06: Change of variable.

* Change of variable technique is a method of finding the distribution of a () of a R.V.
* In many (prob) problms the form of the density () / mass () maybe complex so as to make computation difficult. This technique will provide compact discription of a distri... & it will be relatively easy to compute mean, varience, etc.
* eg → suppose that R.V x take on 3 values $-1, 0, 1$ with (prob) $\frac{11}{32}, \frac{16}{32}, \frac{5}{32}$ respectively. let us transform the R.V x taking $y = 2x+1$

→

x →	-1	0	1
$p(x)$ →	$\frac{11}{32}$	$\frac{16}{32}$	$\frac{5}{32}$

→ discrete (non)

$y = ?$ and $p(y)$?

distri ()

y			
$p(y)$			

$$y = 2x+1$$

$x = -1$

$$y = 2\times -1 + 1 = -1$$

$x = 0$ $\quad y = 2\times 0 + 1 = 1$

$x = 1$ $\quad y = 2\times 1 + 1 = 3$

$P(y)$?.

$P(y=-1) = P(2x+1=-1) = P(2x=-2)$
$= P(2x = -1-1) = P(x=-1)$
$= P$
$= 11/32$ — a.

$P(y=1) = P(2x+1=1) = P(2x=0)$
$= P(2x = 1-1) = P(x=0) = \frac{16}{32}$

$P(y=3) = P(2x+1=3) = P(2x=2)$
$= P(2x = 3-1) = P(x=1)$
$= \frac{5}{32}$

∴ Distribution ( ),

y	-1	1	3	Total
$P(y)$	$\frac{11}{32}$	$\frac{16}{32}$	$\frac{5}{32}$	1

* In the above eg. if we transform the R.V 'X' as $Y = X^2$, The possible values of y are

$Y = x^2$.

$x=1 \rightarrow y=1$

$x=0 \rightarrow y=0$

$x=-1 \rightarrow y=1$ values of $y = 0, 1$

$\therefore P(y=0) = P(x^2=0) = P(x=0) = \frac{16}{32}$ —

$P(y=1) = P(x^2=1) = P(x=1 \text{ or } x=-1)$
$= P(x=1) + P(x=-1)$
$= \frac{5}{32} + \frac{11}{32}$
$= \frac{5+11}{32} = \frac{16}{32}$

∴ Distri— ( ),

y	0	1	Total
$P(y)$	$\frac{16}{32}$	$\frac{16}{32}$	1.

1) A R.V x has density $f(x) = \frac{x+2}{6}$, $0 < x < 2$, let $g(x) = \begin{cases} 0 & 0 < x \le 1 \\ 1 & 1 < x \le 3/2 \\ 2 & x > 3/2 \end{cases}$

we can find the (prob) mass ( ) of $g(x)$ as follows.

A) $f(x) = \frac{x+2}{6}$

$g(x)$	0	1	2
$P(g(x))$			

$g(x) = 0, 1, 2.$

$P(g(x)=0) = \int_0^1 f(x) \cdot dx.$

$= \int_0^1 \frac{x+2}{6} \cdot dx = \frac{1}{6}\int_0^1 x+2 \cdot dx.$

$= \cancel{\frac{1}{2}\left[\frac{x}{2}\right]_0^1}$ $= \frac{1}{6}\left[\frac{x^2}{2} + 2x\right]_0^1$

$= \frac{1}{6}\left[\frac{1}{2} + 2 \times 1 - 0\right] = \frac{1}{6}\left[\frac{1}{2} + 2\right]$

$= \frac{1}{6} \cdot \frac{5}{2} = \frac{5}{12}$

$$P(g(x)=1) = \int_1^{3/2} \frac{x+2}{6}\, dx$$

$$= \frac{1}{6}\int_1^{3/2} x+2 \, dx = \frac{1}{6}\left[\frac{x^2}{2}+2x\right]_1^{3/2}$$

$$= \frac{1}{6}\left[\frac{(3/2)^2}{2} + \cancel{2}\cdot\frac{3}{\cancel{2}} - \left(\frac{1}{2} + 2\right)\right].$$

$$= \frac{1}{6}\left[\frac{9/4}{2} + 3 - \left(\frac{5}{2}\right)\right]$$

$$= \frac{1}{6}\left[\frac{9}{8} + \frac{3\times 8}{1\times 8} - \frac{5\times 4}{2\times 4}\right]$$

$$= \frac{1}{6}\left[\frac{9+24-20}{8}\right] = \frac{1}{6}\left[\frac{13}{8}\right]$$

$$= \frac{13}{48}$$

$0 > x > 2$

$$P(g(x)=2) = \int_{3/2}^{2} \frac{x+2}{6}\, dx.$$

$$= \frac{1}{6}\int_{3/2}^{2} x+2 \, dx = \frac{1}{6}\left[\frac{x^2}{2}+2x\right]_{3/2}^{2}$$

$$= \frac{1}{6}\left[\frac{4}{2} + 2\times 2 - \left(\frac{(3/2)^2}{2} + \cancel{2}\times\frac{3}{\cancel{2}}\right)\right]$$

$$= \frac{1}{6}\left[2+4 - \left(\frac{9}{8} + \frac{3\times 8}{1\times 8}\right)\right]$$

$$= \frac{1}{6}\left[6 - \left(\frac{9+24}{8}\right)\right]$$

$$= \frac{1}{6}\left[\frac{6\times 8}{1\times 8} - \frac{33}{8}\right] = \frac{1}{6}\left[\frac{48-33}{8}\right]$$

$$= \frac{1}{6}\times\frac{15}{8} = \frac{1}{}$$

$\therefore$ pmf.,

$g(x)$	0	1	2
$P(g(x))$	5/12	13/48	15/48

2) If X is contin R.V with pdf.

$$f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

Define $y = 3x+1$.

A) Let $G(y)$, distri ... () of y

$$\boxed{G(Y \le y) = \int_{-\infty}^{y} f(x)\,dx}$$

$G(y) \to$ distri

$g(y) \to$ density

$$G(y) = P(Y \le y)$$

$$= P(3x+1 \le y)$$

$$= P(3x \le y-1)$$

$$= P\left(x \le \frac{y-1}{3}\right)$$

min $\to 0$ $(0<x<1)$

max $\to \frac{y-1}{3}$

$$G(y) = \int_0^{\frac{y-1}{3}} f(x)\, dx. = \int_0^{\frac{y-1}{3}} 2x \, dx.$$

$$= \cancel{2}\cdot\left[\frac{x^2}{\cancel{2}}\right]_0^{\frac{y-1}{3}}$$

$$= [x^2]_0^{\frac{y-1}{3}} = \left(\frac{y-1}{3}\right)^2$$

$$= \frac{(y-1)^2}{9}$$

Relation b/w $g(x)$ & $G(x)$.

$$\boxed{g(y) = G'x}$$

$$g(y) = \frac{d}{dx}\frac{(y-1)^2}{9} = \frac{1}{9}\cdot 2(y-1)\frac{d}{dx}(y-1)$$

$$g(y) = \frac{2}{9}\, y-1$$

3) A random variable x has density $f(x) = kx^2 e^{-x^3}$, $x > 0$, determine k and density of $y = x^3$.

A) $x > 0$ interval $\rightarrow \int$

find $\boxed{k \text{ constant} \rightarrow \int_{-\infty}^{\infty} f(x)\, dx = 1}$

$= \int_0^\infty kx^2 e^{-x^3}\, dx = 1$

$= k \int_0^\infty x^2 e^{-x^3}\, dx = 1$ let $t = x^3$

$t = x^3$

$dt = 3x^2\, dx$

$\frac{dt}{3} = x^2 \cdot dx$

$\Rightarrow k \int_0^\infty e^{-t} \cdot \frac{dt}{3} = 1$

$\int e^{-x} = e^{-x}$

$\therefore \int e^{-t} = e^{-t}$

$= \frac{k}{3} \int_0^\infty e^{-t}\, dt = 1$

$= \frac{k}{3} \left[e^{-t} \cdot \frac{d}{dt}(-t) \right]_0^\infty = 1$

$= \frac{k}{3} \left[e^{-t} \cdot -1 \right]_0^\infty = 1 \Rightarrow \frac{-k}{3} \left[e^{-t} \right]_0^\infty = 1$

$= \frac{-k}{3} \left[e^{-\infty} - e^0 \right] = 1$

$e^{-\infty} = 0$

$e^0 = 1$

$= \frac{-k}{3} [0 - 1] = 1$

$= \frac{-k}{3} \cdot -1 = 1 \Rightarrow \frac{k}{3} = 1$

$\therefore k = 3$

density of $y = x^3 \rightarrow f(y)$?

$(g(x) = \ldots)$ or $\rightarrow \boxed{f(y) = f(x) \cdot \left| \frac{dx}{dy} \right|}$ $f(y)$?

$= kx^2 e^{-x^3} \cdot \left| \frac{dx}{dy} \right|$

$k = 3$

$= 3 \cancel{x^2} e^{-x^3} \cdot \frac{1}{3\cancel{x^2}}$

$y = x^3$

$\frac{dx}{dy} =$

$y = x^3$

$\frac{dy}{dx} = 3x^2$

we need $\rightarrow \frac{dx}{dy} = \frac{1}{3x^2}$

$= e^{-x^3}$

$= e^{-y}$ $y > 0$

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p a

Result $\rightarrow$

Let x be contin. R.V with pdf finite $f(x)$, let $y = g(x)$ be strictly monotone (rising (losing)) ( ) x. Assume $g(x)$ is differentiable for all x, then the pdf of R.V y is givn by,

$f(y) = f(x) \cdot \left| \frac{dx}{dy} \right|$, x is expressed then in terms of y.

1) Let x be a contin R.V with pdf $f(x)$, let $y = x^2$ find pdf and the distri. ( ) of y?

A) $G(y) = ?$ $g(y) = ?$

$g(y) = G'(y)$

$\sqrt{x} = x^{1/2}$

$(x^2)^{1/2} = x$

$$G(y) = P(Y \le y)$$
$$= P(x^2 \le y) \Rightarrow \sqrt{x} \le \sqrt{y}$$
$$-\sqrt{y} \le x \le \sqrt{y} \leftarrow = P(|x| \le \sqrt{y})$$
$$= P(-\sqrt{y} \le x \le \sqrt{y})$$

$x^2 \le 4 \Rightarrow \sqrt{4}$

x must be the +ve –ve.

eg

$|X| \le 5$

$\rightarrow -5 \le x \le 5$

$|x| < 4$

$\rightarrow -4 < x < 4$

$|x-2| < 4$

$\rightarrow -4 < x-2 < 4.$

$$\boxed{P(a \le x \le b) = F(b) - F(a)}$$

$$\Rightarrow F(\sqrt{y}) - F(-\sqrt{y})$$

$$\therefore\ g(y) = G'(y)$$
$$= \frac{d}{dy}\left(F(\sqrt{y}) - F(-\sqrt{y})\right)$$
$$= F'(\sqrt{y})\frac{d}{dy}(\sqrt{y}) - F'(-\sqrt{y})\frac{d}{dy}(-\sqrt{y})$$
$$= F'(\sqrt{y})\frac{1}{2\sqrt{y}} - F'(-\sqrt{y}) - \frac{1}{2\sqrt{y}}$$

$G'(x) = g(x)$

$F'(x) = f(x)$

$$= \frac{1}{2\sqrt{y}}F'(\sqrt{y}) + \frac{1}{2\sqrt{y}}F'(-\sqrt{y})$$
$$= \frac{1}{2\sqrt{y}}\left[F'(\sqrt{y}) + F'(-\sqrt{y})\right].$$
$$= \frac{1}{2\sqrt{y}}\left[f(\sqrt{y}) - f(-\sqrt{y})\right]$$