Linear Algebra (5th edition)

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Chapter 02: Linear transformations and matrices

Assignments

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- **Problems**

1. Label the following statements as true or false. In each part, V, W, and Z denote vector spaces with ordered (finite) bases α, β, and γ, respectively; $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ and $\mathsf{U}\colon \mathsf{W} \to \mathsf{Z}$ denote linear transformations; and A and B denote matrices.

(a) $[\mathsf{UT}]_\alpha^\gamma = [\mathsf{T}]_\alpha^\beta [\mathsf{U}]_\beta^\gamma$.

(b) $[\mathsf{T}(v)]_\beta = [\mathsf{T}]_\alpha^\beta [v]_\alpha$ for all $v \in \mathsf{V}$.

(c) $[\mathsf{U}(w)]_\beta = [\mathsf{U}]_\alpha^\beta [w]_\beta$ for all $w \in \mathsf{W}$.

(d) $[\mathsf{I_V}]_\alpha = I$.

(e) $[\mathsf{T}^2]_\alpha^\beta = ([\mathsf{T}]_\alpha^\beta)^2$.

(f) $A^2 = I$ implies that $A = I$ or $A = -I$.

(g) $\mathsf{T} = \mathsf{L}_A$ for some matrix A.

(h) $A^2 = O$ implies that $A = O$, where O denotes the zero matrix.

(i) $\mathsf{L}_{A+B} = \mathsf{L}_A + \mathsf{L}_B$.

(j) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$.

2.3 Composition of linear transformations and matrix multiplication

- **Problems**

2. **(a)** Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute $A(2B+3C)$, $(AB)D$, and $A(BD)$.

(b) Let

$$A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 4 & 0 & 3 \end{pmatrix}.$$

Compute A^t, A^tB, BC^t, CB, and CA.

2.3 Composition of linear transformations and matrix multiplication

- **Problems**

9. Find linear transformations $\mathsf{U}, \mathsf{T}: \mathsf{F}^2 \to \mathsf{F}^2$ such that $\mathsf{UT} = \mathsf{T}_0$ (the zero transformation) but $\mathsf{TU} \neq \mathsf{T}_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

2.4 Invertibility and isomorphisms

- **Problems**

1. Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectively, $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ is linear, and A and B are matrices.

(a) $\left([\mathsf{T}]_\alpha^\beta\right)^{-1} = [\mathsf{T}^{-1}]_\alpha^\beta$.
(b) T is invertible if and only if T is one-to-one and onto.
(c) $\mathsf{T} = \mathsf{L}_A$, where $A = [\mathsf{T}]_\alpha^\beta$.

(f) $AB = I$ implies that A and B are invertible.
(g) If A is invertible, then $(A^{-1})^{-1} = A$.
(h) A is invertible if and only if L_A is invertible.
(i) A must be square in order to possess an inverse.

2.5 The change of coordinate matrix

- **Problems**

1. Label the following statements as true or false.

(a) Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$.

(b) Every change of coordinate matrix is invertible.

(c) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[\mathsf{T}]_\beta = Q[\mathsf{T}]_{\beta'}Q^{-1}$.

(d) The matrices $A, B \in \mathsf{M}_{n\times n}(F)$ are called similar if $B = Q^tAQ$ for some $Q \in \mathsf{M}_{n\times n}(F)$.

(e) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[\mathsf{T}]_\beta$ is similar to $[\mathsf{T}]_\gamma$.

2.5 The change of coordinate matrix

- **Problems**

2. For each of the following pairs of ordered bases β and β' for R^2, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$
(b) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$
(c) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$
(d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

2.5 The change of coordinate matrix

- **Problems**

6. For each matrix A and ordered basis β, find $[\mathsf{L}_A]_\beta$. Also, find an invertible matrix Q such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

(a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$