04: Mathematical Expectation & Moments.

$\Sigma f(x) = 1$

$\int f(x) = 1$

→ Mathmatical Exp :-

If **X** is (dis) R.V with ~~(prob) ()~~ probability function $p(x)$ as $f(x)$, the expected value of x is,

$$\therefore \mathbf{E}(x) = \sum_x x\, p(x),$$

as long as the sum is absolutly convergnt.

If x is R.V with probability density () $f(x)$, the expected value of x is,

$$E(x) = \int_x x \cdot f(x)\, dx.$$

as long as the $\int$ is absolutly convergnt.

Note

If x is R.V assuming the values $x_1, x_2 \dots x_n$ with corresponding (prob) $p_1, p_2 \dots p_n$, then the (exp) value of x, is defined as,

$$E(x) = x_1 p_1 + x_2 p_2 \dots x_n p_n$$

$$\boxed{E(x) = \sum_{i=1}^{n} x_i p_i} \quad ; \quad \sum p_i = 1.$$

→ Expectation of () of R.V :-

let $g(x)$ be a () of R.V 'X' the expected value of $g(x)$ is,

$$\boxed{E(g(x)) = \sum_x g(x) \cdot f(x)} \rightarrow x \rightarrow (\text{dis})$$

if x ~~is (dis)~~, ⟹

$$\boxed{E\, g(x) = \int_x g(x) \cdot f(x)\, dx} \quad x \rightarrow \text{contin.}$$

→ Properties = (E xp).

1) If x is R.V then,

$$\boxed{E(c) = c}, \qquad c \rightarrow \text{constnt.}$$

Proof

If x is contin R.V hving pdf $f(x)$,

$$E(x) = \int_x x\, f(x) \cdot dx.$$

$$\therefore,\ E(c) = \int_x c\, f(x)\, dx.$$

$$= c \underbrace{\int_x f(x) \cdot dx}_{1}.$$

$$= c \times 1$$

$$E(c) = c$$

2) $E(c\, g(x)) = c\, E(g(x))$. $\qquad c \rightarrow$ constnt.

proof

If x is contin R.V having pdf $f(x)$,

$E(c\,g(x)) = \int_x c\, g(x) \cdot f(x)\, dx.$

$E(x) = \int x\, f(x)\, dx$

$E(g(x)) = \int g(x) \cdot f(x)\, dx$

$= c \int_x \underbrace{g(x) \cdot f(x)\, dx}$.

$= c \cdot E(g(x))$

$E(c\,g(x)) = c\, E(g(x))$.

3) $E[ax+b] = a\,E(x) + E(b)$.

a & $b \to$ constant.

Proof

$E[ax+b] = \int (ax+b) \cdot f(x)\, dx.$

$= \int ax\, f(x)\, dx + \int b\, f(x) \cdot dx.$

$= a \int x \cdot f(x)\, dx + b \int f(x)\, dx.$

$= a\,E(x) + b \times 1$

$\boxed{E(ax+b) = a\,E(x) + b}$

since $\int f(x) = 1$

R → μ_r'

C → μ'

⇒ Moments :-

Let X be a R.V the r^{th} raw moment about a value X_0 is defined as,

(Raw m) → $\boxed{\mu_r'(x_0) = E(X - X_0)^r.}$

$= \sum (x - x_0)^r\, p(x)$ → dis

$= \int (x - x_0)^r\, f(x)\, dx$ → contin

In particular $x_0 = 0$,

~~$\mu_r'(0) = E\ x$~~

$\mu_r'(0) = \mu_r' = E(x - 0)^r$

$\boxed{\mu_r'(0) = E(x)^r}$

$= \sum x^r\, p(x)$ → x → dis

$= \int x^r\, f(x)\, dx$ → x → conti

putting,

$\mu_r' = E(x)^r.$

$r=1$ $\mu_1' = E(x) = \mu$.

$r=2$ $\mu_2' = E(x^2)$

$r=3$ $\mu_3' = E(x^3)$

$r=4$ $\mu_4' = E(x^4)$.

The r^{th} central moment / moments (m) is defined about their (exp) value as,

(central m) → $\boxed{\mu_r = E(X - E(x))^r}$ $E(x) = \mu$.

$= E(X - \mu)^r$

$= \sum (x - \mu)^r\, f(x)$

$= \int (x - \mu)^r\, f(x)\, dx.$

$E(x) = \mu$

$V(x) = \mu_2$.

putting $r=1$

$\mu_1 = E(x - E(x))$

$= E(x) - E(x)$

$\boxed{\mu_1 = 0}$

$r=2$ $\mu_2 = E(x - E(x))^2 = V(x) = \sigma^2$

$r=3$ $\mu_3 = E(x - E(x))^3$

$\mu_4 = E(x - E(x))^4$

Note

$\boxed{\mu_0 = 1}$ $\boxed{\mu_1 = 0}$

→ Relation b/w raw (m) & central (m) :-

we can express the r^{th} central (m) in terms of raw (m) of order r or less. (ie)

$\mu_r = \mu_r' - {}^rC_1 \mu_{r-1}' \mu' + {}^rC_2 \mu_{r-2}' (\mu_1')^2 - \dots + (-1)^r (\mu_1')^r$

proof raw (m) about origin,

$\mu_r = E(x - E(x))^r$

$= E[x^r - {}^rC_1 x^{r-1} (\mu_1') + {}^rC_2 x^{r-2} (\mu_1')^2 + \dots + (-1)^r (\mu_1')^r]$

$= E(x^r) - {}^rC_1 E(x^{r-1}) (\mu_1') + {}^rC_2 E(x^{r-2}) (\mu_1')^2 - \dots (-1)^r (\mu_1')^r$

$$\boxed{\mu_r = \mu_r' - {}^rC_1 \mu_{r-1}' \mu_1' + {}^rC_2 \mu_{r-2}' (\mu_1')^2 - \dots (-1)^r (\mu_1')^r}$$

putting $r = 1, 2, 3, 4 \dots$

$$\boxed{\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= \mu_2' - (\mu_1')^2 \\ \mu_3 &= \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3 \\ \mu_4 &= \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' (\mu_1')^2 - 3(\mu_1')^4 \end{aligned}}$$

Result

$E \times E(x) = E(x)$.

1) $V(x) = E(x^2) - (E(x))^2$.

proof

$V(x) = E(x - E(x))^2$
$(a-b)^2$

$= E[x^2 - 2x \cdot E(x) + (E(x))^2]$

$= E(x^2) - 2E(x) \cdot E(x) + (E(x))^2$ $E \times E(x)^2 = E(x)^2$

$= E(x^2) - 2(E(x))^2 + E(x)^2$

$V(x) = E(x^2) - (E(x))^2$

2) $V(ax \pm b) = a^2 V(x)$ (a & b → constant)

proof

$V(ax \pm b) = E\left(\cancel{x - E(x)}\right)^2$

$= E\left((ax \pm b) - E(ax \pm b)\right)^2$

$= E\left(ax + b) - (E(ax) \pm E(b)\right)^2$

$= E\left[(ax \pm \cancel{b}) - (a E(x) \pm \cancel{b})\right]^2$

$= E\left(ax - a E(x)\right)^2$

$= a^2 E\left(x - E(x)\right)^2$

$V(ax \pm b) = a^2 V(x)$

eg → $V(3x + 4) = a^2 V(x)$

$= 3^2 V(x) = 9 V(x)$

⇒ Measures of Central tendency :-

1) Arithemetic Mean :-

$AM = E(x)$

$= \sum x \, p(x)$ x → dis

$= \int_{-\infty}^{\infty} x \, f(x) \, dx$ x → contin

(i.e) $AM = \mu_1'(0) = E(x)$.

2) Median :-

$\int_{-\infty}^{M} f(x) \, dx = \frac{1}{2}$ (or).

$\int_{M}^{\infty} f(x) \, dx = \frac{1}{2}$

$-\infty \quad M \quad \infty$ ($\frac{1}{2}$, $\frac{1}{2}$)

3) Mode :-

Mode is the value of R.V having max (prob).

here we get 2 condit →

a) $f(x) > f(x-1)$

b) $f(x) > f(x+1)$.

In the case of contin R.V max $f(x)$,

a) $f'(x) = 0$

b) $f''(x) = 0$

4) Geometric Mean =

$\log(GM) = E(\log x)$

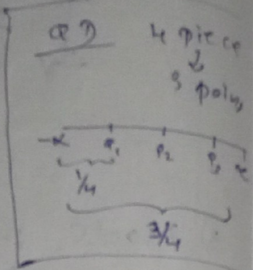
$= \int_x \log x \cdot f(x) \, dx$.

5) Harmonic Mean =

$\frac{1}{HM} = E\left(\frac{1}{x}\right) = \int_x \frac{1}{x} f(x) \, dx$.

→ Measures of Dispersion :-

1) Quartile Deviation :-



$$QD = \frac{Q_3 - Q_1}{2}.$$

$$\int_{-\infty}^{Q_1} f(x)\,dx = \frac{1}{4}$$

&

$$\int_{-\infty}^{Q_3} f(x)\,dx = \frac{3}{4}$$

2) Mean Deviation :-

$$MD = E\,|\,x - E(x)\,| \qquad E(x) = \mu$$

$$= E\,|\,x - \mu\,|$$

$$= \sum_x |x - \mu|\, p(x) \qquad x \to \text{dis.}$$

$$= \int_{-\infty}^{\infty} |x - \mu|\, f(x)\,dx \qquad x \to \text{contin}$$

3) Standard D :-

$$\sigma = \sqrt{V(x)}$$

$$\sigma^2 = V(x) = E(x^2) - (E(x))^2$$

$$= \mu_2' - (\mu_1')^2.$$

$$E(x)^2 = \sum x^2\, p(x) \qquad \text{(dis)}$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx \qquad \text{(conti)}$$

$$E(x) = \sum x\, p(x) \qquad \text{(dis)}$$

$$= \int_{-\infty}^{\infty} x\, f(x)\,dx \qquad \text{(contin)}$$

3) Measures of Skewness :-

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{\mu_3}{\sigma^3}$$

~~Leptokurtic → β > 0~~
~~Mesokurtic → β = 0~~
~~Platykurtic → β < 0~~

4) Measures of kurtosis :-

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$\gamma_2 = \beta_2 - 3.$$

lepto → $\beta_2 > 3$
meso → $\beta_2 = 3$
platy → $\beta_2 < 3$

→ Moment generating ( ) :- (Mgf)

The mgf for R.V x is

$$\boxed{M_x^{(t)} = E\left(e^{tx}\right)}, \quad -\infty < t < \infty, \text{ as long as exp exist.}$$

$$M_x^{(t)} = \sum_x e^{tx}\, p(x)$$

$$M_x^{(t)} = \int_x e^{tx} f(x)\,dx.$$

$M_x^{(t)} = E(e^{tx})$

Mgf से $E(x)$, $E(x^2)$ ... moment निकालते

$\mu_1' = E(x) = \frac{d\,M_x^{(t)}}{dt}\Big|_{t=0}$ (diff 1 time)

$\mu_2' = E(x^2) = \frac{d^2}{dt^2} M_x^{(t)}\Big|_{t=0}$ (diff 2 time)

$\mu_3' = E(x^3) = \frac{d^3}{dt^3} M_x^{(t)}\Big|_{t=0}$ (diff 3 time)

$\vdots$

$\mu_r' = E(x^r) = \frac{d^r}{dt^r} M_x^{(t)}\Big|_{t=0}$

→ Pro of Mgf :-

1) $M_{cx}(t) = M_x(ct)$ (c is any constant)

proof

$M_x^{(t)} = E(e^{tx})$

$M_{cx}(t) = E(e^{tcx})$

$= E\left(e^{(ct)x}\right)$

$\boxed{M_{cx}(t) = M_x(ct)}$

2) $\boxed{M_{ax+b}(t) = e^{bt} M_x(at)}$

Pf

$M_{ax+b}(t) = E\left(e^{t(ax+b)}\right)$

$= E\left(e^{atx+bt}\right)$

$= E\left(e^{atx} \cdot e^{bt}\right)$

$M_x^{(t)} = E(e^{tx})$

$e^{a+b} = e^a \cdot e^b$

(e^{bt} → const)

$= e^{bt} E\left(e^{(at)x}\right)$

$= e^{bt} M_x(at)$

* Corollary :-

$M_{\frac{x-\mu}{\sigma}}(t) = e^{-\mu/\sigma\, t} M_x\left(\frac{1}{\sigma}t\right)$ $a = -\frac{\mu}{\sigma}$ $b = \frac{1}{\sigma}$

by using ② pro we can prove this.

3) $M_{x+y}(t) = M_x(t) \cdot M_y(t)$. x & y independent.

4) $M_x(t) = 1$ at $t=0$

5) $\mu_r' = \frac{d^r}{dt^2} M_x(t)\Big|_{t=0}$ $r = 1, 2, 3 \ldots$

→ Mgf of central moment :-

Let x be R.V having pdf f(x). Let E(x) = μ then the mgf about exp value / mgf about mean is defined as,

$M_{x-\mu}^{(t)} = E\left(e^{t(x-\mu)}\right)$

$= \sum e^{t(x-\mu)} \cdot p(x)$ → (dis)

$= \int e^{t(x-\mu)} \cdot f(x)\, dx$ → (contin)

This () generates the central moment

i.e so it is → central mgf

☆

⇒ Characteristic ( ) :- [$\phi_x(t)$].

$$\boxed{\phi_x(t) = E\left(e^{itx}\right)}$$

$t \to$ any real no.

$i = \sqrt{-1}$ (complex ( ))

If x is R.V with pdf $f(x)$. then,

$\phi_x(t) = E\left(e^{itx}\right)$

$= \sum_x e^{itx} p(x)$ → (dis)

$= \int_x e^{itx} f(x)\, dx$ → contin.

→ Pro) of C. ( ) :-

1) $\phi(t)$ is exp of a complx ( ) of R.V x.

complx no = $a_i + b_i$ conjug $a_i - b$

$a_i - b \to a_i + b$.

2) $\phi(0) = 1$

3) $|\phi(t)| \leq 1$

4) $\phi(t) = \overline{\phi(t)}$, $\overline{\phi(t)}$ is complx Conjugate.

5) $\phi(t)$ is uniformly contin on R (real line)

6) $\left.\dfrac{d^r \phi_x(t)}{dt^r}\right|_{t=0} = \mu_r'$, provided μ_r' exist

Q/

1) If X is a R.V hving pdf.

$f(x) = \dfrac{x+1}{2}$, $-1 \leq x \leq 1$, find $E(x)$ & $V(x)$

A)

$E(x) = \int_x x \cdot f(x) \cdot dx$.

$= \int_{-1}^{1} x \cdot \left(\dfrac{x+1}{2}\right) dx$.

$= \dfrac{1}{2} \int_{-1}^{1} x(x+1)\, dx$.

$= \dfrac{1}{2} \int_{-1}^{1} (x^2 + x)\, dx$.

$= \dfrac{1}{2} \left(\dfrac{x^3}{3} + \dfrac{x^2}{2}\right)_{-1}^{1}$.

$\Rightarrow \dfrac{1}{2}\left[\dfrac{1}{3} + \dfrac{1}{2} - \left(\dfrac{(-1)^3}{3} + \dfrac{(-1)^2}{2}\right)\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{3} + \dfrac{1}{2} - \left[\dfrac{-1}{3} + \dfrac{1}{2}\right]\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{3} + \cancel{\dfrac{1}{2}} + \dfrac{1}{3} - \cancel{\dfrac{1}{2}}\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{3} + \dfrac{1}{3}\right] = \dfrac{1}{\cancel{2}}\left[\dfrac{\cancel{2}}{3}\right] = \dfrac{1}{3}$

$V(x) = E(x^2) - (E(x))^2$.

$E(x^2) = \int_x x^2 \cdot \left(\dfrac{x+1}{2}\right) dx$.

$= \dfrac{1}{2} \int_{-1}^{1} x^2 (x+1)\, dx$.

$= \dfrac{1}{2} \int_{-1}^{1} (x^3 + x^2)\, dx$.

$= \dfrac{1}{2}\left[\dfrac{x^4}{4} + \dfrac{x^3}{3}\right]_{-1}^{1}$.

$= \dfrac{1}{2}\left[\dfrac{1}{4} + \dfrac{1}{3} - \left(\dfrac{(-1)^4}{4} + \dfrac{(-1)^3}{3}\right)\right]$

$= \dfrac{1}{2}\left[\cancel{\dfrac{1}{4}} + \dfrac{1}{3} - \cancel{\dfrac{1}{4}} + \dfrac{1}{3}\right]$

$= \dfrac{1}{\cancel{2}}\left[\dfrac{\cancel{2}}{3}\right] = \dfrac{1}{3}$

$$\therefore \quad V(x) = E(x^2) - (E(x))^2$$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2$$

$$V(x) = \frac{1 \times 3}{3 \times 3} - \frac{1}{9} = \frac{3-1}{9} = \frac{2}{9}$$