B.Sc. in Computer Science and Engineering Thesis

Bar 2-Visibility Representations of 2-Planar Graphs

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December 2019

CANDIDATES' DECLARATION

This is to certify that the work presented in this thesis, titled, "Bar 2-Visibility Representations of 2-Planar Graphs", is the outcome of the investigation and research carried out by us under the supervision of Dr. Shaheena Sultana.

It is also declared that neither this thesis nor any part there of has been submitted anywhere else for the award of any degree, diploma or other qualifications.

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CERTIFICATION

This thesis titled, **"Bar 2-Visibility Representations of 2-Planar Graphs"**, submitted by the group as mentioned below has been accepted as satisfactory in partial fulfillment of the requirements for the degree B.Sc. in Computer Science and Engineering in December 2019.

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ABSTRACT

A bar visibilty representation of a planar graph G is a drawing of G where each vertex is drawn as a horizontal line segment called bars, each edge is drawn as a vertical line segment where the vertical line segment representing an edge must connect the horizontal line segments representing the end vertices. In a bar k-visibility representation of a graph G, a horizontal line corresponds to a vertex called a bar and a vertical line segment corresponding to an edge intersects at most k bars which are not end points of the edge. Thus a bar visibility representation is a bar k-visibility representation for $k = 0$. For $k = 1$, a line segment corresponding to an edge intersects at most one bar which is not an end point of the edge and the representation is called a bar 1-visibility representation. A graph is a bar 1-visibility graph if it admits a bar 1-visibility representation. Similarly, For $k = 2$, a line segment corresponding to an edge intersects at most two bar which is not an end point of the edge and the representation is called a bar 2-visibility representation. A graph is a bar 2-visibility graph if it admits a bar 2-visibility representation.

In this thesis, we study bar 2-visibility Graphs. We give a linear-time algorithm for 2-visibility representation of an embedded 2-planar graph. Since, every 2-planar graph is bar 2-visible and both 2-visible and 2-planar graphs of size n have at most $5n - 10$ edges.

Chapter 1

Introduction

A graph is a diagram showing the relation between variable quantities, typically of two variables, each measured along one of a pair of axes at right angles. Graph theory is the study of graphs, which are mathematical structures used to model pairwise relations between objects from a certain collection.There are thousands of real world problems each of which underlying structure consisting of entities and their relationships, and the representation of these problems as a suitable graph classification can aid in powerful visualization of the concept. Structures that can be represented as graphs are ubiquitous, and many problems of practical interest can be represented by graphs. Graphs can be used to model many types of relations and processes in physical, biological, social and information systems. Many practical problems can be represented by graphs. Emphasizing their application to real-world systems, the term network is sometimes defined to mean a graph in which attributes (e.g. names) are associated with the vertices and edges, and the subject that expresses and understands the real-world systems as a network is called network science.The study of graph theory has its widespread applications in a vast majority of fields ranging from computer network analysis, genetics, bioinformatics, transportation network, social networking concepts to VLSI circuit design, cartography, molecular chemistry and condensed matter physics.

The first paper in the history of graph theory was written by Leonhard Euler [1] on the Seven Bridges of Konigsberg and published in 1736 and the first textbook on graph theory was written by Denes Konig [2] that was published in 1936.

In mathematics, a graph is a collection of points, called vertices, and lines between those points, called edges.Typically,a graph is depicted in diagrammatic form as a set of dots for the vertices,joined by lines or curves for the edges.Graph drawing is a drawing of a graph that is basically a illustrated representation of an embedding of the graph in the plane, usually aimed at a convenient visualization of certain properties of the graph in question or of the object modeled by the graph. A graph structure can be extended by assigning a weight to each edge of the graph.Graphs with weights, or weighted graphs, are used to represent structures in which pairwise connections have some numerical values. Besides the weighted graphs, colored graphs can also be used to focus on certain special attributes of a problem.

Visibility in the plane is a very natural concept but many fundamental problems remain unsolved.Visibility graphs are a much studied approach to these problems. A visibility graph is a graph of intervisible locations, typically for a set of points and obstacles in the Euclidean plane. Each node in the graph represents a point location and each edge represents a visible connection between them.Bar visibility graphs are one of the best understood classes of visibility graphs. In Bar visibility graphs, the vertices correspond to horizontal line segments called bars and visibility runs vertically along lines of sight which connect two bars while being disjoint from all others. These graphs have been completely characterized by Tamassia and Tollis [3].

There is another generalization of a bar visibility graph that has been been introduced by Dean, Evans, Gethner, Laison, Safari and Trotter [6] in 2005. Their idea is that lines of sight are allowed to intersect at most k other bars, where the values of $k = 0, 1, 2, 3, \ldots n$. For the case $k = 0$, a graph is called *visibility graph*. A graph is called *Bar 1-visibility graph* for $k = 1$. Similarly for the case $k = 2$, a graph is called *Bar 2-visibility graph*.

In this thesis, we have focused the visibility representation of 2-planar graphs which is a nonplanar fraph. A visibility representation of a plane graph G is a drawing of G, where the vertices of G are represented by non-overlapping horizontal segments (called vertex segments), and each edge of G is represented by a vertical line segment touching the vertex segments of its end vertices. Tamassia & Tollis [3] have given a linear time algorithm for constructing a visibility representation of a planar graph. In a bar k-visibility representation of a graph a horizontal line corresponding to a vertex and the vertical line segment corresponding to an edge intersects at most k bars which are not end points of the edge. Thus a visibility representation is a bar k-visibility representation for k = 0. Fleshner and Massow have investigated some graph theoretic properties of 1-visibility graphs [7]. Recently Franz J. Brandenburg [8] have developed an algorithm for *1-Visibility Representations of 1-Planar Graphs*.A 1-visibility representation of a graph displays each vertex as a horizontal vertex-segment, called a bar, and each edge as a vertical edge segment between the segments of the vertices, such that each edge-segment crosses at most one vertex-segment and each vertex-segment is crossed by at most one edge-segment. A graph is 1-visible if it has such a representation. 1-visibility is related to 1-planarity where graphs are drawn such that each edge is crossed at most once, and specializes bar 1-visibility where vertex-segments can be crossed many times.He develop a linear time algorithm to compute a 1-visibility representation of an embedded 1-planar graph in $O(n^2)$ area. Hence,

every 1-planar graph is 1-visible. Concerning density, both 1-visible and 1-planar graphs of size n have at most $4n - 8$ edges. For every $n \geq 7$ there are 1-visible graphs with 4n-8 edges, which are not 1-planar.
A bar 1-visibility representation of a graph displays each vertex as a horizontal vertex-segment, called a bar, and each edge as a vertical edge segment between the segments of the vertices, such that each edge-segment crosses at most one vertex-segment and vertex-segment is crossed by many edge-segment.

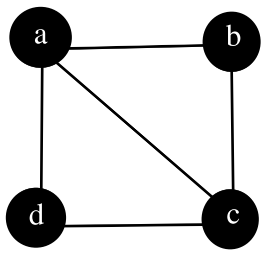
However, there is no algorithm for finding *Bar 2-Visibility Representations of 2-Planar Graphs*. So, In this thesis, we study Visibility Representations,bar 1-Visibility Representations of 1-Planar Graphs. It is easy to see that all Bar Visibility Representations are planar and all Bar 1-visibility Representations are 1-planar.In this thesis, we give a linear time algorithm for finding a *Bar 2-Visibility Representation of 2-Planar Graphs*.We will give the details of the above mentioned algorithm and some of the previous results in this field that have a significant impact on our work.In this chapter, we give some introductory concepts of visibility representation, 1-Visibility Representations and 2-Visibility Representations that will help realizing the concepts presented here. Also, we have presented some applications of this topic in various fields. The rest of this chapter is formed as follows. In section 1.1, we define Visibility Representation of Planar Graphs and in section 1.2, we define 1-Visibility Representations of 1-Planar Graphs and in section 1.3, we define bar 2-Visibility Representations of 2-Planar Graphs which is the central idea of this thesis. Section 1.4 is applications of visibility representations. In section 1.5, we present a brief history of visibility representation,bar 1-visibility representation, 2-Planar Graphs and on the basis of that, in section 1.6, we depict the scope and objective of this thesis. In section 1.7, we present the organization of this thesis.

1.1 Bar Visibility Representations

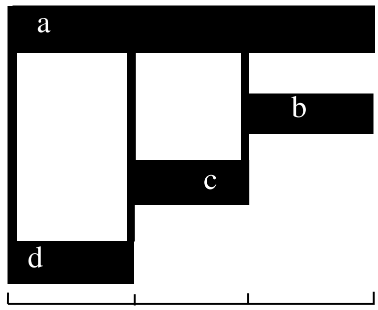
In Bar Visibility Representation [3] the vertices correspond to horizontal line segments, called bars and the edges correspond to vertical lines line segments, called bars. In the Figure 1.1 (a) is a planar graph G, and (b) is the bar visibility representation of G, if there exists a one-to-one correspondence between vertices of G and bars in (b) , such that there is an edge between two vertices in G if and only if there exists an unobstructed vertical line of sight between their corresponding bars.

1.2 Bar 1-Visibility Representations

In bar 1 Visibility Representation , each edge-segment crosses at most one vertex-segment and each vertex-segment is crossed by many edge-segment. Figure 1.2, is showing 1-planar graph

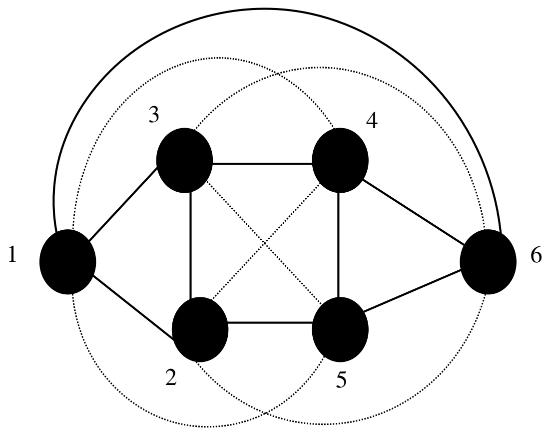


(a)

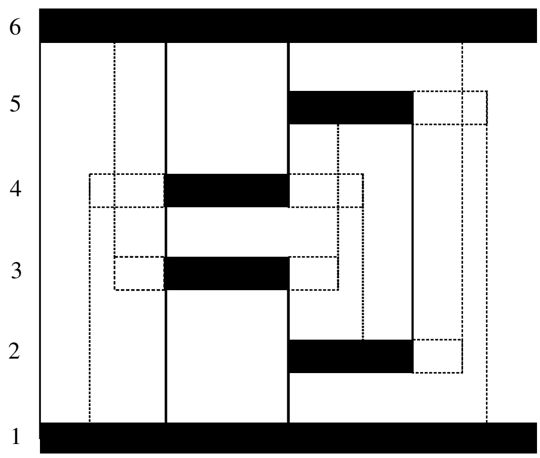


(b)

Figure 1.1: (a) A planar graph ;(b) The visibility representation of the graph G.



(a)



(b)

Figure 1.2: (a) 1-planar graph ; (b) 1-visibility representations of the 1-planar graph.

and its 1-visibility representations,where Dot edges are representing the 1-planar crossing points and dot vertices represents the vertex-expansion for representing 1-visibility representations.

In 1 Visibility Representation [8], each edge-segment crosses at most one vertex-segment and each vertex-segment is crossed by at most one edge-segment and graphs are drawn such that each edge is crossed at most once, and specializes bar 1-visibility where vertex-segments can be crossed many times vertex-segments can be crossed many times.

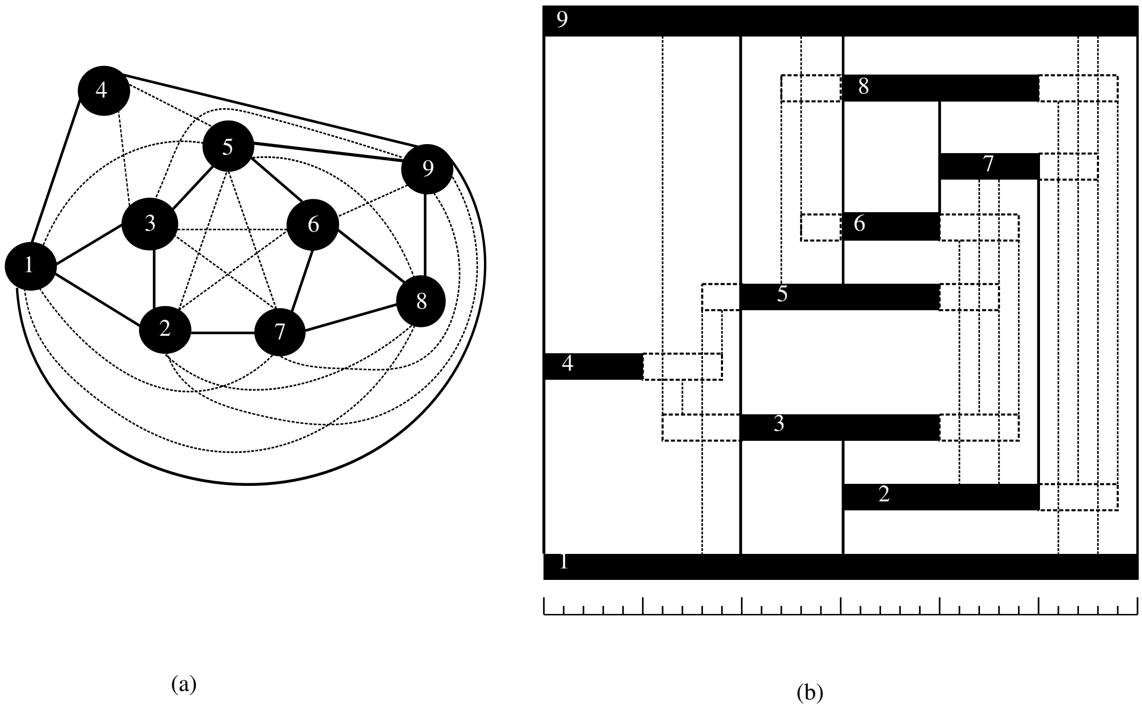


Figure 1.3: (a) A 2-planar graph ; (b) The bar 2-visibility representations of the 2-planar graph.

1.3 Bar 2-Visibility Representations

In Bar 2-Visibility Representations, each edge-segment is crossed at most twice, and vertex-segments can be crossed many times. Figure 1.3, is showing 2-planar graph and its 2-visibility representations,where Dot edges are representing the crossing points and dot vertices represents the vertex-expansion for representing bar 2-visibility representations.

1.4 Applications of bar 2-Visibility Representations of 2-Planar Graphs

The problem of computing a compact bar 2-Visibility Representations of 2-Planar Graphs is important not only in algorithmic graph theory, but also in practical applications such as VLSI layout (Circuit board layout) [10] and Modules and their interconnections of a VLSI circuit are given as a graph where vertices of the graph represents a module of the VLSI circuit and edges represents an interconnection between two modules. From a visibility representation, a planar polyline drawing can be generated with O(1) bends per edge in linear time [13]. Bar 2-Visibility representations can also be used to generate 2-planar orthogonal drawings.It is also used for security systems.

1.5 Previous Results

In this section, we give an outline of the results found in this area. Visibility representation has practical applications in VLSI layout [10] and several researchers concentrated their attention on visibility representations. Otten and Van Wijk [11] shows that every planar graph admits a visibility representation and Tamassia & Tollis [3] develop a linear-time algorithm for Visibility Representation of a planar graph.Alice M. Dean et al. have introduced a generalization of visibility representation for a non-planar graph which is called bar k-visibility representation [6].
In recent years, several works are devoted to this field. Fabrici and Madaras [16] study the existence of subgraphs of bounded degrees in 1-planar graphs which is also called bar 1-visibility graph. It is shown that each 1-planar graph contains a vertex of degree at most 7; they also prove that each 3-connected 1-planar graph contains an edge with both end vertices of degrees at most 20. Sultana, Shaheena and Rahman, Md. Saidur and Roy, Arpita and Tairin, Suraiya [17] generated an algorithm for Bar 1-Visibility Drawings of 1-Planar Graphs. Later Franz J. Brandenburg [8] have developed algorithm for *1-Visibility Representations of 1-Planar Graphs*.Alam, Brandenburg and Kobourov [12] described Straight-Line Grid Drawings of 3-Connected 1-Planar Graphs. Michael, Kaufmann, Chrysanthi N. Raftopoulou [18] describes about Optimal 2- and 3-Planar Graphs.

1.6 Scope of this Thesis

In this section, we give an overview of the basic hunch of the approach we have taken for dealing with the problem of bar 2-Visibility Representation of 2-Planar Graph G. At the end, we list the results obtained by us in this thesis.
If the input graph is 2-Planar Graph , we'll remove all crossing points from the graph G and convert the graph into Planar graph using a technique based on st-numbering. After that , we'll re-insert all the crossing points and again transform the planar graph into 2-Planar Graph. Then we'll convert the 2-Planar Graph into 1-planar graph and then convert this 1-Planar graph into planar graph. After getting the planar graph, we represents the this planar graph into *Visibility Representation* [3].Then we re-insert all 1-Planar crossing points for transforming the graph into 1-Visibility Representations and finally we re-insert all 2-Planar crossing points or deleted edges for transforming 1-Visibility Representations into bar 2-Visibility Representations. Finally, our findings in this thesis is listed here.

- We have developed an algorithm for st-numbering of Non-Planar Graphs .
- we have also developed an algorithm for finding 2-Visibility Representations of 2-Planar Graphs.

1.7 Thesis Organization

The rest of this thesis is organized as follows. In chapter 2, we give some basic terminology of graph theory and graph drawing. In chapter 3, we present previous algorithms on Visibility Representations , Visibility Resentations of 1-Planar Graphs and 2-Planar Graphs. In chapter 4, we mention previous algorithms on constrained visibility representations and our algorithms on bar 2-Visibility Representations of 2-Planar Graphs . Finally, Chapter 5 discusses the open problem in this field and gives this thesis an ending.

Chapter 2

Preliminaries

In this chapter, we define some basic terminology of graph theory, graph drawing and algorithm theory, that we will use the rest of this thesis.Definitions which are not included in this chapter will be introduced as they are needed. We review, in 2.1, some definitions of standard graph-theoretical terms. In 2.2, we discuss about some special classes of graphs that are important for the ideas and concepts used in the later parts of this thesis. We devote 2.3 to define different numbering of planar graph.In 2.4 and 2.5 we define some drawing conventions of planar and non-planar graphs. Finally, we introduce the notion of time complexity in 2.6.

2.1 Basic Terminology

In this section we've given some definitions about graph-theoretical terms that we use throughout this thesis.

2.1.1 Graphs

Graph is a collection of points, called vertices(V), and lines between those points, called edges(E).In Figure 2.1 shows a graph $G = (V, E)$ where each vertex in $V = v1, v2, \ldots, v4$ is drawn as a small circle and each edge in $E = e1, e2, \ldots, e5$ is drawn by a line segment. Vertices are connecting to eachother by edges.

2.1.2 Simple Graphs and Multigraphs

If a graph G has no *multiple edges* or *loops*, then G is said to be a simple graph. In Figure 2.2, (a) is a simple graph.

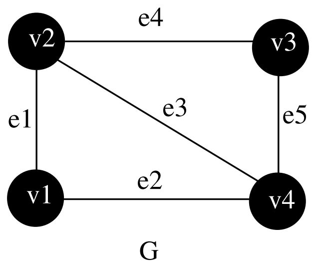


Figure 2.1: A graph G with four vertices and five edges.

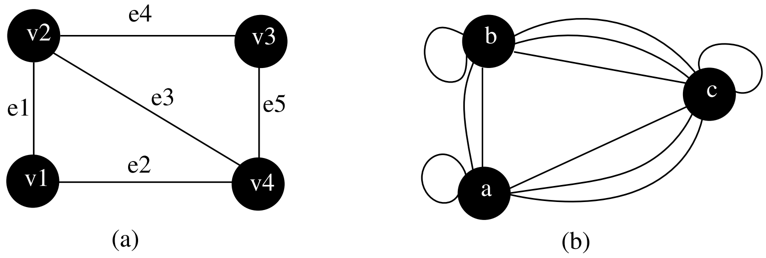


Figure 2.2: (a) A simple graph ; (b) The Multigraph.

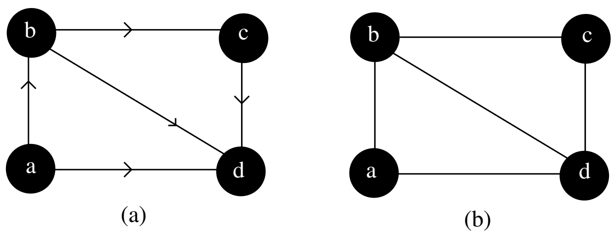


Figure 2.3: (a) A Directed Graph ; (b) A Undirected Graph.

A graph in which loops and multiple edges are allowed is called a Multigraph.It can arise from various applications.Figure 2.2 shows (b) is multigraph.

2.1.3 Directed and Undirected Graphs

A graph in which every edge is directed is called a Directed Graph, and a Graph in which every edge is undirected is called Undirected Graph. Figure 2.3 (a) represents a Directed Graph using arrow sign and (b) shows Undirected Graph where has no direction sign in the edges.

2.1.4 Subgraphs

A subgraph S of a graph G is a graph whose set of vertices and set of edges are all subsets of G. Figure 2.4 (b) and (c) shows the subgraph S for the real Simple graph (a).

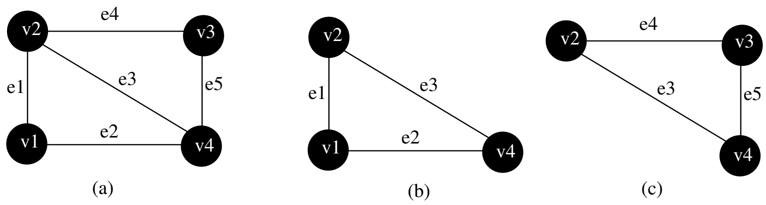


Figure 2.4: (a) A Simple graph G ; (b) and (c) Subgraph S for (a).

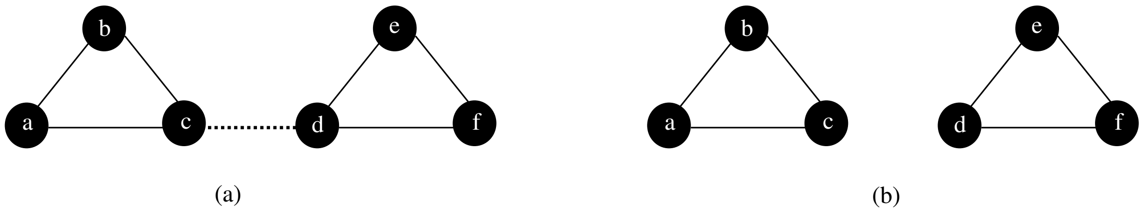


Figure 2.5: (a) A Connected Graph ; (b) A Disconnected Graph.

2.1.5 Connected Graph

A Graph in which there is a path joining each pair of vertices, the graph being undirected. It is always possible to travel in a connected graph between one vertex and any other; no vertex is isolated. If a graph is not connected it will consist of several components, each of which is connected; such a graph is said to be disconnected.Figure 2.5 (a) shows the Connected Graph by dot edge. and (b) shows the graph becomes disconnected when the dot edge is removed.

2.2 Special Classes of Graphs

In this section we have given some definitions of special classes of graphs related to planar graphs and non planar graphs (1-Planar Graphs and 2-Planar Graphs) used in the remainder of the thesis.

2.2.1 Planar Graphs and Plane Graphs

A planar graph is a graph that can be embedded in the plane. It can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. Figure 2.6 shows multiple planar embeddings of the same planar graphs.

A plane graph can be defined as a planar graph with a mapping from every node to a point on a plane, and from every edge to a plane curve on that plane, such that the extreme points of each curve are the points mapped from its end nodes, and all curves are disjoint except on their extreme points.

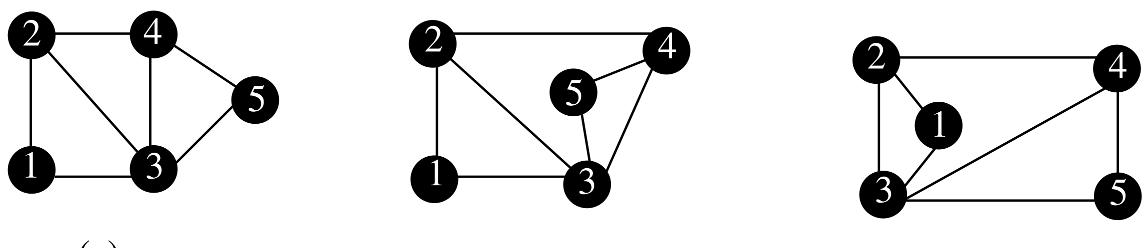


Figure 2.6: Three planar embeddings of the same planar graph.

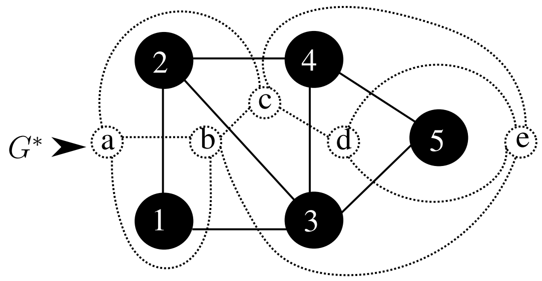


Figure 2.7: Dual graph G^* on the planar graph G.

2.2.2 Dual Graph

For a planar graph (or Plane Graph) G, we often construct another graph called the Dual Graph G^*. The dual graph G^* of a planar graph G is a graph that has a vertex for each face of G. The dual graph has an edge whenever two faces of G are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. In Figure 2.7 the dotted graph represents the Dual Graph G^* for the Planar Graph G where a, b, c, d and e represents the faces of G.

2.2.3 1-Planar Graph

A graph is called a 1-planar graph if it can be drawn in the plane in such a way so that each its edge is crossed by at most one other edge. A 1-planar graph is a graph that has a 1-planar drawing. Figure 2.8 shows 1-Planar Graph.

It is shown by Fabrici and Madaras [16] that each 1-planar graph contains a vertex of degree at most 7; they also proved that each 3-connected 1-planar graph contains an edge with both end vertices of degrees at most 20. The relationship between RAC graphs and 1-planar graphs [19] have also shown by Eades and Liotta.

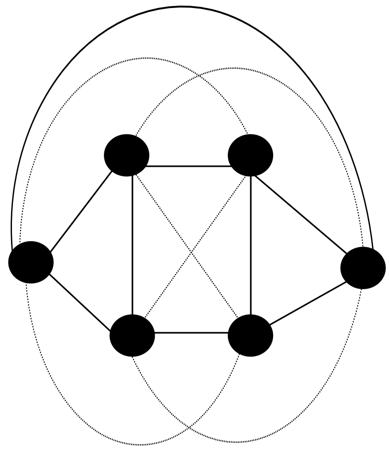
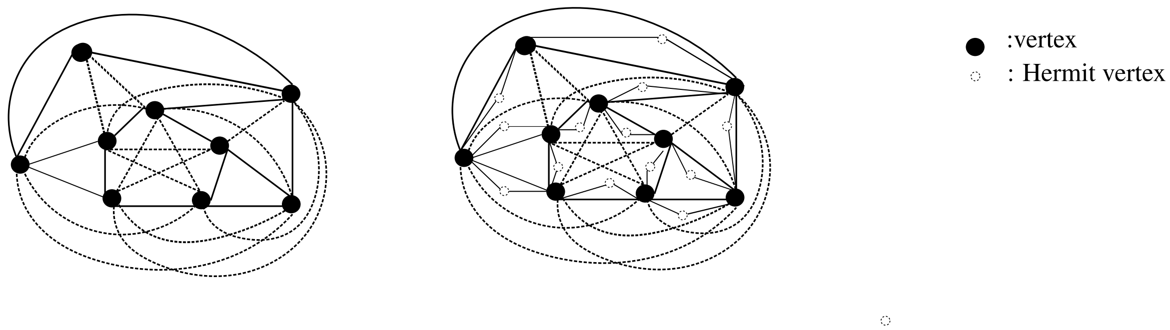


Figure 2.8: 1-Planar Graph.



(a) (b)

Figure 2.9: (a) A 2-Planar Graph ; (b) A 2-Planar Graph with Hermits vertices.

2.2.4 2-Planar Graphs

A graph is called a 2-Planar Graph if it can be drawn in the plane in such a way so that each its edge is crossed by at most two other edge. A 2-planar graph is a graph that has a 2-planar drawing. Pach and Toth have shown that 2-planar graphs with n vertices have at most $5n - 10$ edges and this bound is tight.
Hermits are vertices of degree 1 or 2 which are enclosed by crossing edges and cannot be connected to other vertices. Figure 3.4 (a) shows simple 2-Planar Graph and (b) shows 2-Planar Graph with dot Hermits vertics.

In recent years, several works are devoted to this field. In 2017 , [18] shows the optimal 2- and 3- Planar Graphs.and [14] shows On sparse maximal 2-planar Graphs. However, There is no complete results about 2-Planar Graphs.

2.2.5 Bar k-Visibility Graphs

Bar k-visibility graphs have been introduced by Dean, Evans, Gethner, Laison, Safari and Trotter [6]. Bars are allowed to see through at most k other bars in a bar k-visibility graphs.They

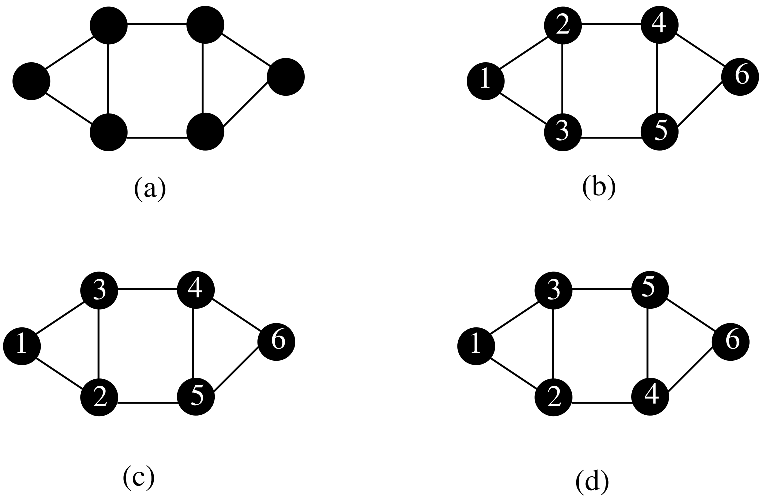


Figure 2.10: (a)Simple Graph G ; (b),(c) and (d) st-Numbering.

seek measurements of closeness to planarity for bar k-visibility graphs since all bar visibility graphs are planar and they obtain an upper bound on the number of edges in a bar k-visibility graph.

2.3 Numbering

In this section we show the techniques of standard graph numbering (st-Numbering) used throughout this thesis.

2.3.1 st-Numbering

Lempel et. al. [20] states that every biconnected graph has an st-numbering. The st-numbering of a graph is not unique, it is like a Simple Graph G.Using an st-numbering of G, we can represent the edges of G from lower numbered vertex to higher-numbered vertex.
Assume, s and t be any two vertices of G. A $st-numbering$ of G is a numbering of its vertices by integers $1, 2, \ldots, n$ such that a vertex s receives number 1, a vertex t receives number n and every other vertex of G is adjacent to at least one lower-numbered vertex and at least one higher numbered vertex. In Figure 2.10 (a) shows the Simple Graph G and (b),(c),(d) shows the st-Numbering for the graph G.

2.4 Drawing Conventions of Planar Graphs

In this section, we introduce some important drawings used in the remainder of the thesis.

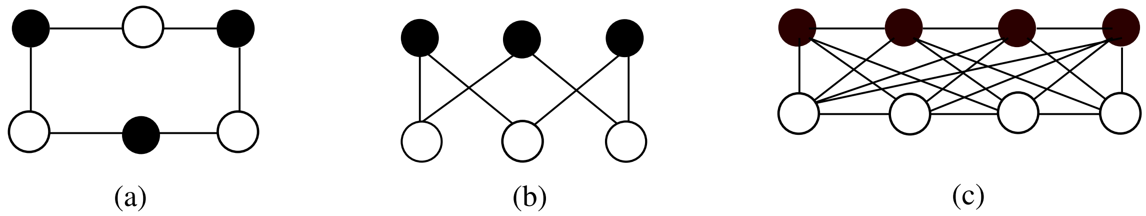


Figure 2.11: (a) A planar drawing ; (b) A non-planar drawing of the graph drawn in (a) ; (c) A graph which has not planar drawing.

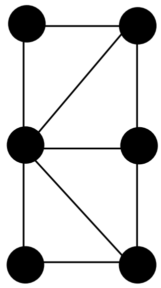


Figure 2.12: A Straight-line Drawing

2.4.1 Planar Drawings

A graph is a Planar Drawings graph if no two edges intersect with each other except at their common end-vertices. Figure 2.11 (a) is a planar drawing, (b)is the non-planar drawing of the graph drawn in (a).

But unfortunately, not all graphs have a planar drawing. Figure 2.11 (c) is an example of one such graph.

2.4.2 Straight-line Drawings

In straight-line drawing of a graph G, each edge is drawn as a straight line segment. Figure 2.12, all edges are represents as straight line segments. Fary [22] and Stein [21] proved that every planar graph has a straight line drawing.

2.4.3 Grid Drawings

A drawing of a graph is called a grid drawing if the vertices are all located at grid points of an integer grid as show in Figure 2.13.

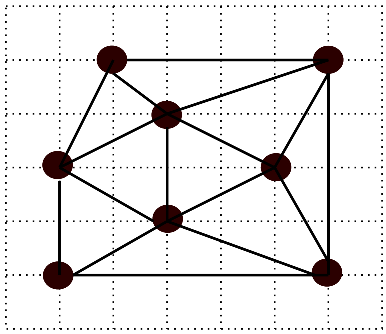


Figure 2.13: A Grid Drawing.

2.4.4 Visibility Drawings

Assume, B is a set of horizontal nonoverlapping segments in the plane. Two segments b, b' of B are said to be visible if they can be joined by a vertical segment not intersecting any other segment of B. Furthermore, b and b' are called ϵ-visible if they can be joined by a vertical band of nonzero width that does not intersect any other segment of B. This is equivalent to saying that b and b' can be joined by two distinct vertical segments not intersecting any other segment of B .

A w-visibility representation for a graph $G = (V, E)$ is a mapping of vertices of G into nonoverlapping horizontal segments (called vertex-segments) and of edges of G into vertical segments (called edge-segments) such that, for each edge $(u, v) \in E$, the associated edge-segment has its endpoints on the vertex-segments corresponding to u and v, and it does not cross any other vertex-segment.Figure 2.14(b)

An ϵ-visibility representation for a graph G is a w-visibility representation with the additional property that two vertex-segments are ϵ -visible if and only if the corresponding vertices of G are adjacent.Figure 2.14(c)

An s-visibility representation for a graph G is a w-visibility representation with additional property that two vertex-segments are visible if and only if the corresponding vertices of G are adjacent. Figure 2.14(d)

2.5 Drawing Conventions of Non-Planar Graphs

In this section we introduce some Non-planar Graphs, which are found suitable in different application domain. In 2.5.1 and 2.5.2 the most important drawing styles are introduced.

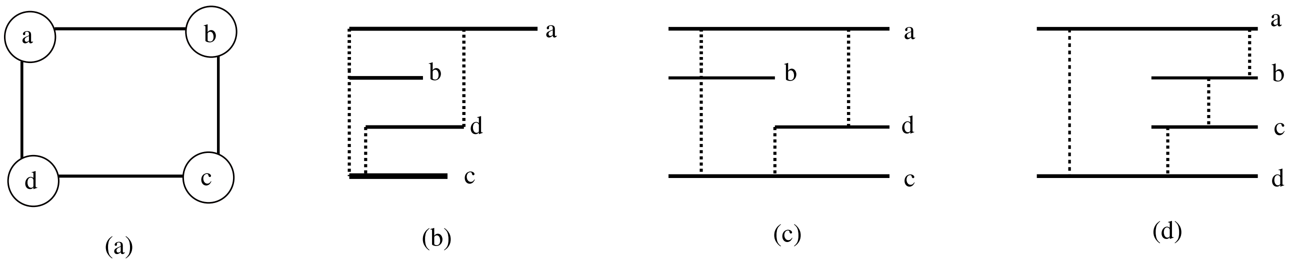


Figure 2.14: (a) A planar Graph G;(b) w-visibility representation; (c)ϵ-visibility representation; (d) s-visibility representation.

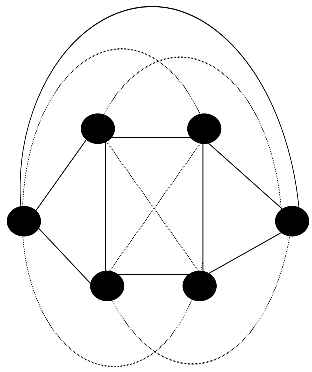


Figure 2.15: 1-planar drawing of a graph.

2.5.1 1-Planar Drawing

A 1-planar drawing is a drawing of a graph where an edge can be crossed by at most one edges (Figure 2.15). A 1-planar graph is a graph that has a 1-planar drawing.

2.5.2 2-Planar Drawing

A 2-planar drawing is a drawing of a graph where an edge can be crossed by at most two edges (Figure 3.4). A 2-planar graph is a graph that has a 2-planar drawing.

2.6 Complexity of Algorithms

Chapter 3

Visibility Representations & 2-Planar Graphs

3.1 Introduction

In this chapter, we discuss about visibility Representations , Visibility Representations of 1 planar graph and 2-planar graph which is related to our thesis.In Section 3.2 we define about visibility representations of planar graph , in Section 3.3 we define 1-visibility representations and bar 1-visibility representations of 1-planar graph and in Section 3.4 we define about 2-planar graphs.

3.2 Visibility Representations of Planar Graphs

In Bar Visibility Representation [3] the vertices correspond to horizontal line segments, called bars and the edges correspond to vertical lines line segments, called bars. In the Figure 1.1 (a) is a planar graph G, and (b) is the bar visibility representation of G, if there exists a one-to-one correspondence between vertices of G and bars in (b) , such that there is an edge between two vertices in G if and only if there exists an unobstructed vertical line of sight between their corresponding bars.
Tamassia & Tollis [3] develop a linear-time algorithm for Visibility Representation of a planar graph.

Theorem 1. *The algorithm VISIBILITY correctly computes a visibility representation of a graph G.*

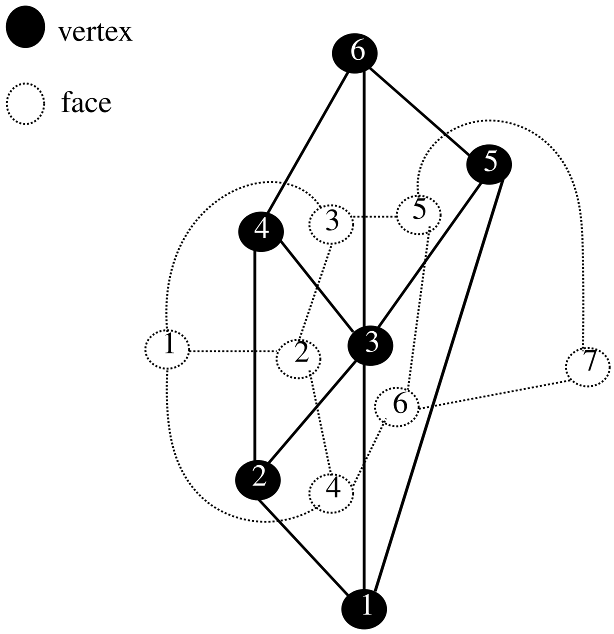


Figure 3.1: Planar Graph G, its dual G^*

Algorithm 1 Visibility Representation

---

input : A planar graph G
output : Visibility Representation of planar graph G

1: Construct a planar st-graph G
2: Assign unit weights to the edges of G and compute st-numbering Y of G.
3: Assign unit weights to the edges of G and compute optimal topological numbering X of G

4: For each vertex v, draw the vertex-segment $T(v)$ at y-coordinate $Y(v)$ and between x-coordinates $X(left(v))$ and $X(right(v) - 1)$.
5: For each edge e, draw the edge-segment $T(e)$ at x-coordinate $X(left(e))$ between y-coordinates $Y\ (orig(e))$ and $Y(dest(e))$.

In Figure 3.1 we draw a planar st-graph G and construct a dual graph G^* on the planar graph so that we can define the inner and outer face of the planar graph.
In Figure 3.2 (a) we draw a straight line drawing for the dual graph G^* for representing the edge segments of planar graph G.
In Figure 3.2 (c) we draw a Grid drawing for planar graph G where vertices are all located at grid points of an integer grid so that we can represent the vertex segment and finally Figure 3.2 (b) is a visibility representation for the planar graph G.

3.3 Visibility Representations of 1-Planar Graphs

In 1 Visibility Representation, each edge-segment crosses at most one vertex-segment and each vertex-segment is crossed by at most one edge-segment and graphs are drawn such that each

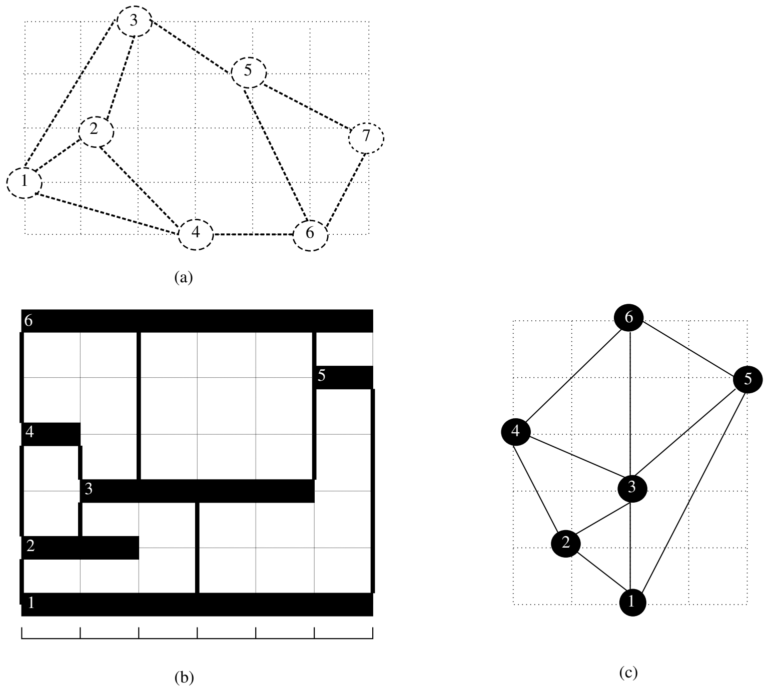


Figure 3.2: (a) A dual graph G^* ; (b) Visibility representation of graph G ; (c) Graph G

edge is crossed at most once [8], and specializes bar 1-visibility where vertex-segments can be crossed many times vertex-segments can be crossed many times. Figure 3.3, is showing 1-planar graph and its bar 1-visibility representations,where dotted edges are representing the 1-planar crossing points and dotted vertices represents the vertex-expansion for representing bar 1-visibility representations.

Franz J. Brandenburg [8] have developed an algorithm for *1-Visibility Representations of 1-Planar Graphs* and given the following theorem.

Theorem 2. *There is a linear time algorithm to construct a 1-visibility representation of an embedded 1-planar graph on a grid of size at most* $(8n - 20) \times (n - 1)$.

3.4 2-Planar Graphs

A graph is called a 2-Planar Graph if it can be drawn in the plane in such a way so that each its edge is crossed by at most two other edge. A 2-planar graph is a graph that has a 2-planar drawing. Pach and Toth have shown that 2-planar graphs with n vertices have at most $5n - 10$ edges and this bound is tight. 2-planar graph is a connected graph.

Hermits are vertices of degree 1 or 2 which are enclosed by crossing edges and cannot be connected to other vertices. Figure 3.4 (a) shows simple 2-Planar Graph and (b) shows 2-Planar Graph with dot Hermits vertics.

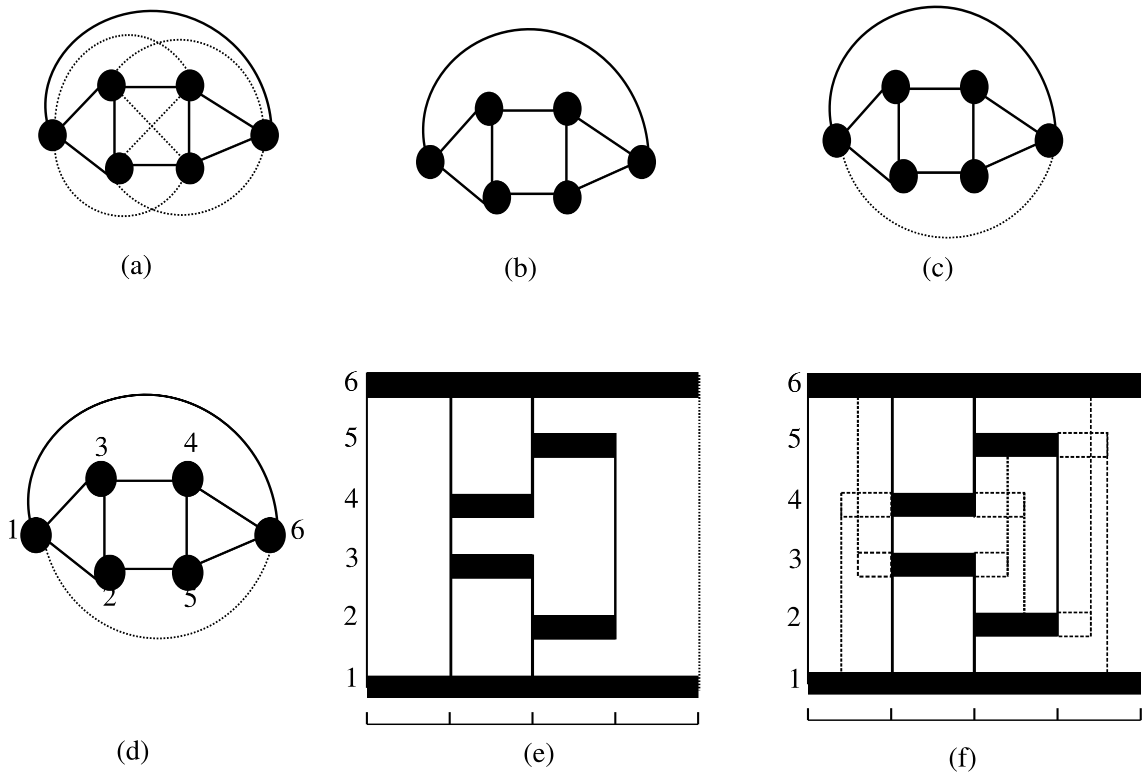


Figure 3.3: (a) A 1-planar graph ; (b) Remove all crossing-points from graph ; (c) Add extra edge according to rule ; (d) st-numbering Planar Graph ; (e) Visibility Representations of the Planar Graph ; (f) Bar 1-Visibility Representations of 1-Planar Graph.

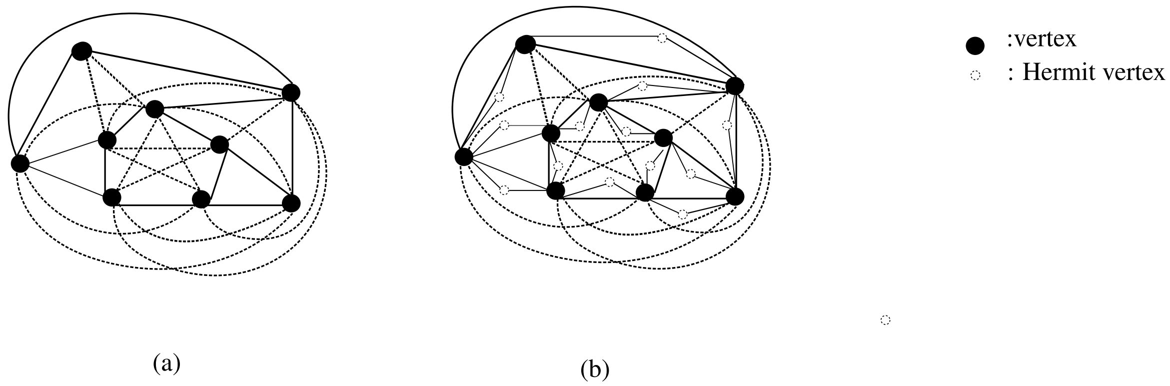
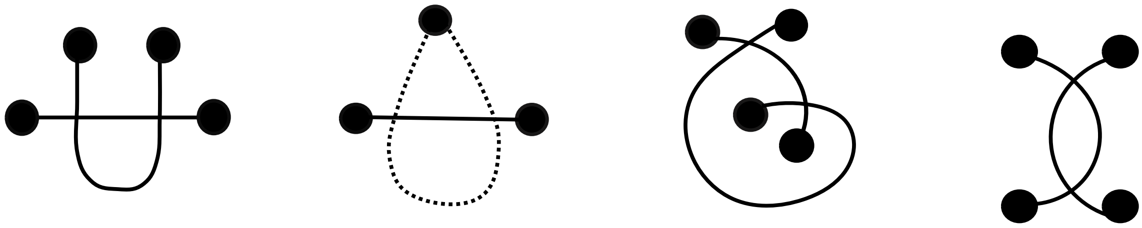


Figure 3.4: (a) A 2-Planar Graph ; (b) A 2-Planar Graph with Hermits vertices.

3.4.1 Configurations of 2-Planar Graphs

Since 2-planar graph is a non-planar graph , its configuration is complex. We know that there are only 3 configurations in 1-planar graph [12]. But 2-planar graphs may have many configurations. we follow [18] [14] and represents the configuration of 2-planar graphs into some specific layer. We represents the 2-planar configurations into following 2 parts.

- Simple Configuration
- Multi-Configuration



(a) (b) (d) (e)

Figure 3.5: 2-planar configuration with 2-edges.

Simple Configuration

After deleting any edge or deleting any crossing node from the graph ,the configuration becomes 1-planar or planar in the Simple Configuration. Simple Configurations can be divided into 3 following parts.

1. **2-planar with 2 edges:**
   In the configuration of 2-planar with 2 edges, Two edges are crossing to each other and both are crossing each other at most 2 times. Figure 3.5 shows that 2-planar crossing occur between 2 edges.

2. **2-planar with 3 edges:**
   In 2-planar with 3 edges crossing, 3 edges are crossed in such a way that either 2 edges are crossed by 1 edges where 2 edges are crossed 1 times and 1 edges are crossed 2 times by 2 edges or 3 edges are crossed by eachother in such a way so that they are crossed eachother at most 2 times.

   Figure 3.6 shows that 2-planar crossing occur between 3 edges.

3. **2-planar with 4 edges:**
   In 2-planar with 4 edges configuration, all edges are connected eachother at most 2 times. Figure 3.7 shows that 2-planar crossing occur between 4 edges.

Multi-Configuration

If the graph is still two planar after removing a crossing point from 2-planar graph's configuration, the configuration will call Multi-Conguration of 2-planar graph(Figure 3.8). Multi-configuration is occured when a face has ≥ 5 edges.

In Figure 3.9 , we can see if we delete crossing node from the configuration Figure 3.9(b) , The configuration is still remain 2-planar (Figure 3.9(c)). Every 2-planar with odd number edges configuration has 2-planar with 3 edges configuration and there has no 2-planar with 3 edges configuration of 2-planar with even number edges configuration.

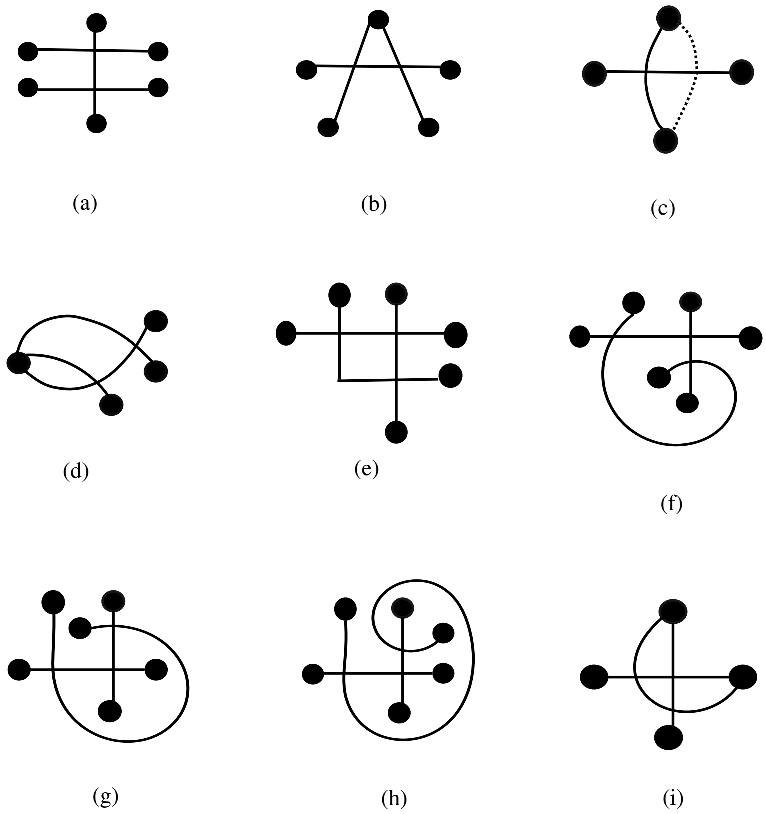


Figure 3.6: 2-planar configuration with 3-edges

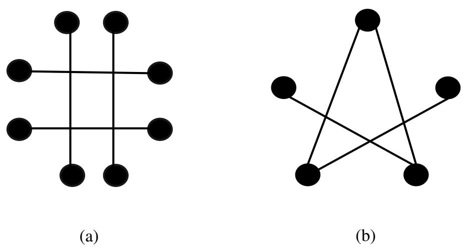


Figure 3.7: 2-planar configuration with 4-edges

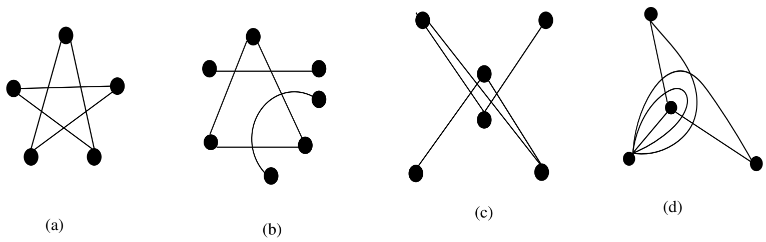


Figure 3.8: Multi-configuration

Lemma 1. *Let $E(G)$ be a maximal 2-planar embedding. Then any edges are crossing at most 2 times.*

Lemma 2. *Let $E(G)$ be a maximal 2-planar embedding. Then with 5 vertices and 5 edges muli-configuration (inner face or outer face) is a complete configuration.*

Proof. If any face is consist of 5 vertices, They can be crossing each other with at most 5 edges

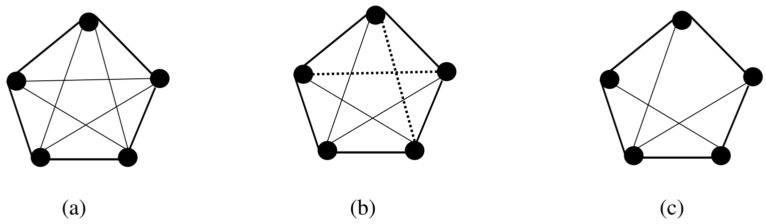


Figure 3.9: Multi-Configuration to Simple Configuration.

and each vertex will have degree 4. Every edges crossing each other 2 times. If we insert any edge then edge crossing 3 times and the graph will become 3-planar graph. Since we know $\geq$ 5 edges occur multi-configuration hence with 5 vertices and 5 edges (inner face or outer face) is a complete configuration which is look like star. Figure 3.9(a) shows the complete configuration. □

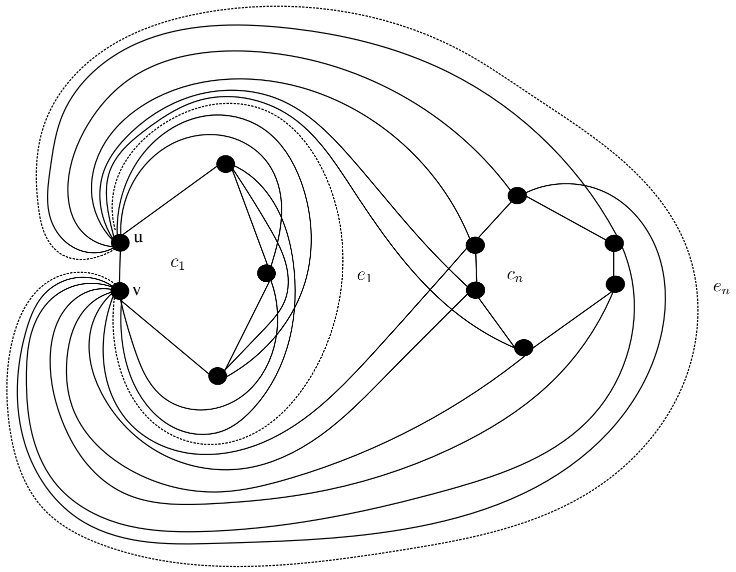
If configuration has n vertex , there will have at most $\frac{n}{2}$ times crossing points. We know that muli-configuration occur for ≥ 5 edges. When edges is <5 then there will at most 2 crossing points. For 5 edges there are 2 crossing nodes and 1 edges and this is the complete configuration. For n vertex , configuration will have at most $\frac{n}{2}$ times crossing points. We work on bar 2-visibility representation of 2-planar graph with at most 5 edges hence a 2-planar configuration will complete configuration with 5 crossing edges.

Lemma 3. *Let $E(G)$ be a maximal 2-planar embedding. Then represent the graph into an ideal form.*

[12] given a theorem of *Normal Form* for 1-planar graph. We follow this theorem and generate a new theorem which is known as Ideal Form of 2-planar Graph which is given below

Theorem 3. *The ideal form of 2-planar graph is, if outer face have any crossing, we will insert an extra edge which will multi-edge. It will not cause any crossing and will remove later. And in 2-planar graphs inner face may have more than one configuration. If inner face have two configuration, we will insert an extra edge which will multi-edge between two configuration.*

Suppose that G is 2-connected with an embedding $E(G)$ with maximal 2-planar components in ideal form. For every separation pair (u, v) there is a sequence $C_1 .... C_n$(Fig**??**) in clockwise order at u. To separate the components at a separation pair (u, v) even further we allow multi-edges and introduce copy edges e_1 to e_n as separation edges. The separation edge e separates all components. The outermost separation edge e encloses all components and the multi-edges from the outer face. This situation is depicted in Fig**??**, where (u, v) is separation pair, the copies of the edge e drawn dotted.



3.5 Conclusion

In this chapter, we have described important algorithm on visibility representation of planar graphs. The approach of this algorithm are very often followed by the researchers in this field. We have mentioned about 2-planar graph and represents its configurations into different part. Finally we generate an ideal form theorem.

Chapter 4

Bar 2-Visibility Representations of 2-Planar Graphs

4.1 Introduction

In Section 3.4 we define about 2-planar graphs. In this chapter we show how to transform a 2-planar graphs into bar 2-visibility representations.

4.2 Numbering of non-planar graph

we know st-numbering is apply on planar graph [23]. There is no algorithm or rules of st-numbering on non-planar graph. We develop a theorem4 and an algorithm2 for st-numbering on non-planar graph.

In Figure 4.1(a) we draw a 2-planar graph which has no number. To represent this graph into st-numbering, we insert temporary vertex at every crossing point on the 2-planar graph. IN Figure 4.1(b), we insert temporary vertex in all crossing points. After that, we delete the temporary vertex so that the graph becomes 2-planar to planar graph. Figure 4.1(c) is now planar graph. Now we apply the rule of st-numbering and insert number to the graph Figure 4.1(d). Finally all the crossing points that had been deleted by temporary vertex(Figure 4.1(b)) will reinserted Figure 4.1(e).

We develop the following theorem.

Theorem 4. *There is a linear time algorithm to represent the non-planar graph into st-numbering.*

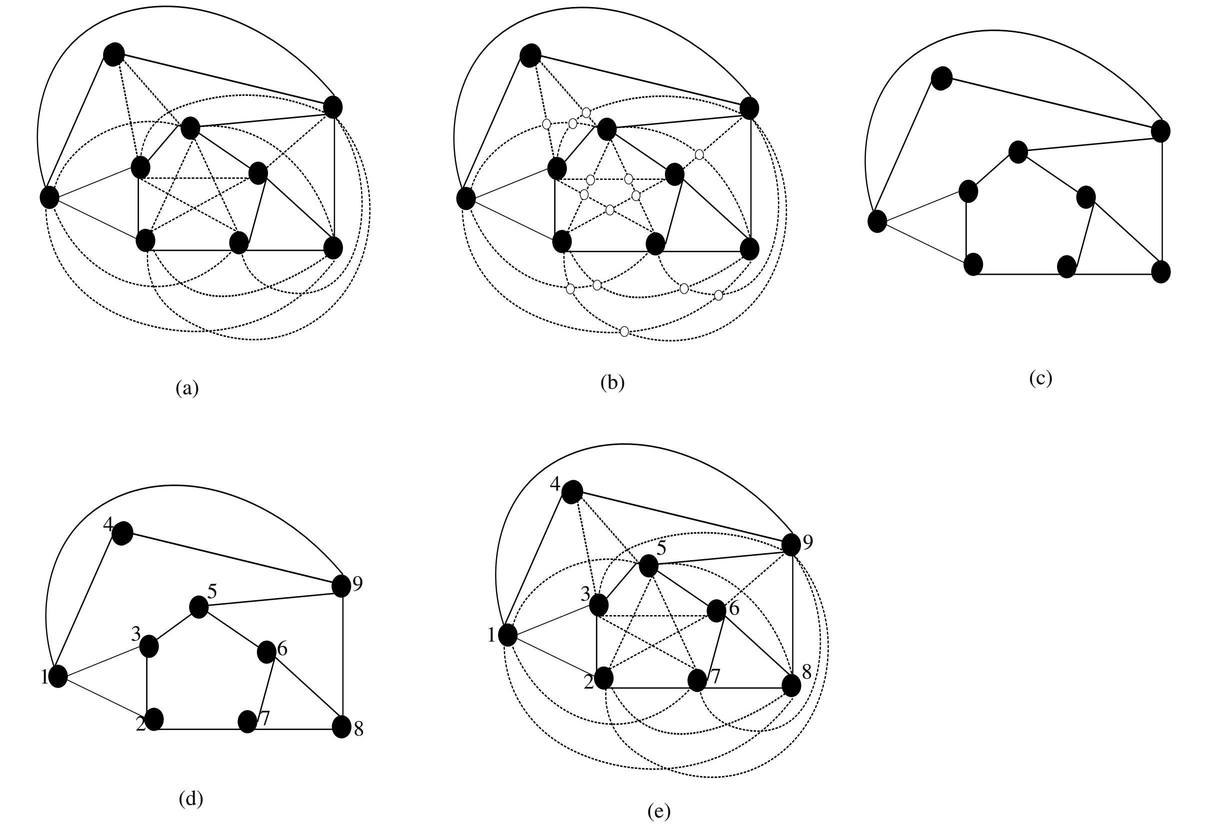


Figure 4.1: (a)2-planar Graph; (b)Insert vertex on crossing point in the 2-planar graph; (c)Planar Graph G; (d)st-numbering on planar graph G; (e)2-planar graph with st-numbering.

Algorithm 2 st-numbering on Non-Planar Graph

Insert temporary vertex in all the crossing points in the graph.
Delete the temporary vertex so that the graph will convert into planar graph.
Insert st-Numbering to the simple graph.
Re-insert all the crossing point to the graph.

4.3 2-Visibility Representations of 2-Planar Graphs

In this section we show that every 2-planar graph G has a bar 2-visibility representation. The result is obtained by the BAR 2-VISIBILITY algorithm, whose input is a 2-connected embedding 2-planar graph $E(G)$ as a witness for 2-planarity. After convert 2-connected embedding 2-planar graph $E(G)$ into maximal 2-connected 2-planar graph $E(G)$ augmentation then we use st-Numbering of Non-planar Graph[4.2] algorithm so that we can insert number into vertices. We cannot directly convert the 2-planar graphs into bar 2-visibility representations. We follow the following steps.

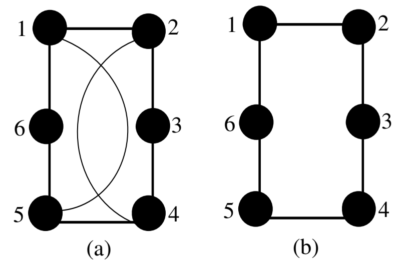


Figure 4.2: 2-planar with 2 edges transformation.

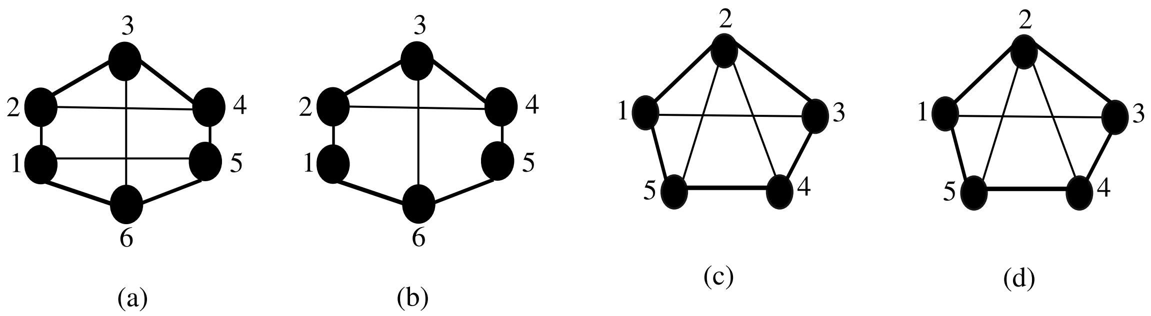


Figure 4.3: 2-planar with 3 edges transformation.

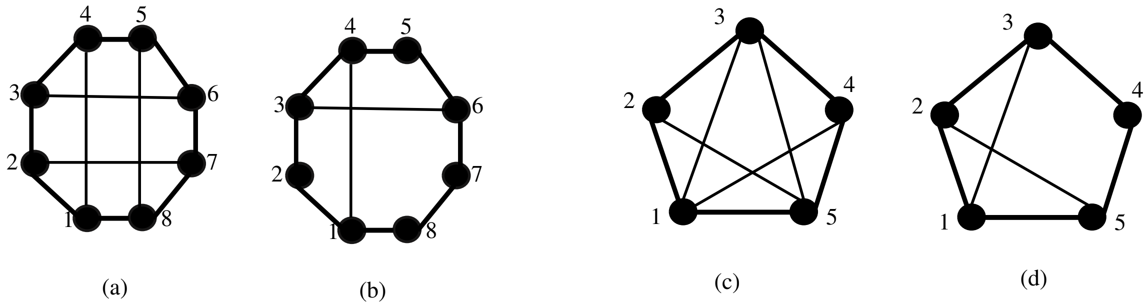


Figure 4.4: 2-planar with 4 edges transformation.

4.3.1 2-planar Graph to Planar Graph

1. Every 2-planar with 2 edges configuration can direct transform into planar by removing the crossing point. If we remove the crossing point from 2-planar with 2 edges configuration, configuration will become Planar Figure 4.2.

2. In 2-planar with 3 edges configuration, the edge pair that contains largest number of vertex by st-numbering will removed for transforming the 2-planar configuration to 1-planar configuration.Figure 4.3

3. In 2-planar with 4 edges configuration, the crossing point that contains highest two largest number of vertex by st-numbering will removed for transforming the 2-planar configuration to 1-planar configuration.Figure 4.4

4. In Multi-configuration, the crossing point that contains highest two largest number of vertices by st-numbering will be removed until the configuration transform into single

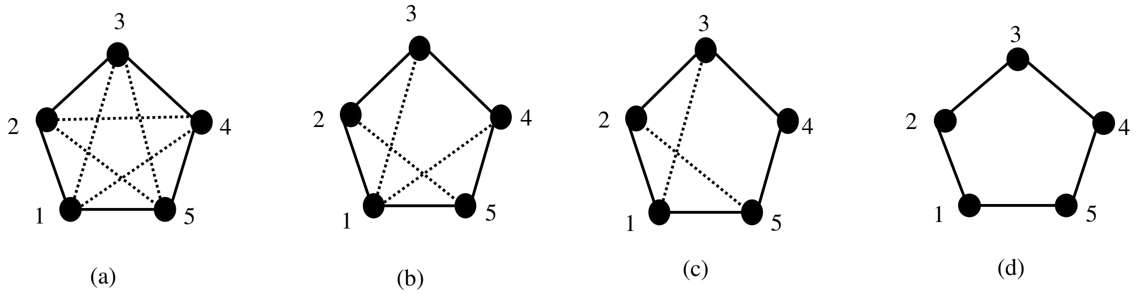


Figure 4.5: Multiconfiguartion Transformation. (a) Multiconfiguration ; (b)Single-Configuration ; (c)x-configuration ; (d) A face

configurations.Figure 4.5

In Figure 4.5(a) the 1st two largest vertex number is 5 and 4. so we remove the crossing ponit of vertex 5 and 4. Edge $5-3$ is crossing with edge $4-2$ and $4-1$. Between edge pair $4-2$ and $4-1$ vertex 2 is greater than 1, So edge $4-2$ will remove with edge $5-3$ as crossing point. After removing the 2-planar crossing point $5-3$ & $4-2$, Figure 4.5(b) becomes single configuration. Edge $1-4$ will deleted between edge $1-4$ & $1-3$ since vertex 4 is higher than vertex 3. In Figure 4.5(c), Now the configuration becomes 1-planar x-configuration.

Lemma 4. *If the configuration is simple configuration and the configuration is not 2-planar with 3 edges, Temporary delete the crossing point which is consist of 1st two highest number of vertices and if the configuration is 2-planar with 3 edges, The highest vertex number of edges will be temporary deleted from the configuration so that configuration becomes 1-planar configuration.*

Lemma 5. *If first two highest number of vertex are connected multiple times with eachother, we will select the edge that has highest number of vertex between two vertex for temporary deleting the crossing points.*

By lemma[4,5] we can transform the embedding 2-planar graphs $E(G)$ into embedding 1-planar graphs $E(G')$. When the 2-planar graphs transform into 1-planar graphs , 1-planar graph will have 1-planar configuration. After transform the embedding 2-planar graph $E(G)$ to 1-planar graph $E(G')$, we will convert the 1-planar graph $E(G')$ to planar graph $E(G'')$

Lemma 6. *If a graph has crossing point of 1-planar configuration, we'll delete the 1-planar crossing point.*

In Figure 4.5(c) has 1-planar crossing point which is X-configuration. We remove the 1-planar crossing point. Figure 4.5(d) shows a face f after deleting 1-planar crossing point.

Now the graph becomes a simple planar graph $E(G'')$.

4.3.2 Visibility Representation To Bar 2-Visibility Representation

We convert the 2-planar graphs to planar graphs. Using Tammassia and Tollis [3] visibility representation algorithm, we transform the planar graph to visibility representations of planar graphs. Then we will re-insert the 1-planar crossing point by the algorithm of 1-planar crossing point insertion2 for representing the graph to bar 1-visibility representation.

Algorithm 3 1-planar crossing point insertion

1: if face is a left-wing,then
At first we'll insert the 1st largest node number edge segment by expanding the vertex segment length 0.40. then we'll insert the 2nd largest node number edge segment by expanding the vertex segment length 0.60.
2: if face is a Right-wing, then
At first we'll insert the 1st largest node number edge segment by expanding the vertex segment length 0.40. then we'll insert the 2nd largest node number edge segment by expanding the vertex segment length 0.60.
3: if face is a diamond, then
At first we'll insert the 1st largest node number edge segment by expanding the vertex segment length 0.40. then we'll insert the 2nd largest node number edge segment by expanding both edge pair vertex segment length 0.60 from that side we insert previous edge-segment.

After converting planar visibility representation to bar 1-visibility representation, we will re-insert all the deleted 2-planar crossing edge and 2-planar crossing point for representing the bar 1-visibility representation to bar 2-visibility representation by the algorithm of 2-planar crossing insertion5.

Algorithm 4 2-planar crossing edge insertion

1: if the deleted crossing edge is from that vertex which has another crossing edge then
The edge segment will inserting by expanding the vertex length 0.20.
2: else
The edge segment will inserting by expanding the vertex length 0.60.

Algorithm 7 2-Planar Crossing Insertion.

1: if there is crossing edge for insertion then insert the edge segment by *2-planar crossing insertion* and go to next step, and if there is no crossing edge insertion then go to next step.
2: If there is no crossing point then stop. If there is crossing point then insert the crossing edge pair of the crossing point by *2-planar crossing point insertion* algorithm.

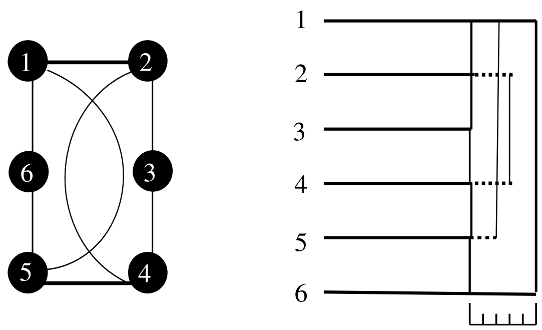
Figure 4.6 is the Visibility Representations of 2-Planar with 2 Edges Configuration. In fig:2planar2edgesvisi(a), edge pair (2,4) and (1,5) crossing with eachother atmost 2 times. fig:2planar2edgesvisi(b) is the visibility representatios of fig:2planar2edgesvisi(a) where edge segment of (1,5) is crossed by 2 vertex segment.

Algorithm 5 2-planar crossing point insertion

1: if face is a left-wing then,
largest node number egde will inserted by expanding the vertex segment 0.60 or 0.80 another edge will inserted between grid 0.20 to 0.80 by expanding the vertex segment.
2: if face is a right-wing then,
largest node number egde will inserted by expanding the vertex segment 0.60 or 0.80 another edge will inserted between grid 0.20 to 0.80 by expanding the vertex segment where expansion priority is 0.20>0.40>0.60>0.80
3: if face is a diamond then,
largest node number edge will inserted in the grid by expanding the vertex segment between 0.60 to 0.80 (expansion priority 0.60>0.80) another edge will inserted in the grid by expanding the vertex segment between 0.20 to 0.80 (expansion priority 0.20>0.40>0.60>0.80)
4: If there is any crossing point remain insert the crossing edge pair of crossing point by the algorithm of *n-crossing point insertion*

Algorithm 6 n-crossing point insertion

1: Largest node number edge will inserted in the grid by expanding the vertex segment 0.20 to 0.80 where expansion priority 0.20>0.40>0.60>0.80
2: Another edge will inserted in the grid by expanding the vertex segment 0.20 to 0.80 where expansion priority 0.20>0.40>0.60>0.80



(a) (b)

Figure 4.6: Visibility Representations of 2-Planar with 2 Edges Configuration.

Figure 4.7 is the Visibility Representations of 2-Planar with 3 Edges Configuration. fig:2planar2edgesvisi(b), is the visibility representations of fig:2planar2edgesvisi(a).In fig:2planar2edgesvisi(a), every vertex has single crossing edge,so in visibility representation of fig:2planar2edgesvisi(a) the crossing edge pair will inserted by expand the vertex length 0.60 which is shown in fig:2planar2edgesvisi(b). In Figure 4.7(a) crossing edge (3,6) is crossed by 2 edges and in Figure 4.7(b), edge segment (3,6) is crossed by two vertex. In the other side, Figure 4.7(c) and Figure 4.7(e) are type because in Figure 4.7(c) vertex no. 2 has two crossing edge pair and in Figure 4.7(e), vertex no. 1 has two crossing edge pair. So the deleted crossing edge will inserted by expand the vertex segment 0.20. Figure 4.7(d) is the visibility representation of Figure 4.7(c) and Figure 4.7(f) is the visibility representation of Figure 4.7(e).

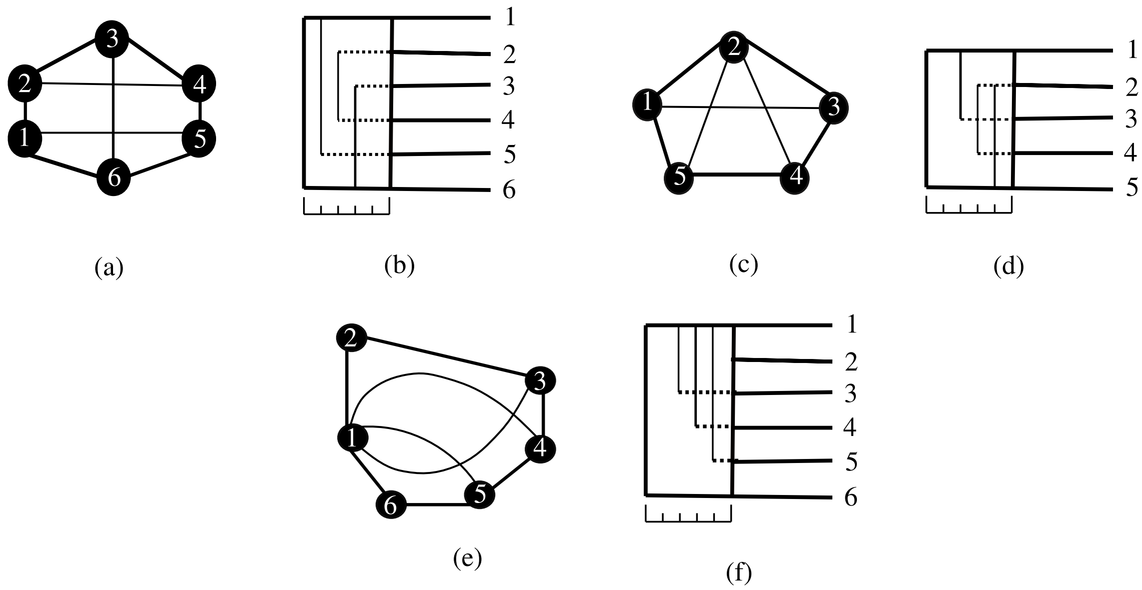


Figure 4.7: Visibility Representations of 2-Planar with 3 Edges Configuration.

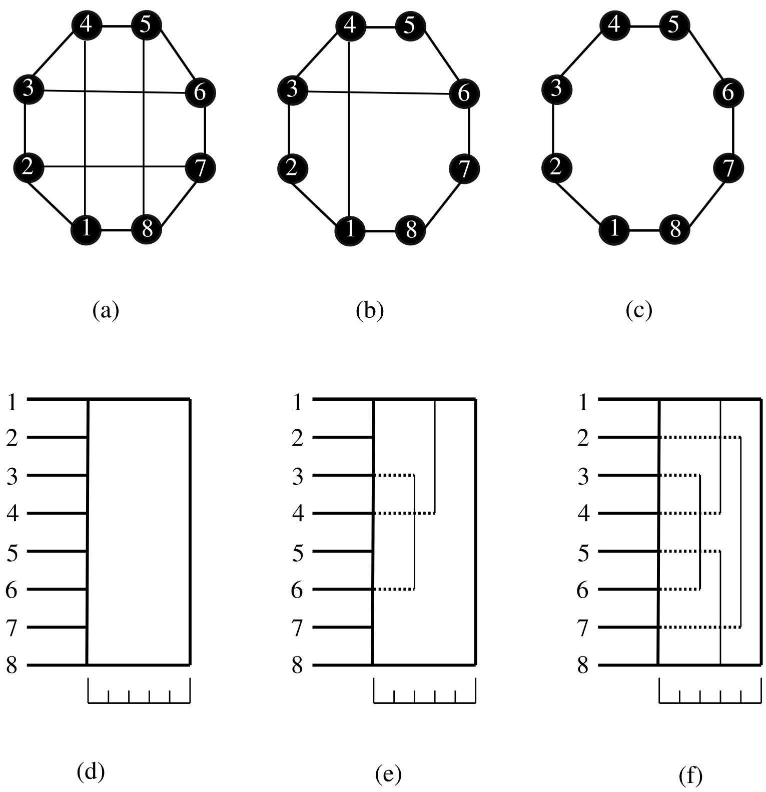


Figure 4.8: Visibility Representations of 2-Planar with 4 Edges Configuration.

Figure 4.8 shows the visibility representations of 2-planar with 4 edges configuration. Figure 4.8(f) is the visibility representation of Figure 4.8(a).

In Figure 4.9

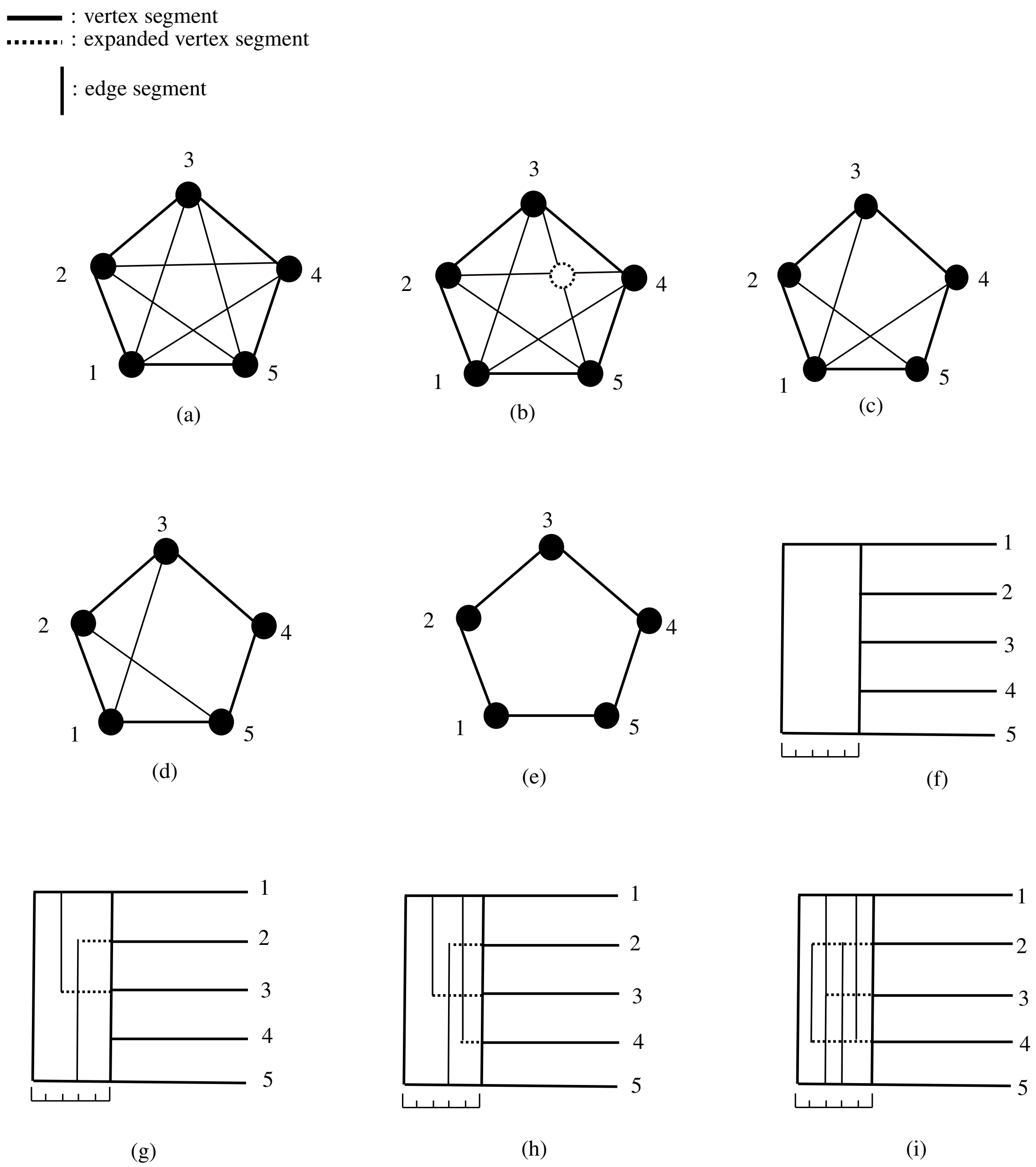


Figure 4.9: Visibility Representations of Multi-Configuration.

4.3.3 Face f in visibility representations

Face f changing depends on st-Numbering. Face can be 3 types: left wing, right wing and diamond.

In left wing Figure 4.9 the vertex expand from left side of the face. Similarly we can draw for right wing where vertex expand from right side in the face Figure 4.10. In Diamond, vertex-segments length expand from both left and right side Figure 4.11.

For face , At fist we apply 1-Planar crossing point insertion**??** algorithm. At first we'll insert the 1st largest node number edge segment by expanding the vertex segment length 0.40. And

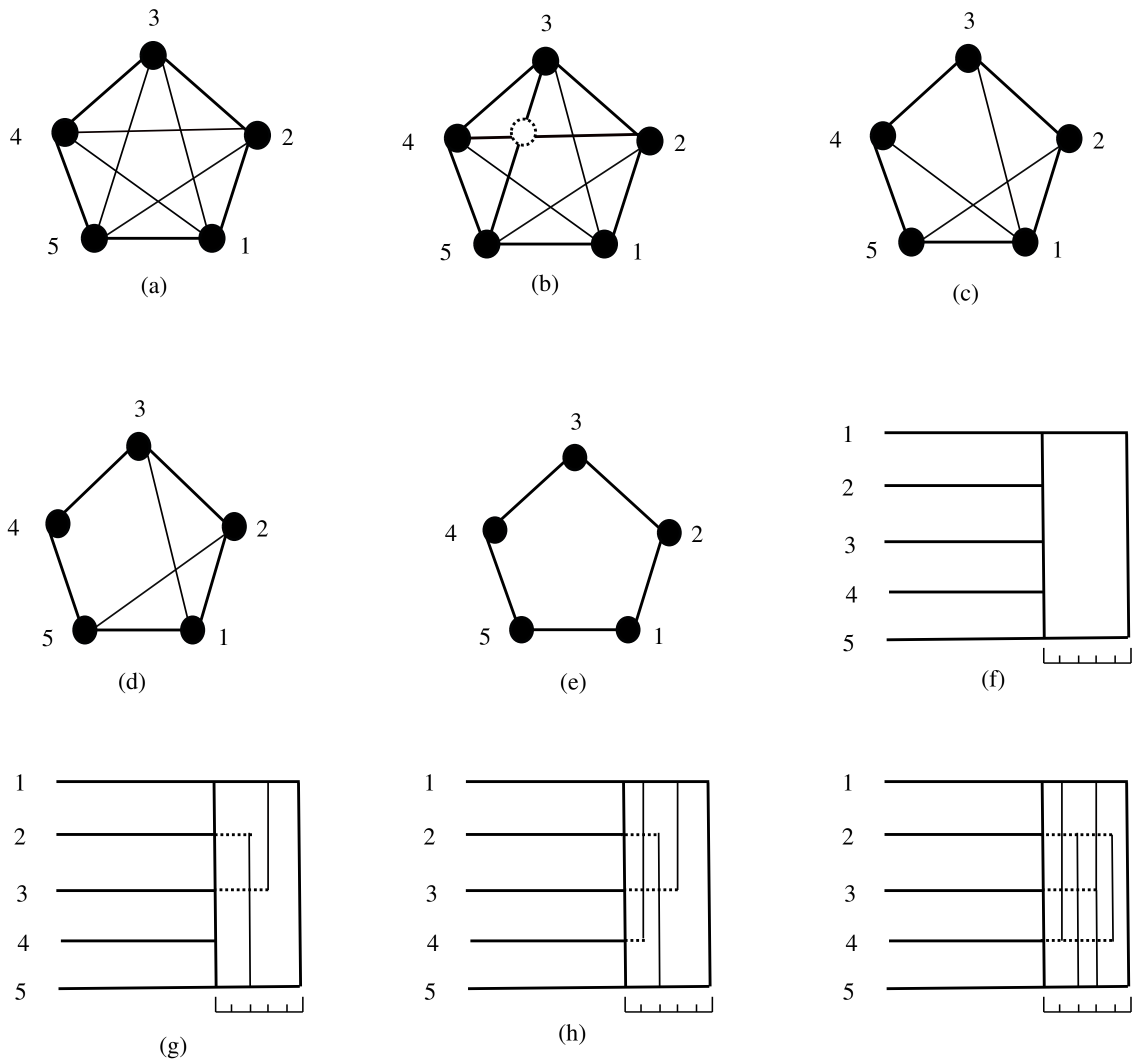


Figure 4.10: Right-wing.

then we'll insert the 2nd largest node number edge segment by expanding the vertex segment length 0.60. Then we apply 2-Planar Crossing Insertion 5 algorithm.

Lemma 7. *If a face $f = (1, 2, 3, 4, 5)$ is drawn by visibility representation, then at first* 1-planar crossing point insertion *adds the pair of crossing edges $(2, 5)$ and $(1, 3)$ and then 2-Planar Crossing Insertion adds the crossing edges $(1, 4)$ and finally again 2-Planar Crossing Insertion adds the pair of crossing edges $(5, 3)$ and $(4, 2)$ inside f with exactly two vertex-edge segment crossing.*

Proof. FigureFigure 4.9 shows the left wing because all vertex are expanding from left side in the face for the configuration. *1-planar crossing point insertion* adds the pair of crossing edges $(2, 5)$ and $(1, 3)$ where edge $(2, 5)$ insert by expanding vertex-segment 0.40 in the grid and edge $(1, 3)$ insert by expanding vertex-segment 0.60 in the grid. *2-Planar Crossing Insertion* adds the crossing edge $(1, 4)$ where edge $(1, 4)$ insert by expanding vertex-segment 0.20 in the grid. Finally we again use *2-Planar Crossing Insertion* algorithm which adds the pair of crossing

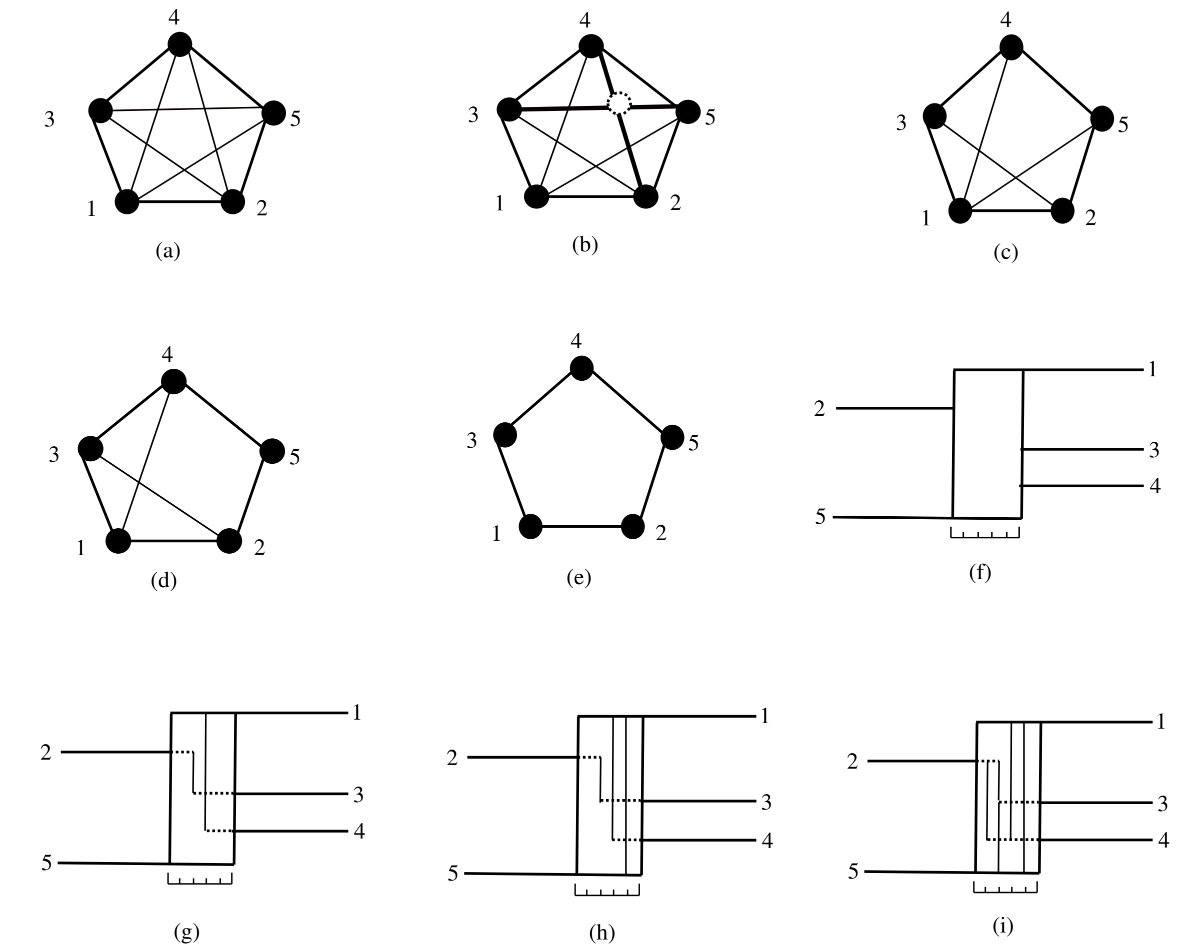


Figure 4.11: Diamond.

edges $(5, 3)$ and $(2, 4)$ where edge $(5, 3)$ insert by expanding vertex-segment 0.60 in the grid and edge $(2, 4)$ insert by expanding vertex-segment 0.80 in the grid. Edge (1, 4) and (2, 5) is crossing two vertex-segment. Now it becomes bar 2-visibility representation. □

We can now establish our main result.

Theorem 5. *There is a linear time algorithm to construct a Bar 2-Visibility Representation of 2-Planar Graphs.*

Proof. First take a embedding 2-connected 2-planar graph. Convert the 2-planar graph into maximal graph. Then we apply the st-Numbering of Non-Planar Graph algorithm. Using separation edge, we decompose the graph. Then we convert the 2-planar graph into 1-planar graph and 1-planar graph to planar graph. By Visibility Representation algorithm we represents the planar graph into visibility. By the algorithm of 1-planar crossing point insertion we represents the visibility representation to bar 1-visibility representation and using the algorithm of 2-planar crossing insertion we represents the bar 1-visibility representation to bar 2-visibility representation. Finally we delete all extra and separation edges and represents the whole process into the algorithm of Bar 2-Visibility. □

Algorithm 8 Bar 2-Visibility.

input : A 2-planar embedding $E(G)$ of a 2-connected 2-planar graph G.
output : A bar 2-visibility representation T on a grid.

Augment $E(G)$ to a maximal 2-connected 2-planar embedding and update G.
Decompose G into components.
foreach separating pair u, v do
In $E(G)$, add a copy of (u, v) as a separating edge to the right of each component at u and update G.
Transform $E(G)$ into ideal form.
Remove 2-planar crossing of $E(G)$. Let $E(G')$ be the remaining 1-planar embedding of the spanning 1-planar subgraph G' of G.
Remove 1-planar crossing point of $E(G')$. Let $E(G'')$ be the remaining planar embedding of the spanning planar subgraph G'' of G'. Assign each pair of crossing edges to the face of $E(G'')$ from which it was extracted.
Construct a planar visibility representation T of G'' by *Visibility Representation*.
Re-insert 1-planar crossing point and represent visibility representations to bar 1-visibility representation.
Re-insert 2-planar crossing and represent bar 1-visibility representations to bar 2-visibility representation.
Remove all the extra edges from T that were added.
Return T.

Figure 4.12,Figure 4.13,Figure 4.14,Figure 4.15 shows the total steps of Bar 2-Visibility Representations of 2-Planar Graphs. Figure 4.12(a) is the embedding 2-connected 2-planar graph $E(G)$. In Figure 4.12(b) we augment the embedding 2-connected 2-planar graph to maximal 2-connected 2-planar graph $E(G)$. Figure 4.12(c)(d)(e)(f) is the steps of st-numbering of non-planar graph so that we can represent our input graph into st-numbering for bar 2-visibility representation. Figure 4.13(a) is the ideal form of $E(G)$ where we separate each components by dotted separation edges which is multi-edge. We transform $E(G)$ to $E(G')$ in Figure 4.13(b) and Figure 4.14(a). Then we transform $E(G')$ to $E(G'')$ in Figure 4.14(b). Figure 4.15(a) is the planar visibility representation of Figure 4.14(b). Figure 4.15(b) is the bar 1-visibility representation of $E(G')$ (Figure 4.14(a)). Figure 4.15(c) is the bar 2-visibility representation of maximal embedding 2-planar graph Figure 4.13(a). Finally Figure 4.15(d) is the exact bar 2-visibility representation of out input graph.

4.4 Conclusion

In this chapter, we reviewe important algorithm on constrained bar 2-visibility representation of 2-planar graphs. We have shown that every 2-planar graph is bar 2-visibility representation.

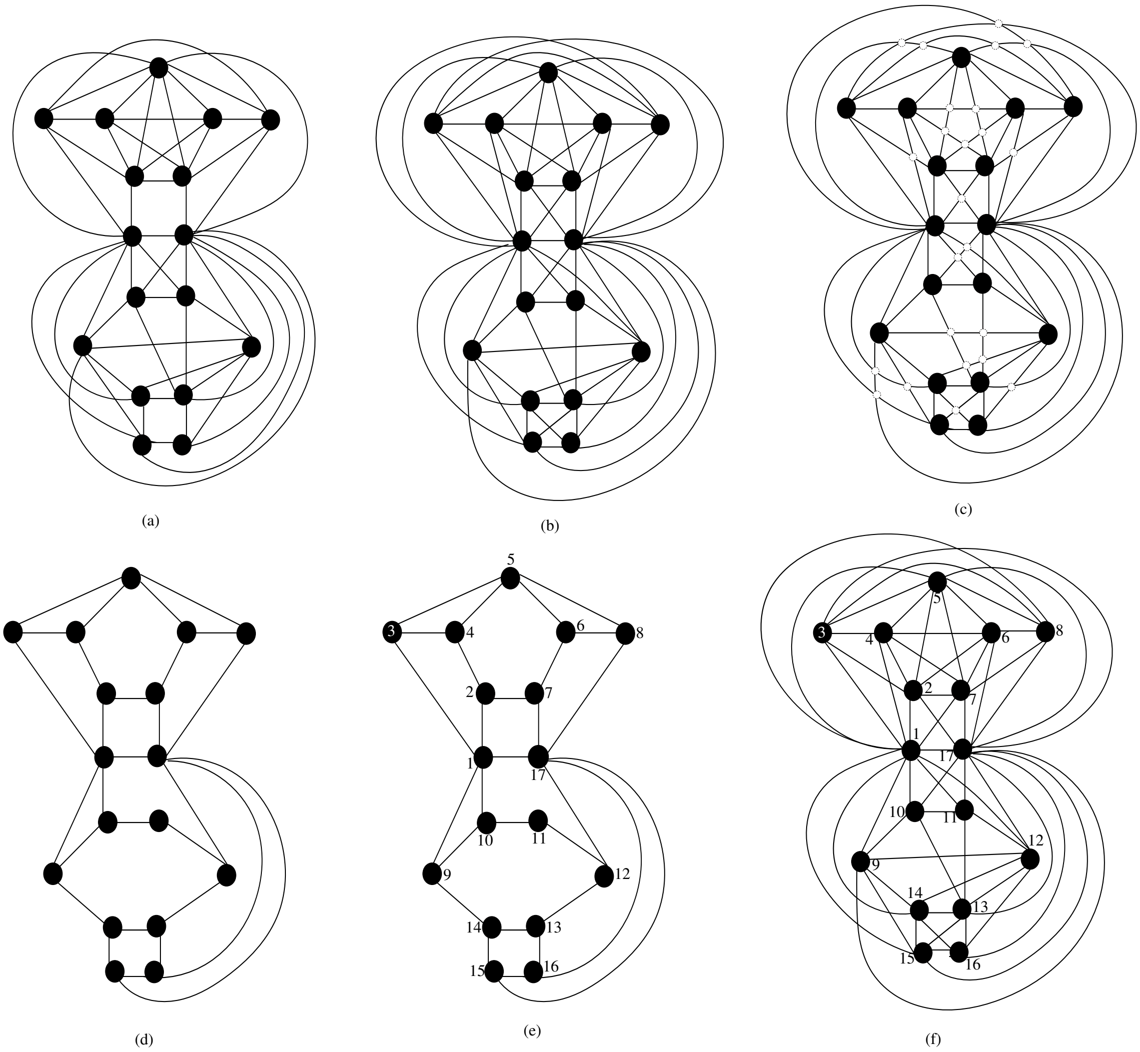
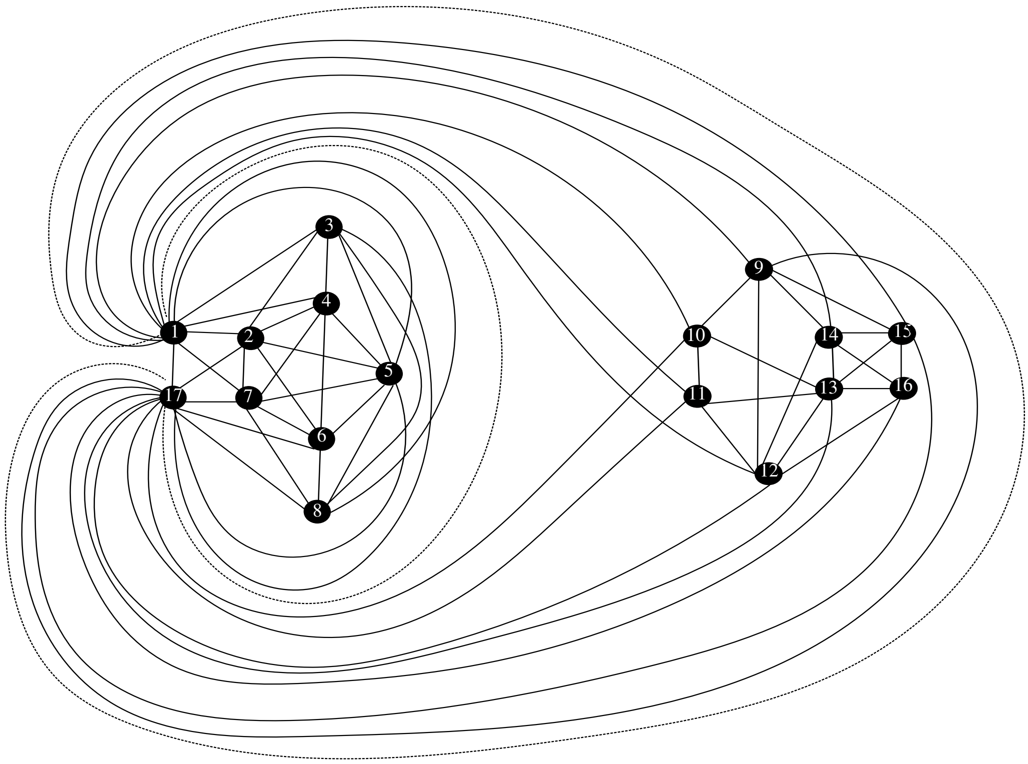
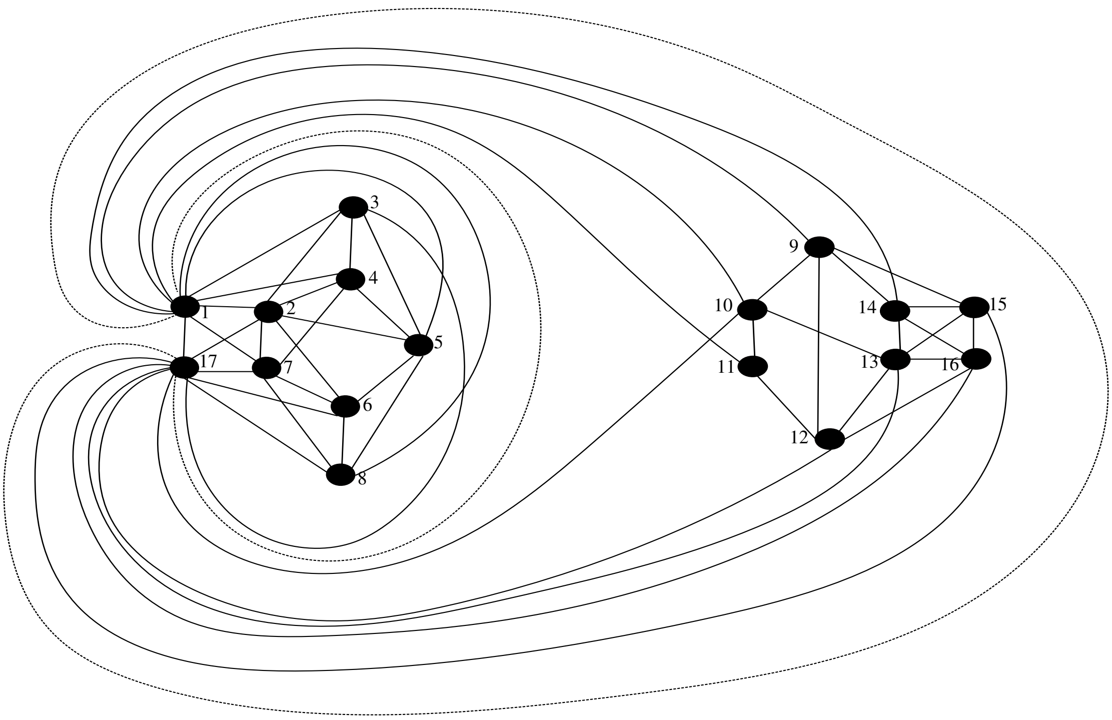


Figure 4.12: Bar 2-Visibility Representations of 2-Planar Graphs(part 1).

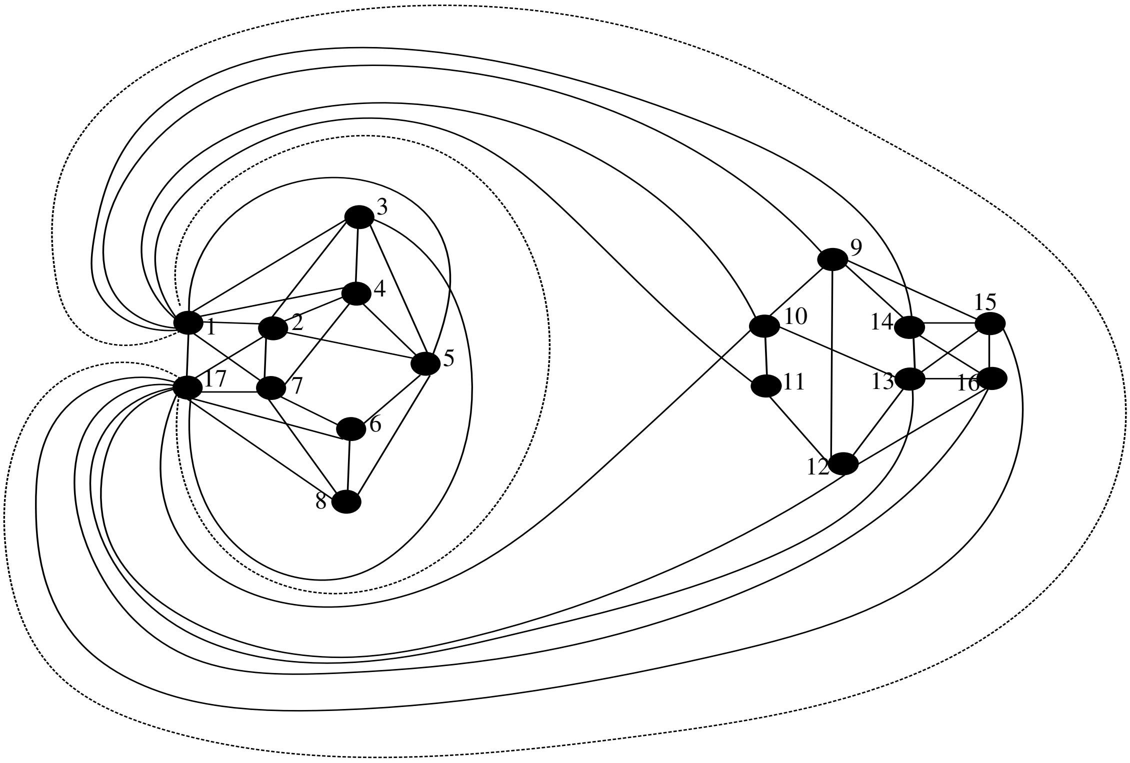


(a)

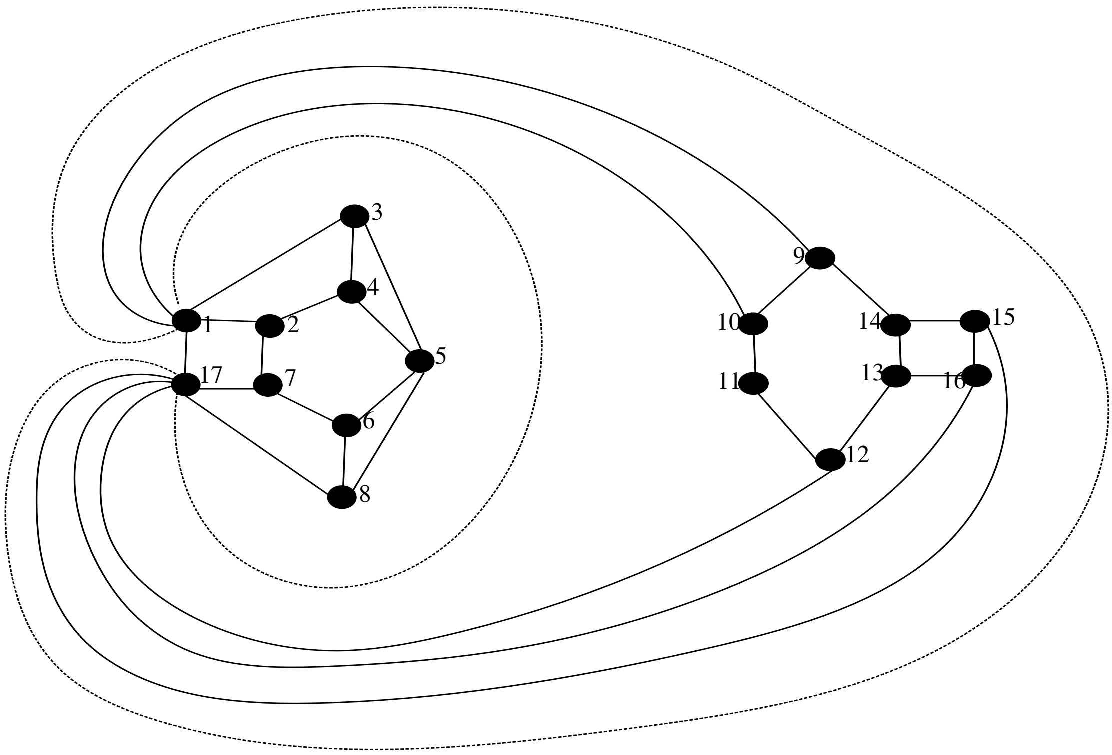


(b)

Figure 4.13: Bar 2-Visibility Representations of 2-Planar Graphs(part 2).



(a)



(b)

Figure 4.14: Bar 2-Visibility Representations of 2-Planar Graphs(part 3).

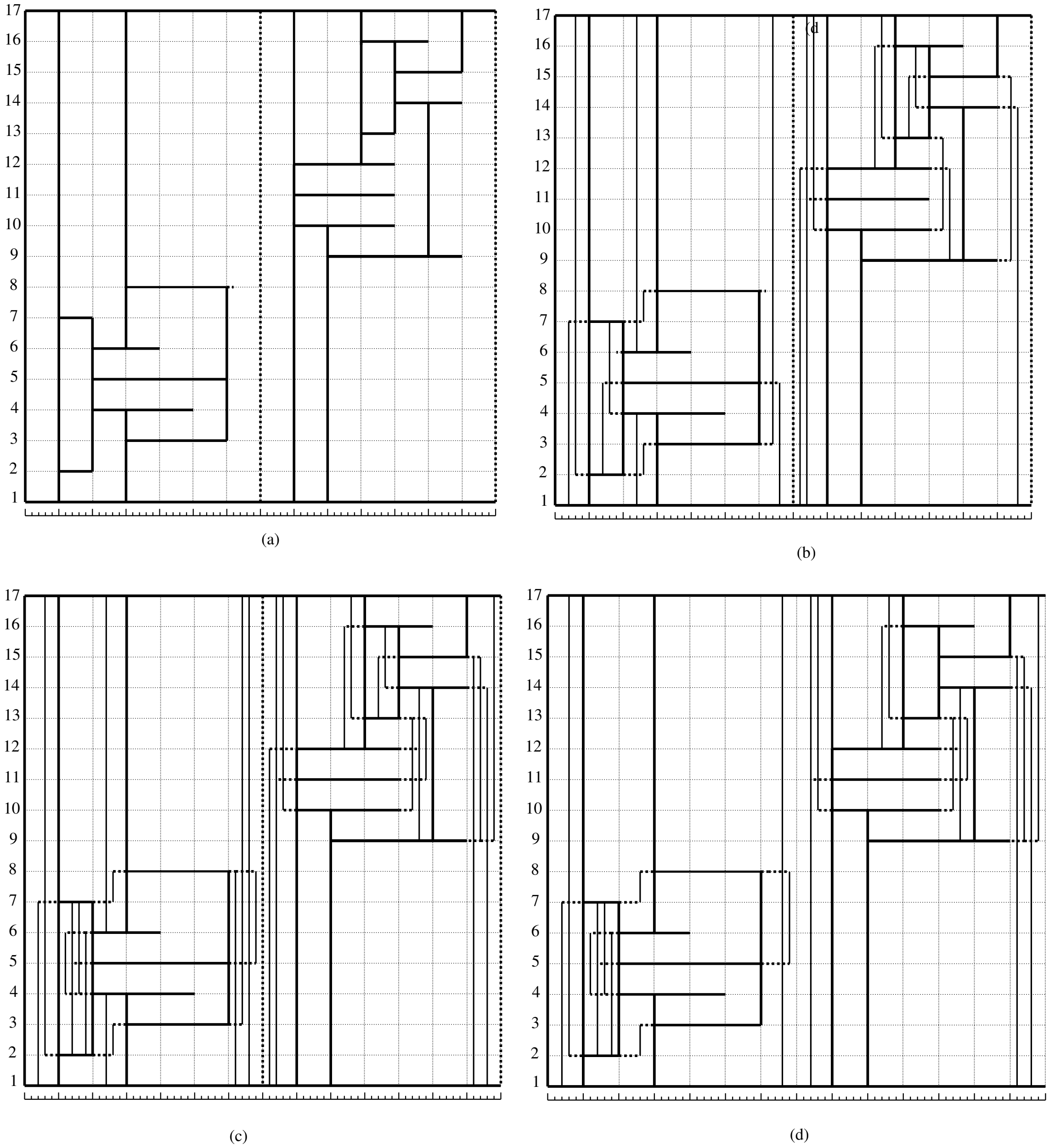


Figure 4.15: Bar 2-Visibility Representations of 2-Planar Graphs(part 4).

Chapter 5

Conclusion

In this chapter, we would like to review the concepts discussed in the earlier sections.
In this thesis, we generate algorithm of bar 2-visibility representations of 2-planar graph such on grid graph. We represent the configuration of 2-planar graphs into specific division. Given an undirected graph G, this thesis gives a drawing algorithm that gives bar 2-visibility representations on grid. We give an algorithm for st-numbering of non-planar graph.
In chapter 1 we presented a brief overview of the basic graph theoretic and drawing concepts. We discussed about visibility representation of planar graph and further defined briefly bar visibility graph. Then we came into the point of bar k-visibility graph and then we discussed about bar 1-visibility graph, bar 1-visibility representation, 1-visibility representation, bar 2-visibility graph and bar 2-visibility representation. We have mentioned some previous works which is related to our thesis.
In chapter 2, we have given some preliminary ideas on graph theory and algorithmic theory on st-numberingn. The chapter includes some basic concepts of graph that may be outside of the visibility representation but helped in our research in this area.
Chapter 3 presents description of visibility representations of planar graph, bar 1-visibility representations of 1-planar graph, 1-visibility representations of 1-planar graph, 2-planar graphs. This chapter gives some the details of the previous results, including the related proofs and descriptions and some part of our thesis work on 2-planar graphs.
Finally, in chapter 3, we present description of bar 2-visibility representations of 2-planar graph. We have described our results on Bar 2-Visibility Representations.

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