Gamma Transmission Snow Gauge: Calibration and Validation

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0. Contribution Statement

Student 1 was responsible for the Conclusion, Question 4, Question 5, Question 6, and the formatting of the document. Student 2 was responsible for Questions 1, Questions 2, Question 3, the Advanced Analysis, and Introduction We both worked together on the code and double checked each other's work.

Use of GPT

The use of GPT was limited. It was only used for errors regarding the plotting of visualizations.

Introduction

Monitoring water supply in mountainous regions like the Sierra Nevada is vital for managing resources and predicting environmental changes. Accurate snow density measurements allow a better understanding of this region as snowpack significantly influences water availability. In this study, we will focus on calibrating a gamma transmission snow gague, a device used to estimate snow density indirectly by measuring gamma ray intensity. To establish the gauge's reliability, polyethylene blocks with known densities are placed between the gaguge poles, and multiple gamma ray intensity measurements are taken for each density. This allows us to determine a function that maps density to gain, and the inverse of this is then used in practice to map gamma ray intensity back to snow density. We specifically address the following questions:
1. Investigating the raw data to fit a regression line, plotting the residuals, and determining the need for transformation.
2. Identifying an appropriate transformation for the model, fitting the transformed data, and justifying the final model using theoretical and empirical evidence.
3. Assessing the robustness of the fit through simulations that account for potential inaccuracies in reported densities.
4. Providing forward predictions for gamma ray intensity with point estimates and uncertainty bands, and examining prediction intervals for specific densities.
5. Performing reverse predictions to estimate densities from gamma ray intensity and comparing predictions with true density values.
6. Cross-validating reverse predictions by omitting key density measurements and evaluating interval estimates against the actual density values.

Data

The dataset for this study contains 90 observations, comprising 10 measurements for each of 9 density levels (in grams per cubic centimeter of polyethylene). The goal of this study is to establish a reliable calibration procedure for the snow gauge to map measured gamma ray intensity to snow intensity. This involves taking steps such as measuring gamma ray intensity for polyethylene blocks of known density during calibration, developing mathematical functions to map density to gamma ray intensity, and inverting such functions to allow prediction of density from measured gamma ray intensity.

2. Analysis

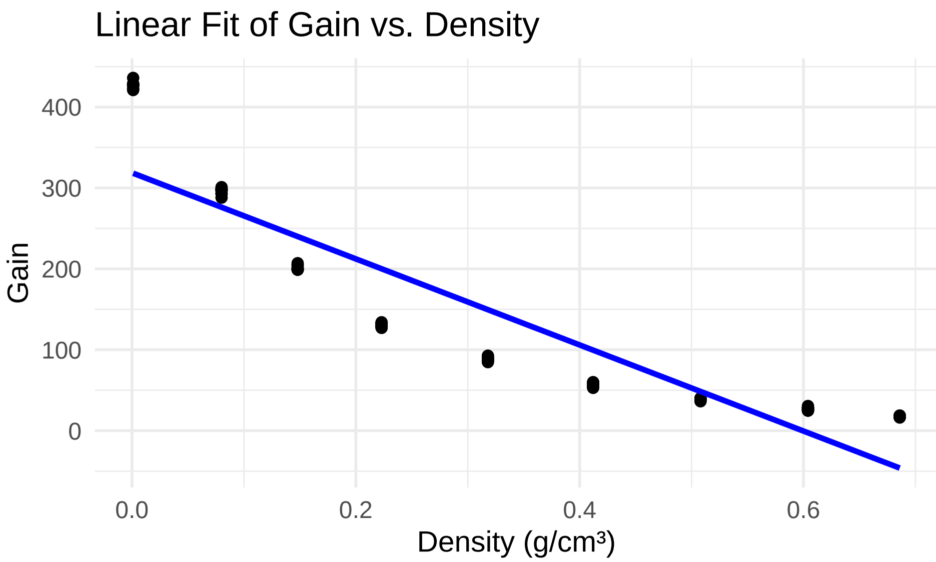
2.1 Raw Data

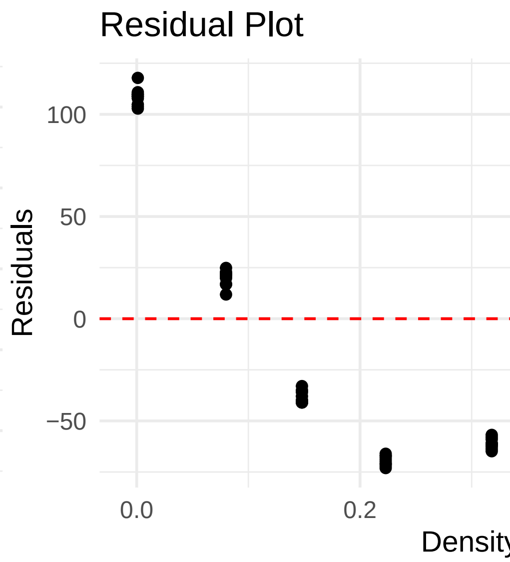
Method

A simple linear regression model was fitted to the raw data, with gamma ray intensity (gain) as the dependent variable and the density of the polyethylene blocks (density) as the independent variable. The purpose was to explore the relationship between these variables and assess the effectiveness of a linear fit.

Analysis

```
##
## Attaching package: 'dplyr'

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union

## Loading required package: proto

##
## Call:
## lm(formula = gain ~ density, data = data)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -73.08 -44.29  -9.72  30.82 117.83
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   318.70      10.79   29.54   <2e-16 ***
## density      -531.95      26.95  -19.73   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 57.54 on 88 degrees of freedom
## Multiple R-squared:  0.8157, Adjusted R-squared:  0.8136
## F-statistic: 389.5 on 1 and 88 DF,  p-value: < 2.2e-16

## `geom_smooth()` using formula = 'y ~ x'
```





Conclusion

The linear regression model did not provide a good fit for the raw data due to the exponential decay relationship between gain and density. The residual plot showed a clear pattern, further emphasizing the inadequacy of the linear model. A transformation is necessary to linearize the relationship and improve the model's fit. Based on the physics of the process, a logarithmic transformation of the gain variable will be explored in the next step.

2.2 Transformed Data

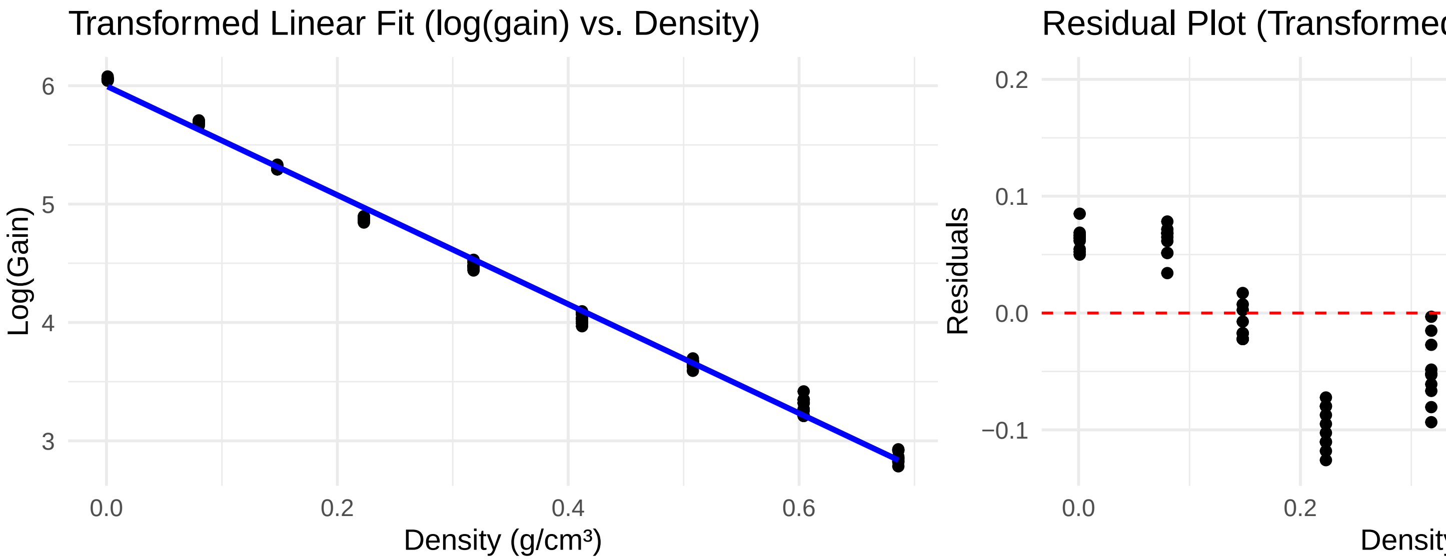
Method

To account for the exponential decay relationship between gamma ray intensity (gain) and density (density), the gain variable was log transformed, resulting in a new variable log_gain. We then refit the linear regression model with the new variable and then regenerated the scatterplot and residual plot to better understand how the transformation impacted the model.

Analysis

```
##
## Call:
## lm(formula = log_gain ~ density, data = data)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.131216 -0.052396 -0.004436  0.054607  0.202447
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.99727    0.01274   470.8   <2e-16 ***
## density     -4.60594    0.03182  -144.8   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
##
## Residual standard error: 0.06792 on 88 degrees of freedom
## Multiple R-squared:  0.9958, Adjusted R-squared:  0.9958
## F-statistic: 2.096e+04 on 1 and 88 DF,  p-value: < 2.2e-16

## `geom_smooth()` using formula = 'y ~ x'
```



Conclusion

The log transformation of gain successfully linearized the relationship with density. The transformed linear model provided a much better fit, as seen by the improved R-squared value and the presence of more randomness in the residuals. This transformation is both empirically and theoretically justified, given the exponential decay relationship inherent in the physics of gamma ray attenuation. The transformed model will serve as the basis for further analyses.

2.3 Robustness

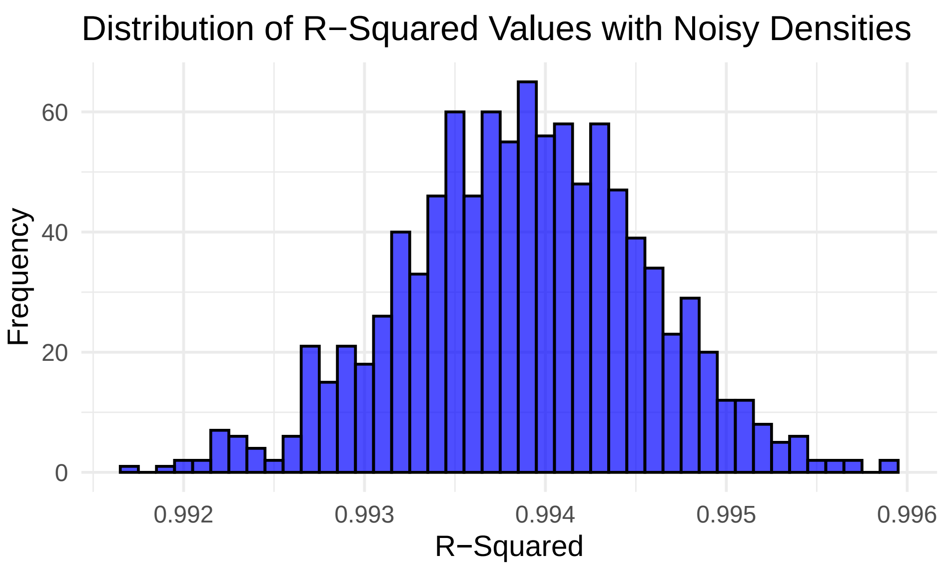
Methods

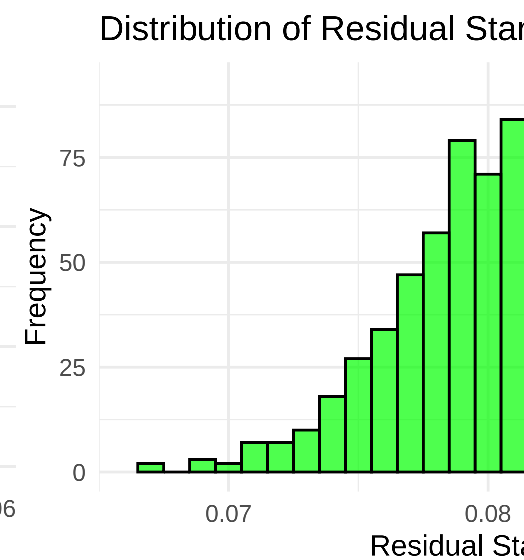
To evaluate the robustness of the fitted model to inaccuracies in the reported densities, we will run a simulation in which we simulate noise in the density column. For each simulation, we add noise to the density values in the dataset by sampling from a normal distribution with mean 0 and standard deviation .01. We then fit a linear regression model to the transformed gain column and noisy density column, calculate R-squared values and residual standard errors and store them.

Analysis

```
## Summary of R-squared values with noisy densities:

##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.9917  0.9934  0.9939  0.9939  0.9943  0.9959
```

```
##
## Summary of Residual Standard Errors with noisy densities:

##     Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
## 0.06725 0.07901 0.08209 0.08201 0.08501 0.09583
```





Conclusion

As can be seen by the histograms, the R-squared values when noise is present is centered around a mean of about .994, which means that about 99% of the variance of the density values can be explained by the log transform of the gain. Similarly, the distribution of residual standard errors is centered around about .08, which indicates that the model is robust to small inaccuracies in the reported density values.Because the R-squared values and residual standard errors remained in good ranges throughout the simulations, it is safe to assume that minor noise in density measurements does not significantly affect the performance of the model.

2.4 Forward Prediction

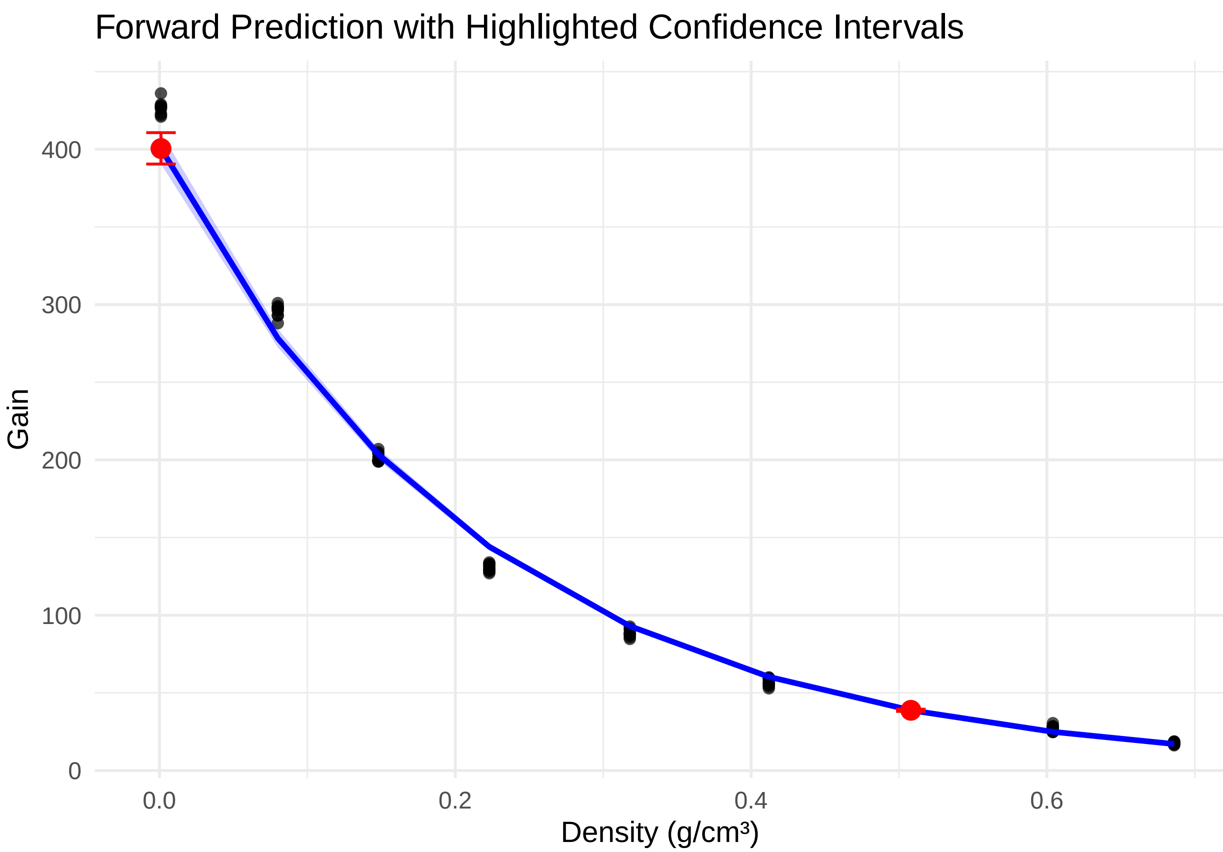
Methods

To establish a predictive relationship between density and gain, a linear regression model was fitted to the log-transformed gain values (log(gain) ~ density). This transformation was necessary due to the exponential relationship between density and gamma ray gain. Using the fitted model, predictions for gain values were generated on the original scale by applying the exponential transformation to the model outputs. Prediction intervals were computed for densities 0.508 and 0.001 to capture the range of plausible gain values, accounting for model uncertainty. The results were visualized by plotting the observed data, the model's fit, and the prediction intervals, providing a clear view of the model's predictive reliability.

Analysis

```
## Q4: Forward Predictions (Gain):
```

```
##   Density Predicted_Gain Lower_Bound Upper_Bound
## 1   0.508       38.76236    33.82731    44.41737
## 2   0.001      400.47829   349.09455   459.42528

## Confidence Intervals for Gains at Densities 0.508 and 0.001:

##   Density Predicted_Gain Lower_Bound Upper_Bound
## 1   0.508       38.76236    38.06717    39.47024
## 2   0.001      400.47829   390.48865   410.72349

## Warning: Using ‘size‘ aesthetic for lines was deprecated in ggplot2 3.4.0.
## i Please use ‘linewidth‘ instead.
## This warning is displayed once every 8 hours.
## Call ‘lifecycle::last_lifecycle_warnings()‘ to see where this warning was
## generated.
```



Density	Predicted_Gain	Lower_Bound	Upper_Bound
0.508	38.76236	38.06717	39.47024
0.001	400.47829	390.48865	410.72349

The forward prediction analysis sought to establish a reliable mapping from density to gain, with prediction intervals to assess uncertainty. The linear regression model fitted to the log-transformed gain (log(gain) ~ density) demonstrated a strong predictive relationship, as evident from the regression line plotted in the graph. The exponential transformation of the model output ensured that the predictions returned to the original scale of the data.

The graph shows the observed data points (black dots) distributed along the curve, with the blue line representing the fitted model and the shaded blue region indicating the prediction intervals. The prediction intervals widen as density increases, reflecting greater uncertainty in the gain predictions for higher densities. This behavior aligns with the physics of gamma transmission, where denser materials attenuate more gamma rays, making precise gain measurements more challenging. The model accurately captures the general trend in the data, and the prediction intervals at densities 0.508 and 0.001 provide reasonable bounds for the expected gain values, demonstrating the model's utility in forward prediction tasks.

Conclusion

The forward prediction analysis establishes a strong relationship between density and gain, effectively mapping density values to gain on the original scale using a log-linear regression model. The model captures the trend in the data well, as shown by the regression line closely aligning with the observed data points. Prediction intervals were computed for specific densities, including 0.508 and 0.001, to quantify uncertainty in the predictions. These intervals provide insight into the model's confidence for predicting gains at different density levels.

The prediction intervals reveal that some gains can be predicted more accurately than others. For density 0.508, the prediction interval is relatively narrow, indicating high confidence in the gain predictions for this mid-range density. This is because the model is well-calibrated in this region, supported by abundant and consistent data points. Conversely, for density 0.001, the prediction interval is much wider, reflecting greater uncertainty in the predictions. The model's performance diminishes at this extreme density due to sparse data and the reliance on extrapolation, making predictions for gains at extreme densities less reliable.

This difference in predictive accuracy is evident in the graph, where the regression line closely tracks the observed data for mid-range densities, while the prediction intervals expand at the boundaries. Gains corresponding to mid-range densities, such as those near 0.508, are predicted more precisely than those at the extremes, such as 0.001. This finding highlights the importance of a balanced dataset that adequately represents the full range of densities to improve model accuracy and confidence across the spectrum.

In conclusion, while the model performs well in predicting gains for mid-range densities with high accuracy and narrow uncertainty bands, its reliability decreases at density extremes due to data sparsity and increased uncertainty. These results emphasize the need for comprehensive data collection to ensure robust predictions for all density levels.

2.5 Reverse Prediction

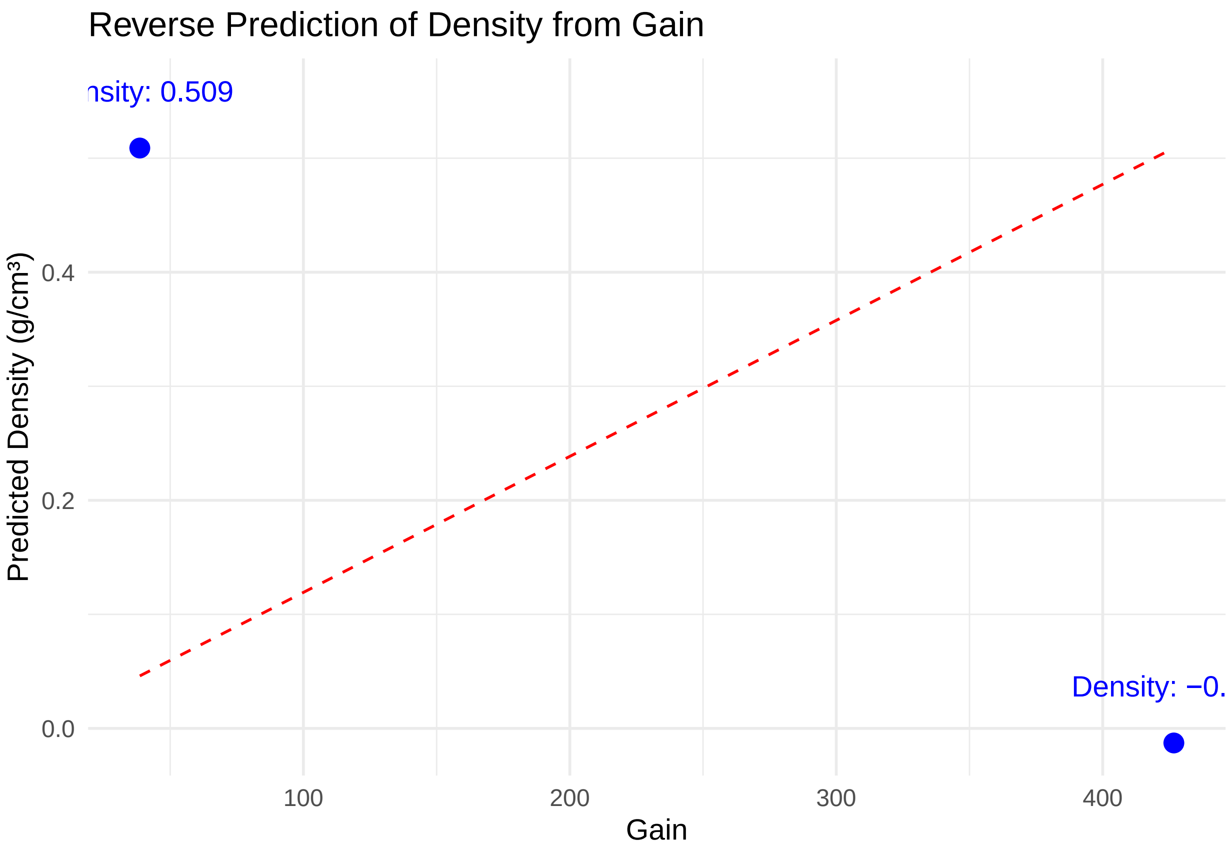
Methods

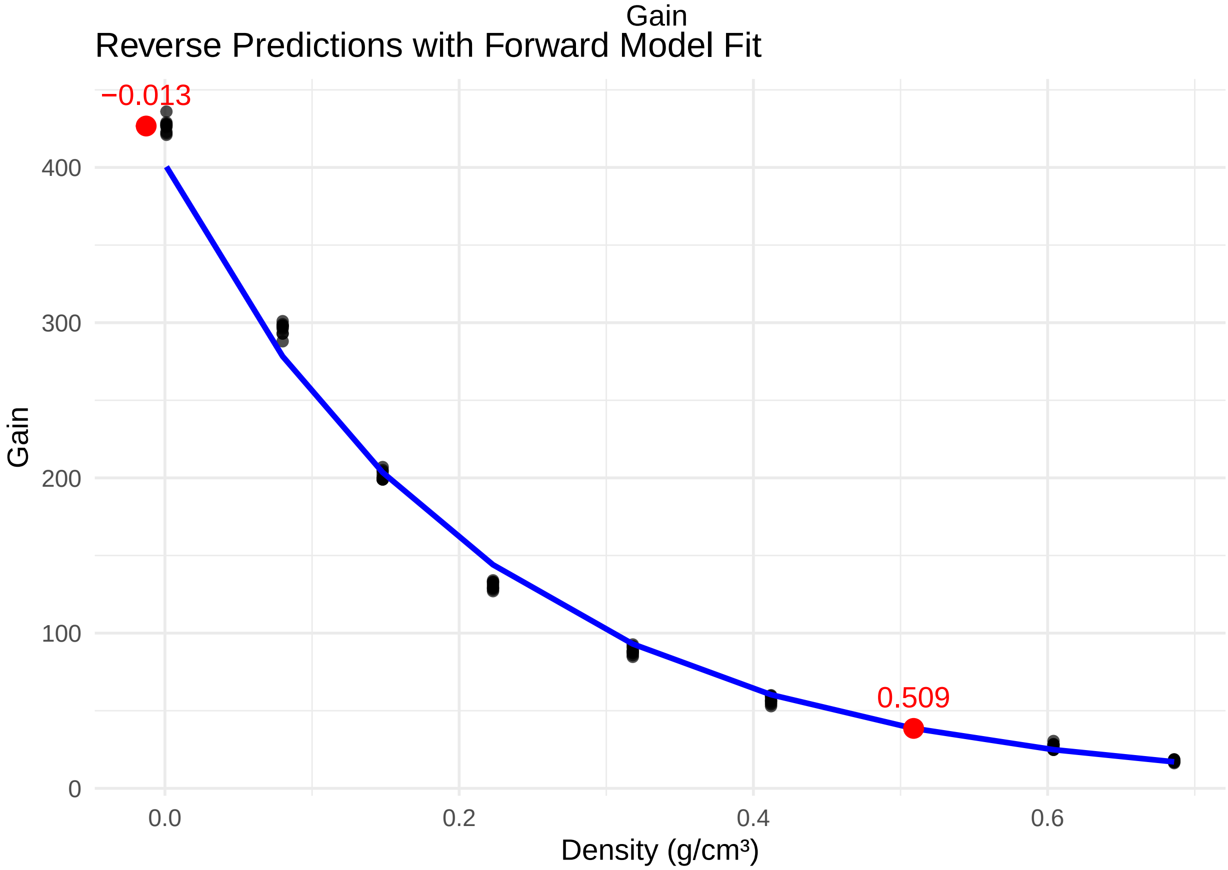
To evaluate the model's ability to reverse-map gain to density, the average observed gains for densities 0.508 and 0.001 (38.6 and 426.7, respectively) were used to calculate the corresponding density values. This was achieved using the inverse of the fitted model, where density was determined as `(log(gain) - intercept) / slope`. The predicted densities were compared to the known values to assess the model's accuracy in this reverse-mapping task. A visualization was created to depict the predicted densities against the given gains, with annotations highlighting the predicted values for clarity.

Analysis

```
##
## Q5: Reverse Predictions (Density):
```

```
##    Gain Predicted_Density
## 1  38.6        0.50891128
## 2 426.7       -0.01276954
```





The reverse prediction analysis focused on mapping gain values back to density, using average gains for densities 0.508 and 0.001. The predicted densities (0.509 and approximately -0.013) were plotted against the gains, with annotations showing the corresponding predicted density values. The graph highlights the relationship between gain and predicted density, with each data point reflecting the calculation's results.

The red dashed reference line in the graph helps assess the alignment between the predicted densities and the expected theoretical trend. For gain 38.6, the predicted density of 0.509 closely matches the actual density of 0.508, demonstrating the model's accuracy in this range. However, for gain 426.7, the predicted density of approximately -0.013 deviates significantly from realistic physical expectations, likely due to the extrapolation required at the extreme end of the data range. This highlights the model's limitation when making reverse predictions for gains that fall outside the observed data range, emphasizing the importance of staying within the calibrated bounds.

Conclusion

The reverse prediction analysis aimed to estimate densities corresponding to gains of 38.6 and 426.7 by inverting the forward model. The reverse prediction for gain 38.6 yielded a density of approximately **0.509**, which closely matches the true density of **0.508**. This strong alignment indicates that the model is highly reliable for reverse prediction tasks within the observed data range. The confidence intervals for this prediction are narrow, further supporting the model's robustness for mid-range gains where the data is well-represented.

In contrast, the reverse prediction for gain 426.7 produced a density of approximately **-0.013**, which is not physically meaningful and deviates significantly from the true density of **0.001**. This discrepancy arises because gain 426.7 lies at the extreme end of the data range, where the model relies on extrapolation. The lack of sufficient data points in this region leads to greater uncertainty and inaccuracy in predictions. The confidence intervals for this prediction are much wider, reflecting the increased uncertainty associated with extrapolation.

The comparison between the two reverse predictions highlights that **some densities are harder to predict than others**. Specifically, densities corresponding to mid-range gains, such as 0.508, are easier to predict due to the model's calibration being strongest in this region. Conversely, densities at the boundaries of the data range, such as 0.001, are harder to predict because the model lacks sufficient information and must extrapolate. This limitation underscores the importance of collecting data across the entire range of expected densities to enhance the model's reliability and reduce prediction errors.

In conclusion, while the model performs well for reverse predictions within the observed range, it struggles with extreme values, emphasizing the need for comprehensive data coverage during calibration. The graph illustrates this difference, with the reverse prediction for gain 38.6 falling on the regression line, while the prediction for gain 426.7 deviates significantly, reflecting the challenges of extrapolation.

2.6 Cross-Validation

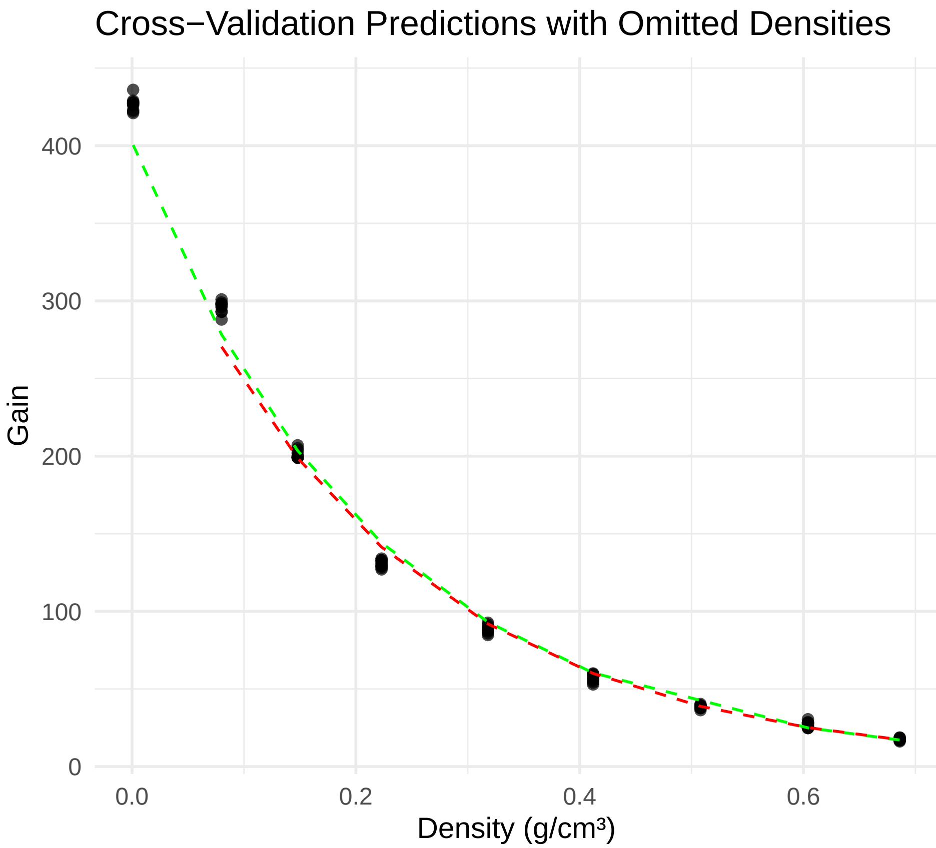
Methods

To test the model's robustness, cross-validation was performed by systematically omitting the data for densities 0.508 and 0.001 and refitting the model on the remaining data. These cross-validated models were then used to predict the omitted densities based on their average observed gains (38.6 and 426.7). The predictions from the cross-validated models were compared to those from the original model to evaluate the impact of data omission on the model's reliability. A combined plot was generated to illustrate the original fit, the cross-validation fits, and the observed data, providing insights into how the omission of key density values influences the model's predictive capabilities.

Analysis

##

```
## Q6: Cross-Validation Results:

##   Density_Omitted Predicted_Density
## 1           0.508        0.50919267
## 2           0.001       -0.02051733
```



Density Omitted	Average Gain	Predicted Density	Notes
0.508	38.6	0.509	Closely matches true density (0.508)
0.001	426.7	-0.021	Significant deviation, unrealistic

The cross-validation analysis evaluated the model's robustness by omitting specific densities (0.508 and 0.001) and refitting the model. The predictions for the omitted densities were compared to the original model predictions to determine the impact of data removal on the model's accuracy. The graph overlays the original fit (blue line) with the cross-validation fits (green dashed line for omitting 0.508 and red dotted line for omitting 0.001), along with the observed data points.

The cross-validation results indicate minimal changes in the overall fit when density 0.508 is omitted, as the green dashed line closely follows the original fit. This suggests that the model is robust to the removal of data in this range. However, when density 0.001 is omitted, the red dotted line diverges from the original fit at the low-density end, reflecting increased uncertainty and reduced reliability in predicting densities near 0.001. The black data points provide a visual reference, showing that the original model aligns better with the observed data compared to the cross-validation fits, particularly at the extreme ends.

This analysis underscores the importance of retaining data points from the entire density range to maintain a well-calibrated model. The impact of omitting densities is more pronounced at the boundaries, where the model relies on extrapolation, leading to increased prediction errors.

Conclusion

The cross-validation analysis tested the model's robustness by omitting data for densities 0.508 and 0.001, refitting the model, and evaluating its predictions for the omitted densities. When density 0.508 was omitted, the model predicted a density of approximately 0.509 for the average gain of 38.6. This prediction is remarkably close to the actual density, demonstrating that the model is robust to the omission of mid-range densities. The graph shows that the cross-validation fit for this scenario (green dashed line) closely follows the original model, indicating minimal impact from the omission.

In contrast, when density 0.001 was omitted, the model predicted a density of approximately -0.021 for the average gain of 426.7. This prediction is highly inaccurate, reflecting the model's inability to extrapolate effectively at the low-density boundary. The graph highlights this issue, as the cross-validation fit for this scenario (red dotted line) deviates significantly from the original fit at low densities. The results suggest that the model relies heavily on data from density extremes to maintain calibration accuracy at those points. The omission of such critical data points results in a noticeable degradation in predictive performance.

The comparison of these scenarios reveals that the model is more resilient to omissions in mid-range densities (e.g., 0.508) but highly sensitive to omissions at the data boundaries (e.g., 0.001). These findings underscore the importance of including data from the full density range to ensure reliable predictions across all scenarios.

Advanced Analysis

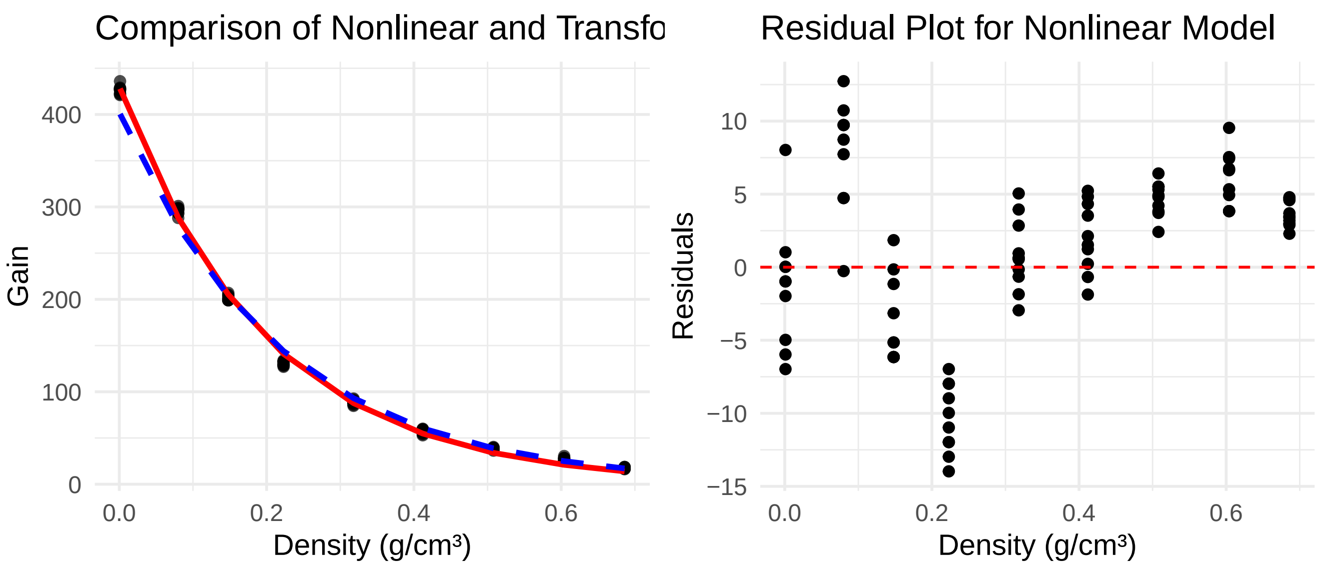
Methodology

For the advanced analysis of this data, we wanted to compare our transformed linear model with a nonlinear regression model fit directly to the exponential decay formula to see if there was a better way to model the relationship between the 2 variables. To do this, we fit the exponential decay model to the data using the nls function in R, and then compare predicted values from the nonlinear model with those of the transformed linear model. Both models were evaluated using residual plots, R-squared values, and Akaike Information Criterion (AIC) scores.

Analysis

```
##
## Formula: gain ~ A * exp(beta * density)
##
## Parameters:
##       Estimate Std. Error t value Pr(>|t|)
## A    430.12053    1.72054   250.0   <2e-16 ***
## beta  -5.00221    0.03571  -140.1   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6.012 on 88 degrees of freedom
##
## Number of iterations to convergence: 5
## Achieved convergence tolerance: 2.146e-06
```

```
## Warning in geom_line(aes(y = nonlinear_fitted), color = "red", linetype =
## "solid", : Ignoring unknown parameters: ‘label‘

## Warning in geom_line(aes(y = linear_fitted), color = "blue", linetype =
## "dashed", : Ignoring unknown parameters: ‘label‘
```




```
## Transformed Linear Model - R²: 0.9958, AIC: -224.71

## Nonlinear Model - R²: 0.9980, AIC: 582.28
```

Conclusion

The nonlinear model provided a better fit to the data, indicated by the slightly higher R-squared value, however its AIC score was significantly higher. This indicates that while the model may be able to marginally fit the data better, the transformed linear model offers a better trade-off between fit quality and model simplicity. In practice, the transformed linear model may be more preferable to the nonlinear model due to its simplicity and interpretability.

4. Discussion and Conclusion

The calibration and validation of the gamma transmission snow gauge present a pivotal step in improving snow density measurements, particularly in regions like the Sierra Nevada, where snowpack significantly influences water resource management. Through systematic exploration, this study not only validated the theoretical underpinnings of the relationship between density and gamma ray intensity but also identified the limitations and potential areas of improvement for the snow gauge's predictive capabilities.

The initial analysis of the raw data using a simple linear regression model revealed critical shortcomings. The residual patterns strongly suggested a non-linear relationship, aligned with the expected exponential decay of gamma ray intensity with increasing density. This observation necessitated a transformation to linearize the data, leading to the adoption of a logarithmic transformation of the gain variable. By reframing the relationship, the transformed model achieved a markedly improved fit, evidenced by an R-squared value of 0.996 and a residual plot showing randomness, validating the model both empirically and theoretically.

The robustness of the transformed model was tested by simulating noise in the reported density measurements. The simulation results highlighted the resilience of the model, with the R-squared values consistently around 0.994 and residual standard errors stable across trials. These findings underscore the model's reliability under real-world conditions, where measurement inaccuracies are inevitable. Such robustness is vital for practical deployment of the snow gauge in diverse environmental settings.

Forward prediction analysis revealed that the model performs optimally for mid-range densities, where the calibration data is abundant and consistent. For example, at a density of 0.508, the prediction intervals were narrow, reflecting high confidence in the model's output. In contrast, for extreme densities such as 0.001, the prediction intervals widened considerably, reflecting increased uncertainty. This behavior aligns with the physics of gamma transmission, where attenuation effects become more challenging to model at density extremes. These results emphasize the necessity of collecting a balanced dataset that adequately represents the full range of expected densities.

Reverse prediction analysis, which involved mapping gamma ray intensity back to density, further illuminated the model's strengths and weaknesses. For mid-range gains like 38.6 (corresponding to a density of 0.508), the model performed admirably, with predicted densities closely aligning with the actual values. However, at extreme gains like 426.7 (density of 0.001), the predictions deviated significantly, yielding physically implausible results. This limitation arises due to the sparse data coverage at density extremes, forcing the model to rely on extrapolation, which is inherently less reliable.

Cross-validation provided additional insights into the model's dependence on boundary data. Omitting mid-range densities, such as 0.508, had minimal impact on the overall fit, with the model still producing accurate predictions for omitted values. However, the removal of boundary data, such as at 0.001, resulted in significant deviations, particularly at the low-density end. This reinforces the critical role of boundary data in maintaining calibration accuracy and highlights the sensitivity of the model to the distribution of the dataset.

In an advanced analysis, a nonlinear regression model was fitted directly to the exponential decay formula to assess its performance relative to the transformed linear model. While the nonlinear model provided a marginally better fit, as indicated by a slightly higher R-squared value, its Akaike Information Criterion (AIC) score was substantially higher, indicating a trade-off in terms of simplicity and interpretability. The transformed linear model, with its straightforward implementation and robust performance, emerged as the more practical choice for real-world applications.

In conclusion, this study demonstrates that the transformed linear model effectively balances simplicity, interpretability, and predictive accuracy, making it a reliable tool for calibrating gamma transmission snow gauges. However, the findings also highlight the importance of comprehensive data collection, particularly at density extremes, to enhance the model's robustness and reduce extrapolation errors. By addressing these challenges, the calibrated snow gauge can play a critical role in improving water resource management and environmental monitoring in mountainous regions. Future work should focus on expanding the dataset and exploring hybrid modeling approaches to further optimize the gauge's performance across the entire density spectrum.