Discrete Structures

1. Mathematical Reasoning → we try to read, comprehend & construct the mathematics logically or mathematically

2. Discrete Structures, Mathematical structures Study on discrete objects & relation b/w objects

   sets, relations, graphs, tree, finite state machine

3. Combinatorial algorithms → Counting & enumerating
4. Modern applied algebra → Group, ring, field
5. Applications & Modeling → communication - networking, Coding, Cryptography

References:

1. Discrete Mathematics and Its Applications — Kenneth H. Rosen & Kamala Kirthivasan, McGraw Hill.

2. Discrete & Combinatorial Mathematics — Ralph P. Grimaldi & B.V. Ramana
   Pearson Education.
3. Algebra — Serge Lang
   Addison Wesley.

4. combinatorics → any books
   ↳ Discrete Mathematical Structures — U.S. Gupta
   Pearson

* Two class Tests — 10
  Midsem — 30 / 25
  Endsem — 60 / 50
  Attendance -

* Fundamentals of Logic

→ Logic → gives the relationship between statem

Ex:

1. All mathematicians wear sandals
2. Anyone who wears sandals is an algebraist
3. All mathematicians are algebraists.

→ Logic will not check any one of the statements 1, 2 and 3 are true (1) or false (0).
But it is like
If 1 & 2 are true, then 3 is true

→ proposition → A statement which is either true or false but not both at the same time.

Ex:

a. The +ve integers that divide 7 are 1 & 7
b. There exists +ve integer n, where also is prime number exists which is greater than → True
c. Earth is the only planet in the universe that contains life. → T/F but not both → It's a proposition
d. Attend all the classes → X not a proposition
e. Rabindranath Tagore won Nobel prize for writing "Sesher Kabita" → False

→ We use variables like p, q, r to represent propositions / statements (primitive)

→ operators → ¬ (negation), ∧ (conjunction), ∨ (disjunction)

→ To form compound propositions

Negation:
→ If p is one statement
¬p → not p

→ Conjunction:
If p & q are two primitive statements
then p ∧ q → p and q

→ Disjunction
p ∨ q → p or q

Then p ∨ q ...

summing operator

p	q	$p \vee q$	$p \veebar q$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	0

→ Operator precedence → $\neg, \wedge, \vee$

→ Implication / conditional proposition:

$p \rightarrow q$ = If p, then q
(or) p is a sufficient condition for q.

→ Bidirectional proposition:

$p \leftrightarrow q$ = p if and only if q
(or) p is necessary and sufficient for q.

p	q	$p \rightarrow q$	$p \leftrightarrow q$
0	0	T (1)	1
0	1	T (1)	0
1	0	F (0)	0
1	1	T (1)	1

→ We donot want a true statement to lead us into believing something that is false

Ex: Develop the truth table for $(p \rightarrow q) \wedge [(q \wedge \neg r) \rightarrow (p \vee r)]$

p	q	r	$p \rightarrow q$	$\neg r$	$q \wedge \neg r$	$p \vee r$	$(q \wedge \neg r) \rightarrow (p \vee r)$	
0	0	0	1	1	0	0	1	1
0	0	1	1	0	0	1	1	1
0	1	0	1	1	1	0	0	0
0	1	1	1	0	0	1	1	1
1	0	0	0	1	0	1	1	0
1	0	1	0	0	0	1	1	0
1	1	0	1	1	1	1	1	1
1	1	1	1	0	0	1	1	1

→ Ex:

s: Amit goes out for a walk

t: The moon is out

u: It is snowing

Symbolic compound statements

(a) $(t \wedge \neg u) \rightarrow s$: If the moon is out & it's not snowing then Amit goes out for a walk

(b) $t \rightarrow (\neg u \rightarrow s)$: If the moon is out, then if it is not snowing, then Amit goes out for a walk

(c) $\neg(s \leftrightarrow (u \vee t))$: It is not the case that Amit goes out for a walk if and only if the moon is out or It is snowing

Statements

(a) Amit goes out walking if and only if the moon is out: $s \leftrightarrow t$

(b) If it is snowing and the moon is not out, then Amit will not go out for a walk : $(u \wedge \neg t) \rightarrow \neg s$

(c) It is snowing but Amit will still go out for a walk : ~~u∧~~ $u \wedge s$

Ex:-

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Ex:

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	0	1	1

→

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0
			↓ Tautology			↓ contradiction

$p \vee T_0 \Leftrightarrow T_0$ } Domination
$p \wedge F_0 \Leftrightarrow F_0$

$(p \wedge q) \Leftrightarrow p$ } Absorption.
$p \vee q) \Leftrightarrow p$

statement containing only ∧ and ∨,
hen the dual of 'S' can be obtained
∧ and ∨ by ∨ and ∧ respectively

statements with ∧ and ∨
$\Leftrightarrow t$, then $s^d \Leftrightarrow t^d$

a tautology
~~and p is a primitive~~

Q. nd p is a
P∧ P. Now if
equival e replace

&

p	q	r	¬u				
0	0	0	1				
0	0	1	1	1			
0	1	0	0	0	0		
0	1	1	0	1	0		
1	0	0	1	1	1	0	
1	0	1	1	1	1		
1	1	0	0	0	0	1	1
1	1	1	0	1	1	0	0

) ∨q)
utology.

Q. $p \veebar q$ and $(p \vee q) \wedge \neg(p \wedge q)$ check

p	q	$p \veebar q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	0

Tautology: If the truth value of a statement is always true it is called a tautology (T_0).

Contradiction: If the truth value of a statement is always false is a contradiction (F_0).

→ $P_1, P_2, \ldots P_n$ → n primitive statements

$q : (P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n)$ → final compound statement → called conclusion (consequence)

set of primitive statements → called premises (antecedent)

p	q	¬p	¬p ∨ q	p → q
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Logical Equivalence

$p \to q \Leftrightarrow \neg p \vee q$

logically equivalent to

* If S_1 & S_2 are two statements which give the same truth value for all assignments of the primitive statements then they are logically equivalent (S_1, S_2)

$S_1 \Leftrightarrow S_2$

Q. Check whether $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are logically equivalent.

&

p	q	r	¬q	¬q ∨ r	p ∧ (¬q ∨ r)	¬r	q ∧ ¬r	p ∨ (q ∧ ¬r)
0	0	0	1	1	0	1	0	0
0	0	1	1	1	0	0	0	0
0	1	0	0	0	0	1	1	1
0	1	1	0	1	0	0	0	0
1	0	0	1	1	1	1	0	1
1	0	1	1	1	1	0	0	1
1	1	0	0	0	0	1	1	1
1	1	1	0	1	1	0	0	1

Q. $p \underline{\vee} q$ and $(p \vee q) \wedge \neg(p \wedge q)$ check

p	q	$p \underline{\vee} q$	p ∨ q	p ∧ q	¬(p ∧ q)	(p ∨ q) ∧ ¬(p ∧ q)
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	0

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ De Morgan's Rule

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

p	q	$p \vee q$	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg p \wedge \neg q$
0	0	0	0	1	1	1	1
0	1	1	0	1	0	1	0
1	0	1	0	0	1	1	0
1	1	1	1	0	0	0	0

* $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

  $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

* If $S_1 \Leftrightarrow S_2$ then the compound statement $S_1 \leftrightarrow S_2$ is a tautology

  If $\neg S_1 \Leftrightarrow \neg S_2$ " "

  $\neg S_1 \leftrightarrow \neg S_2$ " "

Laws of logic

* The primitive statements p, q, r, Tautology (T_0) and contradiction (F_0), with the connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \Leftrightarrow$ form the laws of logic

1. $\neg\neg p \Leftrightarrow p$; Double negation
2. $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ } De Morgan's law
   $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
3. $p \vee q \Leftrightarrow q \vee p$ } Commutative.
   $p \wedge q \Leftrightarrow q \wedge p$
4. $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ } Assosciative
   $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
5. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ } Distributive
   $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
6. $p \vee p \Leftrightarrow p$ } Idempotent
   $p \wedge p \Leftrightarrow p$
7. $p \vee F_0 \Leftrightarrow p$ } Identity
   $p \wedge T_0 \Leftrightarrow p$
8. $p \vee \neg p \Leftrightarrow T_0$ } Inverse
   $p \wedge \neg p \Leftrightarrow F_0$

9. $p \vee T_0 \Leftrightarrow T_0$ } Domination
$p \wedge F_0 \Leftrightarrow F_0$

10. $p \vee (p \wedge q) \Leftrightarrow p$ } Absorption
$p \wedge (p \vee q) \Leftrightarrow p$

Duality
If 'S' is a statement containing only $\wedge$ and $\vee$ connectives, then the dual of 'S' can be obtained by replacing $\wedge$ and $\vee$ by $\vee$ and $\wedge$ respectively
notation 'S^d'

i.e $S: p \vee q$
$\Rightarrow S^d: p \wedge q$

Duality Theorem:
If s & t are two statements with $\wedge$ and $\vee$ connectives only, if $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

Ex: $p \rightarrow q \Leftrightarrow \neg p \vee q$
$\Rightarrow p \rightarrow q \leftrightarrow (\neg p \vee q)$ is a tautology
~~Suppose p is a tautology and p is primitive statement~~
Suppose 'P' is the tautology and p is a primitive statement used in P. Now if p is equivalent to q, then if we replace p by q it is still a tautology

$p: r \vee s \Rightarrow (r \vee s) \rightarrow q \leftrightarrow (\neg(r \vee s) \vee q)$ is a tautology

Q $p: r \wedge s$, $q: \neg t \vee u$.
check whether $P: [p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology or not. Then replace by p, q, the above statements and check

S: $[p \wedge (p \rightarrow q)] \rightarrow q \Leftrightarrow [p \wedge (\neg p \vee q)] \rightarrow q$
$\Leftrightarrow [F_0 \vee (p \wedge q)] \rightarrow q$
$\Leftrightarrow (p \wedge q) \rightarrow q$
$\Leftrightarrow \neg p \vee \neg q \vee q$

$p: r \wedge s$, $q: \neg t \vee u$

$P: [p \wedge (p \to q)] \to q$

$\Leftrightarrow [(r \wedge s) \wedge ((r \wedge s) \to (\neg t \vee u))] \to q$

$p \to q \Leftrightarrow (r \wedge s) \to (\neg t \vee u)$

$\Leftrightarrow (\neg r \vee \neg s) \vee (\neg t \vee u)$

$\Leftrightarrow [(r \wedge s \wedge \neg r) \vee (r \wedge s \wedge \neg s) \vee (r \wedge s \wedge \neg t) \vee (r \wedge s \wedge u)]$

$\Leftrightarrow F_0 \vee F_0 \vee (r \wedge s \wedge \neg t) \vee (r \wedge s \wedge u)$

$\Leftrightarrow (r \wedge s) \wedge (\neg t \vee u)$

$P: (r \wedge s) \wedge (\neg t \vee u) \longrightarrow (\neg t \vee u)$

Q. Simplify:

$(p \vee q) \wedge \neg(\neg p \wedge q)$

Steps	Reasons (usage of laws of logic)
$(p \vee q) \wedge \neg(\neg p \wedge q)$	
$\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	Demorgan's law
$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$	Double negation ~~Distributive law~~
$\Leftrightarrow ((p \vee q) \wedge p) \vee ((p \vee q) \wedge \neg q)$	Distributive law
$\Leftrightarrow ((p \wedge p) \vee (p \wedge q)) \vee ((p \wedge \neg q) \vee (q \wedge \neg q))$	Distributive law
$\Leftrightarrow (p \vee (p \wedge q)) \vee ((p \wedge \neg q) \vee F_0)$	Inverse, Idempotent
$\Leftrightarrow p \vee (p \wedge \neg q)$	Identity, Absorption.
$\Leftrightarrow$ ~~$(p \vee p) \wedge$~~ p	Absorption.

* Contrapositive of a Conditional Proposition

Contrapositive of $p \to q$ is $\neg q \to \neg p$

Converse of $p \to q$ is $q \to p$

Ex: If the network is down, Amit cannot access the internet.

p: The network is down

q: Amit cannot access the internet.

$p \to q$

converse $q \to p$

If Amit cannot access the internet, the network is down

Contrapositive $\neg q \to \neg p$

If Amit can access the internet, then the network is not down.

→ Quantifier

→ $P(x)$ - propositional logic on 'x', x is in the set D.

Ex $P(n)$ - n is an odd integer

→ $P(5)$: True $P(2)$: False

D is the set of natural numbers.

D: Domain of Discourse.

P → is not a proposition

$P(x)$ → is a proposition (either True or False) when x is in set D.

for all x / for every x / for each x, x is in D

$\equiv \underbrace{\forall x \; P(x)}$, x is in D

universal quantifier

universally quantified ~~statement~~ proposition

* $\forall x \; P(x)$ is true if, for each x in the domain of discourse D, $P(x)$ is true

* $\forall x \; P(x)$ is false if for some x in D, $P(x)$ is false

(Counter example).

→ $\forall$ → Universal quantifier

→ $\exists$ → Existential quantifier.

→ Existential quantified proposition

~~$\exists P(x)$ is true~~

$\exists x \; P(x)$ is true if there exists one x in D for which $P(x)$ is true.

$\exists x \; P(x)$ is false if for every x in D $P(x)$ is false.

Ex1: $\forall x\ (x^2 > 0)$ D is the set of real no.s.
→ True.

$\forall x\ (x^2 - 1 > 0)$ " " "
→ False.

* for i := 1 to n
    if ($\neg P(d_i)$)
        return F
return T

d_i in D.
where
$D = \{d_1, d_2, \ldots d_n\}$

Ex: To verify for all real x, if $x > 1$ then $x+1 > 1$

{ $P(x): x+1 > 1$; D is set of real numbers.

Ex 2:

(a) $\exists x \left(\frac{x}{x^2+1} = \frac{2}{5} \right)$; D: R ✓ → $x = 2$ (T)

(b) $\exists x \left(\frac{x}{x^2+1} > 1 \right)$

{ $P(x): \frac{x}{x^2+1} > 1$

$\Rightarrow \neg P(x): \frac{x}{x^2+1} \leq 1$ ~~True~~

[$\because x < x^2 + 1\ \forall x \in R$]

$\therefore \exists x \neg P(x)$ is True

$\Rightarrow \exists x\ P(x)$ is False.

Ex3: for some 'n' if 'n' is prime then
n+1, n+2, n+3, n+4 are not prime
D: set of +ve integers.

{ $\exists x; P(x)$ if x is prime,
$P(x)$: $x+1, x+2, x+3, x+4$ are not prime.
D: set of +ve integers.

$\exists x\ P(x)$ → True.

* $\forall x \; P(x)$ → free variable

* $\exists x \; P(x)$ → bounded variable

* Generalized De Morgan's Law:

If P is propositional function, each pair of propositions below has the same truth values (either both are True or both are false)

$$\left(\neg(\forall x \; P(x)), \; \exists x \; \neg P(x) \right)$$

$$\left(\neg(\exists x \; P(x)), \; \forall x \; \neg P(x) \right)$$

proof:-

Let $\neg(\forall x \; P(x))$ is true

$\Rightarrow (\forall x \; P(x))$ is false.

~~$\Rightarrow \forall x \; \neg P(x)$ is true~~

$\Rightarrow$ atleast one 'x' in D that $P(x)$ is false

$\Rightarrow$ atleast one 'x' in D, $\neg P(x)$ is true

$\Downarrow$

$\exists x \; \neg P(x)$ is true.

let $\neg(\forall x \; P(x))$ is false

$\Rightarrow (\forall x \; P(x))$ is true.

$\Rightarrow (\forall x \; \neg P(x))$ is false

$\Rightarrow$ atleast one 'x' such that $\neg P(x)$ is false

$\Rightarrow \exists x \; \neg P(x)$ is false.

* $P(x): \dfrac{1}{x^2+1} > 1$

$\exists x \; P(x)$ is false by verifying

$\forall x \; \neg P(x)$ is true

* $\forall x\ P(x)$

let for each value of x in D

$P(x)$ takes the values $P_1, P_2, \ldots P_n$

$\Rightarrow \forall x\ P(x) = P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge \ldots \wedge P_n$

$\exists x\ P(x) = P_1 \vee P_2 \vee P_3 \vee P_4 \vee \ldots \vee P_n$

$\neg(P_1 \vee P_2 \vee P_3 \vee \ldots \vee P_n) \Leftrightarrow \neg P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge \ldots \wedge \neg P_n$

$\neg(\exists x\ P(x)) \Leftrightarrow \forall x\ \neg P(x)$

→ Nested Quantifiers

Ex: The sum of any two positive real nos is positive

if $x>0$, $y>0$, then $x+y>0$ D: R

$\forall x\ \forall y\ (((x>0) \wedge (y>0)) \rightarrow (x+y>0))$

Ex 2: $\forall x\ \forall y\ ((x>0) \wedge (y<0) \rightarrow x+y \neq 0)$ False

Ex 3 $\exists x\ \forall y\ (x \leq y)$ D: +ve integers True

Ex 4 $\exists x\ \exists y\ ((x>1) \wedge (y>1) \wedge (xy=6))$ True

Ex 5 $\exists x\ \exists y\ ((x>1) \wedge (y>1) \wedge (xy=7))$ False

D: +ve integers

* $\forall x\ \exists y\ P(x,y)$ → nested quantified proposition

$\neg(\forall x\ \exists y\ P(x,y)) \equiv \exists x\ (\neg(\exists y\ P(x,y)))$

$\equiv \exists x\ \forall y\ \neg P(x,y)$

→ Rules of Inference:

p, q, r, s are propositions.

Rules	Tautology	Name
1. p $p \to q$ $\therefore q$	$[p \wedge (p \to q) \to q]$	Modus Ponens
2. $\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
3. $p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical Syllogism.
4. $p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive Syllogism.
5. p $\therefore p \vee q$	$p \to p \vee q$	Addition
6. $p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
7. p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
8. $p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$	Resolution.

Rules of Interferences for Quantified statements:-

9. $\forall x\ P(x)$ $\therefore P(c)$; abitrary 'c'		Universal Instantiation.
10. $P(c)$ for arbitrary 'c' $\therefore \forall x\ P(x)$		Universal generalization.
11. $\exists x\ P(x)$ $\therefore P(c)$		Existential Instantiation
12. $P(c)$ $\therefore \exists x\ P(x)$		Existential Generalization

Ex: Show that the hypothesis, "if you send me an email, then I will finish writing the program", "if you do not send email, then I will go to sleep early" and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed".

p: send me email
q: I will finish writing the program
r: I will go to sleep early.
s: I will wake up feeling refreshed.

Hypotheses.

$p \to q \Leftrightarrow \neg p \vee q$
$\neg p \to r \Leftrightarrow p \vee r$ } resolution $\Leftrightarrow q \vee r$
$r \to s$

$\neg q \to s$

$q \vee r \Leftrightarrow r \vee q$
$r \to s \quad \neg r \vee s$

$q \vee s$ } resolution

$\neg q \to s$

(or) $p \to q \Leftrightarrow \neg q \to \neg p$

$\neg q \to \neg p$
$\neg p \to r$

$\neg q \to r$

$\neg q \to r$
$r \to s$

$\neg q \to s$ ✓

→ Predicate logic

Q. Everyone loves Microsoft or Apple.
Amit does not love Microsoft
So, Amit loves Apple.

Propositions

M(x): x loves Microsoft : M(x)
A(x): x loves Apple : A(x)

$\Rightarrow$ 1. $\forall x \; M(x) \vee A(x)$ premise

2. $\neg M(Amit)$ premise

3. $M(Amit) \vee A(Amit)$ Instantiation of 1

4. $A(Amit)$ from 2 & 3 by Disjunctive Syllogism

Q. A student in this class has not read the book and everyone in this class passed the 1st class test imply someone who passed the class test has not read the book.

→ Let $C(x)$: x is in this class

$B(x)$: x has read the book.

$P(x)$: x passed class test

steps:

1.	$\exists x \, (C(x) \wedge \neg B(x))$	Premise
2.	$\forall x \, (C(x) \rightarrow P(x))$	Premise
3.	$C(a) \wedge \neg B(a)$	Existential instantiation of 1.
4.	$C(a) \rightarrow P(a)$	Universal instantiation of 2.
5.	$C(a)$	Simplification of 3
6.	$P(a)$	Modus ponens of 4, 5
7.	$\neg B(a)$	Simplification of 3
8.	$P(a) \wedge \neg B(a)$	Conjunction of 6 & 7

Q. Simplify

1. a
2. $\neg a \vee c$
3. $\neg c \vee d$
4. $a \vee c$ → addition on 1
5. c Resolution on 2 & 4
6. d Disjunctive syllogism on 5 & 3

* Notation:

$a \vee \neg b \; c \equiv \left.\begin{matrix} a \vee \neg b \\ a \vee c \end{matrix}\right\}$ 2 premises

Q. 1. $a \vee \neg b$ c
2. $\neg(a \vee d)$

$\therefore \neg b$

{ 1. $a \vee \neg b$ — premise
2. $a \vee c$ — premise
3. $\neg(a \vee d)$ — premise
4. $\neg a \wedge \neg d$ — Demorgan's law on 3
5. $c \wedge \neg d$ — Resolution on 1 & 4
6. $\neg a$ — Simplification of 4
7. $\neg b$ — ~~Resolution~~ Dysjunctive Syllogism on 1 & 6

* Normal Forms :

A well formed formula (wff) or formula is a string involving propositional variables, connectives and parentheses.

Ex: $((p \wedge q) \vee \neg r) \longrightarrow S$

Disjunctive Normal Form (DNF/dnf) – $p \vee q$ sum

Conjunctive Normal Form (CNF/cnf) – $p \wedge q$ product

Product to be false. – $A \wedge F \equiv F$, $P \wedge \neg P \equiv F$

~~Product~~ Sum to be true – $A \vee T \equiv T$, $P \vee \neg P \equiv T$

Q. Obtain DNF (Disjunctive Normal Form) for $(p \rightarrow q) \wedge \neg q$

* We have to replace $\rightarrow, \leftrightarrow$ with $\vee, \wedge, \neg$

Sol $\Rightarrow (p \rightarrow q) \wedge \neg q$

$\Leftrightarrow (\neg p \vee q) \wedge \neg q$

$\Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge \neg q)$ → DNF [∵ $A \vee B$]

DNF is not unique.

Q. $\neg(p \wedge q) \leftrightarrow (p \vee q)$

$\underbrace{\neg(p \wedge q)}_{A} \leftrightarrow \underbrace{(p \vee q)}_{B}$

$A \leftrightarrow B$

$\Leftrightarrow (A \to B) \wedge (B \to A)$

$\Leftrightarrow (\neg A \vee B) \wedge (\neg B \vee A)$

$\Leftrightarrow (\neg A \wedge \neg B) \vee (\neg A \wedge A) \vee (B \wedge \neg B) \vee (B \wedge A)$

$\Leftrightarrow (\neg A \wedge \neg B) \vee (B \wedge A) \vee F_0 \vee F_0$

$\Leftrightarrow (\neg A \wedge \neg B) \vee (A \wedge B)$

$\cancel{\Leftrightarrow (\neg A \vee B) \wedge (\neg A \vee B)}$

$\Leftrightarrow (\neg(\neg(p \wedge q)) \wedge \neg(p \vee q)) \vee (\neg(p \wedge q) \wedge (p \vee q))$

$\Leftrightarrow ((p \wedge q) \wedge (\neg p \wedge \neg q)) \vee ((\neg p \vee \neg q) \wedge (p \vee q))$

$\Leftrightarrow ((p \wedge \neg p) \wedge (q \wedge \neg q)) \vee ((\neg p \vee \neg q) \wedge (p \vee q))$

$\Leftrightarrow (\neg p \vee \neg q) \wedge (p \vee q)$

$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge q) \vee (\neg q \wedge p) \vee (\neg q \wedge q)$

$\Leftrightarrow (\neg p \wedge q) \vee (\neg q \wedge p) \to$ DNF

Q. Obtain CNF

$(p \to q) \wedge \neg q$

$\Leftrightarrow (\neg p \vee q) \wedge \neg q \Leftrightarrow (\neg p \wedge \neg q) \vee (q \wedge \neg q)$

$\searrow$ CNF

$\Leftrightarrow \neg p \wedge \neg q \to$ CNF

Q. $\neg(p \wedge q) \leftrightarrow (p \vee q)$

$\Leftrightarrow (\neg p \vee \neg q) \wedge (p \vee q)$

* Principal CNF (or) canonical POS $\to$ only product of sum.
* Principal DNF $\to$ only sum of product
(or) canonical SOP $\Rightarrow (\ ) \vee (\ )$

$\underbrace{\text{SOP}}_{\text{Sum of Product}}$ $\qquad \underbrace{(\ )}_{\text{Product form}} \vee \underbrace{(\ )}_{\text{Product form}}$

* Product form:

4 conjunctions $\to$ $p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$, $\neg p \wedge \neg q$ $\to$ minterms

4 disjunctions $\to$ $p \vee q$, $p \vee \neg q$, $\neg p \vee q$, $\neg p \vee \neg q$ $\to$ maxterms

If there are 'n' propositional variables,

$\Rightarrow$ there'll be 2^n conjunctions & 2^n disjunctions.

$\downarrow$

each should contain all the variables

* $p \wedge \neg p \equiv F$ $\quad A \vee F \equiv A$
* $p \vee \neg p \equiv T$ $\quad A \wedge T \equiv A$

Q. Obtain principal DNF of $p \vee \neg q$

$p \vee \neg q$ is not principal DNF because the operands of '$\vee$' should contain all the variables

$$\Rightarrow p \vee \neg q \Leftrightarrow \cancel{(p \vee (q \wedge \neg q)) \vee (\neg q \vee (p \wedge \neg p))}$$

$$\Leftrightarrow (p \wedge (q \vee \neg q)) \vee (\neg q \wedge (p \vee \neg p))$$

$$\Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (\neg q \wedge p) \vee (\neg q \wedge \neg p)$$

$$\Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (\neg q \wedge \neg p)$$

Q. $(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$

Q. 1. $p \rightarrow q$
2. $(p \vee q) \wedge \neg p \longrightarrow \neg q$

Q. $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$

$$\Leftrightarrow (\neg p \vee q) \leftrightarrow (\neg q \vee p)$$

$$\Leftrightarrow \neg(\neg p \vee q) \vee (\neg q \vee p)$$

$$\Leftrightarrow (p \wedge \neg q) \vee (\neg q \vee p)$$ De morgan's law

$$\Leftrightarrow ((p \wedge \neg q) \vee \neg q) \vee p$$ Associative law

$$\Leftrightarrow ((p \vee \neg q) \wedge \neg q) \vee p$$

$$\Leftrightarrow (p \vee \neg q \vee p) \wedge (\neg q \vee p)$$

$$\Leftrightarrow (p \vee \neg q) \wedge (p \vee \neg q) \longrightarrow \text{pcnf}$$

$$\Leftrightarrow p \vee \neg q$$

$$\Leftrightarrow (p \wedge \neg q) \vee (\neg q) \vee (p)$$

$$\Leftrightarrow (p \wedge \neg q) \vee (\neg q \wedge (p \vee \neg p)) \vee (p \wedge (q \vee \neg q))$$

$$\Leftrightarrow (p \wedge \neg q) \vee ((\neg q \wedge p) \vee (\neg q \wedge \neg p)) \vee ((p \wedge q) \vee (p \wedge \neg q$$

$$\Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge \neg q) \vee (p \wedge q) \longrightarrow \text{pdnf}$$

* Normal forms for 1^{st} order logic :-

Normal form of 1^{st} order logic is called that is in "Prenex Normal Form" if the formula is of the form

$$(Q x_1)(Q x_2) \cdots (Q x_i) \cdots (Q x_n)\ (M)$$

Here Q is the quantifier either $\forall$ or $\exists$

Domain of discourse is $\{x_1, x_2, \ldots x_n\}$

M is the matrix of the formula that consists of only propositions & connectives but no quantifiers.

→ Two quantifiers $\forall, \exists$

* $\forall$ distributes over $\wedge$ and $\exists$ distributes over $\vee$

* $\forall$ doesnot distribute over $\vee$ and $\exists$ doesnot distribute over $\wedge$.

$$\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\,(P(x) \wedge Q(x))$$

$$\exists x\, P(x) \vee \exists x\, Q(x) \equiv \exists x\,(P(x) \vee Q(x))$$

→ Let P & Q are two propositional functions

Now P depends on x. $P(x)$

Q doesn't depend on x. Q.

$$\forall x\, P(x) \wedge \forall x\, Q \equiv \forall x\,(P(x) \wedge Q)$$

$$\forall x\, P(x) \wedge \forall z\, Q(z) \equiv \forall x\, \forall z\,(P(x) \wedge Q(z))$$

→ $(Qx)\, F(x) \vee G \leftrightarrow (Qx)\,(F(x) \vee G)$

→ $(Qx)\, F(x) \wedge G \leftrightarrow (Qx)\,(F(x) \wedge G)$

Q. $\forall x\, P(x) \longrightarrow \exists x\, Q(x)$, obtain Prenex normal form

Rules:

1. Replace $\rightarrow, \leftrightarrow$ by $\wedge, \vee, \neg$
2. Apply De Morgan.
3. Get all the quantifiers in the left part of the formula.
4. Apply the rules of $\forall$ & $\exists$.

$\S$ $\neg\forall x\, P(x) \rightarrow \exists x\, Q(x)$

$\Leftrightarrow \neg(\forall x\, P(x)) \vee (\exists x\, Q(x))$

$\Leftrightarrow \exists x\, \neg P(x) \vee \exists x\, Q(x)$

$\Leftrightarrow \exists x\, (\neg P(x) \vee Q(x))$

Q. $(\forall x)(\forall y)((\exists z)(P(x,z) \wedge P(y,z)))$
$\rightarrow ((\exists u)\, Q(x,y,u))$

$\S$ $\neg((\forall x)(\forall y)((\exists z)(P(x,z) \wedge P(y,z)))) \vee (\exists u)\, Q(x,y,u)$

$\Leftrightarrow \exists x\, \exists y\, \forall z\, \neg(P(x,z) \wedge P(y,z)) \vee (\exists u) Q(x,y,u)$

$\Leftrightarrow (\exists x\, \exists y\, \forall z\, ((\neg P(x,z)) \vee (\neg P(y,z)))) \vee \exists u\, Q(x,y,u)$

$\Leftrightarrow \exists x\, \exists y\, \forall z\, \exists u\, ((\neg P(x,z) \vee \neg P(y,z) \vee Q(x,y,u))$

* Proofs :-

A mathematical system contains axioms, definitions, terms (defined or undefined)

assumed to be true; new concept

Theorems are propositions. that have already been proved / we have to prove

→ lemma → is a theorem (not so important) but is required to prove other theorems

→ corollary → ~~theorem / part of~~ some (part of) theorem that can be easily derived from the theorem.

An argument that establishes the truth values of theorem → proof

hypothesis — conclusion (T/F)
premises

Proofs

Direct	Indirect
If m is odd & n is even then prove that (m+n) is odd	Proof by contradiction Proof by cases Proof by Existence proof by counterexample

Proof: n is even
$\Rightarrow n = 2k$
m is odd
$\Rightarrow m = 2k' + 1$

$m + n = 2(k + k') + 1$
$= 2k'' + 1$
$\Rightarrow$ Odd

~~For all ($x_1, x_2, \ldots x_n$) if $P(x_1, x_2, \ldots x_n)$ then $q(x_1, x_2, \ldots x_n)$~~

*Theorem:
· For all $(x_1, x_2, \ldots x_n)$, if $p(x_1, x_2, \ldots x_n)$ then $q(x_1, x_2, \ldots x_n)$

$p \to q$
if p is true, q is true. → to be proved
assuming p is true that gives q is false

$p \wedge \neg q$

i.e if $p \wedge \neg q \longrightarrow r \wedge \neg r$ (r is any arbitrary proposition)
$\Rightarrow$ Our assumption is false.
↳ proof by contradiction
i.e $p \to q \equiv (p \wedge \neg q) \to r \wedge \neg r$

p	q	r	$\neg q$	$p \wedge \neg q$	$r \wedge \neg r$	$p \wedge \neg q \to (r \wedge \neg r)$	$p \to q$
0	0	0	1	0	0	1	1
0	0	1	1	0	0	1	1
0	1	0	0	0	0	1	1
0	1	1	0	0	0	1	1
1	0	0	1	1	0	0	0
1	0	1	1	1	0	0	0
1	1	0	0	0	0	1	1
1	1	1	0	0	0	1	1

Q. For some +ve real numbers x & y, prove that if $x + y \geq 2$ then $x \geq 1$ (or) $y \geq 1$

$q: x \geq 1$ or $y \geq 1$
$\neg q: x < 1$ and $y < 1$

if $x < 1$ and $y < 1$ then $x + y < 2$
But Given $x + y \geq 2$

*Proof by Cases

for some real no. 'x' prove 'A'

if $(P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n)$ then q

$\hookrightarrow (P_1 \to q) \vee (P_2 \to q) \vee \ldots \vee (P_n \to q)$

Q. For all real no. x, show that $|x| \geq x$

A. $x \geq 0 \Rightarrow |x| = x \Rightarrow |x| \geq x$ ✓

$x < 0 \Rightarrow |x| = -x$

$x < 0 \Rightarrow 2x < 0$

$x + x < 0$

$x < -x$

$x < |x| \Rightarrow |x| > x$

* Proof by counterexample :-

for a prime no. 'n', $2^n - 1$ is a prime

$(2^n - 1) \to$ Mersenne prime

if $n = 11 \Rightarrow 2^n - 1 = 2389 \to$ not a prime

* Proof by existence :-

for any real no.s a, b $(a<b)$ Show that there is one real x such that $a < x < b$.

Proof: $x = \frac{a+b}{2}$

* Mathematical Induction :-

→ Basis step

→ For trivial value, we check whether the given formula is true.

→ Inductive step

→ Assume the formula is true for some 'n' then show that it is also true for 'n+1'.

⇒ formula is true for all n (in given domain of discourse)

Principle of mathematical induction

→ let $S(n)$ be a propositional function with the domain of discourse of 'n'.

S(n) is true for $n \geq 1$ (n_0) if,
If S(n) is true then S(n+1) is true for $n \geq 1$
then S(n) is true for all values of $n \geq 1$

Ex1:
Use induction to show that $n! \geq 2^{n-1}$ for $n \geq 1$

Basis step:

$S(1): 1! \geq 2^{1-1}$
$1 \geq 1$ (True)

Inductive step:

let us assume that S(n) is true
$\Rightarrow n! \geq 2^{n-1}$

Now $S(n+1): (n+1)! \geq 2^{n}$, we have to show that S(n+1) is true

$n! \times (n+1) \geq 2^{n-1} \times (n+1)$

$(n+1)! \geq 2^{n-1}(n+1)$

$n \geq 1 \Rightarrow (n+1) \geq 2$

$\therefore 2^{n-1}(n+1) \geq 2^{n-1} \times 2 = 2^{n}$

$\therefore (n+1)! \geq 2^{n}$
$\Rightarrow S(n+1)$ is true

Ex2:- $S(n): a + ar + ar^2 + \dots + ar^n = \frac{a(r^{n+1} - 1)}{r-1}, n \geq 0$

Basis step:

$S(0): a = \frac{a(r-1)}{r-1}$
$a = a$ (True)

Inductive step:

Let us assume that S(n) is true.
$\Rightarrow a + ar + ar^2 + \dots + ar^n = \frac{a(r^{n+1} - 1)}{r-1}$

$S(n+1): a + ar + ar^2 + \dots + ar^n + ar^{n+1} = \frac{a(r^{n+2} - 1)}{r-1}$

we have to prove S(n+1) is true

$\Rightarrow (a + ar + ar^2 + \dots + ar^n) + ar^{n+1}$

$= S(n) + ar^{n+1}$

$= \frac{a(r^{n+1} - 1)}{r-1} + ar^{n+1} = \frac{a}{r-1}(\cancel{r^{n+1}} - 1 + r^{n+2} - \cancel{r^{n+1}})$

$\frac{a(r^{n+2} - 1)}{r-1}$

Ex) Use induction to show that $5^n - 1$ is divisible by 4, $n \geq 1$

Basis step:

$S(1)$: $5^1 - 1 = 4$ is divisible by 4

Inductive step:

Let us assume $S(n)$ is true

$\Rightarrow 5^n - 1$ is divisible by 4

$\Rightarrow 5^n - 1 = 4m$, m is an integer

Now, we have to prove, $5^{n+1} - 1$ is divisible by 4

$5^{n+1} - 1 = 5 \cdot 5^n - 1$

$= 5(4m+1) - 1$

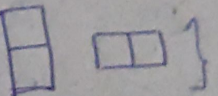
$= 20m + 5 - 1$

$= 4(5m+1)$

$\therefore S(n+1)$ is true.

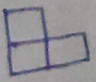
* Polyomino of dimension 'n'

→ is a geometrical figure consisting of 'n' squares joined at edges



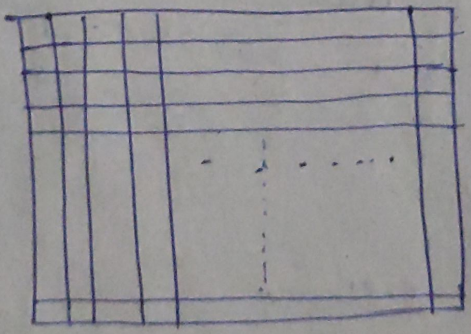
2 squares

domino



3 squares

Triomino/Tromino



nxn square

problem:- Given an nxn square board. To check whether, we can tile the above board with ~~dominoes~~ trominoes.

tiling - exact covering without any overlap or extending outside

nxn square board having one square less is called deficient board

Problem 2: $n \times n$ deficient board can be tiled with trominoes where 'n' is power of '2'

$n = 2^k$

Basis step:

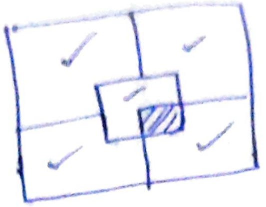
→ $k = 1$

→ 2×2 deficient board

1 tromino

→ $k = 2$

→ $2^2 \times 2^2$ deficient board

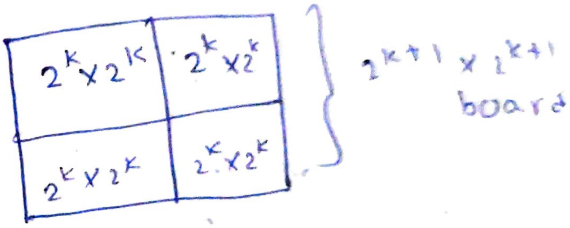


5 trominoes

Inductive step

Assume that deficient board of dimension 2^k can be tiled with trominoes.

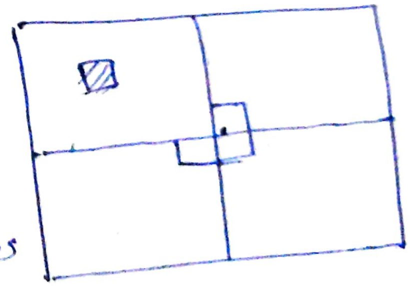
We should prove that $2^{k+1} \times 2^{k+1}$ deficient board can also be tiled.



Assume empty square is in the upper left quadrant.

Now place a tromino at the centre.

Now all the remaining three quadrants needs to be tiled for $2^{2k} - 1$ squares



⇒ By above assumption, all four $2^k \times 2^k$ deficient boards can be tiled

⇒ $2^{k+1} \times 2^{k+1}$ deficient board can be tiled with trominoes.

Q. When 'n' is not a power of '2', if $n \neq 5$, any $n \times n$ deficient board can be tiled with trominoes iff 3 divides $(n^2 - 1)$

Q. prove by induction that sum of first n consecutive odd positive integers is n^2 $\forall n \geq 1$

& Basis step:

$n=1$

$S(1)$ sum of first odd number $=1$

$n^2 = 1^2 = 1$ (True)

Inductive step:

let $S(n)$ is true

i.e sum of first 'n' odd numbers $= n^2$

$1+3+ \cdots + (2n-1) = n^2$

We have to prove $S(n+1)$ is true

sum of first 'n+1' odd numbers

$= 1+3+ \cdots + (2n-1) + (2(n+1)-1)$

$= S(n) + 2n+1$

$= n^2+2n+1$

$= (n+1)^2$

$\therefore S(n+1) = (n+1)^2$

*Harmonic numbers

$H_1 = 1$, $H_2 = 1+\frac{1}{2}$, $H_3 = 1+\frac{1}{2}+\frac{1}{3}$,

$H_n = 1+\frac{1}{2}+\frac{1}{3}+ \cdots + \frac{1}{n}$.

Q) prove, for $n \geq 1$, $\sum_{J=1}^{n} H_J = (n+1)H_n - n$

& Basis step:

$n=1$

$\sum_{J=1}^{1} H_J = H_1 = 1$

$(n+1)H_n - n = (1+1)H_1 - 1 = 2-1 = 1$ (True)

Inductive step:

Assume $S(k)$ is true.

i.e $\sum_{J=1}^{k} H_J = (k+1)H_k - k$

We have to prove $S(k+1)$ is true

$$\sum_{j=1}^{k+1} H_j = \sum_{j=1}^{k} H_j + H_{k+1}$$

$$= (k+1)H_k - k + H_{k+1}$$

$$H_{k+1} = H_k + \frac{1}{k+1} \Rightarrow H_k = H_{k+1} - \frac{1}{k+1}$$

$$\sum_{j=1}^{k+1} H_j = (k+1)\left(H_{k+1} - \frac{1}{k+1}\right) - k + H_{k+1}$$

$$= (k+2)H_{k+1} - (k+1)$$

Q. Show that for all $n \geq 1$, $H_{2^n} \geq 1 + \frac{n}{2}$

$S(n)$: $H_{2^n} \geq 1 + \frac{n}{2}$

Basis Step:

$n = 1 \Rightarrow H_{2^n} = H_2 = 1 + \frac{1}{2}$

$1 + \frac{n}{2} = 1 + \frac{1}{2}$ (True)

Inductive step:

Assume $S(k)$ is true

$\Rightarrow H_{2^k} \geq 1 + \frac{k}{2}$, $k \geq 1$

We have to prove $S(k+1)$ is true

$$\Rightarrow H_{2^{k+1}} = H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k}$$

$$\frac{1}{2^k+1} \geq \frac{1}{2^{k+1}}$$

$$\frac{1}{2^k+2} \geq \frac{1}{2^{k+1}}$$

$$\vdots$$

$$\frac{1}{2^k+2^k-1} \geq \frac{1}{2^{k+1}}$$

$$\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k-1} \geq \frac{1}{2^{k+1}} \times (2^k - 1)$$

$$\Rightarrow \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k-1} + \frac{1}{2^k+2^k} \geq \frac{1}{2^{k+1}}(2^k \cancel{-1} \cancel{+1})$$

$$= \frac{1}{2}$$

$$H_{2^{k+1}} \geq H_{2^k} + \frac{1}{2}$$

$$\Rightarrow H_{2^{k+1}} \geq 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{(k+1)}{2}$$

* Strong form of Mathematical Induction :

Let $S(n)$ be a propositional function with the domain of discourse greater than or equal to n_0.

If $S(n_0)$ is true and $S(k)$ is true for all values which are greater than or equal to n_0 and less than k [i.e $n_0 \leq k < n$]

Then $S(n)$ is true for all values of 'n' for the given domain of discourse

Q. Use mathematical induction to show that postage of Rs.4 or more can be achieved by using only Rs. 2 & Rs. 5 stamps.

S Basis step:

$n = 4$

$\Rightarrow$ We can use $4 = 2+2$

$n = 5$

$\Rightarrow$ $5 = 5$

Inductive step :- $(n \geq 6)$

~~Add Rs. 2 to~~

Consider the inductive assumption as $(n-2)$ postage can be achieved with Rs.2 & Rs.5 stamps

Add Rs.2 stamps to achieve stamps with any value.

Q. Define a sequence where n^{th} term is

$C_n = C_{[\frac{n}{2}]} + n$ for all $n > 1$ &

$C_1 = 0$ $\qquad [\,] \rightarrow GIF$

Show that $C_n < 4n$ for $n \geq 1$

S If any term of a sequence is represented as a function of preceding terms, then we can use the strong form of mathematical induction to prove the properties of that sequence.

Basis step
$n=2$
$c_2 = c_1 + 2$
$c_2 = 2 < 8$ (True)

$n=3$
$c_3 = c_1 + 2$
$= 2 < 12$ (True)

Inductive step:-
let for all integers from 1 to k-1, the above inequality is true
$\Rightarrow S(K)$:
$C_K = C_{\lceil \frac{K}{2} \rceil} + K$

$C_{\lceil \frac{K}{2} \rceil} < 4 \lceil \frac{K}{2} \rceil$

$\Rightarrow C_K < 4 \lceil \frac{K}{2} \rceil + K$

$\lceil \frac{K}{2} \rceil \le K/2$

$\Rightarrow C_K < \frac{4K}{2} + K = 3K$

$C_K < 3K \Rightarrow \boxed{C_K < 4K}$

Ex:- Given a sequence
$a_0 = 1,\ a_1 = 2,\ a_2 = 3,$
$a_n = a_{n-1} + a_{n-2} + a_{n-3}$
Prove that $a_n \le 3^n,\ n \ge 0$

Basis step:
$n=0 \Rightarrow 1 \le 1$ (True)
$n=1 \Rightarrow 2 \le 3$ (True)
$n=2 \Rightarrow 3 \le 9$ (True)
$n=3 \Rightarrow 6 \le 27$ (True).

Inductive step:-
let for all integers from 0 to k-1, $a_i \le 3^i$,
$\Rightarrow S(K)$
$a_K = a_{K-1} + a_{K-2} + a_{K-3}$

$\Rightarrow a_K \le 3^{K-1} + 3^{K-2} + 3^{K-3}$
$< 3^{K-1} + 3^{K-1} + 3^{K-1} = 3^K$

Q Suppose we have 'n' numbers $a_1, a_2, a_3, \ldots, a_n$.
Insert parentheses and then multiply "n" no.s
e.g for $n=4$, $P_4 = (a_1 a_2)(a_3 a_4)$
Prove that if we insert parentheses in any manner whatsoever and then multiply the 'n' numbers, we perform $(n-1)$ multiplications
$\Downarrow N_n = n-1$

L Basis step

$n=1$
$N_1 = 0 = 1-1$ (True)
$n=2$
$N_2 = 1 = 2-1$ (True)

Inductive step :-

Assume some no. $t<n$
Multiplication of n no.s
$= \underbrace{(a_1 a_2 a_3 \ldots a_t)}_{t \text{ terms}} . \underbrace{(a_{t+1}, a_{t+2}, \ldots a_n)}_{(n-t) \text{ term}}$
$\Rightarrow N_n = N_t + N_{n-t} + 1$
Since $t<n$, $n-t<n$

$N_t = t-1$, $N_{n-t} = n-t-1$

$\therefore N_n = \cancel{t}-1 + n - \cancel{t} - \cancel{1} + \cancel{1}$

$\boxed{N_n = n-1}$

* Well-ordering property for non-negative integers :

It states that every non-empty set of non-negative integers has a least value ↘ equivalent to the two forms of mathematical induction.

Strong Induction and well ordering property:

p(n) is a propositional function
we have to prove p(n) is true for all positive integers 'n'

~~Basis step:~~

~~P(n) is true.~~

~~Inductive step:~~

~~If P(K) is true, then P(K+1) is true.~~

~~Then P(n) is true, for all~~

Basis step:

$P(n_0)$ (or) $(P(n_0), P(n_0+1).$ ... $)$ is true

Inductive step:

$P(n_0), P(n_0+1), P(n_0+2), \ldots P(k)$

are all true.

Then we have to show, $P(k+1)$ is true

then $P(n)$ is true for all +ve integer 'n'

Ex: Show that If 'n' is an integer greater than '1', then n can be represented as the product of primes.

Basis step:-

$n = 2$

$P(2)$ is true.

Inductive step: (strong form of induction)

$P(j)$ is true $2 \le j \le k$

⇒ we have to prove

$[P(2) \wedge P(3) \wedge \ldots \wedge P(k)] \rightarrow P(K+1)$

$K+1$ can be (i) prime ⇒ True

(ii) Composite

if $k+1$ is composite. let a, b are factors of $K+1$

such that $K+1 = a \times b$

⇒ $2 \le a \le K$, $2 \le b \le K$

∴ $P(a)$ & $P(b)$ are true

⇒ $P(K+1)$ is also true

* ~~Theorem~~: A simple polygon with 'n' sides, wh
~~positive integers~~ n is a positive integer ≥ 3, c
be triangulated with $(n-2)$ triangles.

Lemma: Every simple polygon has an inte
diagonal. → No two non-consecutive
Sides intersec

To be used, not from the above theorem

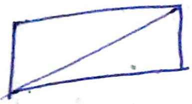
Proof:

Basis step:

$n = 3 \rightarrow$ A triangle

$n-2 = 1$ ✓

$n = 4 \Rightarrow 4 - 2 = 2$ ✓



Inductive step:

· P(n) be the propositional function that states that a polygon with 'n' sides can be triangulated with 'n-2' triangles.

According to the above lemma, ~~the~~ one interior diagonal exists for P polygon

let for $3 \leq j \leq k$, P(j) is true.

$\Rightarrow$ there is an interior diagonal for the k+1 sided polygon.

∴ it divides the polygon into two polygo
of sides ~~less~~ say a, b

now, $a < k+1$, $b < k+1$

& $3 \leq a, b \leq k$, $a + b = k + 1 + 2 = k + 3$

$\rightarrow$ P(a), P(b) is true

~~P(k+1) is also true~~

~~ie~~ ie No. of triangles in 'a' sided polygon : a-
" " " 'b' " " = b-2

Total No. of triangles in k+1 sided polygon

$= a - 2 + b - 2$

$= a + b - 4$

* **Theorem:** A simple polygon with 'n' sides, when ~~three integers~~ n is a positive integer ≥ 3, can be triangulated with $(n-2)$ triangles.

Lemma: Every simple polygon has an interior diagonal.
→ no two non-consecutive sides intersect

to be used, not from the above theorem

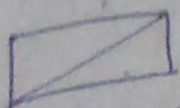
proof:

Basis step:

$n=3 \rightarrow$ A triangle

$n-2 = 1$ ✓

$n=4 \Rightarrow 4-2 = 2$ ✓



Inductive step:

P(n) be the propositional function that states that a polygon with 'n' sides can be triangulated with 'n-2' triangles.

According to the above lemma, one interior diagonal exists for P polygon

let for $3 \leq j \leq k$, $P(j)$ is true.

$\Rightarrow$ there is an interior diagonal for the k+1 sided polygon.

$\therefore$ it divides the polygon into two polygons of sides say a, b

now, $a < k+1$, $b < k+1$

& $3 \leq a, b \leq k$, $a+b = k+1+2 = k+3$

$\rightarrow P(a), P(b)$ is true

$\therefore$ ~~P(k+1) is also true~~

i.e No. of triangles in 'a' sided polygon $= a-2$

" " " 'b' " " $= b-2$

$\therefore$ Total No. of triangles in k+1 sided polygon

$= a-2+b-2$

$= a+b-4$

$= k+3-4$

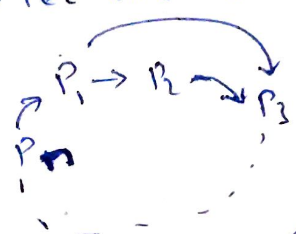
$= k-1$

$= (k+1)-2$

Ex: In a round-robin tournament, every p[illegible] plays with every other exactly once and each match has a winner and a loser. players $P_1, P_2, \ldots, P_m$ are said to form a cycle if P_1 beats P_2, ... P_m beats P_1. show that if there is a cycle of length $m \geq 3$, then there must be a cycle of '3'.

Basis step :
for $m = 3$
It is true

Inductive step:
~~let $S(m)$ is true.~~
consider a cycle length of '3' does not exist.
$\Rightarrow$ let the min. length of a cycle is $n > 3$.



Now if P_1 beats P_3 then we can omit P_2 to get a cycle of length $n-1$ → which is a contradiction.

else
we get a cycle of length '3'
↓ contradiction

$\therefore$ a cycle of length '3' exists.

*Recursive Definition and Mathematical Induction:

$b_0, b_1, \ldots, b_n$ n is a +ve integer

$b_n = 2n$

$\Rightarrow$ compute $b_6 = 2 \times 6$
$\Rightarrow$ b_n is explicitly defined.

$a_0 = 0, \quad a_1 = 1, \quad a_2 = 2$

$a_n = a_{n-1} + a_{n-2} + a_{n-3}, \quad n \geq 3$

$$\begin{aligned} a_6 &= a_5 + a_4 + a_3 \\ &= a_4 + a_3 + a_2 + a_3 + a_2 + a_1 + a_3 \\ &= a_3 + a_2 + a_1 + 3a_3 + 2a_2 + a_1 \\ &= 4(a_2 + a_1 + a_0) + 3a_2 + 2a_1 = 7a_2 + 6a_1 + 4a_0 \end{aligned}$$

$s(1) = 1, \quad s(n) = 2 \cdot s(\lfloor n/2 \rfloor)$, $s(37) = ?$

$s(37) = 2 \cdot s(18)$
$= 2 \cdot 2 \cdot s(9)$
$= 2 \cdot 2 \cdot 2 \cdot s(4)$
$= 2^4 s(2)$
$= 2^5 s(1)$
$= 32$

Ex: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ → explicit formula. Convert this into recursive

$H_1 = 1$
$H_n = H_{n-1} + \frac{1}{n}, \quad n \geq 2$

*Fibonacci Numbers

$F_0 = 0, \quad F_1 = 1$

$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$

Q. Prove $\sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1}$

Basis step:

$n = 0$

$F_0^2 = 0 \qquad F_0 \times F_1 = 0$

$n = 1$

$F_0^2 + F_1^2 = 0 + 1 = 1 \qquad F_1 \times F_2 = 1 \times 1 = 1$

Inductive step:

let $\sum_{i=0}^{k} F_i^2 = F_k \times F_{k+1}$

$\Rightarrow \sum_{i=0}^{k+1} F_i^2 = \sum_{i=0}^{k} F_i^2 + F_{k+1}^2$
$= F_k \times F_{k+1} + F_{k+1}^2$
$= F_{k+1}(F_k + F_{k+1})$
$= F_{k+1} \times F_{k+2}$

* Lucas No.

$L_0 = 2, \quad L_1 = 1$

$L_n = L_{n-1} + L_{n-2}$

Q. prove that
L_n

Basis step:
$n = 1$
$L_1 = 1$
$n = 2$
$L_2 = 2 + 1 = 3$

Inductive s

~~Let $L_k =$~~
~~$\Rightarrow L_{k+1} =$~~

Let the
$\therefore L_{k+1} =$

HW
1. Prove the
2. Prove $5F_n$

1. Basis step:
$n = 1$
$L_1^2 = 1$
$n = 2$
$L_1^2 + L_2^2 = 1^2 + 3^2$
Inductive
let $L_1^2 +$
$\Rightarrow L_1^2 + L_2^2$

2. Basis step
$n = 0$
$5F_2 = 5$
$n = 1$
$5F_3 = $

(37) = ?

Q. prove that for all positive integers n,

$$L_n = F_{n-1} + F_{n+1}$$

Basis step :-

$n = 1$

$L_1 = 1$, $F_0 + F_2 = 1$

$n = 2$

$L_2 = 2+1 = 3$, $F_1 + F_3 = 1 + 2 = 3$.

Inductive step.

~~let $L_k = F_{k-1} + F_{k+1}$~~

~~$\Rightarrow L_{k+1} = L_k + L_{k-1}$~~

let the hypothesis be true for all $s \le k$

$\therefore L_{k+1} = L_{k-1} + L_k$

$= (F_{k-2}) + F_k + (F_{k-1}) + F_{k+1}$

$= F_k + (F_k + F_{k+1})$

$= F_k + F_{k+2}$

explicit formula

convert this into recursive

HW

1. Prove that $L_1^2 + L_2^2 + L_3^2 + \cdots L_n^2 = L_n L_{n+1} - 2$
2. Prove $5F_{n+2} = L_{n+4} - L_n$

1s Basis step :

$n = 1$

$L_1^2 = 1$ $L_1 \times L_2 - 2 = 3 - 2 = 1$

$n = 2$

$L_1^2 + L_2^2 = 1^2 + 3^2 = 10$ $L_2 \times L_3 - 2 = 3 \times 4 - 2 = 10$

Inductive step :-

let $L_1^2 + L_2^2 + \cdots + L_k^2 = L_k L_{k+1} - 2$

$\Rightarrow L_1^2 + L_2^2 + \cdots + L_{k+1}^2 = (L_k L_{k+1} - 2) + L_{k+1}^2$

$= (L_{k+1})(L_k + L_{k+1}) - 2$

$= L_{k+1} L_{k+2} - 2$

2. Basis step :

$n = 0$

$5F_2 = 5$ $L_4 - L_0 = 7 - 2 = 5$

$n = 1$

$5F_3 = 5 \times 2 = 10$ $L_5 - L_1 = 11 - 1 = 10$

Inductive step:

let the hypothesis be true for all $s \leq k$

$5L_{k+3} = 5L_{k+2} + 5L_{k+1}$

$= L_{k+4} - L_k + L_{k+3} - L_{k-1}$

$= (L_{k+4} + L_{k+3}) - (L_k + L_{k+1})$

$= L_{k+5} - L_{k+2}$

$\Rightarrow 5L_{(k+1)+2} = L_{(k+1)+4} - L_{(k+1)}$

Q. Compute the power a^n, a is a real no. & n is a non-negative integer, $a > 0$

A. Recursive:

procedure power(a real, n non-negative integer)

if $(n == 0)$ power $(a, 0) = 1$

else power(a, n) = $a \times$ power$(a, n-1)$

proof:

$n = 0$

$a^0 = 1$, power$(a, 0) = 1$

Recursive step:

let a^k = power(a, k) i.e formula is true for 'k'

$\Rightarrow a^{k+1} = a \times a^k$

$= a \times$ power(a, k)

power$(a, k+1)$ = $a \times$ power(a, k).

$\therefore a^{k+1}$ = power$(a, k+1)$

Q. Compute a^n modulo m:

A. procedure mod-power(a, n, m) a→real, m, n→non-negative integers, $m \geq 2$

if$(n = 0)$ mod-power$(a, 0, m) = 1$

else if n is even

mod-power(a, n, m) = $\left(\text{mod-power}\left(a, \frac{n}{2}, m\right)\right)^2 \bmod m$

else

mod-power(a, n, m) = $\left(\left(\text{mod-power}\left(a, \frac{n-1}{2}, m\right)\right)^2 \times a\right) \bmod m$

Proof:

Basis step:

$n = 0$, $a^0 =$

$m = 2$

mod-p

Recursive step

let it is tr

if k is even,

$\Rightarrow$ mod-pow

$\therefore$ mod-powe

a^{k+1} mo

if k is odd

$\Rightarrow$ mod-pow

$\therefore$ mod-pow

a^{k+1} mod

* Recursively

Basis step

Recursive step

exclusion rul

Ex: consid

A. Basis s

Recursiv

$\Rightarrow S$ is

$S =$

rue for all $j \le k$

$+ 5L_{k+1}$

$L_k + L_{k+3} - L_{k-1}$

$_{k+3}) - (L_k + L_{k+1})$

$_{k+1}$

$L_{k+1})$

a is a real no. & n is

$a > 0$

al, n nonnegative integer)

power $(a, 0) = 1$

$= a \times$ power $(a, n-1)$

rmula is true for 'k'

er (a, k).

$+1)$

a→real

$, m)$ m, n → non-negative integers

$m \ge 2$

$n) = 1$

$\left(\text{mod-power}\left(a, \frac{n}{2}, m\right)\right)^2 \bmod m$

$\text{wer}\left(a, \frac{n-1}{2}, m\right)\Big)^2 \times a\Big) \bmod m$

$\lfloor \frac{n}{2} \rfloor$

Proof:

Basis step:

$n = 0$, $a^0 = 1$ → $a^0 \bmod 2 = 1$

$m = 2$

mod power $(a, 0, 2) = 1$

Recursive step:

let it is true for all $j \le k$

if k is even

$\Rightarrow$ mod power$(a, k, m) = \left(\text{mod power}\left(a, \frac{k}{2}, m\right)\right)^2 m$

$\therefore$ mod power$(a, k+1, m) = (\text{mod power}(a, \ldots, m) \times a$ mo

$a^{k+1} \bmod m = (a^k \times a) \bmod m$

$= ((a^k \bmod m) \times a (\bmod m)) \bmod m$

$= \left(\left(\text{mod-power}\left(a, \frac{k}{2}, m\right)\right)^2 \times a \bmod m\right)$

if k is odd

$\Rightarrow$ mod-power$(a, k, m) = \left(\left(\text{mod-power}\left(a, \frac{k-1}{2}, m\right)\right)^2 \times a \bmod$ mo

$\therefore$ mod-power$(a, k+1, m) = \left\{\text{mod-power}\left(a, \frac{k+1}{2}, m\right)\right\}^2 \bmod m$

$a^{k+1} \bmod m = a^{\frac{k+1}{2}} a^{\frac{k+1}{2}} \bmod m$

$= \left(\text{mod power}\left(a, \frac{k+1}{2}, m\right)\right)^2 \bmod m$

* Recursively Defined sets:

Basis step → initial collection of elements

Recursive step → Rules for forming new elements from the ~~set~~ known elements of the set

exclusion rule → Set consists of the elements nothing b the elements complying with Basis step & Recursive step.

(All the el satisfying B step & Basis may not be in the set)

Ex: consider a subset S of Integers

Basis step: $3 \in S$

Recursive step: If $x \in S$ & $y \in S$ then $x + y \in S$

Basis

$\Rightarrow$ S is set of elements which are multiples of 3

$S = \{3, 6, 9, 12, \ldots\}$

Proof:

Basis step:

$n = 0$, $m = 2$, $a^0 = 1 \rightarrow a^0 \bmod 2 = 1$

mod power $(a, 0, 2) = 1$

Recursive step:

Let it is true for all $j \le k$

if k is even
$\Rightarrow$ mod_power(a, k, m) = $(\text{mod\_power}(a, k/2, m))^2 \bmod m$

$\therefore$ mod_power$(a, k+1, m)$ = $(\text{mod\_power}(a, k/2, m))^2 \times a \bmod m$ mod m

$a^{k+1} \bmod m = (a^k \times a) \bmod m$

$= ((a^k \bmod m) \times a (\bmod m)) \bmod m$

$= ((\text{mod\_power}(a, \frac{k}{2}, m))^2 \times a \bmod m) \bmod m$

if k is odd

$\Rightarrow$ mod_power$(a, k, m) = ((\text{mod_power}(a, \frac{k-1}{2}, m))^2 \times a \bmod m)$ mod m

$\therefore$ mod_power$(a, k+1, m) = \{\text{mod_power}(a, \frac{k+1}{2}, m)\}^2 \bmod m$

$a^{k+1} \bmod m = a^{\frac{k+1}{2}} a^{\frac{k+1}{2}} \bmod m$

$= (\text{mod power}(a, \frac{k+1}{2}, m))^2 \bmod m$

* Recursively Defined sets:

Basis step $\rightarrow$ initial collection of elements

Recursive step $\rightarrow$ Rules for forming new elements from the ~~set~~ known elements of the set

exclusion rule $\rightarrow$ Set consists of the elements nothing but the elements complying with Basis step & Recursive step.

(All the elements satisfying Recursive step & Basis step may not be there in the set)

Basis step

Ex: consider a subset S of Integers

Basis step: $3 \in S$

Recursive step: If $x \in S$ & $y \in S$ then $x + y \in S$

$\Rightarrow S$ is set of elements which are multiples of 3.

$S = \{3, 6, 9, 12, \ldots\}$

* Strings

string over a set of (symbols) alphabet Σ, is a finite sequence of symbols.

Defn: The set Σ^* of strings over the set Σ can be defined (recursively)

Basis step: $\lambda \in \Sigma^*$, λ is the empty string containing no symbols

Recursive step: If $w \in \Sigma^*$ and $x \in \Sigma$ then $wx \in \Sigma^*$

Ex: $\Sigma = \{0, 1\}$

B.S: $\lambda = \{\ \}$

~~R.S: $\Sigma^* = \lambda, \lambda 0, \lambda 1,$~~

~~$\{\lambda, 0, 1\}$~~

R.S: $\Sigma^* = \lambda, 0, 1, 00, 01, 10, 11, \ldots$

If $l(x)$ represents the length of string x

Recursive definition of length of string.

B.S: Length of empty set = 0

R.S: If $l(w) = l \Rightarrow l(wx) = l+1$

*

Σ = S(propositional variable), T, F, $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

Σ^* → set of well-defined formulae

Ex: $p \wedge (\neg q)$ ✓

$(p \wedge q) \rightarrow (q \wedge F)$ ✓

$\neg \vee (p \wedge q)$ ✗

* If T, F, S (set of propositions) and the set o

form the compound syst

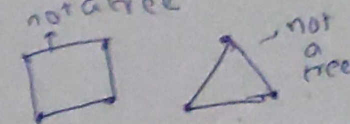
B.S: T, F, S are well f

R.S: If p & q are w

the $\neg p$, $p \wedge q$,

* Graph-Example

Graph → set of vertices (V)

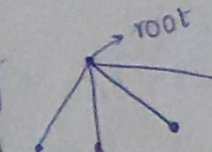


A Graph without c

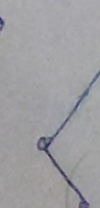
~~* Recursively Defined~~

Non-empty set of

edges (may or m



Binary tree →



Level of a vertex =

* Recursive Definitio

Set of vertices

vertex called r

is defined rec

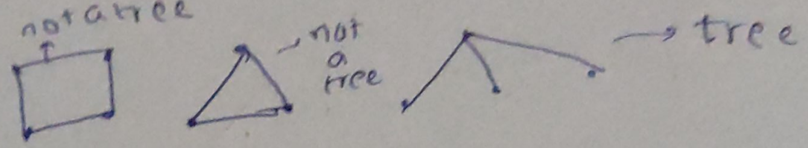
* If T, F, S (set of proposition) and the set of connectives ($\neg, \wedge, \vee,$ …) form the compound statement

B.S: T, F, S are well formed formula

R.S: If p & q are w.f.f. (well formed formula) then $\neg p$, $p \wedge q$, $p \to q$ are w.f.f. (after initial application of recursive rule)

* Graph Example

Graph → set of vertices & edges.
(V) (E)

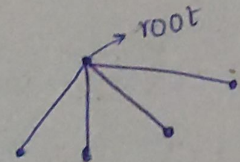


A Graph without a cycle → tree

* ~~Recursively Defined Set~~

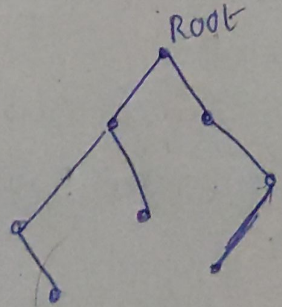
→ Non-empty set of vertices and set of edges (may or may not be empty)

tree



level of Root = 0

Binary tree →



Level of a vertex = length of the unique path from the designated root to the vertex.

* Recursive Definition of a Rooted Tree:

Set of vertices containing a distinguished vertex called root, edges connecting these vertices is defined recursively as below

Basis step

A single vertex called 'r' is the rooted tree

Recursive step?

Suppose $T_1, T_2, \ldots, T_n$ are disjoint rooted trees

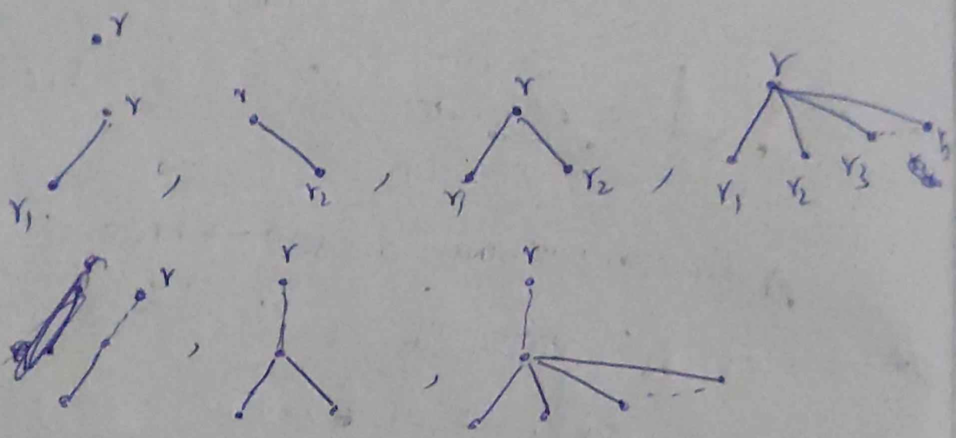
Then the tree formed by connecting root 'r' with the roots of $T_1, T_2, \ldots T_n$ (i.e $r_1, r_2, \ldots r_n$), is also a rooted tree.

⇒ B.S

I.S

Step1:

Step2:



* Binary Tree

* Extended Binary tree

↳ One child can be empty

⇒ (One subtree can be empty)

* Full Binary tree

↳ No child can be empty

Recursive Def^n of Extended Binary tree:-

Basis step:

An empty set is an extended Binary tree

Recursive step:

suppose T_1, T_2 are disjoint extended binary trees.

Then the tree formed by connecting the root with roots of T_1, T_2 is also an extended Binary tree.

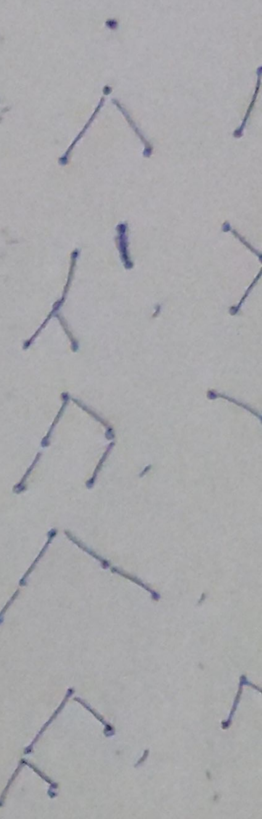
B.S ∅

I.S

Step1:

Step2:

Step3



Recursive Def^n of

Basis step:

One vertex tree.

Recursive step

Suppose T_1, T_2,

trees.

Then the tree

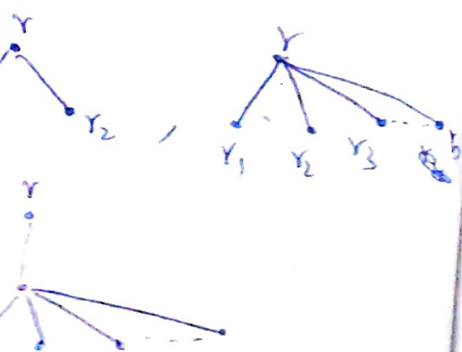
with the roots

Binary tree.

... tree.

... are disjoint rooted

... by connecting root 'r' ... T_n (i.e $r_1, r_2, \ldots r_n$).



... be empty

(can be empty)

... be empty

... y tree:-

... ed Binary tree

... int extended

... necting the

... is

BS $\varnothing$

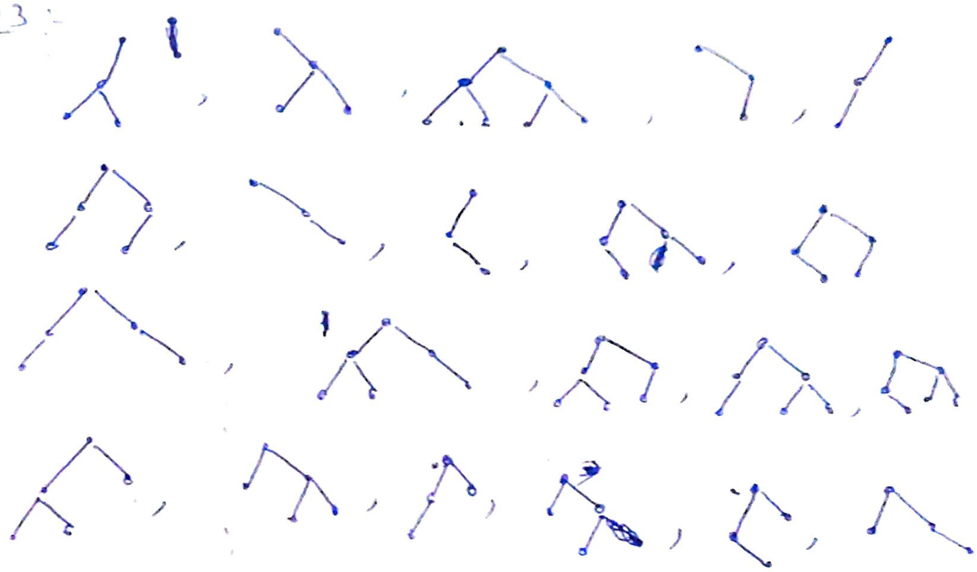
IS

Step 1: .

Step 2:

$\Rightarrow$ # No of trees $= 1 + 2 \times 1$

Step 3:



$\Rightarrow$ No. of trees $= 21$

Recursive Def^n of Full Binary Tree:

Basis step:

One vertex called root is a Full Binary tree.

Recursive step:

Suppose $T_1, T_2, \ldots$ are disjoint Full Binary trees.

Then the tree formed by connecting the root with the roots of $T_1, T_2, \ldots T_n$ is also a Full Binary tree.

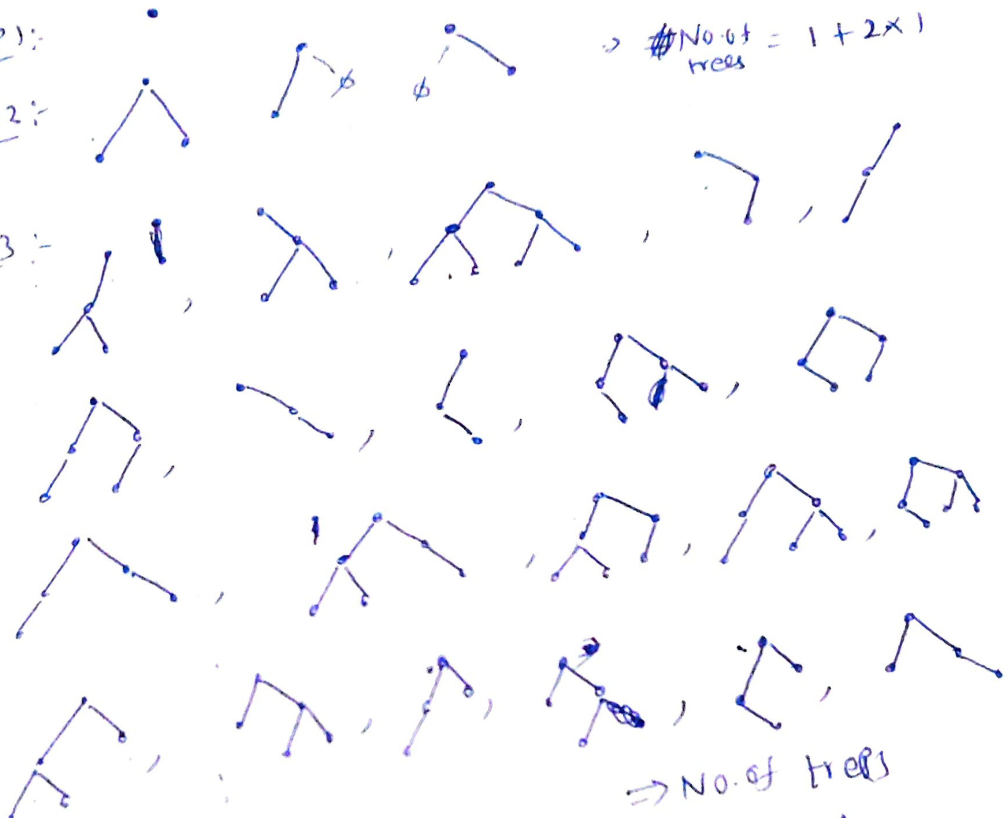
B.S $\emptyset$

I.S

Step 1:-

Step 2:-

Step 3:-



$\Rightarrow$ No. of trees $= 1 + 2 \times 1$

$\Rightarrow$ No. of trees $= 21$

Recursive Def^n of Full Binary Tree:

Basis step:

One vertex called root is a Full Binary tree.

Recursive step

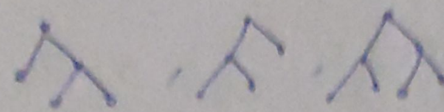
Suppose T_1, T_2, are disjoint Full Binary tree.

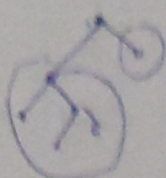
Then the tree formed by connecting the root with the roots of $T_1, T_2, \ldots T_n$ is also a Full Binary tree.

B.S.

I.S.

Step 1: 

Step 2: 

Step 3: 

Ex: Recursive Defn

B.S: $3 \in S$

I.S: if $x \in S$, $y \in S$, then $x+y \in S$

S is the set of multiples of '3'.

Proof: Let A be a set of ~~non-negative~~ positive multiples of 3

To prove $A=S$, we have to prove that

(i) A is a subset of S

(ii) S is a subset of A.

Let p(n) be a proposition that the set 'S' contains first 'n' multiples of '3'.

B.S: $n=1$

$A=\{3\}$

$3 \in S$ (B.S of 'S') True

I.S: let p(k) is true.

$\Rightarrow 3k \in S$

We have to show $3k+3 \in S$

W.K.T $3 \in S$ & $3k \in S$

$\Rightarrow 3k+3 \in S \Rightarrow 3(k+1) \in S$

Hence p(k+1) is true.

(ii) if $x \in A$

$\Rightarrow 3|x,$

$\Rightarrow 3|(x$

S is

Recursive D

Height h(

B.S: heigh

R.S: If T_1

has

Recursive De

B.S: For

R.S: n(

T_1, T

Theorem:

If T is

B.S:

$n(T)=1$

$\Rightarrow h(T)=0$

$2^{h(1)+1}-$

$n(T)$

I.S:

T_1 & T_2

n_1 & n

assum

$\Rightarrow$

(ii) Let

if $x \in A$, $y \in A \Rightarrow x \in S$, $y \in S$ [$\because$ A is ...]

$\Rightarrow 3|x$, $3|y$ [$\because$ A is set of multiples of 3]

$\Rightarrow 3|(x+y)$ $\quad x+y \in S$

$\therefore$ S is a subset of A

Recursive Def^n:

Height $h(T)$ of a full binary tree T recursively

BS: height $h(T)$ of ~~Full Binary Tree~~ one vertex F.B.T is zero

$h(T) = 0$

R.S:- If T_1, T_2 are Full Binary Trees then $T_1 \cdot T_2$ has height $h(T) = 1 + \max(h(T_1), h(T_2))$

Recursive Def^n of No. of vertices $n(T)$ of F.B.T

B.S: For a single root node, $n(T) = 1$

R.S:- $n(T) = 1 + n(T_1) + n(T_2)$

T_1, T_2 are subtrees of the root

Theorem:

If T is a full binary tree, then

$n(T) \leq 2^{h(T)+1} - 1$

B.S:

$n(T) = 1$

$\Rightarrow h(T) = 0$

$2^{h(T)+1} - 1 = 2^1 - 1 = 1$

$n(T) \leq 2^{h(T)+1}$ $\Rightarrow$ True

I.S:

T_1 & T_2 are F.B. Trees having no. of vertices n_1 & n_2 and T is formed with T_1, T_2 connected to the root

assume that the result is true for all $n(T) \leq n_1 + n_2$

$\Rightarrow T_1$ & T_2 is true.

$\therefore n(T) = n_1 + n_2 + 1$

$n_1 \le 2^{h(T_1)+1} - 1$

$n_2 \le 2^{h(T_2)+1} - 1$

$\therefore n(T) \le 1 + 2^{h(T_1)+1} - 1 + 2^{h(T_2)+1} - 1$

$= 2^{h(T_1)+1} + 2^{h(T_2)+1} - 1$

$\le 2 \cdot \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1$

$= 2 \cdot 2^{\max(h(T_1), h(T_2))+1} - 1$

$= 2 \cdot 2^{h(T)} - 1$

$= 2^{h(T)+1} - 1$

$\therefore n(T) \le 2^{h(T)+1} - 1$

~~* well ordering form~~

lexicographic ordering.

(x_1, y_1) is less than ~~or equal to~~ (x_2, y_2)

$\Rightarrow$ either $x_1 < x_2$

(or) $x_1 = x_2$ & $y_1 < y_2$

Ex: Recursive def^n

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1, & \text{if } n = 0 \\ a_{m,n-1} + n, & \text{if } n > 0 \end{cases}$$

$a_{0,0} = 0$

Show that $a_{m,n} = m + \frac{n(n+1)}{2}$ for all $(m,n) \in \mathbb{N} \times \mathbb{N}$

B.S:

$m = 0, n = 0$

$a_{0,0} = 0 \qquad m + \frac{n(n+1)}{2} = 0 + 0 = 0$

I.S:

Assume the result is true for some (m', n') where (m', n') less than (m, n)

$\Rightarrow a_{m',n'}$

if $n = 0$

$a_{m,n} =$

$=$

$=$

if $n > 0$

$a_{m,n} =$

$=$

Set-Funct

* Set :-

A =

Subset:

ele

X

If X

X is a

Number

* X = Y

$\Rightarrow \forall x$ (

* empty

* Univers

A = {a

subsets -

P(A) is

collect

$\rightarrow a_{m', n'}$ $m + \frac{}{2}$

if $n = 0$

$a_{m,n} = a_{m-1,n} + 1$

$= (m-1) + \frac{n(n+1)}{2} + 1$

$= m + \frac{n(n+1)}{2}$ ✓ true

1) - 1

if $n > 0$

$a_{m,n} = a_{m,n-1} + n$

$= m + \frac{(n-1)n}{2} + n$

$= m + \frac{n(n+1)}{2}$ True

Set - Function - Relation

* Set :- unordered collection of objects.

$A = \{1, 2, 3\}$

Subset :- If X & Y are two sets and if each element of X is element of Y, then X is a subset of Y

$X \subseteq Y$

If X is a subset of Y & $X \neq Y$, then X is a proper subset of Y, $X \subset Y$

Number of elements of $X = |X| \rightarrow$ cardinality of 'X'.

* $X = Y$

$\Rightarrow \forall x \left(x \in X \rightarrow x \in Y \right) \wedge \left(x \in Y \rightarrow x \in X \right)$

* empty set $\rightarrow \emptyset, \{\}$
* Universal set $\rightarrow U$

$A = \{a, b, c\}$, $|A| = 3$

subsets $\rightarrow \{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

$P(A)$ is the power set of A, which ~~cont~~ is the collection of all subsets of A.

(x_2, y_2)

$= 0$

> 0

for all

$(m, n) \in \mathbb{N} \times \mathbb{N}$

(m', n')

$|P(A)| = 2^3 = 8$

Subsets containing element 'a'	Subsets not containing 'a'
$\{a\}$	$\emptyset$
$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{c\}$
$\{a, b, c\}$	$\{b, c\}$

$2^{n-1} \leftarrow 4 = 4 \rightarrow 2^{n-1}$

Theorem: If $|X| = n$, then $|P(X)| = 2^n$, wh

X is a set.

proof:-

Basis step:-

$n = 0 \Rightarrow X = \emptyset$

$P(x) = \{\emptyset\} \Rightarrow 2^n = 2^0 = 1$

$|P(x)| = 1$ True

Inductive step:-

Let the result is true for 'n'

Let X be a set of cardinality 'n+1'.

We have to show that $|P(X)| = 2^{n+1}$

Let 'y' be a set obtained from X by dropping one element 'a' from X.

$\Rightarrow |P(y)| = 2^n$

y is a set having elements of X except 'a'.

From the property we have seen above,

$$|P(y)| = \frac{|P(x)|}{2}$$

$$|P(x)| = 2|P(y)|$$

$$\boxed{\therefore |P(x)| = 2^{n+1}}$$

* Operations on a set :

Union, Intersection, Difference

A & B are two sets

$A \cup B \rightarrow$ Collection of all distinct elements that exist in A or B.

$A \cap B \rightarrow$ Intersection is the collection of the elements that exist in both A & B.

$A - B \neq B - A$

Let U be the universal set and A, B, C are the subsets of U then the following properties hold for U, A, B, C and $\cup, \cap, \bar{\ }$ (Difference).

(a) Associative : $(A \cup B) \cup C = A \cup (B \cup C)$,
$(A \cap B) \cap C = A \cap (B \cap C)$

(b) Commutative : $A \cup B = B \cup A$, $A \cap B = B \cap A$

(c) Distributive : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(d) Identity : $A \cup \emptyset = A$, $A \cap U = A$

(e) Complementation : $A \cup \bar{A} = U$, $A \cap \bar{A} = \emptyset$

(f) Idempotent : $A \cup A = A$, $A \cap A = A$

(g) Bound : $A \cup U = U$, $A \cap \emptyset = \emptyset$

(h) Absorption : $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

(i) Involution : $\bar{\bar{A}} = A$

(j) 0/1 laws : $\bar{\emptyset} = U$, $\bar{U} = \emptyset$

(k) Demorgan : $\overline{A \cup B} = \bar{A} \cap \bar{B}$
$\overline{A \cap B} = \bar{A} \cup \bar{B}$

* (c) Distributive

Prove $\underbrace{A \cap (B \cup C)}_{X} = \underbrace{(A \cap B) \cup (A \cap C)}_{Y}$

S: Let x be an element of X. $\Rightarrow x \in X$

$x \in A \cap (B \cup C)$