HANBAT NATIONAL UNIVERSITY

Linear Algebra (5^{th} edition)

Stephen Friedberg, Arnold Insel, Lawrence Spence

Chapter 02: Linear transformations and matrices

Jihwan Moon

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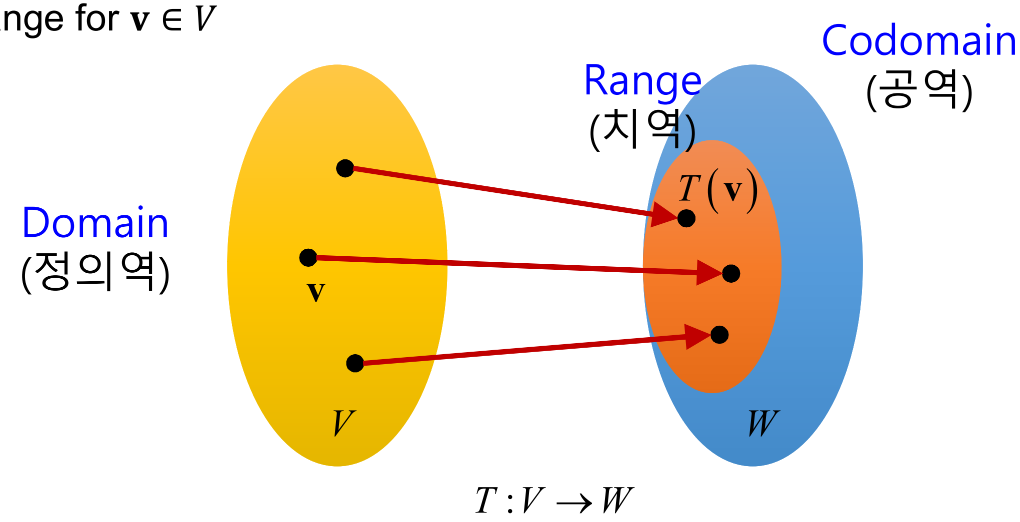
2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - Notation
    - $T: V \rightarrow W$
      - T: A function
      - V: A domain
      - W: A codomain
      - $T(\mathbf{v})$: A range for $\mathbf{v} \in V$



2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**

Linear transformation:
Let V and W be vector spaces over the same field F.
We call a function $T: V \to W$ a linear transformation from V to W (or just linear) if, for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$, we have

(a) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, and
(b) $T(c\mathbf{x}) = cT(\mathbf{x})$

- Properties
  - ① T is linear $\Rightarrow$ $T(\mathbf{0}) = \mathbf{0}$
  - ② T is linear $\Leftrightarrow$ $T(c\mathbf{x}+\mathbf{y}) = cT(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$
  - ③ T is linear $\Rightarrow$ $T(\mathbf{x}-\mathbf{y}) = T(\mathbf{x}) - T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$
  - ④ T is linear $\Leftrightarrow$ $T(\sum_{i=1}^{n} a_i \mathbf{x}_i) = \sum_{i=1}^{n} a_i T(\mathbf{x}_i)$ for $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ and $a_1, \dots, a_n \in F$

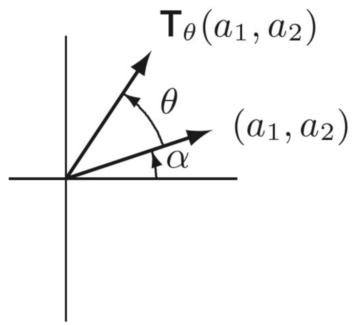
- Generally, property ② often used to prove a given transformation T is linear

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.1*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 + a_2 \\ a_1 \end{bmatrix}$
    - Q: Is function T linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} 2x_1 + x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} c(2x_1 + x_2) + 2y_1 + y_2 \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix}$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

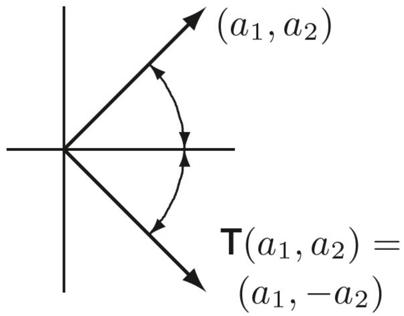
- **Linear transformations**
  - *Example 2.1.2 (Rotation)*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1\cos\theta - a_2\sin\theta \\ a_1\sin\theta + a_2\cos\theta \end{bmatrix}$



    - Q: Is function T linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix} = \begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix} + \begin{bmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{bmatrix} = \begin{bmatrix} c(x_1\cos\theta - x_2\sin\theta) + (y_1\cos\theta - y_2\sin\theta) \\ c(x_1\sin\theta + x_2\cos\theta) + (y_1\sin\theta + y_2\cos\theta) \end{bmatrix} = \begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix}$
      - $\therefore$ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

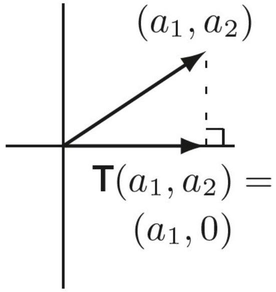
- **Linear transformations**
  - *Example 2.1.3 (Reflection)*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$



    - Q: Is function T linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + y_1 \\ -cx_2 - y_2 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ -cx_2 - y_2 \end{bmatrix}$
      - $\therefore$ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.4 (Projection on the 1st dimension)*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$



    - Q: Is function T linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + y_1 \\ 0 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ 0 \end{bmatrix}$
      - $\therefore$ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.5 (Transpose)*
    - $T: M_{m \times n}(F) \rightarrow M_{n \times m}(F)$ where $T(A) = A^T$
    - Q: Is function T linear?
      - $T(c\mathbf{X} + \mathbf{Y}) = (c\mathbf{X} + \mathbf{Y})^T = c\mathbf{X}^T + \mathbf{Y}^T$
      - $cT(\mathbf{X}) + T(\mathbf{Y}) = c\mathbf{X}^T + \mathbf{Y}^T$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.6 (Derivatives)*
    - $T: V \rightarrow V$ where $T(f) = \frac{df}{dv}$
    - Q: Is function T linear?
      - Letting $g \in V$ and $h \in V$
      - $T(cg + h) = \frac{d}{dv}(cg + h) = c\frac{dg}{dv} + \frac{dh}{dv}$
      - $cT(g) + T(h) = c\frac{dg}{dv} + \frac{dh}{dv}$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.7 (Integration)*
    - $T: \mathbb{R} \rightarrow \mathbb{R}$ where $T(f) = \int_a^b f(t)dt$ for some $a, b \in \mathbb{R}$
    - Q: Is function T linear?
      - Letting $g \in \mathbb{R}$ and $h \in \mathbb{R}$
      - $T(cg + h) = \int_a^b cg(t) + h(t)dt = c\int_a^b g(t)dt + \int_a^b h(t)dt$
      - $cT(g) + T(h) = c\int_a^b g(t)dt + \int_a^b h(t)dt$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example (Identity transformation)*
    - $T: V \to V$ where $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$
    - Q: Is function T linear?
      - Letting $\mathbf{x} \in V$ and $\mathbf{y} \in V$
      - $T(c\mathbf{x} + \mathbf{y}) = c\mathbf{x} + \mathbf{y}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\mathbf{x} + \mathbf{y}$
      - $\therefore$ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example (Zero transformation)*
    - $T: V \to W$ where $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$
    - Q: Is function T linear?
      - Letting $\mathbf{x} \in V$ and $\mathbf{y} \in V$
      - $T(c\mathbf{x} + \mathbf{y}) = \mathbf{0}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}$
      - $\therefore$ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Null space (kernel):
Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
We define null space (or kernel) $N(T)$ of T to be the set of all vectors $\mathbf{x} \in V$ such that

$$N(T) = \{\mathbf{x} \in V | T(\mathbf{x}) = \mathbf{0}\}$$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Range (image) (치역):
Let V and W be vector spaces and let function $T: V \to W$ be linear.
We define range (or image) $R(T)$ of T to be the subset of W containing all images (outputs) under T of vectors in V such that

$$R(T) = \{T(\mathbf{x}) | \mathbf{x} \in V\}$$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.8*
    - $T_1: V \to V$ where $T_1(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$ (identity transformation)
      - Null space
        - $N(T_1) = \{\mathbf{0}\}$
      - Range
        - $R(T_1) = V$
    - $T_2: V \to W$ where $T_2(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$ (zero transformation)
      - Null space
        - $N(T_2) = V$
      - Range
        - $R(T_2) = \{\mathbf{0}\}$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.9*
    - $T: \mathbb{R}^3 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 - a_2 \\ 2a_3 \end{bmatrix}$ for all $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$
      - Null space
        - $N(T) = \left\{\begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \middle| \, x \in \mathbb{R}\right\}$
      - Range
        - $R(T) = \mathbb{R}^2$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.1**:
> Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
> Then, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

- Proof) ($N(T)$ is a subspace of V)
  - Theorem 1.3(a)
    - $\mathbf{0} \in N(T)$ since property ① of linear transformation indicates that $T(\mathbf{0}) = \mathbf{0}$
  - Theorem 1.3(b)
    - $\mathbf{x} + \mathbf{y} \in N(T)$ since $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}$ for $\mathbf{x}, \mathbf{y} \in N(T)$
  - Theorem 1.3(c)
    - $c\mathbf{x} \in N(T)$ since $T(c\mathbf{x}) = cT(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in N(T)$
  - ∴ Subspace!

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.1**:
> Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
> Then, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

- Proof) ($R(T)$ is a subspace of W)
  - Theorem 1.3(a)
    - $\mathbf{0} \in R(T)$ since $\mathbf{0} \in V$ and property ① of linear transformation indicates that $T(\mathbf{0}) = \mathbf{0}$
  - Theorem 1.3(b)
    - $\mathbf{x} + \mathbf{y} \in R(T)$ since $\mathbf{x} + \mathbf{y} = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2) \in R(T)$ for $\mathbf{x} = T(\mathbf{v}_1), \mathbf{y} = T(\mathbf{v}_2) \in R(T)$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$
  - Theorem 1.3(c)
    - $c\mathbf{x} \in R(T)$ since $c\mathbf{x} = cT(\mathbf{v}_1) = T(c\mathbf{v}_1) \in R(T)$ for $\mathbf{x} = T(\mathbf{v}_1) \in R(T)$ and $\mathbf{v}_1 \in V$

  - $\therefore$ Subspace!

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Theorem 2.2:
Let V and W be vector spaces and let function $T: V \to W$ be linear.
$\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V $\quad \Rightarrow \quad R(T) = \text{span}\big(T(\beta)\big) = \text{span}\big(T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\big)$

- Proof)
  - ($\text{span}\big(T(\beta)\big) \subseteq R(T)$)
    - Note that $T(\mathbf{v}_i) \in R(T), \forall i$
    - From Theorem 2.1, $R(T)$ is a subspace
      - $\Rightarrow \text{span}(\{T(\mathbf{v}_i)\}) = \text{span}\big(T(\beta)\big) \in R(T)$ by Theorem 1.5
  - ($\text{span}\big(T(\beta)\big) \supseteq R(T)$)
    - $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ for any $\mathbf{v} \in V$
      - $\Rightarrow T(\mathbf{v}) \in R(T)$
    - $T(\mathbf{v}) = T(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i) \in \text{span}(\{T(\mathbf{v}_i)\}) = \text{span}\big(T(\beta)\big)$ for any $T(\mathbf{v}) \in R(T)$
      - $\Rightarrow R(T) \in \text{span}\big(T(\beta)\big)$
  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.10*
    - $T: V = \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_2 - v_3 & 0 \\ 0 & v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$
      - Linear
    - For a standard basis $\beta = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$
      - $R(T) = \mathrm{span}\big(T(\beta)\big) = \mathrm{span}\left(\left\{T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\right\}\right) = \mathrm{span}\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right) =$ $\mathrm{span}\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right)$
      - $\Rightarrow \dim\big(R(T)\big) = 2$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.10*
    - $T: V = \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_2 - v_3 & 0 \\ 0 & v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$
      - Linear
    - To find a basis for $N(T)$, by letting $T(\mathbf{v}) = \mathbf{0}$,
      - $N(T) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$
      - $\Rightarrow \dim\big(N(T)\big) = 1$
    - Note that $\dim(V) = \dim\big(N(T)\big) + \dim\big(R(T)\big)$
      - (Theorem 2.3 coming soon!)

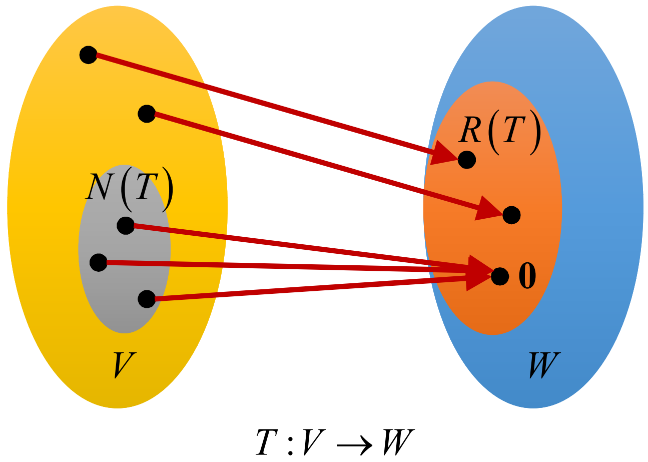
2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Nullity and rank:
Let V and W be vector spaces and let function $T: V \to W$ be linear.
If $N(T)$ and $R(T)$ are finite-dimensional,

$$\text{nullity}(T) \triangleq \dim\big(N(T)\big)$$
$$\text{rank}(T) \triangleq \dim\big(R(T)\big)$$

- Intuition
  - The larger the nullity, the smaller the rank
    - The more vectors carried into $\mathbf{0}$, the smaller the range



2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Theorem 2.3 (Dimension Theorem):
Let V and W be vector spaces and let function $T: V \to W$ be linear.
If V is finite-dimensional,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - Let $n = \dim(V)$ and $k = \dim\big(N(T)\big)$ where $n \geq k$.
  - Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for $N(T)$.
  - Note that $N(T)$ is a subspace of vector space V.
    - $\Rightarrow$ From Corollary 1.11.1, we may extend $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ to a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ for V.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> If V is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - From Theorem 2.2,
    - $R(T) = \text{span}\big(T(\beta)\big) = \text{span}\big(T(\mathbf{v}_1), \dots, T(\mathbf{v}_k), T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\big)$
  - Noting that $T(\mathbf{v}_1) = \cdots = T(\mathbf{v}_k) = \mathbf{0}$
    - $R(T) = \text{span}(\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\})$
    - $\Rightarrow \{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ spans $R(T)$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> If V is finite-dimensional,
> $$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V)$$

- Proof)
  - If $\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ is a linearly independent set,
    - $\sum_{i=k+1}^{n} b_i T(\mathbf{v}_i) = \mathbf{0}$ only when $b_i = 0, i = k+1, \dots, n$
  - From the linear property of T,
    - $\sum_{i=k+1}^{n} b_i T(\mathbf{v}_i) = T(\sum_{i=k+1}^{n} b_i \mathbf{v}_i)$
  - If $T(\sum_{i=k+1}^{n} b_i \mathbf{v}_i) = \mathbf{0}$, then
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i \in N(T)$
  - Hence, $\sum_{i=k+1}^{n} b_i \mathbf{v}_i$ may be expressed as a linear combination of the basis of $N(T)$
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i = \sum_{i=1}^{k} c_i \mathbf{v}_i$ for some c_i's
  - Since $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ are a basis for V,
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i - \sum_{i=1}^{k} c_i \mathbf{v}_i = 0$ only when $b_i = 0, i = k+1, \dots, n$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> If V is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - Hence $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ is a basis for $R(T)$
    - $\Rightarrow \dim(R(T)) = \text{rank}(T) = n - k$
  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.4**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> Then, T is one-to-one $\Leftrightarrow$ $N(T) = \{\mathbf{0}\}$

- Proof)
  - (T is one-to-one $\Rightarrow$ $N(T) = \{\mathbf{0}\}$)
    - From the one-to-one property, there exists only one $\mathbf{x}$ to satisfy $T(\mathbf{x}) = \mathbf{0}$.
    - In the meantime, by the linear property, $T(\mathbf{0}) = \mathbf{0}$.
      - $\Rightarrow \mathbf{x} = \mathbf{0}$
      - $\Rightarrow N(T) = \{\mathbf{0}\}$
  - (T is one-to-one $\Leftarrow$ $N(T) = \{\mathbf{0}\}$)
    - By contradiction, assume T is not one-to-one.
      - $\Rightarrow T(\mathbf{x}) = T(\mathbf{y})$ for some distinct $\mathbf{x}, \mathbf{y} \in V$
    - By the linear property, $\mathbf{0} = T(\mathbf{x}) - T(\mathbf{y}) = T(\mathbf{x} - \mathbf{y})$
      - $\Rightarrow \mathbf{x} - \mathbf{y} \in N(T)$ where $\mathbf{x} - \mathbf{y} \neq \mathbf{0}$
      - $\therefore$ Contradiction
  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.5**:
> Let V and W be finite-dimensional vector spaces of equal dimension, and let function $T: V \rightarrow W$ be linear.
> Then, T is one-to-one $\Leftrightarrow$ T is onto $\Leftrightarrow$ $\mathrm{rank}(T) = \dim(V)$

- "One-to-one" (*Appendix B*)
  - $T(\mathbf{v}_1) = T(\mathbf{v}_2) \Rightarrow \mathbf{v}_1 = \mathbf{v}_2$, for some $\mathbf{v}_1, \mathbf{v}_2 \in V$
  - Equivalently, $\mathbf{v}_1 \neq \mathbf{v}_2 \Rightarrow T(\mathbf{v}_1) \neq T(\mathbf{v}_2)$, for some $\mathbf{v}_1, \mathbf{v}_2 \in V$

- "Onto" (*Appendix B*)
  - $T(\mathbf{v})$ = W, for all $\mathbf{v} \in V$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Theorem 2.5:
Let V and W be finite-dimensional vector spaces of equal dimension, and let function $T: V \rightarrow W$ be linear.
Then, T is one-to-one $\Leftrightarrow$ T is onto $\Leftrightarrow$ $\operatorname{rank}(T) = \dim(V)$

- Proof)
  - From Theorem 2.4, T is one-to-one $\Leftrightarrow$ $N(T) = \{\mathbf{0}\}$ or $\operatorname{nullity}(T) = 0$
  - Also, from Theorem 2.3 (Dimension theorem), $\operatorname{nullity}(T) + \operatorname{rank}(T) = \dim(V)$
    - $\Rightarrow$ T is one-to-one $\Leftrightarrow$ $\operatorname{rank}(T) = \dim(V)$
  - From the equal dimension condition,
    - $\Rightarrow \operatorname{rank}(T) = \dim(V) \Leftrightarrow \operatorname{rank}(T) = \dim(W)$
  - From Theorem 1.11,
    - $\Rightarrow \operatorname{rank}(T) = \dim(W) \Leftrightarrow R(T) = W$

  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.12*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_1 + v_2 \\ v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
      - Linear
    - To find a vector for $N(T)$, by letting $T(\mathbf{v}) = \mathbf{0}$,
      - $N(T) = \{\mathbf{0}\}$
    - By Theorem 2.4
      - One-to-one
    - By Theorem 2.5
      - Onto

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.6**:
> Let V and W be vector spaces of equal dimension, and suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V.
> For $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$, there exists exactly one linear transformation $T: V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$.

- Proof)
  - From the linear property, for $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ with unique scalars $a_1, \dots, a_n$
    - $T(\mathbf{v}) = T(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i) = \sum_{i=1}^{n} a_i \mathbf{w}_i$
  - If there exists another linear function $U: V \to W$ such that $U(\mathbf{v}_i) = \mathbf{w}_i$
    - $U(\mathbf{v}) = U(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i U(\mathbf{v}_i) = \sum_{i=1}^{n} a_i \mathbf{w}_i$
  - Then, we must have $T(\mathbf{v}_i) = U(\mathbf{v}_i), \forall i$.
  - Hence, $U = T$
  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - An implication of Theorem 2.6
    - A linear transformation completely determined by its action on a basis

2.2 The matrix representation of a linear transformation

2.2 The matrix representation of a linear transformation

- **Section 2.1**
  - Studying linear transformations by examining their null spaces and ranges

- **Section 2.2**
  - Representing linear transformations by a matrix
    - Developing a one-to-one correspondence between matrices and linear transformations

2.2 The matrix representation of a linear transformation

- **Ordered basis**

Ordered basis:
Let V be a finite-dimensional vector space.
An ordered basis for V is a basis for V endowed with a specific order.

That is, an ordered basis is a finite sequence of linearly independent vectors in V that spans V.

- *Example 2.2.1*
  - $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
    - A standard ordered basis in $\mathbb{F}^3$ where $\mathbf{e}_i, \forall i$ is a standard basis
  - $\gamma = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$
    - Another ordered basis
  - From the perspective of orders, $\beta \neq \gamma$

2.2 The matrix representation of a linear transformation

- **Ordered basis**

Coordinate vector:
Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis for a finite-dimensional vector space V.
For $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i \in V$, we define the coordinate vector of $\mathbf{v}$ relative to β by

$$[\mathbf{v}]_\beta \triangleq \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

With unique scalars $a_1, \dots, a_n$

- *Example 2.2.2*
  - $V = \mathbb{R}^3$ with $\beta = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$
  - The coordinate vector of $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1\mathbf{e}_1 + 3\mathbf{e}_2 + 6\mathbf{e}_3$
    - $\Rightarrow [\mathbf{v}]_\beta = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - Letting
    - V be a vector space with an ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$
    - W be a vector space with an ordered basis $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$
    - $T: V \rightarrow W$ be a linear function

  - Then, using the ordered basis γ
    - $T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i$ with unique scalars $a_{ij} \in F$ for $i = 1, \dots, m$ for each $j = 1, \dots, n$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **Matrix representation**:
> We call the $m \times n$ matrix $\mathbf{A}$ defined by $[\mathbf{A}]_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ and write $\mathbf{A} = [T]_\beta^\gamma$.
> If $V = W$ and $\beta = \gamma$, then we write $\mathbf{A} = [T]_\beta$.

- For instance, with $n = 2$ and $m = 3$ such that $T: \mathbb{R}^2 \to \mathbb{R}^3$,
  - $$[T(\mathbf{v}_1) \quad T(\mathbf{v}_2)] = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3] \underbrace{\begin{bmatrix} \underbrace{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}}_{[T(\mathbf{v}_1)]_\gamma} & \underbrace{\begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix}}_{[T(\mathbf{v}_2)]_\gamma} \end{bmatrix}}_{} \quad \mathbf{A} = [T]_\beta^\gamma$$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - *Example 2.2.3*
    - If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix}$, $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$, $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \underset{a_{11}}{1} \cdot \mathbf{w}_1 + \underset{a_{21}}{0} \cdot \mathbf{w}_2 + \underset{a_{31}}{2} \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = \underset{a_{12}}{3} \cdot \mathbf{w}_1 + \underset{a_{22}}{0} \cdot \mathbf{w}_2 + (\underset{a_{32}}{-4}) \cdot \mathbf{w}_3$
      - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$
    - For $\gamma_2 = \{\mathbf{w}_1 = \mathbf{e}_3, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_1\}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \underset{a_{11}}{2} \cdot \mathbf{w}_1 + \underset{a_{21}}{0} \cdot \mathbf{w}_2 + \underset{a_{31}}{1} \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = (\underset{a_{12}}{-4}) \cdot \mathbf{w}_1 + \underset{a_{22}}{0} \cdot \mathbf{w}_2 + \underset{a_{32}}{3} \cdot \mathbf{w}_3$
      - $\Rightarrow [T]_\beta^{\gamma_2} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - Letting
    - V be a vector space with an ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$
    - W be a vector space with an ordered basis $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$
    - $T: V \to W$ be a linear function
  - (Zero transformation) Then, using the ordered basis γ
    - $T(\mathbf{v}_j) = \mathbf{0} = 0 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + \cdots + 0 \cdot \mathbf{w}_m$ for $j = 1, \dots, n$
    - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = [\mathbf{0}]$
  - (Identity transformation) For $n = m$ and $\beta = \gamma$,
    - $T(\mathbf{v}_j) = \mathbf{v}_j = 0 \cdot \mathbf{v}_1 + \cdots + 1 \cdot \mathbf{v}_j + \cdots + 0 \cdot \mathbf{v}_n$ for $j = 1, \dots, n$
    - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = \mathbf{I}_\mathrm{n}$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

Kronecker delta:

$$\delta_{ij} \triangleq \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- For instance,
  - $[\mathbf{I}_n]_{ij} = \delta_{ij}, \forall i, j$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

Addition and scalar multiplication of functions:
Let V and W be vector spaces over F.
Let $T, U: V \to W$ be arbitrary functions.
Then, for all $\mathbf{x} \in V$ and $a \in F$,

$$T + U: V \to W \quad \triangleq \quad (T + U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x})$$
$$aT: V \to W \quad \triangleq \quad (aT)(\mathbf{x}) = aT(\mathbf{x})$$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **Theorem 2.7**:
> Let V and W be vector spaces over F.
> Let $T, U: V \rightarrow W$ be linear functions.
> Then, for all $\mathbf{x} \in V$ and $a \in F$,
>
> (a) $aT + U$ is linear, i.e., $(aT + U)(c\mathbf{x} + \mathbf{y}) = c(aT + U)(\mathbf{x}) + (aT + U)(\mathbf{y})$
> (b) The collection of all linear transformations from V to W is a vector space over F.

- Proof)
  - (a)
    - Let $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$.
    - $(aT + U)(c\mathbf{x} + \mathbf{y}) = (aT)(c\mathbf{x} + \mathbf{y}) + U(c\mathbf{x} + \mathbf{y}) = c(aT)(\mathbf{x}) + (aT)(\mathbf{y}) + cU(\mathbf{x}) + U(\mathbf{y}) = c\big((aT)(\mathbf{x}) + U(\mathbf{x})\big) + (aT)(\mathbf{y}) + U(\mathbf{y}) = c(aT + U)(\mathbf{x}) + (aT + U)(\mathbf{y})$
  - (b)
    - (Left as an exercise)

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **The vector space of all linear transformations**:
> Let V and W be vector spaces over F.
> We denote the vector space of all linear transformations from V into W by $\mathcal{L}(V, W)$.
> If $V = W$, we write $\mathcal{L}(V)$.

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

Theorem 2.8:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively.
Let $T, U: V \rightarrow W$ be linear transformations.
Then,

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars c

- Proof)
  - (a)
    - Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$.
    - $T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i$ for unique scalars $a_{ij}, \forall i, j$
    - $U(\mathbf{v}_j) = \sum_{i=1}^{m} b_{ij}\mathbf{w}_i$ for unique scalars $b_{ij}, \forall i, j$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **Theorem 2.8**:
> Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively.
> Let $T, U: V \to W$ be linear transformations.
> Then,
>
> (a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
> (b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars c

- Proof)
  - (a)
    - Then, $(T+U)(\mathbf{v}_j) = T(\mathbf{v}_j) + U(\mathbf{v}_j) = \sum_{i=1}^{m}(a_{ij}+b_{ij})\mathbf{w}_i$
    - Thus, $\left[[T+U]_\beta^\gamma\right]_{ij} = a_{ij} + b_{ij} = \left[[T]_\beta^\gamma\right]_{ij} + \left[[U]_\beta^\gamma\right]_{ij}$
    - $\therefore$ Q.E.D.

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

Theorem 2.8:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively.
Let $T, U: V \to W$ be linear transformations.
Then,

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars c

- Proof)
  - (b)
    - $(cT)(\mathbf{v}_j) = cT(\mathbf{v}_j) = c\sum_{i=1}^m a_{ij}\mathbf{w}_i$
    - Thus, $\left[[cT]_\beta^\gamma\right]_{ij} = ca_{ij} = c\left[[T]_\beta^\gamma\right]_{ij}$
    - $\therefore$ Q.E.D.

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - *Example 2.2.5*
    - Let $T, U: \mathbb{R}^2 \to \mathbb{R}^3$ be linear and $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ for V and $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$ for W
    - If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = 3 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-4) \cdot \mathbf{w}_3$
      - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$
    - If $U\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 - v_2 \\ 2v_1 \\ 3v_1 + 2v_2 \end{bmatrix}$
      - $U(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2 + 3 \cdot \mathbf{w}_3$ and $U(\mathbf{v}_2) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = (-1) \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$
      - $\Rightarrow [U]_\beta^\gamma = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - *Example 2.2.5*
    - Since $(T+U)\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) + U\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ 2v_1 \\ 3v_1 + 2v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 2v_2 \\ 2v_1 \\ 5v_1 - 2v_2 \end{bmatrix}$
      - $(T+U)(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} = 2 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2 + 5 \cdot \mathbf{w}_3$ and $(T+U)(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-2) \cdot \mathbf{w}_3$
      - $\Rightarrow [T+U]_\beta^\gamma = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$
    - Note that
      - $[T]_\beta^\gamma + [U]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix} = [T+U]_\beta^\gamma$

2.3 Composition of linear transformations and matrix multiplication

2.3 Composition of linear transformations and matrix multiplication

- **Section 2.1**
  - Studying linear transformations by examining their null spaces and ranges
- **Section 2.2**
  - Representing linear transformations by a matrix
    - Developing a one-to-one correspondence between matrices and linear transformations
- **Section 2.3**
  - How the matrix representation of a composite of linear transformations is related to the matrix representation of each of the associated linear transformations

2.3 Composition of linear transformations and matrix multiplication

- **Composition**

Theorem 2.9:
Let V, W and Z be vector spaces over the same field F.
Let $T: V \to W$ and $U: W \to Z$ be linear.
Then, $UT: V \to Z$ is linear.

- Proof)
  - Let $\mathbf{x}, \mathbf{y} \in V$ and $a \in F$
  - $(UT)(a\mathbf{x} + \mathbf{y}) = U\big(T(a\mathbf{x} + \mathbf{y})\big) = U\big(aT(\mathbf{x}) + T(\mathbf{y})\big) = aU\big(T(\mathbf{x})\big) + U\big(T(\mathbf{y})\big) = a(UT)(\mathbf{x}) + (UT)(\mathbf{y})$
  - $\therefore$ Q.E.D.

2.3 Composition of linear transformations and matrix multiplication

- **Composition**

Theorem 2.10:
Let V be a vector space.
Let $T, U_1, U_2 \in \mathcal{L}(V)$ (linear)
Then,

(a) $T(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$
(b) $T(U_1U_2) = (TU_1)U_2$
(c) $TI = IT = T$
(d) $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$ for all scalars a

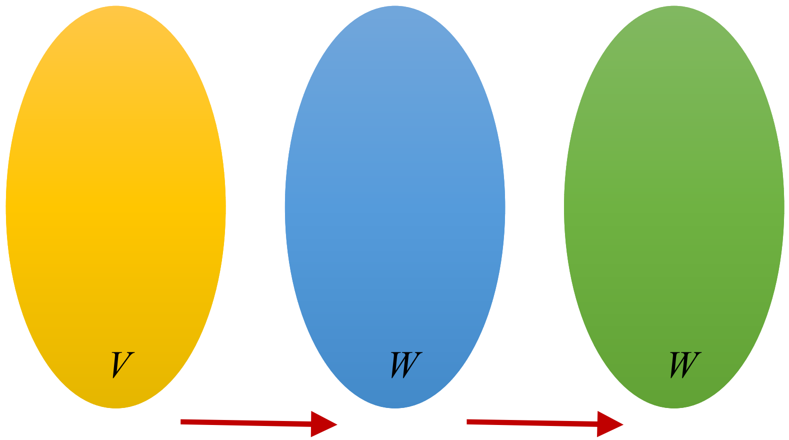
- Proof)
  - (Exercise)

2.3 Composition of linear transformations and matrix multiplication

- **Composition**
  - If $T \in \mathcal{L}(V)$
    - $T^0 \triangleq I$
    - $T^k \triangleq T^{k-1}T$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Let
    - V, W and Z: Finite-dimensional vector spaces
    - $T: V \to W$ linear
    - $U: W \to Z$ linear
    - $\alpha = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ ordered basis for V
    - $\beta = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ ordered basis for W
    - $\gamma = \{\mathbf{z}_1, \dots, \mathbf{z}_p\}$ ordered basis for Z
    - $\mathbf{A} = [U]_\beta^\gamma$
    - $\mathbf{B} = [T]_\alpha^\beta$
  - Then,
    - $(UT)(\mathbf{v}_j) = U\left(T(\mathbf{v}_j)\right) = U\left(\sum_{k=1}^m [\mathbf{B}]_{kj}\mathbf{w}_k\right) = \sum_{k=1}^m [\mathbf{B}]_{kj} U(\mathbf{w}_k) = \sum_{k=1}^m [\mathbf{B}]_{kj}\left(\sum_{i=1}^p [\mathbf{A}]_{ik}\mathbf{z}_i\right) = \sum_{i=1}^p \left(\sum_{k=1}^m [\mathbf{A}]_{ik}[\mathbf{B}]_{kj}\right)\mathbf{z}_i = \sum_{i=1}^p [\mathbf{C}]_{ij}\mathbf{z}_i$



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2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Matrix product:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Let $\mathbf{B}$ be an $n \times p$ matrix.
We define the product of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{AB}$, to be the $m \times p$ matrix such that

$$[\mathbf{AB}]_{ij} = \sum_{k=1}^{n} [\mathbf{A}]_{ik} [\mathbf{B}]_{kj}$$

for $i = 1, \dots, m$ and $j = 1, \dots, p$.

That is, $[\mathbf{AB}]_{ij}$ is the sum of products of corresponding entries from the i-th row of $\mathbf{A}$ and the j-th column of $\mathbf{B}$.

- Caution on the dimension

$$\underbrace{\mathbf{A}}_{(m\times n)} \underbrace{\mathbf{B}}_{(n\times p)} = \underbrace{\mathbf{AB}}_{(m\times p)}$$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - *Example 2.3.1*
    - $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 2 + 1 \cdot 5 \\ 0 \cdot 4 + 4 \cdot 2 + (-1) \cdot 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}$
    - Dimension being $(2 \times 3)(3 \times 1) \rightarrow (2 \times 1)$
  - Not commutative
    - $\mathbf{AB} \neq \mathbf{BA}$
    - e.g.,
      - $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
      - $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Transpose of matrix multiplication
    - For some $\mathbf{A} \in M_{m\times n}(F)$ and $\mathbf{B} \in M_{n\times p}(F)$,
      - $[\mathbf{AB}]_{ij}^T = [\mathbf{AB}]_{ji} = \sum_{k=1}^{n}[\mathbf{A}]_{jk}[\mathbf{B}]_{ki}$
      - $[\mathbf{B}^T\mathbf{A}^T]_{ij} = \sum_{k=1}^{n}[\mathbf{B}^T]_{ik}[\mathbf{A}^T]_{kj} = \sum_{k=1}^{n}[\mathbf{B}]_{ki}[\mathbf{A}]_{jk}$

    - $\therefore (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Theorem 2.11:
Let V, W and Z be finite-dimensional vector spaces with ordered bases α, β and γ, respectively. For linear $T: V \rightarrow W$ and $U: W \rightarrow Z$,

$$[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$$

- *Example 2.3.2*
  - Let $T: V = \mathbb{R}^2 \rightarrow W = \mathbb{R}^3$ and $U: W = \mathbb{R}^3 \rightarrow V = \mathbb{R}^2$
  - Ordered bases
    - $\alpha = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ for V
    - $\beta = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$ for W

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Theorem 2.11:
Let V, W and Z be finite-dimensional vector spaces with ordered bases α, β and γ, respectively. For linear $T: V \to W$ and $U: W \to Z$,

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

- *Example 2.3.2*
  - For $T(\mathbf{v}) = \begin{bmatrix} \frac{1}{2} v_1 \\ v_2 \\ 0 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$
    - $T(\mathbf{v}_1) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3$
    - $\Rightarrow [T]_\alpha^\beta = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

> **Theorem 2.11**:
> Let V, W and Z be finite-dimensional vector spaces with ordered bases α, β and γ, respectively. For linear $T: V \to W$ and $U: W \to Z$,
> $$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

- *Example 2.3.2*
  - For $U(\mathbf{w}) = \begin{bmatrix} 2w_1 \\ w_2 \end{bmatrix}$ for $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3$
    - $U(\mathbf{w}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$, $U(\mathbf{w}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$ and $U(\mathbf{w}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$
    - $\Rightarrow [U]_\beta^\alpha = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

> **Theorem 2.11**:
> Let V, W and Z be finite-dimensional vector spaces with ordered bases α, β and γ, respectively. For linear $T: V \to W$ and $U: W \to Z$,
> $$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

- *Example 2.3.2*
  - For $(UT)(\mathbf{v}) = U\left(\begin{bmatrix} \frac{1}{2}v_1 \\ v_2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2\left(\frac{1}{2}v_1\right) \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^3$
    - $(UT)(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$ and $(UT)(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$
    - $\Rightarrow [UT]_\alpha^\alpha = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - Note that $[U]_\beta^\alpha [T]_\alpha^\beta = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [UT]_\alpha^\alpha$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Corollary 2.11.1:
Let V be a finite-dimensional vector space with an ordered basis β.
Let $T, U \in \mathcal{L}(V)$
Then,

$$[UT]_\beta = [U]_\beta [T]_\beta$$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Theorem 2.12:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Let $\mathbf{B}$ and $\mathbf{C}$ be $n \times p$ matrices.
Let $\mathbf{D}$ and $\mathbf{E}$ be $q \times m$ matrices.
Then,

(a) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$ and $(\mathbf{D}+\mathbf{E})\mathbf{A} = \mathbf{DA}+\mathbf{EA}$
(b) $a(\mathbf{AB}) = (a\mathbf{A})\mathbf{B} = \mathbf{A}(a\mathbf{B})$ for any scalar a
(c) $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$

- Proof)
  - (Exercise)

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Corollary 2.12.1:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Let $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_k$ be $n \times p$ matrices.
Let $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_k$ be $q \times m$ matrices.
Let $a_1, a_2, \dots, a_k$ be scalars.
Then,

(a) $\mathbf{A}\left(\sum_{i=1}^{k} a_i \mathbf{B}_i\right) = \sum_{i=1}^{k} a_i \mathbf{A}\mathbf{B}_i$
(b) $\left(\sum_{i=1}^{k} a_i \mathbf{C}_i\right)\mathbf{A} = \sum_{i=1}^{k} a_i \mathbf{C}_i \mathbf{A}$

- Proof)
  - (Exercise)

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - For an $n \times n$ matrix A,
    - $\mathbf{A}^0 \triangleq \mathbf{I}_n$
    - $\mathbf{A}^k \triangleq \mathbf{A}^{k-1}\mathbf{A}$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Theorem 2.13:
Let $\mathbf{A}$ be an $m \times n$ matrix.
Let $\mathbf{B}$ be $n \times p$ matrices.
Let $\mathbf{u}_j$ be the j-th column of $\mathbf{AB}$.
Let $\mathbf{v}_j$ be the j-th column of $\mathbf{B}$.
Then,

(a) $\mathbf{u}_j = \mathbf{A}\mathbf{v}_j$
(b) $\mathbf{v}_j = \mathbf{B}\mathbf{e}_j$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Proof) (a) $\mathbf{u}_j = \mathbf{A}\mathbf{v}_j$

$$\mathbf{u}_j = \begin{bmatrix} [\mathbf{AB}]_{1j} \\ \vdots \\ [\mathbf{AB}]_{ij} \\ \vdots \\ [\mathbf{AB}]_{nj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{m} [\mathbf{A}]_{1k} [\mathbf{B}]_{kj} \\ \vdots \\ \sum_{k=1}^{m} [\mathbf{A}]_{ik} [\mathbf{B}]_{kj} \\ \vdots \\ \sum_{k=1}^{m} [\mathbf{A}]_{nk} [\mathbf{B}]_{kj} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\mathrm{row},1} \mathbf{v}_j \\ \vdots \\ \mathbf{A}_{\mathrm{row},i} \mathbf{v}_j \\ \vdots \\ \mathbf{A}_{\mathrm{row},n} \mathbf{v}_j \end{bmatrix} = \mathbf{A}\mathbf{v}_j$$

$$\mathbf{A} = \begin{bmatrix} [\mathbf{A}]_{11} & [\mathbf{A}]_{12} & \cdots & [\mathbf{A}]_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ [\mathbf{A}]_{i1} & [\mathbf{A}]_{i2} & \cdots & [\mathbf{A}]_{im} \\ \vdots & \vdots & \ddots & \vdots \\ [\mathbf{A}]_{n1} & [\mathbf{A}]_{n2} & \cdots & [\mathbf{A}]_{nm} \end{bmatrix} \quad \mathbf{A}_{\mathrm{row},i} \qquad \mathbf{B} = \begin{bmatrix} [\mathbf{B}]_{11} & \cdots & [\mathbf{B}]_{1j} & \cdots & [\mathbf{B}]_{1p} \\ [\mathbf{B}]_{21} & \cdots & [\mathbf{B}]_{2j} & \cdots & [\mathbf{B}]_{2p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{B}]_{m1} & \cdots & [\mathbf{B}]_{mj} & \cdots & [\mathbf{B}]_{mp} \end{bmatrix} \quad \mathbf{v}_j$$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Proof) (b) $\mathbf{v}_j = \mathbf{B}\mathbf{e}_j$

$$\mathbf{B}\mathbf{e}_j = \begin{bmatrix} [\mathbf{B}]_{11} & \cdots & [\mathbf{B}]_{1j} & \cdots & [\mathbf{B}]_{1p} \\ [\mathbf{B}]_{21} & \cdots & [\mathbf{B}]_{2j} & \cdots & [\mathbf{B}]_{2p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{B}]_{m1} & \cdots & [\mathbf{B}]_{mj} & \cdots & [\mathbf{B}]_{mp} \end{bmatrix} \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ j\text{-th} \\ \\ \\ \end{array} = \begin{bmatrix} [\mathbf{B}]_{1j} \\ [\mathbf{B}]_{2j} \\ \vdots \\ [\mathbf{B}]_{mj} \end{bmatrix} = \mathbf{v}_j$$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Theorem 2.13
    - Column j of $\mathbf{AB}$ = A linear combination of the columns of $\mathbf{A}$ with column j of $\mathbf{B}$

$$\mathbf{AB} = \begin{bmatrix} [\mathbf{A}]_{11} & [\mathbf{A}]_{12} & \cdots & [\mathbf{A}]_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ [\mathbf{A}]_{i1} & [\mathbf{A}]_{i2} & \cdots & [\mathbf{A}]_{im} \\ \vdots & \vdots & \ddots & \vdots \\ [\mathbf{A}]_{n1} & [\mathbf{A}]_{n2} & \cdots & [\mathbf{A}]_{nm} \end{bmatrix} \begin{bmatrix} [\mathbf{B}]_{11} & \cdots & [\mathbf{B}]_{1j} & \cdots & [\mathbf{B}]_{1p} \\ [\mathbf{B}]_{21} & \cdots & [\mathbf{B}]_{2j} & \cdots & [\mathbf{B}]_{2p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{B}]_{m1} & \cdots & [\mathbf{B}]_{mj} & \cdots & [\mathbf{B}]_{mp} \end{bmatrix} = \begin{bmatrix} [\mathbf{AB}]_{11} & \cdots & [\mathbf{AB}]_{1j} & \cdots & [\mathbf{AB}]_{1p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{AB}]_{i1} & \cdots & [\mathbf{AB}]_{ij} & \cdots & [\mathbf{AB}]_{ip} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ [\mathbf{AB}]_{n1} & \cdots & [\mathbf{AB}]_{nj} & \cdots & [\mathbf{AB}]_{np} \end{bmatrix}$$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**
  - Analogous
    - Row i of $\mathbf{AB}$ = A linear combination of the row i of $\mathbf{A}$ with columns of $\mathbf{B}$

$$\mathbf{AB}=\begin{bmatrix}[\mathbf{A}]_{11} & [\mathbf{A}]_{12} & \cdots & [\mathbf{A}]_{1m}\\ \vdots & \vdots & \ddots & \vdots\\ \boxed{[\mathbf{A}]_{i1} \quad [\mathbf{A}]_{i2} \quad \cdots \quad [\mathbf{A}]_{im}} & & & \\ \vdots & \vdots & \ddots & \vdots\\ [\mathbf{A}]_{n1} & [\mathbf{A}]_{n2} & \cdots & [\mathbf{A}]_{nm}\end{bmatrix}\begin{bmatrix}[\mathbf{B}]_{11} & \cdots & [\mathbf{B}]_{1j} & \cdots & [\mathbf{B}]_{1p}\\ [\mathbf{B}]_{21} & \cdots & [\mathbf{B}]_{2j} & \cdots & [\mathbf{B}]_{2p}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ [\mathbf{B}]_{m1} & \cdots & [\mathbf{B}]_{mj} & \cdots & [\mathbf{B}]_{mp}\end{bmatrix}=\begin{bmatrix}[\mathbf{AB}]_{11} & \cdots & [\mathbf{AB}]_{1j} & \cdots & [\mathbf{AB}]_{1p}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \boxed{[\mathbf{AB}]_{i1}} & \cdots & \boxed{[\mathbf{AB}]_{ij}} & \cdots & \boxed{[\mathbf{AB}]_{ip}}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ [\mathbf{AB}]_{n1} & \cdots & [\mathbf{AB}]_{nj} & \cdots & [\mathbf{AB}]_{np}\end{bmatrix}$$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Theorem 2.14:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. For linear $T: V \to W$,

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- Proof)
  - Let $f: F \to V$ by $f(a) = a\mathbf{u}$ for $a \in F$.
  - An ordered basis
    - $\alpha = \{f_1 = 1\}$ for F
  - $[T(\mathbf{u})]_\gamma = [Tf]_\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [\mathbf{u}]_\beta$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Theorem 2.14:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. For linear $T: V \to W$,

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- *Example 2.3.3*
  - Let $T: V = \mathbb{R}^2 \to W = \mathbb{R}^3$ and $U: W = \mathbb{R}^3 \to V = \mathbb{R}^2$
  - Ordered bases
    - $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ for V
    - $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$ for W

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Theorem 2.14:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. For linear $T: V \to W$,

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- *Example 2.3.3*
  - For $T(\mathbf{v}) = \begin{bmatrix} \frac{1}{2} v_1 \\ v_2 \\ 0 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^3$
    - $T(\mathbf{v}_1) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2 + 0 \cdot \mathbf{w}_3$
    - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

Theorem 2.14:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. For linear $T: V \to W$,

$$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- *Example 2.3.3*
  - Note that $[T]_\beta^\gamma [\mathbf{v}]_\beta = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} v_1 \\ v_2 \\ 0 \end{bmatrix} = [T(\mathbf{v})]_\gamma$

2.3 Composition of linear transformations and matrix multiplication

- **Multiplication of matrices**

> **Theorem 2.14**:
> Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. For linear $T: V \rightarrow W$,
> $$[T(\mathbf{u})]_\gamma = [T]_\beta^\gamma [\mathbf{u}]_\beta, \forall \mathbf{u} \in V$$

- *Example 2.3.3*
  - For $U(\mathbf{w}) = \begin{bmatrix} 2w_1 \\ w_2 \end{bmatrix}$ for $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3$
    - $U(\mathbf{w}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$, $U(\mathbf{w}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$ and $U(\mathbf{w}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$
    - $\Rightarrow [U]_\gamma^\beta = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
  - Note that $[U]_\gamma^\beta [\mathbf{w}]_\gamma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2w_1 \\ w_2 \end{bmatrix} = [U(\mathbf{w})]_\beta$

2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**

Left multiplication:
Let $\mathbf{A}$ be an $m \times n$ matrix with entries from a field F.
We denote by $L_{\mathbf{A}}$ the mapping $L_{\mathbf{A}}: F^n \to F^m$ defined by $L_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ (the matrix product of $\mathbf{A}$ and $\mathbf{x}$) for each column vector $\mathbf{x} \in F^n$.
We call $L_{\mathbf{A}}$ a left-multiplication transformation.

- *Example 2.3.4*
  - Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$
  - Then, $L_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \in F^2$ for some $\mathbf{x} \in F^3$.
  - For example, with $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$,
    - $L_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**

Theorem 2.15:
Let $\mathbf{A}$ be an $m \times n$ matrix with entries from F.
Then, the left-multiplication transformation $L_{\mathbf{A}}: F^n \to F^m$ is linear.
Furthermore, if $\mathbf{B}$ is any other $m \times n$ matrix and β and γ are the standard ordered bases for F^n and F^m, respectively,

(a) $[L_{\mathbf{A}}]_\beta^\gamma = \mathbf{A}$
(b) $L_{\mathbf{A}} = L_{\mathbf{B}} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{B}$
(c) $L_{\mathbf{A}+\mathbf{B}} = L_{\mathbf{A}} + L_{\mathbf{B}}$ and $L_{a\mathbf{A}} = aL_{\mathbf{A}}$ for all $a \in F$
(d) If $T: F^n \to F^m$ is linear, then there exists a unique $m \times n$ matrix $\mathbf{C}$ such that $T = L_{\mathbf{C}}$ or $\mathbf{C} = [T]_\beta^\gamma$.
(e) If $\mathbf{E}$ is an $n \times p$ matrix, then $L_{\mathbf{AE}} = L_{\mathbf{A}} L_{\mathbf{E}}$
(f) If $m = n$, then $L_{\mathbf{I}_n} = \mathbf{I}_{F^n}$

2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**
  - Proof) (a) $[L_{\mathbf{A}}]_\beta^\gamma = \mathbf{A}$
    - The j-th column of $[L_{\mathbf{A}}]_\beta^\gamma$
      - $L_{\mathbf{A}}(\mathbf{e}_j) = \mathbf{A}\mathbf{e}_j$
      - $\Rightarrow$ The j-th column of $\mathbf{A}$
    - $\therefore$ Q.E.D.
  - Proof) (b) $L_{\mathbf{A}} = L_{\mathbf{B}} \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{B}$
    - $(\Rightarrow)$
      - By (a), $\mathbf{A} = [L_{\mathbf{A}}]_\beta^\gamma = [L_{\mathbf{B}}]_\beta^\gamma = \mathbf{B}$
    - $(\Leftarrow)$
      - Trivial
    - $\therefore$ Q.E.D.

2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**
  - Proof) (c) $L_{\mathbf{A}+\mathbf{B}} = L_{\mathbf{A}} + L_{\mathbf{B}}$ and $L_{a\mathbf{A}} = aL_{\mathbf{A}}$ for all $a \in F$
    - (Exercise)
  - Proof) (d) If $T: F^n \to F^m$ is linear, then there exists a unique $m \times n$ matrix $\mathbf{C}$ such that $T = L_{\mathbf{C}}$ or $\mathbf{C} = [T]_\beta^\gamma$.
    - By Theorem 2.14 and (a)
      - $[T(\mathbf{x})]_\gamma = [T]_\beta^\gamma[\mathbf{x}]_\beta = \mathbf{C}[\mathbf{x}]_\beta = [L_{\mathbf{C}}]_\beta^\gamma[\mathbf{x}]_\beta$
    - By (b), $\mathbf{C}$ is unique.
    - $\therefore$ Q.E.D.

2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**
  - Proof) (e) If $\mathbf{E}$ is an $n \times p$ matrix, then $L_{\mathbf{AE}} = L_{\mathbf{A}} L_{\mathbf{E}}$.
    - By Theorem 2.13,
      - $(\mathbf{AE})\mathbf{e}_j = \mathbf{A}(\mathbf{E}\mathbf{e}_j)$
    - Then, $L_{\mathbf{AE}}(\mathbf{e}_j) = (\mathbf{AE})\mathbf{e}_j = \mathbf{A}(\mathbf{E}\mathbf{e}_j) = L_{\mathbf{A}}(\mathbf{E}\mathbf{e}_j) = L_{\mathbf{A}}\left(L_{\mathbf{E}}(\mathbf{e}_j)\right) = (L_{\mathbf{A}} L_{\mathbf{B}})(\mathbf{e}_j)$
    - $\therefore$ Q.E.D.
  - Proof) (f) If $m = n$, then $L_{\mathbf{I}_n} = \mathbf{I}_{F^n}$
    - (Exercise)

2.3 Composition of linear transformations and matrix multiplication

- **Left multiplication**

> **Theorem 2.16**:
> Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be matrices such that $\mathbf{A}(\mathbf{BC})$ is defined.
> Then, $(\mathbf{AB})\mathbf{C}$ is also defined and $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (associative)

- Proof)
  - By Theorem 2.15 (e),
    - $L_{\mathbf{A}(\mathbf{BC})} = L_{\mathbf{A}}L_{\mathbf{BC}} = L_{\mathbf{A}}L_{\mathbf{B}}L_{\mathbf{C}} = (L_{\mathbf{A}}L_{\mathbf{B}})L_{\mathbf{C}} = L_{\mathbf{AB}}L_{\mathbf{C}} = L_{(\mathbf{AB})\mathbf{C}}$
  - By Theorem 2.15 (b),
    - $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
  - $\therefore$ Q.E.D.

2.4 Invertibility and isomorphisms

2.4 Invertibility and isomorphisms

- **Invertibility**

> **Invertibility**:
> Let V and W be vector spaces.
> Let $T: V \rightarrow W$ be linear.
> A function $U: W \rightarrow V$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$.
>
> If T has an inverse, then T is said to be invertible.
> If T is invertible, then the inverse of T is unique and denoted by T^{-1}.

- Facts for invertible functions T and U
  - $(TU)^{-1} = U^{-1}T^{-1}$
  - $(T^{-1})^{-1} = T$
    - T^{-1} being invertible
  - $T: V \rightarrow W$ invertible $\Leftrightarrow$ one-to-one and onto $\Leftrightarrow$ $\operatorname{rank}(T) = \dim(V)$
    - (Theorem 2.5)

2.4 Invertibility and isomorphisms

- **Invertibility**
  - *Example 2.4.1*
    - Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a+b \end{bmatrix}$
    - Let $U: \mathbb{R}^2 \to \mathbb{R}^2$ be $U\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c \\ -c+d \end{bmatrix}$
    - Then, $(TU)\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = T\left(U\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} c \\ -c+d \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$
      - $\Rightarrow TU = I_{\mathbb{R}^2}$
    - Then, $(UT)\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = U\left(T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)\right) = U\left(\begin{bmatrix} a \\ a+b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$
      - $\Rightarrow UT = I_{\mathbb{R}^2}$

    - $\therefore T^{-1} = U$ and $U^{-1} = T$

2.4 Invertibility and isomorphisms

- **Invertibility**

> **Theorem 2.17**:
> Let V and W be vector spaces.
> Let $T: V \to W$ be linear and invertible
> Then, $T^{-1}: W \to V$ is linear.

- Proof)
  - Let $\mathbf{y}_1, \mathbf{y}_2 \in W$ and $c \in F$.
  - T is invertible $\Leftrightarrow$ T is onto and one-to-one
    - $\Rightarrow$ There exist unique vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $T(\mathbf{x}_1) = \mathbf{y}_1$ and $T(\mathbf{x}_2) = \mathbf{y}_2$
    - $\Rightarrow$ $T^{-1}(\mathbf{y}_1) = \mathbf{x}_1$ and $T^{-1}(\mathbf{y}_2) = \mathbf{x}_2$
  - Then,
    - $T^{-1}(c\mathbf{y}_1 + \mathbf{y}_2) = T^{-1}\big(cT(\mathbf{x}_1) + T(\mathbf{x}_2)\big) = T^{-1}\big(T(c\mathbf{x}_1 + \mathbf{x}_2)\big) = c\mathbf{x}_1 + \mathbf{x}_2 = cT^{-1}(\mathbf{y}_1) + T^{-1}(\mathbf{y}_2)$
  - $\therefore$ Q.E.D.

2.4 Invertibility and isomorphisms

- **Invertibility**

> **Corollary 2.17.1**:
> Let V and W be vector spaces.
> Let $T: V \rightarrow W$ be linear and invertible
> Then, V is finite-dimensional $\Leftrightarrow$ W is finite-dimensional.
> In this case, $\dim(V) = \dim(W)$

- Proof) (V is finite-dimensional $\Rightarrow$ W is finite-dimensional)
  - Let $\beta = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for V.
  - By Theorem 2.2,
    - $\Rightarrow T(\beta)$ spans $R(T)$.
  - Since T is onto, $R(T) = W$
    - $\Rightarrow T(\beta)$ spans W.
  - By Theorem 1.9,
    - $\Rightarrow W$ is finite-dimensional.

2.4 Invertibility and isomorphisms

- **Invertibility**

> **Corollary 2.17.1**:
> Let V and W be vector spaces.
> Let $T: V \rightarrow W$ be linear and invertible
> Then, V is finite-dimensional $\Leftrightarrow$ W is finite-dimensional.
> In this case, $\dim(V) = \dim(W)$

- Proof) (V is finite-dimensional $\Leftarrow$ W is finite-dimensional)
  - Let $\gamma = \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m\}$ be a basis for W.
  - By Theorem 2.2,
    - $\Rightarrow T^{-1}(\gamma)$ spans $R(T^{-1})$.
  - Since T^{-1} is onto, $R(T^{-1}) = V$
    - $\Rightarrow T^{-1}(\gamma)$ spans V.
  - By Theorem 1.9,
    - $\Rightarrow V$ is finite-dimensional.
- $\therefore$ Q.E.D.

2.4 Invertibility and isomorphisms

- **Invertibility**

> **Corollary 2.17.1**:
> Let V and W be vector spaces.
> Let $T: V \rightarrow W$ be linear and invertible
> Then, V is finite-dimensional $\Leftrightarrow$ W is finite-dimensional.
> In this case, $\dim(V) = \dim(W)$

- Proof) $(\dim(V) = \dim(W))$
  - Since T is one-to-one
    - $\Rightarrow \text{nullity}(T) = 0$
    - $\Rightarrow \text{rank}(T) = \dim(V)$
  - Since T is onto
    - $\Rightarrow \text{rank}(T) = \dim(W)$
  - $\therefore \dim(V) = \dim(W)$

- $\therefore$ Q.E.D.

2.4 Invertibility and isomorphisms

- **Invertibility**
  - By Theorem 2.5, for a linear transformation T between vector spaces of equal finite dimension,
    - Invertible ⇔ one-to-one ⇔ onto

2.4 Invertibility and isomorphisms

- **Invertibility**

> **Invertible matrix**:
> Let $\mathbf{A}$ be an $n \times n$ matrix.
> Then, $\mathbf{A}$ is invertible if there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$.

- *Example 2.4.2*
  - $\mathbf{A} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$
  - $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$
  - (Section 3.2 for more detail)

2.4 Invertibility and isomorphisms

- **Invertibility**

> **Theorem 2.18**:
> Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively.
> For linear $T: V \rightarrow W$, T is invertible. $\Leftrightarrow$ $[T]_{\beta}^{\gamma}$ is invertible.
> Furthermore,
> $$[T^{-1}]_{\gamma}^{\beta} = \left([T]_{\beta}^{\gamma}\right)^{-1}$$

- Proof) (T is invertible. $\Rightarrow$ $[T]_{\beta}^{\gamma}$ is invertible.)
  - For invertible T, by Corollary 2.17.1,
    - $\dim(V) = \dim(W)$
  - Letting $n = \dim(V) = \dim(W)$, by Theorem 2.11,
    - $\mathbf{I}_n = [I_V]_{\beta} = [T^{-1}T]_{\beta} = [T^{-1}]_{\gamma}^{\beta}[T]_{\beta}^{\gamma}$
    - $\mathbf{I}_n = [I_W]_{\gamma} = [TT^{-1}]_{\gamma} = [T]_{\beta}^{\gamma}[T^{-1}]_{\gamma}^{\beta}$
  - $\therefore \left([T]_{\beta}^{\gamma}\right)^{-1} = [T^{-1}]_{\gamma}^{\beta}$

2.4 Invertibility and isomorphisms

- **Invertibility**

Theorem 2.18:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively.
For linear $T: V \rightarrow W$, T is invertible. $\Leftrightarrow$ $[T]_\beta^\gamma$ is invertible.
Furthermore,

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$$

- Proof) (T is invertible. $\Leftarrow$ $[T]_\beta^\gamma$ is invertible.)
  - Letting the inverse matrix $[U]_\gamma^\beta = [T]_\beta^\gamma$,
    - $[U]_\gamma^\beta [T]_\beta^\gamma = \mathbf{I}_n$ and $[T]_\beta^\gamma [U]_\gamma^\beta = \mathbf{I}_n$
  - By Theorem 2.11,
    - $[U]_\gamma^\beta [T]_\beta^\gamma = [UT]_\beta = \mathbf{I}_n = [I_V]_\beta$ and $[T]_\beta^\gamma [U]_\gamma^\beta = [TU]_\gamma = \mathbf{I}_n = [I_W]_\gamma$
  - $\therefore\ T^{-1} = U$

2.4 Invertibility and isomorphisms

- **Invertibility**
  - *Example 2.4.3*
    - Let $T, U: \mathbb{R}^2 \to \mathbb{R}^2$ be linear and $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ for $V = \mathbb{R}^2$ and $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2\}$ for $W = \mathbb{R}^2$
    - If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_1 + v_2 \end{bmatrix}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2$ and $T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2$
      - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
    - If $U\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \begin{bmatrix} w_1 \\ -w_1 + w_2 \end{bmatrix}$
      - $U(\mathbf{w}_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + (-1) \cdot \mathbf{w}_2$ and $U(\mathbf{w}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2$
      - $\Rightarrow [U]_\gamma^\beta = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

2.4 Invertibility and isomorphisms

- **Invertibility**
  - *Example 2.4.3*
    - Note that (Example 2.4.1)
      - $(TU)\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = T\left(U\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} w_1 \\ -w_1 + w_2 \end{bmatrix}\right) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$
        - $\Rightarrow TU = I_{\mathbb{R}^2}$
      - $(UT)\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = U\left(T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)\right) = U\left(\begin{bmatrix} v_1 \\ v_1 + v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
        - $\Rightarrow UT = I_{\mathbb{R}^2}$
      - $[T]_\beta^\gamma [U]_\gamma^\beta = \mathbf{I}_2$
      - $[U]_\gamma^\beta [T]_\beta^\gamma = \mathbf{I}_2$
    - $\therefore T^{-1} = U$ and $U^{-1} = T$
    - $\therefore \left([T]_\beta^\gamma\right)^{-1} = [U]_\gamma^\beta = [T^{-1}]_\gamma^\beta$ and $\left([U]_\gamma^\beta\right)^{-1} = [T]_\beta^\gamma = [U^{-1}]_\beta^\gamma$

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2.4 Invertibility and isomorphisms

- **Invertibility**

Corollary 2.18.1:
Let V be a finite-dimensional vector space with an ordered bases β.
Let $T: V \to W$ be linear.
Then, T is invertible. $\Leftrightarrow$ $[T]_\beta$ is invertible.
Furthermore,

$$[T^{-1}]_\beta = \left([T]_\beta\right)^{-1}$$

2.4 Invertibility and isomorphisms

- **Invertibility**

Corollary 2.18.2:
Let $\mathbf{A}$ be an $n \times n$ matrix.
Then, T is invertible. $\Leftrightarrow$ $L_{\mathbf{A}}$ is invertible.
Furthermore,

$$(L_{\mathbf{A}})^{-1} = L_{\mathbf{A}^{-1}}$$

2.4 Invertibility and isomorphisms

- **Isomorphisms**

> **Isomorphism**:
> Let V and W be vector spaces.
> Then, V is isomorphic to W if there exists a linear transformation $T: V \rightarrow W$ that is invertible.
> Such a linear transformation is called an isomorphism from V onto W.

- *Example 2.4.4*
  - $T: F^2 \rightarrow M_{2\times1}(F)$
    - $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
  - $\therefore$ Invertible, isomorphism

2.4 Invertibility and isomorphisms

- **Isomorphisms**

> **Isomorphism**:
> Let V and W be vector spaces.
> Then, V is isomorphic to W if there exists a linear transformation $T: V \to W$ that is invertible.
> Such a linear transformation is called an isomorphism from V onto W.

- *Example 2.4.5*
  - $T: F^4 \to M_{2\times 2}(F)$
    - $T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_3 \\ v_2 & v_4 \end{bmatrix}$
  - $\therefore$ Invertible, isomorphism

2.5 The change of coordinate matrix

2.5 The change of coordinate matrix

- **Examples**
  - ① Calculus

    $$\int 2xe^{x^2}\,dx$$

    - By a change of variable $u = x^2$, $du = 2xdx$

      $$\int 2xe^{x^2}\,dx = \int e^u\,du = e^u + c = e^{x^2} + c$$

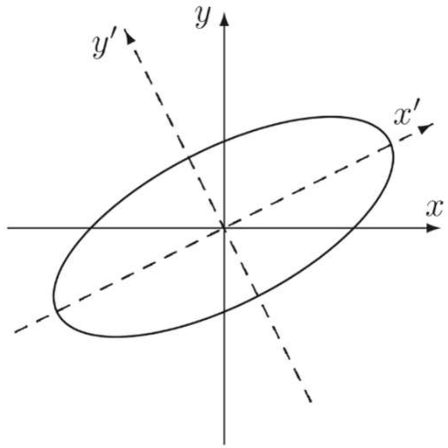
  - ② Geometry

    $$2x^2 - 4xy + 5y^2 = 1$$

    - By change of variables, $\begin{cases} x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y' \\ y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y' \end{cases}$

      $$(x')^2 + 6(y')^2 = 1$$

    - An $x'y'$-coordinate system with coordinate axes rotated from the original xy-coordinate axes
    - $\beta = \{\mathbf{e}_1, \mathbf{e}_2\} \quad \Rightarrow \quad \beta' = \left\{\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}\right\}$



2.5 The change of coordinate matrix

- **Change of coordinates**

Theorem 2.22:
Let β and β' be two ordered bases for a finite-dimensional vector space V.
Let $\mathbf{Q} = [I_V]_{\beta'}^{\beta}$

(a) $\mathbf{Q}$ is invertible.
(b) For any $\mathbf{v} \in V$, $[\mathbf{v}]_{\beta} = \mathbf{Q}[\mathbf{v}]_{\beta'}$

- Proof)
  - (a) (Theorem 2.18)
  - (b)
    - For any $\mathbf{v} \in V$, by Theorem 2.14,
      - $[\mathbf{v}]_{\beta} = [I_V]_{\beta} = [I_V]_{\beta'}^{\beta}[\mathbf{v}]_{\beta'} = \mathbf{Q}[\mathbf{v}]_{\beta'}$

2.5 The change of coordinate matrix

- **Change of coordinates**
  - *Example 2.5.1*
    - Let $\beta = \left\{\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ and $\beta' = \left\{\mathbf{w}_1 = \begin{bmatrix}2\\4\end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix}3\\1\end{bmatrix}\right\}$
    - If $I_V\left(\begin{bmatrix}w_1\\w_2\end{bmatrix}\right) = \begin{bmatrix}w_1\\w_2\end{bmatrix}$
      - $I_V(\mathbf{w}_1) = \begin{bmatrix}2\\4\end{bmatrix} = 3 \cdot \mathbf{v}_1 + (-1) \cdot \mathbf{v}_2$ and $I_V(\mathbf{w}_2) = \begin{bmatrix}3\\1\end{bmatrix} = 2 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$
      - $\Rightarrow [I_V]_{\beta'}^{\beta} = \begin{bmatrix}3 & 2\\-1 & 1\end{bmatrix}$
    - $\therefore \mathbf{Q} = [I_V]_{\beta'}^{\beta} = \begin{bmatrix}3 & 2\\-1 & 1\end{bmatrix}$

2.5 The change of coordinate matrix

- **Linear operator**
    - Mapping a vector space V into itself.

2.5 The change of coordinate matrix

- **Linear operator**

Theorem 2.23:
Let T be a linear operator on a finite-dimensional vector space V
Let β and β' be ordered bases for V.
Suppose that $\mathbf{Q}$ is the change of coordinate matrix that changes β'-coordinates into β-coordinates.
Then,

$$[T]_{\beta'} = \mathbf{Q}^{-1}[T]_{\beta}\mathbf{Q}$$

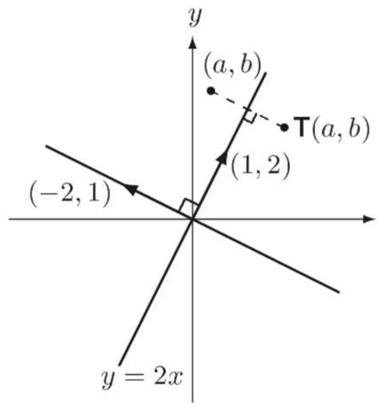
- Proof)
  - By Theorem 2.11,
  - $\mathbf{Q}[T]_{\beta'} = [I_V]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [I_V T]_{\beta'}^{\beta} = [T I_V]_{\beta'}^{\beta} = [T]_{\beta}^{\beta}[I_V]_{\beta'}^{\beta} = [T]_{\beta}\mathbf{Q}$
  - $\Rightarrow [T]_{\beta'} = \mathbf{Q}^{-1}[T]_{\beta}\mathbf{Q}$
  - $\therefore$ Q.E.D.

2.5 The change of coordinate matrix

- **Linear operator**
  - *Example 2.5.2*
    - Let $\beta = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $\beta' = \left\{ \mathbf{w}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$
    - From Example 2.5.1, $\mathbf{Q} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{Q}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$
    - If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 - v_2 \\ v_1 + 3v_2 \end{bmatrix}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 3 \cdot \mathbf{v}_1 + (-1) \cdot \mathbf{v}_2$ and $T(\mathbf{v}_2) = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 1 \cdot \mathbf{v}_1 + 3 \cdot \mathbf{v}_2$
      - $\Rightarrow [T]_\beta^\beta = [T]_\beta = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$
    - Also,
      - $T(\mathbf{w}_1) = \begin{bmatrix} 2 \\ 14 \end{bmatrix} = 4 \cdot \mathbf{w}_1 + (-2) \cdot \mathbf{w}_2$ and $T(\mathbf{w}_2) = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2$
      - $\Rightarrow [T]_{\beta'}^{\beta'} = [T]_{\beta'} = \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}$
    - Then,
      - $\mathbf{Q}^{-1}[T]_\beta \mathbf{Q} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix} = [T]_{\beta'}$

2.5 The change of coordinate matrix

- **Linear operator**
  - *Example 2.5.3* (The reflection T about $y = 2x$)
    - $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$ and $T\left(\begin{bmatrix}-2\\1\end{bmatrix}\right) = \begin{bmatrix}2\\-1\end{bmatrix}$
    - Let $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ and $\beta' = \left\{\mathbf{w}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix}-2\\1\end{bmatrix}\right\}$
      - $T(\mathbf{w}_1) = \begin{bmatrix}1\\2\end{bmatrix} = 1 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2$ and $T(\mathbf{w}_2) = \begin{bmatrix}2\\-1\end{bmatrix} = 0 \cdot \mathbf{w}_1 + (-1) \cdot \mathbf{w}_2$
      - $\Rightarrow [T]_{\beta'}^{\beta'} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
      - $\Rightarrow [T]_{\beta'} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
    - If $I_V\left(\begin{bmatrix}w_1\\w_2\end{bmatrix}\right) = \begin{bmatrix}w_1\\w_2\end{bmatrix}$
      - $I_V(\mathbf{w}_1) = \begin{bmatrix}1\\2\end{bmatrix} = 1 \cdot \mathbf{v}_1 + 2 \cdot \mathbf{v}_2$ and $I_V(\mathbf{w}_2) = \begin{bmatrix}-2\\1\end{bmatrix} = (-2) \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$
      - $\Rightarrow [I_V]_{\beta'}^{\beta} = \begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}$
      - $\Rightarrow \mathbf{Q} = [I_V]_{\beta'}^{\beta} = \begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}$ and $\mathbf{Q}^{-1} = \frac{1}{5}\begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix}$



- $\mathbf{Q}[T]_{\beta'}\mathbf{Q}^{-1} = \begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\frac{1}{5}\begin{bmatrix}1 & 2\\-2 & 1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}-3 & 4\\4 & 3\end{bmatrix} = [T]_\beta$

2.5 The change of coordinate matrix

- **Linear operator**

> **Similar matrices**:
> Let $\mathbf{A}$ and $\mathbf{B}$ be matrices in $M_{n \times n}(F)$.
> We say that $\mathbf{B}$ is similar to $\mathbf{A}$ if there exists an invertible matrix $\mathbf{Q}$ such that
> $$\mathbf{B} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$$

- $[T]_{\beta'}$ similar to $[T]_{\beta}$