02 : Law of large Numbers

⇒ **Chebyshev's inequality :-**

☆ If x is a r.v with $E(x) = \mu$ & $V(x) = \sigma^2$, exists then for any $k > 0$

(a) $P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$ (or)

(b) $P\{|x-\mu| \leq k\sigma\} \geq 1 - \frac{1}{k^2}$

Pf want to prove $P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$

$E(x) = \mu \qquad V(x) = \sigma^2$

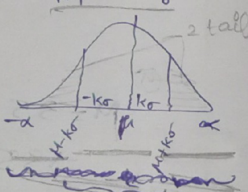
$V(x) = E(x - E(x))^2$

$= E(x-\mu)^2$

☆ $= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$. split $\int$.

$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x) dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x) dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\, dx$.

cancel



2nd term

$V(x) = \sigma^2$

$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\, dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\, dx$.

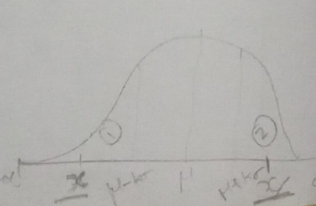
(1) $x \leq \mu - k\sigma$

$x - \mu \leq -k\sigma$

$\langle\rangle$ * with $\Rightarrow -(x-\mu) \geq -(-k\sigma)$

square $\Rightarrow (-(x-\mu))^2 \geq (k\sigma)^2$

$(x-\mu)^2 \geq k^2\sigma^2$ — (A)



$(x-\mu)^2 \geq k^2\sigma^2$ — (B)

$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)\, dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x) dx$

(2) $x \geq \mu + k\sigma$

$x - \mu \geq k\sigma$

$(x-\mu)^2 \geq k^2\sigma^2$ — (B)

(common)

$\sigma^2 \geq k^2\sigma^2 \left[\int_{-\infty}^{\mu-k\sigma} f(x) dx + \int_{\mu+k\sigma}^{\infty} f(x)\, dx\right]$

$\frac{\sigma^2}{k^2\sigma^2} \geq \int_{-\infty}^{\mu-k\sigma} f(x)\, dx + \int_{\mu+k\sigma}^{\infty} f(x)\, dx$.

$\int_a^b f(x) = P(a < x < b)$) in (prob)

only

$\frac{1}{k^2} \geq P\{x \leq \mu - k\sigma\} + P\{x \geq \mu + k\sigma\}$

$\geq P\{x - \mu \leq -k\sigma\} + P\{(x-\mu) \geq k\sigma\}$

* with $-$ → $\geq P\{-(x-\mu) \geq -(-k\sigma)\} + P\{(x-\mu) \geq k\sigma\}$

$= P\{-(x-\mu) \geq k\sigma\} + P\{(x-\mu) \geq k\sigma\}$

$\frac{1}{k^2} \geq P\{|x-\mu| \geq k\sigma\}$

$P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$ on taking compliment,

$P\{|x-\mu| \leq k\sigma\} \geq 1 - \frac{1}{k^2}$

⇒ **Convergence in (prob) :-**

☆ A sequence of r.v $x_1, x_2, \ldots, x_n \ldots$ is said to be converges in (prob) to a constant 'a' if for any $\varepsilon > 0$, however small,

$P\{|x_n - a| < \varepsilon\} \to 1$ as n tends to $\to \infty$,

or equivalently,

$P\{|x_n - a| \geq \varepsilon\} \to 0$ as $n \to \infty$.

i.e. we write $x_n \xrightarrow{P} a$ as $n \to \infty$.

if there exist a r.v 'x' such that

sequence $\{x_n\}$ coverge in (prob) to r.v 'x' if

for any $\varepsilon > 0$, however small.

$P\{|x_n - x| \geq \varepsilon\} \to 0$ as $n \to \infty$.

or, $\lim_{n\to\infty} P\{|x_n - x| \geq \varepsilon\} = 0$.

This is also stochastic cvgnce.

$P(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

⇒ **weak law of large no. →**

Theorem →

let $x_1, x_2 \dots x_n$ be a sequence of independent r.v with $E(x_i) = \mu$ & $V(x_i) = \sigma^2$, $i = 1, 2 \dots n$

define $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$, then any $\varepsilon > 0$,

$P\{|\bar{x}_n - \mu| \geq \varepsilon\} \to 0$ as $n \to \infty$ → so use (a)

Pf

$E(x_i) = \mu$ $\quad$ $V(x_i) = \sigma^2$

$\boxed{\bar{x}_n = \frac{1}{n}\sum x_i}$

$E(\bar{x}_n) = E\left(\frac{1}{n}\sum x_i\right)$

$= \frac{1}{n} E(\sum x_i)$

$= \frac{1}{n}\sum_{i=1}^{n} E(x_i)$

$= \frac{1}{n}\sum_{i=1}^{n} \mu \to \mu\ \mu\ \mu \dots \mu \to n\mu$.

$= \frac{1}{n} n\mu = \mu$.

$\boxed{E(\bar{x}_n) = \mu}$

$V(\bar{x}_n) = V\left(\frac{1}{n}\sum x_i\right)$ $\quad$ $V(ax) = a^2 \cdot V(x)$.

$= \frac{1}{n^2} V(\sum x_i)$

$= \frac{1}{n^2}\sum V(x_i)$

$= \frac{1}{n^2}\sum \sigma^2$

$= \frac{1}{n^2} \cdot n\sigma^2$

$V(\bar{x}_n) = \frac{\sigma^2}{n}$ $\quad$ $\boxed{V(\bar{x}_n) = \frac{\sigma^2}{n}}$

we need SD.

$SD(\bar{x}_n) = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$ $\quad$ $\boxed{SD = \frac{\sigma}{\sqrt{n}}}$

by chebyshev's inequality — (a).

$P\{|x - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$. $\quad$ let $k = t$.

$P\{|\bar{x}_n - \mu| \geq t\frac{\sigma}{\sqrt{n}}\} \leq \frac{1}{t^2}$ $\quad$ $\varepsilon = ?$.

choose $\varepsilon = \frac{t\sigma}{\sqrt{n}}$

$t = \frac{\sqrt{n}\,\varepsilon}{\sigma}$

$t^2 = \left(\frac{\sqrt{n}\,\varepsilon}{\sigma}\right)^2 = \frac{n\varepsilon^2}{\sigma^2}$

$\frac{1}{t^2} = \frac{\sigma^2}{n\varepsilon^2}$ $\quad$ $\lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2}$, $\frac{1}{\infty} = 0$

$\therefore P\{|\bar{x}_n - \mu| \geq t\frac{\sigma}{\sqrt{n}}\} \leq \frac{\sigma^2}{n\varepsilon^2} \to 0$ as $n \to \infty$.

$P\{|\bar{x}_n - \mu| \geq \varepsilon\} \to 0$ as $n \to \infty$

⇒ Bernoulli's law of large no. :-

(binomial)

Theorem → consider n trials of random exp, each trial resulting in a success (failure). If x_n denotes the no. of success in n trials with constant (prob) 'p' of success for each trial, then for any $\epsilon>0$ however small,

$$P\left\{\left|\frac{x_n}{n} - p\right| < \epsilon\right\} \longrightarrow 1 \text{ as } n \to \infty.$$

pf/ Since $x_n \to B(n,p)$

$E(x_n) = np$ $\quad V(x_n) = npq$ $\quad \sigma^2 V(x)$

$E\left(\frac{x_n}{n}\right) = \frac{1}{n} E(x_n)$ $\quad V\left(\frac{x_n}{n}\right) = \frac{1}{n^2} V(x_n)$

$\frac{1}{n}$ outside. $= \frac{1}{n} \cdot np$ $\quad = \frac{1}{n^2} \cdot npq$

$= p$ $\quad = \frac{pq}{n}$

$$SD\left(\frac{x_n}{n}\right) = \sqrt{\frac{pq}{n}} = \frac{\sqrt{pq}}{\sqrt{n}}$$

by Chebyshev's inequality,

$$P\{|x_n - \mu| \le k\sigma\} \ge 1 - \frac{1}{k^2} \qquad k = t$$

$$P\left\{|x_n - p| \le t\frac{\sqrt{pq}}{\sqrt{n}}\right\} \ge 1 - \frac{1}{t^2}$$

Compare,

choose $\epsilon = \frac{t\sqrt{pq}}{\sqrt{n}}$

$t = \frac{\sqrt{n}\,\epsilon}{\sqrt{pq}}$ $\qquad t^2 = \left(\frac{\sqrt{n}\,\epsilon}{\sqrt{pq}}\right)^2 = \frac{n\epsilon^2}{pq}$

$\therefore \frac{1}{t^2} = \frac{pq}{n\epsilon^2}$

$1 - \frac{1}{t^2} = 1 - \frac{pq}{n\epsilon^2}$ — ① $\longrightarrow 1 - \frac{1}{\infty} = 1 - 0 = 1$

~~$P\left\{|x_n - p| \le t\frac{\sqrt{pq}}{\sqrt{n}}\right\} \ge 1 - \frac{pq}{n\epsilon^2} \longrightarrow 1$ as $n \to \infty$~~

~~$P\left\{\left|\frac{x}{n} - p\right| < \epsilon\right\} \longrightarrow 1$ as $n \to \infty$~~

avoid pq ⟶ $0 \le p \le 1$

$p = 1 \quad q = 0$

$p \cdot q = 1 \times 0 = 0$

$p = 0 \quad q = 1, \; pq = 0$

(max) → $p = 1/2 \quad q = 1 - \frac{1}{2} = 1/2$

$p \cdot q = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$\therefore pq \le \frac{1}{4}$

$\frac{pq}{n\epsilon^2} \le \frac{1}{4n\epsilon^2}$

$1 - \frac{pq}{n\epsilon^2} \ge 1 - \frac{1}{4n\epsilon^2}$ ⟶ eg → $\frac{1}{4} < \frac{1}{2}$ & $1 - \frac{1}{4}$, $1 - \frac{1}{2}$; $\frac{3}{4} > \frac{1}{2}$

$\therefore P\left\{\left|\frac{x_n}{n} - p\right| < \frac{t\sqrt{pq}}{\sqrt{n}}\right\} \ge 1 - \frac{1}{t^2}$ — ①

$P\left\{\left|\frac{x_n}{n} - p\right| < \epsilon\right\} \ge 1 - \frac{pq}{n\epsilon^2} \ge 1 - \frac{1}{4n\epsilon^2}$

$P\left\{\left|\frac{x_n}{n} - p\right| < \epsilon\right\} \ge 1 - \frac{1}{4n\epsilon^2}$

as $n \to \infty$

$P\left\{\left|\frac{x_n}{n} - p\right| < \epsilon\right\} \; 1 - \frac{1}{\infty} = 1 - 0 = 1$

$P\left\{\frac{x_n}{n} - p < \epsilon\right\} \to 1$ as $n \to \infty$

2) find least value of (prob), $P(1 \le x \le 7)$ where x is r.v with mean 4, $V(x) = 4$?
(Q → least value / upper value, use Chebyshev's i)

A) $P(1 \le x \le 7)$ → ①

$$\boxed{\begin{array}{l} P\{|x-\mu| \ge k\sigma\} < \frac{1}{k^2} \rightarrow \text{upper.} \\ \text{or } P\{|x-\mu| \le k\sigma\} > 1 - \frac{1}{k^2} \rightarrow \text{least.} \end{array}}$$

use least eq.

$\mu = 4$ $\quad \sigma^2 = 4 \rightarrow \sigma = \sqrt{4} = 2$

by Chebyshev's i,

$$P\{|x-\mu| \le k\sigma\} > 1 - \frac{1}{k^2} \quad ③$$

$$② \; P\{|x-4| \le 2k\} > 1 - \frac{1}{k^2}$$

by ① $x>$, ② $x-4$ so

$$P(1 \le x \le 7) = P(1-4 \le x-4 \le 7-4)$$

$$= P(-3 \le x-4 \le 3) \quad |\,-\,|$$

$$= P(|x-4| \le 3) \quad ④$$

comparing ③ & ④

$2k = 3$, $\quad k = 3/2$

$$1 - \frac{1}{k^2} = 1 - \frac{1}{(3/2)^2} = 1 - \frac{1}{9/4} = 1 - \frac{1}{1} \times \frac{4}{9}$$

$$= 1 - \frac{4}{9} = 5/9$$

least value → 5/9

2) r.v x has mean 50 & $V(x)$ 100 use chebyshe's inequality to obtain appropriate bounds for,
a) $P(|x-50| \geq 15)$
(b) $P(|x-50| < 20)$

a) $\mu = 50$ $\sigma^2 = 100$, $\sigma = \sqrt{100} = 10$.

a) $P(|x-50| \geq 15)$,
by chebyshe's i...,
$P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$
$P\{|x-50| \geq 10k\} \leq \frac{1}{k^2}$ —— ①
Q → $P\{|x-50| \geq 15\}$ —— ②
Compare ① & ②
$10k = 15$
$k = 15/10 = 3/2$

$\therefore P\{|x-50| \geq 15\} \leq \frac{1}{(3/2)^2} = \frac{1}{9/4} = \frac{4}{9}$

$\therefore$ upper value $= 4/9$

(b) $P(|x-50| < 20)$
by chebyshe's i...
$P(|x-\mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$
$P(|x-50| \leq 10k) \geq 1 - \frac{1}{k^2}$ —— ①
Q → $P(|x-50| \leq 20)$ —— ②

① & ②
$10k = 20$
$k = \frac{20}{10} = 2$

$k=2$
$P\{|x-50| \leq 10k\} \geq 1 - \frac{1}{2^2}$
$=20$ $P\{|x-50| \leq 20\} \geq \frac{3}{4}$

Least value $= \cancel{4/9}$ $3/4$

⇒ Central Limit Theorem (CLT) :-

(... distri... normal distri... by $Z = \frac{x-\mu}{\sigma}$)

It states that the sum of large no. of r.v after standisation [(ie) subtraction of mean & deviation by standard deviation] hue approximatly the standard normal distr

Lindeberg – Levy form of CLT (Theorem):-

let $x_1, x_2 \ldots x_n$ be a sequence of independent & identically distributed r.v with $E(x_i) = \mu$ & $V(x_i) = \sigma^2$, $i = 1, 2, \ldots n$ where we assume that $0 < \sigma^2 < \infty$ letting $S_n = x_1 + x_2 + \cdots + x_n$, the normalised r.v

standlising means $\mu=0$ & $\sigma=1$ (a)

$$\boxed{Z = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}} \sim N(0,1) \text{ as } n \to \infty$$

(follows) (a)

($M_Z(t) = e^{t^2/2}$ → prove ... → $z = \frac{S_n - n\mu}{\sqrt{n}\sigma}$ ... normal ...)

Pf/ $E(x_i) = \mu$ $V(x_i) = \sigma^2$

$S_n = x_1 + x_2 + \cdots + x_n$.

$$Z = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$$

general form of mgf,

$$\boxed{M_x(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots} \quad \text{—— (c)}$$

$$S_n = \sum_{i=1}^{n} x_i \quad \text{—— (b)}$$

$$M_z(t) = M_{\frac{S_n - n\mu}{\sqrt{n}\sigma}}(t):$$

$$= M_{S_n - n\mu}(t/\sqrt{n}\sigma)$$

$$M_{ax+b} = e^{bt} \cdot M_{ax}(t)$$

$$\cancel{M_{S_n - n\mu} = e^{(t/\sqrt{n}\sigma)-n\mu} \cdot M_{S_n}(t/\sqrt{n}\sigma)}$$

$$M_{S_n - n\mu}(t/\sqrt{n}\sigma) = e^{\frac{-n\mu t}{\sqrt{n}\sigma}} M_{S_n}(t/\sqrt{n}\sigma)$$

$$= e^{\frac{-n\mu t}{\sqrt{n}\sigma}} \cdot M_{S_n}(t/\sqrt{n}\sigma)$$

$$\frac{n}{\sqrt{n}} = \frac{\sqrt{n}\cdot\sqrt{n}}{\sqrt{n}} = \sqrt{n}$$

$$= e^{-\sqrt{n}\mu t/\sigma} \cdot M_{S_n}(t/\sqrt{n}\sigma)$$

$$= e^{-\sqrt{n}\mu t/\sigma} M_{\sum_{i=1}^{n} x_i}(t/\sqrt{n}\sigma) \quad \text{(b)}$$

$$M_{\sum_{i=1}^{n} x_i} = \prod_{i=1}^{n} M_{x_i}$$

$$= e^{-\sqrt{n}\mu t/\sigma} \prod_{i=1}^{n} M_{x_i}(t/\sqrt{n}\sigma)$$

$$(t = t/\sqrt{n}\sigma)$$

by (c)

$$= e^{-\sqrt{n}\mu t/\sigma} \prod_{i=1}^{n} \left[1 + \frac{t}{\sqrt{n}\sigma}\mu_1' + \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 \mu_2' + \cdots\right]$$

$$M_z(t) = e^{-\sqrt{n}\mu t/\sigma}\left[1 + \frac{t}{\sqrt{n}\sigma}\mu_1' + \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 \mu_2' + \cdots\right]^n$$

taking log on both sides,

$$\log M_z(t) = \log e^{-\sqrt{n}\mu t/\sigma} + \log\left[1 + \frac{t}{\sqrt{n}\sigma}\mu_1' + \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 \mu_2' + \cdots\right]^n$$

$$(\log(a\cdot b) = \log a + \log b)$$

$$(\log x^n = n\log x) \qquad = -\sqrt{n}\frac{\mu t}{\sigma} + n\log\left[1 + \frac{t}{\sqrt{n}\sigma}\mu_1' + \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 \mu_2' + \cdots\right]$$

$$= -\sqrt{n}\frac{\mu t}{\sigma} + n\log\Big[\underbrace{1 + \frac{t}{\sqrt{n}\sigma}\mu_1' + \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 \mu_2' + O\left(\frac{1}{n^{3/2}}\right)}_{x}\Big]$$

unwanted

$$\left(\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots\right) = -\sqrt{n}\frac{\mu t}{\sigma} + n\left[\frac{t}{\sqrt{n}\sigma}\mu_1' + \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2\mu_2' + O\left(\frac{1}{n^{3/2}}\right)\right.$$

$$\left. - \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\mu_1' + \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2\mu_2' + O\left(\frac{1}{n^{3/2}}\right)\right)^2\right]$$

$$= \cancel{-\frac{\sqrt{n}\mu t}{\sigma}} + \cancel{\frac{\sqrt{n}\mu t}{\sigma}} + \frac{t^2}{2\sigma^2}(\mu_2' - {\mu_1'}^2) + O\left(\frac{1}{n^{1/2}}\right)$$

$$= \frac{t^2}{2\sigma^2} \cdot (\sigma^2) + O\left(\frac{1}{n^{1/2}}\right) \qquad (\sigma^2 = \mu_2' - (\mu_1')^2)$$

$$\log M_z(t) = \frac{t^2}{2} + O\left(\frac{1}{n^{1/2}}\right)$$

$$e^{\log M_z(t)} = e^{t^2/2} + e^{O\frac{1}{n^{1/2}}} \quad \text{as } n \to \infty \qquad = \frac{1}{\infty} = 0$$

$$\boxed{M_z(t) = e^{t^2/2}} \quad \text{as } n \to \infty.$$

⇒ Assumptions on CLT :-

1) r.v are independent.
2) All r.v have a common distri-
3) mean & v(x) both exist & finite
4) mean & v(x) of r.v are equal.

we can make use of the following

Corollaries to CLT,

a) $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ asymptotically

b) $\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ asymptotically where $\bar{x} = \frac{S_n}{n}$.

c) $S_n \sim N(n\mu, n\sigma^2)$ asymptotically

d) $\frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$ asymptotically.

upper limit & lower limit ?

(prob) by Chebyshev's ::
(exam — a)

(Exam)
a) Chebyshev's inequality → full.
b) converging (prob) → statement
c) weak law ~ → full
d) central limit → T.

1) Let x have pdf $f(x) = \frac{1}{2\sqrt{3}}, \quad -\sqrt{3} < x < \sqrt{3}$
$= 0$, otherwise

Show by Chebyshev's in $P\{|x| \geq 3/2\}$ has the upper bound $4/9$ whereas the true value $1 - \frac{\sqrt{3}}{2}$

(true value → [illegible] value not by Cheb.)

A) $P\left\{|x| \geq \frac{3}{2}\right\}$

we know,
$P(-a \leq x \leq a) = P(|x| \leq a)$
$\geq \rightarrow \leq$
$1 - P(|x| \leq a)$
(prob) → $\int f(x)\,dx$.

$= 1 - P\{|x| \leq 3/2\}$

$= 1 - P\left\{-\frac{3}{2} \leq x \leq \frac{3}{2}\right\}$

$= 1 - \int_{-3/2}^{3/2} f(x)\,dx$.

$= 1 - \int_{-3/2}^{3/2} \frac{1}{2\sqrt{3}}\,dx = 1 - \frac{1}{2\sqrt{3}} \int 1 \cdot dx$.

$x + x = 2x$

$= 1 - \frac{1}{2\sqrt{3}} (x)_{-3/2}^{3/2} = 1 - \frac{1}{2\sqrt{3}} \left[\frac{3}{2} + \frac{3}{2}\right]$

$\frac{3}{2} + \frac{3}{2} = \frac{2 \cdot 3}{2} = 3$

$= 1 - \frac{1}{2\sqrt{3}} \cdot 3 = 1 - \frac{3}{2\sqrt{3}}$

$3 = \sqrt{3} \cdot \sqrt{3}$

$= 1 - \frac{\sqrt{3} \cdot \cancel{\sqrt{3}}}{2\cancel{\sqrt{3}}} = 1 - \frac{\sqrt{3}}{2}$

To find Chebyshev's, we need $E(x)$ & $V(x)$. We have $f(x)$, we can find $E(x)$ & $V(x)$.

$E(x) = \int_{-\sqrt{3}}^{\sqrt{3}} x \cdot f(x)\,dx. \quad (-\sqrt{3} < x < \sqrt{3})$

$= \int_{-\sqrt{3}}^{\sqrt{3}} x \cdot \frac{1}{2\sqrt{3}}\,dx = \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} x \cdot dx$.

$= \frac{1}{2\sqrt{3}} \left[\frac{x^2}{2}\right]_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{2\sqrt{3}} \cdot \frac{1}{2}\left((\sqrt{3})^2 - (-\sqrt{3})^2\right)$

$= \frac{1}{2\sqrt{3}} \cdot \frac{1}{2}(3 - 3) = \frac{1}{4\sqrt{3}} \cdot 0 = 0$

$E(x) = 0$

$V(x) = E(x^2) - (E(x))^2$

$E(x^2) = \int x^2 \cdot f(x)\,dx$.

$= \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \cdot \frac{1}{2\sqrt{3}}\,dx = \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \cdot dx$.

$= \frac{1}{2\sqrt{3}} \left[\frac{x^3}{3}\right]_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{2\sqrt{3}} \cdot \frac{1}{3}\left[(\sqrt{3})^3 - (-\sqrt{3})^3\right]$

$= \frac{1}{2 \cdot 3 \cdot \sqrt{3}} \left[3\sqrt{3} + 3\sqrt{3}\right] = \frac{1}{6\sqrt{3}} \cdot 2 \cdot 3\sqrt{3}$

$= \frac{1}{\cancel{6\sqrt{3}}} \cdot \cancel{6\sqrt{3}} = 1$

$V(x) = 1 - 0^2 = 1$

$\therefore SD = \sqrt{V(x)} = \sqrt{1} = 1 \qquad \sigma = 1$

by Chebyshev's in

$P\{|x - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$

$P\{|x-0| \geq k\} \leq \frac{1}{k^2}$

$P\{|x| \geq k\} \leq \frac{1}{k^2}$ — ①.

$P\{|x| \geq 3/2\} \leq \frac{1}{k^2}$

$k = \frac{3}{2}$ $\quad \frac{1}{k^2} = \frac{1}{(3/2)^2} = \frac{1}{9/4}$

$= \frac{4}{9}$

upper bound $\frac{4}{9}$ — ②

2) Suppose that a life length of an electronic device has pdf, $f(x) = e^{-x}$, $x > 0$. Determine $P\{|x-1| < 2\}$

a) Exactly & (b) approximately using Chebyshev's i.e.

$x > 0 \longrightarrow \int$.

$\mu = \sigma = ?$

$f(x) = e^{-x}$, $\quad x > 0. \longrightarrow 0 \leq x \leq \infty$.

$E(x) = \int_0^\infty x \cdot f(x)\,dx$.

$= \int_0^\infty x^{2-1} f(x)\,dx$.

$= \int_0^\infty x^{2-1} \cdot e^{-x}\,dx$.

$= \frac{\sqrt{2}}{1^2} = \frac{1\sqrt{1}}{1} = 1$

$\int e^{-mx} \cdot x^{p-1}\,dx = \frac{\sqrt{p}}{m^p}$

$e^{-1x} \cdot x^{2-1}\,dx = \frac{\sqrt{2}}{1^2} = \sqrt{2}$

$\sqrt{n} = (n-1)\sqrt{n-1}$

$\sqrt{2} = (2-1)\sqrt{1}$

$= 1$

$V(x) = E(x^2) - (E(x))^2$

$E(x^2) = \int x^2 \cdot f(x)\,dx$.

$= \int_0^\infty x^2 \cdot e^{-x}\,dx = \int_0^\infty x^{3-1} \cdot e^{-x}\,dx$.

$= \int_0^\infty x^{3-1} \cdot e^{-x}\,dx$.

$\int_0^\infty e^{-mx} \cdot x^{p-1}\,dx = \frac{\sqrt{p}}{m^p}$

$\int_0^\infty e^{-1x} x^{3-1}\,dx = \frac{\sqrt{3}}{1^3} = \sqrt{3}$

$= \sqrt{3} = 2\sqrt{2} = 2$

$\sqrt{2} = 1$

$V(x) = 2 - 1 = 1$

$\therefore \sigma = 1$

a) $P\{|x-1| < 2\} = ?$

$= P\{-2 < x-1 < 2\}$

$= P\{-2+1 < x < 2+1\} = P\{-1 < x < 3\}$

$x > 0$ — ②.

$P(x) = \int_a^b f(x)\,dx$

$= \int_0^3 e^{-x}\,dx = \left[-e^{-x}\right]_0^3$

(exact value) $= \left[e^{-3} + e^0\right] = \left[e^{-3} + 1\right] = 1 - e^{-3}$

$\boxed{\text{Suppose } P\{|x-1| \leq 2\} \ldots\ P\{|x-1| \geq 2\} \ldots\ 1 - P\{|x-1| < 2\}.}$

(b) by Cheb. i.e.

$P\{|x-\mu| < k\sigma\} \geq 1 - \frac{1}{k^2}$ $\quad \mu = 1$, $\sigma = 1$

$P\{|x-1| < k\} \geq 1 - \frac{1}{k^2}$ — ①

Q → $P\{|x-1| < 2\}$ — ②

compare ① & ②

$k = 2$, $\quad \frac{1}{k^2} = \frac{1}{2^2} = \frac{1}{4}$

$1 - \frac{1}{4} = \frac{4-1}{4} = 3/4$

$\therefore P\{|x-1| < 2\} \geq \frac{3}{4}$

3) Let x be a r.v values $-1, 0, 1$ with (resp) (prob) $\frac{1}{8}, \frac{6}{8}, \frac{1}{8}$. find using chebyshev's in (prob) $P\{|x| \geq 1\}$ & upperbound of the (prob)

A) $-1, 0, 1$

$\frac{1}{8}, \frac{6}{8}, \frac{1}{8}$.

$E(x) = \sum x \cdot p(x)$

$= -1 \times \frac{1}{8} + 0 \times \frac{6}{8} + 1 \cdot \frac{1}{8}$

$= -\frac{1}{8} + 0 + \frac{1}{8} = 0$

$V(x) = E(x^2) - (E(x))^2$

$E(x^2) = \sum x^2 \cdot p(x)$.

$= (-1)^2 \times \frac{1}{8} + 0^2 \cdot \frac{6}{8} + 1^2 \cdot \frac{1}{8}$

$= \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$

$V(x) = \frac{1}{4} - 0 \quad = \frac{1}{4}$ $\therefore \sigma = \sqrt{\frac{1}{4}} = \frac{1}{2}$

By che.. in,

$P\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$

$P\{|x| \geq \frac{k}{2}\} \leq \frac{1}{k^2}$ — ①

Q → $P\{|x| \geq 1\}$ — ②

① & ②

$\frac{k}{2} = 1 \quad \therefore k = 1 \times 2 \Rightarrow k = 2$

$\frac{1}{k^2} = \frac{1}{4}$

$P\{|x| \geq 1\} \leq \frac{1}{4}$

4) for geometric distri.. $f(x) = 2^{-x}$, $x = 1, 2, 3 \ldots$ (dev) P.T chebyshev's in.. gives $P\{|x-2| \leq 2\} > \frac{1}{2}$ while actual (prob) is $\frac{15}{16}$?

A) $f(x) = 2^{-x} = \frac{1}{2^x}$

$E(x) = \sum x \cdot f(x)$

$\frac{1}{2^x}$ — $= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + \ldots$

$= \frac{1}{2} + 2 \cdot \frac{2}{2^2} + \frac{3}{2^3} + \ldots$

$= \frac{1}{2}\left[1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2^2}\right]$

$\boxed{1 + 2x + 3x^2 \ldots = \frac{1}{(1-x)^2}}$

$= \frac{1}{2}\left[\frac{1}{(1-\frac{1}{2})^2}\right]$

$= \frac{1}{2} \cdot \frac{1}{1/4}$

$= \frac{1}{2} \cdot \frac{1}{1} \cdot \frac{4}{1} = \frac{4}{2} = 2$

$V(x) = E(x^2) - (E(x))^2$

$\cancel{E(x^2) = \sum x^2 p(x)}$

$\cancel{= 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2^2} + 3^2 \cdot \frac{1}{2^3} + \ldots}$

$\cancel{= \frac{1}{2} + 4 \cdot \frac{1}{4} + 9 \cdot \frac{1}{8} + \ldots}$

$\cancel{= \frac{1}{2}\left[1 + 4 \cdot \frac{1}{2} + 9 \cdot \frac{1}{4}\right]}$

$E(x^2) = E(x(x-1) + x)$

$= E(x(x-1)) + E(x)$.

$= \sum x(x-1) \cdot f(x) + 2$.

$= \left[0 + 2 \cdot 1 \cdot \frac{1}{2^2} + 3 \cdot 2 \frac{1}{2^3} + \ldots\right] + 2$

$= \left[\frac{1}{2} + 3 \cdot \frac{1}{2^2} + \ldots\right] + 2$

$= \frac{1}{2}\left[1 + 3\cdot\frac{1}{2} + 6\left(\frac{1}{2}\right)^2 + \cdots\right]$

$= \frac{1}{2}\left[\frac{1}{(1-x)^3}\right] + 2 \quad = \frac{1}{2}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^3} + 2.$

$\cancel{= \frac{1}{2}\cdot\frac{1}{8} + 2 = \frac{1}{16} + 2 = \frac{1+2}{16}}$

$\cancel{= \frac{3}{16}}$

$\left(1-\frac{1}{2}\right)$
$= \frac{2-1}{2} = \frac{1}{2}$
$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$V(x) = 6 - 2^2 = 2$

$\frac{1}{2}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^3} + 2$

$1 - \frac{1}{2}$
$\frac{1\times 2}{1\times 2}$
$= \frac{2-1}{2} = \left(\frac{1}{2}\right)^3$

$\frac{1}{2}\cdot\frac{1}{\frac{1}{8}} + 2 = \frac{1}{2}\cdot 8 + 2$

$\frac{8}{2} + 2 = 4 + 2 = 6$

$V(x) = 6 - 2^2 = 2$

$\therefore \sigma = \sqrt{2}$

by c.i

$\cancel{P\{|x-2| \le k\sigma\} \ge \frac{1}{k^2}}$

$P\{|x-2| \le \sqrt{2}k\} \ge 1 - \frac{1}{k^2}$

$a \to P\{|x-2| \le 2\} > \frac{1}{2}$

$2 = \sqrt{2}\cdot\sqrt{2}.$

$\sqrt{2}k = 2$, $\quad k = \frac{2}{\sqrt{2}}$

$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

$k = \sqrt{2}$

$P\{|x-2| \le 2\} = 1 - \frac{1}{2}$

$P\{|x-2| \le 2\} = \frac{1}{2}$

actual (prob),

$P\{|x-2| \le 2\} = P\{-2 \le x-2 \le 2\}$

$= P(0 \le X \le 4)$

$= P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$

$= \frac{8+4+3+1}{16} = \frac{15}{16}$