Linear Algebra (5^{th} edition)

Stephen Friedberg, Arnold Insel, Lawrence Spence

Chapter 02: Linear transformations and matrices

Jihwan Moon

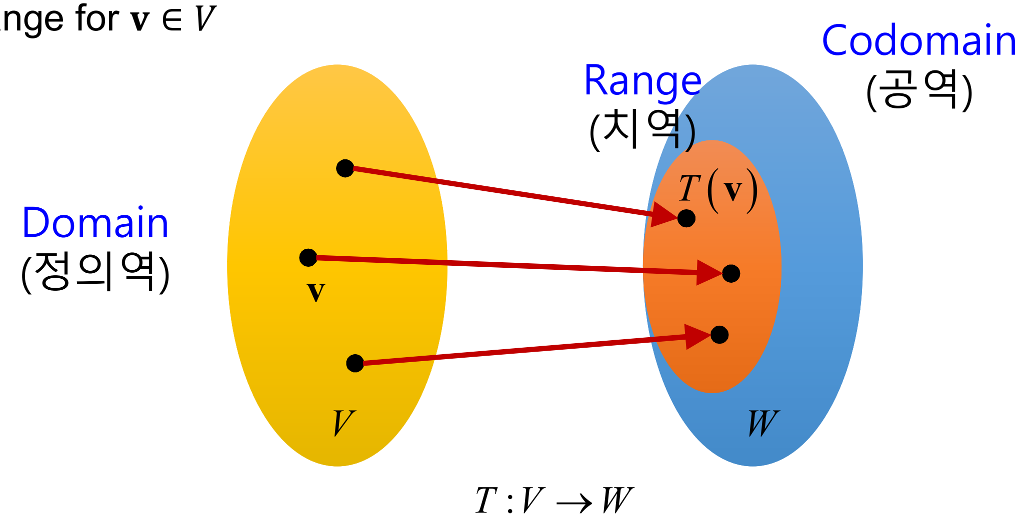
Table of Contents

- **2.1 Linear transformations, null spaces, and ranges**
- **2.2 The matrix representation of a linear transformation**
- **2.3 Composition of linear transformations and matrix multiplication**
- **2.4 Invertibility and isomorphisms**
- **2.5 The change of coordinate matrix**

2.1 Linear transformations, null spaces, and ranges

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - Notation
    - $T: V \rightarrow W$
      - T: A function
      - V: A domain
      - W: A codomain
      - $T(\mathbf{v})$: A range for $\mathbf{v} \in V$



2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**

Linear transformation:
Let V and W be vector spaces over the same field F.
We call a function $T: V \to W$ a linear transformation from V to W (or just linear) if, for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$, we have

(a) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, and
(b) $T(c\mathbf{x}) = cT(\mathbf{x})$

- Properties
  - ① T is linear $\Rightarrow$ $T(\mathbf{0}) = \mathbf{0}$
  - ② T is linear $\Leftrightarrow$ $T(c\mathbf{x}+\mathbf{y}) = cT(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$
  - ③ T is linear $\Rightarrow$ $T(\mathbf{x}-\mathbf{y}) = T(\mathbf{x}) - T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$
  - ④ T is linear $\Leftrightarrow$ $T(\sum_{i=1}^{n} a_i \mathbf{x}_i) = \sum_{i=1}^{n} a_i T(\mathbf{x}_i)$ for $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ and $a_1, \dots, a_n \in F$

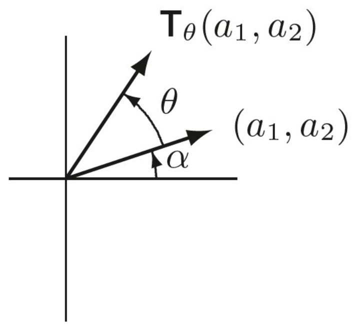
- Generally, property ② often used to prove a given transformation T is linear

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.1*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 + a_2 \\ a_1 \end{bmatrix}$
    - Q: Is function T linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} 2x_1 + x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} c(2x_1 + x_2) + 2y_1 + y_2 \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix}$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.2 (Rotation)*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1\cos\theta - a_2\sin\theta \\ a_1\sin\theta + a_2\cos\theta \end{bmatrix}$

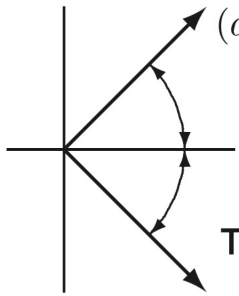


    - Q: Is function T linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix} =$
        $\begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix} + \begin{bmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{bmatrix} =$
        $\begin{bmatrix} c(x_1\cos\theta - x_2\sin\theta) + (y_1\cos\theta - y_2\sin\theta) \\ c(x_1\sin\theta + x_2\cos\theta) + (y_1\sin\theta + y_2\cos\theta) \end{bmatrix} = \begin{bmatrix} (cx_1 + y_1)\cos\theta - (cx_2 + y_2)\sin\theta \\ (cx_1 + y_1)\sin\theta + (cx_2 + y_2)\cos\theta \end{bmatrix}$

      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

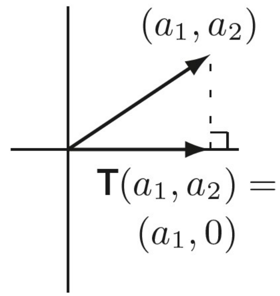
- **Linear transformations**
  - *Example 2.1.3 (Reflection)*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$



    - Q: Is function T linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + y_1 \\ -cx_2 - y_2 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ -cx_2 - y_2 \end{bmatrix}$
      - $\therefore$ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.4 (Projection on the 1st dimension)*
    - $T: \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$



    - Q: Is function T linear?
      - Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$
      - $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + y_1 \\ 0 \end{bmatrix}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ 0 \end{bmatrix}$
      - $\therefore$ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.5 (Transpose)*
    - $T: M_{m\times n}(F) \to M_{n\times m}(F)$ where $T(A) = A^T$
    - Q: Is function T linear?
      - $T(c\mathbf{X} + \mathbf{Y}) = (c\mathbf{X} + \mathbf{Y})^T = c\mathbf{X}^T + \mathbf{Y}^T$
      - $cT(\mathbf{X}) + T(\mathbf{Y}) = c\mathbf{X}^T + \mathbf{Y}^T$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.6 (Derivatives)*
    - $T: V \to V$ where $T(f) = \frac{df}{dv}$
    - Q: Is function T linear?
      - Letting $g \in V$ and $h \in V$
      - $T(cg + h) = \frac{d}{dv}(cg + h) = c\frac{dg}{dv} + \frac{dh}{dv}$
      - $cT(g) + T(h) = c\frac{dg}{dv} + \frac{dh}{dv}$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example 2.1.7 (Integration)*
    - $T: \mathbb{R} \to \mathbb{R}$ where $T(f) = \int_a^b f(t)dt$ for some $a, b \in \mathbb{R}$
    - Q: Is function T linear?
      - Letting $g \in \mathbb{R}$ and $h \in \mathbb{R}$
      - $T(cg + h) = \int_a^b cg(t) + h(t)dt = c\int_a^b g(t)dt + \int_a^b h(t)dt$
      - $cT(g) + T(h) = c\int_a^b g(t)dt + \int_a^b h(t)dt$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example (Identity transformation)*
    - $T: V \rightarrow V$ where $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$
    - Q: Is function T linear?
      - Letting $\mathbf{x} \in V$ and $\mathbf{y} \in V$
      - $T(c\mathbf{x} + \mathbf{y}) = c\mathbf{x} + \mathbf{y}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = c\mathbf{x} + \mathbf{y}$
      - $\therefore$ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Linear transformations**
  - *Example (Zero transformation)*
    - $T: V \to W$ where $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$
    - Q: Is function T linear?
      - Letting $\mathbf{x} \in V$ and $\mathbf{y} \in V$
      - $T(c\mathbf{x} + \mathbf{y}) = \mathbf{0}$
      - $cT(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}$
      - ∴ By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Null space (kernel):
Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
We define null space (or kernel) $N(T)$ of T to be the set of all vectors $\mathbf{x} \in V$ such that

$$N(T) = \{\mathbf{x} \in V | T(\mathbf{x}) = \mathbf{0}\}$$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Range (image) (치역):
Let V and W be vector spaces and let function $T: V \to W$ be linear.
We define range (or image) $R(T)$ of T to be the subset of W containing all images under T of vectors in V such that

$$R(T) = \{T(\mathbf{x}) | \mathbf{x} \in V\}$$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.8*
    - $T_1: V \to V$ where $T_1(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$ (identity transformation)
      - Null space
        - $N(T_1) = \{\mathbf{0}\}$
      - Range
        - $R(T_1) = V$
    - $T_2: V \to W$ where $T_2(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$ (zero transformation)
      - Null space
        - $N(T_2) = V$
      - Range
        - $R(T_2) = \{\mathbf{0}\}$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.9*
    - $T: \mathbb{R}^3 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 - a_2 \\ 2a_3 \end{bmatrix}$ for all $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$
      - Null space
        - $N(T) = \left\{ \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \middle| x \in \mathbb{R} \right\}$
      - Range
        - $R(T) = \mathbb{R}^2$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.1**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> Then, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

- Proof) ($N(T)$ is a subspace of V)
  - Theorem 1.3(a)
    - $\mathbf{0} \in N(T)$ since property ① of linear transformation indicates that $T(\mathbf{0}) = \mathbf{0}$
  - Theorem 1.3(b)
    - $\mathbf{x} + \mathbf{y} \in N(T)$ since $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}$ for $\mathbf{x}, \mathbf{y} \in N(T)$
  - Theorem 1.3(c)
    - $c\mathbf{x} \in N(T)$ since $T(c\mathbf{x}) = cT(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in N(T)$
  - ∴ Subspace!

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.1**:
> Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
> Then, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

- Proof) ($R(T)$ is a subspace of W)
  - Theorem 1.3(a)
    - $\mathbf{0} \in R(T)$ since $\mathbf{0} \in V$ and property ① of linear transformation indicates that $T(\mathbf{0}) = \mathbf{0}$
  - Theorem 1.3(b)
    - $\mathbf{x} + \mathbf{y} \in R(T)$ since $\mathbf{x} + \mathbf{y} = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2) \in R(T)$ for $\mathbf{x} = T(\mathbf{v}_1), \mathbf{y} = T(\mathbf{v}_2) \in R(T)$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$
  - Theorem 1.3(c)
    - $c\mathbf{x} \in R(T)$ since $c\mathbf{x} = cT(\mathbf{v}_1) = T(c\mathbf{v}_1) \in R(T)$ for $\mathbf{x} = T(\mathbf{v}_1) \in R(T)$ and $\mathbf{v}_1 \in V$
  - ∴ Subspace!

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.2**:
> Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
> $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V $\Rightarrow$ $R(T) = \text{span}\big(T(\beta)\big) = \text{span}\big(T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\big)$

- Proof)
  - ($\text{span}\big(T(\beta)\big) \subseteq R(T)$)
    - Note that $T(\mathbf{v}_i) \in R(T), \forall i$
    - From Theorem 2.1, $R(T)$ is a subspace
      - $\Rightarrow \text{span}(\{T(\mathbf{v}_i)\}) = \text{span}\big(T(\beta)\big) \in R(T)$ by Theorem 1.5
  - ($\text{span}\big(T(\beta)\big) \supseteq R(T)$)
    - $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ for any $\mathbf{v} \in V$
      - $\Rightarrow T(\mathbf{v}) \in R(T)$
    - $T(\mathbf{v}) = T(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i) \in \text{span}(\{T(\mathbf{v}_i)\}) = \text{span}\big(T(\beta)\big)$ for any $T(\mathbf{v}) \in R(T)$
      - $\Rightarrow R(T) \in \text{span}\big(T(\beta)\big)$
  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.10*
    - $T: V = \mathbb{R}^3 \to M_{2\times 2}(\mathbb{R})$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_2 - v_3 & 0 \\ 0 & v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$
      - Linear
    - For a standard basis $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
      - $R(T) = \mathrm{span}\big(T(\beta)\big) = \mathrm{span}\left(\left\{ T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right\}\right) = \mathrm{span}\left(\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right\}\right) = \mathrm{span}\left(\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}\right)$
      - $\Rightarrow \dim\big(R(T)\big) = 2$

2.1 Linear transformations, null spaces, and ranges

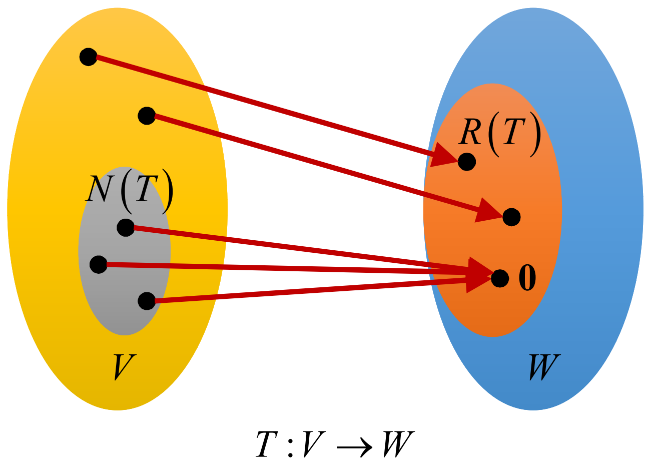
- **Null spaces and ranges**
  - *Example 2.1.10*
    - $T: V = \mathbb{R}^3 \rightarrow M_{2\times 2}(\mathbb{R})$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_2 - v_3 & 0 \\ 0 & v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$
      - Linear
    - To find a basis for $N(T)$, by letting $T(\mathbf{v}) = \mathbf{0}$,
      - $N(T) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$
      - $\Rightarrow \dim(N(T)) = 1$
    - Note that $\dim(V) = \dim(N(T)) + \dim(R(T))$
      - (Theorem 2.3 coming soon!)

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Nullity and rank**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> If $N(T)$ and $R(T)$ are finite-dimensional,
> $$\text{nullity}(T) \triangleq \dim\big(N(T)\big)$$
> $$\text{rank}(T) \triangleq \dim\big(R(T)\big)$$

- Intuition
  - The larger the nullity, the smaller the rank
    - The more vectors carried into $\mathbf{0}$, the smaller the range



2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> If V is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - Let $n = \dim(V)$ and $k = \dim\big(N(T)\big)$ where $n \geq k$.
  - Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for $N(T)$.
  - Note that $N(T)$ is a subspace of vector space V.
    - $\Rightarrow$ From Corollary 1.11.1, we may extend $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ to a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ for V.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> If V is finite-dimensional,
> $$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V)$$

- Proof)
  - From Theorem 2.2,
    - $R(T) = \mathrm{span}\big(T(\beta)\big) = \mathrm{span}\big(T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k), T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\big)$
  - Noting that $T(\mathbf{v}_1) = \cdots = T(\mathbf{v}_k) = \mathbf{0}$
    - $R(T) = \mathrm{span}(\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\})$
    - $\Rightarrow$ $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ spans $R(T)$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> If V is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - If $\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ is a linearly independent set,
    - $\sum_{i=k+1}^{n} b_i T(\mathbf{v}_i) = \mathbf{0}$ only when $b_i = 0, i = k+1, \dots, n$
  - From the linear property of T,
    - $\sum_{i=k+1}^{n} b_i T(\mathbf{v}_i) = T(\sum_{i=k+1}^{n} b_i \mathbf{v}_i)$
  - If $T(\sum_{i=k+1}^{n} b_i \mathbf{v}_i) = \mathbf{0}$, then
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i \in N(T)$
  - Hence, $\sum_{i=k+1}^{n} b_i \mathbf{v}_i$ may be expressed as a linear combination of the basis of $N(T)$
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i = \sum_{i=1}^{k} c_i \mathbf{v}_i$ for some c_i's
  - Since $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ are a basis for V,
    - $\sum_{i=k+1}^{n} b_i \mathbf{v}_i - \sum_{i=1}^{k} c_i \mathbf{v}_i = 0$ only when $b_i = 0, i = k+1, \dots, n$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.3 (Dimension Theorem)**:
> Let V and W be vector spaces and let function $T: V \to W$ be linear.
> If V is finite-dimensional,
> $$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
  - Hence $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ is a basis for $R(T)$
    - $\Rightarrow \dim\big(R(T)\big) = \text{rank}(T) = n - k$
  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.4**:
> Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
> Then, T is one-to-one $\Leftrightarrow$ $N(T) = \{\mathbf{0}\}$

- Proof)
  - (T is one-to-one $\Rightarrow$ $N(T) = \{\mathbf{0}\}$)
    - From the one-to-one property, there exists only one $\mathbf{x}$ to satisfy $T(\mathbf{x}) = \mathbf{0}$.
    - In the meantime, by the linear property, $T(\mathbf{0}) = \mathbf{0}$.
      - $\Rightarrow \mathbf{x} = \mathbf{0}$
      - $\Rightarrow N(T) = \{\mathbf{0}\}$
  - (T is one-to-one $\Leftarrow$ $N(T) = \{\mathbf{0}\}$)
    - By contradiction, assume T is not one-to-one.
      - $\Rightarrow T(\mathbf{x}) = T(\mathbf{y})$ for some distinct $\mathbf{x}, \mathbf{y} \in V$
    - By the linear property, $\mathbf{0} = T(\mathbf{x}) - T(\mathbf{y}) = T(\mathbf{x} - \mathbf{y})$
      - $\Rightarrow \mathbf{x} - \mathbf{y} \in N(T)$ where $\mathbf{x} - \mathbf{y} \neq \mathbf{0}$
      - $\therefore$ Contradiction
  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.5**:
> Let V and W be finite-dimensional vector spaces of equal dimension, and let function $T: V \to W$ be linear.
> Then, T is one-to-one $\Leftrightarrow$ T is onto $\Leftrightarrow$ $\mathrm{rank}(T) = \dim(V)$

- "One-to-one" (*Appendix B*)
  - $T(\mathbf{v}_1) = T(\mathbf{v}_2) \Rightarrow \mathbf{v}_1 = \mathbf{v}_1$, for some $\mathbf{v}_1, \mathbf{v}_2 \in V$
  - Equivalently, $\mathbf{v}_1 \neq \mathbf{v}_2 \Rightarrow T(\mathbf{v}_1) \neq T(\mathbf{v}_2)$, for some $\mathbf{v}_1, \mathbf{v}_2 \in V$

- "Onto" (*Appendix B*)
  - $T(\mathbf{v}) =$ W, for some $\mathbf{v} \in V$

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

> **Theorem 2.5**:
> Let V and W be finite-dimensional vector spaces of equal dimension, and let function $T: V \to W$ be linear.
> Then, T is one-to-one $\Leftrightarrow$ T is onto $\Leftrightarrow$ $\text{rank}(T) = \dim(V)$

- Proof)
  - From Theorem 2.4, T is one-to-one $\Leftrightarrow$ $N(T) = \{\mathbf{0}\}$ or $\text{nullity}(T) = 0$
  - Also, from Theorem 2.3 (Dimension theorem), $\text{nullity}(T) + \text{rank}(T) = \dim(V)$
    - $\Rightarrow$ T is one-to-one $\Leftrightarrow$ $\text{rank}(T) = \dim(V)$
  - From the equal dimension condition,
    - $\Rightarrow \text{rank}(T) = \dim(V) \Leftrightarrow \text{rank}(T) = \dim(W)$
  - From Theorem 1.11,
    - $\Rightarrow \text{rank}(T) = \dim(W) \Leftrightarrow R(T) = W$

  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - *Example 2.1.12*
    - $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where
      - $T(\mathbf{v}) = \begin{bmatrix} v_1 + v_2 \\ v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
      - Linear
    - To find a basis for $N(T)$, by letting $T(\mathbf{v}) = \mathbf{0}$,
      - $N(T) = \{\mathbf{0}\}$
    - By Theorem 2.4
      - One-to-one
    - By Theorem 2.5
      - Onto

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**

Theorem 2.6:
Let V and W be vector spaces of equal dimension, and suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V.
For $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$, there exists exactly one linear transformation $T: V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$.

- Proof)
  - From the linear property, for $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ with unique scalars $a_1, \dots, a_n$
    - $T(\mathbf{v}) = T(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i) = \sum_{i=1}^{n} a_i \mathbf{w}_i$
  - If there exists another linear function $U: V \to W$ such that $U(\mathbf{v}_i) = \mathbf{w}_i$
    - $U(\mathbf{v}) = U(\sum_{i=1}^{n} a_i \mathbf{v}_i) = \sum_{i=1}^{n} a_i U(\mathbf{v}_i) = \sum_{i=1}^{n} a_i \mathbf{w}_i$
  - Then, we must have $T(\mathbf{v}_i) = U(\mathbf{v}_i), \forall i$.
  - Hence, $U = T$
  - $\therefore$ Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- **Null spaces and ranges**
  - An implication of Theorem 2.6
    - A linear transformation completely determined by its action on a basis

2.2 The matrix representation of a linear transformation

2.2 The matrix representation of a linear transformation

- **Section 2.1**
  - Studying linear transformations by examining their null spaces and ranges

- **Section 2.2**
  - Representing linear transformations by a matrix
    - Developing a one-to-one correspondence between matrices and linear transformations

2.2 The matrix representation of a linear transformation

- **Ordered basis**

> **Ordered basis**:
> Let V be a finite-dimensional vector space.
> An ordered basis for V is a basis for V endowed with a specific order.
>
> That is, an ordered basis is a finite sequence of linearly independent vectors in V that spans V.

- *Example 2.2.1*
  - $\beta = \{e_1, e_2, e_3\}$
    - A standard ordered basis in $\mathbb{F}^3$ where $\mathbf{e}_i, \forall i$ is a standard basis
  - $\gamma = \{e_2, e_1, e_3\}$
    - Another ordered basis
  - From the perspective of orders, $\beta \neq \gamma$

2.2 The matrix representation of a linear transformation

- **Ordered basis**

> **Coordinate vector**:
> Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis for a finite-dimensional vector space V.
> For $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i \in V$, we define the coordinate vector of $\mathbf{v}$ relative to β by
> $$[\mathbf{v}]_\beta \triangleq \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$
> With unique scalars $a_1, \dots, a_n$

- *Example 2.2.2*
  - $V = \mathbb{R}^3$ with $\beta = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$
  - The coordinate vector of $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$
    - $\Rightarrow [\mathbf{v}]_\beta = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - Letting
    - V be a vector space with an ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$
    - W be a vector space with an ordered basis $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$
    - $T: V \to W$ be a linear function

  - Then, using the ordered basis γ
    - $T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i$ with unique scalars $a_{ij} \in F$ for $i = 1, \dots, m$ for each $j = 1, \dots, n$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **Matrix representation**:
> We call the $m \times n$ matrix $\mathbf{A}$ defined by $[\mathbf{A}]_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ and write $\mathbf{A} = [T]_\beta^\gamma$.
> If $V = W$ and $\beta = \gamma$, then we write $\mathbf{A} = [T]_\beta$.

- For instance, with $n = 2$ and $m = 3$ such that $T: \mathbb{R}^2 \to \mathbb{R}^3$,
  - $[T(\mathbf{v}_1) \quad T(\mathbf{v}_2)] = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ $\mathbf{A} = [T]_\beta^\gamma$

$[T(\mathbf{v}_1)]_\gamma \quad [T(\mathbf{v}_2)]_\gamma$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - <u>*Example 2.2.3*</u>
    - If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix}$, $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$, $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = 3 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-4) \cdot \mathbf{w}_3$
      - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$

    - For $\gamma_2 = \{\mathbf{w}_1 = \mathbf{e}_3, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_1\}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{w}_3 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_1$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = 3 \cdot \mathbf{w}_3 + 0 \cdot \mathbf{w}_2 + (-4) \cdot \mathbf{w}_1$
      - $\Rightarrow [T]_\beta^{\gamma_2} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - Letting
    - V be a vector space with an ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$
    - W be a vector space with an ordered basis $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$
    - $T: V \to W$ be a linear function
  - Then, using the ordered basis γ
    - $T(\mathbf{v}_j) = \mathbf{0} = 0 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + \cdots + 0 \cdot \mathbf{w}_m$ for $j = 1, \dots, n$
    - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = [\mathbf{0}]$
  - For $n = m$ and $\beta = \gamma$,
    - $T(\mathbf{v}_j) = \mathbf{v}_j = 0 \cdot \mathbf{v}_1 + \cdots + 1 \cdot \mathbf{v}_j + \cdots + 0 \cdot \mathbf{w}_m$ for $j = 1, \dots, n$
    - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = \mathbf{I}_n$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

Kronecker delta:

$$\delta_{ij} \triangleq \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- For instance,
  - $[\mathbf{I}_n]_{ij} = \delta_{ij}, \forall i, j$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

Addition and scalar multiplication of functions:
Let V and W be vector spaces over F.
Let $T, U: V \to W$ be arbitrary functions.
Then, for all $\mathbf{x} \in V$ and $a \in F$,

$$T + U: V \to W \quad \triangleq \quad (T+U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x})$$
$$aT: V \to W \quad \triangleq \quad (aT)(\mathbf{x}) = aT(\mathbf{x})$$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **Theorem 2.7**:
> Let V and W be vector spaces over F.
> Let $T, U: V \to W$ be linear functions.
> Then, for all $\mathbf{x} \in V$ and $a \in F$,
>
> (a) $aT + U$ is linear, i.e., $(aT+U)(c\mathbf{x}+\mathbf{y}) = c(aT+U)(\mathbf{x}) + (aT+U)(\mathbf{y})$
> (b) The collection of all linear transformations from V to W is a vector space over F.

- Proof)
  - (a)
    - Let $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$.
    - $(aT+U)(c\mathbf{x}+\mathbf{y}) = (aT)(c\mathbf{x}+\mathbf{y}) + U(c\mathbf{x}+\mathbf{y}) = c(aT)(\mathbf{x}) + (aT)(\mathbf{y}) + cU(\mathbf{x}) + U(\mathbf{y}) = c\big((aT)(\mathbf{x}) + U(\mathbf{x})\big) + (aT)(\mathbf{y}) + U(\mathbf{y}) = c(aT+U)(\mathbf{x}) + (aT+U)(\mathbf{y})$
  - (b)
    - (Left as an exercise)

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

The vector space of all linear transformations:
Let V and W be vector spaces over F.
We denote the vector space of all linear transformations from V into W by $\mathcal{L}(V,W)$.
If $V = W$, we write $\mathcal{L}(V)$.

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

> **Theorem 2.8**:
> Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively.
> Let $T, U: V \to W$ be linear transformations.
> Then,
>
> (a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
> (b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars c

- Proof)
  - (a)
    - Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$.
    - $T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i$ for unique scalars $a_{ij}, \forall i, j$
    - $U(\mathbf{v}_j) = \sum_{i=1}^{m} b_{ij}\mathbf{w}_i$ for unique scalars $b_{ij}, \forall i, j$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

Theorem 2.8:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively.
Let $T, U: V \to W$ be linear transformations.
Then,

(a) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars c

- Proof)
    - (a)
        - Then, $(T+U)(\mathbf{v}_j) = T(\mathbf{v}_j) + U(\mathbf{v}_j) = \sum_{i=1}^{m}(a_{ij} + b_{ij})\mathbf{w}_i$
        - Thus, $\left[[T+U]_\beta^\gamma\right]_{ij} = a_{ij} + b_{ij} = \left[[T]_\beta^\gamma\right]_{ij} + \left[[U]_\beta^\gamma\right]_{ij}$

        - $\therefore$ Q.E.D.

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**

Theorem 2.8:
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively.
Let $T, U: V \to W$ be linear transformations.
Then,

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$ for all scalars c

- Proof)
  - (b)
    - $(cT)(\mathbf{v}_j) = cT(\mathbf{v}_j) = c\sum_{i=1}^{m} a_{ij}\mathbf{w}_i$
    - Thus, $\left[[cT]_\beta^\gamma\right]_{ij} = ca_{ij} = c\left[[T]_\beta^\gamma\right]_{ij}$

    - ∴ Q.E.D.

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - *Example 2.2.5*
    - Let $T, U: \mathbb{R}^2 \to \mathbb{R}^3$ be linear and $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$, $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$
    - If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix}$
      - $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = 3 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-4) \cdot \mathbf{w}_3$
      - $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$
    - If $U\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 - v_2 \\ 2v_1 \\ 3v_1 + 2v_2 \end{bmatrix}$
      - $U(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2 + 3 \cdot \mathbf{w}_3$ and $U(\mathbf{v}_2) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = (-1) \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$
      - $\Rightarrow [U]_\beta^\gamma = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- **The matrix representation of a linear transformation**
  - *Example 2.2.5*
    - Since $(T+U)\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) + U\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ 2v_1 \\ 3v_1 + 2v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 2v_2 \\ 2v_1 \\ 5v_1 - 2v_2 \end{bmatrix}$
      - $(T+U)(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} = 2 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2 + 5 \cdot \mathbf{w}_3$ and $(T+U)(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-2) \cdot \mathbf{w}_3$
      - $\Rightarrow [T+U]_\beta^\gamma = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$
    - Note that
      - $[T]_\beta^\gamma + [U]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix} = [T+U]_\beta^\gamma$