Flip Graphs on Self-Complementary Ideals of Chain Products

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Motivation

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Our project stems from a generalization of maximal intersecting families. Flip graphs on maximally intersecting families have been studied before, and our goal is to generalize these results.

Ideals

Let $\ell_1, \ldots, \ell_d$ be a sequence of positive integers. Define

$$P = \{1, \ldots, \ell_1\} \times \cdots \times \{1, \ldots, \ell_d\}.$$

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A subset $I \subseteq P$ is an *ideal* if

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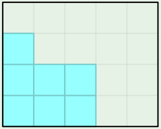

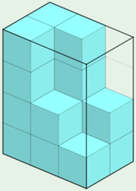

Self-Complementary Ideals

Let $P = \{1, \ldots, \ell_1\} \times \cdots \times \{1, \ldots, \ell_d\}$.

Definition

An ideal $I \subset P$ is *self-complementary* if for every $(a_1, \ldots, a_d) \in P$, exactly one of $(a_1, \ldots, a_d)$ or $(\ell_1 + 1 - a_1, \ldots, \ell_d + 1 - a_d)$ lies in I.

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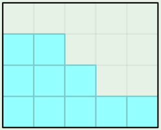

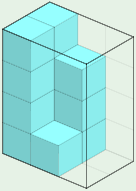

Flips

Let $P = \{1, \ldots, \ell_1\} \times \cdots \times \{1, \ldots, \ell_d\}$, and let I and J be two self-complementary ideals of P.

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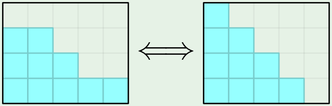

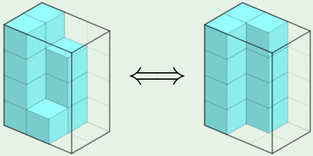

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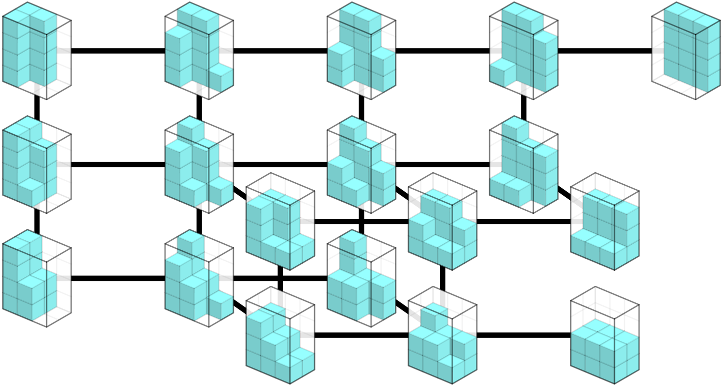
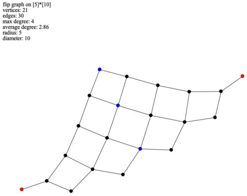
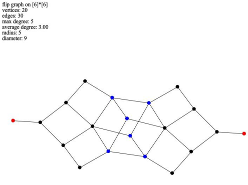
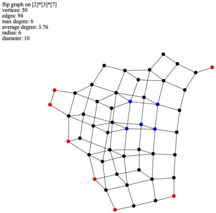
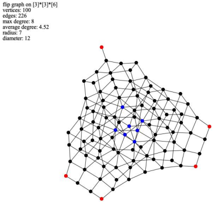


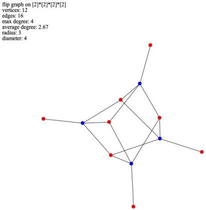
Flip Graph Examples

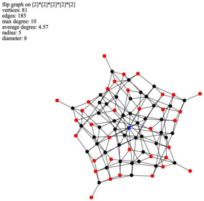






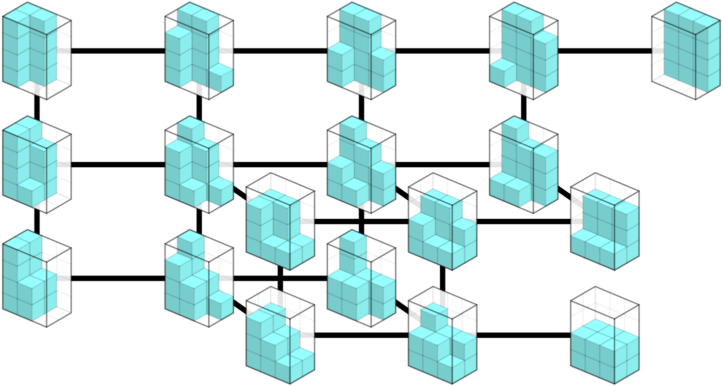






Graph Terminology

Let G be a connected graph.



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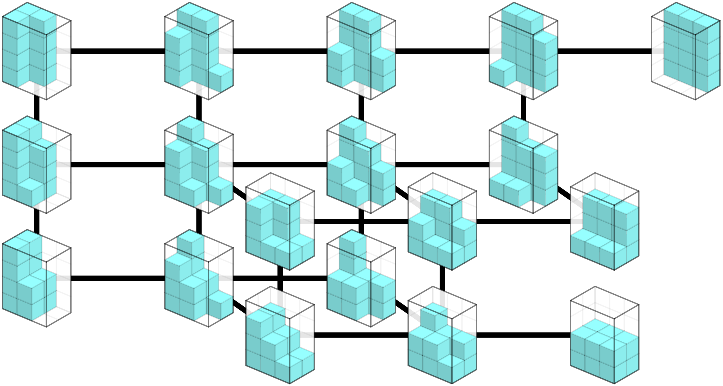
- The *eccentricity* of a vertex v is the maximum distance from v to another vertex.



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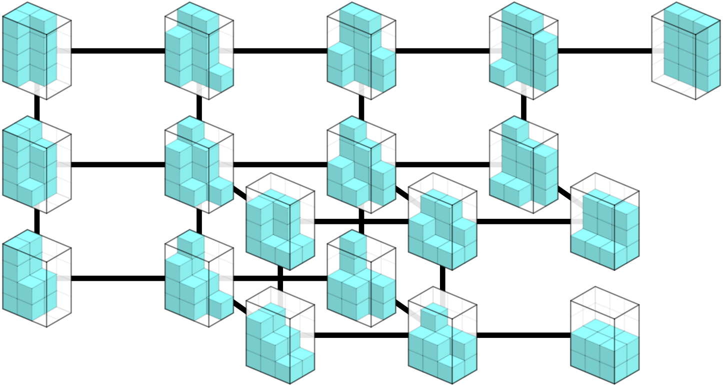

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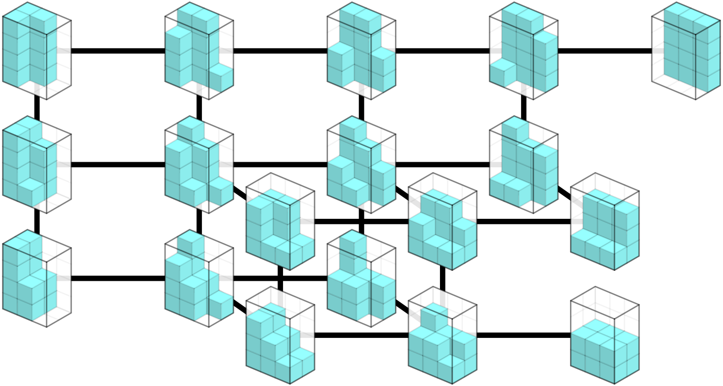
- The *diameter* of G is the maximum eccentricity of a vertex.
- The *radius* of G is the minimum eccentricity of a vertex.



Diameter of Flip Graphs on Self-Complementary Ideals

Let $P = \{1, \ldots, \ell_1\} \times \cdots \times \{1, \ldots, \ell_d\}$, and let G denote the flip graph on self-complementary ideals of P.

Theorem

The diameter of G is

$$\begin{cases} 0 & \text{if all of } \ell_1, \ldots, \ell_d \text{ are odd,} \\ \frac{1}{4}\left|P\right| & \text{if at least two of } \ell_1, \ldots, \ell_d \text{ are even, and} \\ \frac{1}{4}(|P| - \ell_k) & \text{if } \ell_k \text{ is even and the rest are odd.} \end{cases}$$

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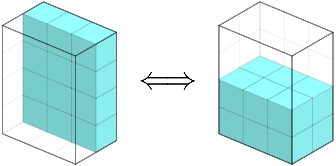

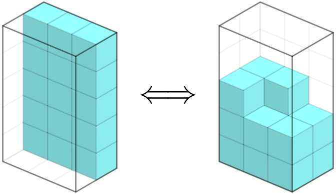

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Suppose $\ell_1, \ldots, \ell_d$ are even. Assuming Chvátal's conjecture, G's radius is

$$\left\lceil \left(\frac{1}{4} - \frac{1}{2^{d+1}} \binom{d-1}{\left\lfloor \frac{1}{2}(d-1) \right\rfloor} \right) |P| \right\rceil.$$

Cyclically Symmetric Self-Complementary Ideals

Let $P = \{1, \ldots, 2r\} \times \{1, \ldots, 2r\} \times \{1, \ldots, 2r\}$.

Definition

A self-complementary ideal $I \subset P$ is *cyclically symmetric* if

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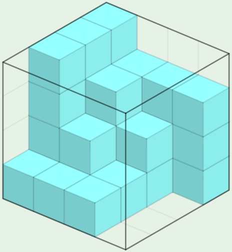

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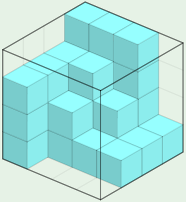

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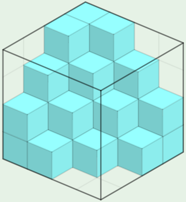

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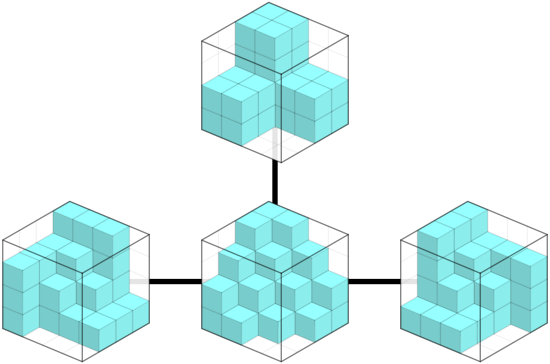

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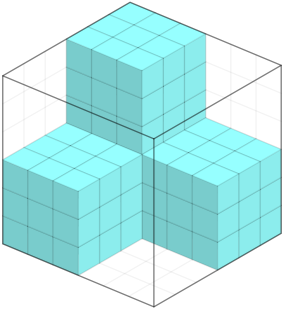

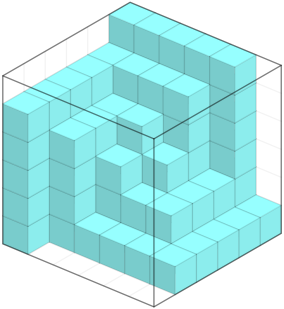

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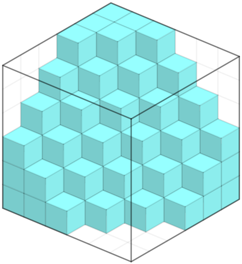

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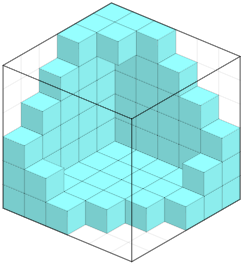

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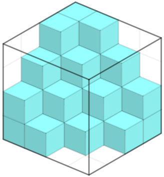

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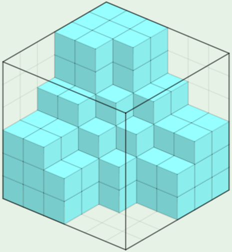

Properties of Flip Graphs on TSSC Ideals

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Theorem

The diameter of G is

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Conjecture

The radius of G is

$$\left\lceil \frac{1}{12}(r-1)(r)(2r-1) \right\rceil.$$

Future Directions

What we studied:

- vertex count
- diameter
- radius

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- vertex count
- diameter
- radius

Other properties of interest:

- maximum degree
- edge count and average degree
- set of vertices with minimum eccentricity (center)
- set of vertices with maximum eccentricity (perimeter)

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