrow \frac{(x-1)^{-2/3+1}}{-2/3+1}$

$\Rightarrow \frac{(x-1)^{1/3}}{1/3} \Rightarrow \underline{\underline{3(x-1)^{1/3}}}$

$\Rightarrow \lim_{\varepsilon\to 0^+} \left(3(x-1)^{1/3}\right)_0^{1-\varepsilon} + \lim_{\delta\to 0^+} \left(3(x-1)^{1/3}\right)_{1+\delta}^{3}$

$\Rightarrow \lim_{\varepsilon\to 0^+} \left(3(1-\varepsilon-1)^{1/3}\right) - 3(0-1)^{1/3} +$

$\lim_{\delta\to 0^+} \left(3(3-1)^{1/3}\right) - \left(3(1+\delta-1)^{1/3}\right).$

$\Rightarrow \lim_{\varepsilon\to 0^+} 3(-\varepsilon)^{1/3} - 3(-1)^{1/3} + \lim_{\delta\to 0^+} 3(2)^{1/3} - 3(\delta)^{1/3}.$

$\varepsilon\to 0$ & $\delta\to 0$

$= 3 + 3\sqrt[3]{2}$

$3(-1)^{1/3} \Rightarrow$

5) Find the area of region that lies b/w the curve $y=\sec x$ & $y=\tan x$ from $x=0$ to $x=\frac{\pi}{2}$

A) $\int_0^{\pi/2} (\sec x - \tan x)\,dx$.

$y = \sec x$, $y=\tan x$, $? = \sec x - \tan x$

$\lim_{\varepsilon\to 0^+} \int_0^{\pi/2-\varepsilon} (\sec x - \tan x)\,dx$. $\quad x = \pi/2 \to \alpha$.

$\lim_{\varepsilon\to 0^+} \left(\ln \sec x + \tan x) + \ln\cos x\right)_0^{\frac{\pi}{2}-\varepsilon}$

$\lim_{\varepsilon\to 0^+} \ln\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) \cdot \cos x\Big)_0^{\frac{\pi}{2}-\varepsilon}$

$\boxed{\log A + \log B = \log(AB)}$

$\lim_{\varepsilon\to 0^+} \ln\left(\frac{1+\sin x}{\cos x} \cdot \cos x\right)_0^{\frac{\pi}{2}-\varepsilon}$

$\lim_{\varepsilon\to 0^+} \ln\left(1+\sin x\right)_0^{\frac{\pi}{2}-\varepsilon}$

$\lim_{\varepsilon\to 0^+} \ln\left(1+\sin\left(\frac{\pi}{2}-\varepsilon\right) - \ln(1+\sin 0)\right)$

$\ln\left(1+\sin\frac{\pi}{2}\right) - \ln(1+\sin 0)$

$\ln(1+1) - \ln 1$

$\ln 2 - \ln 1$

$\boxed{\log A - \log B = \log\left(\frac{A}{B}\right)}$

$\Rightarrow \ln\left(\frac{2}{1}\right) \Rightarrow \underline{\underline{\ln 2}}$

$\boxed{\begin{array}{l}\int \sec x = \ln \sec x + \tan x \\ \int \tan x = \ln \cos x\end{array}}$

6) Find length of the curve $y=(1-x^2)^{1/2}$ $[-1,1]$ interpret the result geometrically?

A) $\boxed{\text{For finding length} = \int_{-1}^{1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}}$

$$L = \int_{-1}^{1} \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$y = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1-x^2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{1-x^2} = \frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$L = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx.$$

$(x = 1 \to \alpha)$ $x=-1$.

$(x = 1, -1)$ infinite

but contin. on $(-1,1)$

Split $\int_{-1}^{0} - + \int_{0}^{1} -$

$$\lim_{\varepsilon \to 0^+} \int_{-1+\varepsilon}^{0} \frac{1}{\sqrt{1-x^2}}\, dx + \lim_{\delta \to 0^+} \int_{0}^{1-\delta} \frac{1}{\sqrt{1-x^2}}\, dx.$$

$$\lim_{\varepsilon \to 0^+} \left[\sin^{-1}x\right]_{-1+\varepsilon}^{0} + \lim_{\delta \to 0^+} \left[\sin^{-1}x\right]_{0}^{1-\delta}$$

$\boxed{\frac{1}{\sqrt{1-x^2}} = \sin^{-1}x}$

$$\lim_{\varepsilon \to 0^+} \sin^{-1}0 - \sin^{-1}(-1+\varepsilon) + \lim_{\delta \to 0^+}\left[\sin^{-1}(1-\delta) - \sin^{-1}0\right].$$

$\Rightarrow 0 - \sin^{-1}(-1) + \sin^{-1}(1) - 0$ $\quad \sin^{-1}(1) \to \frac{\pi}{2}$

$\Rightarrow -\sin^{-1}(-1) + \sin^{-1}(1).$

$$-\left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \underline{\underline{\pi}}$$

circle $\to$ radius $\to$ 1 cm

circumference $= 2\pi$

7) S.T improper $\int \to \int_{0}^{\alpha} \frac{e^{-x}}{\sqrt{x}}\, dx$ cvgs?

$$x=0 \to \int_{0}^{\alpha} \frac{e^{-0}}{\sqrt{0}}\, dx. = \frac{1}{0} = \underline{\underline{\alpha}}$$

$$\int_{0}^{\alpha} \frac{e^{-x}}{\sqrt{x}}\, dx = \int_{0}^{1} \frac{e^{-x}}{\sqrt{x}}\, dx + \int_{1}^{\alpha} \frac{e^{-x}}{\sqrt{x}}\, dx.$$

on $[0,1]$, $e^{-x} \le 1$

$$\therefore \frac{e^{-x}}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

by the def. of $\int$ of unbounded (),

$$\int_{0}^{1} \frac{1}{\sqrt{x}}\, dx = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\, dx.$$

$$= \lim_{\varepsilon \to 0^+} \left[2\sqrt{x}\right]_{\varepsilon}^{1} \Rightarrow \lim_{\varepsilon \to 0^+} \left[2\sqrt{1} - 2\sqrt{\varepsilon}\right]$$

$$= \lim_{\varepsilon \to 0^+} \left[2-\varepsilon\right] = \underline{\underline{2}}$$

RHS $\to$ finite

$\underline{\underline{\text{cvg}}}$

Hence by c. test the 1st improper $\int$ on RHS is cvg.

on $[1,\infty)$,

$\frac{1}{\sqrt{x}} \le 1 \quad \therefore \frac{e^{-x}}{\sqrt{x}} \le e^{-x}$

by def^n of $\int$,

$$\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[-e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[e^{-1} - \underset{0}{\underset{\parallel}{e^{-b}}}\right] = e^{-1}$$

Also cvg

The given improper $\int$ is cvg

8) $\int_a^b \frac{dx}{(x-a)^p}$ P.T con if $p<1$ & divg if $p \ge 1$

A) It is a proper $\int$. when we give a to x deno- become 0.

$$\int_a^b \frac{1}{(x-a)^p}\,dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b \frac{1}{(x-a)^p}\,dx \quad (x-a)^{-p}$$

$$= \lim_{\varepsilon\to 0^+} \left[\frac{(x-a)^{-p+1}}{-p+1}\right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{1-p}\left[(b-a)^{-p+1} - \varepsilon^{-p+1}\right]$$

dt $\lim_{\varepsilon\to 0^+} \varepsilon^{-p+1} = \begin{cases} 0 & \text{if } p<1 \\ \infty & \text{if } p>1 \end{cases}$

$$\int_a^b \frac{1}{(x-a)^p}\,dx = \begin{cases} \frac{1}{1-p}(b-a)^{-p+1} & \text{if } p<1 \\ \infty, & \text{if } p>1. \end{cases}$$

hence $\int_a^b \frac{dx}{(x-a)^p}$ is convg if $p<1$ & dvg if $p>1$.

when $p=1$,

$$\int_a^b \frac{1}{(x-a)^p}\,dx = \int_a^b \frac{1}{x-a}\,dx = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b \frac{1}{x-a}\,dx$$

$$= \lim_{\varepsilon\to 0^+} \left[\ln(x-a)\right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon\to 0^+} \left[\ln(b-a) - \ln\varepsilon\right]$$

$$= \ln(b-a) - \lim_{\varepsilon\to 0^+} \ln\varepsilon.$$

hence the gvn improper $\int$ is cvg if $p<1$ & dvg $p \ge 1$.