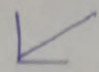
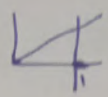
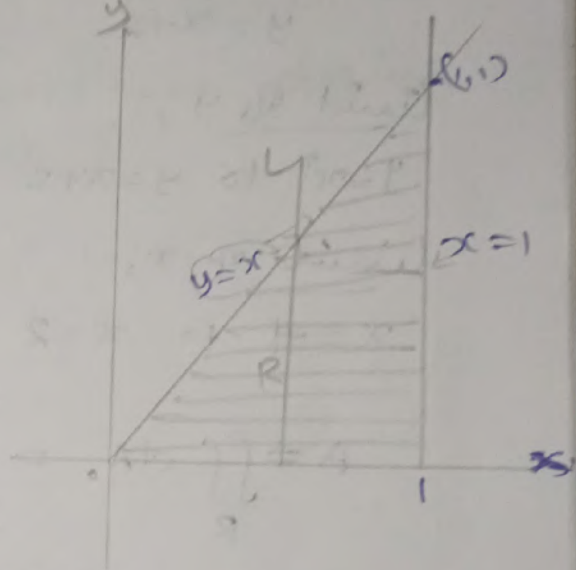
2) Find the vol of prism whose base is the triangle in the xy-plane bounded by x-axis & lines $y=x$ & $x=1$ and whose top lines in the plane $z = f(x,y) = 3-x-y$.

A) $y = x \rightarrow$ 

$x = 1 \rightarrow$ 



limit

If $y = 0$, $x = 0$

$x = 1$ $\quad \int_0^1 \int_1^x$

$$V = \iint_R f(x,y)\, dA$$

$$= \iint_R (3-x-y)\, dA$$

$$= \int_0^1 \int_1^x (3-x-y)\, dy\, dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_1^x dx$$

$$= \int_0^1 \left[3x - x^2 - \frac{x^2}{2} - \left(3 - x - \frac{1}{2}\right)\right] dx$$

$$= \int_0^1 3x - \frac{x^2}{2} - 3 + x + \frac{1}{2}$$

$$= \int_0^1 \left[3x - \frac{3x^2}{2}\right] dx$$

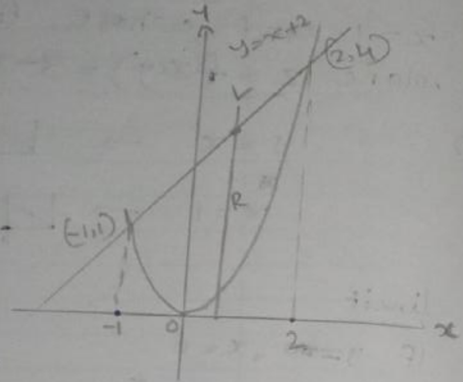
$$= \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1$$

$$x^2 - \frac{x^2}{2} = \frac{2x^2 - x^2}{2} = \frac{x^2}{2}$$

$$3 + \frac{1}{2} = \frac{6+1}{2} = 7/2$$

3) Find area of region enclosed by parabola $y = x^2$ & line $y = x+2$.

A) $A = \iint_R dA$.



$y = x^2$

$y = x+2$

limit of y,

$y = x^2$ to $y = x+2$

limit of x,

$x = -1$ to $x = 2$

$$A = \iint_R dA = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2}\left[\int_{x^2}^{x+2} dy\right]dx$$

$$= \int_{-1}^{2}[y]_{x^2}^{x+2}\,dx = \int_{-1}^{2}\left[x+2-x^2\right]dx$$

$$= \left[\frac{x^2}{2} + 2x + \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

⇒ Laminas with variable Density :-

* If ρ is a constant density (mass per unit area) then mass of lamina coinciding with a region bounded by the graphs of $y = f(x)$, the x-axis & lines $x = a$ & $x = b$,

$$m = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \rho f(x^*_k)\,\Delta x_k.$$

$$\boxed{m = \int_a^b \rho f(x)\,dx}$$

* If a lamina to a region R has a variable density $\rho(x,y)$, ρ is non-ve & continuous on R, then we define its mass m by double $\int$.

(mass). $m = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \rho f(x^*_k, y^*_k)\,\Delta A_k$

$$\boxed{m = \iint_R \rho(x,y)\,dA}$$

* Center of mass of lamina :

with x coordinates, $\boxed{\bar{x} = \frac{M_y}{m}}$

with y " , $\boxed{\bar{y} = \frac{M_x}{m}}$

M_y & M_x are moments,

$$\boxed{\begin{aligned} M_y &= \iint_R x\rho(x,y)\,dA \\ M_x &= \iint_R y\rho(x,y)\,dA \end{aligned}}$$

$\int$-s M_x & M_y also → 1st moments of a lamina about x & y-axes.

2nd moments of a lamina | moments of inertia about x & y-axes, → (I_x), (I_y)

$$\boxed{\begin{aligned} I_x &= \iint_R y^2 \rho(x,y)\, dA \\ I_y &= \iint_R x^2 \rho(x,y)\, dA \end{aligned}}$$

* **Kinetic energy :-**

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m (r\omega)^2$$

$$v = r\omega$$ (cir)

$$= \frac{1}{2} (m r^2) \omega^2$$

$I = mr^2$. moment of inertia

$$\boxed{K = \frac{1}{2} I \omega^2}$$

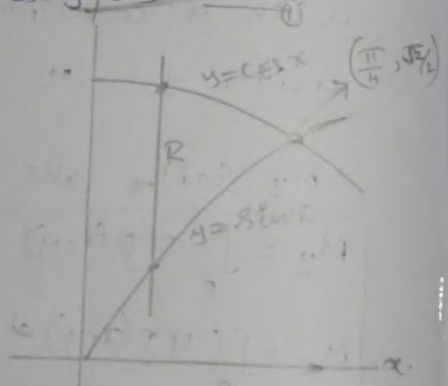
1) A lamina has the shape of the region in 1st quadrant that is bounded by graphs of $y = \sin x$, $y = \cos x$, b/w $x = 0$ & $x = \pi/4$. Find the center of mass if density is $\rho(x,y) = y$?

A) $y = \sin x$

$x = 0$

$y = \cos x$.

$x = \pi/4$.



$$\int_0^{\pi/4} \int_{\sin x}^{\cos x}$$

$$\boxed{m = \iint_R \rho(x,y)\, dA} \to$$ ~~center of mass~~

$$= \int_0^{\pi/4} \int_{\sin x}^{\cos x} y \cdot dy\, dx = \int_0^{\pi/4} \left[\frac{y^2}{2}\right]_{\sin x}^{\cos x}$$

$$= \frac{1}{2} \int_0^{\pi/4} \left[\cos^2 x - \sin^2 x\right] dx.$$

$$= \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx.$$

$$\boxed{\cos^2 x - \sin^2 x = \cos 2x}$$

$$= \frac{1}{2} \left[\frac{\sin 2x}{2}\right]_0^{\pi/4} \Rightarrow \frac{1}{2} \cdot \frac{1}{2} \left[\sin 2 \pi/4 - 0\right]$$

$$= \frac{1}{4} \left[\sin \frac{\pi}{2}\right] = \frac{1}{4} \cdot 1 = \frac{1}{4}$$

M_x, M_y ?

$$\boxed{M_x = \iint_R y\, \rho(x,y)\, dA}$$ → ① ⇒ $y \times y$ dydx

$$= \int_0^{\pi/4} \int_{\sin x}^{\cos x} y^2 \, dy\, dx.$$

$$= \int_0^{\pi/4} \left[\frac{y^3}{3}\right]_{\sin x}^{\cos x} dx. = \frac{1}{3} \int_0^{\pi/4} \left[y^3\right]_{\sin x}^{\cos x} dx.$$

$$= \frac{1}{3} \int_0^{\pi/4} \left[\cos^3 x - \sin^3 x\right] dx.$$

$\cos^2 x \to 1 - \sin^2 x$
$\sin^2 x \to 1 - \cos^2 x$

$$= \frac{1}{3} \int_0^{\pi/4} \left[\cos x (1 - \sin^2 x) - \sin x (1 - \cos^2 x)\right] dx.$$

$(\cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x)$

$$= \frac{1}{3} \int_0^{\pi/4} \cos x - \sin^2 x \cos x - \sin x + \sin x \cos^2 x.$$

$$= \frac{1}{3} \left[\int_0^{\pi/4} \cos x \, dx - \int \sin^2 x \cos x \, dx - \int \sin x \, dx + \int \sin x \cos^2 x \, dx.\right]$$

$$= \frac{1}{3} \left\{ \left[\sin x\right]_0^{\pi/4} - \frac{1}{3}\left[\sin^3 x\right]_0^{\pi/4} - \left[-\cos x\right]_0^{\pi/4} + \frac{1}{3}\left[-\cos^3 x\right]_0^{\pi/4} \right\}.$$

$\cos x \to \sin x$
$\sin^2 x \cos x \to \frac{\sin^3 x}{3}$
$\sin x \to -\cos x$.
$\sin x \cos^2 x \to -\frac{\cos^3 x}{3}$

$= \frac{1}{3}\left\{ \left[\frac{1}{\sqrt2} - 0\right] - \frac{1}{3}\left[\frac{1}{2\sqrt2} - 0\right] - \left[1 - \frac{1}{\sqrt2}\right] + \frac{1}{3}\left[1 - \frac{1}{2\sqrt2}\right] \right\}$

$= \frac{1}{3}\left\{ \frac{1}{\sqrt2} - \frac{1}{6\sqrt2} - 1 + \frac{1}{\sqrt2} + \frac{1}{3} - \frac{1}{6\sqrt2} \right\}$. (cm)

$= \dfrac{6 - 1 - 6\sqrt2 + 6 + 2\sqrt2 - 1}{18\sqrt2} \Rightarrow \dfrac{10 - 4\sqrt2}{18\sqrt2}$

$\therefore M_x = \dfrac{5\sqrt2 - 4}{18}$

$\boxed{M_y = \iint_R x\,\rho(x,y)\,dA} = \int_0^{\pi/4}\int_{\sin x}^{\cos x} xy\,dy\,dx$. $\rho = \rho(x,y) = y$

$= \int_0^{\pi/4}\left[x \cdot \frac{y^2}{2}\right]_{\sin x}^{\cos x} dx = \frac{1}{2}\int_0^{\pi/4}\left[x\,y^2\right]_{\sin x}^{\cos x} dx$.

$= \frac{1}{2}\int_0^{\pi/4}\left[x \cdot (\cos^2 x - \sin^2 x)\right] dx$.

$\boxed{\cos^2 x - \sin^2 x = \cos 2x}$ $= \frac{1}{2}\int_0^{\pi/4}\left[x \cdot \cos 2x\right] dx$.

(p.rule) $\rightarrow = \frac{1}{2}\left[x \cdot \frac{\sin 2x}{2}\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \sin 2x\,dx$.

integrating by parts.

$= \frac{1}{2}\left\{\frac{\pi}{8} - \left[\frac{-1}{4}\cos 2x\right]_0^{\pi/4}\right\}$

$= \frac{1}{2}\left\{\frac{\pi}{8} - \frac{1}{4}\right\} = \dfrac{\pi - 2}{16} \Rightarrow \frac{1}{2}\left[\frac{4\pi - 8}{32}\right] \Rightarrow \frac{4\pi - 8}{64}\frac{\div 4}{\div 4} = \frac{\pi - 2}{16}$

hence coordinates of center of mass,

$\bar{x} = \dfrac{M_y}{m} = \dfrac{(\pi - 2)/16}{1/4} = \dfrac{\pi - 2}{4}$

≈ 0.29

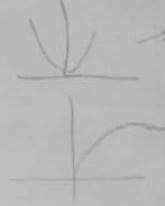
$\bar{y} = \dfrac{M_x}{m} = \dfrac{(5\sqrt2 - 4)/18}{1/4} = \dfrac{10\sqrt2 - 8}{9}$

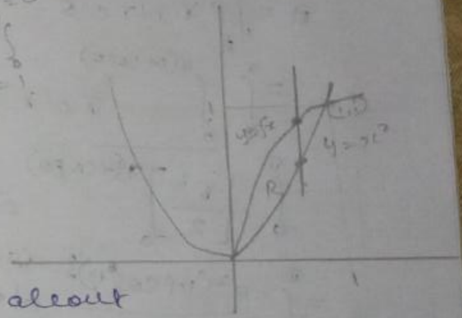
≈ 0.68

coordinates $(0.29, 0.68)$

2) find moment of inertia about x-axis of lamina that has slope of region bounded by graphs of $y = x^2$ & $y = \sqrt{x}$ & density $\rho(x,y) = x^2$?

A).





moment of inertia about x-axis. $\rightarrow$

$I_x = \iint_R y^2 \rho(x,y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} y^2 x^2\,dy\,dx$.

$= \int_0^1 x^2\left[\frac{y^3}{3}\right]_{x^2}^{\sqrt{x}} dx = \frac{1}{3}\int_0^1 x^2\left[x^{3/2} - x^6\right] dx$.

$y^3 \rightarrow (\sqrt{x})^3 \rightarrow (x^{1/2})^3 = x^{3/2}$

$y^3 \rightarrow (x^2)^3 \Rightarrow x^6$

$= \frac{1}{3}\int_0^1\left[x^{7/2} - x^8\right] dx$.

$x^2 \cdot x^{3/2}$, $x^{\frac{4+3}{2}}$, $\Rightarrow x^{7/2}$

$= \frac{1}{3}\left[\frac{x^{9/2}}{9/2} - \frac{x^9}{9}\right] = \frac{1}{3}\left[\frac{2}{9} - \frac{1}{9}\right] = \frac{1}{27}$

⇒ **Area in polar coordinates :-**

The area of a closed bounded R in the polar coordinates,

$$\boxed{A = \iint_R r \, dr \, d\theta}$$

polar coordinates

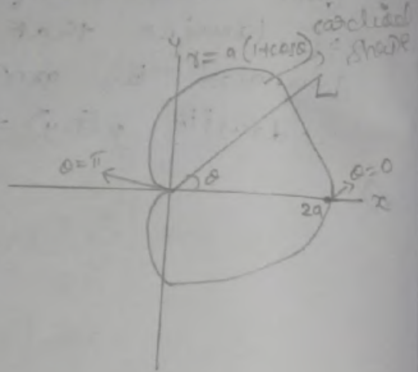
want to evaluate $\iint_R f(r,\theta) \, dA$.

$$\boxed{\iint_R f(r,\theta) \, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{g_2(\theta)} f(r,\theta) r \, dr \, d\theta}$$

Q) 1) find area enclosed by cardioid

$r = a(1+\cos\theta)$

A) $x = \pi \to 0$.



$A = \iint_R r \, dr \, d\theta$.

$= \int_0^{\pi} \int_0^{a(1+\cos\theta)} r \, dr \, d\theta$.

$= \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$.

$= \int_0^{\pi} \frac{a^2(1+\cos\theta)^2}{2} d\theta$. $(a+b)^2$. $= \int_0^{\pi} \frac{a^2 + (\cos^2\theta + 2\cos\theta + 1)}{2} d\theta$.

$= a^2 \int_0^{\pi} \frac{\cos^2\theta + 2\cos\theta + 1}{2} d\theta = a^2 \int_0^{\pi} \frac{\cos^2\theta}{2} + \frac{\cancel{2}\cos\theta}{\cancel{2}} + \frac{1}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$= a^2 \int_0^{\pi} \left[\frac{1+\cos 2\theta}{2 \times 2} + \cos\theta + \frac{1}{2}\right] d\theta = a^2 \int_0^{\pi} \frac{1+\cos 2\theta}{4} + \cos\theta + \frac{1}{2}$

$= a^2 \left[\frac{3\theta}{4} + \frac{\sin 2\theta}{8} + \sin\theta\right]_0^{\pi} = a^2\left[\frac{3\pi}{4} + 0 - 0\right]$

$= \frac{3\pi a^2}{4}$

∴ Area enclosed by cardioid $=$

$2 \times R = \cancel{2} \cdot \frac{3\pi a^2}{\cancel{4}_2} = \frac{3\pi a^2}{2}$

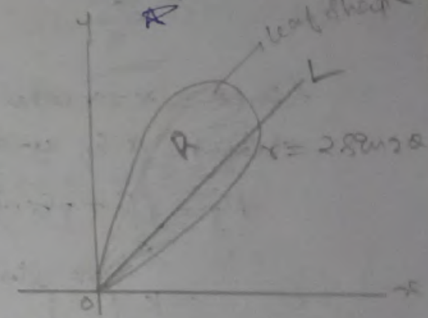
2) find center of mass of lamina that corresponds to the region bounded by 1 leaf of rose $r = 2\sin 2\theta$ in the 1st quadrant if the density at a point P in the lamina is directly proportional to the distance from pole?

A) y limits

$0 \to 2\sin 2\theta$

x limits

$0 \longrightarrow \frac{\pi}{2}$.



$m = \iint_R \rho(x,y) \, dA = \iint_R k|r| \, dA$

$= k \int_0^{\pi/2} \int_0^{2\sin 2\theta} r \cdot r \, dr \cdot d\theta$.

$= k \int_0^{\pi/2} \int_0^{2\sin 2\theta} r^2 \cdot dr \, d\theta = k \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\sin 2\theta} d\theta$.

$= \frac{k}{3} \int_0^{\pi/2} 8\sin^3 2\theta \cdot d\theta = \frac{8k}{3} \int_0^{\pi/2} (1-\cos^2 2\theta)\sin 2\theta \, d\theta$.

Let $\cos 2\theta = u$.

$-2\sin 2\theta \, d\theta = du$.

$\sin 2\theta \, d\theta = -du/2$

Limits
$\theta = 0 \quad , u = 1$
$\theta = \pi/2 \quad , u = \cos\pi = -1 \quad .$

$$\therefore, m = \frac{8k}{3}\int_1^{-1}(1-u^2)\left(-\frac{du}{2}\right)$$
$$= \frac{4k}{3}\int_1^{-1}(1-u^2)\,du .$$
$$= \frac{4k}{3}\left[u - \frac{u^3}{3}\right]_1^{-1} = \frac{4k}{3}\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$$
$$= \frac{16k}{9}.$$

Since $x = r\cos\theta$ & $y = r\sin\theta$, M_x & M_y about x & y-axes,

$$M_y = \iint_R x\,\rho(x,y)\,dA = \iint_R r\cos\theta \cdot k|r|\,dA.$$
$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^2\cos\theta \cdot r\,dr\cdot d\theta$$
$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^3\cos\theta\,dr\,d\theta$$
$$= k\int_0^{\pi/2}\cos\theta\left[\frac{r^4}{4}\right]_0^{2\sin 2\theta} d\theta.$$
$$= 4k\int_0^{\pi/2}\sin^4 2\theta\cos\theta\,d\theta .$$
$$= 4k\int_0^{\pi/2}(2\sin\theta\cos\theta)^4\cos\theta\,d\theta.$$
$$= 64k\int_0^{\pi/2}\sin^4\theta\cos^5\theta\,d\theta.$$
$$= 64k\int_0^{\pi/2}\sin^4\theta(1-\sin^2\theta)^2\cos\theta\,d\theta.$$
$$= 64k\int_0^{\pi/2}\left[\sin^4\theta - 2\sin^6\theta + \sin^8\theta\right]\cos\theta\,d\theta$$

Sub. $\sin\theta = u$.
$\cos\theta\,d\theta = du$
$\sin 0 = 0$, $\sin\pi/2 = 1$ } limit

$$= 64k\int_0^1\left[u^4 - 2u^6 + u^8\right]du$$
$$= 64k\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right]$$
$$= 64k\left[\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right] = \frac{512}{315}k.$$

$$M_x = \iint_R y\,\rho(x,y)\,dA = \iint_R r\sin\theta\cdot k|r|\,dA$$
$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^2\sin\theta\cdot r\cdot dr\,d\theta.$$
$$= k\int_0^{\pi/2}\int_0^{2\sin 2\theta} r^3\sin\theta\,dr\,d\theta.$$
$$= k\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\sin 2\theta}\sin\theta\cdot d\theta.$$
$$= 4k\int_0^{\pi/2}\sin^4 2\theta\sin\theta\,d\theta.$$
$$= 4k\int_0^{\pi/2}(2\sin\theta\cos\theta)^4\sin\theta\,d\theta$$
$$= 64k\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta.$$
$$= 64k\int_0^{\pi/2}\cos^4\theta(1-\cos^2\theta)^2\sin\theta\,d\theta.$$
$$= 64k\int_0^{\pi/2}\left[\cos^4\theta - 2\cos^6\theta + \cos^8\theta\right]\sin\theta\,d\theta$$

$\cos\theta = u.$
$-\sin\theta\,d\theta = du$
$\sin\theta\,d\theta = -du$

$$= 64k\int_0^1(u^4 - 2u^6 + u^8)\,(du)$$
$$= 64k\left[\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right]_0^1$$
$$= 64k\left[\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right] = \frac{512}{315}k$$

Centres of mass,

$\bar{x} = \frac{M_y}{m} = \frac{512k/315}{16k/9}$

$= \frac{32}{35}$

$\bar{y} = \frac{M_x}{m} = \frac{512k/315}{16k/9} = \frac{32}{35}$

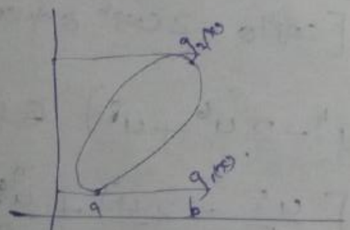
Coordinates $\left(\frac{32}{35}, \frac{32}{35}\right)$

⇒ Green's Theorem in the Plane:-

Suppose that 'C' is a piecewise smooth simple closed curve ($\oint$) bounding a simply connected region 'R', if $P, Q, \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ are contin- on R.

then,

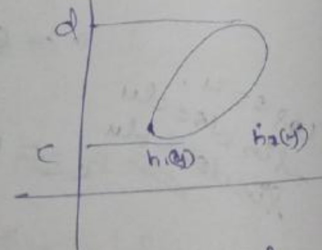
$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$



Type I

$a \le x \le b$

$g_1(x) \le y \le g_2(x)$



Type II

$c \le y \le d$

$h_1(y) \le x \le h_2(y)$

$\iint_R \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\, dy\, dx$

$\boxed{\text{Type I}}$

$\boxed{\int \frac{\partial P}{\partial y}\, dy \quad P(x,y)}$

$= \int_a^b \left[P(x,y)\right]_{g_1(x)}^{g_2(x)} dx$

$= \int_a^b P(x, g_2(x)) - P(x, g_1(x))\, dx$

$= -\int_a^b P(x, g_1(x))\, dx + \int_a^b P(x, g_2(x))\, dx$

$= -\oint_C P(x, g(x))\, dx = -\oint P(x,y)\, dx$ — ①

$\boxed{\text{Type II}}$

$\iint_R \frac{\partial Q}{\partial x}\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x}\, dx\, dy$

$= \int_c^d \left[Q(x,y)\right]_{h_1(y)}^{h_2(y)} dy$

$= \int_c^d \left[Q(h_2(y), y) - Q(h_1(y), y)\right] dy$

$= -\int_c^d Q(h_1(y), y) + Q(h_2(y), y)\, dy$

$= \oint_C Q(x,y)\, dy$ —— ②

from ① & ②

$\Rightarrow \iint_R \frac{\partial P}{\partial y}\, dA - \iint_R \frac{\partial Q}{\partial x}\, dA$

$= -\oint P(x,y)\, dx - \oint Q(x,y)\, dy$

$\Rightarrow \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA = \iint_R \left(Q(x,y) - P(x,y)\right) dA$

$\iint_R \frac{\partial P}{\partial y} dA - \iint_R \frac{\partial Q}{\partial x} dA = \int P\,dx + \int Q\,dy$

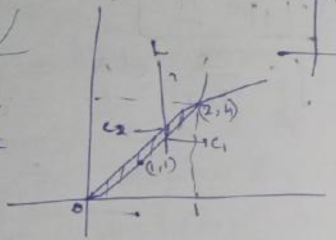
$$\int P\,dx + \int Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

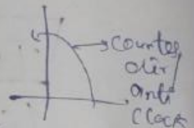
Q/ 1) Verify Green's Theorem for line $\int$ $\oint_C (xy\,dx + x^2 dy)$ where C is the curve enclosed in the region bounded by the parabola & line $y = x^2$, $y = x$ is counter (clis)

A)

C_1 — $y = x^2$
$x = 0,\ y = 0$
$x = 1,\ y = 1$
$x = 2,\ y = 4$

C_2 — $y = x$
$x = 0\ \ y = 0$
$x = 1\ \ y = 1$
$x = 2\ \ y = 2$





$C = C_1 \cup C_2$ (para) eq

on C_1 let $x = t$, $dx = dt$ (with respect to t)

$y = t^2$
$dy = 2t\,dt$ → $y = t^2$, $\frac{dy}{dt} = 2t$, $dy = 2t\,dt$

$\frac{dx}{dt} = \frac{dt}{dx}$

$0 \le t \le 1$

Green's T.
$0 \to 1$
$1 \to 0$ (oppo).

on C_2, $y = t$

$\oint_C P\,dx + Q\,dy$ $1 \le t \le 0$.

$\therefore x = t \quad y = t$
$dx = dt \quad dy = dt$.

$P = xy$.
$Q = x^2$. $C = C_1 \cup C_2$.
$\Rightarrow \int_{C_1 \cup C_2} xy\,dx + x^2 dy$

$\therefore = \int_{C_1} xy\,dx + \int x^2 dy \ \cancel{+ \int_{C_2} xy\,dx + x^2 dy}$

$= \int_0^1 xy\,dx + \int_0^1 x^2 dy + \int_1^0 xy\,dx + x^2 dy$

$= \int_0^1 t \cdot t^2\,dt + \int_0^1 t^2 \cdot 2t\,dt + \int_1^0 t \cdot t^2\,dt +$

$= \int_0^1 t^3\,dt - \int_0^1 2t^3\,dt$ $\int_1^0 \Rightarrow 0 \ldots$ so $-\int_0^1$

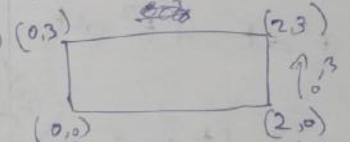
$= \left[\frac{t^4}{4}\right]_0^1 - 2\left[\frac{t^4}{\frac{4}{2}}\right]_0^1$

$= \frac{1}{4} - 0 - \left(0 - \frac{1}{2}\right) = \frac{1}{4} \cancel{\left(\frac{-1}{2}\right)}$

$= \frac{1}{4} - \frac{1}{2} = \cancel{-2}$

2) using evaluate $\oint x^2 e^y dx + y^2 e^x dy$ by Green's theorem, counter clock wise around the bounded in C, where C → ▭ with vertices (0,0) (2,0) (2,3) (0,3)

$0 \le x \le 2$
$0 \le y \le 3$

A) 

$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int P\,dx + Q\,dy$

$P = x^2 e^y$
$Q = y^2 e^x$ $\Rightarrow \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$\frac{\partial Q}{\partial x} = y^2 e^x$ $\Rightarrow \iint_R$

$\frac{\partial P}{\partial y} = x^2 e^y$ $\Rightarrow \int_0^2 \int_0^3 (y^2 e^x - x^2 e^y)\,dy\,dx$

$\Rightarrow \int_0^2 \left[\frac{y^3}{3} e^x - x^2 e^y\right]_0^3 dx.$

$$= \int_0^2 \left[\frac{27}{3} e^x - x^2 e^3 + x^2\right] dx$$

$$= \int_0^2 9e^x - x^2 e^3 + x^2 \, dx = \left(9e^x - \frac{x^3 e^3}{3} + \frac{x^3}{3}\right)_0^2$$

$$= 9e^2 - \frac{8e^3}{3} + \frac{8}{3} - \frac{9}{1}$$

$$= 9e^2 - \frac{8e^3}{3} - \frac{19}{3}$$

① → continuous.

$$\oint_{C_1} p\,dx + q\,dy \qquad 1 \le t \le 0$$

$$P = xy \qquad C = C_1 \cup C_2$$
$$Q = x^2$$

$$= \int_{C_1 \cup C_2} xy\,dx + x^2\,dy$$

$$= \int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy$$

Sub

$$= \int_0^1 t \cdot t^2\,dt + t^2 \cdot 2t\,dt + \int_1^0 t \cdot t\,dt + t^2\,dt$$

$$= \int_0^1 t^3\,dt + 2t^3\,dt + \int_1^0 t^2\,dt + t^2\,dt$$

$$= \int_0^1 t^3 + 2t^3\,dt + \int_1^0 t^2 + t^2\,dt$$

$$= \int_0^1 3t^3\,dt + \int_1^0 2t^2\,dt$$

$$\cancel{= 3\left[\frac{t^4}{4}\right]_0^1 + 2\left[\frac{t^3}{3}\right]_1^0 = \frac{3}{4} + \frac{2}{3}}$$

$$= \int_0^1 3t^3\,dt - \int_0^1 2t^2\,dt$$

$$\left(\frac{3t^4}{4}\right)_0^1 \times \left(\frac{2t^3}{3}\right)_0^1 \times \left(0 - \frac{2}{3}\right) \times \left(-\frac{2}{3}\right)$$

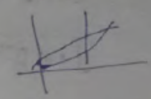
$\frac{3}{4}$ $\frac{3}{4}$ $\frac{3}{4}$ $= \frac{1}{12}$

$$= \left[3\frac{t^4}{4} - 2\frac{t^3}{3}\right]_0^1$$

$$= \frac{3 \times 3}{4 \times 3} - \frac{2 \times 4}{3 \times 4} = \frac{1}{12}(9 - 8) = \frac{1}{12}$$

$y = x^2$, $y = x$

$x^2 \le y \le x$.



$P = xy$ $Q = x^2$

x: $0 \to 1$

y: $x^2 \to x$.

$$\frac{\partial Q}{\partial x} = 2x \qquad \frac{\partial P}{\partial y} = x.$$

$$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_0^1 \int_{x^2}^{x} (2x - x)\,dy\,dx.$$

$$= \int_0^1 \int_{x^2}^{x} x\,dy\,dx = \int_0^1 \left[xy\right]_{x^2}^{x} dx.$$

$$\cancel{= \int_0^1 \frac{x^3}{2} - \frac{x^4}{2}\,dx} = \int_0^1 x \cdot x - x \cdot x^2\,dx.$$

$$= \int_0^1 x^2 - x^3\,dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

LHS = RHS

3) $\oint_C y^2\,dx + x^2\,dy$

C is the boundary of Δ bounded

$x = 0$, $x + y = 1$, $y = 0$ in counter clockwise

(dis)

4) $y = 2x + 1$, $-1 \le x \le 0$

$G(x,y) = 3x^2 + 6y^2$ $\int_C G(x,y)\,ds$?

4) $\int_C G(x,y)\, ds = \int_a^b G(x, f(x)) \sqrt{1+(f'(x))^2}\, dx$ — ①

$f(x) = 2x+1$ $\quad (f'(x))^2 = 4$

$f'(x) = 2$

$1+(f'(x))^2 = 1+4 = 5$

$\sqrt{1+(f'(x))^2} = \sqrt{5}$

$G(x, f(x)) = G(x, 2x+1)$

$= 3x^2 + 6(2x+1)^2 \quad \leftarrow G(x,y) = 3x^2 + 6y^2$

$= 3x^2 + 6[(2x)^2 + 2\cdot 2x \cdot 1 + 1^2]$

$= 3x^2 + 6[4x^2 + 4x + 1]$

$= 3x^2 + 24x^2 + 24x + 6.$

$= 27x^2 + 24x + 6$

→ ①

$= \int_{-1}^{0} (27x^2 + 24x + 6)\sqrt{5}\, dx.$

$= \sqrt{5}\int_{-1}^{0} (27x^2 + 24x + 6\; \cancel{dx}\; dx.$

$= \sqrt{5}\left[\cancel{27}^{9}\frac{x^3}{\cancel{3}} + \cancel{24}^{12}\frac{x^2}{\cancel{2}} + 6x \right]_{-1}^{0}$

$= \sqrt{5}\left[9x^3 + 12x^2 + 6x \right]_{-1}^{0}$

$= \sqrt{5}\left[0 - \left[(9(-1)^3 + 12(-1)^2 + 6\times(-1)\right]\right]$

$= \sqrt{5}\, [9 - 12 + 6] = \sqrt{5}\, [3]$

$= 3\sqrt{5}$

5) $x = \left(\frac{1}{3}\right) t^3$, $\quad y = t^2$, $\quad z = 2t$ $\quad 0 \le t \le 1$

$\int_C 4xyz\, dy$?

A) $f(t) = \frac{1}{3} t^3$ $\quad h(t) = 2t$ $\quad dx = f'(t)\,dt = t^2$

$g(t) = t^2$ $\quad dy = g'(t)\,dt = 2t\, dt$

$f'(t) = \frac{1}{3}\cdot 3t^2 = t^2$ $\quad dz = h'(t)\,dt = 2\, dt$

$g'(t) = 2t$ $\quad h'(t) = 2.$

$\int_C 4xyz\, dy$ $\int_C G(x,y,z)\, dx = \int_a^b G(f(t), g(t), h(t))\, g'(t)\, dt.$

$= \int_0^1 4\left(\frac{t^3}{3}\right)\cdot t^2 \cdot 2t \cdot 2t\, dt$

$= \frac{16}{3}\int_0^1 t^7\, dt \quad = 16\left[\frac{t^8}{8}\right]_0^1$

$= 2/3$