04: Infinite Series

$a_i = \frac{1}{2^{i-1}}$ $\quad \Sigma a_i = ?$

$a_1 = \frac{1}{2^{1-1}} = 1$

$a_2 = \frac{1}{2}$

$1 + \frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2^{i-1}}$

AP → Arithmetic progression

$(1, 2, 3 \ldots)$ $S_n = \frac{n}{2}(2a + (n-1)d)$

GP

$S_n = \frac{a(1-r^n)}{1-r}$

(1.5)

→ Infinite series & partial sums:-

Let sequence $\{a_n\}$ be a sequence of real no. then sum of the Infinite no. of terms of these sequence $a_1 + a_2 + \cdots + a_n + \cdots$ is defined as an Infinite series, denoted by

$$\sum_{i=1}^{\infty} a_i \quad \text{or} \quad \Sigma a_i,$$

$a_i \rightarrow$ the i^{th} term of series

Let $a_1 + a_2 + \cdots a_i + \cdots$ be an I.S. If S_n denote sum of the 1^{st} n terms of this series, we have

$$S_n = a_1 + a_2 + \cdots + a_n.$$

sequence $\{S_n\}$ → sequence of partial sum of given series.

eg → consider the series $\sum a_i$, $a_i = \frac{1}{2^{i-1}}$

$a_1 = \frac{1}{2^{1-1}} = \frac{1}{2^0} = \frac{1}{1} = 1$

$a_2 = \frac{1}{2}$

$a_3 = \frac{1}{2^{3-1}} = \frac{1}{2^2} = \frac{1}{4}$ $\qquad$ a → 1st term ↓ 1.

⋮

$\boxed{S = \text{eq} \to \frac{a(1-r^n)}{1-r}}$

$= \frac{1\left(1-\frac{1}{2}^n\right)}{1-\frac{1}{2}} = \frac{1-\frac{1}{2}^n}{\frac{1\times 2-1}{2}} = \frac{1-\frac{1}{2}^n}{\frac{1}{2}}$

$= 1-\frac{1}{2}^n \times \frac{2}{1} = 2\left(1-\frac{1}{2}^n\right)$

$\sum_{k}^{\infty} a_k = -1+1-1+\dots(-1)^k+\dots$

Q) $a_k = (-1)^k = S_n = \sum_{k=1}^{n} a_k = \begin{cases} -1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$

Q) write down the 1st 4 partial sum of follow series.

1) $1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \dots$ $\qquad$ (S_n = partial sum)

A) $S_1 = 1$

$S_2 = \frac{1\times 7}{1\times 7} + \frac{1}{7} = \frac{7+1}{7} = \frac{8}{7}$

$S_3 = \frac{1\times 49}{1\times 49} + \frac{1\times 7}{7\times 7} + \frac{1}{49}$ = ~~$\frac{7}{49}$~~ = ~~$\frac{7}{49}$~~

$= \frac{49+7+1}{49} = \frac{57}{49}$

$S_4 = \frac{1\times 343}{1\times 343} + \frac{1\times 49}{7\times 49} + \frac{1\times 7}{49\times 7} + \frac{1}{343}$

$= \frac{343+49+7+1}{343} = \frac{400}{343}$

2) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

$S_1 = 1$

$S_2 = \frac{1\times 2}{1\times 2} - \frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$

$S_3 = \frac{1\times 6}{1\times 6} - \frac{1\times 3}{2\times 3} + \frac{1\times 2}{3\times 2} = \frac{6-3+2}{6} = \frac{5}{6}$

$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$= \frac{5\times 4}{6\times 4} - \frac{1\times 6}{4\times 6} = \frac{20-6}{24} =$ ~~$\frac{18}{24}$~~ $\frac{14}{24}$

$= \frac{7}{12}$

~~3) $\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1}{3}, \frac{2}{6}, \frac{3}{9} \dots$, $S_1 = \frac{1}{3}$, $S_2 = \frac{1\times 2}{3\times 2} + \frac{2}{6} = \frac{2+2}{6} = \frac{4}{6}$~~

~~$S_3 = \frac{1}{3} + \frac{2}{6} + \frac{3}{9} = \frac{4\times 3+3}{6\times 3} = \frac{12+3}{9} = \frac{15}{9}$~~

(P.Sum → partial Sum)

→ Convergence and Divergence :-

Let $\sum a_i$ be a series of real no. with partial sum S_n $a_1+a_2 \dots a_n$. If the limit sequence S_n of P.sums convergence to a limit S, the series $\sum a_i$ cvg. and that the sum of series is S.

$\boxed{\begin{aligned} S_n &= a_1 + a_2 + \dots a_n \\ \lim_{n\to\infty} S_n &= S. \end{aligned}}$

→ cvg

→ Dvg.

If the sequence of P.Sum of the series does not cvg, → Dvg.

Q/

1) Discuss the cvg of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} + \cdots$$

A) $a_k = \frac{1}{k(k+1)}$ ~~add~~ $+k$ and $-k$ in numrtr.

$$= \frac{k+1-k}{k(k+1)} = \frac{\cancel{k+1}}{k\cancel{(k+1)}} - \frac{\cancel{k}}{\cancel{k}(k+1)}$$

split

$$a_k = \frac{1}{k} - \frac{1}{k+1}.$$

$a_2 = \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$ (split)

$a_1 = \frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2}$

$a_3 = \frac{1}{3\cdot 4} = \frac{1}{3} - \frac{1}{4}$

$$S_n = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} \cdots + \frac{1}{k} - \frac{1}{k+1} \cdots \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n+\cancel{1}-\cancel{1}}{n+1} = \frac{n}{n+1} \rightarrow \div \Rightarrow \frac{n/n}{n/n + 1/n} = \frac{1}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty} = \frac{1}{1} = 1$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \quad \text{(or)} \quad \rightarrow \frac{\infty}{\infty+1} \qquad \lim_{n\to\infty} S_n = 1$$

2) S.T the series $1 + \frac{1}{2} + \frac{1}{2^2} \cdots$ cvgs

3) S.T the series $1^2 + 2^2 + 3^2 \cdots n^2 + \cdots$ Dvgs.

→ Geometric series :- (GS)

$$S_n = \frac{a(1-r^n)}{1-r}.$$

* If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

$$\boxed{\lim_{n\to\infty} S_n = \frac{a}{1-r}} \quad \boxed{= \sum_{n=1}^{\infty} ar^{n-1}}$$

* If $|r| > 1$, then $|r|^n \longrightarrow \infty$ then G.S dvgs.

eg → G.S $a = \frac{1}{9}$, $r = \frac{1}{3}$

$$\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{\frac{1}{9}}{2/3}$$

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}}$$

$$= \frac{1}{\cancel{9}_3} \cdot \frac{\cancel{3}}{2} = \frac{1}{6}$$

Q) $a = -5$, $r = -\frac{1}{2}$.

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

$$\sum_{n=1} -5\cdot\left(-\frac{1}{2}\right)^{n-1} = \frac{-5}{1+\frac{1}{2}} = \frac{-5}{3/2} = \frac{-5\times 2}{3} = \frac{-10}{3}$$

Q) $\sum_{n=1}^{\infty} \frac{n}{3^n}$ write 1st 4 partial sums ?.

A) $\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots \frac{k}{3^k}.$

$\therefore S_1 = \frac{1}{3}$

$S_2 = \frac{1\times3}{3\times3} + \frac{2}{9} = \frac{3+2}{9} = \frac{5}{9}$

$S_3 = \frac{1}{3} + \frac{2}{9} + \frac{3}{27} = \frac{5\times3}{9\times3} + \frac{3}{27}$

$= \frac{15+3}{27} = \frac{18}{27} = \frac{2}{3}$

$S_4 = \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81}$

$= \frac{2\times27}{3\times27} + \frac{4}{81} = \frac{54+4}{81} = \frac{58}{81}$

Q) S.T series $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$ cvg.

A) $S_n \Rightarrow 1 + \frac{1}{2} + \frac{1}{4} + \cancel{\frac{1}{8}} \cdots$

by GP, $\frac{\frac{1}{2}}{1} + \frac{\frac{1}{4}}{\frac{1}{2}} + \cancel{\ } \cdots$

$\frac{1}{2} + \frac{1}{2} + \cancel{\ } \cdots$

$\sum_{n=1}^{\infty} ar^{n-1}$

$\rightarrow S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(\frac{1}{2}^{k-1}\right)$

$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \frac{1}{2^{n-1}}$

$S_n = \frac{a(1-r^n)}{1-r}$

$= \frac{1-(1/2)^n}{1-1/2} = \frac{1-\frac{1}{2}^n}{\frac{1}{2}} = 1 - \frac{1}{2^{n}} \cdot 2$

$= 2 \cdot 1 - \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(2 - \frac{1}{2^{n-1}}\right) = 2$

$\{S_n\}$ cvg to 2 and it's cvgnt & sum is 2

Q) S.T $1^2 + 2^2 + 3^2 + \cdots n^2 + \cdots$ Dvg.

A) $S_n = \sum_{k=1}^{n} a_k = 1^2 + 2^2 + 3^2 + \cdots n^2$

$= \frac{n(n+1)(2n+1)}{6}$

Clearly $\{S_n\}$ is an rising sequence which tends to ∞ as n tends to ∞ hence the given series Dvg.

1) You wish to draw ~~dollars~~ 10 thousand out of a SIS bank acc at 865 and there after at age 65 and there after you want to draw $\frac{5}{6}$ as much each yr as the preceding one, Assuming that the acc earns no intrest, how much money must you start with to be prepared for and arbitrary large lifespan?

A) $S_n = 10{,}000 \cancel{\left(\frac{5}{6}\right)} + 10{,}000 \cdot \frac{5}{6} + 10{,}000 \left(\frac{5}{6}\right)^2 + 10{,}000 \left(\frac{5}{6}\right)^3 \cdots$

$\therefore a = 10{,}000$

$r = \frac{5}{6}$

If $|r| < 1$, $\therefore S = \frac{a}{1-r} = \frac{10{,}000}{1-\frac{5}{6}} = \frac{10{,}000}{1/6}$

$\frac{5}{6} < 1$

$= 60{,}000$

2) Express the repeating decimal 5.232323... as the ratio of 2 integrs.

2) $5.232323\ldots$

$23/10000$

$23 \to 0.23 \to \frac{23}{100}$.

$5+\frac{23}{100}+\frac{23}{10,000}+\frac{23}{100,000}+\cdots$

Common $\to$ $5+\frac{23}{100}\left[1+\frac{1}{100}+\frac{1}{10000}+\cdots\right]$ — ①

GP $a=1$, $r=\frac{1}{100}$

$|r|<1 \to r=\frac{1}{100}<1$

$\therefore S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{100}}$

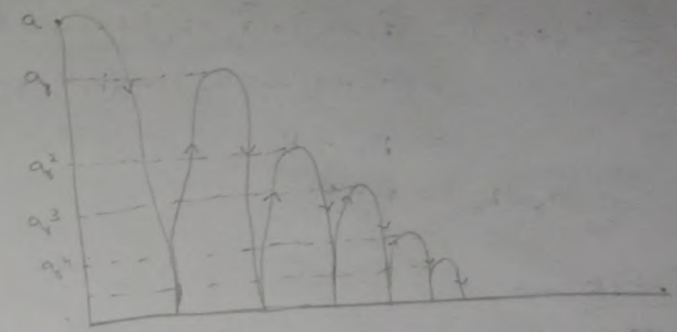
$=\frac{1}{\frac{99}{100}}$, $S=\frac{100}{99}$ — ②

Rule ② in ①,

$5+\frac{23}{100}\times\frac{100}{99} = 5+\frac{23}{99}$

$=\frac{518}{99}$

3) A ball is droped from a height of 4m. Each time it strikes the pavement after falling from a height of h m it rebounds to the height of $0.75\,h$ m. Find the total vertical distance the ball travels up and down.



→ Theorems of convergence :-

1) Algebraic rule :-

If $\sum_{k=1}^{\infty} a_k$ & $\sum_{k=1}^{\infty} b_k$ cgs $\to$ one of limit. and.

$\sum_{k=1}^{\infty} a_k = A$ & $\sum_{k=1}^{\infty} b_k = B$. then,

(1) $\sum_{k=1}^{\infty} a_k+b_k$ cgs and $\sum_{k=1}^{\infty} a_k+b_k$

$=\sum_{k=1}^{\infty} a_k+\sum_{k=1}^{\infty} b_k = A+B$ (Sum rule)

(2) $\sum_{k=1}^{\infty} Ca_k$ cgs and $\sum_{k=1}^{\infty} Ca_k = C\sum_{k=1}^{\infty} a_k = CA.$ (C. multiple rule)

proof $\sum_{k=1}^{\infty} ca_k = ca_1+ca_2+\cdots ca_n$.

$= c\,(a_1+a_2+\cdots a_n)$

$\sum_{k=1}^{\infty} ca_k = cA_n$.

let $\{A_n\}$ $\{B_n\}$ be partial sums of $\sum_{k=1}^{\infty} a_k$ $\sum_{k=1}^{\infty} b_k$

$A_n = a_1+a_2+\ldots$

$B_n = b_1+b_2\ldots$

$\{A_n\}$ $\{B_n\}$ cgs so $A_n \to A$, $B_n \to b$.

(i) $\sum_{k=1}^{\infty}(a_k+b_k) = (a_1+b_1)+(a_2+b_2)+\cdots(a_n+b_n)$
$= (a_1+a_2 \cdots a_n) + (b_1+b_2 \cdots b_n)$
$= A_n + B_n$

$\sum_{k=1}^{\infty}(a_k+b_k) = A+B$

Rmrk →
1) Every non-zero multiple of a dvg series divergence.
2) If $\sum a_n$ cvgs & $\sum b_n$ dvgs then $\sum a_n + b_n$ and $\sum a_n - b_n$ both dvg.

→ Therum → i^{th} term test :-
If $\sum a_i$ cvgs then $\boxed{\lim_{i\to\infty} a_i = 0}$.

Pf/ ~~$\sum a_i$~~ let S_i denote the i^{th} partial sum of given series, then
$S_i = a_1 + a_2 + \cdots a_{i-1} + a_i$
$a_i = S_i - S_{i-1}$.
$\sum a_i$ cvg, $\lim_{i\to\infty} a_i = \lim_{i\to\infty} S_i - \lim_{i\to\infty} S_{i-1}$.
$= S - S$
$= 0$

$\{S_n\}$ cvg,
$\lim_{n\to\infty} S_n = S$.

→ The i^{th} term test for dvgs:-
If $\lim_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^{\infty} a_i$ dvgs

(1) $\sum_{i=1}^{\infty} i^2$ dvg, $a_i = i^2 \to \infty$.
(2) $\sum_{i=1}^{\infty} \frac{i+1}{i}$ dvg, bcz $\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{i+1}{i}$
$= \lim_{i\to\infty} \frac{i}{i} + \frac{1}{i}$
$= \lim_{i\to\infty}\left(1+\frac{1}{i}\right) = 1+0$
$= 1 \neq 0$

(3) $\sum_{i=1}^{\infty} (-1)^{i+1}$ dvgs, bcz $\lim_{i\to\infty} a_i = \lim_{i\to\infty} (-1)^{i+1}$ does not exist.

(4) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ dvgs bcz
$\lim_{n\to\infty} \frac{-n}{2n+5} = \lim \frac{n(-1)}{n(2+\frac{5}{n})}$
$= \lim_{n\to\infty} \frac{-1}{2+5/n} = \frac{-1}{2} \neq 0$

3.A) Total vertical distance travelled by ball is $a + 2ar + 2ar^2 + 2ar^3 \cdots$ which after the 1^{st} term is a geometric series with 1^{st} term $2ar$ & common ratio r. Hence total vertical distance travelled by the ball is,
$S = a + 2ar + 2ar^2 + 2ar^3 + \cdots$
$= a + \frac{2ar}{1-r} = a\,\frac{1+r}{1-r}$

$a = 4m \quad r = 0.75 = \frac{3}{4} < 1$.
∴ total (dis) travelled by the ball is,

$= 4 \cdot \frac{1 + 3/4}{1 - 3/4} = 28\ m$

5) find the sum of the following series

$\sum_{i=0}^{\infty} \frac{3^i - 2^i}{6^i}$ (split)

A) $\sum_{i=0}^{\infty} \frac{3^i}{6^i} - \frac{2^i}{6^i}$

$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i - \left(\frac{1}{3}\right)^i = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i - \sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i$

$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}}$

$= \frac{1}{1/2} - \frac{1}{2/3}$

$= 2 - \frac{3}{2} = \frac{4-3}{2} = \frac{1}{2}$

6) $\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=1}^{\infty} \frac{2^n}{3^n} - \frac{1}{3^n}$

$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

$\Rightarrow S_n = \frac{a}{1-r}$ $\quad \frac{1}{2} < 1$

$= \frac{2/3}{1-\frac{2}{3}}$

$= \frac{2/3}{\frac{3-2}{3}}$

$= \frac{2/3}{1/3} = \frac{2}{3} \cdot \frac{3}{1} = 2$

$\frac{1}{3} < 1 \rightarrow S_n = \frac{a}{1-r} = \frac{1/3}{1-\frac{1}{3}} = \frac{1/3}{\frac{3-1}{3}} = \frac{1/3}{2/3}$

$= \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$ — ②

① − ② $\quad 2 - \frac{1}{2} = \frac{3}{2}$

7) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$

$(-1)^{1+1} \frac{3}{2} = (-1)^2 \frac{3}{2} = \frac{3}{2}$

$(-1)^{2+1} \frac{3}{2^2} = -1 \cdot \frac{3}{4} = -\frac{3}{4}$

$(-1)^{3+1} \frac{3}{2^3} = 1 \cdot \frac{3}{8} = \frac{3}{8}$

$\frac{3}{2},\ -\frac{3}{4},\ \frac{3}{8} \ldots$

$r = \frac{-3}{4} \cdot \frac{2}{3} = -\frac{1}{2}$

$S_n = \frac{a}{1-r} = \frac{3/2}{1-(-1/2)} = \frac{3/2}{1+1/2} = \frac{3/2}{\frac{2+1}{2}} = \frac{3/2}{3/2} = 1$

8) S.T following series Dvgs.

$1\frac{1}{2} + 3\frac{3}{4} + 7\frac{7}{8} + 15\frac{15}{16} + \ldots$

A) $\left(2 - \frac{1}{2}\right) + \left(4 - \frac{1}{4}\right) + \left(8 - \frac{1}{8}\right) \ldots$

$\sum_{i=1}^{\infty} 2^i - \left(\frac{1}{2}\right)^i$

The series $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$ is a GP with $r = \frac{1}{2} < 1$ and hence is cvg.

Now consider the series $\sum_{i=1}^{\infty} 2^i - \left(\frac{1}{2}\right)^i$ if it were cvgnt we could add to it the cvgnt series $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i$

the result cvg by sum rule.

the resulting series is, (add $\sum(\frac{1}{2})^i$)

$\sum_{i=1}^{\infty} 2^i - (\frac{1}{2})^i + (\frac{1}{2})^i$ $\quad \sum 2^i \to 2+4+8+\ldots$ $\quad a=2$, $r=2$ $\quad |r|<1$

$\sum_{i=1}^{\infty} 2^i \to$ Dvg.

which is a GP with common ratio $2>1$ i.e cvgnt.

$\therefore$ the series $\sum_{i=1}^{\infty} 2^i - (\frac{1}{2})^i$ is Dvgnt.

6) S.T the series $1+2+3+4+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^4}+\ldots$ is cvgnt & find its sum ?

A) $1+2+3+4+\sum_{i=1}^{\infty}(\frac{1}{4})^i$

$\sum_{i=1}^{\infty}(\frac{1}{4})^i$ $\quad a=\frac{1}{4}$ $\quad r=\frac{1}{4}<1$

$S_n = \frac{a}{1-r} = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3}$

$\therefore 1+2+3+4+\frac{1}{3}$

$S_n \Rightarrow 10+\frac{1}{3} = \frac{30+1}{3} = \frac{31}{3} \to$ cvg

~~GP are always cvgnt~~

7) S.T following series Dvgs. (HM-series)

$\sum_{i=1}^{\infty}\frac{1}{i} = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{i}$

$\to$ Test for cvgnt of a series :-

1) Integral test :-

let sequence $\{a_i\}$ be a seq. of +ve terms, suppose that $a_i = f(i)$ where $f(x)$ is continuous +ve, using ( ) & x, $\forall$ $x \geq 1$

$\int_1^{\infty} f(x)\,dx$ cvgs so $\sum a_i$ cvgs.

$\int_1^{\infty} f(x)\,dx$ dvgs so $\sum a_i$ dvgs

Q) using integral test P.T the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\ldots$ cvgs.

A) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\ldots\ldots$ $\quad f(i)=a_i$

$f(i) = \sum_{i=1}^{\infty}\left(\frac{1}{i}\right)^2 = \sum_{i=1}^{\infty}\frac{1}{i^2}$ $\quad f(x)=\frac{1}{x^2}$

$f(i) = \frac{1}{i^2} = a_i$ $\quad f(i)=\frac{1}{i^2}=a_i$

$\int_1^{\infty}\frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$.

$= \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$

$= \lim_{b\to\infty}\left[-\frac{1}{b}+1\right] = \lim_{b\to\infty}\left[1-\frac{1}{b}\right]$

$= 1-\frac{1}{\infty} = 1-0 = 1$

$\therefore$ cvgs

Q) S.T the P-series,

$$\sum_{i=1}^{\infty} \frac{1}{i^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} \cdots + \frac{1}{i^p},$$

where P is a real constnt $\to$ cvgs if $P>1$ & dvg if $P \le 1$.

A) $f(x) = \frac{1}{x^p}$ $\qquad f(i) = \frac{1}{i^p}$

$P>1$ — cvg — ①
$P \le 1$ — dvg — ②
$P=1$ — dvg — ③

$\int_1^{\infty} f(x)\,dx = $ cvg.

$P>1$

$$= \int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$
$$= \lim_{b\to\infty} \left[x^{-p}\right]_1^b$$
$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$
$$= \frac{1}{-p+1} \lim_{b\to\infty} \left[x^{-p+1}\right]_1^b$$
$$= \frac{1}{-p+1} \lim_{b\to\infty} \left[b^{-p+1} - 1\right]$$
$$= \frac{1}{-p+1} \left[\infty - 1\right]$$
$$= \frac{1}{-p+1} \left[-1\right]$$

$P>1$, Let's take value $p=2$

$$= \frac{1}{-2+1}[-1] = \frac{1}{-1}(-1) = 1$$

$P>1 \to$ cvg

$P>1$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$
$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$
$$= \frac{1}{-p+1} \lim_{b\to\infty} \left[x^{-p+1}\right]_1^b$$
$$= \frac{1}{-p+1} \lim_{b\to\infty} \left[b^{-p+1} - 1\right] = \infty \to \text{dvg}$$

$P=1$ $\quad \frac{1}{-p+1} \lim_{b\to\infty}\left[b^{-p+1} - 1\right] = \infty \to$ dvg.

H.W Q) S.T series, $\sum_{m=2}^{\infty} \frac{1}{m(\ln m)^p}$, $P>0$.
cgs if $p>1$, dgs if $0<P\le 1$.

* Rmrk
The error in approximating P series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ by its n^{th} partial sum, $\sum_{n=1}^{N} \frac{1}{n^p}$ can be calculated by using the principle behind integral tst.

$$\text{Error} = \sum_{n=1}^{\infty} \frac{1}{n^p} - \sum_{n=1}^{N} \frac{1}{n^p} = \sum_{n=N+1}^{\infty} \frac{1}{n^p} \le \int_N^{\infty} \frac{1}{x^p}\,dx$$

$$\boxed{\text{Error} = \frac{1}{(P-1)N^{P-1}}}$$

* It is not that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ use this eq to cal $\frac{\pi^2}{6}$ with an error < 0.05.

$\frac{1}{20} = 0.05$

$0.05 \Rightarrow \frac{1}{20}$

$\therefore N = 20 \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad n = 20.$

$1 \to 1$

$2 \to 1 + \frac{1}{4} = 1.25$

$3 \to 1 + \frac{1}{4} + \frac{1}{9} = 1.36$

$4 \to 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots \frac{1}{400} = 1.596$

→ Comparison Test :-

1) Direct comparison test :- (b_i - cvg, a_i cvg)

Let $\sum a_i$ and $\sum_{i=1}^{\infty} b_i$ be series such that $\boxed{|a_i| \le b_i}$. if $\sum_{i=1}^{\infty} b_i$ is cnvrgnt then so is $\sum_{i=1}^{\infty} a_i$

Q) S.T the following series is cvgnt

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i\,3^{i+1}}$$

A) $a_i = \sum_{i=1}^{\infty} \frac{(-1)^i}{i\,3^{i+1}}$

$b_i = \frac{1}{3^i}$

$\sum_{i=1}^{\infty} b_i = \underbrace{\sum_{i=1}^{\infty} \frac{1}{3^i}}_{GP}$

$a = 1/3$

$r = 1/3 \underline{<1} \to$ cvg.

$S = \frac{a}{1-r}$

$\sum b_i$ is cvgnt so is $\sum a_i$

(2) Ratio Comparison test :-

Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be series with $b_i > 0 \ \forall i$, if

1) $|a_i| \le b_i \ \forall i$ or if $\lim_{i\to\infty} (|a_i|/b_i) < \infty$ and $\sum b_i$ is cvgnt then $\sum a_i$ is cvgnt.

2) $a_i \ge b_i \ \forall i$ or if $\lim_{i\to\infty} \frac{a_i}{b_i} > 0$ $\sum b_i$ is dvgnt then $\sum b_i$ is dvgnt.

Rmrk

* GP $\sum_{k=1}^{\infty} ar^{k-1}$ which cvg if $|r|<1$ & dvgs if $|r| \ge 1$.
* P-series → $\sum_{k=1}^{\infty} \frac{1}{k^p}$ cgs if $p>1$, dvg if $p \le 1$.

Q 1) Test for cvg of following series →

a) $\sum_{i=1}^{\infty} \frac{1}{2^i - i}$ $\qquad a_i = \frac{1}{2^i - i} \qquad b_i = \frac{1}{2^i}$ we choose

$$\lim_{i\to\infty} \frac{a_i}{b_i} = \lim_{i\to\infty} \frac{\frac{1}{2^i - i}}{1/2^i}$$

$$= \lim_{i\to\infty} \frac{1}{2^i - i} \times 2^i$$

$$= \lim_{i\to\infty} \frac{2^i}{2^i - i} \quad (\div 2^i \text{ throughout})$$

$$= \lim_{i\to\infty} \frac{2^i/2^i}{\frac{2^i}{2^i} - \frac{i}{2^i}} = \lim_{i\to\infty} \frac{1}{1 - \frac{i}{2^i}} \quad \text{—(1)}$$

$\frac{d}{dx}(2^x)$

$\lim_{i\to\infty} \frac{i}{2^i} = \frac{\infty}{\infty}$ (L'Hopital's rule) $= \lim_{i\to\infty} 2^i \ln 2 = 0$

Sub - (1)

$\lim_{i\to\infty} \frac{1}{1-0} = 1 < \infty$

$\sum b_i = \sum_{i=1}^{\infty} \frac{1}{2^i} \rightarrow \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$

$a = 1/2 \quad r = 1/2$

2) Test for convergence of the follow series.

a) $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \frac{7}{4\cdot5\cdot6} + \ldots$

A) (n, n) → General form $\sum_{i=1}^{\infty} 2n-1$

$1 \rightarrow 2\times1-1 = 1$

$2\times2-1 = 3$

$2\times3-1 = 5$

General form $n(n+1)(n+2)$ $\Rightarrow \sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)} = a_n$

$\sum_{i=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)} = a_n$

we consider $b_n = \frac{1}{n\cdot n} = \frac{1}{n^2}$

$b_n = \frac{1}{n\cdot n} = \frac{1}{n^2}$

$\sum_{i=1}^{\infty} \frac{a_n}{b_n} = \lim \frac{\frac{2n-1}{n(n+1)(n+2)}}{1/n^2} = \lim_{i\to\infty} \frac{(2n-1)\,n^2}{n(n+1)(n+2)}$

$= \lim_{i\to\infty} \frac{2n^3-n^2}{(n^2+n)(n+2)} = \frac{2n^3-n^2}{n^3+3n^2+2n}$ $\div n^3$ throughout.

$= \lim_{i\to\infty} \frac{\frac{2n^3}{n^3} - \frac{n^2}{n^3}}{\frac{n^3}{n^3} + \frac{3n^2}{n^3} + \frac{2n}{n^3}} = \lim_{i\to\infty} \frac{2 - \frac{1}{n}}{1 + \frac{3}{n} + \frac{2}{n^2}}$

$= \frac{2}{1} = 2 > 1$

by Ratio test, $\sum a_i$ is also cvg

3) $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} \ldots$ test for the cvg.

A) we can write this as,

$\frac{1}{2^i-1} = a_i$

$i = 1 \rightarrow \frac{1}{2^1-1} = \frac{1}{1}$

$b_i = \frac{1}{2^i}$

by ratio test → $\lim_{i\to\infty} \frac{a_i}{b_i} = \frac{\frac{1}{2^i-1}}{1/2^i} = \lim_{i\to\infty} \frac{2^i}{2^i-1}$ $(\div 2^i)$

$= \lim_{i\to\infty} \frac{\frac{2^i}{2^i}}{\frac{2^i}{2^i} - \frac{1}{2^i}} = \lim_{i\to\infty} \frac{1}{1-\frac{1}{2^i}} = \frac{1}{1-0} = 1$

$b_i = \frac{1}{2^i}$

$\rightarrow \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$a = 1/2 \quad r = 1/2$

$|r| < 1$

a_i also cgnt

4) $\frac{1}{3} + \frac{\sqrt{2}}{5} + \frac{\sqrt{3}}{7} + \frac{\sqrt{5}}{9} \ldots \rightarrow a_n = \frac{\sqrt{n}}{2n+1} \quad b_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}\cdot\sqrt{n}}{2n+1}$

$= \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}} = \frac{1}{2} \neq 0$

also $\sum_{i=1}^{\infty} b_n = \sum_{i=1}^{\infty} \frac{1}{n^{1/2}}$ with $p = \frac{1}{2} < 1$ dvg

5) $\sum_{i=1}^{\infty} \sqrt{\frac{1+4^n}{1+5^n}}$

→ Ratio test :-

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r.$

$r > 1 \rightarrow$ cgs

$r < 1 \rightarrow$ dvg

$r = \infty \rightarrow$ dvg.

$r = 1 \rightarrow$ inconclusive

3) Test for cvgnt of the following series.

a) $\sum_{i=1}^{\infty} \frac{2^i}{i^8}$

A) $a_i = \frac{2^i}{i^8}$ $\quad a_{i+1} = \frac{2^{i+1}}{(i+1)^8}$

$\lim_{n\to\infty} \left|\frac{a_{i+1}}{a_i}\right| = \left|\frac{\frac{2^{i+1}}{(i+1)^8}}{\frac{2^i}{i^8}}\right|$

$a^{b+c} = a^b \cdot a^c$

$2^{i+1} \to 2^i \cdot 2^1 = 2^i \cdot 2$

$(i+1)^8 \to i^8 \cdot 1$

$= \lim_{n\to\infty} \frac{\frac{2^i \cdot 2}{(i+1)^8}}{\frac{2^i}{i^8}}$ $\quad \frac{\frac{2}{(i+1)^8}}{i^8} = \lim_{n\to\infty} \frac{2 i^8}{(i+1)^8}$

$= 2 > 1 \to$ cvg

b) $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\quad$ ($! \to$ factorial $\hookrightarrow$ 1×2×…×n.)

$a_n = \frac{1}{n!}$ $\quad a_{n+1} = \frac{1}{(n+1)!}$

$\cancel{\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \frac{1}{n!} \cdot \frac{(n+1)!}{1}}$

$\cancel{= \lim_{n\to\infty} \frac{(n+1)!}{n!}}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{(n+1)!} \cdot \frac{n!}{1}$

$\lim_{n\to\infty} \frac{n!}{(n+1)!}$ $\quad \frac{1\times2\times3\cdots\times n}{1\times2\times\cdots\times(n+1)}$

$= \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = \frac{1}{\infty} = 0$

c) $\sum_{i=1}^{\infty} \frac{b^n}{n!}$ $\quad a_n = \frac{b^n}{n!}$ $\quad a_{n+1} = \frac{b^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{b^{n+1}}{(n+1)!}}{\frac{b^n}{n!}} = \frac{b^{n+1}}{(n+1)!} \cdot \frac{n!}{b^n}$ $\quad b^n \times b$

$\cancel{\lim_{n\to\infty} \frac{n! \, (b^{n+1})}{b^n (n+1)!}} = \lim_{n\to\infty} \frac{b}{n+1} = \frac{b}{\infty+1}$

$= \frac{b}{\infty} = 0$

cvg

4) Examine the cvgnce of series

$\frac{1}{3} + \frac{1\cdot2}{3\cdot5} + \frac{1\cdot2\cdot3}{3\cdot5\cdot7} + \frac{1\cdot2\cdot3\cdot4}{3\cdot5\cdot7\cdot9} + \cdots$

A) $a_n = \frac{1\times2\times\cdots\times n}{3\times5\times7\cdots 2n+1}$ $\quad \cancel{a_{n+1} = \frac{n+1}{2(n+1)}}$

$\cancel{a_n = \frac{n}{2n+1}}$ $\quad \cancel{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$

$a_{n+1} = \frac{1\cdot2\cdot3\cdots n(n+1)}{3\cdot5\cdot7\cdots 2n+1(2n+3)}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1\cdot2\cdot3\cdots n(n+1)}{3\cdot5\cdot7\cdots(2n+1)(2n+3)}}{\frac{1\cdot2\cdot3\cdots n}{3\cdot5\cdot7\cdots 2n+1}}$

$\lim_{n\to\infty} \frac{n+1}{2n+3} \div n \lim_{n\to\infty} \frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}+\frac{3}{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}}$

$= \frac{1}{2} < 1 \rightarrow$ cvg

3) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$a_n = \frac{n^n}{n!}$ $\quad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$

$(n+1)! = (n+1)n!$

$\lim_{n\to\infty} \frac{\frac{(n+1)^n \times (n+1)}{n+1}}{n^n} = \lim_{n\to\infty} \frac{n^n \cdot 1^n \times (n+1)}{n+1}$ $\quad (n+1)^{n+1} = (n+1)^n \cdot (n+1)$

$\lim_{n\to\infty} 1^n = 1^\infty = e \rightarrow e \geq 1 =$ dvg

4) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ $\quad a_n = \frac{(2n)!}{(n!)^2}$ $\quad n! = n(n-1)$, $(2n+2)! = (2n+2)(2n+1)(2n)!$

$a_{n+1} = \frac{2(n+1)!}{((n+1)!)^2}$

(next pg)

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$\lim_{n\to\infty} \frac{\frac{2(n+1)!}{((n+1)!)^2 (n+1)^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \frac{2(n+1)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$

$= \lim_{n\to\infty} \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$

$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = \frac{4n+2}{n+1}$

$= \lim_{n\to\infty} \frac{4n+2}{n+1} \div n = \lim_{n\to\infty} \frac{4 + 2/n}{1 + 1/n} = \frac{4}{1} = 4 > 1 \rightarrow$ dvg

5) $\sum_{n=1}^{\infty} \frac{n^3+1}{2^n+1}$

6) $\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!} \Rightarrow a_n = \frac{4^n (n!)^2}{(2n)!}$

$a_{n+1} = \frac{4^{n+1}((n+1)!)^2}{2(n+1)!} = \frac{4^{n+1}((n+1)!)^2}{(2n+2)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{4^n(n!)^2} =$ $\quad (4^{n+1} = 4^n \cdot 4)$

$= \frac{4^n \cdot 4((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{4^n (n!)^2} = \frac{4(n+1)(n+1)}{(2n+1)(2n+2)} = \frac{2(n+1)}{2n+1}$

$\lim_{n\to\infty} \frac{2n+1}{2n+1} \div n = \lim_{n\to\infty} \frac{\frac{2n}{n} + \frac{1}{n}}{\frac{2n}{n} + \frac{1}{n}} = \frac{2}{2} = 1$

ratio test fails

5 A) $\sum_{n=1}^{\infty} \frac{n^3+1}{2^n+1} \Rightarrow a_n = \frac{n^3+1}{2^n+1}$ $\quad a_{n+1} = \frac{(n+1)^3+1}{2^{n+1}+1}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3+1}{2^{n+1}+1} \cdot \frac{2^n+1}{n^3+1}$

$= \frac{(n+1)^3+1}{n^3+1} \cdot \frac{2^n+1}{2^{n+1}+1}$

$= \frac{\left[\frac{n}{n} + \frac{1}{n}\right]^3 + \left(\frac{1}{n^3}\right)}{1 + \left(\frac{1}{n^3}\right)} \cdot \frac{1 + \frac{1}{2^n}}{2 + \frac{1}{2^n}}$

$\lim_{n\to\infty} \left[\frac{\left[1 + \frac{1}{n}\right]^3 + \left(\frac{1}{n}\right)^3}{1 + \frac{1}{n^3}} \cdot \frac{1 + \frac{1}{2^n}}{2 + \frac{1}{2^n}}\right] = \frac{1}{2} < 1$ cvg

4) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ $\quad a_n = \frac{(2n)!}{(n!)^2}$

$a_{n+1} = \frac{(2(n+1))!}{((n+1)!)^2}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2(n+1)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$

$n! = n(n-1)$

$2(n+1)! = (2n+2)! \to (2n+2)(2n+1)(2n)$

$(n+1)! = (n+1)\,n!$

$= \frac{(2n+2)(2n+1)(2n) \cdot (n!)^2}{((n+1)\,n!)^2\,(2n)!}$

$= \frac{(2n+2)(2n+1)\cancel{(2n)} \cdot \cancel{(n!)^2}}{(n+1)^2\,\cancel{(n!)^2}\,\cancel{(2n)!}}$

$-(n+1)^2 = (n+1)(n+1)$

$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$ $\quad n^2+1$

$= \lim_{n\to\infty} \frac{\cancel{n}(2+2/n)\cancel{n}(2+1/n)}{\cancel{n^2}(1+1/n)^2} = \lim_{n\to\infty} \frac{(2+2/n)(2+1/n)}{(1+\frac{1}{n})^2}$

$= \frac{2\cdot 2}{1} = 4 > 0$

dvg

To find error $\to$ (E)

If $\frac{a_{n+1}}{a_n} < r < 1$ $\forall n > N$ error is approximatly $\sum_{n=1}^{\infty} a_n$ by $\sum_{n=1}^{N} a_n \le |a_N| \frac{r}{1-r}$.

Q/ 1) what is the error in approximating $\sum_{n=1}^{\infty} \frac{1}{n!}$ by its 4th partial sum $\sum_{n=1}^{4} \frac{1}{n!}$

A) $a_n = \frac{1}{n!}$ $\quad a_{n+1} = \frac{1}{(n+1)!}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \frac{\frac{1}{1\times\cdots\times n(n+1)}}{\frac{1}{1\times\cdots\times n}} = \frac{1}{n+1}$

by its 4th partial sum $\to \frac{1}{n+1} = \frac{1}{4+1} = \frac{1}{5}$

$\frac{1}{n+1} < \frac{1}{5} = r$. $\quad r = \frac{1}{5}$

$N=4$ $\therefore |a_N| \frac{r}{1-r}$ $\quad a_n = \frac{1}{n!}$ $\quad n=4 = \frac{1}{4!}$ $\quad r = \frac{1}{5}$

$E \le \frac{\frac{1}{4!}\cdot\frac{1}{5}}{1-\frac{1}{5}}$

$= \frac{\frac{1}{24} \times \frac{1}{5}}{4/5} = \frac{\frac{1}{24\times5}}{4/5} = \frac{1}{24\times\cancel{5}} \cdot \frac{\cancel{5}}{4} = \frac{1}{96}$.

0.0105

$\to$ Root test :-

let $\sum a_n$ infinite series,

Suppose, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$

$r < 1 \to$ cvg.

$r > 1 \to$ dvg $\qquad r = 1 \to$ inconclusive

$r > \infty$

Q/ 1) Test follow... series —

$\sum_{n=1}^{\infty} \frac{1}{n^n}$

$a_n = \frac{1}{n^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

cvg

$\sqrt{n} = n^{1/2}$

$\sqrt[n]{\frac{1}{n^n}} = \left(\frac{1}{n^n}\right)^{1/n}$

$(2^2)^{1/2} = \sqrt{4} = 2$

$(n^n)^{1/n} = n$

2) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ $a_n = \frac{2^n}{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{(2^n)^{1/n}}{(n^2)^{1/n}} = \lim_{n\to\infty} = \frac{2}{1} = 2$$

$\cancel{2}\ 2 > 1$

dvg

$(n^2)^{1/n}$

$n^{2\cdot\frac{1}{n}} \to n^{\frac{2}{n}}$

give limit $\to n^{\frac{2}{\infty}}$

$\to n^0 = 1$

3) $\sum_{n=1}^{\infty} \frac{n^{n^2}}{(n+1)^{n^2}}$ ~~a_n~~ $a_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n^{n^2}}{(n+1)^{n^2}}\right)^{1/n} = \left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{1/n}$$

$(n^2)^{1/n} = n$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \div n$$

$$\lim_{n\to\infty} \left(\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{1^{\infty}}$$

$$= \frac{1}{e} < 1, \text{ cvg}$$

4) $\sum_{n=1}^{\infty} \left(\sqrt[n]{n}-1\right)^n = a_n = \sqrt[n]{a_n} = \sqrt[n]{n}-1$, $\lim_{n\to\infty} \sqrt[n]{a_n}$

$= \lim_{n\to\infty} (\sqrt[n]{n}-1) = 1-1 = 0 < 1$

cvg

→ Alternating series :-

An infinite series $\sum_{i=1}^{\infty} a_i$ is → alternating, if the terms a_i, a_i — are alternately +ve & -ve i.e. if absolute values $|a_i|$ are ↓sing to 0.

(eg)

condition

1) $a_1 > 0,\ a_2 < 0,\ a_3 > 0,\ a_4 < 0 \ldots$

$a_1 < 0;\ a_2 > 0,\ a_3 < 0,\ a_4 > 0 \ldots$

2) $|a_1| \geq |a_2| \geq |a_3| \geq |a_4| \ldots$

3) $\lim_{i\to\infty} |a_i| = 0.$

(1) —, +, —, + ~

eg:-

AS * $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots (-1)^{n+1}\frac{1}{n} + \ldots$

→ HM

AS * $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots \frac{(-1)^n 4}{2^n} + \ldots$ cvg.

not AS) $1 - 2 + 3 + 4 + 5 \ldots (-1)^n n + \ldots$

(not A.S)

→ Alternating series test :-

* If $\sum_{i=1}^{\infty} a_i$ is a series such that the terms $|a_i|$ alternate in sign are ↓sing in absolute value & tend to 0 then it cvgs,

* The error made in approximating an alternate series $\sum_{i=1}^{\infty} a_i$ by its n^{th} partial sum, $S_n = \sum_{i=1}^{n} a_i$ is not > than $|a_{n+1}|$.

Q/ 1) find the partial sum $\sum_{i=1}^{3} \frac{(-1)^i}{i3^{i+1}}$ & estimate difference b/w the partial sum & the entire series.

A) $\sum_{i=1}^{3} \frac{(-1)^i}{i3^{i+1}} = \frac{-1}{1\cdot 3^2} + \frac{1}{2\cdot 3^3} - \frac{1}{3\cdot 3^4}$

$= \frac{-1 \times 54}{9 \times 54} + \frac{1 \times 9}{54 \times 9} - \frac{1}{243} \times 2$

$= \frac{-54 + 9 - 2}{486} = \frac{-47}{486}$

(1) $S_n = -0.096$

alternating series test
2 condition
① $\lim_{i\to\infty} |a_i| = 0 \to$ cvg
② find error

$a_i = \frac{(-1)^i}{i3^{i+1}}$, $|a_i| = \frac{(1)^i}{i3^{i+1}} > \frac{1}{(i+1)3^{i+2}} = a_{i+1}$

$\lim_{i\to\infty} |a_i| = \lim_{i\to\infty} \frac{(1)^i}{i3^{i+1}} = \frac{1}{\infty} = 0$

(2) by alternating test,

error

$\left| \sum_{i=1}^{\infty} \frac{(-1)^i}{i3^{i+1}} - \sum_{i=1}^{3} \frac{(-1)^i}{i3^{i+1}} \right|$ $\quad \infty \xrightarrow[\text{diff.}]{} 3$

$= \left| \sum_{i=1}^{\infty} \frac{(-1)^i}{i3^{i+1}} - -0.096 \right| \le |a_4|$ $\quad n=3$, $a_n = a_3$, $a_{n+1} = a_4$

$= \frac{1}{4\cdot 3^5} = 0.001$

∴ conclude that the sum of the entire series lies in the interval

$[-0.096 - 0.001, -0.096 + 0.001]$

$[-0.097, -0.095]$

→ Absolute cvgs & conditional cvgs →

let $\sum_{n=1}^{\infty} a_n$ be a series of real no. then

a) If $\sum_{n=1}^{\infty} |a_n|$ cvgs, we say that, $\sum_{n=1}^{\infty} a_n$ cgs absolutely

b) If $\sum_{n=1}^{\infty} a_n$ cvgs, but $\sum_{n=1}^{\infty} |a_n|$ cvgs conditionally

→ Theorem → Every absolutely cvgs series is cvgs.

let $\sum a_n$ be an absolutely cvg series. now by definition, the series $\sum |a_n|$ for any the integer n

$a_n \le |a_n|$ so

$-|a_n| \le a_n \le |a_n|$

$+|a_n|$ $\quad -|a_n| + |a_n| \le a_n + |a_n| \le |a_n| + |a_n|$

$0 \le a_n + |a_n| \le 2|a_n|$

Since $\sum |a_n|$ cvgs, $\sum 2|a_n|$ cvgs & hence by comparison test, the non-ve series $\sum a_n + |a_n|$ cvgs

$\sum a_n = \sum a_n + |a_n| - |a_n|$

$= \sum a_n + |a_n| - |a_n|$ cgs

Q/ i) S.T the following series are conditionally cvg.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)}$

A) $a_1 = \frac{(-1)^0}{1} = \frac{1}{2-1} = \frac{1}{1}$

$n=2$ $\quad \frac{(-1)^{2-1}}{2\times2-1} = \frac{(-1)^1}{3} = \frac{-1}{3}$

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

$a_n = \frac{(-1)^{n-1}}{2n-1}$ $\quad |a_n| = \frac{(1)}{2n-1} > \frac{1}{2n+1} = |a_{n+1}|$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{2n-1} = \frac{1}{\infty} = 0 \to$ cvg.

$b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \frac{1}{2n-1} n$ $\quad \div n$ throughout.

$\lim_{n\to\infty} \frac{n/n}{2n/n - 1/n} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$ cvg

2) $\sum_{n=1}^{\infty} (-1)^n \sqrt{n^2+1} - n$. (P.S)

→ **power series:-** It about $x=0$, is a series of the form

power series form → $\boxed{\sum_{n=0}^{\infty} a_n x^n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots$

$x = x_0$ (centre)

$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \dots + a_n (x-x_0)^n + \dots$

($a_0, a_1, a_2 \dots a_n \to$ coefficient)

→ **Ratio test for power series :-**

let $\sum_{n=0}^{\infty} a_n x^n$ be a gvn power series assume that $\boxed{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = l}$ exist.

let $\boxed{R = 1/l}$ if $l=0$, let $R=\infty$ & if $l=\infty$, let $R=0$, then,

R → radius
l → length

1) If $|x| < R$ powerseries cvg absolutely
2) If $|x| > R$ " dvg
3) If $x = \pm R$ could. cvg / dvg.

Suppose a power series $\sum a_n x^n$ cvgs with in a finite interval, the grtest interval (centered at 0) such that the power series $\sum a_n x^n$ cvgs at every point with in it → interval of cvgnce of the series & half of the length of the interval of cvgnce → radius of cvg of series.

• If a P.S cvgs absolutely for all values of x, its R of cvgnce is infinite.

• If a P.S cvgs only at its centre & dvgs else where its R of cvgnce is 0.

→ **Root test for P.S :-**

let $\sum_{n=0}^{\infty} a_n x^n$ be a P.S, assume that $\lim_{n\to\infty} |a_n|^{1/n} = \rho$, then the R of cvgnce $R = 1/\rho$

Q/1) for wht value of x ~~diff~~ the follo. P.S cvg.

a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

$a_n x^n \to$ form

$a_n = (-1)^{n-1} \cdot \frac{1}{n}$

$a_{n+1} = (-1)^{n+1-1} \cdot \frac{1}{n+1}$

$= (-1)^n \cdot \frac{1}{n+1}$

$l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n \cdot \frac{1}{n+1}}{(-1)^{n-1} \frac{1}{n}}\right|$ (avoid -ve)

$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1}$

$= \lim_{n\to\infty} \frac{n}{n+1} \quad \div n$

$= \lim_{n\to\infty} \frac{n/n}{n/n + 1/n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1} = 1$

$l = 1 \qquad R = 1/l = 1/1 = 1$

$x = 1$ no dvg/cvg

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = (-1)^{1-1} \frac{1}{1} = 1$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$

$x^{n=2}$

Alternating HM

If $x = -1$

$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots$

negtive of HM.

$\sum_{n=0}^{\infty} a_n x^n$, cgs, whn. $-1 < x \le 1$

2) $\sum_{n=1}^{\infty} n!\, x^n$

$a_n = n!$

$a_{n+1} = n+1\,!$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1\,!}{n!}\right|$ ~~$= n+1$~~

$= \lim_{n\to\infty} n+1.$

~~$\Rightarrow n$~~ $= \infty + 1 = \infty.$

~~$l = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{} \Rightarrow \frac{1}{} = 1$~~

~~$l = 1 \qquad R = 1/l$~~

3) $\sum_{n=1}^{\infty} \frac{x^n}{2^n + 4^n}$ (4) $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\, 3^n}$

(5) Determine the R of cvgnce of P.S.

a) $\sum_{n=1}^{\infty} \frac{n^n x^n}{(2n+1)^n}$

$a_n = \frac{n^n}{(2n+1)^n}$

zero power n sensitive we can use root test.

$\lim_{n\to\infty} |a_n|^{1/n} \Rightarrow$ power cancelled.

$= \lim_{n\to\infty} \frac{n}{2n+1} \quad \div n \Rightarrow \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} = \rho$

$\rho = 1/2 \qquad R = 1/\rho = 1/\tfrac{1}{2} = 2$

R of cvgncl is 2

(6) $\sum_{k=0}^{\infty} \frac{k^5}{(k+1)!} x^k$ using ratio test.

$a_{k+1} = \frac{(k+1)^5}{(k+2)!}$

$\lim_{n\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{n\to\infty} \left[\frac{(k+1)^5}{(k+2)!} \cdot \frac{(k+1)!}{k^5}\right] = \lim_{n\to\infty} \left(1+\frac{1}{k}\right)^5$

$= \lim_{k\to\infty} \frac{1}{k+2} = 1 \cdot 0 = 0$

→ Term by term diff^n & $\int^{n}$:-

To diff^n / $\int$ a P.S within its R of cvg^nce 'R' diff / integrate term by term if $|x-x_0| < R$.

$$\frac{d}{dx}\sum_{n=1}^{\infty} a_n (x-x_0)^n \Rightarrow \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$\int \sum_{n=1}^{\infty} a_n (x-x_0)^n = \sum_{n=1}^{\infty} \frac{a(x-x_0)^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{a}{n+1}(x-x_0)^{n+1} + C$$

The resulting series cvgs if $|x-x_0|<R$.

Q/ 1) If $f(x) = \frac{1}{1-x}$ find series for $f'(x)$ & $f''(x)$

A) $f(x) = \frac{1}{1-x}$ $\qquad f'(x) = \frac{-1}{(1-x)^2}$

$$f''(x) = \frac{-1}{(1-x)^2} = -(1-x)^{-2} = -2(1-x)^{-3}$$

$$= 2 \cdot \frac{1}{(1-x)^3}$$

$$= \frac{2}{(1-x)^3}$$

Series

$f'(x)$

$1 + x + x^2 + x^3 \dots$

$\Rightarrow 1 + 2x + 3x^2 + \dots, \quad = \sum_{n=0}^{\infty} n x^{n-1}$

$f''(x)$

$2 + 6x + \dots \quad = \sum_{n=0}^{\infty} n(n-1)x^{n-1-1}$

$$= \sum_{n=0}^{\infty} n(n-1)x^{n-2}$$

HW

a) $\sum_{n=1}^{\infty} \frac{x^n}{2^n+4^n}$ $\qquad a_n = \frac{1}{2^n+4^n}$

$$a_{n+1} = \frac{1}{2^{n+1}+4^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^n+4^n}{2^{n+1}+4^{n+1}}$$

$$\lim_{n\to\infty} \frac{(2/4)^n+1}{(2/4)^n+4} = \lim_{n\to\infty} \frac{(1/2)^n+1}{2(1/2)^n+4} = \frac{1}{4}.$$

$l = 1/4 \qquad R = 1/l = 4$

cvgs absolutely, if $|x| < R = 4$ & dvg if $|x| > 4$.

when $x = 4$,

$$\sum_{n=1}^{\infty} \frac{4^n}{2^n+4^n} = \sum_{n=1}^{\infty} \frac{2^n}{1+2^n}$$

n^{th} term,

$$= \frac{2^n}{1+2^n} = \frac{1}{(\frac{1}{2})^n+1} \to 1 \text{ as } n\to\infty.$$

when $x = -4$,

$$\sum_{n=1}^{\infty} \frac{(-4)^n}{2^n+4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{2^n+4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{1+2^n}$$

n^{th} term, $\frac{(-1)^n 2^n}{1+2^n} = \frac{(-1)^n}{(\frac{1}{2})^n+1}$

The above series dvg,

$\sum_{n=1}^{\infty} a_n x^n$ cvgs for $-4<x<4$ & dvg elsewhere.

b) $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$ $\quad = a_n \qquad a_{n+1} = \frac{1}{3^{n+1}(n+1)\sqrt{n+1}}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n\sqrt{n}\,3^n}{(n+1)\sqrt{n+1}\,3^{n+1}}$

$= \lim_{n\to\infty} \frac{1}{3} \frac{n^{3/2}}{(1+n)^{3/2}} = \frac{1}{3} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-3/2}$

$= \frac{1}{3}$

$l = 1/3 \quad \underline{R=3}$

cvg absolutely by ratio test, if $|x| < R = 3$ & dvg if $|x| > 3$.

when $x = 3$,

$\sum_{n=1}^{\infty} \frac{3^n}{3^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

which is a p series with $p = \frac{3}{2} > 1$ & cvg.

when $x = -3$

$\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$

$\sum_{n=1}^{\infty} a_n x^n$ cvg for $-3 \le x \le 3$ & dvg elsewhere.

⇒ Algebraic operations on p.s :-

let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with radius of cvgs R, let $g(x) = \sum_{n=0}^{\infty} b_n x^n$, with radius of cgs S, if T is the smaller of R and S, then ~~f(x)+g(x)~~

* $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n \quad |x| < T$

* $c f(x) = \sum_{n=0}^{\infty} (c \cdot a_n) x^n \quad |x| < R.$

* $f(x) \cdot g(x) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_m b_{n-m}\right) x^n, \quad |x| < T$

* If $b_0 \ne 0$, $\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} c_n x^n \quad x \to 0$

Q 1) write down p.s of the form $\sum_{n=0}^{\infty} a_n x^n$ for $\frac{2}{3-x}$, wht are their R of their cvg?

A) $\frac{2}{3-x}$ take $\frac{3}{3}$ out.

$\frac{2}{3}\left[\frac{1}{1-\frac{x}{3}}\right] = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$

$= \frac{2}{3^{n+1}} \sum_{n=0}^{\infty} x^n \quad |x^n| < 1$

$a_n x^n$ form $\quad a_n = \frac{2}{3^{n+1}}$

$a_{n+1} = \frac{2}{3^{n+2}} \quad = \frac{3^{n+1}}{3^{n+2}}$

$l = \frac{a_{n+1}}{a_n} = \frac{2/3^{n+2}}{2/3^{n+1}} = \frac{3^{n+2}}{3^{n+1}} \wedge = \frac{3^n \cdot 3^2}{3^n \cdot 3} = \frac{1}{(3)}$

$l = \frac{1}{3}$, $R = 1/l = 3$ radius of cvgs 3

2) $\frac{5}{4-x}$ (3) $\frac{23-7x}{(3-x)(4-x)}$

4) multiply Geometric series

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots x^n + \dots = \frac{1}{1-x}$, for $|x| < 1$

by itself to get a p.s for $\frac{1}{(1-x)^2}$ for $|x| < 1$

A) $A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}$

$B(x) = \sum_{n=0}^{\infty} b_n x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$

$a_n = b_n = 1$.

$A(x) \cdot B(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$

$C_n = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m b_{n-m}\right)$

$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots = 1 + 1 + \cdots n+1$

$\frac{1}{(1-x)^2} = A(x) \cdot B(x)$

$= \sum_{n=0}^{\infty} (n+1) x^n = 1 + 2x + 3x^2 + \cdots$

Que

2.1) $\frac{5}{4-x} = \frac{5}{4}\left[\frac{1}{1-\frac{x}{4}}\right] = \sum_{n=1}^{\infty} \frac{5}{4^{n+1}} x^n$,

$\left|\frac{x}{4}\right| < 1$.

series of ratio, $a_{n+1} = \frac{5}{4^{n+2}}$.

$\frac{a_{n+1}}{a_n} = \frac{\frac{5}{4^{n+2}}}{\frac{5}{4^{n+1}}} = \frac{4^{n+1}}{4^{n+2}} = \frac{1}{4}$.

R of cvg is $\underline{4}$

3.A) $\frac{23-7x}{(3-x)(4-x)} = \frac{A}{3-x} + \frac{B}{4-x}$

$23 - 7x = A(4-x) + B(3-x)$

Putting $x=4$, $23-28 = -B$ & $B = 5$

" $x=3$, $23-21 = A$ & $A = 2$

$\therefore \frac{23-7x}{(3-x)(4-x)} = \frac{2}{3-x} + \frac{5}{4-x}$

using addition of 2 P's,

$\frac{23-7x}{(3-x)(4-x)} = \frac{2}{3-x} + \frac{5}{4-x} = \sum_{n=1}^{\infty} \frac{2}{3^{n+1}} x^n + \sum_{n=1}^{\infty} \frac{5}{4^{n+1}} x^n$

$= \sum_{n=1}^{\infty} \left(\frac{2}{3^{n+1}} + \frac{5}{4^{n+1}}\right) x^n$.

R of cvg for $\frac{2}{3-x}$ is 3 & R of cvg for $\frac{5}{4-x}$ is 4, R of cvg of their sum is min $\{3, 4\} = 3$

⇒ Taylor and maclaurin series :-

* $f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + f''(x_0)(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^2 + \cdots$

* Let f be a (?) with deri. of all orders throughout some interval containing x_0 as an interior point. Then taylor series by f at $x = x_0$ is.

$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$

* Maclaurin series generated by f is taylor series generated by f at $x=0$,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots \frac{f^{(n)}(0)}{n!} x^n + \dots$$

Taylor polynomials :-

at $x = x_0$,

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Q/
1) find Taylor. S generated by $f(x) = e^x$ at $x=1$ where it anywhere does the series cvg ?

A) $f(x) = e^x$ $\quad x_0 = 1$

$f(x) = e^x$. $\quad f(1) = e$

$f'(x) = e^x$. $\quad f'(1) = e$

$f''(x) = e^x$ $\quad f''(1) = e$

$\dots \quad \dots$

general $\to f^{(n)}(x) = e^x$. $\quad f^{(n)}(1) = e$.

$$f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots \frac{f^{(n)}(1)}{n!}(x-1)^n + \dots$$

$$= e + \frac{e}{1!}(x-1) + \frac{e}{2!}(x-1)^2 + \dots \frac{e}{n!}(x-1)^n + \dots$$

$$= e\left[1 + \frac{1}{1!}(x-1) + \frac{1}{2!}(x-1)^2 - \dots + \frac{1}{n!}(x-1)^n + \dots\right]$$

$$= e \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)(x-1)^n.$$

P.S. coefficient of $(x-1)^n$ is $a_n = \frac{1}{n!}$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{(n+1)!} \cdot \frac{n!}{1}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$l = 0 \quad R = \frac{1}{l} = \infty$.

$f(x) = e^x$ at $x=1$ cvg for all values of x.

→ **cvg of Taylor's series :-**

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0) + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{n+1}(t)\, dt.$$

↳ Taylor's formula with remainder in $\int$ form

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{n+1}(c)}{(n+1)!}(x-x_0)^{n+1} \dots$$

↳ Taylor's formula with remainder in deri^n form.

→ **Taylor series test :-**

To prove that a fn $f(x) =$ its T.S,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \text{ on } I$$, it is sufficient to show,

(a) f is infinitely differentiable on I.

(b) deri^n of f grow no faster than a constant c times the powers of a constant M, that is for x in I.

$$\left|f^{(n)}(x)\right| \le CM^n, \quad n = 0, 1, 2, 3 \dots$$

→ **Some imp Taylor & maclaurin S :-**

1) Geometric $= \frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n$.

$R = 1$

2) Binomial $= (1+x)^{\alpha} = 1 + \alpha x + \dfrac{\alpha(\alpha-1)}{2!}x^2 + \dots$

$= \sum_{n=0}^{\infty} \dfrac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n$, $R = 1$

3) Sine $=$ $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$

$R = \infty$.

4) Cos $\rightarrow \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dots \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$

$R = \infty$.

5) Exponential $\rightarrow e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} \dots = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

$R = \infty$.

6) log $\rightarrow \ln x = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} + \dots$

$= \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} (x-1)^n$ $R > 1$

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} \dots = \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} x^n$

$R = 1$.

Q
1) Expand $f(x) = \dfrac{1}{(1+x)^2}$ in M. Series using this series find $f^{(5)}(0)$ & $f^{(6)}(0)$ without calculation of f directly?

A) $\dfrac{1}{(1+x)^2}$ $\rightarrow$ Geometric S

$= \dfrac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots$

$= 1 - x^2 + x^4 - x^6 + \dots$

$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$ //

cvg for all $|-x^2| < 1$ i.e. $|x| < 1$.

above series is M.S for $f(x) = \dfrac{1}{(1+x)^2}$ in $(-1,1)$

we know $\dfrac{f^{(5)}(0)}{5!}$ is coefficient of x^5 in M.S

for $f(x) = \dfrac{1}{(1+x^2)}$, since coefficient is 0,

$f^{(5)}(0) = 0$

$\dfrac{f^{(6)}(0)}{6!}$, $f(x) = \dfrac{1}{(1+x^2)}$

since coeff is $(-1)^3$,

$f^{(6)}(0) = 6! \times (-1)^3 = -6! = \underline{\underline{-720}}$