

Linear Algebra (5th edition)

Stephen Friedberg, Arnold Insel, Lawrence Spence

Chapter 02: Linear transformations and matrices

Jihwan Moon

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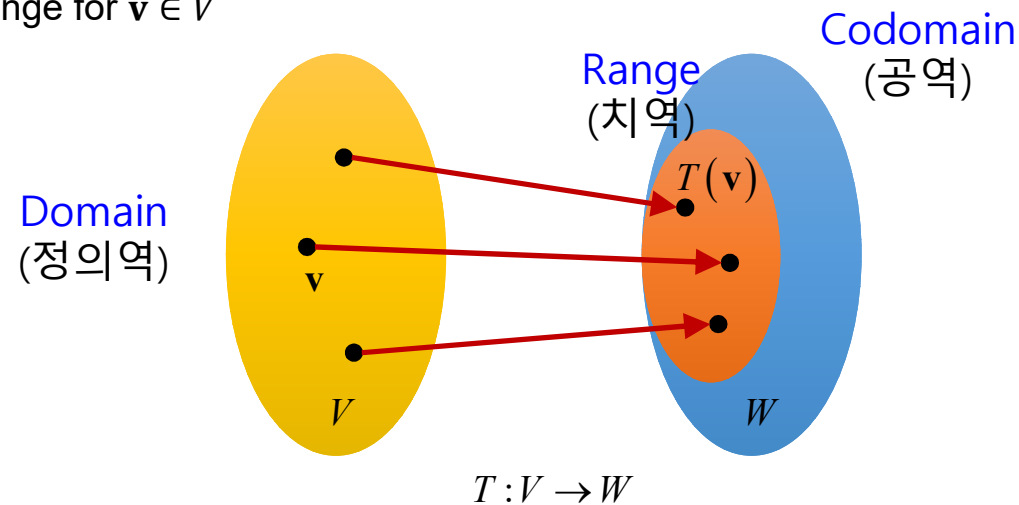
2.1 Linear transformations, null spaces, and ranges

2.1 Linear transformations, null spaces, and ranges

- Linear transformations

- Notation

- $T: V \rightarrow W$
 - T : A function
 - V : A domain
 - W : A codomain
 - $T(\mathbf{v})$: A range for $\mathbf{v} \in V$



2.1 Linear transformations, null spaces, and ranges

- Linear transformations

Linear transformation:

Let V and W be vector spaces over the same field F .

We call a function $T: V \rightarrow W$ a **linear transformation from V to W** (or just **linear**) if, for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$, we have

(a) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, and

(b) $T(c\mathbf{x}) = cT(\mathbf{x})$

- Properties

- ① T is linear $\Rightarrow T(\mathbf{0}) = \mathbf{0}$
- ② T is linear $\Leftrightarrow T(c\mathbf{x} + \mathbf{y}) = cT(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$
- ③ T is linear $\Rightarrow T(\mathbf{x} - \mathbf{y}) = T(\mathbf{x}) - T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in V$
- ④ T is linear $\Leftrightarrow T(\sum_{i=1}^n a_i \mathbf{x}_i) = \sum_{i=1}^n a_i T(\mathbf{x}_i)$ for $\mathbf{x}_1, \dots, \mathbf{x}_n \in V$ and $a_1, \dots, a_n \in F$

- Generally, **property ②** often used to prove a given transformation T is linear

2.1 Linear transformations, null spaces, and ranges

- Linear transformations

- Example 2.1.1

- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 + a_2 \\ a_1 \end{bmatrix}$

- Q: Is function T linear?

- Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

- $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix}$

- $cT(\mathbf{x}) + T(\mathbf{y}) = c \begin{bmatrix} 2x_1 + x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} c(2x_1 + x_2) + 2y_1 + y_2 \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 2(cx_1 + y_1) + (cx_2 + y_2) \\ cx_1 + y_1 \end{bmatrix}$

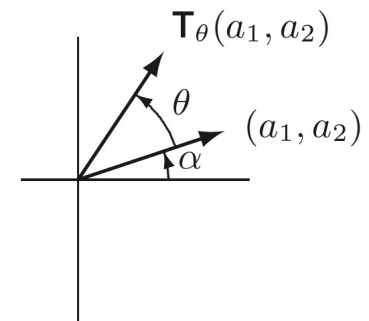
- \therefore By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- Linear transformations

- Example 2.1.2 (Rotation)

- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \cos \theta - a_2 \sin \theta \\ a_1 \sin \theta + a_2 \cos \theta \end{bmatrix}$



- Q: Is function T linear?

- Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

- $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} (cx_1 + y_1) \cos \theta - (cx_2 + y_2) \sin \theta \\ (cx_1 + y_1) \sin \theta + (cx_2 + y_2) \cos \theta \end{bmatrix} =$
 $\begin{bmatrix} (cx_1 + y_1) \cos \theta - (cx_2 + y_2) \sin \theta \\ (cx_1 + y_1) \sin \theta + (cx_2 + y_2) \cos \theta \end{bmatrix}$

- $cT(\mathbf{x}) + T(\mathbf{y}) = c \begin{bmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{bmatrix} + \begin{bmatrix} y_1 \cos \theta - y_2 \sin \theta \\ y_1 \sin \theta + y_2 \cos \theta \end{bmatrix} =$
 $\begin{bmatrix} c(x_1 \cos \theta - x_2 \sin \theta) + (y_1 \cos \theta - y_2 \sin \theta) \\ c(x_1 \sin \theta + x_2 \cos \theta) + (y_1 \sin \theta + y_2 \cos \theta) \end{bmatrix} = \begin{bmatrix} (cx_1 + y_1) \cos \theta - (cx_2 + y_2) \sin \theta \\ (cx_1 + y_1) \sin \theta + (cx_2 + y_2) \cos \theta \end{bmatrix}$

- \therefore By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- Linear transformations

- Example 2.1.3 (Reflection)

- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$

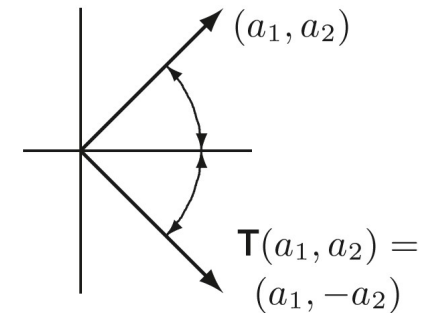
- Q: Is function T linear?

- Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

- $T(c\mathbf{x} + \mathbf{y}) = T\left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} cx_1 + y_1 \\ -cx_2 - y_2 \end{bmatrix}$

- $cT(\mathbf{x}) + T(\mathbf{y}) = c \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ -cx_2 - y_2 \end{bmatrix}$

- \therefore By property ②, linear!



2.1 Linear transformations, null spaces, and ranges

- Linear transformations

- Example 2.1.4 (Projection on the 1st dimension)

- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $T \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$

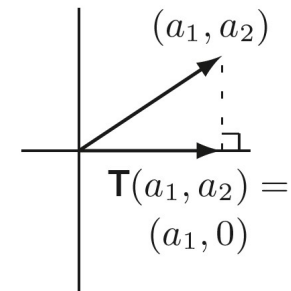
- Q: Is function T linear?

- Letting $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

- $T(c\mathbf{x} + \mathbf{y}) = T \left(\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix} \right) = \begin{bmatrix} cx_1 + y_1 \\ 0 \end{bmatrix}$

- $cT(\mathbf{x}) + T(\mathbf{y}) = c \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 + y_1 \\ 0 \end{bmatrix}$

- \therefore By property ②, linear!



2.1 Linear transformations, null spaces, and ranges

- Linear transformations
 - Example 2.1.5 (Transpose)
 - $T: M_{m \times n}(F) \rightarrow M_{n \times m}(F)$ where $T(A) = A^T$
 - Q: Is function T linear?
 - $T(c\mathbf{X} + \mathbf{Y}) = (c\mathbf{X} + \mathbf{Y})^T = c\mathbf{X}^T + \mathbf{Y}^T$
 - $cT(\mathbf{X}) + T(\mathbf{Y}) = c\mathbf{X}^T + \mathbf{Y}^T$
 - \therefore By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- Linear transformations

- Example 2.1.6 (Derivatives)

- $T: V \rightarrow V$ where $T(f) = \frac{df}{dv}$

- Q: Is function T linear?

- Letting $g \in V$ and $h \in V$

- $T(cg + h) = \frac{d}{dv}(cg + h) = c \frac{dg}{dv} + \frac{dh}{dv}$

- $cT(g) + T(h) = c \frac{dg}{dv} + \frac{dh}{dv}$

- \therefore By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- Linear transformations

- Example 2.1.7 (Integration)

- $T: \mathbb{R} \rightarrow \mathbb{R}$ where $T(f) = \int_a^b f(t)dt$ for some $a, b \in \mathbb{R}$

- Q: Is function T linear?

- Letting $g \in \mathbb{R}$ and $h \in \mathbb{R}$

- $T(cg + h) = \int_a^b cg(t) + h(t)dt = c \int_a^b g(t)dt + \int_a^b h(t)dt$

- $cT(g) + T(h) = c \int_a^b g(t)dt + \int_a^b h(t)dt$

- \therefore By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- Linear transformations
 - Example (*Identity transformation*)
 - $T: V \rightarrow V$ where $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$
 - Q: Is function T linear?
 - Letting $\mathbf{x} \in V$ and $\mathbf{y} \in V$
 - $T(c\mathbf{x} + \mathbf{y}) = c\mathbf{x} + \mathbf{y}$
 - $cT(\mathbf{x}) + T(\mathbf{y}) = c\mathbf{x} + \mathbf{y}$
 - \therefore By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- Linear transformations
 - Example (Zero transformation)
 - $T: V \rightarrow W$ where $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$
 - Q: Is function T linear?
 - Letting $\mathbf{x} \in V$ and $\mathbf{y} \in V$
 - $T(c\mathbf{x} + \mathbf{y}) = \mathbf{0}$
 - $cT(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}$
 - \therefore By property ②, linear!

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Null space (kernel):

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.

We define **null space** (or kernel) $N(T)$ of T to be the set of all vectors $\mathbf{x} \in V$ such that

$$N(T) = \{\mathbf{x} \in V | T(\mathbf{x}) = \mathbf{0}\}$$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Range (image) (치역):

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.

We define **range** (or image) $R(T)$ of T to be the **subset of W** containing **all images under T** of vectors in V such that

$$R(T) = \{T(\mathbf{x}) | \mathbf{x} \in V\}$$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

- Example 2.1.8

- $T_1: V \rightarrow V$ where $T_1(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$ (identity transformation)

- Null space

- $N(T_1) = \{\mathbf{0}\}$

- Range

- $R(T_1) = V$

- $T_2: V \rightarrow W$ where $T_2(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$ (zero transformation)

- Null space

- $N(T_2) = V$

- Range

- $R(T_2) = \{\mathbf{0}\}$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

- Example 2.1.9

- $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ where $T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 - a_2 \\ 2a_3 \end{bmatrix}$ for all $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$

- Null space

- $N(T) = \left\{ \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$

- Range

- $R(T) = \mathbb{R}^2$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.1:

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear. Then, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W .

- Proof) ($N(T)$ is a subspace of V)
 - Theorem 1.3(a)
 - $\mathbf{0} \in N(T)$ since property ① of linear transformation indicates that $T(\mathbf{0}) = \mathbf{0}$
 - Theorem 1.3(b)
 - $\mathbf{x} + \mathbf{y} \in N(T)$ since $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}$ for $\mathbf{x}, \mathbf{y} \in N(T)$
 - Theorem 1.3(c)
 - $c\mathbf{x} \in N(T)$ since $T(c\mathbf{x}) = cT(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} \in N(T)$
- \therefore Subspace!

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.1:

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear. Then, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W .

- Proof) ($R(T)$ is a subspace of W)
 - Theorem 1.3(a)
 - $\mathbf{0} \in R(T)$ since $\mathbf{0} \in V$ and property ① of linear transformation indicates that $T(\mathbf{0}) = \mathbf{0}$
 - Theorem 1.3(b)
 - $\mathbf{x} + \mathbf{y} \in R(T)$ since $\mathbf{x} + \mathbf{y} = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2) \in R(T)$ for $\mathbf{x} = T(\mathbf{v}_1), \mathbf{y} = T(\mathbf{v}_2) \in R(T)$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$
 - Theorem 1.3(c)
 - $c\mathbf{x} \in R(T)$ since $c\mathbf{x} = cT(\mathbf{v}_1) = T(c\mathbf{v}_1) \in R(T)$ for $\mathbf{x} = T(\mathbf{v}_1) \in R(T)$ and $\mathbf{v}_1 \in V$
- \therefore Subspace!

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.2:

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.

$\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of $V \Rightarrow R(T) = \text{span}(T(\beta)) = \text{span}(T(\mathbf{v}_1), \dots, T(\mathbf{v}_n))$

- Proof)
 - $(\text{span}(T(\beta)) \subseteq R(T))$
 - Note that $T(\mathbf{v}_i) \in R(T), \forall i$
 - From Theorem 2.1, $R(T)$ is a subspace
 - $\Rightarrow \text{span}(\{T(\mathbf{v}_i)\}) = \text{span}(T(\beta)) \in R(T)$ by Theorem 1.5
 - $(\text{span}(T(\beta)) \supseteq R(T))$
 - $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i$ for any $\mathbf{v} \in V$
 - $\Rightarrow T(\mathbf{v}) \in R(T)$
 - $T(\mathbf{v}) = T(\sum_{i=1}^n a_i \mathbf{v}_i) = \sum_{i=1}^n a_i T(\mathbf{v}_i) \in \text{span}(\{T(\mathbf{v}_i)\}) = \text{span}(T(\beta))$ for any $T(\mathbf{v}) \in R(T)$
 - $\Rightarrow R(T) \in \text{span}(T(\beta))$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

- Example 2.1.10

- $T: V = \mathbb{R}^3 \rightarrow M_{2 \times 2}(\mathbb{R})$ where

- $T(\mathbf{v}) = \begin{bmatrix} v_2 - v_3 & 0 \\ 0 & v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

- Linear

- For a standard basis $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

- $R(T) = \text{span}(T(\beta)) = \text{span} \left(\left\{ T \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right), T \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right), T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \right\} \right) = \text{span} \left(\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \right) =$
 $\text{span} \left(\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \right)$

- $\Rightarrow \dim(R(T)) = 2$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

- Example 2.1.10

- $T: V = \mathbb{R}^3 \rightarrow M_{2 \times 2}(\mathbb{R})$ where

- $T(\mathbf{v}) = \begin{bmatrix} v_2 - v_3 & 0 \\ 0 & v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

- Linear

- To find a basis for $N(T)$, by letting $T(\mathbf{v}) = \mathbf{0}$,

- $N(T) = \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$

- $\Rightarrow \dim(N(T)) = 1$

- Note that $\dim(V) = \dim(N(T)) + \dim(R(T))$

- (Theorem 2.3 coming soon!)

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

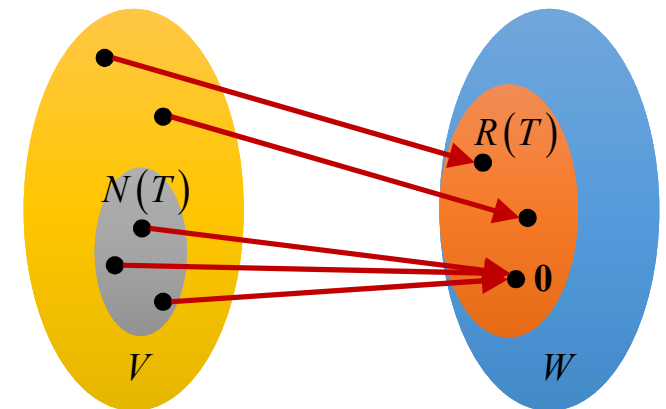
Nullity and rank:

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
If $N(T)$ and $R(T)$ are finite-dimensional,

$$\text{nullity}(T) \triangleq \dim(N(T))$$

$$\text{rank}(T) \triangleq \dim(R(T))$$

- Intuition
 - The **larger** the nullity, the **smaller** the rank
 - The **more vectors** carried into $\mathbf{0}$, the **smaller** the range



$$T: V \rightarrow W$$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.3 (Dimension Theorem):

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
If V is finite-dimensional,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)

- Let $n = \dim(V)$ and $k = \dim(N(T))$ where $n \geq k$.
- Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for $N(T)$.
- Note that $N(T)$ is a subspace of vector space V .
 - \Rightarrow From [Corollary 1.11.1](#), we may extend $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ to a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ for V .

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.3 (Dimension Theorem):

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
If V is finite-dimensional,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
 - From [Theorem 2.2](#),
 - $R(T) = \text{span}(T(\beta)) = \text{span}(T(\mathbf{v}_1), \dots, T(\mathbf{v}_k), T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n))$
 - Noting that $T(\mathbf{v}_1) = \dots = T(\mathbf{v}_k) = \mathbf{0}$
 - $R(T) = \text{span}(\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\})$
 - $\Rightarrow \{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ spans $R(T)$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.3 (Dimension Theorem):

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
If V is finite-dimensional,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)

- If $\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ is a linearly independent set,
 - $\sum_{i=k+1}^n b_i T(\mathbf{v}_i) = \mathbf{0}$ only when $b_i = 0, i = k + 1, \dots, n$
- From the linear property of T ,
 - $\sum_{i=k+1}^n b_i T(\mathbf{v}_i) = T(\sum_{i=k+1}^n b_i \mathbf{v}_i)$
- If $T(\sum_{i=k+1}^n b_i \mathbf{v}_i) = \mathbf{0}$, then
 - $\sum_{i=k+1}^n b_i \mathbf{v}_i \in N(T)$
- Hence, $\sum_{i=k+1}^n b_i \mathbf{v}_i$ may be expressed as a linear combination of the basis of $N(T)$
 - $\sum_{i=k+1}^n b_i \mathbf{v}_i = \sum_{i=1}^k c_i \mathbf{v}_i$ for some c_i 's
- Since $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ are a basis for V ,
 - $\sum_{i=k+1}^n b_i \mathbf{v}_i - \sum_{i=1}^k c_i \mathbf{v}_i = \mathbf{0}$ only when $b_i = 0, i = k + 1, \dots, n$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.3 (Dimension Theorem):

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
If V is finite-dimensional,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

- Proof)
 - Hence $\{T(\mathbf{v}_{k+1}), \dots, T(\mathbf{v}_n)\}$ is a basis for $R(T)$
 - $\Rightarrow \dim(R(T)) = \text{rank}(T) = n - k$
 - \therefore Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.4:

Let V and W be vector spaces and let function $T: V \rightarrow W$ be linear.
Then, T is one-to-one $\Leftrightarrow N(T) = \{\mathbf{0}\}$

- Proof)

- (T is one-to-one $\Rightarrow N(T) = \{\mathbf{0}\}$)
 - From the one-to-one property, there exists only one \mathbf{x} to satisfy $T(\mathbf{x}) = \mathbf{0}$.
 - In the meantime, by the linear property, $T(\mathbf{0}) = \mathbf{0}$.
 - $\Rightarrow \mathbf{x} = \mathbf{0}$
 - $\Rightarrow N(T) = \{\mathbf{0}\}$
- (T is one-to-one $\Leftarrow N(T) = \{\mathbf{0}\}$)
 - By contradiction, assume T is not one-to-one.
 - $\Rightarrow T(\mathbf{x}) = T(\mathbf{y})$ for some distinct $\mathbf{x}, \mathbf{y} \in V$
 - By the linear property, $\mathbf{0} = T(\mathbf{x}) - T(\mathbf{y}) = T(\mathbf{x} - \mathbf{y})$
 - $\Rightarrow \mathbf{x} - \mathbf{y} \in N(T)$ where $\mathbf{x} - \mathbf{y} \neq \mathbf{0}$
 - \therefore Contradiction

- \therefore Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.5:

Let V and W be finite-dimensional vector spaces of **equal dimension**, and let function $T: V \rightarrow W$ be linear.

Then, T is **one-to-one** $\Leftrightarrow T$ is **onto** $\Leftrightarrow \text{rank}(T) = \text{dim}(V)$

- “**One-to-one**” (*Appendix B*)

- $T(\mathbf{v}_1) = T(\mathbf{v}_2) \Rightarrow \mathbf{v}_1 = \mathbf{v}_2$, for some $\mathbf{v}_1, \mathbf{v}_2 \in V$
- Equivalently, $\mathbf{v}_1 \neq \mathbf{v}_2 \Rightarrow T(\mathbf{v}_1) \neq T(\mathbf{v}_2)$, for some $\mathbf{v}_1, \mathbf{v}_2 \in V$

- “**Onto**” (*Appendix B*)

- $T(\mathbf{v}) = W$, for all $\mathbf{v} \in V$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.5:

Let V and W be finite-dimensional vector spaces of **equal dimension**, and let function $T: V \rightarrow W$ be linear.

Then, T is **one-to-one** $\Leftrightarrow T$ is **onto** $\Leftrightarrow \text{rank}(T) = \text{dim}(V)$

- Proof)
 - From **Theorem 2.4**, T is one-to-one $\Leftrightarrow N(T) = \{\mathbf{0}\}$ or $\text{nullity}(T) = 0$
 - Also, from **Theorem 2.3** (Dimension theorem), $\text{nullity}(T) + \text{rank}(T) = \text{dim}(V)$
 - $\Rightarrow T$ is **one-to-one** $\Leftrightarrow \text{rank}(T) = \text{dim}(V)$
 - From the equal dimension condition,
 - $\Rightarrow \text{rank}(T) = \text{dim}(V) \Leftrightarrow \text{rank}(T) = \text{dim}(W)$
 - From **Theorem 1.11**,
 - $\Rightarrow \text{rank}(T) = \text{dim}(W) \Leftrightarrow R(T) = W$
- \therefore Q.E.D.

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

- Example 2.1.12

- $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where

- $T(\mathbf{v}) = \begin{bmatrix} v_1 + v_2 \\ v_1 \end{bmatrix}$ for $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

- Linear

- To find a vector for $N(T)$, by letting $T(\mathbf{v}) = \mathbf{0}$,

- $N(T) = \{\mathbf{0}\}$

- By Theorem 2.4

- One-to-one

- By Theorem 2.5

- Onto

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges

Theorem 2.6:

Let V and W be vector spaces of equal dimension, and suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V .

For $\mathbf{w}_1, \dots, \mathbf{w}_n \in W$, there exists **exactly one linear** transformation $T: V \rightarrow W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$.

- Proof)

- From the linear property, for $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i$ with unique scalars a_1, \dots, a_n
 - $T(\mathbf{v}) = T(\sum_{i=1}^n a_i \mathbf{v}_i) = \sum_{i=1}^n a_i T(\mathbf{v}_i) = \sum_{i=1}^n a_i \mathbf{w}_i$
- If there exists another linear function $U: V \rightarrow W$ such that $U(\mathbf{v}_i) = \mathbf{w}_i$
 - $U(\mathbf{v}) = U(\sum_{i=1}^n a_i \mathbf{v}_i) = \sum_{i=1}^n a_i U(\mathbf{v}_i) = \sum_{i=1}^n a_i \mathbf{w}_i$
- Then, we must have $T(\mathbf{v}_i) = U(\mathbf{v}_i), \forall i$.
- Hence, $U = T$

2.1 Linear transformations, null spaces, and ranges

- Null spaces and ranges
 - An implication of [Theorem 2.6](#)
 - A [linear transformation](#) completely determined by its action on a [basis](#)

2.2 The matrix representation of a linear transformation

2.2 The matrix representation of a linear transformation

- **Section 2.1**
 - Studying linear transformations by examining their **null spaces** and **ranges**
- **Section 2.2**
 - Representing linear transformations by a **matrix**
 - Developing a **one-to-one correspondence** between matrices and linear transformations

2.2 The matrix representation of a linear transformation

- Ordered basis

Ordered basis:

Let V be a finite-dimensional vector space.

An **ordered basis** for V is a basis for V endowed with a **specific order**.

That is, an **ordered basis** is a finite sequence of **linearly independent** vectors in V that **spans** V .

- Example 2.2.1

- $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
 - A **standard ordered basis** in \mathbb{F}^3 where $\mathbf{e}_i, \forall i$ is a **standard basis**
- $\gamma = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$
 - Another ordered basis
- From the perspective of orders, $\beta \neq \gamma$

2.2 The matrix representation of a linear transformation

- Ordered basis

Coordinate vector:

Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an **ordered basis** for a finite-dimensional vector space V . For $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i \in V$, we define the **coordinate vector** of \mathbf{v} **relative to** β by

$$[\mathbf{v}]_{\beta} \triangleq \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

With unique scalars a_1, \dots, a_n

- Example 2.2.2

- $V = \mathbb{R}^3$ with $\beta = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$

- The coordinate vector of $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1\mathbf{e}_1 + 3\mathbf{e}_2 + 6\mathbf{e}_3$

- $\Rightarrow [\mathbf{v}]_{\beta} = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation
 - Letting
 - V be a vector space with an **ordered basis** $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$
 - W be a vector space with an **ordered basis** $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$
 - $T: V \rightarrow W$ be a **linear** function
 - Then, using the ordered basis γ
 - $T(\mathbf{v}_j) = \sum_{i=1}^m a_{ij} \mathbf{w}_i$ with **unique** scalars $a_{ij} \in F$ for $i = 1, \dots, m$ for each $j = 1, \dots, n$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

Matrix representation:

We call the $m \times n$ matrix \mathbf{A} defined by $[\mathbf{A}]_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ and write $\mathbf{A} = [T]_{\beta}^{\gamma}$.

If $V = W$ and $\beta = \gamma$, then we write $\mathbf{A} = [T]_{\beta}$.

- For instance, with $n = 2$ and $m = 3$ such that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$,

$$[T(\mathbf{v}_1) \quad T(\mathbf{v}_2)] = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad \mathbf{A} = [T]_{\beta}^{\gamma}$$

$[T(\mathbf{v}_1)]_{\gamma} \quad [T(\mathbf{v}_2)]_{\gamma}$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

- Example 2.2.3

- If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix}$, $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$, $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$

- $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = 3 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-4) \cdot \mathbf{w}_3$

- $\Rightarrow [T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$

- For $\gamma_2 = \{\mathbf{w}_1 = \mathbf{e}_3, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_1\}$

- $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{w}_3 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_1$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = (-4) \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 3 \cdot \mathbf{w}_3$

- $\Rightarrow [T]_{\beta}^{\gamma_2} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

- Letting

- V be a vector space with an **ordered basis** $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$
 - W be a vector space with an **ordered basis** $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$
 - $T: V \rightarrow W$ be a **linear** function

- (**Zero transformation**) Then, using the ordered basis γ

- $T(\mathbf{v}_j) = \mathbf{0} = 0 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + \dots + 0 \cdot \mathbf{w}_m$ for $j = 1, \dots, n$

- $\Rightarrow [T]_{\beta}^{\gamma} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} = [\mathbf{0}]$

- (**Identity transformation**) For $n = m$ and $\beta = \gamma$,

- $T(\mathbf{v}_j) = \mathbf{v}_j = 0 \cdot \mathbf{v}_1 + \dots + 1 \cdot \mathbf{v}_j + \dots + 0 \cdot \mathbf{v}_m$ for $j = 1, \dots, n$

- $\Rightarrow [T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = \mathbf{I}_n$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

Kronecker delta:

$$\delta_{ij} \triangleq \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- For instance,
 - $[\mathbf{I}_n]_{ij} = \delta_{ij}, \forall i, j$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

Addition and scalar multiplication of functions:

Let V and W be vector spaces over F .

Let $T, U: V \rightarrow W$ be arbitrary functions.

Then, for all $\mathbf{x} \in V$ and $a \in F$,

$$T + U: V \rightarrow W \triangleq (T + U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x})$$

$$aT: V \rightarrow W \triangleq (aT)(\mathbf{x}) = aT(\mathbf{x})$$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

Theorem 2.7:

Let V and W be vector spaces over F .

Let $T, U: V \rightarrow W$ be linear functions.

Then, for all $\mathbf{x} \in V$ and $a \in F$,

- (a) $aT + U$ is linear, i.e., $(aT + U)(c\mathbf{x} + \mathbf{y}) = c(aT + U)(\mathbf{x}) + (aT + U)(\mathbf{y})$
- (b) The collection of all linear transformations from V to W is a vector space over F .

- Proof)
 - (a)
 - Let $\mathbf{x}, \mathbf{y} \in V$ and $c \in F$.
 - $(aT + U)(c\mathbf{x} + \mathbf{y}) = (aT)(c\mathbf{x} + \mathbf{y}) + U(c\mathbf{x} + \mathbf{y}) = c(aT)(\mathbf{x}) + (aT)(\mathbf{y}) + cU(\mathbf{x}) + U(\mathbf{y}) = c((aT)(\mathbf{x}) + U(\mathbf{x})) + (aT)(\mathbf{y}) + U(\mathbf{y}) = c(aT + U)(\mathbf{x}) + (aT + U)(\mathbf{y})$
 - (b)
 - (Left as an exercise)

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

The vector space of all linear transformations:

Let V and W be vector spaces over F .

We denote the **vector space of all linear transformations** from V into W by $\mathcal{L}(V, W)$.

If $V = W$, we write $\mathcal{L}(V)$.

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

Theorem 2.8:

Let V and W be finite-dimensional vector spaces with ordered bases β and γ , respectively.

Let $T, U: V \rightarrow W$ be linear transformations.

Then,

(a) $[T + U]_{\beta}^{\gamma} = [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}$

(b) $[cT]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma}$ for all scalars c

- Proof)

- (a)

- Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$.
 - $T(\mathbf{v}_j) = \sum_{i=1}^m a_{ij} \mathbf{w}_i$ for unique scalars $a_{ij}, \forall i, j$
 - $U(\mathbf{v}_j) = \sum_{i=1}^m b_{ij} \mathbf{w}_i$ for unique scalars $b_{ij}, \forall i, j$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

Theorem 2.8:

Let V and W be finite-dimensional vector spaces with ordered bases β and γ , respectively.

Let $T, U: V \rightarrow W$ be linear transformations.

Then,

(a) $[T + U]_{\beta}^{\gamma} = [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}$

(b) $[cT]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma}$ for all scalars c

- Proof)

- (a)

- Then, $(T + U)(\mathbf{v}_j) = T(\mathbf{v}_j) + U(\mathbf{v}_j) = \sum_{i=1}^m (a_{ij} + b_{ij})\mathbf{w}_i$

- Thus, $[T + U]_{\beta}^{\gamma}_{ij} = a_{ij} + b_{ij} = [T]_{\beta}^{\gamma}_{ij} + [U]_{\beta}^{\gamma}_{ij}$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

Theorem 2.8:

Let V and W be finite-dimensional vector spaces with ordered bases β and γ , respectively.

Let $T, U: V \rightarrow W$ be linear transformations.

Then,

(a) $[T + U]_{\beta}^{\gamma} = [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}$

(b) $[cT]_{\beta}^{\gamma} = c[T]_{\beta}^{\gamma}$ for all scalars c

- Proof)

- (b)

- $(cT)(\mathbf{v}_j) = cT(\mathbf{v}_j) = c \sum_{i=1}^m a_{ij} \mathbf{w}_i$

- Thus, $[cT]_{\beta}^{\gamma} = c a_{ij} = c [T]_{\beta}^{\gamma}$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

- Example 2.2.5

- Let $T, U: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be linear and $\beta = \{\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \mathbf{e}_2\}$ for T and $\gamma = \{\mathbf{w}_1 = \mathbf{e}_1, \mathbf{w}_2 = \mathbf{e}_2, \mathbf{w}_3 = \mathbf{e}_3\}$ for U

- If $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix}$

- $T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$ and $T(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = 3 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-4) \cdot \mathbf{w}_3$

- $\Rightarrow [T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$

- If $U\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 - v_2 \\ 2v_1 \\ 3v_1 + 2v_2 \end{bmatrix}$

- $U(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2 + 3 \cdot \mathbf{w}_3$ and $U(\mathbf{v}_2) = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = (-1) \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 2 \cdot \mathbf{w}_3$

- $\Rightarrow [U]_{\beta}^{\gamma} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$

2.2 The matrix representation of a linear transformation

- The matrix representation of a linear transformation

- Example 2.2.5

- Since $(T + U)\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) + U\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 + 3v_2 \\ 0 \\ 2v_1 - 4v_2 \end{bmatrix} + \begin{bmatrix} v_1 - v_2 \\ 2v_1 \\ 3v_1 + 2v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 2v_2 \\ 2v_1 \\ 5v_1 - 2v_2 \end{bmatrix}$

- $(T + U)(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} = 2 \cdot \mathbf{w}_1 + 2 \cdot \mathbf{w}_2 + 5 \cdot \mathbf{w}_3$ and $(T + U)(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2 \cdot \mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + (-2) \cdot \mathbf{w}_3$

- $\Rightarrow [T + U]_{\beta}^{\gamma} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$

- Note that

- $[T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix} = [T + U]_{\beta}^{\gamma}$