Quiz3 Q1: Jacobian Matrix

(x, y) denotes the original coordinate of the point in the original image I, while $(x^{'}, y^{'})$ the coordinate of the point in the transformed image $I^{'}$.

1 Translation

$$\begin{bmatrix} x^{'} \\ y^{'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} \tag{1}$$

It only has parameters t_x and t_y, and the corresponding Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial x^{'}}{\partial t_x} & \frac{\partial x^{'}}{\partial t_y} \\ \frac{\partial y^{'}}{\partial t_x} & \frac{\partial y^{'}}{\partial t_y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

$I^{'}(x, y) = I(x + 3, y - 2)$ defines the translation, and there are 2 non-zero elements in J.

2 Euclidean Transformation

$$\begin{bmatrix} x^{'} \\ y^{'} \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) & t_x \\ sin(\theta) & cos(\theta) & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \cdot cos(\theta) - y \cdot sin(\theta) + t_x \\ x \cdot sin(\theta) + y \cdot cos(\theta) + t_y \end{bmatrix} \tag{3}$$

Generally, Euclidean transform has 3 parameters: rotation angle θ, translation t_x, t_y.
The Jacobian matrix of Euclidean transform is

$$J = \begin{bmatrix} \frac{\partial x^{'}}{\partial t_x} & \frac{\partial x^{'}}{\partial t_y} & \frac{\partial x^{'}}{\partial \theta} \\ \frac{\partial y^{'}}{\partial t_x} & \frac{\partial y^{'}}{\partial t_y} & \frac{\partial y^{'}}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -x \cdot sin(\theta) - y \cdot cos(\theta) \\ 0 & 1 & x \cdot cos(\theta) - y \cdot sin(\theta) \end{bmatrix} \tag{4}$$

Note: $I^{'}(x, y) = I(-y, x)$ defines the Euclidean transformation without translation (only parameter θ). The Jacobian matrix should be

$$J = \begin{bmatrix} \frac{\partial x^{'}}{\partial \theta} \\ \frac{\partial y^{'}}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -x \cdot sin(\theta) - y \cdot cos(\theta) \\ x \cdot cos(\theta) - y \cdot sin(\theta) \end{bmatrix} \tag{5}$$

So there are only 2 non-zero elements in the Jacobian matrix.

3 Similarity

$$\begin{bmatrix} x^{'} \\ y^{'} \end{bmatrix} = \begin{bmatrix} s \cdot cos(\theta) & -s \cdot sin(\theta) & t_x \\ s \cdot sin(\theta) & s \cdot cos(\theta) & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s \cdot x \cdot cos(\theta) - s \cdot y \cdot sin(\theta) + t_x \\ s \cdot x \cdot sin(\theta) + s \cdot y \cdot cos(\theta) + t_y \end{bmatrix} \tag{6}$$

Generally, similarity transform has 4 parameters: scale factor s, rotation angle θ, translation tx, ty.
The Jacobian matrix of similarity transformation is

$$J = \begin{bmatrix} \frac{\partial x^{'}}{\partial t_x} & \frac{\partial x^{'}}{\partial t_y} & \frac{\partial x^{'}}{\partial s} & \frac{\partial x^{'}}{\partial \theta} \\ \frac{\partial y^{'}}{\partial t_x} & \frac{\partial y^{'}}{\partial t_y} & \frac{\partial y^{'}}{\partial s} & \frac{\partial y^{'}}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & x \cdot cos(\theta) - y \cdot sin(\theta) & -s \cdot x \cdot sin(\theta) - s \cdot y \cdot cos(\theta) \\ 0 & 1 & x \cdot sin(\theta) + y \cdot cos(\theta) & s \cdot x \cdot cos(\theta) - s \cdot y \cdot sin(\theta) \end{bmatrix} \tag{7}$$

Note: $I^{'}(x, y) = I(0.3x + 3, 0.3y - 2)$ defines the similarity transformation without rotation (only parameter s,t_x,t_y). The Jacobian matrix should be

$$J = \begin{bmatrix} \frac{\partial x^{'}}{\partial t_x} & \frac{\partial x^{'}}{\partial t_y} & \frac{\partial x^{'}}{\partial s} \\ \frac{\partial y^{'}}{\partial t_x} & \frac{\partial y^{'}}{\partial t_y} & \frac{\partial y^{'}}{\partial s} \end{bmatrix} = \begin{bmatrix} 1 & 0 & x \cdot cos(\theta) - y \cdot sin(\theta) \\ 0 & 1 & x \cdot sin(\theta) + y \cdot cos(\theta) \end{bmatrix} \tag{8}$$

So there are only 4 non-zero elements in the Jacobian matrix.

4 Affine

$$\begin{bmatrix} x^{'} \\ y^{'} \end{bmatrix} = \begin{bmatrix} p_1 & p_3 & t_x \\ p_2 & p_4 & t_y \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 \cdot x + p_3 \cdot y + t_x \\ p_2 \cdot x + p_4 \cdot y + t_y \end{bmatrix} \tag{9}$$

Generally, affine transform has 6 parameters: p_1, p_2, p_3, p_4, and translation tx, ty.
The Jacobian matrix of affine transformation is

$$J = \begin{bmatrix} \frac{\partial x^{'}}{\partial t_x} & \frac{\partial x^{'}}{\partial t_y} & \frac{\partial x^{'}}{\partial p_1} & \frac{\partial x^{'}}{\partial p_2} & \frac{\partial x^{'}}{\partial p_3} & \frac{\partial x^{'}}{\partial p_4} \\ \frac{\partial y^{'}}{\partial t_x} & \frac{\partial y^{'}}{\partial t_y} & \frac{\partial y^{'}}{\partial p_1} & \frac{\partial y^{'}}{\partial p_2} & \frac{\partial y^{'}}{\partial p_3} & \frac{\partial y^{'}}{\partial p_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 \\ 0 & 1 & 0 & x & 0 & y \end{bmatrix} \tag{10}$$

Note:

(1) $I^{'}(x, y) = I(0.3x + 3, 0.5y - 2)$ defines the affine transformation with all the 6 parameters. The Jacobian matrix should be

$$J = \begin{bmatrix} \frac{\partial x^{'}}{\partial t_x} & \frac{\partial x^{'}}{\partial t_y} & \frac{\partial x^{'}}{\partial p_1} & \frac{\partial x^{'}}{\partial p_4} \\ \frac{\partial y^{'}}{\partial t_x} & \frac{\partial y^{'}}{\partial t_y} & \frac{\partial y^{'}}{\partial p_1} & \frac{\partial y^{'}}{\partial p_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & y \end{bmatrix} \tag{11}$$

So there are only 4 non-zero elements in the Jacobian matrix.

(2) $I^{'}(x, y) = I(x + 0.3y + 3, 0.5x + y - 2)$ defines the affine transformation without parameters p_2 and p_3. The Jacobian matrix should be

$$J = \begin{bmatrix} \frac{\partial x^{'}}{\partial t_x} & \frac{\partial x^{'}}{\partial t_y} & \frac{\partial x^{'}}{\partial p_1} & \frac{\partial x^{'}}{\partial p_2} & \frac{\partial x^{'}}{\partial p_3} & \frac{\partial x^{'}}{\partial p_4} \\ \frac{\partial y^{'}}{\partial t_x} & \frac{\partial y^{'}}{\partial t_y} & \frac{\partial y^{'}}{\partial p_1} & \frac{\partial y^{'}}{\partial p_2} & \frac{\partial y^{'}}{\partial p_3} & \frac{\partial y^{'}}{\partial p_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & x & 0 & y & 0 \\ 0 & 1 & 0 & x & 0 & y \end{bmatrix} \tag{12}$$

So there are only 6 non-zero elements in the Jacobian matrix.