Linear Algebra (5th edition)

Stephen Friedberg, Arnold Insel, Lawrence Spence

Chapter 01: Vector spaces

Jihwan Moon

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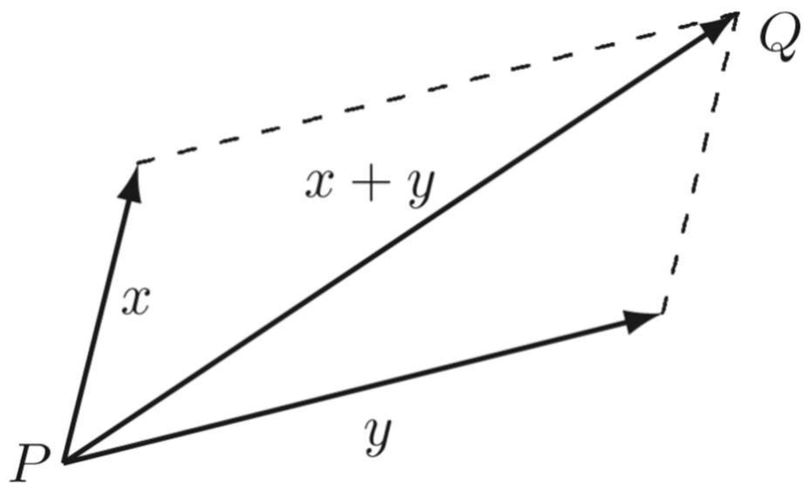
1.1 Introduction

1.1 Introduction

- **Vector**
  - An entity involving both magnitude and direction

  - Represented by an arrow
    - Length of the arrow = Magnitude of the vector
    - Direction of the arrow = Direction of the vector

  - Irrespective of the position

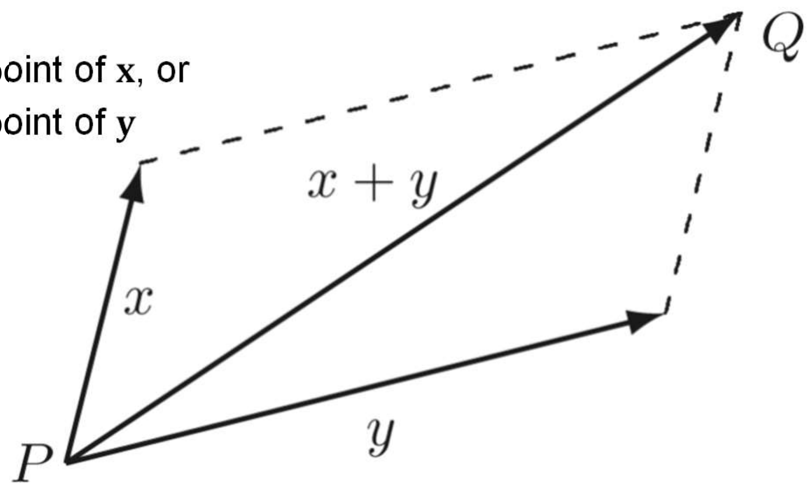


1.1 Introduction

- **Vector addition**

Parallelogram law for vector addition:
The sum of two vectors $\mathbf{x}$ and $\mathbf{y}$ that act at the same point P is the vector beginning at P that is represented by the diagonal of a parallelogram having $\mathbf{x}$ and $\mathbf{y}$ as adjacent sides.

- Geometrically obtaining the endpoint Q, i.e., $\mathbf{x}+\mathbf{y}$
  - ① Allowing $\mathbf{x}$ to act at P and then $\mathbf{y}$ to act at the end point of $\mathbf{x}$, or
  - ② Allowing $\mathbf{y}$ to act at P and then $\mathbf{x}$ to act at the end point of $\mathbf{y}$
  - "Tail-to-head" addition

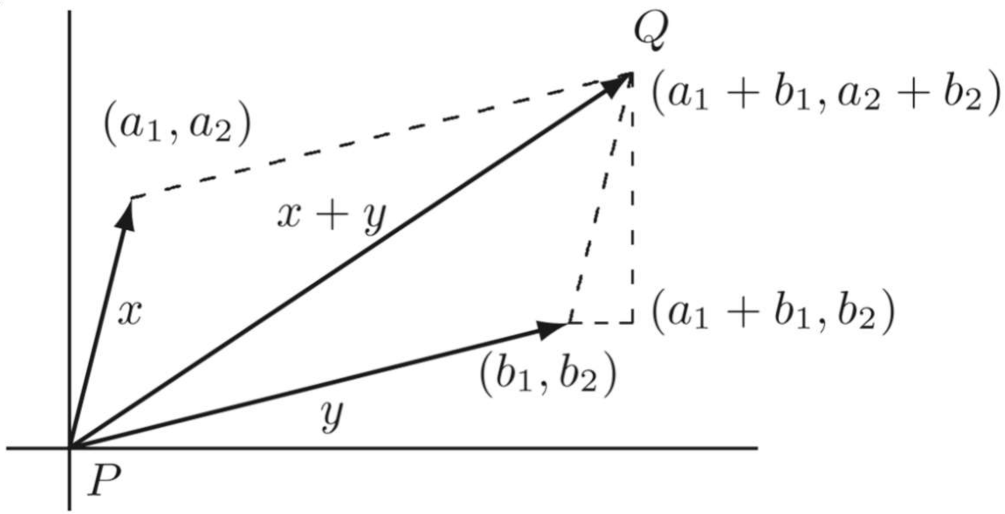


1.1 Introduction

- **Vector addition**

Parallelogram law for vector addition:
The sum of two vectors $\mathbf{x}$ and $\mathbf{y}$ that act at the same point P is the vector beginning at P that is represented by the diagonal of a parallelogram having $\mathbf{x}$ and $\mathbf{y}$ as adjacent sides.

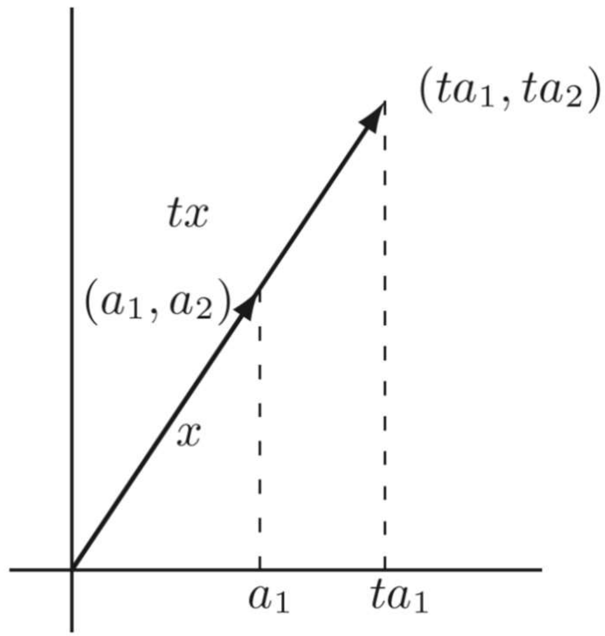
- Algebraically obtaining the endpoint Q, i.e., $\mathbf{x}+\mathbf{y}$
    - (a_1, a_2): The endpoint of $\mathbf{x}$
    - (b_1, b_2): The endpoint of $\mathbf{y}$
    - (a_1+b_1, a_2+b_2): The end point of $\mathbf{x}+\mathbf{y}$
    - Assumed to emanate from the origin



- Often refer to "the point $\mathbf{x}$"
    - rather than "the endpoint of the vector $\mathbf{x}$"

1.1 Introduction

- **Scalar multiplication**
  - Multiplying the vector by a real number
  - Geometrically,
    - For $t > 0$
      - $t\mathbf{x}$ in the same direction of $\mathbf{x}$
    - For $t < 0$
      - $t\mathbf{x}$ in the opposite direction from $\mathbf{x}$
    - Length (magnitude) of $t\mathbf{x}$ = $|t|$ times the length (magnitude) of $\mathbf{x}$
    - $\mathbf{x}$ and $\mathbf{y}$ in parallel if $\mathbf{y} = t\mathbf{x}$ for some non-zero real number t
  - Algebraically,
    - (a_1, a_2): The endpoint of $\mathbf{x}$
    - (ta_1, ta_2): The endpoint of $t\mathbf{x}$
    - Assumed to emanate from the origin



1.1 Introduction

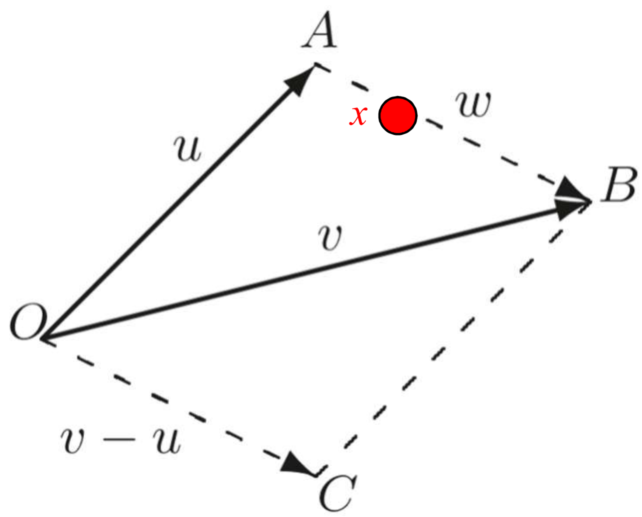
- **Properties regarding vector addition and scalar multiplication**
  - ① For all vectors $\mathbf{x}$ and $\mathbf{y}$, (commutativity)
    - $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$
  - ② For all vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, (associativity)
    - $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$
  - ③ There exists a vector denoted $\mathbf{0}$ such that (existence of identity)
    - $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for each vector $\mathbf{x}$
  - ④ For each vector $\mathbf{x}$, there is a vector $\mathbf{y}$ such that (existence of inverse)
    - $\mathbf{x}+\mathbf{y}=\mathbf{0}$

1.1 Introduction

- **Properties regarding vector addition and scalar multiplication**
  - ⑤ For each vector $\mathbf{x}$, (existence of identity)
    - $1\mathbf{x} = \mathbf{x}$
  - ⑥ For each pair of real numbers a and b and each vector $\mathbf{x}$, (associativity)
    - $(ab)\mathbf{x} = a(b\mathbf{x})$
  - ⑦ For each real number a and each pair of vectors $\mathbf{x}$ and $\mathbf{y}$, (distributivity of scalars)
    - $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$
  - ⑧ For each pair of real numbers a and b and each vector $\mathbf{x}$, (distributivity of vectors)
    - $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$

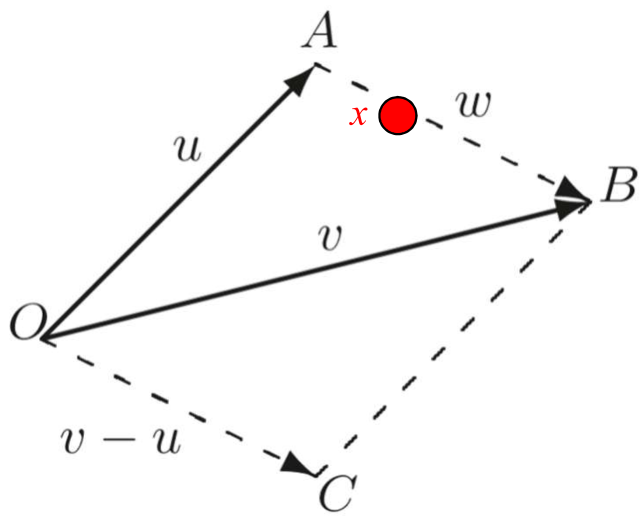
1.1 Introduction

- **An equation of the line through 2 distinct points**
  - Vectors pointing at two points A and B
    - $\mathbf{u}$: Vector from O to A
    - $\mathbf{v}$: Vector from O to B
  - Vector $\mathbf{w}$ from the two points A and B
    - From "tail-to-head" addition,
      - $\mathbf{u} + \mathbf{w} = \mathbf{v}$
      - $\Rightarrow \mathbf{w} = \mathbf{v} - \mathbf{u}$
  - Any point x on the line joining A and B
    - Obtained by the endpoint of $t\mathbf{w}$ beginning at A for some real number t
    - $\Rightarrow \mathbf{u} + t\mathbf{w} = \mathbf{u} + t(\mathbf{v} - \mathbf{u}) = (1 - t)\mathbf{u} + t\mathbf{v}$ for some real number t
  - (Recall) Irrespective of the position
    - e.g.) The coordinates of the endpoint C $(\mathbf{v} - \mathbf{u})$ = The difference between the coordinates of B and A



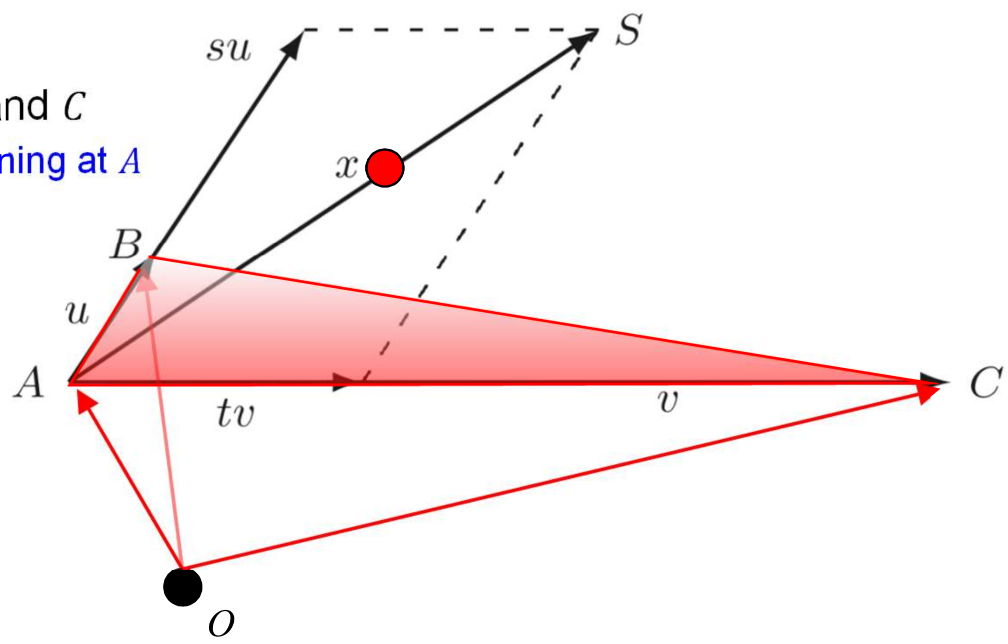
1.1 Introduction

- **An equation of the line through 2 distinct points**
  - *Example 1.1*
    - The coordinate of A: $(-2,0,1)$
    - The coordinate of B: $(4,5,3)$
    - Then,
      - Coordinates of C: $(4,5,3) - (-2,0,1) = (6,5,2)$
      - The equation of the line through A and B:
        - $\mathbf{x} = \mathbf{u} + t(\mathbf{v} - \mathbf{u}) = (-2,0,1) + t(6,5,2)$



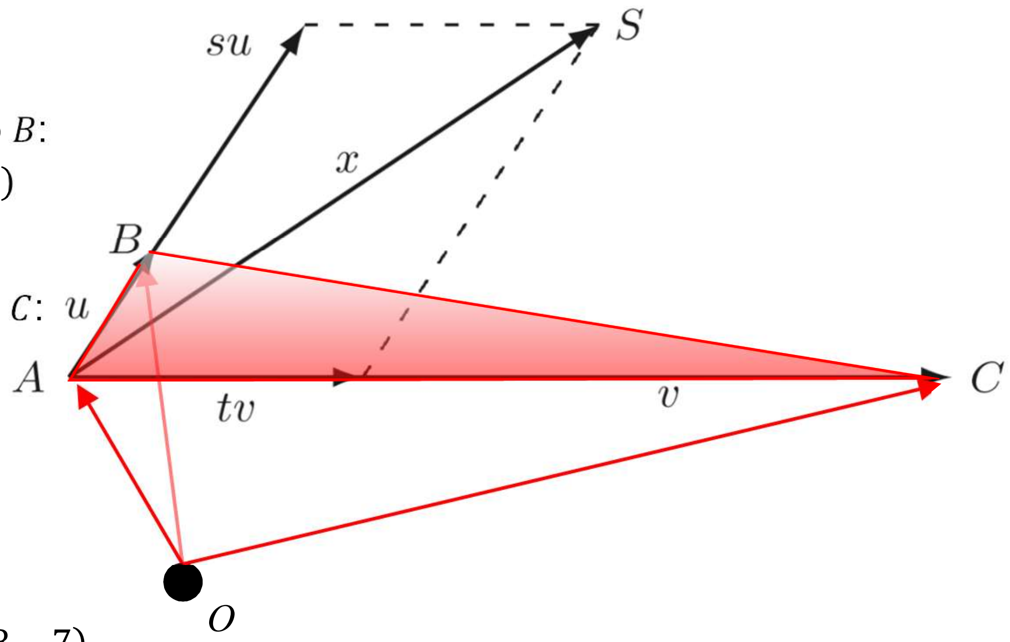
1.1 Introduction

- **An equation of the plane through 3 distinct points**
  - Vectors beginning at A and ending at two points A and B
    - $\mathbf{u}$: Vector from A to B
    - $\mathbf{v}$: Vector from A to C
  - Any point x on the plane containing A, B and C
    - Obtained by the endpoint of $s\mathbf{u} + t\mathbf{v}$ beginning at A for some real number s and t
    - $\Rightarrow A + s\mathbf{u} + t\mathbf{v}$ for some real number s and t



1.1 Introduction

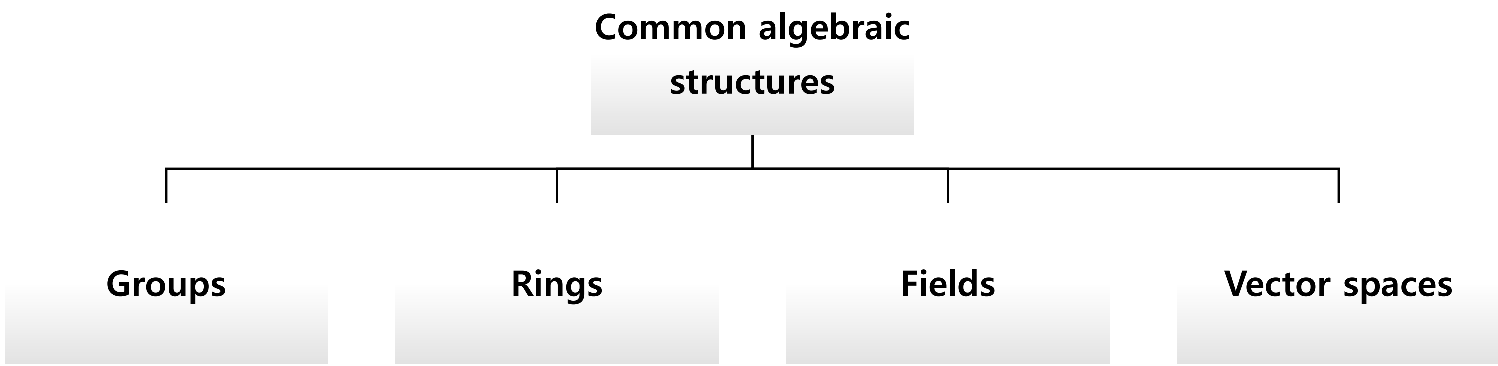
- **An equation of the plane through 3 distinct points**
  - *Example 1.2*
    - The coordinate of A: $(1,0,2)$
    - The coordinate of B: $(-3,-2,4)$
    - The coordinate of C: $(1,8,-5)$
    - Then,
      - Coordinates of the vector $\mathbf{u}$ from A to B:
        - $(-3,-2,4)-(1,0,2)=(-4,-2,2)$
      - Coordinates of the vector $\mathbf{v}$ from A to C:
        - $(1,8,-5)-(1,0,2)=(0,8,-7)$
      - The equation of the plane through A, B and C:
        - $x=A+s\mathbf{u}+t\mathbf{v}$
          $=(1,0,2)+s(-4,-2,2)+t(0,8,-7)$



1.2 Vector spaces

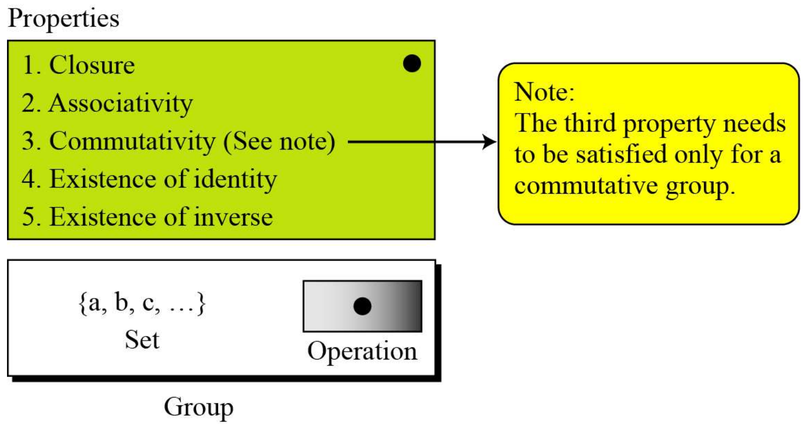
1.2 Vector spaces

- **Algebraic structures (대수 구조)**
  - The combination of the set (집합) and the operations (연산) that are applied to the elements of the set
  - Common algebraic structures:
    - Groups (군)
    - Rings (환)
    - Fields (체)
    - Vector space (벡터 공간)



1.2 Vector spaces

- **Groups (군), $\mathbf{G}$**
  - A set of elements with a binary operation "•" that satisfies four properties (성질) or axioms (공리)
    - Property ①: Closure (닫힘)
      - If $a, b \in \mathbf{G}$, then $a \bullet b \in \mathbf{G}$
    - Property ②: Associativity (결합)
      - If $a, b, c \in \mathbf{G}$, then $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
      - Any order of the operation yielding the same result
    - Property ③: Existence of identity (항등원의 존재)
      - Existence of e for all $a \in \mathbf{G}$ such that $e \bullet a = a \bullet e = a$
    - Property ④: Existence of inverse (역원의 존재)
      - Existence of $\acute{a}$ for each $a \in \mathbf{G}$ such that $\acute{a} \bullet a = a \bullet \acute{a} = e$



- Commutative group (가환군), or abelian group, if commutativity also holds
  - Property ⑤: Commutativity (교환 법칙)
    - For all $a, b \in \mathbf{G}$, $a \bullet b = b \bullet a$

1.2 Vector spaces

- **Groups (군), G**
  - Application
    - A single operation involved in a group
      - +, -, x, /
    - A pair of operations, as long as they are inverse, also involved in a group
      - (+, -) and (x, /)
        - Only one pair supported at a time

1.2 Vector spaces

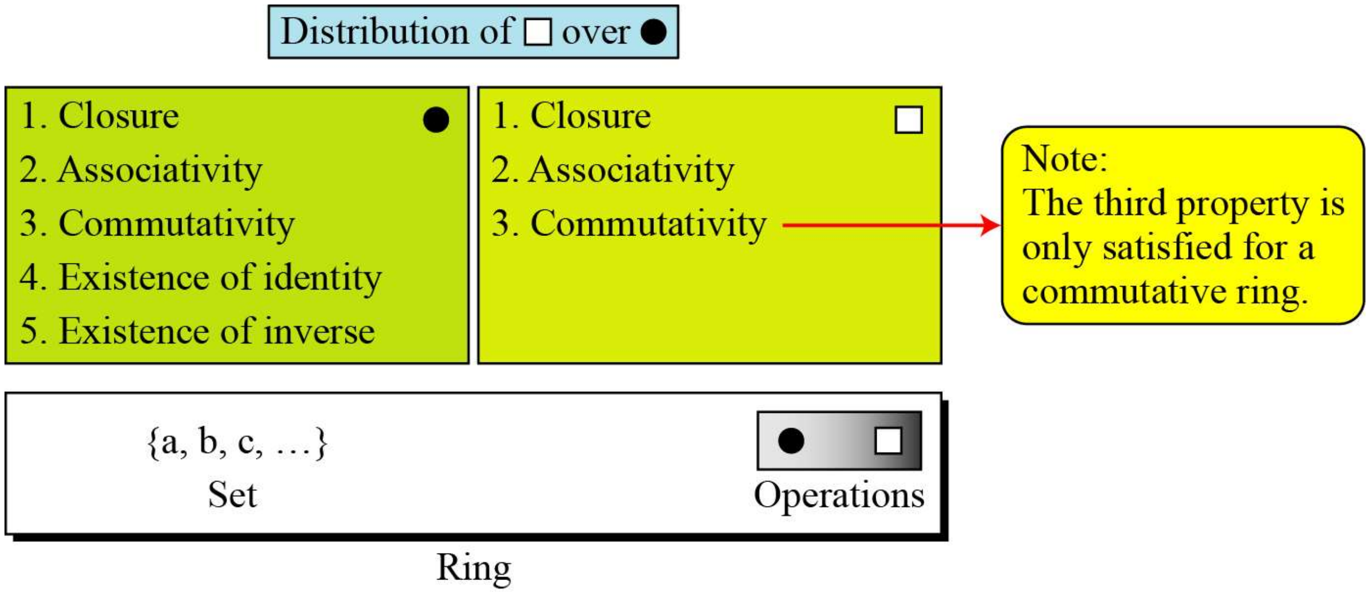
- **Groups (군), $\mathbf{G}$**
  - *Example* Modulo addition!
    - The set of residue integers with the addition operator, $\mathbf{G} = \langle \mathbf{Z}_n, + \rangle$
      - Closure?
        - $(a+b) \bmod n \in \mathbf{Z}_n$ for any $a, b \in \mathbf{Z}_n$, Yes
      - Associative?
        - $\big((a+b)+c\big) \bmod n = \big(a+(b+c)\big) \bmod n$ for any $a, b, c \in \mathbf{Z}_n$, Yes
      - Existence of identity?
        - $e = 0$
        - $(a+0) \bmod n = (0+a) \bmod n = a \bmod n$, Yes
      - Existence of inverse?
        - $\acute{a} = -a$ or equivalently, $\acute{a} = n - a$
        - $\big(a+(-a)\big) \bmod n = \big((-a)+a\big) \bmod n = 0 \bmod n = e$, Yes
      - Commutativity?
        - $(a+b) \bmod n = (b+a) \bmod n$, Yes

1.2 Vector spaces

- **Rings (환), $\mathbf{R}$**
  - An algebraic structure (대수 구조) with two operations, denoted as $\mathbf{R} = \langle \{\ldots\}, \bullet, \square \rangle$
  - First operation $\bullet$ satisfying
    - Closure (닫힘)
    - Associativity (결합)
    - Existence of identity (항등원의 존재성)
    - Existence of inverse (역원의 존재성)
    - Commutativity (교환 법칙)
  - Second operation $\square$ satisfying
    - Closure (닫힘)
    - Associativity (결합)
  - Distributivity (분배 법칙) of the second operation $\square$ over the first operation $\bullet$
    - For all $a, b, c \in \mathbf{R}$,
      - $a \square (b \bullet c) = (a \square b) \bullet (a \square c)$
      - $(a \bullet b) \square c = (a \square c) \bullet (b \square c)$
  - Commutative ring (가환 환) if the second operation $\square$ also satisfies commutativity

1.2 Vector spaces

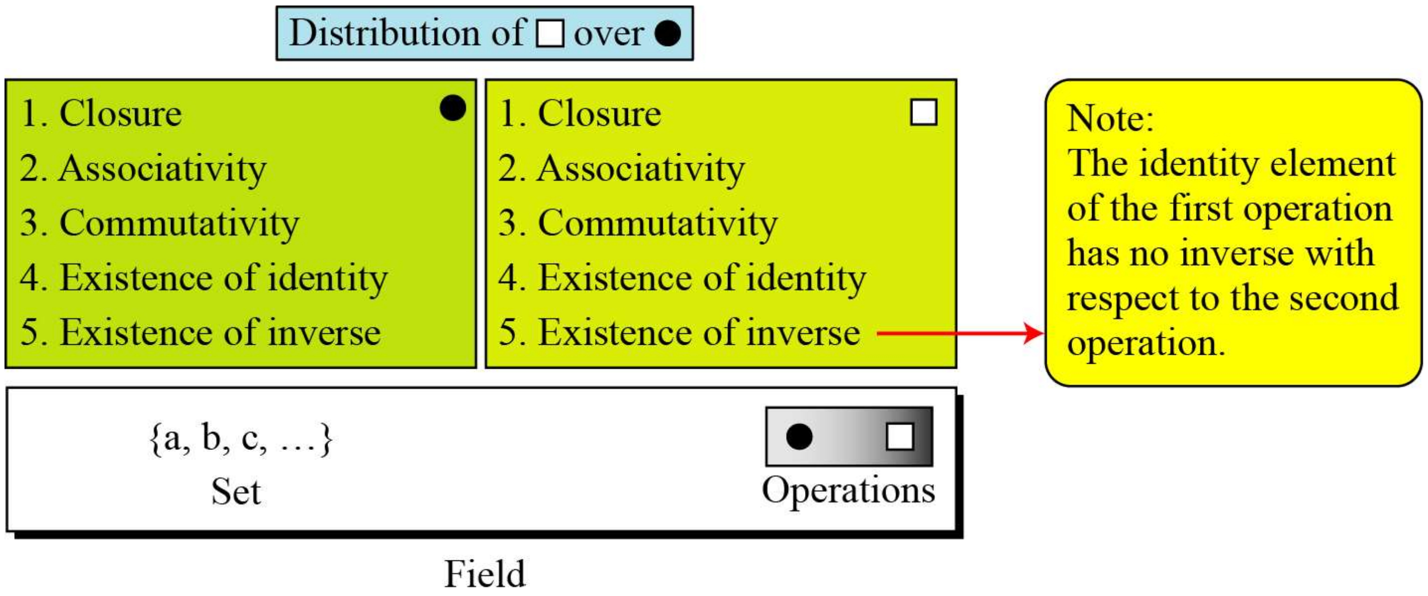
- **Rings (환), R**



  - *Example*
    - $\mathbf{R} = \langle \mathbf{Z}, +, \times \rangle$

1.2 Vector spaces

- **Field (체), F**
  - $\mathbf{F} = \langle\{\ldots\}, \bullet, \square\rangle$
  - A commutative ring (가환 환) in which …
    - The second operation satisfies all five properties
    - The identity (항등원) of the first operation has no inverse with respect to the second operation



1.2 Vector spaces

- **Vector space (벡터 공간), V**

Vector space (linear space) V over field F:
- Elements of V are called "vectors"
- Elements of F are called "scalars"

- Two operations
  - ① Vector addition ($V \times V \rightarrow V$)
    - For each pair of elements $\mathbf{x}$ and $\mathbf{y}$ in V, there is a unique element $\mathbf{x}+\mathbf{y}$ in V
  - ② Scalar multiplication ($F \times V \rightarrow V$)
    - For each element a in F and each element $\mathbf{x}$ in V, there is a unique $a\mathbf{x}$ in V

1.2 Vector spaces

- **Vector space (벡터 공간), V**

Vector space (linear space) V over field F:
- The following 8 conditions hold:

	Axiom	Meaning
①	Associativity of vector addition	$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
②	Commutativity of vector addition	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
③	Existence of identity of vector addition	There exists an element $\mathbf{0}\in V$ such that $\mathbf{v}+\mathbf{0}=\mathbf{v}$ for all $\mathbf{v}\in V$
④	Existence of inverse of vector addition	For every $\mathbf{v}\in V$, there exists an element $-\mathbf{v}\in V$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$

1.2 Vector spaces

- **Vector space (벡터 공간), V**

Vector space (linear space) V over field F:
- The following 8 conditions hold:

	Axiom	Meaning
⑤	Associativity of scalar multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$
⑥	Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$
⑦	Distributivity of scalar multiplication w.r.t. vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
⑧	Distributivity of scalar multiplication w.r.t. field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

1.2 Vector spaces

- **Vector space (벡터 공간), V**
  - Possible scalar fields F
    - Real numbers, $\mathbb{R}$
    - Complex numbers, $\mathbb{C}$
    - Etc.

  - The representation of a n-tuple vector

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$$

    - $a_1, \ldots, a_n \in F$: Entries or components
    - F^n: The set of all n-tuple vectors with entries from a field F

1.2 Vector spaces

- **Vector space (벡터 공간), V**
  - Vector addition and scalar multiplication

$$\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n, \ \mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in F^n$$

$$\Rightarrow \quad \mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

$$\Rightarrow \quad c\mathbf{u} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$$

  - $\mathbf{u}$ and $\mathbf{v}$ equal if $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$

1.2 Vector spaces

- **Vector space (벡터 공간), V**
  - *Example 1.2.1*
    - For $F = \mathbb{R}$ and $V = \mathbb{R}^3$,

$$\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$-5\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \\ 0 \end{bmatrix}$$

    - For $F = \mathbb{C}$ and $V = \mathbb{C}^2$,

$$\begin{bmatrix} 1+j \\ 2 \end{bmatrix} + \begin{bmatrix} 2-j3 \\ j4 \end{bmatrix} = \begin{bmatrix} 3-j2 \\ 2+j4 \end{bmatrix}$$

$$j\begin{bmatrix} 1+j \\ 2 \end{bmatrix} = \begin{bmatrix} -1+j \\ j2 \end{bmatrix}$$

1.2 Vector spaces

- **Matrices**
  - An $m \times n$ matrix with entries from a field F

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(F)$$

  - $a_{k\ell} \in F$:
    - Entries or components
  - $a_{k\ell} \in F$ for $k = \ell$:
    - Diagonal entries
  - $[a_{k1} \quad \cdots \quad a_{kn}]$:
    - The k-th row vector in F^n
  - $\begin{bmatrix} a_{1\ell} \\ \vdots \\ a_{m\ell} \end{bmatrix}$:
    - The ℓ-th column vector in F^m

1.2 Vector spaces

- **Matrices**
  - An $m \times n$ matrix with entries from a field F

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(F)$$

- Zero matrix
  - $a_{k\ell} = 0$ for all k, ℓ
- Square matrix
  - $m = n$

1.2 Vector spaces

- **Matrices**
  - Matrix addition and scalar multiplication

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}(F),\ \mathbf{B}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \in M_{m\times n}(F)$$

$$\Rightarrow [\mathbf{A}+\mathbf{B}]_{k\ell}=[\mathbf{A}]_{k\ell}+[\mathbf{B}]_{k\ell}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

$$\Rightarrow [c\mathbf{A}]_{k\ell}=c[\mathbf{A}]_{k\ell}=\begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

- where $[\mathbf{A}]_{k\ell}=a_{k\ell}$ and $[\mathbf{B}]_{k\ell}=b_{k\ell}$

1.2 Vector spaces

- **Matrices**
  - *Example 1.2.2*
    - For $M_{2\times 3}(\mathbb{R})$

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$

$$-3\begin{bmatrix} 1 & 0 & -2 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 6 \\ 9 & -6 & -9 \end{bmatrix}$$

1.2 Vector spaces

- **Theorems**

Theorem 1.1 (Cancellation Law for Vector Addition):
If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in a vector space V such that $\boxed{\mathbf{x}+\mathbf{z}=\mathbf{y}+\mathbf{z}}$, then $\mathbf{x}=\mathbf{y}$.

- Proof)
  - From property ④ of vector space, there exists a vector $\mathbf{v}$ such that $\boxed{\mathbf{z}+\mathbf{v}=\mathbf{0}}$.
  - Then,
    - $\mathbf{x}=\mathbf{x}+\boxed{\mathbf{0}}=\mathbf{x}+\boxed{(\mathbf{z}+\mathbf{v})}\underset{\text{Property ①}}{=}\boxed{(\mathbf{x}+\mathbf{z})}+\mathbf{v}=\boxed{(\mathbf{y}+\mathbf{z})}+\mathbf{v}\underset{\text{Property ①}}{=}\mathbf{y}+\boxed{(\mathbf{z}+\mathbf{v})}=\mathbf{y}+\boxed{\mathbf{0}}=\mathbf{y}$

1.2 Vector spaces

- **Theorems**

Corollary 1.1.1
The vector $\mathbf{0}$ in property ③ is unique

Corollary 1.1.2
The vector $\mathbf{y}$ in property ④ is unique

1.2 Vector spaces

- **Theorems**

Theorem 1.2:
In any vector space V, the following statements are true:
(a) $0\mathbf{x} = \mathbf{0}$ for each $\mathbf{x} \in V$
(b) $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$ for each $a \in F$ and each $x \in V$
(c) $a\mathbf{0} = \mathbf{0}$ for each $a \in F$

- Proof) (a)
  - $\boxed{0\mathbf{x} + 0\mathbf{x}} = (0+0)\mathbf{x} = 0\mathbf{x} = \boxed{0\mathbf{x} + \mathbf{0}}$

    Property ⑧

  - From Theorem 1.1, $0\mathbf{x} = \mathbf{0}$

1.2 Vector spaces

- **Theorems**

> **Theorem 1.2**:
> In any vector space V, the following statements are true:
> (a) $0\mathbf{x} = \mathbf{0}$ for each $\mathbf{x} \in V$
> (b) $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$ for each $a \in F$ and each $x \in V$
> (c) $a\mathbf{0} = \mathbf{0}$ for each $a \in F$

- Proof) (b)
  - From Corollary 1.1.2
    - Vector $-(a\mathbf{x}) \in V$ is the unique element such that $a\mathbf{x} + \big(-(a\mathbf{x})\big) = \mathbf{0}$.
  - From Theorem 1.2 (a) and property ⑧,
    - $\mathbf{0} = 0\mathbf{x} = \big(a + (-a)\big)\mathbf{x} = a\mathbf{x} + (-a)\mathbf{x}$
  - From Theorem 1.1,
    - $(-a)\mathbf{x} = -(a\mathbf{x})$
  - From property ⑤,
    - $(-a)\mathbf{x} = \big(a \cdot (-1)\big)\mathbf{x} = a\big((-1)\mathbf{x}\big) = a(-\mathbf{x})$

1.2 Vector spaces

- **Theorems**

> **Theorem 1.2**:
> In any vector space V, the following statements are true:
> (a) $0\mathbf{x} = \mathbf{0}$ for each $\mathbf{x} \in V$
> (b) $(-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$ for each $a \in F$ and each $x \in V$
> (c) $a\mathbf{0} = \mathbf{0}$ for each $a \in F$

- Proof) (c)
  - From property ③,
    - $\boxed{a\mathbf{0} + \mathbf{0}} = a\mathbf{0}$
  - From property ③ and property ⑧,
    - $a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) = \boxed{a\mathbf{0} + a\mathbf{0}}$
  - From Theorem 1.1, $a\mathbf{0} = \mathbf{0}$

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1.3 Subspaces

1.3 Subspaces

- **Subspace W**

> **Subspace W of vector space V over field F**:
> A vector space over F with operations of vector addition and scalar multiplication defined on V

  - e.g., $\{\mathbf{0}\}$ and V as subspaces of V
  - Vector space property ①, ②, ⑤, ⑥, ⑦ and ⑧ automatically satisfied for all vectors in V
  - Only needed to check ③ and ④, or the following 4 conditions:
    - Closure under vector addition
      - $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$
    - Closure under scalar multiplication
      - $c\mathbf{x} \in W$ whenever $c \in F$ and $\mathbf{x} \in W$
    - $\mathbf{0} \in W$
    - Each vector in W has an additive inverse in W

1.3 Subspaces

- **Subspace W**

Theorem 1.3: (Existence of the additive inverse need not be checked)
Let V be a vector space and W a subset of V.
W is a subspace of V if and only if the following 3 conditions hold for the operations defined in V:
(a) $\mathbf{0} \in W$
(b) $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$
(c) $c\mathbf{x} \in W$ whenever $c \in F$ and $\mathbf{x} \in W$

- Proof) "if" part (W is a subspace of V $\Leftarrow$ 3 conditions hold)
  - Assume (a), (b), and (c) hold true.
  - Then, from the previous slide, only the existence of the additive inverse needs to be verified.
  - From condition (c)
    - If $\mathbf{x} \in W$, then $(-1)\mathbf{x} \in W$
  - From Theorem 1.2 (b),
    - $(-1)\mathbf{x} = -\mathbf{x}$
    - $\therefore$ The additive inverse $-\mathbf{x} \in W$ exists for each $\mathbf{x} \in W$.

1.3 Subspaces

- **Subspace W**

> **Theorem 1.3**: (Existence of the additive inverse need not be checked)
> Let V be a vector space and W a subset of V.
> W is a subspace of V if and only if the following 3 conditions hold for the operations defined in V:
> (a) $\mathbf{0} \in W$
> (b) $\mathbf{x} + \mathbf{y} \in W$ whenever $\mathbf{x} \in W$ and $\mathbf{y} \in W$
> (c) $c\mathbf{x} \in W$ whenever $c \in F$ and $\mathbf{x} \in W$

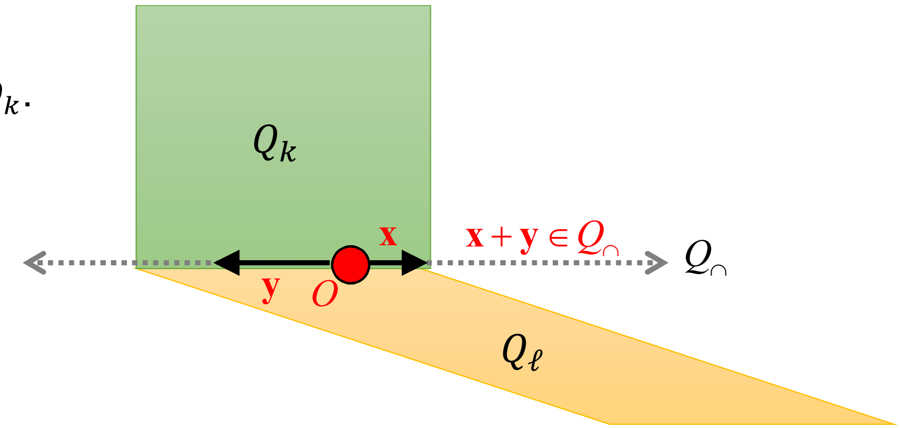
- Proof) "only if" part (W is a subspace of $V \Rightarrow$ 3 conditions hold)
  - Assume W is a subspace of V.
    - A vector space over F with operations of vector addition and scalar multiplication defined on V
  - Then, (b) and (c) automatically hold true.
  - Also, there must exist $\mathbf{z} \in W$ such that $\mathbf{x} + \boxed{\mathbf{z}} = \mathbf{x}$ for $\mathbf{x} \in W$
  - Meanwhile, since $\mathbf{x} \in V$ as well, we have $\mathbf{x} + \boxed{\mathbf{0}} = \mathbf{x}$ where $\mathbf{0} \in V$ is the zero vector of V.
  - From Theorem 1.1,
    - $\mathbf{z} = \mathbf{0}$, and (a) holds true.

1.3 Subspaces

- **Subspace W**

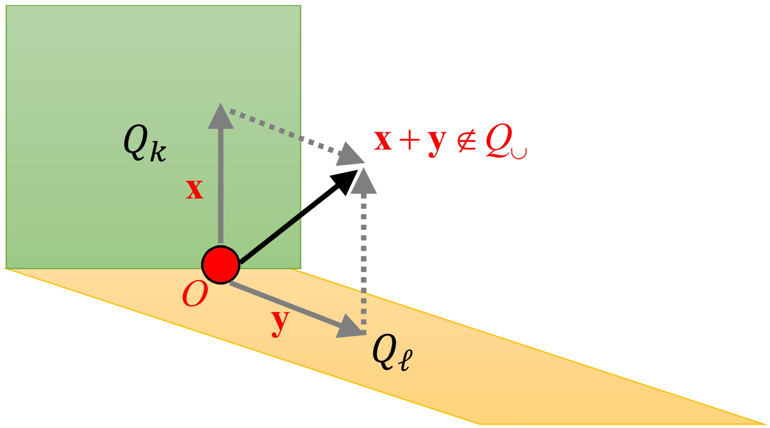
> **Theorem 1.4**:
> Any intersection of subspaces of a vector space V is a subspace of V.

- Proof)
  - Let $Q_{\cap} = \bigcap\{Q_1, \dots, Q_n\}$ be the intersection of subspaces $Q_1, \dots, Q_n$ of V.
  - Since every subspace contains the zero vector, we have $\mathbf{0} \in Q_{\cap}$.
    - (Theorem 1.3(a))
  - Let $a \in F$, $\mathbf{x} \in Q_k$, $\mathbf{y} \in Q_\ell$ and $\mathbf{x}, \mathbf{y} \in Q_{\cap}$.
  - Since $\mathbf{x}, \mathbf{y} \in Q_{\cap}$ it is also true that $\mathbf{x} \in Q_\ell$, $\mathbf{y} \in Q_k$.
  - Then, $\mathbf{x} + \mathbf{y} \in Q_{\cap}$ and $a\mathbf{x} \in Q_{\cap}$ (or $a\mathbf{y} \in Q_{\cap}$) because Q_k and Q_ℓ are subspaces where $\mathbf{x}$ and $\mathbf{y}$ simultaneously belong to.
    - (Theorem 1.3(b) and (c))
  - $\therefore$ Subspace!



1.3 Subspaces

- **Subspace W**
  - Any union of subspaces of a vector space V is not a subspace of V.
  - Proof)
    - Let $Q_\cup = \cup\{Q_1, \ldots, Q_n\}$ be the union of subspaces $Q_1, \ldots, Q_n$ of V.
    - Since every subspace contains the zero vector, we have $\mathbf{0} \in Q_\cup$.
      - (Theorem 1.3(a))
    - Let $a \in F$, $\mathbf{x} \in Q_k$, $\mathbf{y} \in Q_\ell$
    - Then, it is not guaranteed that $\mathbf{x} + \mathbf{y} \in Q_\cup$
      - Possibly in another subspace in V
    - $\therefore$ Not a subspace



1.3 Subspaces

- **Transpose, $\mathbf{A}^T$**
  - Obtained by interchanging the rows with the columns
    - $[\mathbf{A}^T]_{k\ell} = [\mathbf{A}]_{\ell k}$
  - The transpose of an $m \times n$ matrix $\mathbf{A} \Rightarrow$ An $n \times m$ matrix
  - e.g.)

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

1.3 Subspaces

- **Types of matrices**
  - Symmetric matrix
    - $\mathbf{A}^T = \mathbf{A}$

    - Square matrix

    - The set W of all symmetric matrices = A subspace of $M_{n\times n}(F)$?
      - Theorem 1.3(a)
        - Zero matrix $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in W$
      - Theorem 1.3(b): closure under addition
        - $\mathbf{A} + \mathbf{B} \in W$ since $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T = \mathbf{A} + \mathbf{B}$ for $\mathbf{A}, \mathbf{B} \in W$
      - Theorem 1.3(c): closure under scalar multiplication
        - $a\mathbf{A} \in W$ since $(a\mathbf{A})^T = a\mathbf{A}^T = a\mathbf{A}$ for $\mathbf{A} \in W$
      - $\therefore$ Subspace!

1.3 Subspaces

- **Types of matrices**
  - Upper triangular matrix
    - $[\mathbf{A}]_{k\ell} = 0$ for $k > \ell$
    - e.g.)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{bmatrix}$$

  - Diagonal matrix
    - $[\mathbf{A}]_{k\ell} = 0$ for $k \neq \ell$
    - e.g.)

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

1.3 Subspaces

- **Types of matrices**
  - *Example 1.3.3*
    - The set W of all diagonal matrices = A subspace of $M_{n\times n}(F)$?
      - Theorem 1.3(a)
        - Zero matrix $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in W$
      - Theorem 1.3(b): closure under addition
        - $\mathbf{A}+\mathbf{B} \in W$ since $[\mathbf{A}+\mathbf{B}]_{k\ell} = 0$ for $k \neq \ell$ for $\mathbf{A}, \mathbf{B} \in W$
      - Theorem 1.3(c): closure under scalar multiplication
        - $a\mathbf{A} \in W$ since $[a\mathbf{A}]_{k\ell} = 0$ for $k \neq \ell$ for $\mathbf{A} \in W$
      - $\therefore$ Subspace!

1.3 Subspaces

- **Types of matrices**
  - *Example 1.3.5*
    - The set W of $M_{m\times n}(\mathbb{R})$ matrices with nonnegative entries
      - Theorem 1.3(a)
        - Zero matrix $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in W$
      - Theorem 1.3(b): closure under addition
        - $\mathbf{A}+\mathbf{B} \in W$ since $[\mathbf{A}+\mathbf{B}]_{k\ell} \geq 0$ for all k, ℓ for $\mathbf{A}, \mathbf{B} \in W$
      - Theorem 1.3(c): closure under scalar multiplication
        - $a\mathbf{A} \notin W$ since $[a\mathbf{A}]_{k\ell} < 0$ for $a < 0$ for $\mathbf{A} \in W$
        - $\therefore$ Not a subspace

1.3 Subspaces

- **Trace,** $\mathrm{tr}(\mathbf{A})$
  - Obtained by summing the diagonal entries of an $n \times n$ square matrix
    - $\mathrm{tr}(\mathbf{A}) = [\mathbf{A}]_{11} + [\mathbf{A}]_{22} + \cdots + [\mathbf{A}]_{nn}$

1.4 Linear combinations and systems of linear equations

1.4 Linear combinations and systems of linear equations

- **Linear combination**

Linear combination:
Let V be a vector space and S a non-empty subset of V.
A vector $\mathbf{v} \in V$ is called a linear combination of vectors of S if there exist a finite number of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ in S and scalar $a_1, a_2, \ldots, a_n$ in F such that $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$.

$a_1, a_2, \ldots, a_n$: The coefficients of the linear combination

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1.4 Linear combinations and systems of linear equations

- **Linear combination**
  - *Example 1.4.1*
    - Each row showing vitamin content
      - e.g.) Apple butter

$$\begin{bmatrix} 0.00 \\ 0.01 \\ 0.02 \\ 0.20 \\ 2.00 \end{bmatrix}$$

- Represented in $\mathbb{R}^5$
    - Raw wild rice as a linear combination

$$\begin{bmatrix} 0.00 \\ 0.05 \\ 0.06 \\ 0.30 \\ 0.00 \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.02 \\ 0.02 \\ 0.40 \\ 0.00 \end{bmatrix} + \begin{bmatrix} 0.00 \\ 0.34 \\ 0.05 \\ 4.70 \\ 0.00 \end{bmatrix} + 2\begin{bmatrix} 0.00 \\ 0.02 \\ 0.25 \\ 0.40 \\ 0.00 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 0.45 \\ 0.63 \\ 6.20 \\ 0.00 \end{bmatrix}$$

TABLE 1.1 Vitamin Content of 100 Grams of Certain Foods

	A (units)	B_1 (mg)	B_2 (mg)	Niacin (mg)	C (mg)
Apple butter	0	0.01	0.02	0.2	2
Raw, unpared apples (freshly harvested)	90	0.03	0.02	0.1	4
Chocolate-coated candy with coconut center	0	0.02	0.07	0.2	0
Clams (meat only)	100	0.10	0.18	1.3	10
Cupcake from mix (dry form)	0	0.05	0.06	0.3	0
Cooked farina (unenriched)	(0)[a]	0.01	0.01	0.1	(0)
Jams and preserves	10	0.01	0.03	0.2	2
Coconut custard pie (baked from mix)	0	0.02	0.02	0.4	0
Raw brown rice	(0)	0.34	0.05	4.7	(0)
Soy sauce	0	0.02	0.25	0.4	0
Cooked spaghetti (unenriched)	0	0.01	0.01	0.3	0
Raw wild rice	(0)	0.45	0.63	6.2	(0)

Source: Bernice K. Watt and Annabel L. Merrill, *Composition of Foods* (Agriculture Handbook Number 8), Consumer and Food Economics Research Division, U.S. Department of Agriculture, Washington, D.C., 1963.

[a]Zeros in parentheses indicate that the amount of a vitamin present is either none or too small to measure.

1.4 Linear combinations and systems of linear equations

- **Linear combination**
  - *Example 1.4.1*
    - Clams as a linear combination

$$2\begin{bmatrix}0.00\\0.01\\0.02\\0.20\\2.00\end{bmatrix}+\begin{bmatrix}90.00\\0.03\\0.02\\0.10\\4.00\end{bmatrix}+\begin{bmatrix}0.00\\0.02\\0.07\\0.20\\0.00\end{bmatrix}+\begin{bmatrix}0.00\\0.01\\0.01\\0.10\\0.00\end{bmatrix}+\begin{bmatrix}10.00\\0.01\\0.03\\0.20\\2.00\end{bmatrix}+\begin{bmatrix}0.00\\0.01\\0.01\\0.30\\0.00\end{bmatrix}=\begin{bmatrix}100.00\\0.10\\0.18\\1.30\\10.00\end{bmatrix}$$

TABLE 1.1 Vitamin Content of 100 Grams of Certain Foods

	A (units)	B_1 (mg)	B_2 (mg)	Niacin (mg)	C (mg)
Apple butter	0	0.01	0.02	0.2	2
Raw, unpared apples (freshly harvested)	90	0.03	0.02	0.1	4
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Clams (meat only)	100	0.10	0.18	1.3	10
Cupcake from mix (dry form)	0	0.05	0.06	0.3	0
Cooked farina (unenriched)	(0)[a]	0.01	0.01	0.1	(0)
Jams and preserves	10	0.01	0.03	0.2	2
Coconut custard pie (baked from mix)	0	0.02	0.02	0.4	0
Raw brown rice	(0)	0.34	0.05	4.7	(0)
Soy sauce	0	0.02	0.25	0.4	0
Cooked spaghetti (unenriched)	0	0.01	0.01	0.3	0
Raw wild rice	(0)	0.45	0.63	6.2	(0)

Source: Bernice K. Watt and Annabel L. Merrill, *Composition of Foods* (Agriculture Handbook Number 8), Consumer and Food Economics Research Division, U.S. Department of Agriculture, Washington, D.C., 1963.

[a]Zeros in parentheses indicate that the amount of a vitamin present is either none or too small to measure.

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - Necessary to determine whether a vector can be expressed as a linear combination
    - (A general method in Chapter 03)
  - e.g.) $\begin{bmatrix}2\\6\\8\end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-2\\-4\\-2\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}0\\2\\3\end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix}2\\0\\-3\end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix}-3\\8\\16\end{bmatrix}$
    - Coefficients to be determined: a_1, a_2, a_3, a_4, a_5

$$\begin{bmatrix}2\\6\\8\end{bmatrix} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + a_4\mathbf{u}_4 + a_5\mathbf{u}_5$$

$$\Rightarrow \quad \begin{array}{cccccccc} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ 2a_1 & -4a_2 & +2a_3 & & +8a_5 & = & 6 \\ a_1 & -2a_2 & +3a_3 & -3a_4 & +16a_5 & = & 8 \end{array}$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - Necessary to determine whether a vector can be expressed as a linear combination
    - (A general method in Chapter 03)
  - e.g.) $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} -3 \\ 8 \\ 16 \end{bmatrix}$
    - Coefficients to be determined: a_1, a_2, a_3, a_4, a_5

$$\Rightarrow \begin{array}{ccccccc} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & 2a_3 & -4a_4 & +14a_5 & = & 2 \\ & & 3a_3 & -5a_4 & +19a_5 & = & 6 \end{array} \Rightarrow \begin{array}{ccccccc} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & a_3 & -2a_4 & +7a_5 & = & 1 \\ & & 3a_3 & -5a_4 & +19a_5 & = & 6 \end{array}$$

(Row2) ← (Row2)-2×(Row1)
(Row3) ← (Row3)-(Row1)

(Row2) ← (Row2)÷2

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - Necessary to determine whether a vector can be expressed as a linear combination
    - (A general method in Chapter 03)
  - e.g.) $\begin{bmatrix}2\\6\\8\end{bmatrix}$ as a linear combination of $\mathbf{u}_1=\begin{bmatrix}1\\2\\1\end{bmatrix}, \mathbf{u}_2=\begin{bmatrix}-2\\-4\\-2\end{bmatrix}, \mathbf{u}_3=\begin{bmatrix}0\\2\\3\end{bmatrix}, \mathbf{u}_4=\begin{bmatrix}2\\0\\-3\end{bmatrix}, \mathbf{u}_5=\begin{bmatrix}-3\\8\\16\end{bmatrix}$
    - Coefficients to be determined: a_1, a_2, a_3, a_4, a_5

$$\Rightarrow \begin{array}{ccccccc} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & a_3 & -2a_4 & +7a_5 & = & 1 \\ & & & a_4 & -2a_5 & = & 3 \end{array} \Rightarrow \begin{array}{cccccc} a_1 & -2a_2 & & & +a_5 & = & -4 \\ & & a_3 & & +3a_5 & = & 7 \\ & & & a_4 & -2a_5 & = & 3 \end{array}$$

(Row3) ← (Row3)-3×(Row2)

(Row1) ← (Row1)-2×(Row3)
(Row2) ← (Row2)+2×(Row3)

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - Necessary to determine whether a vector can be expressed as a linear combination
    - (A general method in Chapter 03)
  - e.g.) $\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ -4 \\ -2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} -3 \\ 8 \\ 16 \end{bmatrix}$
    - For any a_2, a_5,

$$a_1 = 2a_2 - a_5 - 4$$
$$a_2 = a_2$$
$$a_3 = -3a_5 + 7$$
$$a_4 = 2a_5 + 3$$
$$a_5 = a_5$$

    - For instance, if $a_2 = 0, a_5 = 0$,

$$\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} = -4\mathbf{u}_1 + 0\mathbf{u}_2 + 7\mathbf{u}_3 + 3\mathbf{u}_4 + 0\mathbf{u}_5$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - 3 types of operations to simplify the original system
    - ① Interchanging the order of any two equations in the system
      - e.g.)

$$\begin{array}{rrrrrrr} & & a_3 & -2a_4 & +7a_5 & = & 1 \\ a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & & a_4 & -2a_5 & = & 3 \end{array} \quad \xRightarrow{\text{(Row1)} \leftrightarrows \text{(Row2)}} \quad \begin{array}{rrrrrrr} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & a_3 & -2a_4 & +7a_5 & = & 1 \\ & & & a_4 & -2a_5 & = & 3 \end{array}$$

    - ② Multiplying any equation in the system by a non-zero constant
      - e.g.)

$$\begin{array}{rrrrrrr} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & 2a_3 & -4a_4 & +14a_5 & = & 2 \\ & & 3a_3 & -5a_4 & +19a_5 & = & 6 \end{array} \quad \xRightarrow{\text{(Row2)} \leftarrow 0.5\times\text{(Row2)}} \quad \begin{array}{rrrrrrr} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & a_3 & -2a_4 & +7a_5 & = & 1 \\ & & 3a_3 & -5a_4 & +19a_5 & = & 6 \end{array}$$

    - ③ Adding a constant multiple of any equation to another equation in the system
      - e.g.)

$$\begin{array}{rrrrrrr} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ & & a_3 & -2a_4 & +7a_5 & = & 1 \\ & & & a_4 & -2a_5 & = & 3 \end{array} \quad \Rightarrow \quad \begin{array}{rrrrrr} a_1 & -2a_2 & & +a_5 & = & -4 \\ & & a_3 & +3a_5 & = & 7 \\ & & a_4 & -2a_5 & = & 3 \end{array}$$

(Row1) ← (Row1)-2×(Row3)
(Row2) ← (Row2)+2×(Row3)

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - Properties for the final simplified system to have
    - ① The first non-zero coefficient in each equation equal to 1
    - ② If an unknown is the first unknown with a non-zero coefficient in some equation, then that unknown occurring with a 0 coefficient in all the other equations
    - ③ The first unknown with a non-zero coefficient in any equation having a larger subscript than the first unknown with a non-zero coefficient in preceding equations

$$
\begin{array}{cccccccc}
a_1 & -2a_2 & & & +a_5 & = & -4 \\
 & & a_3 & & +3a_5 & = & 7 \\
 & & & a_4 & -2a_5 & = & 3
\end{array}
$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - *Example 1.4.2*

$$\begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

  - Coefficients to be determined: a_1, a_2

$$\begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rrcr} a_1 & +3a_2 & = & 2 \\ -2a_1 & -5a_2 & = & -2 \\ -5a_1 & -4a_2 & = & 12 \\ -3a_1 & -9a_2 & = & -6 \end{array}$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - *Example 1.4.2*

$$\begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

- Coefficients to be determined: a_1, a_2

$$\Rightarrow \begin{array}{cccc} a_1 & +3a_2 & = & 2 \\ & a_2 & = & 2 \\ & 11a_2 & = & 22 \\ & 0 & = & 0 \end{array} \quad \Rightarrow \begin{array}{cccc} a_1 & +3a_2 & = & 2 \\ & a_2 & = & 2 \\ & a_2 & = & 2 \\ & 0 & = & 0 \end{array}$$

(Row2) ← (Row2)+2×(Row1)
(Row3) ← (Row3)+5×(Row1)
(Row4) ← (Row4)+3×(Row1)

(Row3) ← (Row3)÷11

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - *Example 1.4.2*

$$\begin{bmatrix} 2 \\ -2 \\ 12 \\ -6 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

    - Coefficients to be determined: a_1, a_2

$$\Rightarrow \begin{aligned} a_1 \quad &= -4 \\ 0 &= 0 \\ a_2 &= 2 \\ 0 &= 0 \end{aligned}$$

(Row1) ← (Row1)-3×(Row3)
(Row2) ← (Row2)-(Row3)

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - *Example 1.4.2*

$$\begin{bmatrix} 3 \\ -2 \\ 7 \\ 8 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

- Coefficients to be determined: a_1, a_2

$$\begin{bmatrix} 3 \\ -2 \\ 7 \\ 8 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rrcr} a_1 & +3a_2 & = & 3 \\ -2a_1 & -5a_2 & = & -2 \\ -5a_1 & -4a_2 & = & 7 \\ -3a_1 & -9a_2 & = & 8 \end{array}$$

1.4 Linear combinations and systems of linear equations

- **Systems of linear equations**
  - *Example 1.4.2*

$$\begin{bmatrix} 3 \\ -2 \\ 7 \\ 8 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 1 \\ -2 \\ -5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -5 \\ -4 \\ -9 \end{bmatrix}$$

- Coefficients to be determined: a_1, a_2

$$\Rightarrow \begin{array}{cccc} a_1 & +3a_2 & = & 3 \\ & a_2 & = & 4 \\ & 11a_2 & = & 22 \\ & \boxed{0} & \boxed{=} & \boxed{17} \end{array}$$

Indicating no solution!

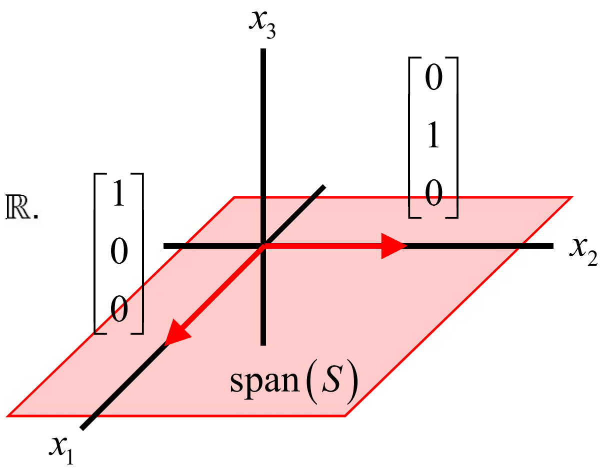
(Row2) ← (Row2)+2×(Row1)
(Row3) ← (Row3)+5×(Row1)
(Row4) ← (Row4)+3×(Row1)

1.4 Linear combinations and systems of linear equations

- **Span**

> **Span**:
> Let S be a nonempty subset of a vector space V.
> The span of S, denoted $\mathrm{span}(S)$, is the set consisting of all linear combinations of the vectors in S.
> For convenience, we define $\mathrm{span}(\emptyset) = \{\mathbf{0}\}$.

- e.g.) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $V = \mathbb{R}^3$
  - $\mathrm{span}(S)$ consisting all vectors $a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for some real numbers $a, b \in \mathbb{R}$.
  - $\mathrm{span}(S)$ = A subspace of $V = \mathbb{R}^3$



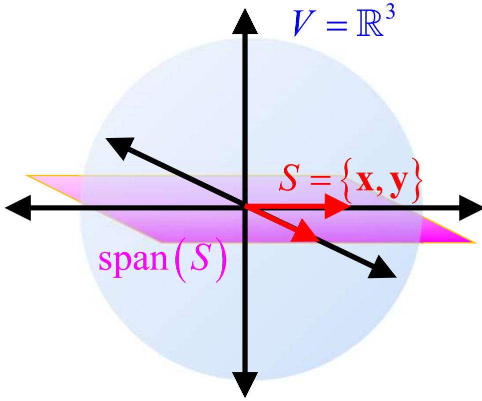
1.4 Linear combinations and systems of linear equations

- **Span**

> **Theorem 1.5**:
> The span of any subset S of a vector space V is a subspace of V containing S. Moreover, any subspace of V that contains S must also contain the span of S.

- Proof) $\mathrm{span}(S)$ = A subspace of V that contains S
  - If $S = \emptyset$
    - $\mathrm{span}(S) = \{\mathbf{0}\}$ is a subspace of V.
    - $\mathrm{span}(S) = \{\mathbf{0}\}$ contains $S = \emptyset$.

    - $\therefore$ $\mathrm{span}(S)$ is a subspace that contains S for $S = \emptyset$!

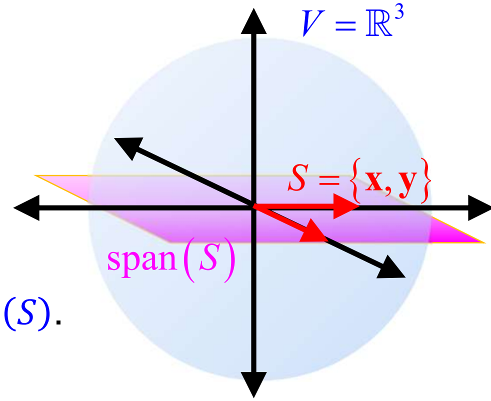


1.4 Linear combinations and systems of linear equations

- **Span**

Theorem 1.5:
The span of any subset S of a vector space V is a subspace of V containing S. Moreover, any subspace of V that contains S must also contain the span of S.

- Proof) $\mathrm{span}(S)$ = A subspace of V that contains S
  - If $S \neq \emptyset$
    - S containing a vector $\mathbf{z}$
    - Theorem 1.3(a)
      - Zero vector $0\mathbf{z} = \mathbf{0} \in \mathrm{span}(S)$
    - Theorem 1.3(b): closure under addition
      - Let $\mathbf{x}, \mathbf{y} \in \mathrm{span}(S)$.
      - Then, $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m$ and $\mathbf{y} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$ for some $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n \in S$ and $a_1, \ldots, a_m, b_1, \ldots, b_n \in F$.
      - Thus, $\mathbf{x} + \mathbf{y} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m + b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n \in \mathrm{span}(S)$.

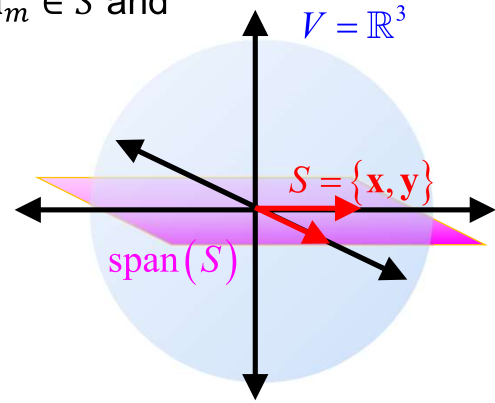


1.4 Linear combinations and systems of linear equations

- **Span**

> **Theorem 1.5**:
> The span of any subset S of a vector space V is a subspace of V containing S. Moreover, any subspace of V that contains S must also contain the span of S.

- Proof) $\mathrm{span}(S)$ = A subspace of V that contains S
  - If $S \neq \emptyset$
    - Theorem 1.3(c): closure under scalar multiplication
      - Let $\mathbf{x} \in \mathrm{span}(S)$ such that $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m$ for some $\mathbf{u}_1, \dots, \mathbf{u}_m \in S$ and $a_1, \dots, a_m \in F$.
      - Then, $c\mathbf{x} = (ca_1)\mathbf{u}_1 + (ca_2)\mathbf{u}_2 + \cdots + (ca_m)\mathbf{u}_m \in \mathrm{span}(S)$.
    - S contained in $\mathrm{span}(S)$
      - If $\mathbf{v} \in S$, it is also $\mathbf{v} \in \mathrm{span}(S)$ since $\mathbf{v} = 1\mathbf{v}$ (linear combination).
      - Since it is true for all arbitrary $\mathbf{v} \in S$, we have $S \subseteq \mathrm{span}(S)$.



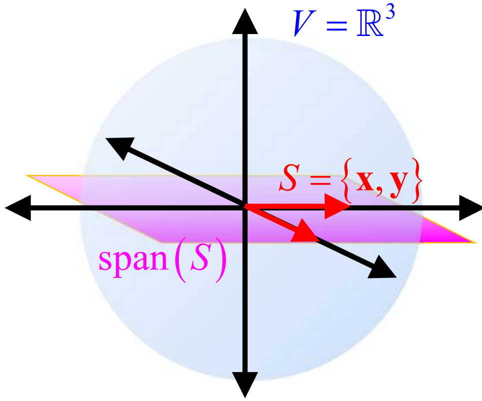
    - $\therefore$ $\mathrm{span}(S)$ is a subspace that contains S for $S \neq \emptyset$!

1.4 Linear combinations and systems of linear equations

- **Span**

Theorem 1.5:
The span of any subset S of a vector space V is a subspace of V containing S. Moreover, any subspace of V that contains S must also contain the span of S.

- Proof) $\mathrm{span}(S) \subseteq$ A subspace of V that contains S
  - Let W be a subspace of V that contains S.
  - Let $\mathbf{x} \in \mathrm{span}(S)$.
  - Then $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_m\mathbf{u}_m$ for some $\mathbf{u}_1, \ldots, \mathbf{u}_m \in S$ and $a_1, \ldots, a_m \in F$.
  - Also, since $S \subseteq W$, it is true that $\mathbf{u}_1, \ldots, \mathbf{u}_m \in W$.
  - Thus, $\mathbf{x} \in W$.
  - Since it is true for all arbitrary $\mathbf{x} \in \mathrm{span}(S)$, we have $\mathrm{span}(S) \subseteq W$.



1.4 Linear combinations and systems of linear equations

- **Span**

> **Spanning (or generating)**:
> A subset S of a vector space V spans (or generates) V if $\operatorname{span}(S) = V$.
> In this case, we also say that the vectors of S span (or generate) V.

- *Example 1.4.3*
  - Vectors $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ spanning (or generating) $V = \mathbb{R}^3$

- *Example 1.4.5*
  - Matrices $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$ spanning (or generating) $V = M_{2\times 2}(\mathbb{R})$
  - Matrices $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ not spanning (or generating) $V = M_{2\times 2}(\mathbb{R})$
    - Not every 2×2 matrix as a linear combination of these 3 matrices

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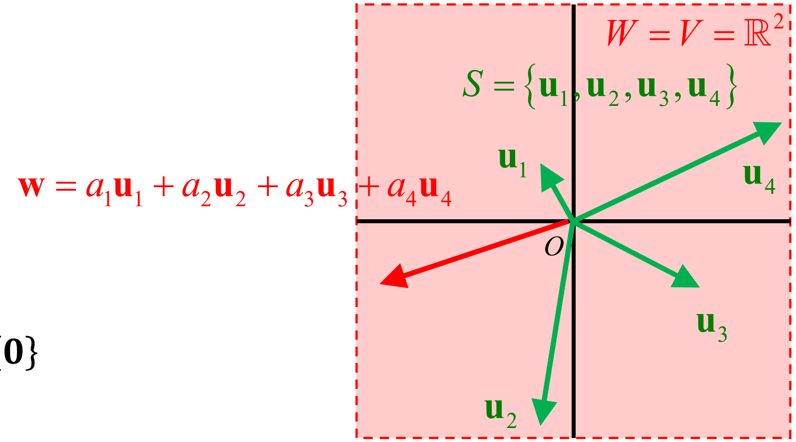
1.5 Linear dependence and linear independence

1.5 Linear dependence and linear independence

- **A finite subset S spanning a subspace W**
  - Supposing
    - V: A vector space over an infinite field F
    - W: A subspace of V
  - Then,
    - W an infinite set unless W is the zero subspace, $\{\mathbf{0}\}$
  - Desirable to find a "small" finite subset S of W that spans W
    - Able to describe each vector in W as a linear combination of the finite number of vectors in S
    - Smaller S $\Rightarrow$ Fewer number of computations required to represent vectors in W



Not a "small" subset S to span W

1.5 Linear dependence and linear independence

- **A finite subset S spanning a subspace W**
  - e.g.) Subspace W of $\mathbb{R}^3$ spanned by $S = \left\{\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}\right\}$
    - Q: Is it a "minimal" subset of S that also spans W?
      - A just enough number of vectors to span W
      - No need to have a vector that is a linear combination of the others in S
    - Checking whether $\mathbf{u}_4$ is a linear combination of the others:

$$\underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4} = a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_3}$$

      - No solution! $\Rightarrow$ Not a linear combination of the others

1.5 Linear dependence and linear independence

- **A finite subset S spanning a subspace W**
  - e.g.) Subspace W of $\mathbb{R}^3$ spanned by $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$
    - Q: Is it a "minimal" subset of S that also spans W?
      - A just enough number of vectors to span W
      - No need to have a vector that is a linear combination of the others in S
    - Checking whether $\mathbf{u}_3$ is a linear combination of the others:

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_3} = a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_4 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4}$$

      - Solution $a_1 = 2, a_2 = -3, a_4 = 0$
      - $\therefore$ The current set S having redundant vectors for spanning W

1.5 Linear dependence and linear independence

- **A finite subset S spanning a subspace W**
  - e.g.) Subspace W of $\mathbb{R}^3$ spanned by $S = \left\{\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}\right\}$
    - Q: Is it a "minimal" subset of S that also spans W?
      - A just enough number of vectors to span W
      - No need to have a vector that is a linear combination of the others in S
    - Writing differently,

$$a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_3} + a_4 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4} = \mathbf{0}$$

      - Solution $a_1 = -2, a_2 = 3, a_3 = 1, a_4 = 0$

Not "small" enough subset S for spanning subspace W

=

Some vectors being a linear combination of the other vectors in S

=

Non-zero solution to yield the zero vector $\mathbf{0}$ by a linear combination

1.5 Linear dependence and linear independence

- **Linear dependence**

> **Linear dependence**:
> A subset S of a vector space V is called linearly dependent if there exist a finite number of distinct vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ in S and scalars $a_1, a_2, \dots, a_n$, not all zero, such that
> $$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \mathbf{0}$$

- Trivial representation
  - $a_1 = a_2 = \cdots = a_n = 0$

- Required to have a nontrivial representation for linear dependence
  - At least one coefficient being non-zero

- Any subset containing the zero vector $\mathbf{0} \Rightarrow$ Linearly dependent subset
  - E.g.) A linear combination of itself $\mathbf{0} = 1 \cdot \mathbf{0}$

1.5 Linear dependence and linear independence

- **Linear dependence**
  - *Example 1.5.1*
    - Considering

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

    - Linearly dependent since for $a_1 = 4, a_2 = -3, a_3 = 2, a_4 = 0$

$$a_1 \begin{bmatrix} 1 \\ 3 \\ -4 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 2 \\ -4 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix} + a_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{array}{ccccccc} a_1 & +2a_2 & +a_3 & -a_4 & = & 0 \\ 3a_1 & +2a_2 & -3a_3 & & = & 0 \\ -4a_1 & -4a_2 & +2a_3 & +a_4 & = & 0 \\ 2a_1 & & -4a_3 & & = & 0 \end{array}$$

      - i.e., non-zero solution existing for the zero vector

1.5 Linear dependence and linear independence

- **Linear dependence**
  - *Example 1.5.2*
    - Considering

$$S = \left\{ \begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix}, \begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} \right\}$$

    - Linearly dependent since for $a_1 = 5, a_2 = 3, a_3 = -2$

$$a_1 \begin{bmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{bmatrix} + a_2 \begin{bmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{bmatrix} + a_3 \begin{bmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{bmatrix} = [\mathbf{0}]$$

      - i.e., non-zero solution existing for the zero matrix

1.5 Linear dependence and linear independence

- **Linear independence**

> **Linear independence**:
> A subset S of a vector space V is called linearly independent if there does not exist a finite number of distinct vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ in S and scalars $a_1, a_2, \dots, a_n$, not all zero, such that
> $$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \mathbf{0}$$

- Facts about linear independence
  - ① The empty set ⇒ Linearly independent
    - The linearly dependence required to be non-empty
  - ② A set consisting of a single non-zero vector ⇒ Linearly independent
  - ③ Linearly independent if and only if the only representation of the zero vector $\mathbf{0}$ is the trivial representation

1.5 Linear dependence and linear independence

- **Linear independence**
  - *Example 1.5.3*
    - Considering

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

    - Linearly independent since only $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$ is the solution

$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{array}{ccccc} a_1 & & & & = 0 \\ & a_2 & & & = 0 \\ & & a_3 & & = 0 \\ -a_1 & -a_2 & -a_3 & +a_4 & = 0 \end{array}$$

1.5 Linear dependence and linear independence

- **Linear independence**

Theorem 1.6:
Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

Corollary:
Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

1.5 Linear dependence and linear independence

- **A finite subset S spanning a subspace W (revisited)**
  - e.g.) Subspace W of $\mathbb{R}^3$ spanned by $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$
    - Q: Is it a "minimal" subset of S that also spans W?
      - A just enough number of vectors to span W
      - No need to have a vector that is a linear combination of the others in S
      - ✚ Linearly independent
    - Recalling $\mathbf{u}_3$ was a linear combination of the other vectors since for $a_1 = -2, a_2 = 3, a_3 = 1, a_4 = 0$

$$a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_3 \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{u}_3} + a_4 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4} = \mathbf{0}$$

      - $\therefore$ $\mathbf{u}_3$ being a redundant vector in set S for spanning W
      - $\Rightarrow$ Set S being linearly dependent

1.5 Linear dependence and linear independence

- **A finite subset S spanning a subspace W (revisited)**
  - e.g.) Subspace W of $\mathbb{R}^3$ spanned by $S = \left\{ \mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \cancel{\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$
    - Q: Is it a "minimal" subset of S that also spans W?
      - A just enough number of vectors to span W
      - No need to have a vector that is a linear combination of the others in S
      - Linearly independent
    - By removing the redundant $\mathbf{u}_3$ from S

$$a_1 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}}_{\mathbf{u}_1} + a_2 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}_2} + a_4 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}}_{\mathbf{u}_4} = \mathbf{0}$$

$$\Rightarrow \begin{array}{rrrcl} 2a_1 & +a_2 & +a_4 & = & 0 \\ -a_1 & -a_2 & -2a_4 & = & 0 \\ 4a_1 & +3a_2 & -a_4 & = & 0 \end{array} \quad \Rightarrow \begin{array}{rrrcl} 2a_1 & +a_2 & +a_4 & = & 0 \\ & -a_2 & -3a_4 & = & 0 \\ & +a_2 & -3a_4 & = & 0 \end{array} \quad \Rightarrow \begin{array}{rrrcl} 2a_1 & +a_2 & +a_4 & = & 0 \\ & -a_2 & -3a_4 & = & 0 \\ & & -6a_4 & = & 0 \end{array}$$

  - The only solution to the system: $a_1 = a_2 = a_4 = 0$
  - $\therefore$ Linearly independent

1.5 Linear dependence and linear independence

- **A finite subset S spanning a subspace W (revisited)**

> **Theorem 1.7**:
> Let S be a linearly independent subset of a vector space V, and let $\mathbf{v}$ be a vector in V that is not in S.
> Then, $S \cup \{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \mathrm{span}(S)$

- Proof) $S \cup \{\mathbf{v}\}$ linearly dependent $\Longrightarrow \mathbf{v} \in \mathrm{span}(S)$
  - $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is linearly independent while $S \cup \{\mathbf{v}\}$ is linearly dependent.
    - $\Rightarrow \mathbf{v}$ is a redundant vector
    - $\Rightarrow \mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$ in which not every coefficient is zero.
  - Note that $\mathrm{span}(S) = \{a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n | a_1, \dots, a_n \in F\}$.
    - $\therefore \mathbf{v} \in \mathrm{span}(S)$

1.5 Linear dependence and linear independence

- **A finite subset S spanning a subspace W (revisited)**

> **Theorem 1.7**:
> Let S be a linearly independent subset of a vector space V, and let $\mathbf{v}$ be a vector in V that is not in S.
> Then, $S \cup \{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \mathrm{span}(S)$

- Proof) $S \cup \{\mathbf{v}\}$ linearly dependent $\Leftarrow$ $\mathbf{v} \in \mathrm{span}(S)$
  - $\mathrm{span}(S) = \{a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n | a_1, \ldots, a_n \in F\}$ for $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$
    - $\Rightarrow \mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$
    - $\Rightarrow$ $\mathbf{v}$ is a redundant vector for the set S.
    - $\therefore S \cup \{\mathbf{v}\} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{v}\}$ is linearly dependent.

1.6 Bases and dimension

1.6 Bases and dimension

- **Bases**

Basis:
A basis β for a vector space V is a linearly independent subset of V that spans V.

- *Example 1.6.1*
  - $\emptyset$ being a basis for the zero vector space $\{\mathbf{0}\}$

- *Example 1.6.2*
  - The standard basis for n-dimensional field F^n:

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

1.6 Bases and dimension

- **Bases**

> **Basis**:
> A basis β for a vector space V is a linearly independent subset of V that spans V.

- *Example 1.6.3*
  - $\{E^{ij} | 1 \le i \le m, 1 \le j \le n\}$ being a basis for $M_{m \times n}(F)$

$$E^{ij} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \boxed{1} & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

row i, column j

- Note: Not every vector space having a finite basis

1.6 Bases and dimension

- **Bases**

Theorem 1.8:
Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be distinct vectors in V.
Then, $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V if and only if each $\mathbf{v} \in V$ can be uniquely expressed as a linear combination of vectors of β as

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$$

with unique scalars $a_1, a_2, \dots, a_n$.

- Proof) $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for $V \Rightarrow$ each $\mathbf{v} \in V$ can be uniquely expressed
  - Let β be a basis for V.
    - $\Rightarrow \mathrm{span}(\beta) = V$
    - $\Rightarrow \mathbf{v} \in \mathrm{span}(\beta)$
  - By contradiction, assume $\mathbf{v} \in V$ is not uniquely expressed.
    - $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$
    - $\mathbf{v} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_n\mathbf{u}_n$
    - Here, there exist some i's such that $a_i \neq b_i$

1.6 Bases and dimension

- **Bases**

Theorem 1.8:
Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be distinct vectors in V.
Then, $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V if and only if each $\mathbf{v} \in V$ can be uniquely expressed as a linear combination of vectors of β as

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$$

with unique scalars $a_1, a_2, \dots, a_n$.

- Proof) $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for $V \Rightarrow$ each $\mathbf{v} \in V$ can be uniquely expressed
  - By subtracting one from the other,
    - $\mathbf{0} = (a_1 - b_1)\mathbf{u}_1 + (a_2 - b_2)\mathbf{u}_2 + \cdots + (a_n - b_n)\mathbf{u}_n$
  - Since $a_i \neq b_i$ for some i's, this is a non-zero solution for the zero vector $\mathbf{0}$.
    - $\Rightarrow$ Contradicting the fact that $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are linearly independent
    - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **Bases**

Theorem 1.8:
Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be distinct vectors in V.
Then, $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V if and only if each $\mathbf{v} \in V$ can be uniquely expressed as a linear combination of vectors of β as

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$$

with unique scalars $a_1, a_2, \dots, a_n$.

- Proof) $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V $\Leftarrow$ each $\mathbf{v} \in V$ can be uniquely expressed
  - By contradiction, assume β is not a basis.
    - $\Rightarrow$ Linearly dependent set that spans V.
  - Then there exists a non-zero solution $b_1, b_2, \dots, b_n$ such that
    - $\mathbf{0} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_n\mathbf{u}_n$
  - Note that for any scalar c,
    - $\mathbf{0} = cb_1\mathbf{u}_1 + cb_2\mathbf{u}_2 + \cdots + cb_n\mathbf{u}_n$

1.6 Bases and dimension

- **Bases**

Theorem 1.8:
Let V be a vector space and $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be distinct vectors in V.
Then, $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V if and only if each $\mathbf{v} \in V$ can be uniquely expressed as a linear combination of vectors of β as

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$$

with unique scalars $a_1, a_2, \dots, a_n$.

- Proof) $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a basis for V $\Leftarrow$ each $\mathbf{v} \in V$ can be uniquely expressed
  - By adding $\mathbf{v}$ on both sides,
    - $\mathbf{v} = cb_1\mathbf{u}_1 + cb_2\mathbf{u}_2 + \cdots + cb_n\mathbf{u}_n + \mathbf{v} = (cb_1 + a_1)\mathbf{u}_1 + (cb_2 + a_2)\mathbf{u}_2 + \cdots + (cb_n + a_n)\mathbf{u}_n$
  - This equation holds true for any scalar c
    - $\Rightarrow$ Contradicting $\mathbf{v}$ is uniquely expressed
    - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **Bases**

> **Theorem 1.9**:
> If a vector space V is spanned by a finite set S, then some subset of S is a basis for V.
> Hence, V has a finite basis.

- Proof)
  - If $S = \emptyset$,
    - The only subset of S
      - $\emptyset$: Linearly independent
    - Note that a linear combination of no vectors is, by convention, $\mathbf{0}$.
      - $\Rightarrow \emptyset$ spans $V = \{\mathbf{0}\}$
    - $\therefore$ The subset $\emptyset$ is a basis for $V = \{\mathbf{0}\}$.

1.6 Bases and dimension

- **Bases**

Theorem 1.9:
If a vector space V is spanned by a finite set S, then some subset of S is a basis for V.
Hence, V has a finite basis.

- Proof)
  - If $S = \{\mathbf{0}\}$,
    - The subsets of S
      - $\emptyset$: Linearly independent
      - $\{\mathbf{0}\}$: Linearly dependent (can't be a basis!)
    - Note that a linear combination of no vectors is, by convention, $\mathbf{0}$.
      - $\Rightarrow \emptyset$ spans $V = \{\mathbf{0}\}$
    - $\therefore$ The subset $\emptyset$ is a basis for $V = \{\mathbf{0}\}$.

1.6 Bases and dimension

- **Bases**

Theorem 1.9:
If a vector space V is spanned by a finite set S, then some subset of S is a basis for V.
Hence, V has a finite basis.

- Proof)
  - If S is a non-empty set other than $\{\mathbf{0}\}$,
    - It is possible to find a maximal linearly independent set $\beta = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \subseteq S$.
      - By including vectors one by one and check for linearly independence for each inclusion.
    - If $\beta = S$,
      - $\text{span}(\beta) = \text{span}(S) = V$
      - $\therefore$ The subset β is a basis for $V = \text{span}(S)$.

1.6 Bases and dimension

- **Bases**

> **Theorem 1.9**:
> If a vector space V is spanned by a finite set S, then some subset of S is a basis for V.
> Hence, V has a finite basis.

- Proof)
  - If S is a non-empty set other than $\{\mathbf{0}\}$,
    - If $\beta \subset S$,
      - For any $\mathbf{v}$ such that $\mathbf{v} \in S, \mathbf{v} \notin \beta$, the union $\beta \cup \{\mathbf{v}\}$ is linearly dependent
      - By Theorem 1.7, $\mathbf{v} \in \mathrm{span}(\beta)$
      - $\Rightarrow S \subseteq \mathrm{span}(\beta)$
      - $\Rightarrow \mathrm{span}(S) \subseteq \mathrm{span}(\beta)$
      - Also, $\beta \subset S$ implies $\mathrm{span}(\beta) \subset \mathrm{span}(S)$
      - $\Rightarrow \mathrm{span}(S) \subseteq \mathrm{span}(\beta) \subset \mathrm{span}(S)$
      - $\Rightarrow \mathrm{span}(\beta) = \mathrm{span}(S) = V$
      - $\therefore$ The subset β is a basis for $V = \mathrm{span}(S)$.

1.6 Bases and dimension

- **Bases**
  - A finite spanning set for V able to be reduced to a basis for V

  - *Example 1.6.6*
    - $S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} \right\}$
    - Q1: *Does S span $V = \mathbb{R}^3$?*
      - System of linear equations for an arbitrary vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $V = \mathbb{R}^3$

$$a_1 \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + a_2 \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + a_5 \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rrrrrcl} 2a_1 & +8a_2 & +a_3 & & +7a_5 & = & x_1 \\ -3a_1 & -12a_2 & & 2a_4 & +2a_5 & = & x_2 \\ 5a_1 & +20a_2 & -2a_3 & -a_4 & & = & x_3 \end{array}$$

1.6 Bases and dimension

- **Bases**
  - A finite spanning set for V able to be reduced to a basis for V

  - *Example 1.6.6*
    - $S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} \right\}$
    - Q1: *Does S span $V = \mathbb{R}^3$?*
      - By simplifying the equations,

$$\begin{array}{lccccccc} & 2a_1 & +8a_2 & +a_3 & & +7a_5 & = & x_1 \\ \Rightarrow & & & 15a_3 & & -45a_5 & = & -2x_1 - 2x_2 - 4x_3 \\ & & & & 5a_4 & 20a_5 & = & 2x_1 + 3x_2 + x_3 \end{array}$$

      - Letting $a_2 = a_5 = 0$,

$$a_1 = \frac{1}{2}(-a_3 + x_1) = \frac{1}{2}\left(-\frac{1}{15}(-2x_1 - 2x_2 - 4x_3) + x_1\right) = \frac{17}{30}x_1 + \frac{1}{15}x_2 + \frac{2}{15}x_3$$

$$a_3 = \frac{1}{15}(-2x_1 - 2x_2 - 4x_3), \quad a_4 = \frac{1}{5}(2x_1 + 3x_2 + x_3)$$

      - $\therefore S$ spans $V = \mathbb{R}^3$

1.6 Bases and dimension

- **Bases**
  - A finite spanning set for V able to be reduced to a basis for V
  - *Example 1.6.6*
    - $S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ -12 \\ 20 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} \right\}$
    - Q2: *Is there any subset of S that is a basis for $V = \mathbb{R}^3$?*
      - Yes there is!

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

1.6 Bases and dimension

- **Bases**

Theorem 1.10 (Replacement theorem):
Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ spans V.

- Proof)
  - If $m = 0$,
    - $L = \emptyset$
    - We may set $H = G$ and $L \cup H = G$ which spans V.

    - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **Bases**

Theorem 1.10 (Replacement theorem):
Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ spans V.

- Proof)
  - If $m = n$,
    - By Theorem 1.8, L itself is a basis for V.
    - Since $n - m = 0$, we have $H = \emptyset$, and $L \cup H = L$ spans V.

    - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **Bases**

Theorem 1.10 (Replacement theorem):
Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ spans V.

- Proof)
  - If $m < n$,
    - Assume true for $0 < m < n$.
    - Let $L_m = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a linearly independent subset of V.
    - Let $H_m = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{n-m}\}$ be a subset of G such that $m \leq n$ and $L_m \cup H_m$ spans V.

1.6 Bases and dimension

- **Bases**

Theorem 1.10 (Replacement theorem):
Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ spans V.

- Proof)
  - If $m < n$,
    - Now, consider the case of $m+1$.
    - Let $L_{m+1} = L_m \cup \{\mathbf{v}_{m+1}\} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}\}$ be a linearly independent subset of V.
    - Recall that $L_m \cup H_m$ spanned V.
      - $\Rightarrow \mathbf{v}_{m+1} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m + b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_{n-m}\mathbf{u}_{n-m}$
    - Also note that if all b_i's are zero, it contradicts the fact that L_{m+1} is linearly independent.
    - Without loss of generality, assume $b_{n-m} \neq 0$.
      - $\Rightarrow \mathbf{u}_{n-m} = -\frac{a_1}{b_{n-m}}\mathbf{v}_1 - \cdots - \frac{a_m}{b_{n-m}}\mathbf{v}_m + \frac{1}{b_{n-m}}\mathbf{v}_{m+1} - \frac{b_1}{b_{n-m}}\mathbf{u}_1 - \cdots - \frac{b_{n-(m+1)}}{b_{n-m}}\mathbf{u}_{n-(m+1)}$

1.6 Bases and dimension

- **Bases**

Theorem 1.10 (Replacement theorem):
Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ spans V.

- Proof)
  - If $m < n$,
    - Let $H_{m+1} = H_m \setminus \mathbf{u}_{n-m} = \{\mathbf{u}_1, \dots, \mathbf{u}_{n-(m+1)}\}$.
      - $\Rightarrow \mathbf{u}_{n-m} \in \operatorname{span}(L_{m+1} \cup H_{m+1})$.
      - $\Rightarrow \{\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}, \mathbf{u}_1, \dots, \mathbf{u}_{n-(m+1)}, \mathbf{u}_{n-m}\} = L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\} \subseteq \operatorname{span}(L_{m+1} \cup H_{m+1})$

1.6 Bases and dimension

- **Bases**

Theorem 1.10 (Replacement theorem):
Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ spans V.

- Proof)
  - If $m < n$,
    - By the second part of Theorem 1.5,
      - $\Rightarrow \mathrm{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\}) \subseteq \mathrm{span}(L_{m+1} \cup H_{m+1})$
    - Since $L_{m+1} \cup H_{m+1} \subseteq L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\}$,
      - $\Rightarrow \mathrm{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\}) \subseteq \mathrm{span}(L_{m+1} \cup H_{m+1}) \subseteq \mathrm{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\})$
      - $\Rightarrow \mathrm{span}(L_{m+1} \cup H_{m+1}) = \mathrm{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\})$

1.6 Bases and dimension

- **Bases**

Theorem 1.10 (Replacement theorem):
Let V be a vector space, spanned by a set G containing exactly n vectors.
Let L be a linearly independent subset of V containing exactly m vectors.
Then, $m \leq n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ spans V.

- Proof)
  - If $m < n$,
    - Recall that $\mathbf{v}_{m+1} \in \text{span}(L_m \cup H_m)$
      - $\Rightarrow \text{span}(L_m \cup H_m \cup \{\mathbf{v}_{m+1}\}) = \text{span}(L_m \cup H_m) = V$
    - Note that
      - $L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\} = L_m \cup H_m \cup \{\mathbf{v}_{m+1}\}$
    - Thus,
      - $\text{span}(L_{m+1} \cup H_{m+1}) = \text{span}(L_{m+1} \cup H_{m+1} \cup \{\mathbf{u}_{n-m}\}) = \text{span}(L_m \cup H_m \cup \{\mathbf{v}_{m+1}\}) = V$
  - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **Bases**

> **Corollary 1.10.1**:
> Let V be a vector space having a finite basis.
> Then, all bases for V are finite, and every basis for V contains the same number of vectors.

- Proof)
  - By contradiction, suppose:
    - β_1 is a finite basis for V of n vectors.
    - β_2 is another finite basis for V of m vectors where $m > n$.
  - Now, obviously, V is spanned by β_1 with n vectors.
  - From Theorem 1.10, any linearly independent subsets with ℓ number of vectors must satisfy $\ell \leq n$.
  - However, β_2 is a linearly independent subset of V of m vectors where $m > n$

  - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **Dimension**

Dimension, $\mathbf{dim}(\boldsymbol{V})$:
The unique integer n such that every basis for V contains exactly n elements

- Finite-dimensional
  - Having a basis consisting of a finite number of vectors

- Infinite-dimensional
  - Having a basis consisting of an infinite number of vectors

1.6 Bases and dimension

- **Dimension**
  - *Example 1.6.7*
    - (from *Example 1.6.1*)
    - $\emptyset$ being a basis for the zero vector space $\{0\}$
    - $\emptyset$ having no elements
    - $\Rightarrow \dim(\{0\}) = 0$
  - *Example 1.6.8*
    - (from *Example 1.6.2*)
    - The standard basis for n-dimensional field F^n:

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

    - $\Rightarrow \dim(F^n) = n$

1.6 Bases and dimension

- **Dimension**
  - *Example 1.6.9*
    - (from *Example 1.6.3*)
    - $\{E^{ij}|1 \leq i \leq m, 1 \leq j \leq n\}$ being a basis for $M_{m\times n}(F)$

$$E^{ij} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \boxed{1} & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

row i, column j

    - $\Rightarrow \dim(M_{m\times n}(F)) = mn$

1.6 Bases and dimension

- **Dimension**

Corollary 1.10.2:
Let V be a vector space with dimension n.

(a) A spanning set for V contains exactly n vectors is a basis for V.

(b) Any linearly independent subset of V that contains exactly n vectors is a basis for V.

(c) Every linearly independent subset of V can be extended to a basis for V. That is, if L is a linearly independent subset of V, then there is a basis β of V such that $L \subseteq \beta$.

1.6 Bases and dimension

- **Dimension**
  - Proof) (a) A spanning set for V that contains exactly n vectors $\Rightarrow$ A basis for V
    - Let $G = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a finite spanning set for V.
    - By Theorem 1.9, there exists a subset $H \subseteq G$ that is a basis for V.
    - By Corollary 1.10.1, H has exactly n linearly independent vectors.
    - Now, if $m = n$, we must have $G = H$.
    - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **Dimension**
  - Proof) (b) Any linearly independent subset of V that contains exactly n vectors ⇒ A basis for V
    - A vector $\mathbf{v}$ is uniquely expressed by a linearly independent subset $L = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$
      - ∴ Unique expression ⇔ Linearly independence
    - By Theorem 1.8, L being able to express a vector uniquely implies that it is a basis for V
    - ∴ Q.E.D.

1.6 Bases and dimension

- **Dimension**
  - Proof) (c) L is a linearly independent subset of V. $\Rightarrow$ There is a basis β of V such that $L \subseteq \beta$.
    - Let V be spanned by a basis β with n vectors
    - Let L be a linearly independent subset of V with m vectors.
    - By Theorem 1.10, there is a subset H of β containing $n - m$ vectors such that $L \cup H$ spans V.
      - $\Rightarrow L \cup H$ has at most n vectors.
    - By Theorem 1.9, since $L \cup H$ spans V, there exists a subset $\Phi \subseteq L \cup H$ that is a basis for V.
    - By Corollary 1.10.1, Φ has exactly n vectors
      - $\Rightarrow L \cup H$ has at least n vectors.

1.6 Bases and dimension

- **Dimension**
  - Proof) (c) L is a linearly independent subset of V. $\Rightarrow$ There is a basis β of V such that $L \subseteq \beta$.
    - Thus, $L \cup H$ has exactly n vectors.
    - By Corollary 1.10.2 (a), $L \cup H$ is a basis, i.e., $L \cup H = \beta$
      - $\Rightarrow L \subseteq \beta$
    - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **Dimension**
  - *Example 1.6.15*
    - (from *Example 1.4.5*)
    - 4 matrices $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ spanning or generating $V = M_{2\times 2}(\mathbb{R})$
    - $\Rightarrow$ A basis for $M_{2\times 2}(\mathbb{R})$ since $\dim\left(M_{2\times 2}(\mathbb{R})\right) = 4$
  - *Example 1.6.16*
    - (from *Example 1.5.3*)

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

    - Linearly independent set that contains exactly 4 vectors
    - $\Rightarrow$ A basis for $\mathbb{R}^4$ since $\dim(\mathbb{R}^4) = 4$

1.6 Bases and dimension

- **The dimension of subspaces**

> **Theorem 1.11**:
> Let W be a subspace of a finite-dimensional vector space V.
> Then, W is finite-dimensional and $\dim(W) \leq \dim(V)$.
> Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

- Proof)
  - Let $\dim(V) = n$.
  - If $W = \{0\}$,
    - $\emptyset$ is a linearly independent basis
    - $\Rightarrow \dim(W) = 0 \leq n$
  - If $W = \mathrm{span}(\mathbf{w}_1)$, for some non-zero $\mathbf{w}_1$
    - $\mathbf{w}_1$ alone is linearly independent.
    - $\Rightarrow \dim(W) = 1 \leq n$

1.6 Bases and dimension

- **The dimension of subspaces**

Theorem 1.11:
Let W be a subspace of a finite-dimensional vector space V.
Then, W is finite-dimensional and $\dim(W) \leq \dim(V)$.
Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

- Proof)
  - If $W = \mathrm{span}(\{\mathbf{w}_1, \ldots, \mathbf{w}_k\})$, by adding one by one so as to remain linearly independent,
    - By Corollary 1.10.1, no linearly independent subset of V can contain more than n vectors.
    - $\Rightarrow \dim(W) = k \leq n$
  - If $\dim(W) = n$,
    - A basis for W is a linearly independent subset of V containing n vectors
    - From Corollary 1.10.2 (b), that basis is also a basis for V.
    - $\Rightarrow V = W$
  - $\therefore$ Q.E.D.

1.6 Bases and dimension

- **The dimension of subspaces**
  - *Example 1.6.18*
    - $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \in V = F^5 \middle| a_1 + a_3 + a_5 = 0, a_2 = a_4 \right\}$
    - A possible basis is

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

    - $\Rightarrow \dim(W) = 3 \leq \dim(V) = 5$

1.6 Bases and dimension

- **The dimension of subspaces**
  - *Example 1.6.19*
    - $\{E^{ij} | 1 \leq i \leq n, 1 \leq j \leq n\}$ being a basis for square matrices $V = M_{n\times n}(F)$

$$E^{ij} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \boxed{1} & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \text{row } i, \text{ column } j$$

    - For the set of diagonal $n \times n$ matrices $W = \{M_{n\times n}(F) | [A]_{k\ell} = 0 \text{ for } k \neq \ell\}$,
      - A possible basis being $\{E^{11}, E^{22}, \ldots, E^{nn}\}$
    - $\Rightarrow \dim(W) = n \leq \dim(V) = n^2$

1.6 Bases and dimension

- **The dimension of subspaces**
  - *Example 1.6.20*
    - $\{E^{ij} | 1 \le i \le n, 1 \le j \le n\}$ being a basis for square matrices $V = M_{n \times n}(F)$

$$E^{ij} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \boxed{1} & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

row i, column j

    - For the set of symmetric $n \times n$ matrices $W = \{M_{n\times n}(F) | [A]_{k\ell} = [A]_{\ell k}\}$,
      - A possible basis being $\{E^{11}, E^{12}, \dots, E^{1n}, E^{22}, E^{23}, \dots, E^{2n}, E^{33}, E^{34}, \dots, E^{nn}\}$
    - $\Rightarrow \dim(W) = n + (n-1) + \cdots + 1 = \frac{n(n\_1)}{2} \le \dim(V) = n^2$

$$\begin{bmatrix} a & b & c & d \\ b & e & f & h \\ c & f & g & i \\ d & h & i & j \end{bmatrix}$$

This side determines the other side automatically!
$\Rightarrow [A]_{k\ell} = [A]_{\ell k}$

1.6 Bases and dimension

- **The dimension of subspaces**

> **Corollary 1.11.1**:
> If W is a subspace of a finite-dimensional vector space V, then, any basis for W can be extended to a basis for V.

- Proof)
  - Let S be a basis for W.
  - Note that S is a linearly independent subset of V
  - By Corollary 1.10.2 (c) implies S can be extended to a basis for V.
  - $\therefore$ Q.E.D.