HANBAT NATIONAL UNIVERSITY

# Linear Algebra (5th edition)

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Chapter 02: Linear transformations and matrices

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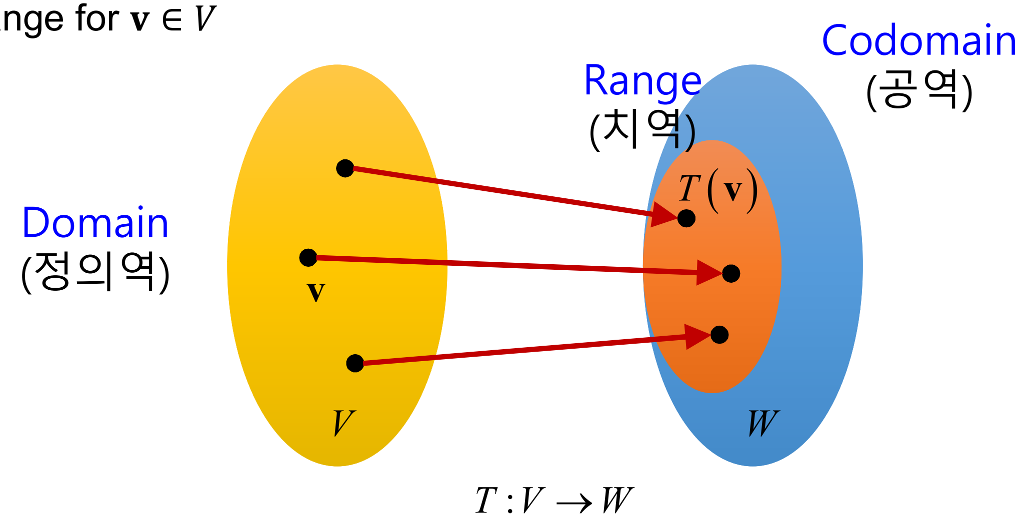
# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - Notation
    - $T : V \rightarrow W$
      - $T$: A function
      - $V$: A domain
      - $W$: A codomain
      - $T(\mathbf{v})$: A range for $\mathbf{v} \in V$



# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**

> **Linear transformation**:
> Let $V$ and $W$ be vector spaces over the same field $F$.
> We call a function $T: V \to W$ a linear transformation from $V$ to $W$ (or just linear) if, for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$, we have
>
> (a) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, and
> (b) $T(c\mathbf{x}) = cT(\mathbf{x})$

- Properties
  - ① $T$ is linear $\Rightarrow$ $T(\mathbf{0}) = \mathbf{0}$
  - ② $T$ is linear $\Leftrightarrow$ $T(c\mathbf{x}+\mathbf{y}) = cT(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$
  - ③ $T$ is linear $\Rightarrow$ $T(\mathbf{x}-\mathbf{y}) = T(\mathbf{x}) - T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$
  - ④ $T$ is linear $\Leftrightarrow$ $T(\sum_{i=1}^{n} a_i \mathbf{x}_i) = \sum_{i=1}^{n} a_i T(\mathbf{x}_i)$ for $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ and $a_1, \dots, a_n \in F$

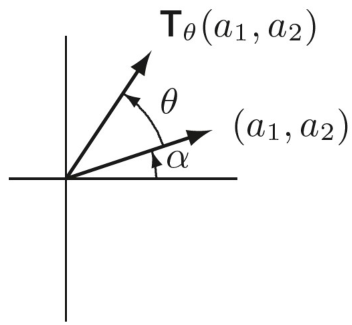
- Generally, property ② often used to prove a given transformation $T$ is linear

# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.1*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 + a_2 \\ a_1 \end{bmatrix}$
    - Q: Is function $T$ linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} 2x_1 + x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} c(2x_1 + x_2) + 2y_1 + y_2 \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix}$
      - $\therefore$ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

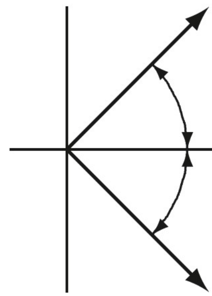
- **Linear transformations**
  - *Example 2.1.2 (Rotation)*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1\cos\theta - a_2\sin\theta \\ a_1\sin\theta + a_2\cos\theta \end{bmatrix}$




    - Q: Is function $T$ linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix} =$ $\begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix} + \begin{bmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{bmatrix} =$ $\begin{bmatrix} c(x_1\cos\theta - x_2\sin\theta) + (y_1\cos\theta - y_2\sin\theta) \\ c(x_1\sin\theta + x_2\cos\theta) + (y_1\sin\theta + y_2\cos\theta) \end{bmatrix} = \begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix}$
      - ∴ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

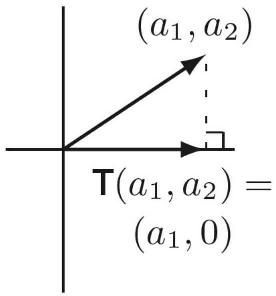
- **Linear transformations**
  - *Example 2.1.3 (Reflection)*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$




    - Q: Is function $T$ linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + y_1 \\ -cx_2 - y_2 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ -cx_2 - y_2 \end{bmatrix}$
      - ∴ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.4 (Projection on the 1st dimension)*
    - $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$




    - Q: Is function $T$ linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + y_1 \\ 0 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ 0 \end{bmatrix}$
      - ∴ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.5 (Transpose)*
    - $T: M_{m\times n}(F) \rightarrow M_{n\times m}(F)$ where $T(A) = A^T$
    - Q: Is function $T$ linear?
      - $T(c\mathbf{X} + \mathbf{Y}) = (c\mathbf{X} + \mathbf{Y})^T = c\mathbf{X}^T + \mathbf{Y}^T$
      - $cT(\mathbf{X}) + T(\mathbf{Y}) = c\mathbf{X}^T + \mathbf{Y}^T$
      - ∴ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.6 (Derivatives)*
    - $T: V \rightarrow V$ where $T(f) = \frac{df}{dv}$
    - Q: Is function $T$ linear?
      - Letting $g \in V$ and $h \in V$
      - $T(cg + h) = \frac{d}{dv}(cg + h) = c\frac{dg}{dv} + \frac{dh}{dv}$
      - $cT(g) + T(h) = c\frac{dg}{dv} + \frac{dh}{dv}$
      - ∴ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.7 (Integration)*
    - $T: \mathbb{R} \to \mathbb{R}$ where $T(f) = \int_a^b f(t)dt$ for some $a, b \in \mathbb{R}$
    - Q: Is function $T$ linear?
      - Letting $g \in \mathbb{R}$ and $h \in \mathbb{R}$
      - $T(cg + h) = \int_a^b cg(t) + h(t)dt = c\int_a^b g(t)dt + \int_a^b h(t)dt$
      - $cT(g) + T(h) = c\int_a^b g(t)dt + \int_a^b h(t)dt$
      - ∴ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example (Identity transformation)*
    - $T: V \to V$ where $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$
    - Q: Is function $T$ linear?
      - Letting $\mathbf{x} \in V$ and $\mathbf{y} \in V$
      - $T(c\mathbf{x} + \mathbf{y}) = c\mathbf{x} + \mathbf{y}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\mathbf{x} + \mathbf{y}$
      - $\therefore$ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example (Zero transformation)*
    - $T: V \to W$ where $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$
    - Q: Is function $T$ linear?
      - Letting $\mathbf{x} \in V$ and $\mathbf{y} \in V$
      - $T(c\mathbf{x} + \mathbf{y}) = \mathbf{0}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}$
      - $\therefore$ By property ②, linear!

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

**Null space (kernel)**:
Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
We define null space (or kernel) $N(T)$ of $T$ to be the set of all vectors $\mathbf{x} \in V$ such that

$$N(T) = \{\mathbf{x} \in V \mid T(\mathbf{x}) = \mathbf{0}\}$$

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

**Range (image) (치역)**:
Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
We define range (or image) $R(T)$ of $T$ to be the subset of $W$ containing all images (outputs) under $T$ of vectors in $V$ such that

$$R(T) = \{T(\mathbf{x}) | \mathbf{x} \in V\}$$

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - <u>*Example 2.1.8*</u>
    - $T_1: V \to V$ where $T_1(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$ (identity transformation)
      - Null space
        - $N(T_1) = \{\mathbf{0}\}$
      - Range
        - $R(T_1) = V$
    - $T_2: V \to W$ where $T_2(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$ (zero transformation)
      - Null space
        - $N(T_2) = V$
      - Range
        - $R(T_2) = \{\mathbf{0}\}$

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.9*
    - $T: \mathbb{R}^3 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 - a_2 \\ 2a_3 \end{bmatrix}$ for all $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$
      - Null space
        - $N(T) = \left\{ \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \middle| x \in \mathbb{R} \right\}$
      - Range
        - $R(T) = \mathbb{R}^2$

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.1**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> Then, $N(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.

- Proof) ($N(T)$ is a subspace of $V$)
  - Theorem 1.3(a)
    - $\mathbf{0} \in N(T)$ since property ① of linear transformation indicates that $T(\mathbf{0}) = \mathbf{0}$
  - Theorem 1.3(b)
    - $\mathbf{x} + \mathbf{y} \in N(T)$ since $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}$ for $\mathbf{x}, \mathbf{y} \in N(T)$
  - Theorem 1.3(c)
    - $c\mathbf{x} \in N(T)$ since $T(c\mathbf{x}) = cT(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in N(T)$
  - ∴ Subspace!

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.1**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> Then, $N(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.

- Proof) ($R(T)$ is a subspace of $W$)
  - Theorem 1.3(a)
    - $\mathbf{0} \in R(T)$ since $\mathbf{0} \in V$ and property ① of linear transformation indicates that $T(\mathbf{0}) = \mathbf{0}$
  - Theorem 1.3(b)
    - $\mathbf{x} + \mathbf{y} \in R(T)$ since $\mathbf{x} + \mathbf{y} = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2) \in R(T)$ for $\mathbf{x} = T(\mathbf{v}_1), \mathbf{y} = T(\mathbf{v}_2) \in R(T)$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$
  - Theorem 1.3(c)
    - $c\mathbf{x} \in R(T)$ since $c\mathbf{x} = cT(\mathbf{v}_1) = T(c\mathbf{v}_1) \in R(T)$ for $\mathbf{x} = T(\mathbf{v}_1) \in R(T)$ and $\mathbf{v}_1 \in V$
  - ∴ Subspace!

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.2**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of $V$ $\Rightarrow$ $R(T) = \mathrm{span}\big(T(\beta)\big) = \mathrm{span}\big(T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\big)$

- Proof)
  - ($\mathrm{span}\big(T(\beta)\big) \subseteq R(T)$)
    - Note that $T(\mathbf{v}_i) \in R(T), \forall i$
    - From Theorem 2.1, $R(T)$ is a subspace
      - $\Rightarrow \mathrm{span}(\{T(\mathbf{v}_i)\}) = \mathrm{span}\big(T(\beta)\big) \in R(T)$ by Theorem 1.5
  - ($\mathrm{span}\big(T(\beta)\big) \supseteq R(T)$)
    - $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ for any $\mathbf{v} \in V$
      - $\Rightarrow T(\mathbf{v}) \in R(T)$
    - $T(\mathbf{v}) = T(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i) \in \mathrm{span}(\{T(\mathbf{v}_i)\}) = \mathrm{span}\big(T(\beta)\big)$ for any $T(\mathbf{v}) \in R(T)$
      - $\Rightarrow R(T) \in \mathrm{span}\big(T(\beta)\big)$
  - $\therefore$ Q.E.D.

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.10*
    - $T: V = \mathbb{R}^3 \rightarrow M_{2\times 2}(\mathbb{R})$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_2 - v_3 & 0 \\ 0 & v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$
      - Linear
    - For a standard basis $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
      - $R(T) = \mathrm{span}\big(T(\beta)\big) = \mathrm{span}\left(\left\{ T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right\}\right) = \mathrm{span}\left(\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right\}\right) = \mathrm{span}\left(\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}\right)$
      - $\Rightarrow \dim\big(R(T)\big) = 2$

# 2.1 Linear transformations, null spaces, and ranges

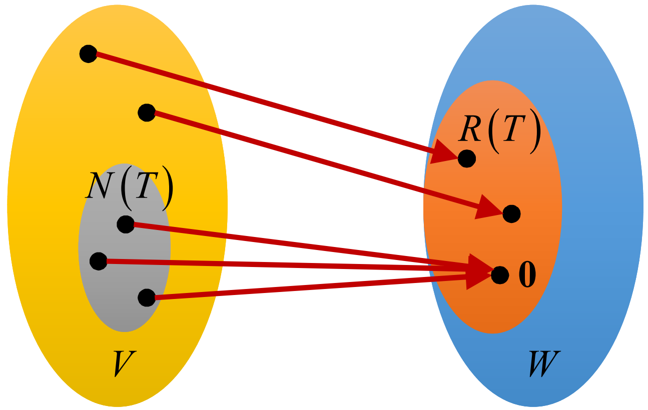
- **Null spaces and ranges**
  - *Example 2.1.10*
    - $T: V = \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_2 - v_3 & 0 \\ 0 & v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$
      - Linear
    - To find a basis for $N(T)$, by letting $T(\mathbf{v}) = \mathbf{0}$,
      - $N(T) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$
      - $\Rightarrow \dim(N(T)) = 1$
    - Note that $\dim(V) = \dim(N(T)) + \dim(R(T))$
      - (Theorem 2.3 coming soon!)

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Nullity and rank**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> If $N(T)$ and $R(T)$ are finite-dimensional,
> $$\text{nullity}(T) \triangleq \dim\big(N(T)\big)$$
> $$\text{rank}(T) \triangleq \dim\big(R(T)\big)$$

- Intuition
  - The larger the nullity, the smaller the rank
    - The more vectors carried into $\mathbf{0}$, the smaller the range




$T: V \to W$

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> If $V$ is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - Let $n = \dim(V)$ and $k = \dim\big(N(T)\big)$ where $n \geq k$.
  - Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis for $N(T)$.
  - Note that $N(T)$ is a subspace of vector space $V$.
    - $\Rightarrow$ From Corollary 1.11.1, we may extend $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ to a basis $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for $V$.

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> If $V$ is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - From Theorem 2.2,
    - $R(T) = \text{span}\big(T(\beta)\big) = \text{span}\big(T(\mathbf{v}_1), \dots, T(\mathbf{v}_k), T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\big)$
  - Noting that $T(\mathbf{v}_1) = \cdots = T(\mathbf{v}_k) = \mathbf{0}$
    - $R(T) = \text{span}(\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\})$
    - $\Rightarrow \{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ spans $R(T)$

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> If $V$ is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - If $\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ is a linearly independent set,
    - $\sum_{i=k+1}^{n} b_i T(\mathbf{v}_i) = \mathbf{0}$ only when $b_i = 0, i = k+1, \dots, n$
  - From the linear property of $T$,
    - $\sum_{i=k+1}^{n} b_i T(\mathbf{v}_i) = T(\sum_{i=k+1}^{n} b_i \mathbf{v}_i)$
  - If $T(\sum_{i=k+1}^{n} b_i \mathbf{v}_i) = \mathbf{0}$, then
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i \in N(T)$
  - Hence, $\sum_{i=k+1}^{n} b_i \mathbf{v}_i$ may be expressed as a linear combination of the basis of $N(T)$
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i = \sum_{i=1}^{k} c_i \mathbf{v}_i$ for some $c_i$'s
  - Since $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ are a basis for $V$,
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i - \sum_{i=1}^{k} c_i \mathbf{v}_i = 0$ only when $b_i = 0, i = k+1, \dots, n$

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> If $V$ is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - Hence $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ is a basis for $R(T)$
    - $\Rightarrow \dim(R(T)) = \text{rank}(T) = n - k$
  - $\therefore$ Q.E.D.

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.4**:
> Let $V$ and $W$ be vector spaces and let function $T: V \to W$ be linear.
> Then, $T$ is one-to-one $\Leftrightarrow$ $N(T) = \{\mathbf{0}\}$

- Proof)
  - ($T$ is one-to-one $\Rightarrow$ $N(T) = \{\mathbf{0}\}$)
    - From the one-to-one property, there exists only one $\mathbf{x}$ to satisfy $T(\mathbf{x}) = \mathbf{0}$.
    - In the meantime, by the linear property, $T(\mathbf{0}) = \mathbf{0}$.
      - $\Rightarrow \mathbf{x} = \mathbf{0}$
      - $\Rightarrow N(T) = \{\mathbf{0}\}$
  - ($T$ is one-to-one $\Leftarrow$ $N(T) = \{\mathbf{0}\}$)
    - By contradiction, assume $T$ is not one-to-one.
      - $\Rightarrow T(\mathbf{x}) = T(\mathbf{y})$ for some distinct $\mathbf{x}, \mathbf{y} \in V$
    - By the linear property, $\mathbf{0} = T(\mathbf{x}) - T(\mathbf{y}) = T(\mathbf{x} - \mathbf{y})$
      - $\Rightarrow \mathbf{x} - \mathbf{y} \in N(T)$ where $\mathbf{x} - \mathbf{y} \neq \mathbf{0}$
      - $\therefore$ Contradiction
  - $\therefore$ Q.E.D.

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.5**:
> Let $V$ and $W$ be finite-dimensional vector spaces of equal dimension, and let function $T: V \to W$ be linear.
> Then, $T$ is one-to-one $\Leftrightarrow$ $T$ is onto $\Leftrightarrow$ $\mathrm{rank}(T) = \dim(V)$

- "One-to-one" (*Appendix B*)
  - $T(\mathbf{v}_1) = T(\mathbf{v}_2) \;\Rightarrow\; \mathbf{v}_1 = \mathbf{v}_2$ , for some $\mathbf{v}_1, \mathbf{v}_2 \in V$
  - Equivalently, $\mathbf{v}_1 \neq \mathbf{v}_2 \;\Rightarrow\; T(\mathbf{v}_1) \neq T(\mathbf{v}_2)$ , for some $\mathbf{v}_1, \mathbf{v}_2 \in V$

- "Onto" (*Appendix B*)
  - $T(\mathbf{v}) =$ W, for all $\mathbf{v} \in V$

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

**Theorem 2.5**:
Let $V$ and $W$ be finite-dimensional vector spaces of equal dimension, and let function $T: V \to W$ be linear.
Then, $T$ is one-to-one $\Leftrightarrow$ $T$ is onto $\Leftrightarrow$ $\text{rank}(T) = \dim(V)$

- Proof)
  - From Theorem 2.4, $T$ is one-to-one $\Leftrightarrow$ $N(T) = \{\mathbf{0}\}$ or $\text{nullity}(T) = 0$
  - Also, from Theorem 2.3 (Dimension theorem), $\text{nullity}(T) + \text{rank}(T) = \dim(V)$
    - $\Rightarrow$ $T$ is one-to-one $\Leftrightarrow$ $\text{rank}(T) = \dim(V)$
  - From the equal dimension condition,
    - $\Rightarrow \text{rank}(T) = \dim(V) \Leftrightarrow \text{rank}(T) = \dim(W)$
  - From Theorem 1.11,
    - $\Rightarrow \text{rank}(T) = \dim(W) \Leftrightarrow R(T) = W$
  - $\therefore$ Q.E.D.

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.12*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_1 + v_2 \\ v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
      - Linear
    - To find a vector for $N(T)$, by letting $T(\mathbf{v}) = \mathbf{0}$,
      - $N(T) = \{\mathbf{0}\}$
    - By Theorem 2.4
      - One-to-one
    - By Theorem 2.5
      - Onto

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.6**:
> Let $V$ and $W$ be vector spaces of equal dimension, and suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for $V$.
> For $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$, there exists exactly one linear transformation $T: V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$.

- Proof)
  - From the linear property, for $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ with unique scalars $a_1, \dots, a_n$
    - $T(\mathbf{v}) = T(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i) = \sum_{i=1}^{n} a_i \mathbf{w}_i$
  - If there exists another linear function $U: V \to W$ such that $U(\mathbf{v}_i) = \mathbf{w}_i$
    - $U(\mathbf{v}) = U(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i U(\mathbf{v}_i) = \sum_{i=1}^{n} a_i \mathbf{w}_i$
  - Then, we must have $T(\mathbf{v}_i) = U(\mathbf{v}_i), \forall i$.
  - Hence, $U = T$
  - $\therefore$ Q.E.D.

# 2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - An implication of Theorem 2.6
    - A linear transformation completely determined by its action on a basis

# 2.2 The matrix representation of a linear transformation

# 2.2 The matrix representation of a linear transformation

- **Section 2.1**
  - Studying linear transformations by examining their null spaces and ranges
- **Section 2.2**
  - Representing linear transformations by a matrix
    - Developing a one-to-one correspondence between matrices and linear transformations

# 2.2 The matrix representation of a linear transformation

- **Ordered basis**

> **Ordered basis**:
> Let $V$ be a finite-dimensional vector space.
> An ordered basis for V is a basis for V endowed with a specific order.
>
> That is, an ordered basis is a finite sequence of linearly independent vectors in $V$ that spans $V$.

- *Example 2.2.1*
  - $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
    - A standard ordered basis in $\mathbb{F}^3$ where $\mathbf{e}_i, \forall i$ is a standard basis
  - $\gamma = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$
    - Another ordered basis
  - From the perspective of orders, $\beta \neq \gamma$

# 2.2 The matrix representation of a linear transformation

- **Ordered basis**

**Coordinate vector**:
Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis for a finite-dimensional vector space $V$.
For $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i \in V$, we define the coordinate vector of $\mathbf{v}$ relative to $\beta$ by

$$[\mathbf{v}]_\beta \triangleq \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

With unique scalars $a_1, \dots, a_n$

- *Example 2.2.2*
  - $V = \mathbb{R}^3$ with $\beta = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$
  - The coordinate vector of $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1\mathbf{e}_1 + 3\mathbf{e}_2 + 6\mathbf{e}_3$
    - $\Rightarrow [\mathbf{v}]_\beta = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - Letting
    - $V$ be a vector space with an ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$
    - $W$ be a vector space with an ordered basis $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$
    - $T: V \to W$ be a linear function
  - Then, using the ordered basis $\gamma$
    - $T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i$ with unique scalars $a_{ij} \in F$ for $i = 1, \dots, m$ for each $j = 1, \dots, n$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **Matrix representation**:
> We call the $m \times n$ matrix $\mathbf{A}$ defined by $[\mathbf{A}]_{ij} = a_{ij}$ the matrix representation of $T$ in the ordered bases $\beta$ and $\gamma$ and write $\mathbf{A} = [T]_\beta^\gamma$.
> If $V = W$ and $\beta = \gamma$, then we write $\mathbf{A} = [T]_\beta$.

- For instance, with $n = 2$ and $m = 3$ such that $T: \mathbb{R}^2 \to \mathbb{R}^3$,
  - $$[T(\mathbf{v}_1) \quad T(\mathbf{v}_2)] = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3] \underbrace{\begin{bmatrix} \underbrace{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}}_{[T(\mathbf{v}_1)]_\gamma} & \underbrace{\begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix}}_{[T(\mathbf{v}_2)]_\gamma} \end{bmatrix}}_{} \quad \mathbf{A} = [T]_\beta^\gamma$$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - *Example 2.2.3*
    - If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix}$, $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$, $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \underset{a_{11}}{1} \cdot \mathbf{w}_1 + \underset{a_{21}}{0} \cdot \mathbf{w}_2 + \underset{a_{31}}{2} \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = \underset{a_{12}}{3} \cdot \mathbf{w}_1 + \underset{a_{22}}{0} \cdot \mathbf{w}_2 + (\underset{a_{32}}{-4}) \cdot \mathbf{w}_3$
      - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$
    - For $\gamma_2 = \{\mathbf{w}_1 = \mathbf{e}_3, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_1\}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \underset{a_{11}}{2} \cdot \mathbf{w}_1 + \underset{a_{21}}{0} \cdot \mathbf{w}_2 + \underset{a_{31}}{1} \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = (\underset{a_{12}}{-4}) \cdot \mathbf{w}_1 + \underset{a_{22}}{0} \cdot \mathbf{w}_2 + \underset{a_{32}}{3} \cdot \mathbf{w}_3$
      - $\Rightarrow [T]_\beta^{\gamma_2} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - Letting
    - $V$ be a vector space with an ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$
    - $W$ be a vector space with an ordered basis $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$
    - $T: V \to W$ be a linear function
  - (Zero transformation) Then, using the ordered basis $\gamma$
    - $T(\mathbf{v}_j) = \mathbf{0} = 0 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + \cdots + 0 \cdot \mathbf{w}_m$ for $j = 1, \dots, n$
    - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = [\mathbf{0}]$
  - (Identity transformation) For $n = m$ and $\beta = \gamma$,
    - $T(\mathbf{v}_j) = \mathbf{v}_j = 0 \cdot \mathbf{v}_1 + \cdots + 1 \cdot \mathbf{v}_j + \cdots + 0 \cdot \mathbf{w}_m$ for $j = 1, \dots, n$
    - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = \mathbf{I}_\mathrm{n}$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

**Kronecker delta**:

$$\delta_{ij} \triangleq \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- For instance,
  - $[\mathbf{I}_n]_{ij} = \delta_{ij}, \forall i, j$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

**Addition and scalar multiplication of functions**:
Let $V$ and $W$ be vector spaces over $F$.
Let $T, U: V \rightarrow W$ be arbitrary functions.
Then, for all $\mathbf{x} \in V$ and $a \in F$,

$$T + U: V \rightarrow W \quad \triangleq \quad (T + U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x})$$
$$aT: V \rightarrow W \quad \triangleq \quad (aT)(\mathbf{x}) = aT(\mathbf{x})$$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **Theorem 2.7**:
> Let $V$ and $W$ be vector spaces over $F$.
> Let $T, U: V \to W$ be linear functions.
> Then, for all $\mathbf{x} \in V$ and $a \in F$,
>
> (a) $aT + U$ is linear, i.e., $(aT+U)(c\mathbf{x}+\mathbf{y}) = c(aT+U)(\mathbf{x}) + (aT+U)(\mathbf{y})$
> (b) The collection of all linear transformations from $V$ to $W$ is a vector space over $F$.

- Proof)
  - (a)
    - Let $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$.
    - $(aT+U)(c\mathbf{x}+\mathbf{y}) = (aT)(c\mathbf{x}+\mathbf{y}) + U(c\mathbf{x}+\mathbf{y}) = c(aT)(\mathbf{x}) + (aT)(\mathbf{y}) + cU(\mathbf{x}) + U(\mathbf{y}) = c\big((aT)(\mathbf{x}) + U(\mathbf{x})\big) + (aT)(\mathbf{y}) + U(\mathbf{y}) = c(aT+U)(\mathbf{x}) + (aT+U)(\mathbf{y})$
  - (b)
    - (Left as an exercise)

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **The vector space of all linear transformations**:
> Let $V$ and $W$ be vector spaces over $F$.
> We denote the vector space of all linear transformations from $V$ into $W$ by $\mathcal{L}(V, W)$.
> If $V = W$, we write $\mathcal{L}(V)$.

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

**Theorem 2.8**:
Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively.
Let $T, U: V \to W$ be linear transformations.
Then,

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars $c$

- Proof)
  - (a)
    - Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$.
    - $T(\mathbf{v}_j) = \sum_{i=1}^m a_{ij}\mathbf{w}_i$ for unique scalars $a_{ij}, \forall i, j$
    - $U(\mathbf{v}_j) = \sum_{i=1}^m b_{ij}\mathbf{w}_i$ for unique scalars $b_{ij}, \forall i, j$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

**Theorem 2.8**:
Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively.
Let $T, U: V \to W$ be linear transformations.
Then,

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars $c$

- Proof)
  - (a)
    - Then, $(T+U)(\mathbf{v}_j) = T(\mathbf{v}_j) + U(\mathbf{v}_j) = \sum_{i=1}^{m}(a_{ij}+b_{ij})\mathbf{w}_i$
    - Thus, $\left[[T+U]_\beta^\gamma\right]_{ij} = a_{ij} + b_{ij} = \left[[T]_\beta^\gamma\right]_{ij} + \left[[U]_\beta^\gamma\right]_{ij}$

    - $\therefore$ Q.E.D.

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

**Theorem 2.8**:
Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively.
Let $T, U: V \to W$ be linear transformations.
Then,

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars $c$

- Proof)
  - (b)
    - $(cT)(\mathbf{v}_j) = cT(\mathbf{v}_j) = c\sum_{i=1}^{m} a_{ij}\mathbf{w}_i$
    - Thus, $\left[[cT]_\beta^\gamma\right]_{ij} = ca_{ij} = c\left[[T]_\beta^\gamma\right]_{ij}$



    - $\therefore$ Q.E.D.

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - *Example 2.2.5*
    - Let $T, U: \mathbb{R}^2 \to \mathbb{R}^3$ be linear and $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ for $T$ and $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$ for $U$
    - If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = 3 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-4) \cdot \mathbf{w}_3$
      - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$
    - If $U\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 - v_2 \\ 2v_1 \\ 3v_1 + 2v_2 \end{bmatrix}$
      - $U(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2 + 3 \cdot \mathbf{w}_3$ and $U(\mathbf{v}_2) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = (-1) \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$
      - $\Rightarrow [U]_\beta^\gamma = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$

# 2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - *Example 2.2.5*
    - Since $(T+U)\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) + U\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ 2v_1 \\ 3v_1 + 2v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 2v_2 \\ 2v_1 \\ 5v_1 - 2v_2 \end{bmatrix}$
      - $(T+U)(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} = 2 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2 + 5 \cdot \mathbf{w}_3$ and $(T+U)(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-2) \cdot \mathbf{w}_3$
      - $\Rightarrow [T+U]_\beta^\gamma = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$
    - Note that
      - $[T]_\beta^\gamma + [U]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix} = [T+U]_\beta^\gamma$

# 2.3 Composition of linear transformations and matrix multiplication

# 2.3 Composition of linear transformations and matrix multiplication

- **Section 2.1**
  - Studying linear transformations by examining their null spaces and ranges

- **Section 2.2**
  - Representing linear transformations by a matrix
    - Developing a one-to-one correspondence between matrices and linear transformations

- **Section 2.3**
  - How the matrix representation of a composite of linear transformations is related to the matrix representation of each of the associated linear transformations

# 2.3 Composition of linear transformations and matrix multiplication

- **Composition**

> **Theorem 2.9**:
> Let $V, W$ and $Z$ be vector spaces over the same field $F$.
> Let $T: V \to W$ and $U: W \to Z$ be linear.
> Then, $UT: V \to Z$ is linear.

- Proof)
  - Let $\mathbf{x}, \mathbf{y} \in V$ and $a \in F$
  - $(UT)(a\mathbf{x} + \mathbf{y}) = U\big(T(a\mathbf{x} + \mathbf{y})\big) = U\big(aT(\mathbf{x}) + T(\mathbf{y})\big) = aU\big(T(\mathbf{x})\big) + U\big(T(\mathbf{y})\big) = a(UT)(\mathbf{x}) + (UT)(\mathbf{y})$
  - $\therefore$ Q.E.D.

# 2.3 Composition of linear transformations and matrix multiplication

- **Composition**

**Theorem 2.10**:
Let $V$ be a vector space.
Let $T, U_1, U_2 \in \mathcal{L}(V)$ (linear)
Then,

(a) $T(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$
(b) $T(U_1U_2) = (TU_1)U_2$
(c) $T\mathbf{I} = \mathbf{I}T = T$
(d) $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$ for all scalars $a$

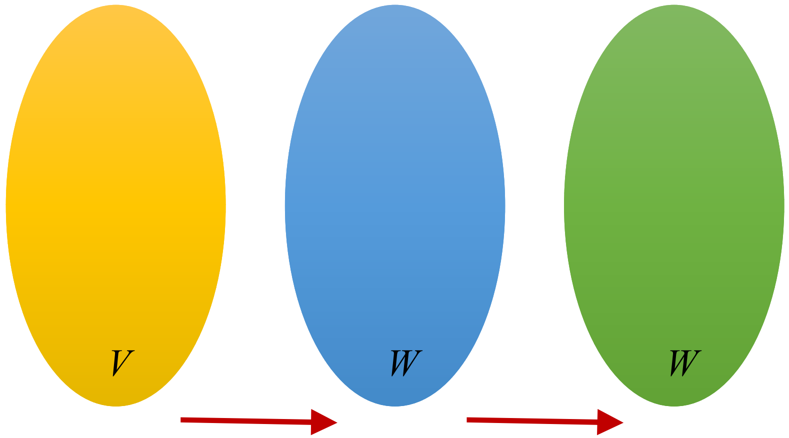
- Proof)
  - (Exercise)

# 2.3 Composition of linear transformations and matrix multiplication

- **Composition**
  - If $T \in \mathcal{L}(V)$
    - $T^0 \triangleq \mathbf{I}$
    - $T^k \triangleq T^{k-1}T$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Let
    - $V, W$ and $Z$: Finite-dimensional vector spaces
    - $T: V \to W$ linear
    - $U: W \to Z$ linear
    - $\alpha = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ ordered basis for $V$
    - $\beta = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ ordered basis for $W$
    - $\gamma = \{\mathbf{z}_1, \ldots, \mathbf{z}_p\}$ ordered basis for $Z$
    - $\mathbf{A} = [U]_\beta^\gamma$
    - $\mathbf{B} = [T]_\alpha^\beta$
  - Then,
    - $(UT)(\mathbf{v}_j) = U\left(T(\mathbf{v}_j)\right) = U\left(\sum_{k=1}^m [\mathbf{B}]_{kj}\mathbf{w}_k\right) = \sum_{k=1}^m [\mathbf{B}]_{kj} U(\mathbf{w}_k) = \sum_{k=1}^m [\mathbf{B}]_{kj}\left(\sum_{i=1}^p [\mathbf{A}]_{ik}\mathbf{z}_i\right) = \sum_{i=1}^p \left(\sum_{k=1}^m [\mathbf{A}]_{ik}[\mathbf{B}]_{kj}\right)\mathbf{z}_i = \sum_{i=1}^p [\mathbf{C}]_{ij}\mathbf{z}_i$



# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Matrix product**:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Let $\mathbf{B}$ be an $n \times p$ matrix.
We define the product of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{AB}$, to be the $m \times p$ matrix such that

$$[\mathbf{AB}]_{ij} = \sum_{k=1}^{n} [\mathbf{A}]_{ik}[\mathbf{B}]_{kj}$$

for $i = 1, \dots, m$ and $j = 1, \dots, p$.

That is, $[\mathbf{AB}]_{ij}$ is the sum of products of corresponding entries from the $i$-th row of $\mathbf{A}$ and the $j$-th column of $\mathbf{B}$.

- Caution on the dimension

$$\underbrace{\mathbf{A}}_{(m\times n)}\underbrace{\mathbf{B}}_{(n\times p)} = \underbrace{\mathbf{AB}}_{(m\times p)}$$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - *Example 2.3.1*
    - $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + 2\cdot 2 + 1\cdot 5 \\ 0\cdot 4 + 4\cdot 2 + (-1)\cdot 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}$
    - Dimension being $(2\times 3)(3\times 1) \rightarrow (2\times 1)$

  - Not commutative
    - $\mathbf{AB} \neq \mathbf{BA}$
    - e.g.,
      - $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
      - $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Transpose of matrix multiplication
    - For some $\mathbf{A} \in M_{m\times n}(F)$ and $\mathbf{B} \in M_{n\times p}(F)$,
      - $[\mathbf{AB}]^T_{ij} = [\mathbf{AB}]_{ji} = \sum_{k=1}^{n}[\mathbf{A}]_{jk}[\mathbf{B}]_{ki}$
      - $[\mathbf{B}^T\mathbf{A}^T]_{ij} = \sum_{k=1}^{n}[\mathbf{B}^T]_{ik}[\mathbf{A}^T]_{kj} = \sum_{k=1}^{n}[\mathbf{B}]_{ki}[\mathbf{A}]_{jk}$

    - $\therefore (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

> **Theorem 2.11**:
> Let $V, W$ and $Z$ be finite-dimensional vector spaces with ordered bases $\alpha, \beta$ and $\gamma$, respectively. For linear $T: V \to W$ and $U: W \to Z$,
> $$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

- *Example 2.3.2*
  - Let $T: V = \mathbb{R}^2 \to W = \mathbb{R}^3$ and $U: W = \mathbb{R}^3 \to V = \mathbb{R}^2$
  - Ordered bases
    - $\alpha = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ for $V$
    - $\beta = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$ for $W$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Theorem 2.11**:
Let $V, W$ and $Z$ be finite-dimensional vector spaces with ordered bases $\alpha, \beta$ and $\gamma$, respectively. For linear $T: V \to W$ and $U: W \to Z$,

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

- *Example 2.3.2*
  - For $T(\mathbf{v}) = \begin{bmatrix} \frac{1}{2} v_1 \\ v_2 \\ 0 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$
    - $T(\mathbf{v}_1) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3$
    - $\Rightarrow [T]_\alpha^\beta = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

> **Theorem 2.11**:
> Let $V, W$ and $Z$ be finite-dimensional vector spaces with ordered bases $\alpha, \beta$ and $\gamma$, respectively. For linear $T: V \to W$ and $U: W \to Z$,
> $$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

- *Example 2.3.2*
  - For $U(\mathbf{w}) = \begin{bmatrix} 2w_1 \\ w_2 \end{bmatrix}$ for $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3$
    - $U(\mathbf{w}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$, $U(\mathbf{w}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$ and $U(\mathbf{w}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$
    - $\Rightarrow [U]_\beta^\alpha = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

> **Theorem 2.11**:
> Let $V, W$ and $Z$ be finite-dimensional vector spaces with ordered bases $\alpha, \beta$ and $\gamma$, respectively. For linear $T: V \to W$ and $U: W \to Z$,
> $$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

- *Example 2.3.2*
  - For $(UT)(\mathbf{v}) = U\left(\begin{bmatrix} \frac{1}{2}v_1 \\ v_2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2\left(\frac{1}{2}v_1\right) \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^3$
    - $(UT)(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$ and $(UT)(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$
    - $\Rightarrow [UT]_\alpha^\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - Note that $[U]_\beta^\alpha [T]_\alpha^\beta = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [UT]_\alpha^\alpha$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Corollary 2.11.1**:
Let $V$ be a finite-dimensional vector space with an ordered basis $\beta$.
Let $T, U \in \mathcal{L}(V)$
Then,

$$[UT]_\beta = [U]_\beta [T]_\beta$$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Theorem 2.12**:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Let $\mathbf{B}$ and $\mathbf{C}$ be $n \times p$ matrices.
Let $\mathbf{D}$ and $\mathbf{E}$ be $q \times m$ matrices.
Then,

(a) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ and $(\mathbf{D}+\mathbf{E})\mathbf{A} = \mathbf{DA} + \mathbf{EA}$
(b) $a(\mathbf{AB}) = (a\mathbf{A})\mathbf{B} = \mathbf{A}(a\mathbf{B})$ for any scalar $a$
(c) $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{AI}_n$

- Proof)
  - (Exercise)

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Corollary 2.12.1**:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Let $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_k$ be $n \times p$ matrices.
Let $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_k$ be $q \times m$ matrices.
Let $a_1, a_2, \ldots, a_k$ be scalars.
Then,

(a) $\mathbf{A}\left(\sum_{i=1}^{k} a_i \mathbf{B}_i\right) = \sum_{i=1}^{k} a_i \mathbf{A}\mathbf{B}_i$
(b) $\left(\sum_{i=1}^{k} a_i \mathbf{C}_i\right)\mathbf{A} = \sum_{i=1}^{k} a_i \mathbf{C}_i \mathbf{A}$

- Proof)
  - (Exercise)

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - For an $n \times n$ matrix $A$,
    - $\mathbf{A}^0 \triangleq \mathbf{I}_n$
    - $\mathbf{A}^k \triangleq \mathbf{A}^{k-1}\mathbf{A}$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Theorem 2.13**:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Let $\mathbf{B}$ be $n \times p$ matrices.
Let $\mathbf{u}_j$ be the $j$-th column of $\mathbf{AB}$.
Let $\mathbf{v}_j$ be the $j$-th column of $\mathbf{B}$.
Then,

(a) $\mathbf{u}_j = \mathbf{A}\mathbf{v}_j$
(b) $\mathbf{v}_j = \mathbf{B}\mathbf{e}_j$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Proof) (a) $\mathbf{u}_j = \mathbf{A}\mathbf{v}_j$

$$\mathbf{u}_j = \begin{bmatrix} [\mathbf{AB}]_{1j} \\ \vdots \\ [\mathbf{AB}]_{ij} \\ \vdots \\ [\mathbf{AB}]_{nj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{m} [\mathbf{A}]_{1k} [\mathbf{B}]_{kj} \\ \vdots \\ \sum_{k=1}^{m} [\mathbf{A}]_{ik} [\mathbf{B}]_{kj} \\ \vdots \\ \sum_{k=1}^{m} [\mathbf{A}]_{nk} [\mathbf{B}]_{kj} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\text{row},1} \mathbf{v}_j \\ \vdots \\ \mathbf{A}_{\text{row},i} \mathbf{v}_j \\ \vdots \\ \mathbf{A}_{\text{row},n} \mathbf{v}_j \end{bmatrix} = \mathbf{A}\mathbf{v}_j$$

$$\mathbf{A} = \begin{array}{c} \\ \\ \\ \mathbf{A}_{\text{row},i} \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} [\mathbf{A}]_{11} & [\mathbf{A}]_{12} & \cdots & [\mathbf{A}]_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ [\mathbf{A}]_{i1} & [\mathbf{A}]_{i2} & \cdots & [\mathbf{A}]_{im} \\ \vdots & \vdots & \ddots & \vdots \\ [\mathbf{A}]_{n1} & [\mathbf{A}]_{n2} & \cdots & [\mathbf{A}]_{nm} \end{bmatrix} \qquad \mathbf{B} = \begin{array}{c} \begin{array}{ccccc} & & \mathbf{v}_j & & \end{array} \\ \begin{bmatrix} [\mathbf{B}]_{11} & \cdots & [\mathbf{B}]_{1j} & \cdots & [\mathbf{B}]_{1p} \\ [\mathbf{B}]_{21} & \cdots & [\mathbf{B}]_{2j} & \cdots & [\mathbf{B}]_{2p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{B}]_{m1} & \cdots & [\mathbf{B}]_{mj} & \cdots & [\mathbf{B}]_{mp} \end{bmatrix} \end{array}$$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Proof) (b) $\mathbf{v}_j = \mathbf{B}\mathbf{e}_j$

$$\mathbf{B}\mathbf{e}_j = \begin{bmatrix} [\mathbf{B}]_{11} & \cdots & [\mathbf{B}]_{1j} & \cdots & [\mathbf{B}]_{1p} \\ [\mathbf{B}]_{21} & \cdots & [\mathbf{B}]_{2j} & \cdots & [\mathbf{B}]_{2p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{B}]_{m1} & \cdots & [\mathbf{B}]_{mj} & \cdots & [\mathbf{B}]_{mp} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \boxed{1} \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ j\text{-th} \\ \\ \\ \end{array} = \begin{bmatrix} [\mathbf{B}]_{1j} \\ [\mathbf{B}]_{2j} \\ \vdots \\ [\mathbf{B}]_{mj} \end{bmatrix} = \mathbf{v}_j$$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Theorem 2.13
    - Column $j$ of $\mathbf{AB}$ = A linear combination of the columns of $\mathbf{A}$ with column $j$ of $\mathbf{B}$

$$\mathbf{AB}=\begin{bmatrix}[\mathbf{A}]_{11} & [\mathbf{A}]_{12} & \cdots & [\mathbf{A}]_{1m}\\ \vdots & \vdots & \ddots & \vdots\\ [\mathbf{A}]_{i1} & [\mathbf{A}]_{i2} & \cdots & [\mathbf{A}]_{im}\\ \vdots & \vdots & \ddots & \vdots\\ [\mathbf{A}]_{n1} & [\mathbf{A}]_{n2} & \cdots & [\mathbf{A}]_{nm}\end{bmatrix}\begin{bmatrix}[\mathbf{B}]_{11} & \cdots & [\mathbf{B}]_{1j} & \cdots & [\mathbf{B}]_{1p}\\ [\mathbf{B}]_{21} & \cdots & [\mathbf{B}]_{2j} & \cdots & [\mathbf{B}]_{2p}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ [\mathbf{B}]_{m1} & \cdots & [\mathbf{B}]_{mj} & \cdots & [\mathbf{B}]_{mp}\end{bmatrix}=\begin{bmatrix}[\mathbf{AB}]_{11} & \cdots & [\mathbf{AB}]_{1j} & \cdots & [\mathbf{AB}]_{1p}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ [\mathbf{AB}]_{i1} & \cdots & [\mathbf{AB}]_{ij} & \cdots & [\mathbf{AB}]_{ip}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ [\mathbf{AB}]_{n1} & \cdots & [\mathbf{AB}]_{nj} & \cdots & [\mathbf{AB}]_{np}\end{bmatrix}$$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Analogous
    - Row $i$ of $\mathbf{AB}$ = A linear combination of the row $i$ of $\mathbf{A}$ with columns of $\mathbf{B}$

$$\mathbf{AB} = \begin{bmatrix} [\mathbf{A}]_{11} & [\mathbf{A}]_{12} & \cdots & [\mathbf{A}]_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{[\mathbf{A}]_{i1} \quad [\mathbf{A}]_{i2} \quad \cdots \quad [\mathbf{A}]_{im}} \\ \vdots & \vdots & \ddots & \vdots \\ [\mathbf{A}]_{n1} & [\mathbf{A}]_{n2} & \cdots & [\mathbf{A}]_{nm} \end{bmatrix} \begin{bmatrix} [\mathbf{B}]_{11} & \cdots & [\mathbf{B}]_{1j} & \cdots & [\mathbf{B}]_{1p} \\ [\mathbf{B}]_{21} & \cdots & [\mathbf{B}]_{2j} & \cdots & [\mathbf{B}]_{2p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{B}]_{m1} & \cdots & [\mathbf{B}]_{mj} & \cdots & [\mathbf{B}]_{mp} \end{bmatrix} = \begin{bmatrix} [\mathbf{AB}]_{11} & \cdots & [\mathbf{AB}]_{1j} & \cdots & [\mathbf{AB}]_{1p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \boxed{[\mathbf{AB}]_{i1}} & \cdots & \boxed{[\mathbf{AB}]_{ij}} & \cdots & \boxed{[\mathbf{AB}]_{ip}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{AB}]_{n1} & \cdots & [\mathbf{AB}]_{nj} & \cdots & [\mathbf{AB}]_{np} \end{bmatrix}$$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Theorem 2.14**:
Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively. For linear $T: V \to W$,

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- Proof)
  - Let $f: F \to V$ by $f(a) = a\mathbf{u}$ for $a \in F$.
  - An ordered basis
    - $\alpha = \{f_1 = 1\}$ for $F$
  - $[T(\mathbf{u})]_\gamma = [Tf]_\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [\mathbf{u}]_\beta$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

> **Theorem 2.14**:
> Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively. For linear $T: V \to W$,
> $$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- *Example 2.3.3*
  - Let $T: V = \mathbb{R}^2 \to W = \mathbb{R}^3$ and $U: W = \mathbb{R}^3 \to V = \mathbb{R}^2$
  - Ordered bases
    - $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ for $V$
    - $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$ for $W$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

> **Theorem 2.14**:
> Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively. For linear $T: V \to W$,
> $$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- *Example 2.3.3*
  - For $T(\mathbf{v}) = \begin{bmatrix} \frac{1}{2} v_1 \\ v_2 \\ 0 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^3$
    - $T(\mathbf{v}_1) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3$
    - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Theorem 2.14**:
Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively. For linear $T: V \rightarrow W$,

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- *Example 2.3.3*
  - Note that $[T]_\beta^\gamma [\mathbf{v}]_\beta = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} v_1 \\ v_2 \\ 0 \end{bmatrix} = [T(\mathbf{v})]_\gamma$

# 2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

**Theorem 2.14**:
Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively. For linear $T: V \to W$,

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- *Example 2.3.3*
  - For $U(\mathbf{w}) = \begin{bmatrix} 2w_1 \\ w_2 \end{bmatrix}$ for $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3$
    - $U(\mathbf{w}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$, $U(\mathbf{w}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$ and $U(\mathbf{w}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$
    - $\Rightarrow [U]_\gamma^\beta = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
  - Note that $[U]_\gamma^\beta [\mathbf{w}]_\gamma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2w_1 \\ w_2 \end{bmatrix} = [U(\mathbf{w})]_\beta$

# 2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**

**Left multiplication**:
Let $\mathbf{A}$ be an $m \times n$ matrix with entries from a field $F$.
We denote by $L_{\mathbf{A}}$ the mapping $L_{\mathbf{A}}: F^n \to F^m$ defined by $L_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ (the matrix product of $\mathbf{A}$ and $\mathbf{x}$) for each column vector $\mathbf{x} \in F^n$.
We call $L_{\mathbf{A}}$ a left-multiplication transformation.

- *Example 2.3.4*
  - Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$
  - Then, $L_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \in F^2$ for some $\mathbf{x} \in F^3$.
  - For example, with $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$,
    - $L_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

# 2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**

**Theorem 2.15**:
Let $\mathbf{A}$ be an $m \times n$ matrix with entries from $F$.
Then, the left-multiplication transformation $L_{\mathbf{A}}: F^n \to F^m$ is linear.
Furthermore, if $\mathbf{B}$ is any other $m \times n$ matrix and $\beta$ and $\gamma$ are the standard ordered bases for $F^n$ and $F^m$, respectively,

(a) $[L_{\mathbf{A}}]_\beta^\gamma = \mathbf{A}$
(b) $L_{\mathbf{A}} = L_{\mathbf{B}} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{B}$
(c) $L_{\mathbf{A}+\mathbf{B}} = L_{\mathbf{A}} + L_{\mathbf{B}}$ and $L_{a\mathbf{A}} = aL_{\mathbf{A}}$ for all $a \in F$
(d) If $T: F^n \to F^m$ is linear, then there exists a unique $m \times n$ matrix $\mathbf{C}$ such that $T = L_{\mathbf{C}}$ or $\mathbf{C} = [T]_\beta^\gamma$.
(e) If $\mathbf{E}$ is an $n \times p$ matrix, then $L_{\mathbf{AE}} = L_{\mathbf{A}} L_{\mathbf{E}}$
(f) If $m = n$, then $L_{\mathbf{I}_n} = \mathbf{I}_{F^n}$

# 2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**
  - Proof) (a) $[L_{\mathbf{A}}]_{\beta}^{\gamma} = \mathbf{A}$
    - The $j$-th column of $[L_{\mathbf{A}}]_{\beta}^{\gamma}$
      - $L_{\mathbf{A}}(\mathbf{e}_j) = \mathbf{A}\mathbf{e}_j$
      - $\Rightarrow$ The j-th column of $\mathbf{A}$
    - $\therefore$ Q.E.D.
  - Proof) (b) $L_{\mathbf{A}} = L_{\mathbf{B}} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{B}$
    - $(\Rightarrow)$
      - By (a), $\mathbf{A} = [L_{\mathbf{A}}]_{\beta}^{\gamma} = [L_{\mathbf{B}}]_{\beta}^{\gamma} = \mathbf{B}$
    - $(\Leftarrow)$
      - Trivial
    - $\therefore$ Q.E.D.

# 2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**
  - Proof) (c) $L_{\mathbf{A}+\mathbf{B}} = L_{\mathbf{A}} + L_{\mathbf{B}}$ and $L_{a\mathbf{A}} = aL_{\mathbf{A}}$ for all $a \in F$
    - (Exercise)

  - Proof) (d) If $T: F^n \to F^m$ is linear, then there exists a unique $m \times n$ matrix $\mathbf{C}$ such that $T = L_{\mathbf{C}}$ or $\mathbf{C} = [T]_\beta^\gamma$.
    - By Theorem 2.14 and (a)
      - $[T(\mathbf{x})]_\gamma = [T]_\beta^\gamma[\mathbf{x}]_\beta = \mathbf{C}[\mathbf{x}]_\beta = [L_{\mathbf{C}}]_\beta^\gamma[\mathbf{x}]_\beta$
    - By (b), $\mathbf{C}$ is unique.

    - ∴ Q.E.D.

# 2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**
  - Proof) (e) If $\mathbf{E}$ is an $n \times p$ matrix, then $L_{\mathbf{AE}} = L_{\mathbf{A}} L_{\mathbf{E}}$.
    - By Theorem 2.13,
      - $(\mathbf{AE})\mathbf{e}_j = \mathbf{A}(\mathbf{E}\mathbf{e}_j)$
    - Then, $L_{\mathbf{AE}}(\mathbf{e}_j) = (\mathbf{AE})\mathbf{e}_j = \mathbf{A}(\mathbf{E}\mathbf{e}_j) = L_{\mathbf{A}}(\mathbf{E}\mathbf{e}_j) = L_{\mathbf{A}}\left(L_{\mathbf{E}}(\mathbf{e}_j)\right) = (L_{\mathbf{A}} L_{\mathbf{B}})(\mathbf{e}_j)$
    - $\therefore$ Q.E.D.
  - Proof) (f) If $m = n$, then $L_{\mathbf{I}_n} = \mathbf{I}_{F^n}$
    - (Exercise)

# 2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**

> **Theorem 2.16**:
> Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be matrices such that $\mathbf{A}(\mathbf{BC})$ is defined.
> Then, $(\mathbf{AB})\mathbf{C}$ is also defined and $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (associative)

- Proof)
  - By Theorem 2.15 (e),
    - $L_{\mathbf{A}(\mathbf{BC})} = L_{\mathbf{A}}L_{\mathbf{BC}} = L_{\mathbf{A}}L_{\mathbf{B}}L_{\mathbf{C}} = (L_{\mathbf{A}}L_{\mathbf{B}})L_{\mathbf{C}} = L_{\mathbf{AB}}L_{\mathbf{C}} = L_{(\mathbf{AB})\mathbf{C}}$
  - By Theorem 2.15 (b),
    - $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

  - $\therefore$ Q.E.D.