05: Random Variable and probability Distribution

A R.V is a real value () defined over the sample space, so its domain of definition is the sample space 'S' & range is the real line extending from $-\infty$ to $+\infty$.

In other words R.V is a mapping from sample space to real no.

In symbols $X : S \longrightarrow (-\infty, \infty)$
(from)

R.V is also called chance variable / stochastic variable. (X or $X(\omega)$).
(omega)

eg → Consider a R.V consisting of 2 tosses of a coin, then

$$S = \{HH, HT, TT, TH\}$$

Then,

(no. of head) $X = 2, 1, 0.$

eg → In coin tossing exp $S = \{\omega_1, \omega_2\}$ where $\omega_1 = \text{head}$, $\omega_2 = \text{Tail}$.

Now define $X(\omega) = \begin{cases} 0, & \text{if } \omega = T \\ 1, & \text{if } \omega = H \end{cases}$

here $x(\omega)$ takes only 2 values → 0 & 1.

Such a R.V → Bernoulli R.V

→ (Pro) distributions :-

1) ~~Discrete~~ R.V :- let X be a (dis) R.V assuming the values $x_1, x_2 \dots, x_n$ from

the real line. let the corresponding (pro) be $f(x_1), f(x_2) \dots f(x_n)$ then,

(pmf) $p(X = x_i) = f(x_i) \rightarrow$ (pro) mass ( ) /

(pro) ( ) of X, provided it satisfy the

cond~ a) ~~condn~~ $f(x_i) \geq 0$ for all i

b) $\sum f(x_i) = 1$. (Total pro) = 1)

2) Contin R.V :- If X is a contin R.V

i.e if $P(x \leq X \leq x + dx) = f(x)\, dx$, then

(pdf) $f(x) \rightarrow$ (pro) density ( ), provided

it satisfy the condi~

a) $f(x) \geq 0$ for all x.

b) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

real line

* Result :-

$P(a < X < b) = P(a < X \leq b) = P(a \leq X \leq b) =$

$P(a \leq X \leq b) = \int_a^b f(x)\, dx$ = the area under the curve $y = f(x)$ enclosed b/w $x = a$ & $x = b$.

pdf = f(x)

=> Distribution ( ) :- $(F(x))$

If X is a (dis) R.V with pmf f(x) then the cumilative distribution ( ) is defined as $\boxed{F_X(x) = P(X \leq x) = \sum_{-\infty}^{x} p(x)}$

If X is a contin~ R.V with pdf f(x), then the distri~ ( ) is defined as

$$\boxed{F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x) \cdot dx}$$

-> (pro) of distri~ ( ) :-

1) F is defined for all real value of X

2) $F(-\infty) = 0$, $F(\infty) = 1$

3) $0 \leq F(x) \leq 1$

4) $F(a) \leq F(b)$ if $a < b$,

F(x) is non ↓sing.

5) If X is (dis), $F(b) - F(a) = P(a < X \leq b)$

6) If X is conti~, $F(b) - F(a) = P(a \leq X \leq b)$

= Area under the (prob) curve.

7) $\dfrac{dF(x)}{dx} = f(x)$

☆ $\boxed{f(x)\,? \rightarrow \frac{d}{dx}F(x) = \text{we get } f(x). \quad F(x)\,? \rightarrow \int f(x) \rightarrow F(x)}$

$\rightarrow P(X \leq a) = F(a)$.

-> Moments of contin~ (pro) distri~ =

1) A M of $\boxed{X = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$ = E(X)

2) Median, M is determined by solving eq,

$\boxed{M = \int_{-\infty}^{M} f(x) \cdot dx = \frac{1}{2}}$ or $\boxed{\int_{M}^{\infty} f(x)\, dx = \frac{1}{2}}$

3) Mode (Z) ->

$\boxed{f'(x) = 0 \quad f''(x) < 0}$ i.e varifying the condi~ at the mode.

4) GM ->

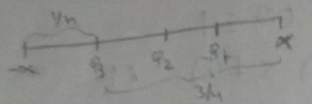
$$\boxed{\log G = \int_{-\infty}^{\infty} (\log x) \cdot f(x)\, dx}$$

5) HM $\rightarrow$

$$\boxed{\frac{1}{H} = \int_{-\alpha}^{\alpha} \frac{1}{x} f(x)\, dx}$$

6) quartiles $\rightarrow$

$$\boxed{\int_{-\alpha}^{q_1} f(x)\, dx = \frac{1}{4}}, \quad \boxed{\int_{-\alpha}^{q_3} f(x)\, dx = \frac{3}{4}}$$



7) The r^{th} central moment $\rightarrow$

$$\boxed{\mu_r = \int_{-\alpha}^{\alpha} (x-\mu)^r f(x)\, dx} \quad (r = 1, 2, \ldots)$$

8) $v(x) = \mu_2 = \int_{-\alpha}^{\alpha} (x-\mu)^2 f(x)\, dx$.

9) $MD = \int_{-\alpha}^{\alpha} |x - \text{mean}|\, f(x)\, dx$.

Q/ 1) Obtain the prob distri... of the no. of heads when 3 coins are tossed together? $p(x)$?

$2^n = 2^3 = 8$

A) 3 H together.

$S = \{HHH, TTT, HHT, HTH, HTT, THH, TTH, THT\}$.

$X \rightarrow$ no. of H.

$\therefore X = 0, 1, 2, 3$

$x=0$

$p(x=0) = P\{TTT\} = \frac{1}{8}$ — fav case / total

$x=1$ $p(x=1) = P\{HTT, THT, TTH\} = \frac{3}{8}$.

$x=2$ $p(x=2) = P\{HHT, THH, HTH\} = 3/8$.

$x=3$ $p(x=3) = P\{HHH\} = 1/8$.

$X = x$	0	1	2	3	Total
$p(X=x)$	1/8	3/8	3/8	1/8	1

((closet) [illegible] box [illegible])

2) from a lot containing 25 items 5 of which are deffectivess, 4 items are choosen at random, find the (prob) distri... of the no. of deffectiveis obtained?

A) 4 items choosen (combination) may be; 0,1,2,3,4

20 not deff, 5 - def

$X \rightarrow$ no. of deffectives.

$X : 0, 1, 2, 3, 4$.

$x=0$

$$p(x=0) = \frac{20C_4 \times 5C_0}{25C_4} = \frac{4845 \times 1}{12650} \div 5 = \frac{969}{2530}$$

$x=1$

$$p(x=1) = \frac{5C_1 \times 20C_3}{25C_4} = \frac{5 \times 1140}{12650} \div 5 = \frac{1140}{2530}$$

$x=2$

$$p(x=2) = \frac{5C_2 \times 20C_2}{25C_4} = \frac{10 \times 190}{12650} \div 5 = \frac{380}{2530}$$

$x=3$

$$P(x=3) = \frac{5C_3 \times 20C_1}{25C_4} = \frac{10 \times 20}{12650} \frac{\div 5}{\div 5} = \frac{40}{2530}$$

$x=4$

$$P(x=4) = \frac{5C_4 \times 20C_0}{25C_4} = \frac{\cancel{5} \times 1}{\cancel{12650}\ 2530} = \frac{1}{2530}$$

(Prob). distri--

$X=x$	0	1	2	3	4	Total
$P(X=x)$	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$	1

3) Examine whether $f(x)$ ~~as~~ defined below is pdf.,

$$f(x) = \begin{cases} 0, & x<2 \\ \frac{1}{18}(3+2x), & 2 \le x < 4 \to \int \\ 0, & x>4. \end{cases}$$

A) $\boxed{\text{pdf} \Rightarrow \int_{-\alpha}^{\alpha} f(x)\,dx = 1.}$

$$\int_2^4 f(x)\,dx = \int_2^4 \frac{1}{18}(3+2x)$$

$$= \frac{1}{18}\int_2^4 3+2x = \frac{1}{18}\left[3x + 2\frac{x^2}{2}\right]_2^4$$

$$= \frac{1}{18}\left[3\times4 + 2\times\frac{4^2}{2} - \left(3\times2 + 2\times\frac{2^2}{2}\right)\right]$$

$$= \frac{1}{18}\left[12 + 16 - (6+4)\right]$$

$$= \frac{1}{18}\left[28 - 10\right] = \frac{1}{\cancel{18}}\left[\cancel{18}\right] = 1$$

4) If distri-- $F(x)$ ( ), $F(x)$ is gvn,

$$F(x) = \begin{cases} \frac{2x^2}{5}, & 0<x\le1 \\ -\frac{3}{5} + \frac{2\left(3x - \frac{x^2}{2}\right)}{5}, & 1<x\le2 \\ 1, & x>2 \end{cases}$$

find density ( ) ? $f(x)$

A) $\boxed{f(x) = \frac{d}{dx}F(x)}$

$$= \frac{d}{dx}\begin{cases} \frac{2x^2}{5} \\ -\frac{3}{5} + \frac{2\left(3x - \frac{x^2}{2}\right)}{5} \\ 1 \end{cases}$$

$$= \begin{cases} \frac{2}{5}\cdot 2x, & 0<x\le1 \\ 0 + \frac{2}{5}\cdot 3 - \frac{1}{\cancel{2}}\cdot\cancel{2}x, & 1<x\le2 \\ 0, & x>2. \end{cases}$$

$$= \begin{cases} \frac{4x}{5} & 0<x\le1 \\ \frac{2}{5}(3-x) & 1<x\le2 \\ 0 & x>2 \end{cases}$$

5) Given $F(x) = \begin{cases} 0 & , x < 0 \\ x^2 & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases}$

Determine ⓐ $P(X \le 0.5)$ ⓑ $P(0.5 \le X < 0.8)$

ⓒ $P(X > 0.9)$

A) Pro)

$$\boxed{\begin{aligned} &P(X \le a) = F(a) \\ &P(a \le X \le b) = F(b) - F(a) \\ &P(X > a) = 1 - P(X \le a) = 1 - F(a) \end{aligned}}$$

$F(x) = x^2$, $0 \le x \le 1$

a) $P(X \le 0.5) = P(0.5)$

$= (0.5)^2 = 0.25$

b) $P(0.5 \le x \le 0.8) = F(0.8) - F(0.5)$

$= (0.8)^2 - (0.5)^2$

$= 0.64 - 0.25$

$= 0.39$

c) $P(X > 0.9) = 1 - F(0.9)$

$= 1 - (0.9)^2$

$= 1 - 0.81 = 0.19$

6) A R.V x has the density ( )

$f(x) = \begin{cases} k \dfrac{1}{1+x^2} & , -\infty < x < \infty \\ 0 & , \text{elsewhere} \end{cases}$

Determine k and distri – ( )

A) $f(x)$ is a pdf,

$k = ?$

$\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$\int_{-\infty}^{\infty} k \cdot \frac{1}{1+x^2}\,dx = 1.$

$\int \frac{1}{1+x^2} = \tan^{-1}x$

$k \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 1$

$k\,(\tan^{-1}x)_{-\infty}^{\infty} = 1$

$\tan^{-1}(-\infty)$
$= -\tan^{-1}\infty$
$= -\pi/2$

$k\,(\tan^{-1}\infty - \tan^{-1}(-\infty)) = 1$

$k\left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = 1$

$k\left(\frac{\cancel{2}\pi}{\cancel{2}}\right) = 1$

$k\pi = 1$

$k = \frac{1}{\pi}$

tan [illegible]

$\tan = \frac{\sin}{\cos}$

$\tan\frac{\pi}{2} = \frac{\sin \pi/2}{\cos \pi/2}$

$= \frac{1}{0} = \infty$

$\boxed{\tan\frac{\pi}{2} = \infty}$

$F(x)$?

$$\boxed{F(x) = \int_{-\infty}^{x} f(x)\,dx}$$

$F(x) = \int_{-\infty}^{x} k \frac{1}{1+x^2}\,dx.$

$= k \int_{-\infty}^{x} \frac{1}{1+x^2}\,dx.$

$= \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+x^2}\,dx.$

$= \frac{1}{\pi} \cdot \left[\tan^{-1}x\right]_{-\infty}^{x}$ $= \frac{1}{\pi}\left[\tan^{-1}x - \tan^{-1}(-\infty)\right]$

[illegible]

$F(x) = \frac{1}{\pi}\left[\tan^{-1}x + \frac{\pi}{2}\right]$

6) Let x be a R.V with (prob) density ( ) $f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ -ax+3a & 2 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$

a) Determine the constant a ?

b) compute $P(X \le 1.5)$?

A) $f(x)$ is pdf.

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. → for finding constant

$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax+3a)\,dx + 0 = 1.$

$= \left[a\frac{x^2}{2}\right]_0^1 + [ax]_1^2 + \left[-a\frac{x^2}{2} + 3ax\right]_2^3 = 1$

$= a\frac{1}{2} + 2a - a + \left[-a.\frac{9}{2} + 9a - (-a.2 + 6a)\right] = 1$

$= \frac{1}{2}a + a - \frac{9}{2}a + 9a + 2a - 6a = 1$

$= \cancel{\left[\frac{1}{2}a - \frac{9}{2}a\right] + [a + 9a] + [2a - 6a] = 1}$

$= -4a + 6a = 1 \Rightarrow 6a - 4a = 1$

$\cancel{4a - 6} \Rightarrow 2a = 1$

$a = \frac{1}{2}$

b) $P(X \le 1.5) = \int_{-\infty}^{1.5} f(x)\,dx$. a, $1 \le x \le 2$.

$= \int_0^1 ax\,dx + \int_1^{1.5} a\,dx$.

$= \int_0^1 \frac{1}{2}x\,dx + \int_1^{1.5} \frac{1}{2}\,dx$.

$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^{1.5}$

$= \frac{1}{2}\left[\frac{1}{2}\right] + \frac{1}{2}[1.5 - 1]$

$= \frac{1}{4} + \frac{1}{2}[0.5]$ $\cancel{= \frac{2}{4} = \frac{1}{2}}$

$= \frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

7) The (prob) mass ( ) of R.V x —

x	0	1	2	3	4	5
$p(x)$	k^2	$\frac{k}{4}$	$\frac{5k}{2}$	$\frac{k}{4}$	$2k^2$	k^2

find (a) k (b) Distri ( ) of x.

A) (a) $\sum p(x) = 1$

$k^2 + \frac{k}{4} + \frac{5k}{2} + \frac{k}{4} + 2k^2 + k^2 = 1.$

$4k^2 + \frac{2k + 20k + 2k}{8} - 1 = 0$

(Qua) $4k^2 + 3k - 1 = 0$

$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4 \times 4 \times -1}}{8}$

$= \frac{-3 \pm \sqrt{25}}{8} \Rightarrow \frac{-3+5}{8}, \frac{-3-5}{8}$

$\Rightarrow \frac{2}{8} = \frac{1}{4}$, $\frac{-8}{8} = -1$

so (prob) is > 0 we have $k = \frac{1}{4}$

(b) $k = \frac{1}{4}$

Sub k in table,

x	0	1	2	3	4	5	Total
$p(x)$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{10}{16}$	$1/16$	$\frac{2}{16}$	$\frac{1}{16}$	1

$(1/4)^2 = \frac{1}{16}$

$\frac{\frac{1}{4}}{4} = \frac{1}{16}$

$\frac{5 \times \frac{1}{4}}{2} = \frac{10}{16}$

∴ distribution (f) of x is,

$$F(x) = p(X \leq x) = \begin{cases} 0 & , x < 0 \\ 1/16 & , 0 \leq x < 1 \\ 2/16 & , 1 \leq x < 2 \\ 12/16 & , 2 \leq x < 3 \\ 13/16 & , 3 \leq x < 4 \\ 15/16 & , 4 \leq x < 5 \\ 1 & , x \geq 5 \end{cases}$$