Linear Algebra (5th edition)

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Chapter 03: Elementary matrix operations and systems of linear equations

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3.1 Elementary matrix operations and elementary matrices

3.1 Elementary matrix operations and systems of linear equations

- **Objectives**
  - ① The study of certain "rank-preserving" operations in matrices
    - Obtaining a simple method for computing the rank of a linear transformation between finite-dimensional vector spaces
  - ② The application of these operations and the theory of linear transformations to the solution of systems of linear equations
    - The most important application of linear algebra
    - 3 Elementary row operations to simplify the system
      - Interchanging any two equations in the system
      - Multiplying any equation in the system by a non-zero constant
      - Adding a multiple of one equation to another

3.1 Elementary matrix operations and systems of linear equations

- **Elementary operations**

Elementary operations:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Any one of the following 3 operations on the rows (columns) of $\mathbf{A}$ is called an elementary row (column) operation:

① Interchanging any two rows (columns) of $\mathbf{A}$
② Multiplying any row (column) of $\mathbf{A}$ by a non-zero scalar
③ Adding any scalar multiple of a row (column) of $\mathbf{A}$ to another row (column)

- *Example 3.1.1*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{bmatrix} \overset{\text{Row1} \rightleftarrows \text{Row2}}{\Rightarrow} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{bmatrix} \overset{\text{Row2} \leftarrow \text{Row2} + 4 \times \text{Row3}}{\Rightarrow} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 17 & 2 & 7 & 12 \\ 4 & 0 & 1 & 2 \end{bmatrix}$$

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- *Example 3.1.1*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{bmatrix} \overset{\text{Col1} \rightleftarrows \text{Col2}}{\Rightarrow} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 2 & -1 & 3 \\ 0 & 4 & 1 & 2 \end{bmatrix} \overset{\text{Col2} \leftarrow \text{Col2} + 4 \times \text{Col3}}{\Rightarrow} \begin{bmatrix} 2 & 13 & 3 & 4 \\ 1 & -2 & -1 & 3 \\ 0 & 8 & 1 & 2 \end{bmatrix}$$

3.1 Elementary matrix operations and systems of linear equations

- **Elementary operations**

Elementary matrix:
An $n \times n$ elementary matrix is a matrix obtained by performing an elementary operation on $\mathbf{I}_n$.
The elementary matrix is said to be of type ①, ②, or ③ according to whether the elementary operation performed on $\mathbf{I}_n$ is a type ①, ②, or ③ operation, respectively.

- *Example*
  - Type ①: Interchanging the first two rows of $\mathbf{I}_3$

$$\mathbf{E} = \begin{bmatrix} 0 & \boxed{1} & 0 \\ \boxed{1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{EA} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{bmatrix}$$

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- *Example*
  - Type ①: Interchanging the first two columns of $\mathbf{I}_4$

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{AE} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 2 & -1 & 3 \\ 0 & 4 & 1 & 2 \end{bmatrix}$$

3.1 Elementary matrix operations and systems of linear equations

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- *Example*
  - Type ③: Row1←Row1-2×Row3 of $\mathbf{I}_3$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{EA} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 2 & 1 & 0 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{bmatrix}$$

3.1 Elementary matrix operations and systems of linear equations

- **Elementary operations**

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An $n \times n$ elementary matrix is a matrix obtained by performing an elementary operation on $\mathbf{I}_n$.
The elementary matrix is said to be of type ①, ②, or ③ according to whether the elementary operation performed on $\mathbf{I}_n$ is a type ①, ②, or ③ operation, respectively.

- *Example*
  - Type ③: Col1←Col1-2×Col3 of $\mathbf{I}_4$

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{AE} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 2 & 3 & 4 \\ 4 & 1 & -1 & 3 \\ 2 & 0 & 1 & 2 \end{bmatrix}$$

3.1 Elementary matrix operations and systems of linear equations

- **Elementary row operations**

Theorem 3.1:
Let $\mathbf{A} \in M_{m\times n}(F)$ and suppose that $\mathbf{B}$ is obtained from $\mathbf{A}$ by performing an elementary row (column) operation.
Then there exists an $m \times m$ ($n \times n$) elementary matrix $\mathbf{E}$ such that $\mathbf{B} = \mathbf{EA}$ ($\mathbf{B} = \mathbf{AE}$).

In fact, $\mathbf{E}$ is obtained from $\mathbf{I}_m$ ($\mathbf{I}_n$) by performing the same elementary row (column) operation as that which was performed on $\mathbf{A}$ to obtain $\mathbf{B}$.

Conversely, if $\mathbf{E}$ is an elementary $m \times m$ ($n \times n$) matrix, then $\mathbf{EA}$ ($\mathbf{AE}$) is the matrix obtained from $\mathbf{A}$ by performing the same elementary row (column) operation as that which produces $\mathbf{E}$ form $\mathbf{I}_m$ ($\mathbf{I}_n$).

3.1 Elementary matrix operations and systems of linear equations

- **Elementary row operations**

Theorem 3.2:
Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type.

- Proof)
  - Let $\mathbf{E}$ be an elementary $n \times n$ matrix.
  - Then, $\mathbf{E}$ can be obtained by an elementary row operation on $\mathbf{I}_n$.
  - By reversing the steps, we can transform $\mathbf{E}$ back into $\mathbf{I}_n$.
  - By Theorem 3.1,
    - There exists an elementary matrix $\bar{\mathbf{E}}$ such that $\bar{\mathbf{E}}\mathbf{E} = \mathbf{I}_n$.
  - $\therefore \mathbf{E}^{-1} = \bar{\mathbf{E}}$
  - $\therefore$ Q.E.D.