

Linear Algebra (5th edition)

Stephen Friedberg, Arnold Insel, Lawrence Spence

Chapter 01: Vector spaces

Jihwan Moon

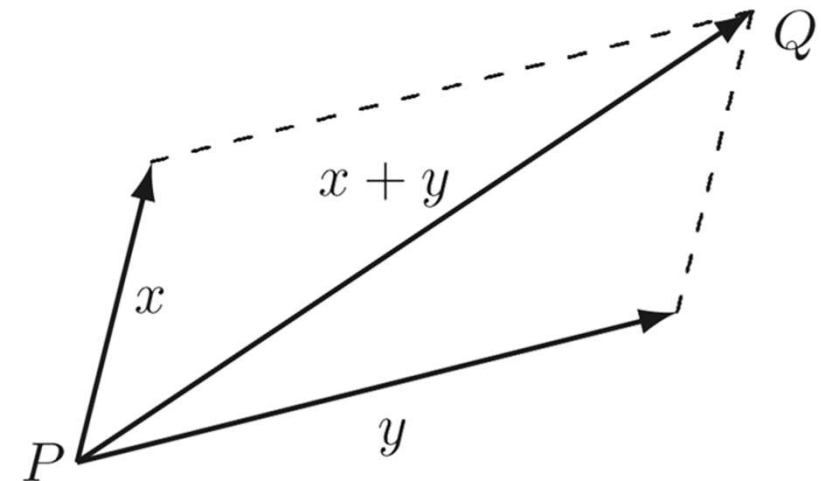
Table of Contents

- **1.1 Introduction**
- **1.2 Vector spaces**
- **1.3 Subspaces**
- **1.4 Linear combinations and systems of linear equations**
- **1.5 Linear dependence and linear independence**
- **1.6 Bases and dimension**

1.1 Introduction

1.1 Introduction

- **Vector**
 - An entity involving both **magnitude** and **direction**
 - Represented by an arrow
 - **Length** of the arrow = Magnitude of the vector
 - **Direction** of the arrow = Direction of the vector
 - **Irrespective** of the position



1.1 Introduction

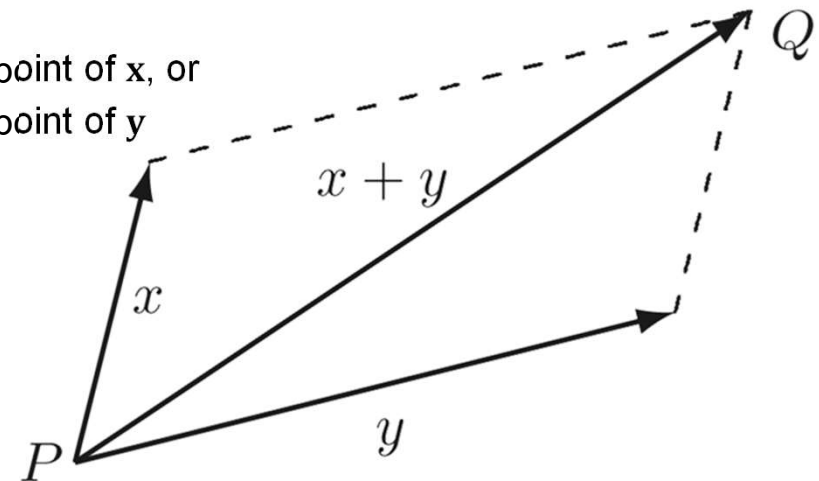
- Vector addition

Parallelogram law for vector addition:

The **sum** of two vectors \mathbf{x} and \mathbf{y} that act at the same point P is the vector beginning at P that is represented by the **diagonal of a parallelogram** having \mathbf{x} and \mathbf{y} as adjacent sides.

- **Geometrically** obtaining the endpoint Q , i.e., $\mathbf{x} + \mathbf{y}$

- ① Allowing \mathbf{x} to act at P and then \mathbf{y} to act at the end point of \mathbf{x} , or
- ② Allowing \mathbf{y} to act at P and then \mathbf{x} to act at the end point of \mathbf{y}
- “**Tail-to-head**” addition



1.1 Introduction

• Vector addition

Parallelogram law for vector addition:

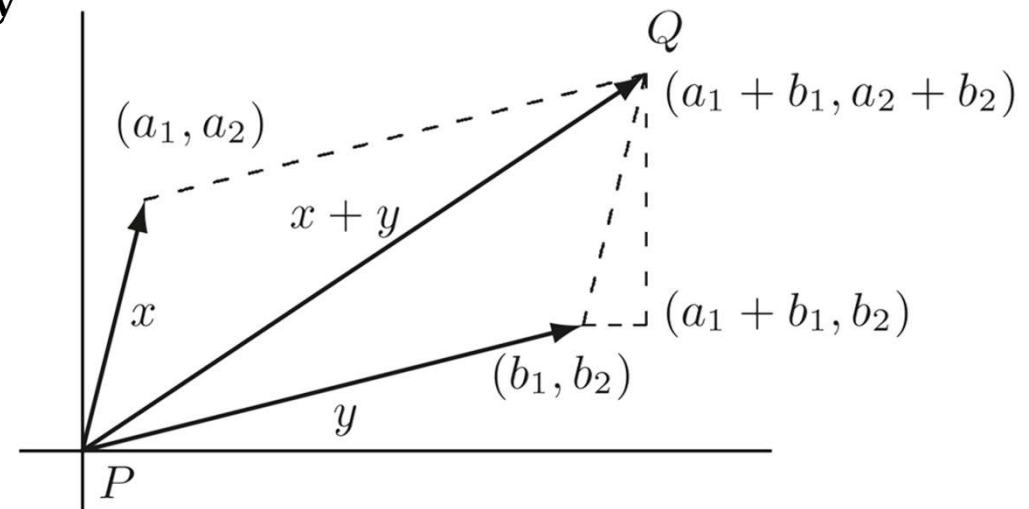
The **sum** of two vectors \mathbf{x} and \mathbf{y} that act at the same point P is the vector beginning at P that is represented by the **diagonal of a parallelogram** having \mathbf{x} and \mathbf{y} as adjacent sides.

• Algebraically obtaining the endpoint Q , i.e., $\mathbf{x} + \mathbf{y}$

- (a_1, a_2) : The endpoint of \mathbf{x}
- (b_1, b_2) : The endpoint of \mathbf{y}
- $(a_1 + b_1, a_2 + b_2)$: The end point of $\mathbf{x} + \mathbf{y}$
- Assumed to emanate **from the origin**

• Often refer to “the point \mathbf{x} ”

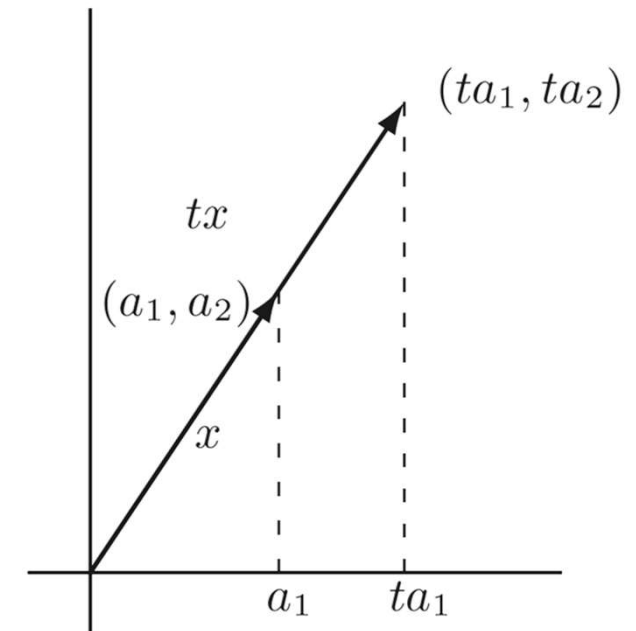
- rather than “the endpoint of the vector \mathbf{x} ”



1.1 Introduction

- **Scalar multiplication**

- Multiplying the vector by a **real** number
- **Geometrically**,
 - For $t > 0$
 - tx in the **same** direction of x
 - For $t < 0$
 - tx in the **opposite** direction from x
 - Length (magnitude) of $tx = |t|$ **times the length** (magnitude) of x
 - x and y in **parallel** if $y = tx$ for some non-zero real number t
- **Algebraically**,
 - (a_1, a_2) : The endpoint of x
 - (ta_1, ta_2) : The endpoint of tx
 - Assumed to emanate **from the origin**



1.1 Introduction

- **Properties regarding vector addition and scalar multiplication**

- ① For all vectors x and y , ([commutativity](#))
 - $x + y = y + x$
- ② For all vectors x , y and z , ([associativity](#))
 - $(x + y) + z = x + (y + z)$
- ③ There exists a vector denoted 0 such that (existence of [identity](#))
 - $x + 0 = x$ for each vector x
- ④ For each vector x , there is a vector y such that (existence of [inverse](#))
 - $x + y = 0$

1.1 Introduction

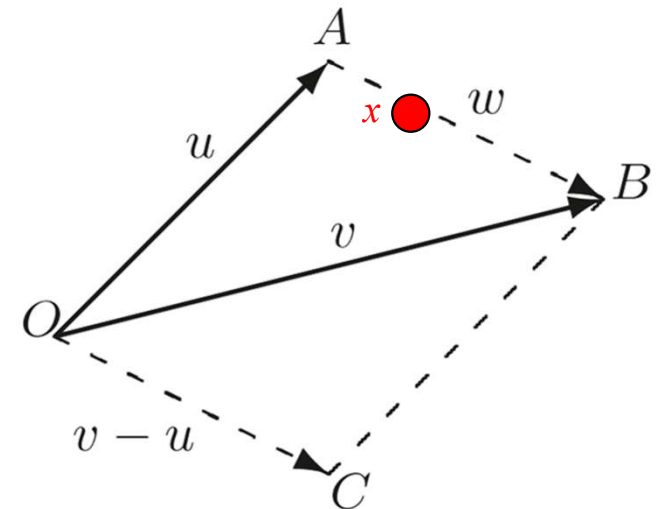
- **Properties regarding vector addition and scalar multiplication**
 - ⑤ For each vector \mathbf{x} , (existence of **identity**)
 - $1\mathbf{x} = \mathbf{x}$
 - ⑥ For each pair of real numbers a and b and each vector \mathbf{x} , (**associativity**)
 - $(ab)\mathbf{x} = a(b\mathbf{x})$
 - ⑦ For each real number a and each pair of vectors \mathbf{x} and \mathbf{y} , (**distributivity** of scalars)
 - $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$
 - ⑧ For each pair of real numbers a and b and each vector \mathbf{x} , (**distributivity** of vectors)
 - $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$

1.1 Introduction

- **An equation of the line through 2 distinct points**

- Vectors pointing at two points A and B
 - \mathbf{u} : Vector from O to A
 - \mathbf{v} : Vector from O to B

- Vector \mathbf{w} from the two points A and B
 - From “tail-to-head” addition,
 - $\mathbf{u} + \mathbf{w} = \mathbf{v}$
 - $\Rightarrow \mathbf{w} = \mathbf{v} - \mathbf{u}$



- Any point x on the line joining A and B
 - Obtained by the endpoint of $t\mathbf{w}$ beginning at A for some real number t
 - $\Rightarrow \mathbf{u} + t\mathbf{w} = \mathbf{u} + t(\mathbf{v} - \mathbf{u}) = (1 - t)\mathbf{u} + t\mathbf{v}$ for some real number t

- (Recall) Irrespective of the position

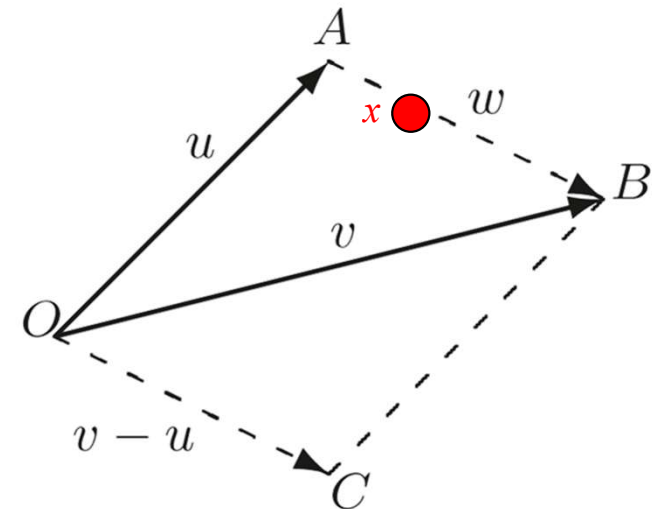
- e.g.) The coordinates of the endpoint C ($\mathbf{v} - \mathbf{u}$) = The difference between the coordinates of B and A

1.1 Introduction

- An equation of the line through 2 distinct points

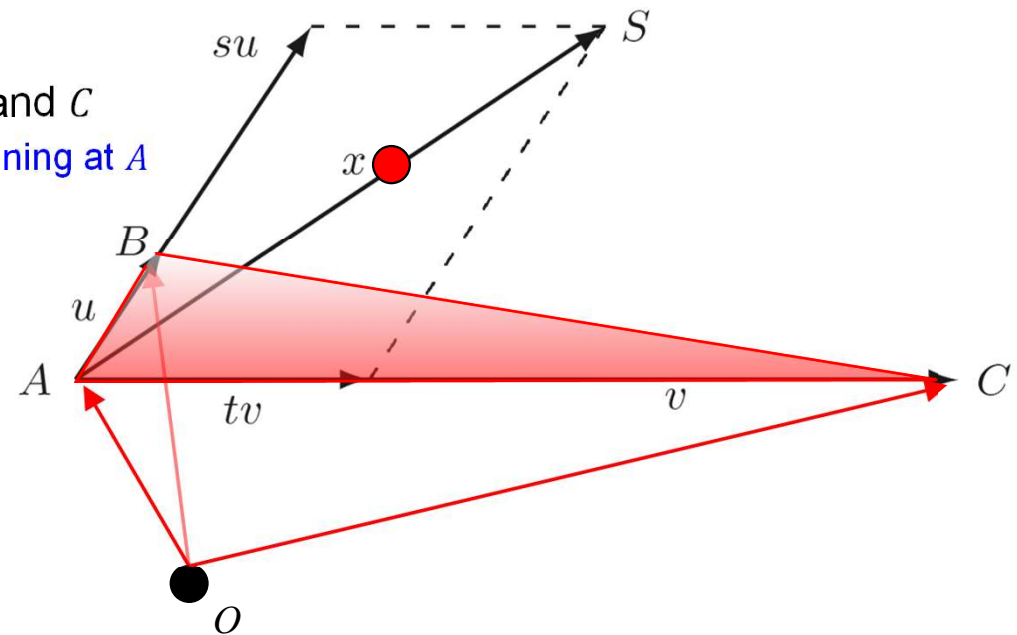
- Example 1.1

- The coordinate of A: $(-2,0,1)$
- The coordinate of B: $(4,5,3)$
- Then,
 - Coordinates of C: $(4,5,3) - (-2,0,1) = (6,5,2)$
 - The equation of the line through A and B:
 - $\mathbf{x} = \mathbf{u} + t(\mathbf{v} - \mathbf{u}) = (-2,0,1) + t(6,5,2)$



1.1 Introduction

- **An equation of the plane through 3 distinct points**
 - Vectors beginning at A and ending at two points A and B
 - \mathbf{u} : Vector from A to B
 - \mathbf{v} : Vector from A to C
- Any point x on the plane containing A , B and C
 - Obtained by the endpoint of $s\mathbf{u} + t\mathbf{v}$ beginning at A for some real number s and t
 - $\Rightarrow A + s\mathbf{u} + t\mathbf{v}$ for some real number s and t

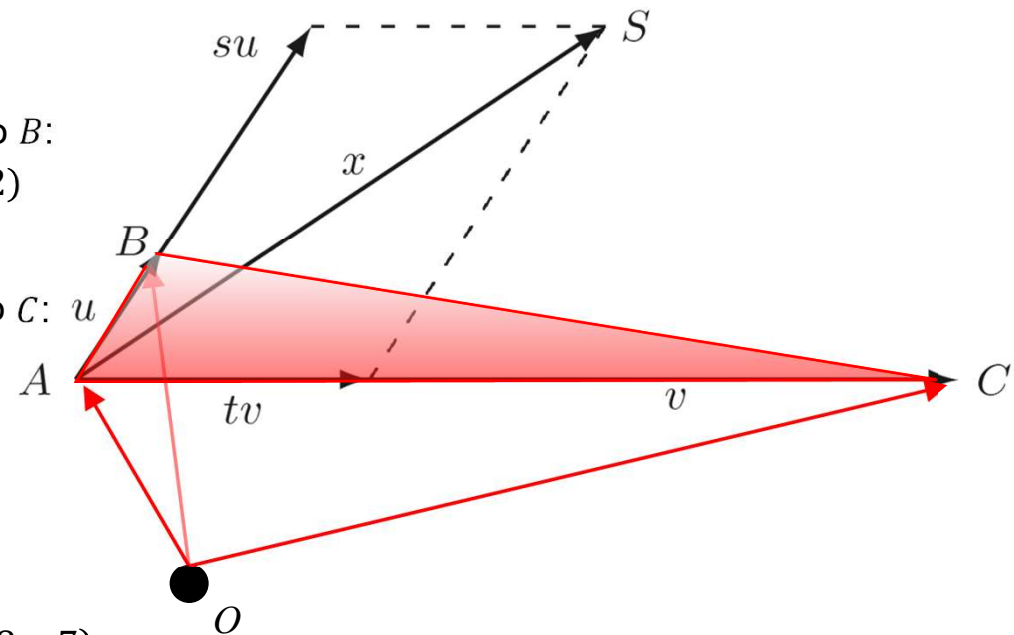


1.1 Introduction

• An equation of the plane through 3 distinct points

• Example 1.2

- The coordinate of A: $(1,0,2)$
- The coordinate of B: $(-3,-2,4)$
- The coordinate of C: $(1,8,-5)$
- Then,
 - Coordinates of the vector \mathbf{u} from A to B:
 - $(-3,-2,4) - (1,0,2) = (-4,-2,2)$
 - Coordinates of the vector \mathbf{v} from A to C: u
 - $(1,8,-5) - (1,0,2) = (0,8,-7)$
 - The equation of the plane through A, B and C:
 - $x = A + s\mathbf{u} + t\mathbf{v}$
 $= (1,0,2) + s(-4,-2,2) + t(0,8,-7)$

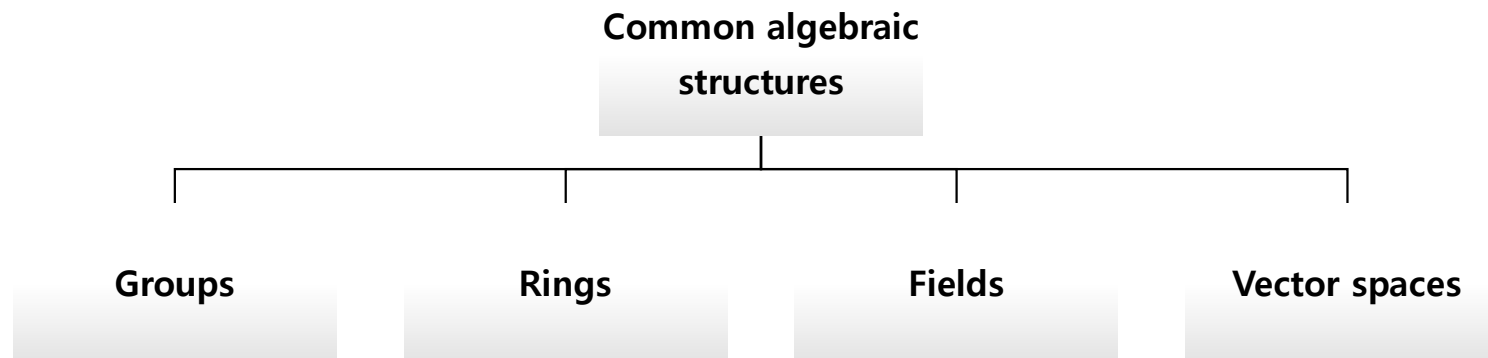


1.2 Vector spaces

1.2 Vector spaces

- **Algebraic structures (대수 구조)**

- The combination of the **set** (집합) and the **operations** (연산) that are applied to the elements of the set
- Common algebraic structures:
 - **Groups** (군)
 - **Rings** (환)
 - **Fields** (체)
 - **Vector space** (벡터 공간)



1.2 Vector spaces

• Groups (군), G

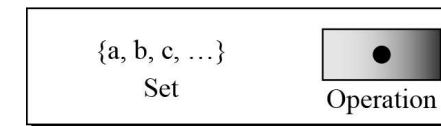
- A set of elements with a binary operation “ \bullet ” that satisfies four properties (성질) or axioms (공리)

- Property ①: Closure (닫힘)
 - If $a, b \in G$, then $a \bullet b \in G$
- Property ②: Associativity (결합)
 - If $a, b, c \in G$, then $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
 - Any order of the operation yielding the same result
- Property ③: Existence of identity (항등원의 존재)
 - Existence of e for all $a \in G$ such that $e \bullet a = a \bullet e = a$
- Property ④: Existence of inverse (역원의 존재)
 - Existence of a^{-1} for each $a \in G$ such that $a^{-1} \bullet a = a \bullet a^{-1} = e$

Properties

1. Closure
2. Associativity
3. Commutativity (See note)
4. Existence of identity
5. Existence of inverse

Note:
The third property needs to be satisfied only for a commutative group.



Group

- Commutative group (가환군), or abelian group, if commutativity also holds
 - Property ⑤: Commutativity (교환 법칙)
 - For all $a, b \in G$, $a \bullet b = b \bullet a$

1.2 Vector spaces

- **Groups (\bar{G}), G**
 - Application
 - A single operation involved in a group
 - $+$, $-$, \times , $/$
 - A pair of operations, as long as they are inverse, also involved in a group
 - $(+, -)$ and $(\times, /)$
 - Only one pair supported at a time

1.2 Vector spaces

- Groups ($\langle \cdot \rangle$), G

- Example

Modulo addition!

- The set of residue integers with the addition operator, $G = \langle \mathbb{Z}_n, + \rangle$
 - Closure?
 - $(a + b) \bmod n \in \mathbb{Z}_n$ for any $a, b \in \mathbb{Z}_n$, Yes
 - Associative?
 - $((a + b) + c) \bmod n = (a + (b + c)) \bmod n$ for any $a, b, c \in \mathbb{Z}_n$, Yes
 - Existence of identity?
 - $e = 0$
 - $(a + 0) \bmod n = (0 + a) \bmod n = a \bmod n$, Yes
 - Existence of inverse?
 - $\acute{a} = -a$ or equivalently, $\acute{a} = n - a$
 - $(a + (-a)) \bmod n = ((-a) + a) \bmod n = 0 \bmod n = e$, Yes
 - Commutativity?
 - $(a + b) \bmod n = (b + a) \bmod n$, Yes

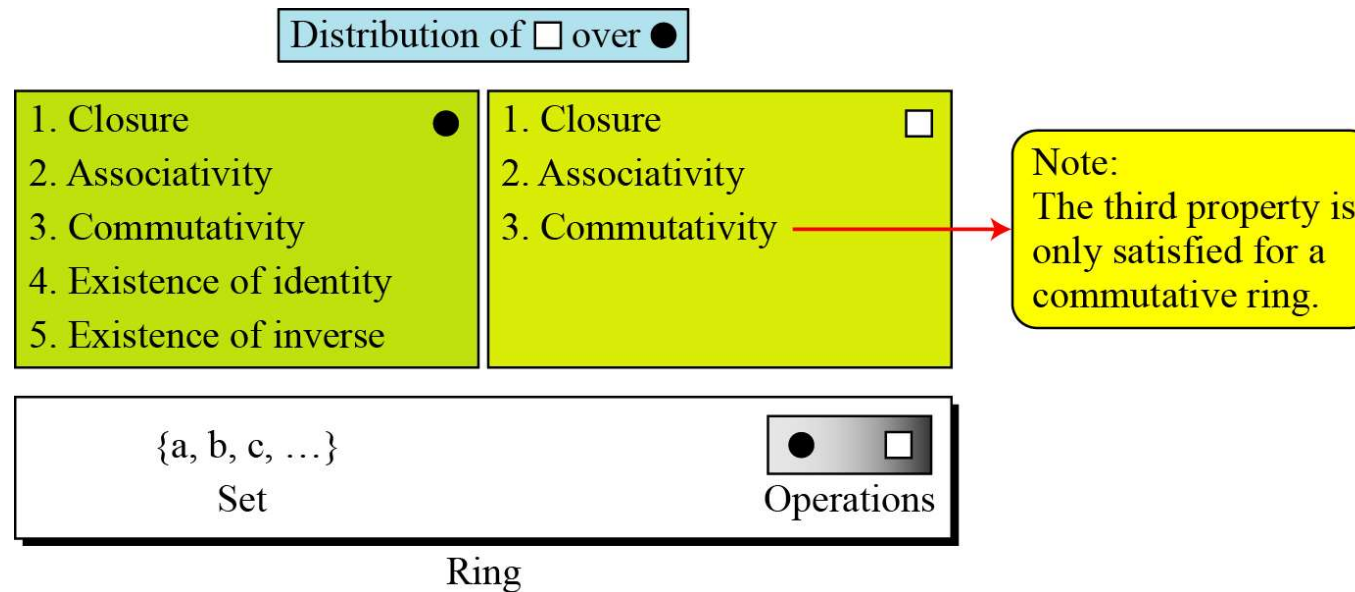
1.2 Vector spaces

- Rings (환), R

- An algebraic structure (대수 구조) with **two operations**, denoted as $R = \langle \{...\}, \bullet, \square \rangle$
- **First operation \bullet** satisfying
 - Closure (닫힘)
 - Associativity (결합)
 - Existence of identity (항등원의 존재성)
 - Existence of inverse (역원의 존재성)
 - Commutativity (교환 법칙)
- **Second operation \square** satisfying
 - Closure (닫힘)
 - Associativity (결합)
- **Distributivity (분배 법칙)** of the **second operation \square** over the **first operation \bullet**
 - For all $a, b, c \in R$,
 - $a \square (b \bullet c) = (a \square b) \bullet (a \square c)$
 - $(a \bullet b) \square c = (a \square c) \bullet (b \square c)$
- **Commutative ring (가환 환)** if the second operation \square also satisfies **commutativity**

1.2 Vector spaces

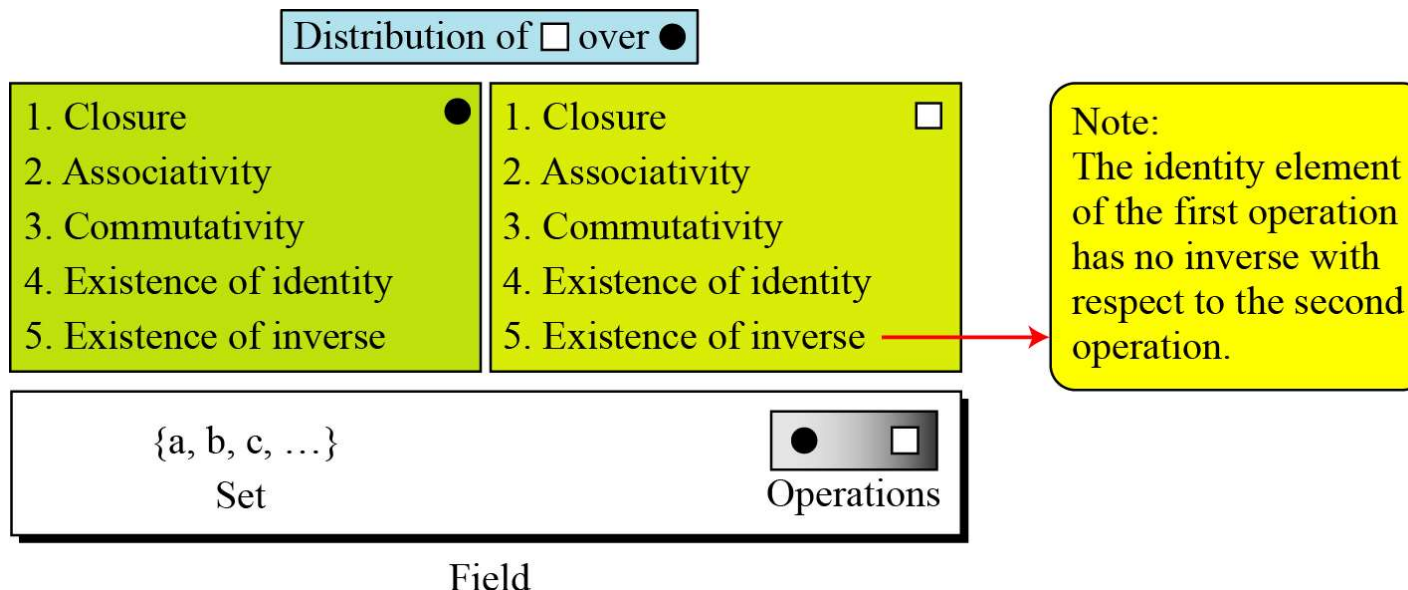
- Rings (환), R



- Example
 - $R = \langle \mathbb{Z}, +, \times \rangle$

1.2 Vector spaces

- Field (체), F
 - $F = \langle \{ \dots \}, \bullet, \square \rangle$
 - A commutative ring (가환 환) in which ...
 - The second operation satisfies all five properties
 - The identity (항등원) of the first operation has no inverse with respect to the second operation



1.2 Vector spaces

- Vector space (벡터 공간), V

Vector space (linear space) V over field F :

- Elements of V are called "vectors"
- Elements of F are called "scalars"
- Two operations
 - ① **Vector addition** ($V \times V \rightarrow V$)
 - For each pair of elements \mathbf{x} and \mathbf{y} in V ,
there is a unique element $\mathbf{x} + \mathbf{y}$ in V
 - ② **Scalar multiplication** ($F \times V \rightarrow V$)
 - For each element a in F and each element \mathbf{x} in V ,
there is a unique $a\mathbf{x}$ in V

1.2 Vector spaces

- Vector space (벡터 공간), V

Vector space (linear space) V over field F :

- The following 8 conditions hold:

	Axiom	Meaning
①	Associativity of vector addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
②	Commutativity of vector addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
③	Existence of identity of vector addition	There exists an element $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$
④	Existence of inverse of vector addition	For every $\mathbf{v} \in V$, there exists an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

1.2 Vector spaces

- Vector space (벡터 공간), V

Vector space (linear space) V over field F :

- The following 8 conditions hold:

	Axiom	Meaning
⑤	Associativity of scalar multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$
⑥	Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$
⑦	Distributivity of scalar multiplication w.r.t. vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
⑧	Distributivity of scalar multiplication w.r.t. field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

1.2 Vector spaces

- **Vector space (벡터 공간), V**
 - Possible scalar fields F
 - Real numbers, \mathbb{R}
 - Complex numbers, \mathbb{C}
 - Etc.
 - The representation of a n -tuple vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$$

- $a_1, \dots, a_n \in F$: Entries or components
- F^n : The set of all n -tuple vectors with entries from a field F

1.2 Vector spaces

- Vector space (벡터 공간), V
 - Vector addition and scalar multiplication

$$\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n, \mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in F^n$$

$$\Rightarrow \mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

$$\Rightarrow c\mathbf{u} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

- \mathbf{u} and \mathbf{v} equal if $a_1 = b_1, a_2 = b_2, \dots, a_n = b_n$

1.2 Vector spaces

- Vector space (벡터 공간), V

- Example 1.2.1

- For $F = \mathbb{R}$ and $V = \mathbb{R}^3$,

$$\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$-5 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \\ 0 \end{bmatrix}$$

- For $F = \mathbb{C}$ and $V = \mathbb{C}^2$,

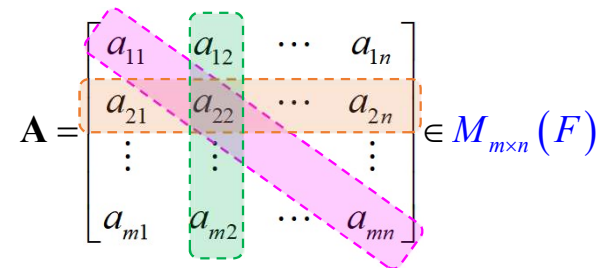
$$\begin{bmatrix} 1+j \\ 2 \end{bmatrix} + \begin{bmatrix} 2-j3 \\ j4 \end{bmatrix} = \begin{bmatrix} 3-j2 \\ 2+j4 \end{bmatrix}$$

$$j \begin{bmatrix} 1+j \\ 2 \end{bmatrix} = \begin{bmatrix} -1+j \\ j2 \end{bmatrix}$$

1.2 Vector spaces

• Matrices

- An $m \times n$ matrix with entries from a field F

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(F)$$


The diagram shows a matrix A with entries a_{kl}. A horizontal orange dashed box highlights the second row (k=2), a vertical green dashed box highlights the second column (l=2), and a pink dashed diagonal line highlights the main diagonal (k=l). The matrix is labeled as belonging to M_{m×n}(F).

- $a_{k\ell} \in F$:
 - Entries or components
- $a_{k\ell} \in F$ for $k = \ell$:
 - Diagonal entries
- $[a_{k1} \ \cdots \ a_{kn}]$:
 - The k -th row vector in F^n
- $\begin{bmatrix} a_{1\ell} \\ \vdots \\ a_{m\ell} \end{bmatrix}$:
 - The ℓ -th column vector in F^m

1.2 Vector spaces

- **Matrices**

- An $m \times n$ matrix with entries from a field F

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(F)$$

- **Zero** matrix
 - $a_{k\ell} = 0$ for all k, ℓ
 - **Square** matrix
 - $m = n$

1.2 Vector spaces

• Matrices

- Matrix addition and scalar multiplication

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(F), \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \in M_{m \times n}(F)$$

$$\Rightarrow [\mathbf{A} + \mathbf{B}]_{k\ell} = [\mathbf{A}]_{k\ell} + [\mathbf{B}]_{k\ell} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

$$\Rightarrow [c\mathbf{A}]_{k\ell} = c[\mathbf{A}]_{k\ell} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

- where $[\mathbf{A}]_{k\ell} = a_{k\ell}$ and $[\mathbf{B}]_{k\ell} = b_{k\ell}$

1.2 Vector spaces

- **Matrices**

- Example 1.2.2

- For $M_{2 \times 3}(\mathbb{R})$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$
$$-3 \begin{bmatrix} 1 & 0 & -2 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 6 \\ 9 & -6 & -9 \end{bmatrix}$$

1.2 Vector spaces

• Theorems

Theorem 1.1 (Cancellation Law for Vector Addition):

If \mathbf{x} , \mathbf{y} , and \mathbf{z} are vectors in a vector space V such that $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$, then $\mathbf{x} = \mathbf{y}$.

• Proof)

- From property ④ of vector space, there exists a vector \mathbf{v} such that $\mathbf{z} + \mathbf{v} = \mathbf{0}$.

• Then,

$$\mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x} + (\mathbf{z} + \mathbf{v}) = (\mathbf{x} + \mathbf{z}) + \mathbf{v} = (\mathbf{y} + \mathbf{z}) + \mathbf{v} = \mathbf{y} + (\mathbf{z} + \mathbf{v}) = \mathbf{y} + \mathbf{0} = \mathbf{y}$$

↑
↑
 Property ① Property ①

1.2 Vector spaces

- Theorems

Corollary 1.1.1

The vector $\mathbf{0}$ in property ③ is unique

Corollary 1.1.2

The vector \mathbf{y} in property ④ is unique

1.2 Vector spaces

- Theorems

Theorem 1.2:

In any vector space V , the following statements are true:

- (a) $0\mathbf{x} = \mathbf{0}$ for each $\mathbf{x} \in V$
- (b) $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$ for each $a \in F$ and each $x \in V$
- (c) $a\mathbf{0} = \mathbf{0}$ for each $a \in F$

- Proof) (a)

- $0\mathbf{x} + 0\mathbf{x} = (0 + 0)\mathbf{x} = 0\mathbf{x} = 0\mathbf{x} + \mathbf{0}$

↑
Property ⑧

- From Theorem 1.1, $0\mathbf{x} = \mathbf{0}$

1.2 Vector spaces

- Theorems

Theorem 1.2:

In any vector space V , the following statements are true:

- (a) $0\mathbf{x} = \mathbf{0}$ for each $\mathbf{x} \in V$
- (b) $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$ for each $a \in F$ and each $\mathbf{x} \in V$
- (c) $a\mathbf{0} = \mathbf{0}$ for each $a \in F$

- Proof) (b)

- From [Corollary 1.1.2](#)
 - Vector $-(a\mathbf{x}) \in V$ is the unique element such that $a\mathbf{x} + (-(a\mathbf{x})) = \mathbf{0}$.
- From [Theorem 1.2 \(a\)](#) and property ⑧,
 - $\mathbf{0} = 0\mathbf{x} = (a + (-a))\mathbf{x} = a\mathbf{x} + (-a)\mathbf{x}$
- From [Theorem 1.1](#),
 - $(-a)\mathbf{x} = -(a\mathbf{x})$
- From property ⑤,
 - $(-a)\mathbf{x} = (a \cdot (-1))\mathbf{x} = a((-1)\mathbf{x}) = a(-\mathbf{x})$

1.2 Vector spaces

- Theorems

Theorem 1.2:

In any vector space V , the following statements are true:

(a) $0\mathbf{x} = \mathbf{0}$ for each $\mathbf{x} \in V$

(b) $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$ for each $a \in F$ and each $x \in V$

(c) $a\mathbf{0} = \mathbf{0}$ for each $a \in F$

- Proof) (c)

- From property ③,

- $a\mathbf{0} + \mathbf{0} = a\mathbf{0}$

- From property ③ and property ⑧,

- $a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) = a\mathbf{0} + a\mathbf{0}$

- From Theorem 1.1, $a\mathbf{0} = \mathbf{0}$

1.3 Subspaces

1.3 Subspaces

- Subspace W

Subspace W of vector space V over field F :

A vector space over F with operations of vector addition and scalar multiplication defined on V

- e.g., $\{0\}$ and V as subspaces of V
- Vector space property ①, ②, ⑤, ⑥, ⑦ and ⑧ automatically satisfied for all vectors in V
- Only needed to check ③ and ④, or the following 4 conditions:
 - Closure under vector addition
 - $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$
 - Closure under scalar multiplication
 - $c\mathbf{x} \in W$ whenever $c \in F$ and $\mathbf{x} \in W$
 - $\mathbf{0} \in W$
 - Each vector in W has an additive inverse in W

1.3 Subspaces

- Subspace W

Theorem 1.3: (Existence of the additive inverse need not be checked)

Let V be a vector space and W a subset of V .

W is a subspace of V if and only if the following 3 conditions hold for the operations defined in V :

(a) $\mathbf{0} \in W$

(b) $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$

(c) $c\mathbf{x} \in W$ whenever $c \in F$ and $\mathbf{x} \in W$

- Proof) “if” part (W is a subspace of $V \iff 3$ conditions hold)
 - Assume (a), (b), and (c) hold true.
 - Then, from the previous slide, only the existence of the additive inverse needs to be verified.
 - From condition (c)
 - If $\mathbf{x} \in W$, then $(-1)\mathbf{x} \in W$
 - From Theorem 1.2 (b),
 - $(-1)\mathbf{x} = -\mathbf{x}$
 - \therefore The additive inverse $-\mathbf{x} \in W$ exists for each $\mathbf{x} \in W$.

1.3 Subspaces

- Subspace W

Theorem 1.3: (Existence of the additive inverse need not be checked)

Let V be a vector space and W a subset of V .

W is a subspace of V if and only if the following 3 conditions hold for the operations defined in V :

(a) $\mathbf{0} \in W$

(b) $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$

(c) $c\mathbf{x} \in W$ whenever $c \in F$ and $\mathbf{x} \in W$

- Proof) “only if” part (W is a subspace of $V \Rightarrow$ 3 conditions hold)

- Assume W is a subspace of V .

- A vector space over F with operations of vector addition and scalar multiplication defined on V

- Then, (b) and (c) automatically hold true.

- Also, there must exist $\mathbf{z} \in W$ such that $\mathbf{x} + \mathbf{z} = \mathbf{x}$ for $\mathbf{x} \in W$

- Meanwhile, since $\mathbf{x} \in V$ as well, we have $\mathbf{x} + \mathbf{0} = \mathbf{x}$ where $\mathbf{0} \in V$ is the zero vector of V .

- From Theorem 1.1,

- $\mathbf{z} = \mathbf{0}$, and (a) holds true.

1.3 Subspaces

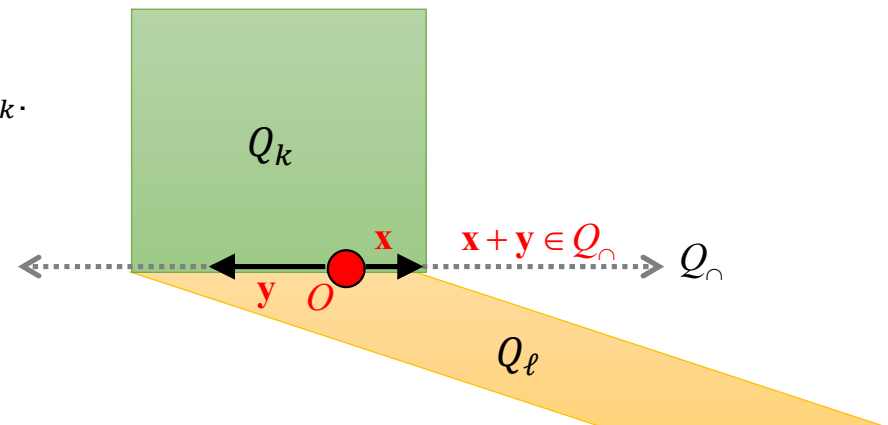
• Subspace W

Theorem 1.4:

Any **intersection** of subspaces of a vector space V is a **subspace** of V .

• Proof)

- Let $Q_{\cap} = \cap\{Q_1, \dots, Q_n\}$ be the intersection of subspaces Q_1, \dots, Q_n of V .
- Since every subspace contains the zero vector, we have $\mathbf{0} \in Q_{\cap}$.
 - (Theorem 1.3(a))
- Let $a \in F$, $\mathbf{x} \in Q_k$, $\mathbf{y} \in Q_{\ell}$ and $\mathbf{x}, \mathbf{y} \in Q_{\cap}$.
- Since $\mathbf{x}, \mathbf{y} \in Q_{\cap}$ it is also true that $\mathbf{x} \in Q_{\ell}$, $\mathbf{y} \in Q_k$.
- Then, $\mathbf{x} + \mathbf{y} \in Q_{\cap}$ and $a\mathbf{x} \in Q_{\cap}$ (or $a\mathbf{y} \in Q_{\cap}$) because Q_k and Q_{ℓ} are subspaces where \mathbf{x} and \mathbf{y} simultaneously belong to.
 - (Theorem 1.3(b) and (c))
- \therefore Subspace!



1.3 Subspaces

• Subspace W

- Any **union** of subspaces of a vector space V is **not** a subspace of V .

- Proof)

- Let $Q_U = U\{Q_1, \dots, Q_n\}$ be the union of subspaces Q_1, \dots, Q_n of V .

- Since every subspace contains the zero vector, we have $\mathbf{0} \in Q_U$.

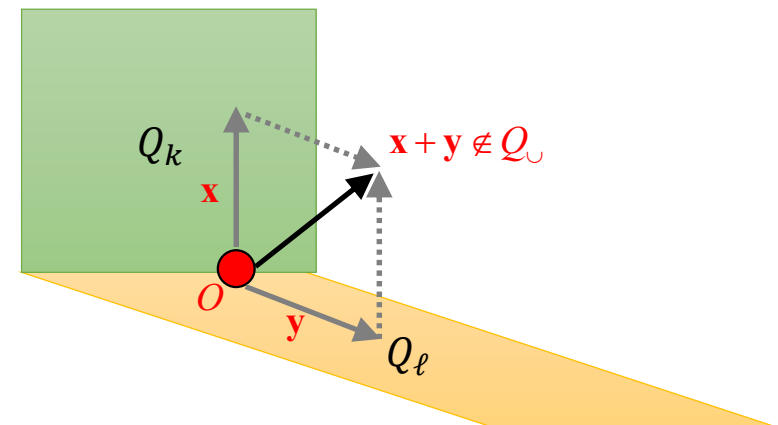
- (Theorem 1.3(a))

- Let $a \in F$, $\mathbf{x} \in Q_k$, $\mathbf{y} \in Q_\ell$

- Then, it is not guaranteed that $\mathbf{x} + \mathbf{y} \in Q_U$

- Possibly in another subspace in V

- \therefore **Not** a subspace



1.3 Subspaces

- **Transpose, A^T**
 - Obtained by **interchanging** the **rows** with the **columns**
 - $[A^T]_{k\ell} = [A]_{\ell k}$
 - The transpose of an $m \times n$ matrix $A \Rightarrow$ An $n \times m$ matrix
 - e.g.)

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

1.3 Subspaces

- **Types of matrices**

- Symmetric matrix

- $\mathbf{A}^T = \mathbf{A}$

- Square matrix

- The set W of all symmetric matrices = A subspace of $M_{n \times n}(F)$?

- Theorem 1.3(a)

- Zero matrix $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in W$

- Theorem 1.3(b): closure under addition

- $\mathbf{A} + \mathbf{B} \in W$ since $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T = \mathbf{A} + \mathbf{B}$ for $\mathbf{A}, \mathbf{B} \in W$

- Theorem 1.3(c): closure under scalar multiplication

- $a\mathbf{A} \in W$ since $(a\mathbf{A})^T = a\mathbf{A}^T = a\mathbf{A}$ for $\mathbf{A} \in W$

- \therefore Subspace!

1.3 Subspaces

- **Types of matrices**

- Upper triangular matrix

- $[A]_{k\ell} = 0$ for $k > \ell$
 - e.g.)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{bmatrix}$$

- Diagonal matrix

- $[A]_{k\ell} = 0$ for $k \neq \ell$
 - e.g.)

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

1.3 Subspaces

- **Types of matrices**

- Example 1.3.3

- The set W of all diagonal matrices = A subspace of $M_{n \times n}(F)$?

- Theorem 1.3(a)

- Zero matrix $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in W$

- Theorem 1.3(b): closure under addition

- $\mathbf{A} + \mathbf{B} \in W$ since $[\mathbf{A} + \mathbf{B}]_{k\ell} = 0$ for $k \neq \ell$ for $\mathbf{A}, \mathbf{B} \in W$

- Theorem 1.3(c): closure under scalar multiplication

- $a\mathbf{A} \in W$ since $[a\mathbf{A}]_{k\ell} = 0$ for $k \neq \ell$ for $\mathbf{A} \in W$

- \therefore Subspace!

1.3 Subspaces

- **Types of matrices**

- Example 1.3.5

- The set W of $M_{m \times n}(\mathbb{R})$ matrices with nonnegative entries

- Theorem 1.3(a)

- Zero matrix $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in W$

- Theorem 1.3(b): closure under addition

- $\mathbf{A} + \mathbf{B} \in W$ since $[\mathbf{A} + \mathbf{B}]_{k\ell} \geq 0$ for all k, ℓ for $\mathbf{A}, \mathbf{B} \in W$

- Theorem 1.3(c): closure under scalar multiplication

- $a\mathbf{A} \notin W$ since $[a\mathbf{A}]_{k\ell} < 0$ for $a < 0$ for $\mathbf{A} \in W$
 - \therefore Not a subspace

1.3 Subspaces

- **Trace, $\text{tr}(\mathbf{A})$**
 - Obtained by **summing the diagonal entries** of an $n \times n$ **square** matrix
 - $\text{tr}(\mathbf{A}) = [\mathbf{A}]_{11} + [\mathbf{A}]_{22} + \cdots + [\mathbf{A}]_{nn}$

1.4 Linear combinations and systems of linear equations

1.4 Linear combinations and systems of linear equations

- Linear combination

Linear combination:

Let V be a vector space and S a non-empty subset of V .

A vector $\mathbf{v} \in V$ is called a **linear combination** of vectors of S if there exist a finite number of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ in S and scalar a_1, a_2, \dots, a_n in F such that $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_n\mathbf{u}_n$.

a_1, a_2, \dots, a_n : The **coefficients** of the linear combination

1.4 Linear combinations and systems of linear equations

- Linear combination

- Example 1.4.1

- Each row showing vitamin content
 - e.g.) Apple butter

$$\begin{bmatrix} 0.00 \\ 0.01 \\ 0.02 \\ 0.20 \\ 2.00 \end{bmatrix}$$

- Represented in \mathbb{R}^5

- Raw wild rice as a linear combination

$$\begin{bmatrix} 0.00 \\ 0.05 \\ 0.06 \\ 0.30 \\ 0.00 \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.02 \\ 0.02 \\ 0.40 \\ 0.00 \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.34 \\ 0.05 \\ 4.70 \\ 0.00 \end{bmatrix} + 2 \begin{bmatrix} 0.00 \\ 0.02 \\ 0.02 \\ 0.40 \\ 0.00 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.45 \\ 0.63 \\ 6.20 \\ 0.00 \end{bmatrix}$$

TABLE 1.1 Vitamin Content of 100 Grams of Certain Foods

	A (units)	B ₁ (mg)	B ₂ (mg)	Niacin (mg)	C (mg)
Apple butter	0	0.01	0.02	0.2	2
Raw, unpared apples (freshly harvested)	90	0.03	0.02	0.1	4
Chocolate-coated candy with coconut center	0	0.02	0.07	0.2	0
Clams (meat only)	100	0.10	0.18	1.3	10
Cupcake from mix (dry form)	0	0.05	0.06	0.3	0
Cooked farina (unenriched)	(0) ^a	0.01	0.01	0.1	(0)
Jams and preserves	10	0.01	0.03	0.2	2
Coconut custard pie (baked from mix)	0	0.02	0.02	0.4	0
Raw brown rice	(0)	0.34	0.05	4.7	(0)
Soy sauce	0	0.02	0.25	0.4	0
Cooked spaghetti (unenriched)	0	0.01	0.01	0.3	0
Raw wild rice	(0)	0.45	0.63	6.2	(0)

Source: Bernice K. Watt and Annabel L. Merrill, *Composition of Foods* (Agriculture Handbook Number 8), Consumer and Food Economics Research Division, U.S. Department of Agriculture, Washington, D.C., 1963.

^aZeros in parentheses indicate that the amount of a vitamin present is either none or too small to measure.

1.4 Linear combinations and systems of linear equations

- Linear combination

- Example 1.4.1

- Clams as a linear combination

$$\begin{array}{c}
 \begin{bmatrix} 0.00 \\ 0.01 \\ 0.02 \\ 0.20 \\ 2.00 \end{bmatrix} + \begin{bmatrix} 90.00 \\ 0.03 \\ 0.02 \\ 0.10 \\ 4.00 \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.02 \\ 0.07 \\ 0.20 \\ 0.00 \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.01 \\ 0.01 \\ 0.10 \\ 0.00 \end{bmatrix} \\
 2 \begin{bmatrix} 10.00 \\ 0.01 \\ 0.03 \\ 0.20 \\ 2.00 \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.01 \\ 0.01 \\ 0.30 \\ 0.00 \end{bmatrix} = \begin{bmatrix} 100.00 \\ 0.10 \\ 0.18 \\ 1.30 \\ 10.00 \end{bmatrix}
 \end{array}$$

TABLE 1.1 Vitamin Content of 100 Grams of Certain Foods

	A (units)	B ₁ (mg)	B ₂ (mg)	Niacin (mg)	C (mg)
Apple butter	0	0.01	0.02	0.2	2
Raw, unpared apples (freshly harvested)	90	0.03	0.02	0.1	4
Chocolate-coated candy with coconut center	0	0.02	0.07	0.2	0
Clams (meat only)	100	0.10	0.18	1.3	10
Cupcake from mix (dry form)	0	0.05	0.06	0.3	0
Cooked farina (unenriched)	(0) ^a	0.01	0.01	0.1	(0)
Jams and preserves	10	0.01	0.03	0.2	2
Coconut custard pie (baked from mix)	0	0.02	0.02	0.4	0
Raw brown rice	(0)	0.34	0.05	4.7	(0)
Soy sauce	0	0.02	0.25	0.4	0
Cooked spaghetti (unenriched)	0	0.01	0.01	0.3	0
Raw wild rice	(0)	0.45	0.63	6.2	(0)

Source: Bernice K. Watt and Annabel L. Merrill, *Composition of Foods* (Agriculture Handbook Number 8), Consumer and Food Economics Research Division, U.S. Department of Agriculture, Washington, D.C., 1963.

^aZeros in parentheses indicate that the amount of a vitamin present is either none or too small to measure.

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**

- Necessary to determine whether a vector can be expressed as a linear combination

- (A general method in Chapter 03)

- e.g.) $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} -3 \\ 8 \\ 16 \end{bmatrix}$

- Coefficients to be determined: a_1, a_2, a_3, a_4, a_5

$$\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + a_4 \mathbf{u}_4 + a_5 \mathbf{u}_5$$

$$\Rightarrow \begin{array}{rrrrrcl} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ 2a_1 & -4a_2 & +2a_3 & & +8a_5 & = & 6 \\ a_1 & -2a_2 & +3a_3 & -3a_4 & +16a_5 & = & 8 \end{array}$$

1.4 Linear combinations and systems of linear equations

• Systems of linear equations

- Necessary to determine whether a vector can be expressed as a linear combination

- (A general method in Chapter 03)

- e.g.) $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} -3 \\ 8 \\ 16 \end{bmatrix}$

- Coefficients to be determined: a_1, a_2, a_3, a_4, a_5

$$\begin{array}{rcl} \Rightarrow & \begin{array}{rrrrr} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & 2a_3 & -4a_4 & +14a_5 & = & 2 \\ & & 3a_3 & -5a_4 & +19a_5 & = & 6 \end{array} & \Rightarrow & \begin{array}{rrrrr} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & a_3 & -2a_4 & +7a_5 & = & 1 \\ & & 3a_3 & -5a_4 & +19a_5 & = & 6 \end{array} \end{array}$$

(Row2) \leftarrow (Row2)-2×(Row1)
 (Row3) \leftarrow (Row3)-(Row1)

(Row2) \leftarrow (Row2)÷2

1.4 Linear combinations and systems of linear equations

• Systems of linear equations

- Necessary to determine whether a vector can be expressed as a linear combination

- (A general method in Chapter 03)

- e.g.) $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} -3 \\ 8 \\ 16 \end{bmatrix}$

- Coefficients to be determined: a_1, a_2, a_3, a_4, a_5

$$\begin{array}{rclcl} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & a_3 & -2a_4 & +7a_5 & = & 1 \\ & & & a_4 & -2a_5 & = & 3 \end{array}$$

 \Rightarrow


(Row3) \leftarrow (Row3)-3×(Row2)

 \Rightarrow


(Row1) \leftarrow (Row1)-2×(Row3)

(Row2) \leftarrow (Row2)+2×(Row3)

$$\begin{array}{rclcl} a_1 & -2a_2 & & & +a_5 & = & -4 \\ & & a_3 & & +3a_5 & = & 7 \\ & & & a_4 & -2a_5 & = & 3 \end{array}$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**

- Necessary to determine whether a vector can be expressed as a linear combination

- (A general method in Chapter 03)

- e.g.) $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} -3 \\ 8 \\ 16 \end{bmatrix}$

- For any a_2, a_5 ,

$$a_1 = 2a_2 - a_5 - 4$$

$$a_2 = a_2$$

$$a_3 = -3a_5 + 7$$

$$a_4 = 2a_5 + 3$$

$$a_5 = a_5$$

- For instance, if $a_2 = 0, a_5 = 0$,

$$\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} = -4\mathbf{u}_1 + 0\mathbf{u}_2 + 7\mathbf{u}_3 + 3\mathbf{u}_4 + 0\mathbf{u}_5$$

1.4 Linear combinations and systems of linear equations

• Systems of linear equations

- 3 types of operations to **simplify** the original system

- ① **Interchanging** the order of any two equations in the system

• e.g.)

$$\begin{array}{rcl}
 \begin{array}{rcl} a_3 & -2a_4 & +7a_5 = 1 \\ a_1 & -2a_2 & +2a_4 -3a_5 = 2 \end{array} & \Rightarrow & \begin{array}{rcl} a_1 & -2a_2 & +2a_4 -3a_5 = 2 \\ a_3 & -2a_4 & +7a_5 = 1 \end{array} \\
 a_4 & -2a_5 = 3 & \text{(Row1) } \leftrightarrow \text{(Row2)} & & a_4 & -2a_5 = 3
 \end{array}$$

- ② **Multiplying** any equation in the system by a non-zero constant

• e.g.)

$$\begin{array}{rcl}
 \begin{array}{rcl} a_1 & -2a_2 & +2a_4 -3a_5 = 2 \\ 2a_3 & -4a_4 & +14a_5 = 2 \end{array} & \Rightarrow & \begin{array}{rcl} a_1 & -2a_2 & +2a_4 -3a_5 = 2 \\ a_3 & -2a_4 & +7a_5 = 1 \end{array} \\
 3a_3 & -5a_4 & +19a_5 = 6 & \text{(Row2) } \leftarrow 0.5 \times \text{(Row2)} & 3a_3 & -5a_4 & +19a_5 = 6
 \end{array}$$

- ③ **Adding** a constant multiple of any equation to another equation in the system

• e.g.)

$$\begin{array}{rcl}
 \begin{array}{rcl} a_1 & -2a_2 & +2a_4 -3a_5 = 2 \\ a_3 & -2a_4 & +7a_5 = 1 \\ a_4 & -2a_5 = 3 \end{array} & \Rightarrow & \begin{array}{rcl} a_1 & -2a_2 & +a_5 = -4 \\ a_3 & & +3a_5 = 7 \\ a_4 & -2a_5 = 3 \end{array} \\
 & & \text{(Row1) } \leftarrow \text{(Row1)} - 2 \times \text{(Row3)} \\
 & & \text{(Row2) } \leftarrow \text{(Row2)} + 2 \times \text{(Row3)}
 \end{array}$$

1.4 Linear combinations and systems of linear equations

• Systems of linear equations

- Properties for the **final simplified** system to have

- ① The **first non-zero coefficient** in each equation equal to 1
- ② If an unknown is the **first unknown with a non-zero coefficient** in some equation, then that unknown occurring with a 0 coefficient in all the other equations
- ③ The **first unknown with a non-zero coefficient** in any equation having a **larger subscript** than the first unknown with a non-zero coefficient in **preceding equations**

$$\begin{array}{rcl}
 \boxed{a_1} - 2a_2 & \boxed{a_3} & +a_5 = -4 \\
 \boxed{a_1} & \boxed{a_3} & +3a_5 = 7 \\
 \boxed{a_1} & \boxed{a_4} & -2a_5 = 3
 \end{array}$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**

- Example 1.4.2

$$\begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

- Coefficients to be determined: a_1, a_2

$$\begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rcl} a_1 & +3a_2 & = 2 \\ -2a_1 & -5a_2 & = -2 \\ -5a_1 & -4a_2 & = 12 \\ -3a_1 & -9a_2 & = -6 \end{array}$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**

- Example 1.4.2

$$\begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

- Coefficients to be determined: a_1, a_2

$$a_1 + 3a_2 = 2$$

$$a_2 = 2$$

$$11a_2 = 22$$

$$0 = 0$$

$$a_1 + 3a_2 = 2$$

$$a_2 = 2$$

$$a_2 = 2$$

$$0 = 0$$



$(\text{Row2}) \leftarrow (\text{Row2}) + 2 \times (\text{Row1})$ $(\text{Row3}) \leftarrow (\text{Row3}) \div 11$
 $(\text{Row3}) \leftarrow (\text{Row3}) + 5 \times (\text{Row1})$
 $(\text{Row4}) \leftarrow (\text{Row4}) + 3 \times (\text{Row1})$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**

- Example 1.4.2

$$\begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

- Coefficients to be determined: a_1, a_2

$$\begin{array}{rcl} a_1 & = & -4 \\ 0 & = & 0 \\ \Rightarrow \uparrow & & a_2 = 2 \\ & & 0 = 0 \end{array}$$

$$(\text{Row1}) \leftarrow (\text{Row1}) - 3 \times (\text{Row3})$$

$$(\text{Row2}) \leftarrow (\text{Row2}) - (\text{Row3})$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**

- Example 1.4.2

$$\begin{bmatrix} 3 \\ -2 \\ 7 \\ 8 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

- Coefficients to be determined: a_1, a_2

$$\begin{bmatrix} 3 \\ -2 \\ 7 \\ 8 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rcl} a_1 & +3a_2 & = 3 \\ -2a_1 & -5a_2 & = -2 \\ -5a_1 & -4a_2 & = 7 \\ -3a_1 & -9a_2 & = 8 \end{array}$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**

- Example 1.4.2

$$\begin{bmatrix} 3 \\ -2 \\ 7 \\ 8 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

- Coefficients to be determined: a_1, a_2

$$a_1 + 3a_2 = 3$$

$$a_2 = 4$$

$$11a_2 = 22$$

$$0 = 17$$



Indicating no solution!

$$(\text{Row2}) \leftarrow (\text{Row2}) + 2 \times (\text{Row1})$$

$$(\text{Row3}) \leftarrow (\text{Row3}) + 5 \times (\text{Row1})$$

$$(\text{Row4}) \leftarrow (\text{Row4}) + 3 \times (\text{Row1})$$

1.4 Linear combinations and systems of linear equations

- Span

Span:

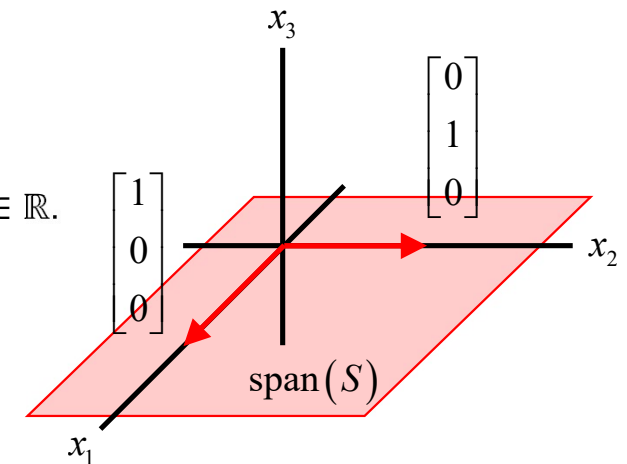
Let S be a nonempty subset of a vector space V .

The **span** of S , denoted $\text{span}(S)$, is the set consisting of **all linear combinations** of the vectors in S .

For convenience, we define $\text{span}(\emptyset) = \{\mathbf{0}\}$.

- e.g.) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $V = \mathbb{R}^3$

- $\text{span}(S)$ consisting **all vectors** $a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for some real numbers $a, b \in \mathbb{R}$.
- $\text{span}(S) =$ A **subspace** of $V = \mathbb{R}^3$



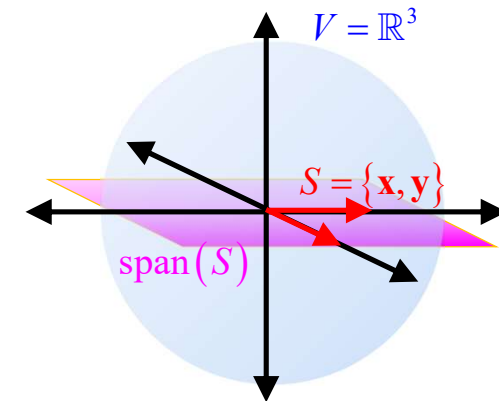
1.4 Linear combinations and systems of linear equations

- Span

Theorem 1.5:

The span of any subset S of a vector space V is a subspace of V containing S . Moreover, any subspace of V that contains S must also contain the span of S .

- Proof) $\text{span}(S)$ = A subspace of V that contains S
 - If $S = \emptyset$
 - $\text{span}(S) = \{\mathbf{0}\}$ is a subspace of V .
 - $\text{span}(S) = \{\mathbf{0}\}$ contains $S = \emptyset$.
 - $\therefore \text{span}(S)$ is a subspace that contains S for $S = \emptyset$!



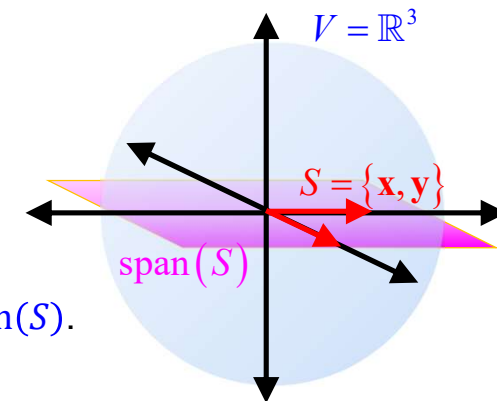
1.4 Linear combinations and systems of linear equations

• Span

Theorem 1.5:

The span of any subset S of a vector space V is a subspace of V containing S . Moreover, any subspace of V that contains S must also contain the span of S .

- Proof) $\text{span}(S)$ = A subspace of V that contains S
 - If $S \neq \emptyset$
 - S containing a vector \mathbf{z}
 - Theorem 1.3(a)
 - Zero vector $0\mathbf{z} = \mathbf{0} \in \text{span}(S)$
 - Theorem 1.3(b): closure under addition
 - Let $\mathbf{x}, \mathbf{y} \in \text{span}(S)$.
 - Then, $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m$ and $\mathbf{y} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$ for some $\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_n \in S$ and $a_1, \dots, a_m, b_1, \dots, b_n \in F$.
 - Thus, $\mathbf{x} + \mathbf{y} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m + b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n \in \text{span}(S)$.



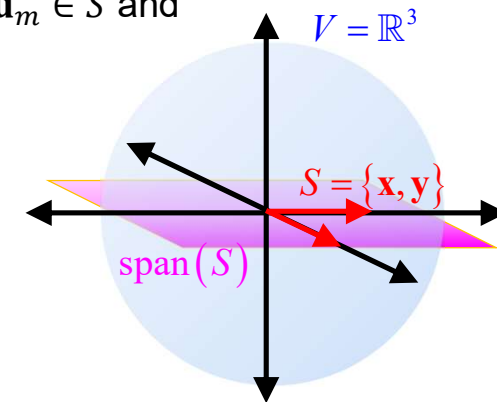
1.4 Linear combinations and systems of linear equations

• Span

Theorem 1.5:

The span of any subset S of a vector space V is a subspace of V containing S . Moreover, any subspace of V that contains S must also contain the span of S .

- Proof) $\text{span}(S)$ = A subspace of V that contains S
 - If $S \neq \emptyset$
 - Theorem 1.3(c): closure under scalar multiplication
 - Let $\mathbf{x} \in \text{span}(S)$ such that $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m$ for some $\mathbf{u}_1, \dots, \mathbf{u}_m \in S$ and $a_1, \dots, a_m \in F$.
 - Then, $c\mathbf{x} = (ca_1)\mathbf{u}_1 + (ca_2)\mathbf{u}_2 + \cdots + (ca_m)\mathbf{u}_m \in \text{span}(S)$.
 - S contained in $\text{span}(S)$
 - If $\mathbf{v} \in S$, it is also $\mathbf{v} \in \text{span}(S)$ since $\mathbf{v} = 1\mathbf{v}$ (linear combination).
 - Since it is true for all arbitrary $\mathbf{v} \in S$, we have $S \subseteq \text{span}(S)$.
- $\therefore \text{span}(S)$ is a subspace that contains S for $S \neq \emptyset$!



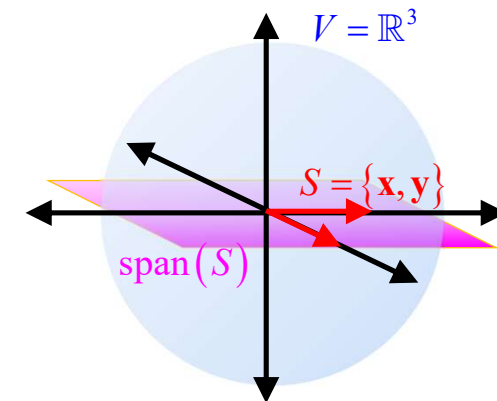
1.4 Linear combinations and systems of linear equations

- Span

Theorem 1.5:

The span of any subset S of a vector space V is a subspace of V containing S . Moreover, any subspace of V that contains S must also contain the span of S .

- Proof) $\text{span}(S) \subseteq W$ A subspace of V that contains S
 - Let W be a subspace of V that contains S .
 - Let $\mathbf{x} \in \text{span}(S)$.
 - Then $\mathbf{x} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_m \mathbf{u}_m$ for some $\mathbf{u}_1, \dots, \mathbf{u}_m \in S$ and $a_1, \dots, a_m \in F$.
 - Also, since $S \subseteq W$, it is true that $\mathbf{u}_1, \dots, \mathbf{u}_m \in W$.
 - Thus, $\mathbf{x} \in W$.
 - Since it is true for all arbitrary $\mathbf{x} \in \text{span}(S)$, we have $\text{span}(S) \subseteq W$.



1.4 Linear combinations and systems of linear equations

- Span

Spanning (or generating):

A subset S of a vector space V **spans** (or **generates**) V if $\text{span}(S) = V$.
In this case, we also say that the vectors of S **span** (or **generate**) V .

- Example 1.4.3

- Vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ **spanning** (or **generating**) $V = \mathbb{R}^3$

- Example 1.4.5

- Matrices $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **spanning** (or **generating**) $V = M_{2 \times 2}(\mathbb{R})$
- Matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ **not** spanning (or generating) $V = M_{2 \times 2}(\mathbb{R})$
 - Not every 2×2 matrix as a linear combination of these 3 matrices

1.5 Linear dependence and linear independence

1.5 Linear dependence and linear independence

- A finite subset S spanning a subspace W

- Supposing

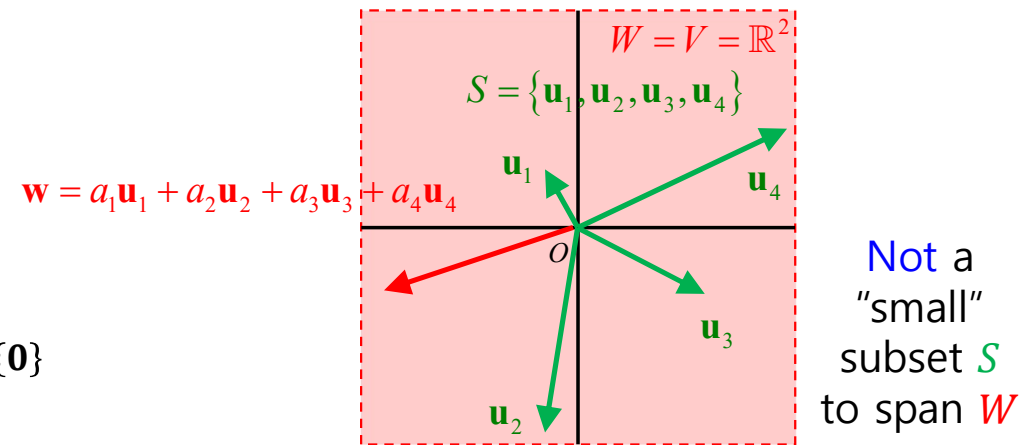
- V : A vector space over an infinite field F
 - W : A subspace of V

- Then,

- W an infinite set unless W is the zero subspace, $\{0\}$

- Desirable to find a “small” finite subset S of W that spans W

- Able to describe each vector in W as a linear combination of the finite number of vectors in S
 - Smaller $S \Rightarrow$ Fewer number of computations required to represent vectors in W



1.5 Linear dependence and linear independence

- A finite subset S spanning a subspace W

- e.g.) Subspace W of \mathbb{R}^3 spanned by $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$

- Q: Is it a “minimal” subset of S that also spans W ?
 - A **just enough** number of vectors to span W
 - **No need** to have a vector that is a linear combination of the others in S
 - Checking whether \mathbf{u}_4 is a linear combination of the others:

$$\underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4} = a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_3}$$

- No solution! \Rightarrow Not a linear combination of the others

1.5 Linear dependence and linear independence

- A finite subset S spanning a subspace W

- e.g.) Subspace W of \mathbb{R}^3 spanned by $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$

- Q: Is it a “minimal” subset of S that also spans W ?
 - A **just enough** number of vectors to span W
 - **No need** to have a vector that is a linear combination of the others in S
 - Checking whether \mathbf{u}_3 is a linear combination of the others:

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_3} = a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_4 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4}$$

- Solution $a_1 = 2, a_2 = -3, a_4 = 0$
 - \therefore The current set S having **redundant** vectors for spanning W

1.5 Linear dependence and linear independence

- A finite subset S spanning a subspace W

- e.g.) Subspace W of \mathbb{R}^3 spanned by $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$

- Q: Is it a “minimal” subset of S that also spans W ?
 - A just enough number of vectors to span W
 - No need to have a vector that is a linear combination of the others in S
 - Writing differently,

$$a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_3} + a_4 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4} = \mathbf{0}$$

- Solution $a_1 = -2, a_2 = 3, a_3 = 1, a_4 = 0$

Not “small” enough subset S
for spanning subspace W

=

Some vectors being a linear
combination of the other
vectors in S

=

Non-zero solution to yield
the zero vector $\mathbf{0}$ by a linear
combination

1.5 Linear dependence and linear independence

- Linear dependence

Linear dependence:

A subset S of a vector space V is called **linearly dependent** if there exist a finite number of distinct vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ in S and scalars a_1, a_2, \dots, a_n , not all zero, such that

$$a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \dots + a_n \mathbf{u}_n = \mathbf{0}$$

- **Trivial** representation
 - $a_1 = a_2 = \dots = a_n = 0$
- **Required** to have a **nontrivial** representation for linear dependence
 - **At least one** coefficient being **non-zero**
- Any subset containing the **zero** vector $\mathbf{0} \Rightarrow$ **Linearly dependent** subset
 - E.g.) A linear combination of itself $\mathbf{0} = 1 \cdot \mathbf{0}$

1.5 Linear dependence and linear independence

- Linear dependence

- Example 1.5.1

- Considering

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Linearly dependent since for $a_1 = 4, a_2 = -3, a_3 = 2, a_4 = 0$

$$a_1 \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix} + a_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{array}{rrrr} a_1 & +2a_2 & +a_3 & -a_4 = 0 \\ 3a_1 & +2a_2 & -3a_3 & = 0 \\ -4a_1 & -4a_2 & +2a_3 & +a_4 = 0 \\ 2a_1 & & -4a_3 & = 0 \end{array}$$

- i.e., **non-zero solution** existing for the **zero** vector

1.5 Linear dependence and linear independence

- Linear dependence

- Example 1.5.2

- Considering

$$S = \left\{ \begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix}, \begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} \right\}$$

- Linearly dependent since for $a_1 = 5, a_2 = 3, a_3 = -2$

$$a_1 \begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix} + a_2 \begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix} + a_3 \begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} = [\mathbf{0}]$$

- i.e., **non-zero** solution existing for the **zero** matrix

1.5 Linear dependence and linear independence

- Linear independence

Linear independence:

A subset S of a vector space V is called **linearly independent** if there does **not** exist a finite number of distinct vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ in S and scalars a_1, a_2, \dots, a_n , not all zero, such that

$$a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \dots + a_n \mathbf{u}_n = \mathbf{0}$$

- Facts about linear independence

- ① The **empty** set \Rightarrow Linearly independent
 - The linearly dependence required to be non-empty
- ② A set consisting of **a single non-zero vector** \Rightarrow Linearly independent
- ③ Linearly independent if and only if the **only representation of the zero** vector $\mathbf{0}$ is the **trivial** representation

1.5 Linear dependence and linear independence

- Linear independence

- Example 1.5.3

- Considering

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Linearly independent since only $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$ is the solution

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{array}{rcl} a_1 & & = 0 \\ & a_2 & = 0 \\ & & a_3 = 0 \\ -a_1 & -a_2 & -a_3 + a_4 = 0 \end{array}$$

1.5 Linear dependence and linear independence

- Linear independence

Theorem 1.6:

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly **dependent**, then S_2 is linearly **dependent**.

Corollary:

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly **independent**, then S_1 is linearly **independent**.

1.5 Linear dependence and linear independence

- A finite subset S spanning a subspace W (revisited)

- e.g.) Subspace W of \mathbb{R}^3 spanned by $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$

- Q: Is it a “minimal” subset of S that also spans W ?

- A **just enough** number of vectors to span W

- **No need** to have a vector that is a linear combination of the others in S

- **Linearly independent**

- Recalling \mathbf{u}_3 was a linear combination of the other vectors since for $a_1 = -2, a_2 = 3, a_3 = 1, a_4 = 0$

$$a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_3} + a_4 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4} = \mathbf{0}$$

- $\therefore \mathbf{u}_3$ being a **redundant** vector in set S for spanning W

- \Rightarrow Set S being **linearly dependent**

1.5 Linear dependence and linear independence

- A finite subset S spanning a subspace W (revisited)

- e.g.) Subspace W of \mathbb{R}^3 spanned by $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$

- Q: Is it a “minimal” subset of S that also spans W ?
 - A **just enough** number of vectors to span W
 - **No need** to have a vector that is a linear combination of the others in S

+ • **Linearly independent**

- By removing the redundant \mathbf{u}_3 from S

$$a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_4 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4} = \mathbf{0}$$

$$\begin{array}{rclclclcl} \Rightarrow & 2a_1 & +a_2 & +a_4 & = & 0 & \Rightarrow & 2a_1 & +a_2 & +a_4 & = & 0 & \Rightarrow & 2a_1 & +a_2 & +a_4 & = & 0 \\ & -a_1 & -a_2 & -2a_4 & = & 0 & \Rightarrow & -a_2 & -3a_4 & = & 0 & \Rightarrow & -a_2 & -3a_4 & = & 0 \\ & 4a_1 & +3a_2 & -a_4 & = & 0 & \Rightarrow & +a_2 & -3a_4 & = & 0 & \Rightarrow & -6a_4 & = & 0 \end{array}$$

- The only solution to the system: $a_1 = a_2 = a_4 = 0$

- **∴ Linearly independent**

1.5 Linear dependence and linear independence

- A finite subset S spanning a subspace W (revisited)

Theorem 1.7:

Let S be a linearly independent subset of a vector space V , and let \mathbf{v} be a vector in V that is not in S .

Then, $S \cup \{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \text{span}(S)$

- Proof) $S \cup \{\mathbf{v}\}$ linearly dependent $\Rightarrow \mathbf{v} \in \text{span}(S)$
 - $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is linearly independent while $S \cup \{\mathbf{v}\}$ is linearly dependent.
 - $\Rightarrow \mathbf{v}$ is a redundant vector
 - $\Rightarrow \mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_n\mathbf{u}_n$ in which not every coefficient is zero.
 - Note that $\text{span}(S) = \{a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_n\mathbf{u}_n \mid a_1, \dots, a_n \in F\}$.
 - $\therefore \mathbf{v} \in \text{span}(S)$

1.5 Linear dependence and linear independence

- A finite subset S spanning a subspace W (revisited)

Theorem 1.7:

Let S be a linearly independent subset of a vector space V , and let \mathbf{v} be a vector in V that is not in S .

Then, $S \cup \{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \text{span}(S)$

- Proof) $S \cup \{\mathbf{v}\}$ linearly dependent $\iff \mathbf{v} \in \text{span}(S)$
 - $\text{span}(S) = \{a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n \mid a_1, \dots, a_n \in F\}$ for $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$
 - $\Rightarrow \mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$
 - $\Rightarrow \mathbf{v}$ is a redundant vector for the set S .
 - $\therefore S \cup \{\mathbf{v}\} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \mathbf{v}\}$ is linearly dependent.

1.6 Bases and dimension

1.6 Bases and dimension

- Bases

Basis:

A **basis** β for a vector space V is a linearly **independent** subset of V that **spans** V .

- Example 1.6.1

- \emptyset being a basis for the zero vector space $\{0\}$

- Example 1.6.2

- The **standard basis** for n -dimensional field F^n :

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

1.6 Bases and dimension

- Bases

Basis:

A **basis** β for a vector space V is a linearly **independent** subset of V that **spans** V .

- Example 1.6.3

- $\{E^{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ being a basis for $M_{m \times n}(F)$

$$E^{ij} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \boxed{1} & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \text{ row } i, \text{ column } j$$

- Note: **Not** every vector space having a **finite** basis

1.6 Bases and dimension

- Bases

Theorem 1.8:

Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be distinct vectors in V . Then, $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V if and only if each $\mathbf{v} \in V$ can be **uniquely expressed** as a linear combination of the vectors of β as

$$\mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \dots + a_n \mathbf{u}_n$$

with **unique scalars** a_1, a_2, \dots, a_n .

- Proof) $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for $V \Rightarrow$ each $\mathbf{v} \in V$ can be uniquely expressed
 - Let β be a basis for V .
 - $\Rightarrow \text{span}(\beta) = V$
 - $\Rightarrow \mathbf{v} \in \text{span}(\beta)$
 - By contradiction, assume $\mathbf{v} \in V$ is not uniquely expressed.
 - $\mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \dots + a_n \mathbf{u}_n$
 - $\mathbf{v} = b_1 \mathbf{u}_1 + b_2 \mathbf{u}_2 + \dots + b_n \mathbf{u}_n$
 - Here, there exist some i 's such that $a_i \neq b_i$

1.6 Bases and dimension

- Bases

Theorem 1.8:

Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be distinct vectors in V . Then, $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V if and only if each $\mathbf{v} \in V$ can be **uniquely expressed** as a linear combination of vectors of β as

$$\mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \dots + a_n \mathbf{u}_n$$

with unique scalars a_1, a_2, \dots, a_n .

- Proof) $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for $V \Rightarrow$ each $\mathbf{v} \in V$ can be uniquely expressed
 - By subtracting one from the other,
 - $\mathbf{0} = (a_1 - b_1)\mathbf{u}_1 + (a_2 - b_2)\mathbf{u}_2 + \dots + (a_n - b_n)\mathbf{u}_n$
 - Since $a_i \neq b_i$ for some i 's, this is a non-zero solution for the zero vector $\mathbf{0}$.
 - \Rightarrow Contradicting the fact that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are linearly independent
 - \therefore Q.E.D.

1.6 Bases and dimension

- Bases

Theorem 1.8:

Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be distinct vectors in V . Then, $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V if and only if each $\mathbf{v} \in V$ can be **uniquely expressed** as a linear combination of vectors of β as

$$\mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \dots + a_n \mathbf{u}_n$$

with unique scalars a_1, a_2, \dots, a_n .

- Proof) $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for $V \Leftrightarrow$ each $\mathbf{v} \in V$ can be uniquely expressed
 - By contradiction, assume β is not a basis.
 - \Rightarrow Linearly dependent set that spans V .
 - Then there exists a non-zero solution b_1, b_2, \dots, b_n such that
 - $\mathbf{0} = b_1 \mathbf{u}_1 + b_2 \mathbf{u}_2 + \dots + b_n \mathbf{u}_n$
 - Note that for any scalar c ,
 - $\mathbf{0} = cb_1 \mathbf{u}_1 + cb_2 \mathbf{u}_2 + \dots + cb_n \mathbf{u}_n$

1.6 Bases and dimension

- Bases

Theorem 1.8:

Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be distinct vectors in V . Then, $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V if and only if each $\mathbf{v} \in V$ can be **uniquely expressed** as a linear combination of vectors of β as

$$\mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \dots + a_n \mathbf{u}_n$$

with unique scalars a_1, a_2, \dots, a_n .

- Proof) $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for $V \Leftrightarrow$ each $\mathbf{v} \in V$ can be uniquely expressed
 - By adding \mathbf{v} on both sides,
 - $\mathbf{v} = cb_1 \mathbf{u}_1 + cb_2 \mathbf{u}_2 + \dots + cb_n \mathbf{u}_n + \mathbf{v} = (cb_1 + a_1) \mathbf{u}_1 + (cb_2 + a_2) \mathbf{u}_2 + \dots + (cb_n + a_n) \mathbf{u}_n$
 - This equation holds true for any scalar c
 - \Rightarrow Contradicting \mathbf{v} is uniquely expressed
 - \therefore Q.E.D.

1.6 Bases and dimension

- Bases

Theorem 1.9:

If a vector space V is spanned by a finite set S , then **some subset** of S is a **basis** for V .

Hence, V has **a finite basis**.

- Proof)
 - If $S = \emptyset$,
 - The only subset of S
 - \emptyset : **Linearly independent**
 - Note that a linear combination of no vectors is, by convention, $\mathbf{0}$.
 - $\Rightarrow \emptyset$ spans $V = \{\mathbf{0}\}$
 - \therefore **The subset \emptyset is a basis for $V = \{\mathbf{0}\}$.**

1.6 Bases and dimension

- Bases

Theorem 1.9:

If a vector space V is spanned by a finite set S , then **some subset** of S is a **basis** for V .

Hence, V has **a finite basis**.

- Proof)
 - If $S = \{\mathbf{0}\}$,
 - The subsets of S
 - \emptyset : **Linearly independent**
 - $\{\mathbf{0}\}$: Linearly dependent (can't be a basis!)
 - Note that a linear combination of no vectors is, by convention, $\mathbf{0}$.
 - $\Rightarrow \emptyset$ spans $V = \{\mathbf{0}\}$
 - \therefore **The subset \emptyset is a basis for $V = \{\mathbf{0}\}$.**

1.6 Bases and dimension

- Bases

Theorem 1.9:

If a vector space V is spanned by a finite set S , then **some subset** of S is a **basis** for V .

Hence, V has **a finite basis**.

- Proof)
 - If S is a non-empty set other than $\{\mathbf{0}\}$,
 - It is possible to find a **maximal linearly independent** set $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \subseteq S$.
 - By including vectors one by one and check for linearly independence for each inclusion.
 - If $\beta = S$,
 - $\text{span}(\beta) = \text{span}(S) = V$
 - \therefore **The subset β is a basis for $V = \text{span}(S)$.**

1.6 Bases and dimension

- Bases

Theorem 1.9:

If a vector space V is spanned by a finite set S , then **some subset** of S is a **basis** for V .

Hence, V has **a finite basis**.

- Proof)

- If S is a non-empty set other than $\{0\}$,
 - If $\beta \subset S$,
 - For any \mathbf{v} such that $\mathbf{v} \in S, \mathbf{v} \notin \beta$, the union $\beta \cup \{\mathbf{v}\}$ is linearly dependent
 - By Theorem 1.7, $\mathbf{v} \in \text{span}(\beta)$
 - $\Rightarrow S \subseteq \text{span}(\beta)$
 - $\Rightarrow \text{span}(S) \subseteq \text{span}(\beta)$
 - Also, $\beta \subset S$ implies $\text{span}(\beta) \subset \text{span}(S)$
 - $\Rightarrow \text{span}(S) \subseteq \text{span}(\beta) \subset \text{span}(S)$
 - $\Rightarrow \text{span}(\beta) = \text{span}(S) = V$
 - \therefore The subset β is a basis for $V = \text{span}(S)$.

1.6 Bases and dimension

- **Bases**

- A **finite spanning set** for V able to be **reduced** to a **basis** for V

- Example 1.6.6

- $S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} \right\}$

- Q1: Does S span $V = \mathbb{R}^3$?

- System of linear equations for an arbitrary vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $V = \mathbb{R}^3$

$$a_1 \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + a_2 \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + a_5 \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rrrrr} 2a_1 & +8a_2 & +a_3 & & +7a_5 & = & x_1 \\ -3a_1 & -12a_2 & & 2a_4 & +2a_5 & = & x_2 \\ 5a_1 & +20a_2 & -2a_3 & -a_4 & & = & x_3 \end{array}$$

1.6 Bases and dimension

• Bases

- A **finite spanning set** for V able to be **reduced** to a **basis** for V

• Example 1.6.6

- $S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} \right\}$

- Q1: Does S span $V = \mathbb{R}^3$?

- By simplifying the equations,

$$\begin{aligned} 2a_1 + 8a_2 + a_3 + 7a_5 &= x_1 \\ \Rightarrow 15a_3 - 45a_5 &= -2x_1 - 2x_2 - 4x_3 \\ 5a_4 + 20a_5 &= 2x_1 + 3x_2 + x_3 \end{aligned}$$

- Letting $a_2 = a_5 = 0$,

$$a_1 = \frac{1}{2}(-a_3 + x_1) = \frac{1}{2} \left(-\frac{1}{15}(-2x_1 - 2x_2 - 4x_3) + x_1 \right) = \frac{17}{30}x_1 + \frac{1}{15}x_2 + \frac{2}{15}x_3$$

$$a_3 = \frac{1}{15}(-2x_1 - 2x_2 - 4x_3), \quad a_4 = \frac{1}{5}(2x_1 + 3x_2 + x_3)$$

- $\therefore S$ spans $V = \mathbb{R}^3$

1.6 Bases and dimension

- **Bases**

- A **finite spanning set** for V able to be **reduced** to a **basis** for V

- Example 1.6.6

- $S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} \right\}$

- Q2: *Is there any subset of S that is a basis for $V = \mathbb{R}^3$?*

- Yes there is!

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

1.6 Bases and dimension

- Bases

Theorem 1.10 (Replacement theorem):

Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors
such that $L \cup H$ spans V .

- Proof)
 - If $m = 0$,
 - $L = \emptyset$
 - We may set $H = G$ and $L \cup H = G$ which spans V .
 - \therefore Q.E.D.

1.6 Bases and dimension

- Bases

Theorem 1.10 (Replacement theorem):

Let V be a vector space, spanned by a set G containing exactly n vectors. Let L be a linearly independent subset of V containing exactly m vectors. Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ spans V .

- Proof)

- If $m = n$,
 - By Theorem 1.8, L itself is a basis for V .
 - Since $n - m = 0$, we have $H = \emptyset$, and $L \cup H = L$ spans V .

- \therefore Q.E.D.

1.6 Bases and dimension

- Bases

Theorem 1.10 (Replacement theorem):

Let V be a vector space, spanned by a set G containing exactly n vectors. Let L be a linearly independent subset of V containing exactly m vectors. Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ spans V .

- Proof)

- If $m < n$,
 - Assume true for $0 < m < n$.
 - Let $L_m = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a linearly independent subset of V .
 - Let $H_m = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-m}\}$ be a subset of G such that $m \leq n$ and $L_m \cup H_m$ spans V .

1.6 Bases and dimension

• Bases

Theorem 1.10 (Replacement theorem):

Let V be a vector space, spanned by a set G containing exactly n vectors. Let L be a linearly independent subset of V containing exactly m vectors. Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ spans V .

• Proof)

- If $m < n$,
 - Now, consider the case of $m + 1$.
 - Let $L_{m+1} = L_m \cup \{v_{m+1}\} = \{v_1, v_2, \dots, v_m, v_{m+1}\}$ be a linearly independent subset of V .
 - Recall that $L_m \cup H_m$ spanned V .
 - $\Rightarrow v_{m+1} = a_1 v_1 + a_2 v_2 + \dots + a_m v_m + b_1 u_1 + b_2 u_2 + \dots + b_{n-m} u_{n-m}$
 - Also note that if all b_i 's are zero, it contradicts the fact that L_{m+1} is linearly independent.
 - Without loss of generality, assume $b_{n-m} \neq 0$.

$$\Rightarrow u_{n-m} = -\frac{a_1}{b_{n-m}} v_1 - \dots - \frac{a_m}{b_{n-m}} v_m + \frac{1}{b_{n-m}} v_{m+1} - \frac{b_1}{b_{n-m}} u_1 - \dots - \frac{b_{n-(m+1)}}{b_{n-m}} u_{n-(m+1)}$$

1.6 Bases and dimension

- Bases

Theorem 1.10 (Replacement theorem):

Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors
such that $L \cup H$ spans V .

- Proof)

- If $m < n$,

- Let $H_{m+1} = H_m \setminus \mathbf{u}_{n-m} = \{\mathbf{u}_1, \dots, \mathbf{u}_{n-(m+1)}\}$.

- $\Rightarrow \mathbf{u}_{n-m} \in \text{span}(L_{m+1} \cup H_{m+1})$.

- $\Rightarrow \{\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}, \mathbf{u}_1, \dots, \mathbf{u}_{n-(m+1)}, \mathbf{u}_{n-m}\} = L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\} \subseteq \text{span}(L_{m+1} \cup H_{m+1})$

1.6 Bases and dimension

• Bases

Theorem 1.10 (Replacement theorem):

Let V be a vector space, spanned by a set G containing exactly n vectors.
 Let L be a linearly independent subset of V containing exactly m vectors.
 Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors
 such that $L \cup H$ spans V .

• Proof)

- If $m < n$,
 - By the second part of Theorem 1.5,
 - $\Rightarrow \text{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\}) \subseteq \text{span}(L_{m+1} \cup H_{m+1})$
 - Since $L_{m+1} \cup H_{m+1} \subseteq L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\}$,
 - $\Rightarrow \text{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\}) \subseteq \text{span}(L_{m+1} \cup H_{m+1}) \subseteq \text{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\})$
 - $\Rightarrow \text{span}(L_{m+1} \cup H_{m+1}) = \text{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\})$

1.6 Bases and dimension

• Bases

Theorem 1.10 (Replacement theorem):

Let V be a vector space, spanned by a set G containing exactly n vectors.
 Let L be a linearly independent subset of V containing exactly m vectors.
 Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors
 such that $L \cup H$ spans V .

• Proof)

- If $m < n$,
 - Recall that $\mathbf{v}_{m+1} \in \text{span}(L_m \cup H_m)$
 - $\Rightarrow \text{span}(L_m \cup H_m \cup \{\mathbf{v}_{m+1}\}) = \text{span}(L_m \cup H_m) = V$
 - Note that
 - $L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\} = L_m \cup H_m \cup \{\mathbf{v}_{m+1}\}$
 - Thus,
 - $\text{span}(L_{m+1} \cup H_{m+1}) = \text{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\}) = \text{span}(L_m \cup H_m \cup \{\mathbf{v}_{m+1}\}) = V$
- \therefore Q.E.D.

1.6 Bases and dimension

- Bases

Corollary 1.10.1:

Let V be a vector space having a finite basis.

Then, **all bases** for V are finite, and every basis for V contains **the same number of vectors**.

- Proof)

- By contradiction, suppose:

- β_1 is a finite basis for V of n vectors.

- β_2 is another finite basis for V of m vectors where $m > n$.

- Now, obviously, V is spanned by β_1 with n vectors.

- From **Theorem 1.10**, any linearly independent subsets with ℓ number of vectors must satisfy $\ell \leq n$.

- However, β_2 is a linearly independent subset of V of m vectors where $m > n$

- \therefore Q.E.D.

1.6 Bases and dimension

- Dimension

Dimension, $\dim(V)$:

The unique integer n such that every basis for V contains exactly n elements

- Finite-dimensional
 - Having a basis consisting of a finite number of vectors
- Infinite-dimensional
 - Having a basis consisting of an infinite number of vectors

1.6 Bases and dimension

- **Dimension**

- Example 1.6.7

- (from Example 1.6.1)
 - \emptyset being a basis for the zero vector space $\{0\}$
 - \emptyset having no elements
 - $\Rightarrow \dim(\{0\}) = 0$

- Example 1.6.8

- (from Example 1.6.2)
 - The **standard basis** for n-dimensional field F^n :

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

- $\Rightarrow \dim(F^n) = n$

1.6 Bases and dimension

- **Dimension**

- Example 1.6.9

- (from Example 1.6.3)
 - $\{E^{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ being a basis for $M_{m \times n}(F)$

$$E^{ij} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \boxed{1} & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \text{ row } i, \text{ column } j$$

- $\Rightarrow \dim(M_{m \times n}(F)) = mn$

1.6 Bases and dimension

- Dimension

Corollary 1.10.2:

Let V be a vector space with dimension n .

- (a) A spanning set for V contains exactly n vectors is a basis for V .
- (b) Any linearly independent subset of V that contains exactly n vectors is a basis for V .
- (c) Every linearly independent subset of V can be extended to a basis for V . That is, if L is a linearly independent subset of V , then there is a basis β of V such that $L \subseteq \beta$.

1.6 Bases and dimension

- **Dimension**

- Proof) (a) A spanning set for V that contains exactly n vectors \Rightarrow A basis for V
 - Let $G = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a finite spanning set for V .
 - By Theorem 1.9, there exists a subset $H \subseteq G$ that is a basis for V .
 - By Corollary 1.10.1, H has exactly n linearly independent vectors.
 - Now, if $m = n$, we must have $G = H$.
 - \therefore Q.E.D.

1.6 Bases and dimension

- **Dimension**

- Proof) (b) Any linearly independent subset of V that contains exactly n vectors \Rightarrow A basis for V
 - A vector \mathbf{v} is uniquely expressed by a linearly independent subset $L = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$
 - \therefore Unique expression \Leftrightarrow Linearly independence
 - By Theorem 1.8, L being able to express a vector uniquely implies that it is a basis for V
- \therefore Q.E.D.

1.6 Bases and dimension

- **Dimension**

- Proof) (c) L is a linearly independent subset of V . \Rightarrow There is a basis β of V such that $L \subseteq \beta$.
 - Let V be spanned by a basis β with n vectors
 - Let L be a linearly independent subset of V with m vectors.
 - By Theorem 1.10, there is a subset H of β containing $n - m$ vectors such that $L \cup H$ spans V .
 - $\Rightarrow L \cup H$ has at most n vectors.
 - By Theorem 1.9, since $L \cup H$ spans V , there exists a subset $\Phi \subseteq L \cup H$ that is a basis for V .
 - By Corollary 1.10.1, Φ has exactly n vectors.
 - $\Rightarrow L \cup H$ has at least n vectors.

1.6 Bases and dimension

- **Dimension**

- Proof) (c) L is a linearly independent subset of V . \Rightarrow There is a basis β of V such that $L \subseteq \beta$.
 - Thus, $L \cup H$ has exactly n vectors.
- By Corollary 1.10.2 (a), $L \cup H$ is a basis, i.e., $L \cup H = \beta$
 - $\Rightarrow L \subseteq \beta$
- \therefore Q.E.D.

1.6 Bases and dimension

- **Dimension**

- Example 1.6.15

- (from Example 1.4.5)

- 4 matrices $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ spanning or $V = M_{2 \times 2}(\mathbb{R})$

- \Rightarrow A basis for $M_{2 \times 2}(\mathbb{R})$ since $\dim(M_{2 \times 2}(\mathbb{R})) = 4$ ([Corollary 1.10.2 \(a\)](#))

- Example 1.6.16

- (from Example 1.5.3)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- **Linearly independent** set that contains exactly 4 vectors

- \Rightarrow A basis for \mathbb{R}^4 since $\dim(\mathbb{R}^4) = 4$ ([Corollary 1.10.2 \(b\)](#))

1.6 Bases and dimension

- The dimension of subspaces

Theorem 1.11:

Let W be a subspace of a finite-dimensional vector space V .
Then, W is finite-dimensional and $\dim(W) \leq \dim(V)$.
Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

- Proof)
 - Let $\dim(V) = n$.
 - (W is finite-dimensional and $\dim(W) \leq \dim(V)$)
 - If $W = \{\mathbf{0}\}$,
 - \emptyset is a linearly independent basis
 - $\Rightarrow \dim(W) = 0 \leq n$
 - If $W = \text{span}(\{\mathbf{w}_1, \dots, \mathbf{w}_k\})$, by adding one by one so as to remain linearly independent,
 - By [Corollary 1.10.2 \(c\)](#), no linearly independent subset of V can contain more than n vectors.
 - $\Rightarrow \dim(W) = k \leq n$

1.6 Bases and dimension

- The dimension of subspaces

Theorem 1.11:

Let W be a subspace of a finite-dimensional vector space V .
Then, W is finite-dimensional and $\dim(W) \leq \dim(V)$.
Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

- Proof)
 - (If $\dim(W) = \dim(V)$, then $V = W$)
 - If $\dim(W) = n$,
 - A basis for W is a linearly independent subset of V containing n vectors
 - From Corollary 1.10.2 (b), that basis is also a basis for V .
 - $\Rightarrow V = W$
- \therefore Q.E.D.

1.6 Bases and dimension

- The dimension of subspaces

- Example 1.6.18

- $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \in V = F^5 \mid a_1 + a_3 + a_5 = 0, a_2 = a_4 \right\}$

- A possible basis is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- $\Rightarrow \dim(W) = 3 \leq \dim(V) = 5$

1.6 Bases and dimension

- The dimension of subspaces

- Example 1.6.19

- $\{E^{ij} | 1 \leq i \leq n, 1 \leq j \leq n\}$ being a basis for square matrices $V = M_{n \times n}(F)$

$$E^{ij} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \boxed{1} & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \text{ row } i, \text{ column } j$$

- For the set of **diagonal** $n \times n$ matrices $W = \{M_{n \times n}(F) | [A]_{k\ell} = 0 \text{ for } k \neq \ell\}$,
 - A possible basis being $\{E^{11}, E^{22}, \dots, E^{nn}\}$
 - $\Rightarrow \dim(W) = n \leq \dim(V) = n^2$

1.6 Bases and dimension

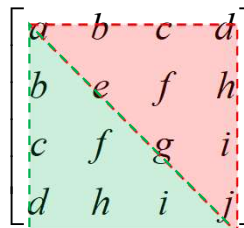
• The dimension of subspaces

• Example 1.6.20

- $\{E^{ij} | 1 \leq i \leq n, 1 \leq j \leq n\}$ being a basis for square matrices $V = M_{n \times n}(F)$

$$E^{ij} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \boxed{1} & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \text{ row } i, \text{ column } j$$

- For the set of **symmetric** $n \times n$ matrices $W = \{M_{n \times n}(F) | [A]_{k\ell} = [A]_{\ell k}\}$,
 - A possible basis being $\{E^{11}, E^{12}, \dots, E^{1n}, E^{22}, E^{23}, \dots, E^{2n}, E^{33}, E^{34}, \dots, E^{nn}\}$
- $\Rightarrow \dim(W) = n + (n-1) + \cdots + 1 = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} \leq \dim(V) = n^2$



$$\begin{bmatrix} a & b & c & d \\ b & e & f & h \\ c & f & g & i \\ d & h & i & j \end{bmatrix}$$

This side determines the other side automatically!
 $\Rightarrow [A]_{k\ell} = [A]_{\ell k}$

1.6 Bases and dimension

- The dimension of subspaces

Corollary 1.11.1:

If W is a subspace of a finite-dimensional vector space V , then, any basis for W can be **extended** to a **basis** for V .

- Proof)
 - Let S be a basis for W .
 - Note that S is a linearly independent subset of V
 - By **Corollary 1.10.2 (c)** implies S can be extended to a basis for V .
 - **\therefore Q.E.D.**