Math Samples

CS 4243 Computer Vision & Pattern Recognition

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Eigendecomposition of 2x2 Matrices

The **eigenvectors** of a matrix **A** are the vectors **x** that satisfy: $$Ax = \lambda x$$

The scalar λ is the **eigenvalue** corresponding to **x.** $$det(A - \lambda I) = 0$$

The eigenvalues are found by solving:

$$det \begin{bmatrix} h_{11} - \lambda & h_{12} \\ h_{21} & h_{22} - \lambda \end{bmatrix} = 0$$

For us, ***A*** = ***H*** is a 2x2 matrix; we can directly solve for the eigenvalues:

$$\lambda_{\pm} = \frac{1}{2}\left[(h_{11} + h_{22}) \pm \sqrt{4h_{12}h_{21} + (h_{11} - h_{22})^2}\right]$$

Once you know λ, you find **x** by solving

$$\begin{bmatrix} h_{11} - \lambda & h_{12} \\ h_{21} & h_{22} - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

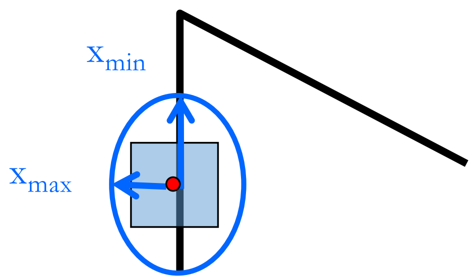
The Math Behind Corner Detection

$$E(u,v) \approx \begin{bmatrix} u & v \end{bmatrix} \underbrace{\begin{bmatrix} A & B \\ B & C \end{bmatrix}}_{H} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$A = \sum_{(x,y)\in W} I_x^2$$

$$B = \sum_{(x,y)\in W} I_x I_y$$

$$C = \sum_{(x,y)\in W} I_y^2$$



$$H x_{\max} = \lambda_{\max} x_{\max}$$

$$H x_{\min} = \lambda_{\min} x_{\min}$$

Eigenvalues and eigenvectors of H define shift directions with the smallest and largest change in error

- x_{max} direction of largest increase in E
- λ_{max} related to amount of increase in direction x_{max}
- x_{min} direction of smallest increase in E
- λ_{min} related to amount of increase indirection x_{min}

If corners represent having large increase in all directions, this suggests that both λ_{max} and λ_{min} should be sufficiently large!

Adapted from: S. Seitz, D. Frolova, D. Simakov

The Maths of Mean Shift (1)

Data is d-dimensional; so density function is also d-dimensional.

Given n data points $\mathbf{x}_i \in \mathbb{R}^d$, the multivariate kernel density estimate using a radially symmetric kernel[1] (e.g., Epanechnikov and Gaussian kernels), $K(\mathbf{x})$, is given by,

$$\hat{f}_K = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{\mathbf{x} - \mathbf{x}_i}{h}\right), \quad (1)$$

Approximated density is a summation of the kernels centered on each point x_i.

where h (termed the *bandwidth* parameter) defines the radius of kernel. The radially symmetric kernel is defined as,

$$K(\mathbf{x}) = c_k k(\|\mathbf{x}\|^2), \quad (2)$$

where c_k represents a normalization constant.

https://courses.csail.mit.edu/6.869/handouts/PAMIMeanshift.pdf

Comaniciu & Meer, 2002

The Maths of Mean Shift (2)

Taking the gradient ("derivative") of: $\hat{f}_K = \frac{1}{nh^d} \sum_{i=1}^{n} K\left(\frac{\mathbf{x}-\mathbf{x}_i}{h}\right)$

$$\nabla \hat{f}(\mathbf{x}) = \underbrace{\frac{2c_{k,d}}{nh^{d+2}} \left[\sum_{i=1}^{n} g\left(\left\|\frac{\mathbf{x}-\mathbf{x}_i}{h}\right\|^2\right)\right]}_{\text{term 1}} \underbrace{\left[\frac{\sum_{i=1}^{n} \mathbf{x}_i g\left(\left\|\frac{\mathbf{x}-\mathbf{x}_i}{h}\right\|^2\right)}{\sum_{i=1}^{n} g\left(\left\|\frac{\mathbf{x}-\mathbf{x}_i}{h}\right\|^2\right)} - \mathbf{x}\right]}_{\text{term 2}}, \qquad (3)$$

We want this to be equal to 0

where $g(x) = -k'(x)$ denotes the derivative of the selected kernel profile.

Proportional to density estimate at x; Unlikely to be 0

Only option is for this term to be 0.

$$x = \frac{\sum_{i=1}^{n} \mathbf{x}_i g\left(\left\|\frac{\mathbf{x}-\mathbf{x}_i}{h}\right\|^2\right)}{\sum_{i=1}^{n} g\left(\left\|\frac{\mathbf{x}-\mathbf{x}_i}{h}\right\|^2\right)}$$

But how do we solve this? Incrementally!

https://courses.csail.mit.edu/6.869/handouts/PAMIMeanshift.pdf

Comaniciu & Meer, 2002

Determining the homography matrix w/ DLT

Given a set of matched key points $\{p_i, p'_i\}$ find the best estimate of H s.t. $P' = H \cdot P$

Write out linear equation for each correspondence:

$$P' = H \cdot P \qquad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Expand matrix multiplication:

$$x' = \alpha(h_1 x + h_2 y + h_3)$$
$$y' = \alpha(h_4 x + h_5 y + h_6)$$
$$1 = \alpha(h_7 x + h_8 y + h_9)$$

Divide out unknown scale factor:

$$x'(h_7 x + h_8 y + h_9) = (h_1 x + h_2 y + h_3)$$
$$y'(h_7 x + h_8 y + h_9) = (h_4 x + h_5 y + h_6)$$

Slide credit: K. Kitani

Determining the homography matrix w/ DLT

Stack together constraints from multiple point correspondences:

$$\mathbf{A}\boldsymbol{h} = \mathbf{0}$$

Homogeneous linear least squares problem.
We want to avoid the trivial solution $h = 0$.

$$\begin{bmatrix} -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \\ -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \\ & & & & \vdots & & & & \\ -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Use singular value decomposition (SVD) on A. Set $\boldsymbol{h}$ equal to the (right) singular vector corresponding to smallest singular value. (detailed proof)

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$

Singular values found on diagonal elements of Σ.

Columns of V form (right) singular vectors.

Slide credit: K. Kitani

Solving LK Alignment

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p})) + \nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \Delta \boldsymbol{p} - T(\boldsymbol{x}) \right]^2$$

re-arrange terms

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[\underbrace{\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}}}_{\text{vector of constants}} \underbrace{\Delta \boldsymbol{p}}_{\text{vector of variables}} - \underbrace{\{T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))\}}_{\text{constant}} \right]^2$$

Yet another example of least-squares approximation!

$$\min_{\Delta \boldsymbol{p}} \sum_{\boldsymbol{x}} \left[\mathbf{Ax} - \mathbf{b} \right]^2$$

Least squares approximation

$$\hat{x} = \arg\min_{x} ||Ax - b||^2$$

is solved by

$$x = (A^\top A)^{-1} A^\top b$$

The LK objective is minimized when

$$\Delta \boldsymbol{p} = H^{-1} \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top [T(\boldsymbol{x}) - I(\mathbf{W}(\boldsymbol{x};\boldsymbol{p}))]$$

Hessian Matrix

$$H = \sum_{\boldsymbol{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]^\top \left[\nabla I \frac{\partial \mathbf{W}}{\partial \boldsymbol{p}} \right]$$

For successful inversion, H should be well-conditioned, i.e. eigenvalues are both large and approx. similar in magnitude → "corner" region!

In our derivation, we will use Bhattacharyya coefficient for comparing p and q, i.e.

$$\rho[\mathbf{p}(\mathbf{y}), \mathbf{q}] = \sum_m \sqrt{p_m(\mathbf{y}) q_m}$$

A large ρ means p and q are similar.

Objective function $\max_{\boldsymbol{y}} \rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}]$

Assuming a good initial guess

$$\rho[\boldsymbol{p}(\boldsymbol{y}_0 + \boldsymbol{y}), \boldsymbol{q}]$$

$$p_m = C_h \sum_n k\left(\left\|\frac{\boldsymbol{y} - \boldsymbol{y}_n}{h}\right\|^2\right) \delta[b(\boldsymbol{y}_n) - m]$$

Linearize around the initial guess (Taylor series expansion)

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m} + \frac{1}{2} \sum_m p_m(\boldsymbol{y}) \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}}$$

function at specified value

derivative

Fully expanded

$$\rho[\boldsymbol{p}(\boldsymbol{y}), \boldsymbol{q}] \approx \frac{1}{2} \sum_m \sqrt{p_m(\boldsymbol{y}_0) q_m} + \frac{1}{2} \sum_m \left\{ C_h \sum_n k\left(\left\|\frac{\boldsymbol{y} - \boldsymbol{y}_n}{h}\right\|^2\right) \delta[b(\boldsymbol{y}_n) - m] \right\} \sqrt{\frac{q_m}{p_m(\boldsymbol{y}_0)}}$$