Prove that a ~~3×3~~

($3m \times 2n$) checker board can be completely tiled using triominoes.

prove that a 5×5 deficient board can be tiled

Then prove that a 7×7 deficient board can be tiled.

Set function relation.

Set is an unordered collection of objects

$A = \{1, 2, 3\}$

element :

$A = \{x \mid x$ is a +'ve even integer$\}$

Subset.

if x and y are 2 sets.
and if each element of X is the element of Y
then X is a subset of Y.

$X \subseteq Y$,

$X = Y$

$\forall x((x \in X \rightarrow x \in Y) \wedge (x \in Y \rightarrow x \in X))$

if X is a subset of Y and $X \neq Y$

then $X \subseteq Y$

negation i.e $X \neq Y$

$\exists x(((x \in X) \wedge (x \notin Y)) \vee ((x \in Y)$

No. of elements of x, $|X|$ = cardinality of X.

empty set : ϕ, $\{\}$

Universal set : U is the universal set

set of all ~~[illegible]~~ elements.

$A = \{a, b, c\}$, $|A| = 3$

ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$,

$\{a, c\}$, $\{a, b, c\}$

$P(A)$ is the set of all subsets of A.

$|A| = 3$

$|P(A)| = 2^3 = 8$

Subsets containing a	Subsets not containing a
$\{a\}$	ϕ
$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{c\}$
$\{a, b, c\}$	$\{b, c\}$
2^{n-1}	2^{n-1}

Theorem : if $|x| = n$, prove that $|P(x)| = 2^n$, where x is a set.

Proof : B.S $n = 0$, $2^0 = 1$

Power set consists of one subset, which is ϕ, i.e. the empty set.

$|P(x)| = 1$

$A\ (x \notin X)$

I.S

Let the result is true for n.

Let X be a set of cardinality $(n+1)$,
we have to show that $|P(X)| = 2^{n+1}$

Let Y be a set obtained from X by dropping one element from X.

$$|P(Y)| = 2^n$$

now $\frac{|P(X)|}{2} = |P(Y)|$

$$\therefore |P(X)| = 2 \cdot |P(Y)| = 2^{n+1}$$

Operations on set

Union ; Intersection , Difference

A and B are two sets

A U B : Collection of all distinct elements that exists in A or B

$A = \{1, 2, 3\}$

$B = \{3, 7, 8\}$

$A \cup B = \{1, 2, 3, 7, 8\}$

$A \cap B = \{3\}$

$A - B = \{1, 2\}$

$B - A = \{7, 8\}$

$\therefore A - B \neq B - A$

Let U be the universal set and

A, B, C are subsets of U

then following properties hold for U, A, B, C and $\cup, \cap$

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap C$$

Identity → $A \cup \phi = A, \quad A \cap U = A$

Complementation → $A \cup \bar{A} = U; \quad A \cap \bar{A} = \phi$

Idempotent → $A \cup A = A, \quad A \cap A = A$

Bound → $A \cup U = U, \quad A \cap \phi = \phi$

Prove: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$X = Y$ means,

we, $\forall x((x \in X \to x \in Y) \wedge (x \in Y \to x \in X))$ ...

Absorption

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Involution $\bar{\bar{A}} = A$

$\bar{U} = \phi, \quad \bar{\phi} = U$

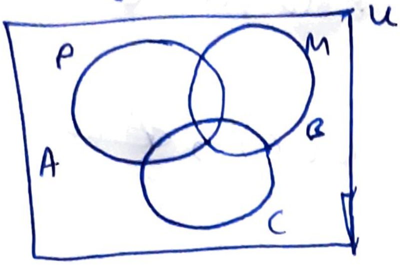
De Morgan

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Set - Function - Relation:

Venn Diagram is a pictorial representation of set



Among 165 students of a class, say 8 students.

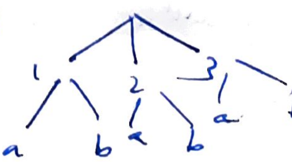
$A, B \subseteq U$

If X and Y are 2 sets, $x \in X, y \in Y$

Cartesian product is defined as $X \times Y$, which is the set of all ordered pairs of (x, y)

$X = \{1, 2, 3\}$ $X \times Y$ →

$Y = \{a, b\}$



$\{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$Y \times X \rightarrow \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

note: $X \times Y \neq Y \times X$

$|X \times Y| = |Y \times X|$

$X \times X = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$Y \times Y = \{(a,a), (a,b), (b,a), (b,b)\}$

Let X & Y be 2 sets and a function f is defined as the subsets of the cartesian product $X \times Y$ such that for all $x \in X$, & $y \in Y$ $(x, y) \in f$

$$\boxed{f : X \to Y}$$

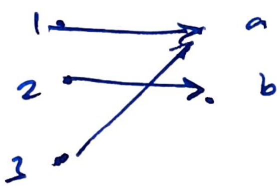
$X = \{1, 2, 3\}$

$Y = \{a, b\}$

subsets of $X \times Y$ $= \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$f : X \to Y$

$\{(1,a), (3,a), (2,b)\}$



Injective:

f is injective if for every $y \in Y$, ~~$x \in X$~~ there is ~~exactly~~ at most one ~~y~~ $x \in X$ such that $(x, y) \in f$

for all $x \in X$ there is exactly one $y \in Y$

Surjective: if range is Y.

Composition

Let g be a function from X to Y and f be a function from Y to Z then we can define composition of the factors as.

$$(f o g)(x) : \cancel{X} \to Z$$

$$g : X \to Y = \{(x, y) \mid x, y \in g\}$$

$$f : Y \to Z = \{(y, z) \mid (y, z) \in f\}$$

$$X = \{1, 2, 3\}, \quad Y = \{a, b\}, \quad Z = \{x, y, z\}$$

$\frac{1}{1} + \dots + \frac{1}{(n+1)^2}$

$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$

$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots < \frac{n^2 - n}{n^2}$

$\frac{4n - 1}{4n}$

$< 2 - \frac{1}{n+1}$

$\frac{n-2}{4}$

$\frac{n-1}{n^2} < \frac{n-1}{n}$

$< 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$

$< 2 + \frac{1}{n+1} - \frac{1}{n}$

$\frac{1}{2^2} \dots \frac{1}{n^2} < 1 - \frac{1}{n}$

$\frac{n - (n-1)}{n(n+1)}$

$< \frac{n-1}{n}$

$\frac{3}{4} - \frac{1}{n}$

$$\boxed{\begin{aligned} f(n) &= n^4 \\ g(n) &= \log_3 n \\ g o f(n) &= 4 \log_3 n \\ f o g(n) &= (\log_3 n)^4 \end{aligned}}$$

Sequence & String

Sequence is a special type of function where the domain consists of consecutive integers

$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$

$\frac{1}{\sqrt{3n+4}}$

$\cdot \frac{1}{\sqrt{3n+1}} \cdot \frac{2n+1}{2n+2}$

A string over X is a finite sequence of elements of X, X is a finite set.

$X = \{a, b, c, d\}$

$\beta_1 = a$ $\quad \beta_2 \beta_1 \beta_3 = \{bac\}$

$\beta_2 = b$ $\quad b^2 a^4 c = bbaaaac$

$\beta_3 = c$

$\frac{2n+1}{2n+2} \cdot$

$\underbrace{(2n+1)}_{a}{}^2 (\cancel{3n+4})$ (with $3n$ and $\cancel{b+3}$ above)

$< \underbrace{(3n+1)}_{a} (\cancel{2n+2})^2$ (with $2n+2$ above and $a+1$ below)

$(4n^2+4n+1)(3n) \cancel{(2n+1)}^2$

$\cancel{12n^3} + \cancel{12n^2} + \cancel{3n} + 16n^2 + \cancel{16n} + 4$ $\quad a^2 b + a^2 3 < ba^2 + 2ba + b$

$< 4(3n+1)(2n+n^2+1)$ $\quad \cancel{12n^3} + \cancel{24n^2} \quad n$

$< (12n+4)(n^2+2n+1)$ $\quad + \cancel{12n} + 4n^2 + \cancel{8n} + 4$

Given a sequence $S_n = 2^n + 4 \cdot 3^n \quad n \geq 0$

find S_0

find S_1

find a function for S_i $\longrightarrow S_i = 2^i + 4 \cdot 3^i$

$S_{n-1} \longrightarrow S_{n-1} = 2^{n-1} + 4 \cdot 3^{n-1}$

S_{n-2}

Prove that S_n satisfies $S_n = 5 S_{n-1} - 6 S_{n-2}$ $\boxed{n \geq 2}$

R.H.S

$$5(2^{n-1} + 4 \cdot 3^{n-1}) - 6(2^{n-2} + 4 \cdot 3^{n-2})$$

$$= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} + 4(5 \cdot 3^{n-1} - 6 \cdot 3^{n-2})$$

$$= 2^n + 4 \cdot 3^n$$

$S_1, S_2, \ldots S_i \ldots S_n, S_{n+1}, \ldots$

if $S_n < S_{n+1}$ $\longrightarrow$ increasing sequence

$S_n \leq S_{n+1}$ — nondecreasing sequence

Relation :

Defn : A relation (binary) R from X to Y is a subset of the cartesian product $X \times Y$, if $(x, y) \in R$, we call xRy or x is related to y. If $X = Y$, R is on X.

$\{x \in X \mid (x, y) \in R \text{ for } y \in Y\}$ domain

$y \in Y \mid (x, y) \in R, \text{ for } x \in X\}$ Range.

Function :

(a) the domain should be X

(b) for all $x \in X$, this is exactly one $y \in Y$

• Function is a special type of relation.

Example :

$R = \{(x, y) \mid x \leq y\}$.

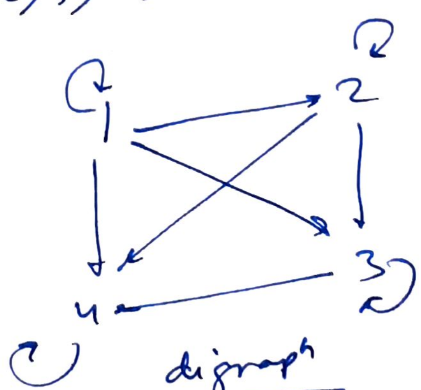
$x = \{2, 3, 4\}$

$Y = \{3, 4, 5, 6\}$

$R = \{(2,3) (2,4) (2,5) (2,6), (3,3) (3,4) (3,5) (3,6), (4,4) (4,5), (4,6)\}$

$X = \{1, 2, 3, 4\}, x \leq y, x, y \in X\}$

R =



digraph

Reflexive Yes! $\{(1,1), (2,2), (3,3) (4,4)\}$

Symmetric NO !

$R = \{(a,a), (b,c), (c,b), (d,d)\}$

not reflexive

symmetric,

Anti symmetric

If $(x,y) \in R$ and $x \neq y$ then $(y,x) \notin R$ then R is antisymmetric

$X = \{2,3,4\}$ x divides y : $R = \{(2,6),(3,3),(3,6)\}$

$Y = \{3,5,6\}$

for a relation to be symmetric & anti symmetric at the same time.

$R = \{(a,a),(b,b),(c,c)\ldots\}$

The pair having property if $(x,y) \in R$, $x \neq y$ doesn't belong to R, then $(x,y) \in R$ & $x \neq y$, then $(y,x) \in R$. ∴ it is symmetric and antisymmetric at the same time.

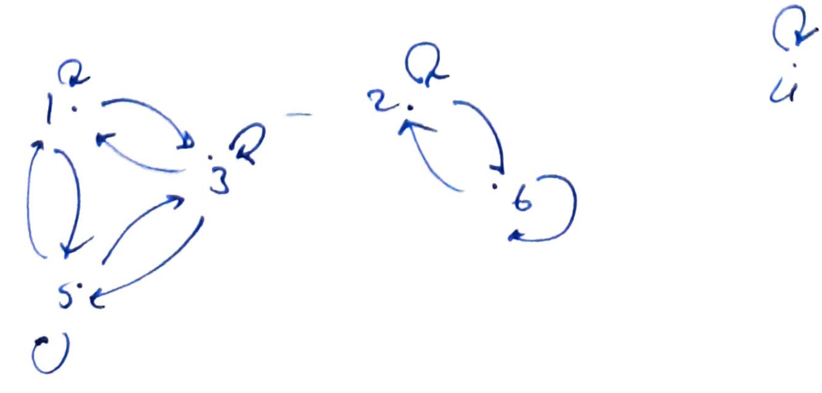
Relation R is transitive if $(x,y) \in R$ & $(y,z) \in R$ then $(x,z) \in R$

$R : \{(x,y) \mid x \leq y\}$

$R : \{(x,y) \mid x \text{ divides } y\}$

$x \leq y, \; y \leq z$

$x \leq z$



$R : \{ (1,1), (1,3), (1,5), (3,1), (3,3), (3,5),$
$(5,5), (5,1), (5,3), (2,2), (2,6),$
$(6,2), (6,6), (4,4) \}$

Reflexive symmetric, not antisymmetric,
transitive

Partial Order :

if a relation is reflexive, antisymmetric and transitive then the relation is called a partial order

Equivalence Relation :

if a relation is reflexive, symmetric and transitive then the relation is called an equivalence relation.

R is on X. (Total order)

Consider a subset S of X

$x \in X$, $x \in S$

xRa. $a \in X$, $a \in S$

$(x, a) \in X$ $(x, a) \in S$

$(xRa \in S)$

Partition

R is an equivalence relation on X, $a \in X$

$[a] = \{x \in X \mid x R a\}$ — all elements that are related to a

$S = \{[a] \mid a \in X\}$ is a partition on X

All sets of [a] are called the equivalence classes.

$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$[1] = (1,1)\ (1,4)\ (1,7)\ (1,10)$

$[2] = (2,2)\ (2,5)\ (3,8)$

$[3] = 3,3)\ (3,6)\ (3,9)$

Recurrence Relations :

Defn : A recurrence relation of a sequence is an equation that relates the nth term of the sequence to certain predecessors.

e.g ⇒ $\boxed{a_n = a_{n-1} + 3}$

e.g. Some one invests Rs 10,000/- @ 15% compound Interest

$a_0 = 10,000$

a_i : ~~10,000~~ B. $\boxed{a_{i-1}\left(1 + \frac{15}{100}\right) = a_i}$

$p \to (q \to r)$

$\neg q \to \neg p$

p

mod

$\neg p \quad q$

$p \to (q \to$

$q \to r$

$r.$

$p \to q$

$\neg q$

$\neg p$

$\therefore$ $\boxed{a_n = (1.15)^n a_0}$ Explicit formula

$\boxed{a_n = (1.15)\, a_{n-1}}$ Recurrence relation

Example: The no. of subsets S_n of a set of n elements.

Recurrence relation :

$$S_n = 2 S_{n-1}$$

now $S_0 = 1$

$S_1 = 2^1$

$S_2 = 2^2$

$\therefore \boxed{S_n = 2^n}$

Q

Find the number of n-bit strings which does not contain '111'

1 1 1 1 1 1

111

a_4 = 1110

$(n-2)/(2^{n-3})$

→ strings begin with either 0 or 1

for beginning with '0',

1) 0 $\boxed{S_{n-1}}$

for beginning with '10',

2) 1, 0. $\boxed{S_{n-2}}$

for beginning with '1,1'
we need the 3rd as '0' (as it cannot be 1)

∴ 3) 1, 1, 0 $\boxed{S_{n-3}}$

∴ $(S_n = S_{n-1} + S_{n-2} + S_{n-3})$ (Ans)

$$\boxed{\begin{aligned} S_1 &= 2 \\ S_2 &= 4 \\ S_3 &= 7 \end{aligned}}$$

Q Find the no. of n-bit strings which has no 2 consecutive 0's.

→ If it begins with '0' then next must be 1

0 1 $\boxed{S_{n-2}}$

If it begins with '1'

then 1 $\boxed{S_{n-1}}$

$$\boxed{S_n = S_{n-1} + S_{n-2}}$$

Initial value

$$\boxed{\begin{aligned} S_1 &= 2 \\ S_2 &= 3 \end{aligned}}$$

Code Enumeration

n bit code words are formed from decimal digits 0, 1, . . . , 9

A valid code word contains even no. of 0's

Find the number of n bit valid codes.

$9^n + 9^{n-2}\,{}^nC_2 + {}^nC_4\,9^{n-4} + {}^nC_6 + {}^nC_8 + {}^nC$

$$\boxed{\frac{10^n + 8^n}{2}}$$

Let S_n be the no of valid n-bit codewords

then case 1:

a) Let no. of valid codes of size $n-1$ be S_{n-1}

then the newest digit added can be from 1 to 9

$\therefore \; 9S_{n-1}$

Case. 2

a) Let no. of invalid cases of size $n-1$

is $10^{n-1} - S_{n-1}$

then if we add just one zero, we get a valid code

$10^{n-1} - S_{n-1}$

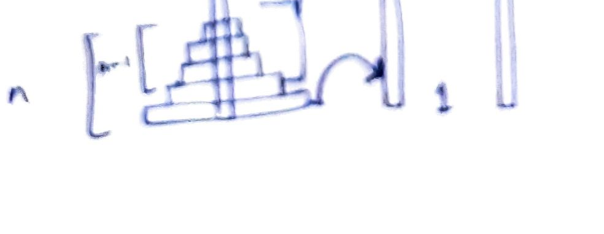
$$\therefore \boxed{S_n = 10^{n-1} + 8S_{n-1}}$$

3 pegs

N no.

Tower of Hanoi

$$\boxed{C_n = 2C_{n-1} + 1}$$



$$\boxed{2C_{n-1} + 1 = C_n}$$

$$\therefore \boxed{C_n = 2^n - 1}$$

Prove that the soln is optimal. Let d_n be the optimal soln

We have to show that $C_n = d_n$

B.S $n = 1$ $C_1 = 1 = d_n$

I.S Let the result is true for $(n-1)$ disk

i.e. $\boxed{C_{n-1} = d_{n-1}}$

now $d_n \geq 2d_{n-1} + 1$

$d_n \geq 2C_{n-1} + 1$

$d_n \geq C_n$

also $C_n \leq d_n$

$\boxed{C_n = d_n}$

Solving recurrence relation To, linear Homogeneous eqn/ recurrence relation

Linear : $y = ax + b$

Non linear : $y = ax^2 + bx + C$. Linear non homogeneous

$= ax^2 + bx^3 + C$ Non linear

check ??

$a_n = a_{n-1} + a_{n-2}$ — linear

$a_n = 5n\, a_{n-1}$ — Non linear

$a_n^2 = 2a_{n-1} + 6$ — Non linear

$a_n = 2a_{n-1}\, a_{n-2}$ — Non homogeneous

Definition :

A linear homogeneous recurrence relation of order k with constant coefficients is of the form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$

with k - initial conditions $a_0 = C_0, a_1 = C_1$

c_i = constant coefficients

$a_n = 5a_{n-1} - 6a_{n-2}$; $a_0 = 7$, $a_1 = 16$

1st order R.R.

$S_n = 2S_{n-1}$ | $C_n = 2C_{n-1} + 1$

Sol^n $\boxed{S_n = 2^n}$ | $\boxed{C_n = 2^n - 1}$

$a_n = 5a_{n-1} - 6a_{n-2}$; $a_0 = 7$, $a_1 = 16$

→ Let the sol^n be of the form t^n ;

$S_n = t^n$

$t^n = 5t^{n-1} - 6t^{n-2}$

$t^2 - 5t + 6 = 0$

$t = 2$; $t = 3$

Let S and T be the sol^n s ,

$S = 2^n$ and $T = 3^n$

$U = bS + dT$ is the solution.

$U_n = bS_n + dT_n$ is the sol^n of

$a_0 = b + d = 7$

$a_1 = 2b + 3d = 16$

$b = 5$, $d = 2$

$\therefore a_n = 5 \cdot 2^n + 2 \cdot 3^n$

Theorem:

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a 2nd order linear homogeneous recurrence relation with constant coefficients If S and T are solutions then $U = bS + dT$ is also a solution of the recurrence relation.

If r is a root of $t^2 - c_1 t - c_2 = 0$ then the sequence r^n, $n = 0, 1, \ldots n$ is a solution. If 'a' is a sequence defined by the conditions $a_0 = C_0$, $a_1 = C_1 \ldots$ and r_1 & r_2 are roots with $r_1 \neq r_2$ then there exists b and d such that

$$a_n = b r_1^n + d r_2^n, \; n = 0, 1 \ldots$$

(R.R) of

Proof:

If S & T are the solutions of the recurrence relation 2nd order

$$S_n = c_1 S_{n-1} + c_2 S_{n-2}$$

$$T_n = c_1 T_{n-1} + c_2 T_{n-2}$$

$$U_n = b(S_n) + d T_n = c_1(b S_{n-1} + d T_{n-1}) + c_2(b S_{n-2} + d T_{n-2})$$

$$= c_1 a_{n-1} + c_2 a_{n-2}$$

$\therefore$ U is the solution

Since r is a root $r^2 = c_1 r + c_2$

$$c_1 r^{n-1} + c_2 r^{n-2} = r^{n-2}(c_1 r + c_2) = r^{n-2} \cdot r^2$$

$$= r^n$$

Deer population

$S_n = 200$

$S_n = (1.01)^n$

Assume that the deer population
200 at time $n=0$
220 at time $n=1$
and increase from $(n-1)$ to n
is twice that of $(n-2)$ to $(n-1)$

R.R

$$d_n - d_{n-1} = 2(d_{n-1} - d_{n-2})$$

$$\boxed{\begin{aligned} d_0 &= 200 \\ d_1 &= 220 \end{aligned}}$$

Let S_n be t^n:

$$t^2 - 3t + 2 = 0$$

$$t = 1, 2$$

$$\cancel{at + bt^2}$$

$$a \cdot 1^n + b \cdot 2^n$$

$$a + b = 200$$

$$a + 2b = 220$$

$$b = 20$$

$$a = 180$$

$$\boxed{d_n = 180 + 20 \cdot 2^n}$$

∴ exponential growth

Fibonacci:

$$f_n = f_{n-1} + f_{n-2}$$

$$t^2 - t - 1 = 0$$

$$\left(t = \frac{1+\sqrt{5}}{2}\right), \left(\frac{1-\sqrt{5}}{2}\right)$$

$$U_n = b\left(\frac{1+\sqrt5}{2}\right)^n + d\left(\frac{1-\sqrt5}{2}\right)^n \qquad f_1 = 1, \quad f_2 = 1$$

$$b\left(\frac{1+\sqrt5}{2}\right) + d\left(\frac{1-\sqrt5}{2}\right) = 1$$

$$b\left(\frac{1+\sqrt5}{2}\right)^2 + d\left(\frac{1-\sqrt5}{2}\right)^2 = 1$$

$$\boxed{b = \frac{1}{\sqrt5}, \quad d = \frac{-1}{\sqrt5}}$$

$$f_n = \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n$$

Theorem:

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a 2nd order linear homogeneous r.r. with initial conditions $a_0 = C_0$, $a_1 = C_1$. If both the roots are equal then the soln is $a_n = br^n + dnr^n$

Pf

$$t^n - c_1 t^{n-1} - c_2 t^{n-2} = 0$$

$$t^2 - c_1 t - c_2 = 0$$

$$\therefore (t-r)^2 = 0$$

$$t^2 - 2tr + r^2 = 0 = t^2 - c_1 t - c_2$$

$$\boxed{\begin{aligned} c_1 &= 2r \\ c_2 &= -r^2 \end{aligned}}$$

$$c_1[(n-1)r^{n-1}] + c_2[(n-2)r^{n-2}]$$

$$= 2r.(n-1)r^{n-1} - r^2(n-2)r^{n-2}$$

$$= r^n(2n-2-n+2) = \underline{nr^n}$$

$\therefore U_n = b r^n + d n r^n$

Ex: $d_n = 4(d_{n-1} - d_{n-2})$ $\quad d_0 = 1$, $d_1 = 1$

$t^2 - 4t + 4 = 0$

$t = 2$

$U_n = d_n = b 2^n + d n 2^n$

$d_0 = b = 1$

$d_1 = 2b + 2d = 1$

$d = -\frac{1}{2}$

$$\boxed{\therefore d_n = 2^n - \frac{n}{2} \cdot 2^n}$$

Solⁿ:

A linear homogeneous r.r. of order k

$t^k - c_1 t^{k-1} - c_2 t^{k-2} \dots - c_k = 0$

if r is a root of multiplicity 'm' then the solution contain the term

$r^n + d_1 n r^n + d_2 n^2 r^n + \dots d_{m-1} n^{m-1} r^n$

Linear nonhomogeneous recurrence relation

$a_n = 3a_{n-1} + 2n$

$a_n = 3a_{n-2} + (n^2 + n + 1)$

$a_n = 3a_{n-1} + 4a_{n-2} + 3^n$ $\quad \alpha \cdot 3^n$

$a_n = 3a_{n-1} + 5a_{n-2} + 6a_{n-3} + 7^n + 5$ $\quad \alpha_1 7^n + \alpha_2$

In general

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$

Solution is the sum of solutions of 2 separate parts of R.R.

$$a_n = \{ a_n^{(P)} + a_n^{(h)} \}$$

particular sol^n

sol^n for homogeneous part.

Theorem : if $\{ a_n^{(P)} \}$ is a particular sol^n of non-homogeneous linear r.r. with constant coefficients.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$$

then every sol^n is of the form $\{ a_n^{(P)} + a_n^{(h)} \}$

$\{ a_n^{(P)} \}$ – particular sol^n

$\{ a_n^{(h)} \}$ – sol^n for the associated homogeneous part

$$a_n^{(P)} = c_1 a_{n-1}^{(P)} + c_2 a_{n-2}^{(P)} + \dots + c_k a_{n-k}^{(P)} + F(n)$$

Let the 2nd sol^n be $b_n = c_1 b_{n-1} + c_2 b_{n-2} + \dots + c_k b_{n-k} + F(n)$

$$b_n - a_n^{(P)} = c_1 \left(b_{n-1} - a_{n-1}^{(P)} \right) + c_2 \left(b_{n-2} - a_{n-2}^{(P)} \right) + \dots + c_k \left(b_{n-k} - a_{n-k}^{(P)} \right)$$

$$\therefore \boxed{b_n - a_n^{(P)} = a_n^{(h)}}$$

$$\boxed{\therefore b_n = \{ a_n^{(P)} + a_n^{(h)} \}}$$

$a_n = 3a_{n-1} + 2n$, $a_1 = 3$

sol^n of homogeneous part

$a_n^{(h)} = \alpha 3^n$

$a_n^{(p)} = cn + d$

putting in the r r

$cn + d = 3(c(n-1) + d) + 2(n)$

$\therefore n(2c+2) + (2d - 3c) = 0$

$c = -1$

$d = -\frac{3}{2}$

$\therefore$ the formula is $a_n = \alpha 3^n - n - \frac{3}{2}$

for $n = 1$

$3 = \alpha 3 - \frac{5}{2}$

$3\alpha = \frac{11}{2}$

$\alpha = \frac{11}{6}$

$\therefore \boxed{a_n = \frac{11}{6} \cdot 3^n - n - \frac{3}{2}}$

find all the sol^n of the r.r

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

$\alpha 2^n + \beta \cdot 3^n + c \cdot 7^n$

$c \cdot 7^n = 5c\, 7^{n-1} - 6c\, 7^{n-2} + 7^n$

$c \cdot 7^2 = 5c \cdot 7 - 6c + 7^2$

$20c = 7^2$

$$\boxed{c = \frac{49}{20}}$$

solve by substitution

$a_n^2 - 2a_{n-1}^2 = 1 \quad n \geq 1, \ a_0 = 1$

Non linear

$a_0^2 = 1 = b_0$

$a_n^2 = b_n$

$b_n - 2b_{n-1} = 1$

$b_n = 2b_{n-1} + 1$

$$\boxed{b_n = \alpha 2^n + C}$$

$b_0 = \alpha + C = 1$

$b_1 = 2\alpha + C = 3$

$\alpha = 2$

$c = -1$

$b_n = 2 \cdot 2^n - 1$

$= 2^{n+1} - 1$

$$\boxed{a_n = \sqrt{2^{n+1} - 1}}$$

$\frac{49}{20}$

D.S Tut

Q:1

$P(n) = P(n-5) + P(n-1)$

n dollars

consider as a string n-1 or n-5 (depending on whether first element is 1 or 5)

\$5, \$1, or \$1 coin

\$1 coin : a_{n-1}

\$1 bill : a_{n-1}

\$5 bill : a_{n-5}

$$\boxed{a_n = 2a_{n-1} + a_{n-5} \quad n \geq 5}$$

recurrence

base cases

$a_0 = 1$ (no. of ways of selecting no coin / dollar)

1 b
1 c
5 b

$a_1 = 2$

$a_2 = 4$

$a_3 = 8$

$a_4 = 16$

a_5 = ~~HAH~~ ~~BB~~ ~~+~~ ~~HA~~ 25

(2)

01
10
02

0 1 (n-2)
or 2

1 0 (n-2)
or 2

2 0
or 1
or 2

0/1/2

012/102

0102/1012

01012/10102

0,1,2

0010

00
11

$P(n) = 2P(n-2) + 2P(n-2) + 3P(n-1)$

P(n-2)

1 0

2 | n − 1 |

02 | n − 2 |

12 | n − 2 |

0102 | n − 3 |

1012 | n − 3 |

0, 1, 2

1 − 0 − 1

0 − 1 − 0 − 1

1 − 2

0 − 2

2 (1 − 2^(n−3)

2^(n−2)

2 − ^(n−1)

$a(n-1) + a(n-2) + \dots + a(1)$

$a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3} + \dots 2a_0 + 2$

$a_{n-1} = \cancel{2a}\,a_{n-2} + 2a_{n-3} + 2a_{n-4} + \dots 2a_0 + 2$

$a_n - a_{n-1} = a_{n-1} + a_{n-2}$

$a_0 = 2$

01010101 - - - - - - 2 $\boxed{X}$

1010 - - - - - 2

$\boxed{a_n = 2a_{n-1} + a_{n-2}}$ (Ans)

Partial ordering :

if a relation R on a set S is reflexive, antisymmetric and transitive, then the relation is called the partial ordering on the set S with relation R is called partial ordered set or poset.

$(x, x) \in R$

$(x, y) \in R \; ; \; x \neq y$

$(y, x) \notin R$

$(x, y) \in R$

$(y, t) \in R$

$(x, t) \in R$

R : $\leq$ $S = \{1, 2, 3, 4, 5, 10\}$

$(a, a) \in R, a \in S$ $(S, \leq)$ partial ordering

R : $a \mid b$

$(S, \mid)$ partial order

as if $a \mid b$

then $b \nmid a$ (unless $a = b$)

$(a, b) \in S$

R : comparable or incomparable either (a, b) or (b, a) it is a partial order then ?? (a, b) is comparable

e.g. $S = (1, 2, 3, 4, 5, 7, 10)$

R : | $(2, 4)$ comparable as $2 \mid 4$

$(5, 7)$ not comparable

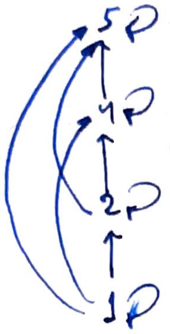
neither $5 \mid 7$ nor $7 \mid 5$

If all the elements of a set S with relation $\leq$ i.e. $(S, \leq)$ are comparable then S is a total order.

$S = \{1, 2, 4, 5\}$

$R : \leq$

Digraph :



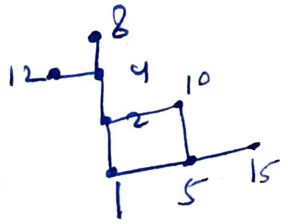
Minimized digraph for poset is Hasse Diagram

Note we know it is reflexive :
∴ remove self loop

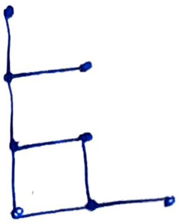
we know it is transitive.
∴ remove extra arrows

$S = \{1, 2, 4, 5, 8, 10, 12, 15\}$

$R = |$



:



Ex:

$S = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

$R \subseteq$

$S = \{a, b, c\}$

$R : \subseteq$

reflexive

$A \subseteq A$

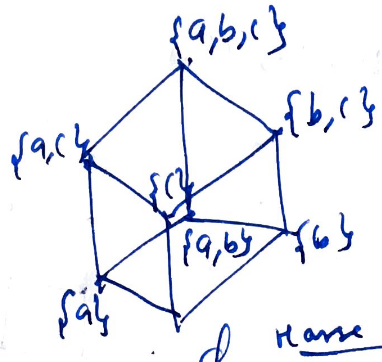
anti-symmetric:

$a \subseteq b,\ b \subseteq a$

$a = b$

transitive: $a \subseteq b,\ b \subseteq c,\ a \subseteq c$

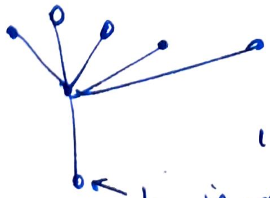
$\therefore$ it is a partial ordering set.



Hasse diagram

Maximal and minimal element of a poset $(S, \leq)$

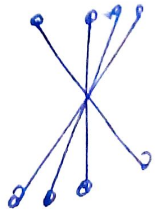
Consider elements $a, b \in S$ there is no b such that $a \leq b$, then a is a maximal element. and there is no b such that $b \leq a$, a is minimal



a_i is maximal

$1 \leq i \leq n$

b_1 is minimal



Consider a subset A of S :

upper bounds and lower bounds :

for all b, where $b \in S$, $a \in A$

$a \leq b$, a is a lower bound of A.

for all b

$b \leq a$, a is upper bound of A.

least upper bound (lup)

greatest lower bound (glp)

POSET and its Properties :

Let A be the subset of S let $a \in A$, and $u \in S$

if for all $a \in A$ and $u \in S$

if for all $a \in A$ and $z \in S$

$a \leq u$ and $u \leq z$, z is the upper bound.

if u exists and $u \leq z$. z is the upper bound

least upper bound (lub).

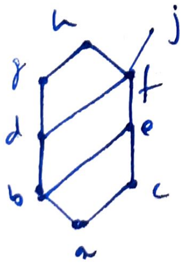
for all $a \in A$, $l \in S$, $l \geq z$, $z \in S$

$l \leq a$,

glb = greatest lower bound (A)

Q) Find the lower and upper bounds of the subset

{a, b, c}, {f, h}, {a, c, d, f}



{a, b, c}

l.b – a

u.b – e, f, j, h

{b, d, g}

l.b – a, b

u.b – h, g

{f, h}

l.b – a, b, c, d, e, f

u.b – h

glb – b

lub – g

{a, c, d, f}

l.b – a

u.b – h, j

Ex. Find the glb & lub of $\{3, 9, 12\}$ & $\{1, 2, 4, 5, 10\}$, if they exist in the poset $\{Z^+, |\}$ → divides relation

$\{3, 9, 12\}$ – lb – 1, 3

glb – 3

ub – 36 & 72, ...

llub – 36

$\{1, 2, 4, 5, 10\}$.

glb – 1 (greatest element which divides all of these)

lub – 20 (LCM of all)

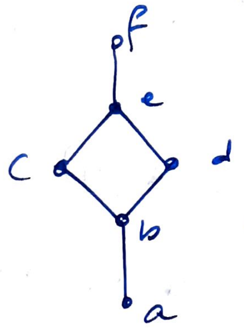
$(P(S), \subseteq)$, $S = \{a, b, c\}$

glb – ϕ

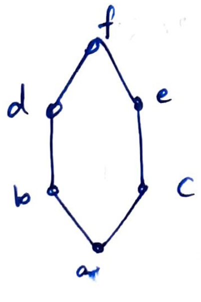
lub – S

Imp: lattice is a poset in which every pair of elements has a glb ~~&~~ lub.

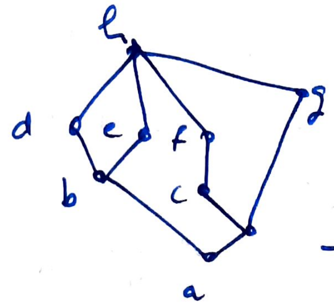
Hasse.



It is lattice



Yes

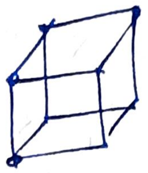


Yes

$\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}$

Lattice

No



$(Z^+, |)$ is it a lattice? **Yes** (Check)

Q) $(\{1,2,3,4,5\}, |)$? is it a lattice? **No**

for $(2, 3)$, 6 is no there. $\therefore$ **No**

Q) $(\{1,2,4,8,16\}, |)$ $\therefore$ **Yes** i a lattice

Applications: Boolean algebra, lattice based info security

Theorem: If any poset $(P, \leq)$ if the meets and joins exist they satisfy the following rules.

Here, lub of (a,b) is defined as join, notation $a \vee b$.

glb of (a,b) is defined as meet notation $a \wedge b$

L1: $x \wedge x = x$; $x \vee x = x$

L2: $x \wedge y = y \wedge x$; $x \vee y = y \vee x$

L3: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$; $(x \vee y) \vee z = x \vee (y \vee z)$

L4: $x \wedge (x \vee y) = x$; $x \vee (x \wedge y) = x$

L5: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

L6: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Pigeonhole - Principles

Dirichlet Principle / Shoebox Principle

Formulation :

Let X and Y be 2 finite sets

Let $f: X \to Y$

if X has more elements than Y then

f is not one to one. & if f is one-one

if X & Y have same elements, then

f is onto

if X & Y have same elements & if f is onto,

then f is one-one.

Appl^n. 1

Among 13 people there are 2 who have their birthdays in the same month.

$$\underset{\text{people}}{n+1} \longrightarrow \underset{\text{months}}{n}$$

Appl^n. 2

There are n married couples. How many of the 2^n people must be selected in order to guarantee that one has selected a married couple.

Select n people such that n no. of husbands or n no. of wives

Select one more from the rest of the people.

$(n+1)$ people should be selected to get atleast one married couple

Appl^n. 3.

Given m integers $a_1, a_2, \ldots a_m$ there exist integers k and l with $0 \le k < l \le m$ such that

$a_{k+1} + a_{k+2} \ldots + a_l$ is divisible by m.

Let 'holes' be the remainder

$0, 1, \ldots m-1$

$\square\, 0$
$\square\, 1$
$\vdots$
$\square\, m-1$

note : let $S_1, S_2, \ldots S_n$ are the sums of the consequtive terms of $a_1, a_2, \ldots a_m$

$\therefore$ Then, $a_{k+1} + a_{k+2} + \cdots + a_l = \boxed{S_l - S_k}$

now divide all S_i by m

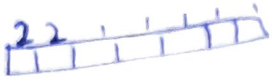
and assign each to box of given remainder

now if the remainder is '0'. then that sum is the answer.

else if none of the remainders is '0', there are 'm' sums and '$m-1$' (remainder) boxes. Thus 1 box will have 2 'sums'. If we subtract the 2 sums, the result is divisible by m.

Applⁿ - 4 :

A chess master who has 11 weeks to prepare for a tournament, decides to play at least one game everyday, but not to play more than 12 games in a week. Show that there exists a succession (consecutive) days during which the chess master will have played exactly 21 games.



a_i — no. of games played till i^{th} day

$a_i = b_1 + b_2 + \cdots + b_i$

$a_1 \ge 1$

$1 \le a_1 < a_2 < a_3 \cdots < a_{77} \le 11 \times 12 \; (=132)$

strictly increasing (as at least one game on each day)

$1+21 \le a_1 + 21 < a_2 + 21 < \cdots < a_{77} + 21 \le 153$

Total 154 terms in the 2 sequences, but there are 153 games.

Thus there are some j & i so that

$$\boxed{a_j = a_i + 21}$$

$$\boxed{b_{i+1} + b_{i+2} + \cdots + b_j = 21}$$

Appl^n :-

From the integers 1, 2, . . . , 200, we choose 101 integers, show that among the integers chosen, there are two such that one of them is divisible by the other.

note if we choose 100 no.s,
we can choose 101, . . . , 200.
it does not satisfy.

→ ~~Soln~~ Any integer can be represented as
$2^k \times a$, where a is odd. $k \geq 0$

from 1 to 200, how many odd no.s — 100

Here a are pigeonholes.

There are 2 no.s for which a is same

$I_1 : 2^r \cdot a$

$I_2 : 2^s \cdot a$

if $r > s$, I_1 is divisible by I_2
~~then~~ I_2 is divisible by I_1
if $r < s$, then

Previously :

P.

Th -1 :
If $(n+1)$ objects are put in n boxes, then atleast one box contains two or more objects

if n objects are put into n boxes and no box is empty, then each box contains atleast one object

if n objects are put into n boxes and no box gets ~~.~~ more than one object, then each box has an object in ~~it~~ it

Contd...

Let m and n are relatively prime positive integers and let a, b be integers where $0 \le a \le m-1$, $0 \le b \le n-1$, then there is a positive integer x, such that remainder when x is divided by $\underline{m}$ is $\underline{a}$ and remainder when x is divided by $\underline{n}$ is $\underline{b}$.

Chinese remainder Theorem.

??

Let n integers be considered as follows.

$0 \cdot m + a$, $m + a$, $2m + a$, ... $(n-1)m + a$

Suppose; 2 of these ~~are~~ ~~[illegible]~~ have same remainder, when divided by 'n'.

Let $im + a$ and $jm + a$ are 2 such integers, $0 \le i < j \le (n-1)$

$q_i n + b$

$q_j n + b$

$\therefore$

$(j-i)m = (q_j - q_i) \cdot n$, n is not a factor of m

n cannot be a factor of $(j-i)$

thus our assumption is not correct

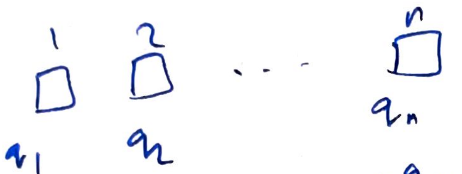
$\therefore$ each integer has a different remainder when it is divided by n.

$\therefore$ There exist some b, so that one of the integers among this list is $q \cdot n + b$, $0 \le b \le n-1$.

Theorem 2 (Strong form of pigeonhole Principle)

Let $q_1, q_2 \cdots q_n$ be the integers, if

$q_1 + q_2 + \cdots + q_n - n + 1$ objects are put into n ~~objects~~ boxes, then either the 1st box contains atleast q_1 objects or the 2nd box contains at least q_2 objects or ... or the n^{th} box contains at least q_n objects.



$q_1 + q_2 + \cdots + q_n - n + 1$

if $q_1 = q_2 = \cdots = q_n = 2$

$2n - n + 1 = n+1$ objects, n boxes

Let each box contain fewer objects (one less)
So total objects are

$(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1)$

$= q_1 + q_2 + \cdots + q_n - n$

Now, if I add one more objects that will put to any one of the box so that box contains $(q_i - 1) + 1 = q_i$ objects.

\# if $n(r-1)+1$ objects are put into n boxes, then one of the boxes are r or more of the objects.

$q_1 = q_2 \cdots q_n = r$

\# if the ~~avg~~ avg. of n non-negative no-s $m_1, m_2, \cdots, m_n$ is greater than $(r-1)$

$$\frac{m_1 + m_2 + \cdots + m_n}{n} > r - 1$$

then atleast one of the integer is greater than or equal to r

If the average of n nonnegative integers $m_1, m_2, \ldots m_n$ is less than ~~less~~ $(r+1)$

$$\frac{m_1 + m_2 + \cdots + m_n}{n} < r+1$$

then at least one of the integers is less than $(r+1)$

if the avg. of $m_1, m_2, \ldots m_n$ is atleast equal to r then atleast one of the integers $m_1, m_2, \ldots m_n$ satisfies $m_i \geq r$

TUTORIAL

Q.1)

a) NOT

b) YES

$((a,b),(c,d)) \in R$ only if $a+d = b+c$

Q.2) Reflexive:

$a+b = a+b$

Symmetric:

$a+d = b+c$

$$\frac{((a,b),(c,d))}{((c,d),(a,b))}$$

$c+b = d+a$

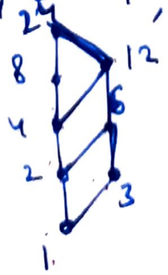
Transitive:

$a+d = b+c \quad ((a,b),(c,d))$

$c+f = d+e \quad ((c,d),(e,f))$

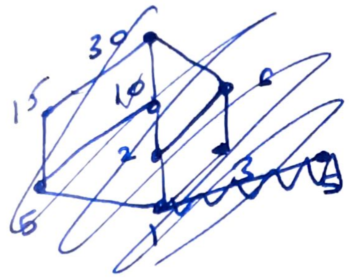
$a+f = b+e \quad ((a,b),(e,f))$

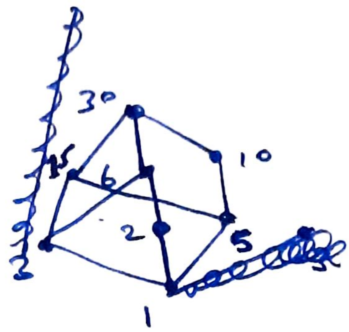
Q.3) $S_{24} = 1, 2, 3, 4, 6, 8, 12, 24$



• 6

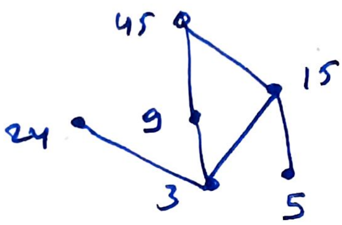
$S_{30} = 1, 2, 3, 5, 6, 10, 15, 30$





(4) $(\{3, 5, 9, 15, 24, 45\}, |)$

(a)



4(a) 24, 45

(b) 3, 5

(c) NO

(d) NO

(e) 15, 45

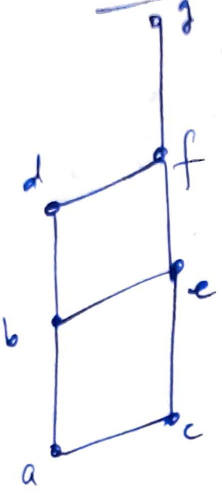
(f) 15

(g) 3, 5, 15

(h) 15

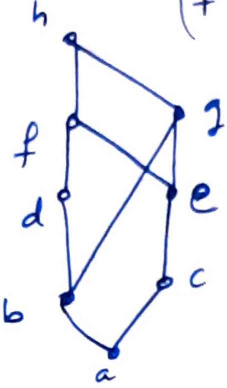
(5)

(a) lattice



(b) NOT lattice

(f, g)
glb ?? not there



(c) lattice.

(d) non - lattice

Pigeonhole contd. (~~first part done in other copy~~ (~~most probably~~ common copy))

Appl. 7 :

A basket of fruits is being arranged out of apples, bananas and oranges. What is the smallest no. of fruits that can be put in a basket so that either there are atleast 8 apples or at least 6 bananas. or atleast 9 oranges are there

→ use strong form :

$q_1 = 8$
$q_2 = 6$
$q_3 = 9$

$8+6+9-3+1 = 21$ fruits are required

Appln 8. Two disks, one smaller than the other are divided into 200 congruent sectors. In the larger disk 100 of the sectors are chosen arbitrarily and painted red, then the other 100 are painted blue. In the smaller disk, each sector is painted either red or blue with no stipulation on the no. of red & blue sectors. The small disk is placed on the larger disk, so that the centers coincide. Show that it is possible to align the two disks so that the no. of sectors of small disk whose color matches to the corresponding sector of the larger disk is atleast 100.

If the average of $m_1, m_2, \ldots, m_n$ is at least equal r, then at least one of the integers $m_1, m_2, \ldots, m_n$ satisfies

$m_i \geq r$

Total color matches : 200×100

average no. of color matches per position

$$\frac{200 \times 100}{200} = 100$$

??

Appln 9. Show that every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of n^2+1 real nos contains either an increasing subsequence of length $(n+1)$ or decreasing subsequence of length $(n+1)$

$a_1, a_2, \ldots, a_k$ — sequence

$a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ — subsequence

$1 \leq i_1 < i_2 < \ldots < i_k \leq n$

$a_{i_1} \leq a_{i_2} \leq a_{i_3} \leq a_{i_k}$ — increasing subsequence

<u>note</u> : $a_1\, a_4\, a_7\, a_5$ is <u>not</u> a subsequence

Suppose, the sequence does not have any increasing subsequence of length $(n+1)$, then we have to show there is a decreasing subsequence of length, atleast, $(n+1)$. Suppose, m_k is the length of the longest increasing subsequence that begins with a_k.

$a_1, a_2, \ldots a_{n^2+1}$

$\boxed{m_k \leq n,\ m_k \geq 1}$

$\boxed{\therefore\ 1 \leq m_k \leq n}$ — pigeon holes

$a_1, a_2, \ldots a_{n^2+1}$ — n^2+1 no.s (pigeons)

m_k — length of longest increasing sub-sequence starting with a_k

$1 \leq m_k \leq n$

values of m_k lie between 1 & n. .. n holes

$n^2+1 = n(n+1) - n + 1$

$\therefore$ $(n+1)$ no. of increasing subsequences whose lengths are same.

$m_{k_1} = m_{k_2} = m_{k_3} = \ldots = m_{k_{n+1}}$

<u>suppose</u> :

$a_{k_i} < a_{k_{i+1}}$

starting from $a_{k_{i+1}}$ we have an s.s of length 'n'. if we add a_{k_i} to it we get an SS of length $(n+1)$.

$\therefore$ <u>CONTRADICTION</u>.

$a_{ki} \geq a_{ki+1}$

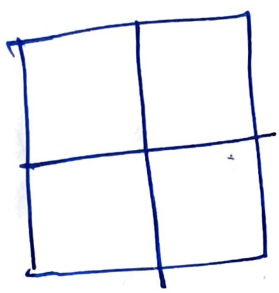
or it is true for

$i = 1, 2, \ldots, n+1$

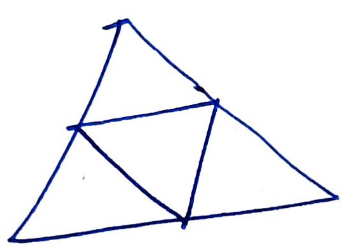
$a_{i_1} \geq a_{i_2} \geq a_{i_3} \ldots \geq a_{i_{n+1}}$

decreasing S.S. of length at least $(n+1)$.

Prove that any five points chosen with a square of side length 2, there are two whose distance apart is at most $\sqrt{2}$.



Prove that 5 points



Counting & Permulations _ Combinations

Four Basic counting principles :

1. **Addition** :

If $S_1, S_2 \ldots S_m$ are partitions of a set S
then the size of S, $|S| = |S_1| + |S_2| + \ldots + |S_m|$

S :- all the courses offered

S_i :- the courses offered by i^{th} dept.

$S = S_1 \cup S_2 \cup \ldots \cup S_m$

$S_i \cap S_j = \phi \; (i \neq j)$

multiplication principle :

Let S be a set of ordered pairs (a, b), of objects where the first object a comes from ~~a set~~ size of p, and for each choice of object a there are q choices of object. b then

$|S| = p \times q$

Ex. The no. of way a man, a woman, a boy and a girl can be selected for 5 men, 6 women, 2 boys & 4 girls $= 5 \times 6 \times 2 \times 4$

Ex. determine the no. of +ve integers that are factors of the no.

$3^4 \times 5^2 \times 11^7 \times 13^8$

$= \boxed{5 \times 3 \times 8 \times 9}$

Subtraction principle :

if A be a set and U be a larger set then complement of A is define as

$\bar{A} = \{ x \in U, x \notin A \}$

$|A| = |U| - |\bar{A}|$

Ex. Computer passwords are to consist of 6 symbols from 0, 1, . . . , 9 and the lowercase letters a, b, c, . . . , z. How many computer passwords have a repeated symbol?

$= 36^6 - {}^{36}P_6$

Division Principle:

Let S be a finite set that is positioned with k parts with the same no. of objects.

part → S_i

Then $k = \dfrac{|S|}{\text{no. of objects in parts}}$

Count the ordered & unordered arrangements.

i) without repetition – set S, elements are distinct $S = \{a, b, c\}$

ii) with repetition

multiset, $S = \{5.a, 6.b, 1.c\}$

$= \{a, a, a, a, a, b, b, b, b, b, b, c\}$

Example. How many odd numbers b/w 1000 & 9999 which have distinct digits

units place = 1, 3, 5, 7, 9

thousands place 1 to 9 (−1) digit which was taken in units place

= 8

hundreds place 0 to 9 (−2) = 8.

ten's place 0 to 9 − (3) = 7

∴ 8 × 8 × 7 × 5

Permutation of a set

An ordered arrangement of r of the n elements
- r - permutation of a set of n elements

$n = 3 : \{a, b, c\}$

$r = 1$, 1-p $\underline{3}$ a, b, c

$r = 2$, 2-p $\underline{6}$ ab, ac, ba, bc, ca, cb

$r = 3$, 3-p $\underline{6}$ abc, acb, bac, bca, cab, cba

Counting of Permutations - Combinations :

The no. of r-permutations of an n-element set

$P(n, r)$

$P(n, r) = 0$, if $r > n$

$P(n, 1) = n$

$P(n, n)$: permutation of S.

for n and r be two integers with $r < n$

$$\boxed{P(n, r) = n \times (n-1)(n-2) \cdots (n-r+1)}$$

$$P(n, r) = \frac{n!}{(n-r)!}$$

Qn. what is the no. of ways to order the 26 letters so that the no two vowels occur consequently

$\bigcirc \; C_1 \; \bigcirc \; C_2 \; \bigcirc \; C_3 \; \ldots \; C_{21} \; \bigcirc$

spaces = 22

$\therefore P(22, 5) \times 21!$

Permutation of a set

An ordered arrangement of r of the n elements

– r-permutation of a set of n elements

$n = 3$: $\{a, b, c\}$

$r = 1$, 1-p $\underline{3}$ a, b, c

$r = 2$, 2-p $\underline{6}$ ab, ac, ba, bc, ca, cb

$r = 3$, 3-p $\underline{6}$ abc, acb, bac, bca, cab, cba

Counting of Permutations – Combinations:

The no. of r-permutations of an n-element set

$P(n, r)$

$P(n, r) = 0$, if $r > n$

$P(n, 1) = n$

$P(n, n)$ = permutation of S.

for n and r be +'ve integers with $r \le n$

$$\boxed{P(n, r) = n \times (n-1)(n-2) \cdots (n-r+1)}$$

$$P(n, r) = \frac{n!}{(n-r)!}$$

Qn. what is the no. of ways to order the 26 letters so that no two vowels occur consequently

[illegible]

$\circ \; C_1 \; \circ \; C_2 \; \circ \; C_3 \; \ldots \; C_{21} \; \circ$

spaces $= 22$

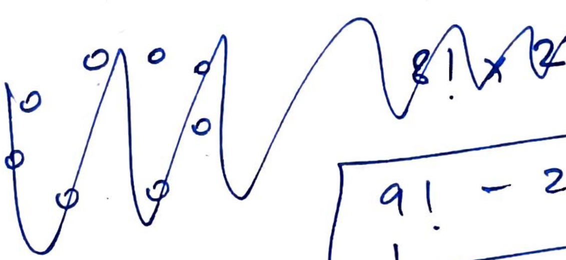
$\therefore P(22, 5) \times 21!$

Theorem: no. of circular r-permutations of a set of n-elements $= \dfrac{n!}{r(n-r)!} = \boxed{\dfrac{{}^nP_r}{r}}$

A B C D E F

Since each part contains r linear r-permutation, so, the no. of circular r-permutations equals the no. of parts.

Ex. 10 people, including two who do not wish to sit next to each other, are to be at a round table. How many circular seating arrangements are there



$8! \times 2$

$\boxed{9! - 2 \times 8!}$

all distinct — take 2 together

Ex. Combinations of a set:

An unordered arrangement of a set of n elements

$S = \{a, b, c, d\}$.

$\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Theorem

for $0 \le r \le n$

$P(n, r) = r! \times {}^nC_r$

${}^nC_r = \dfrac{P(n, r)}{r!} = \dfrac{n!}{r!(n-r)!}$

Proof: S be an n-element set, each r-permutation

(i) choose r elements from S

(ii) Arrange the chosen r elements in some order.

$$\boxed{P(n,r) = {}^nC_r \times r!}$$

$$^nC_r = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

Permutation of multisets

$S = \{2.a, 3.b, 1.c\}$ $2+3+1 = 6$

multiset for infinite repetition:

$S = \{\infty.a, \infty.b, \infty.c\}$

Theorem:

Let S be a multiset with objects of k different types, where each has an infinite repetition number

Then the no. of r-permutations is k^r

each filled in k possible ways:

_ _ _ _ _

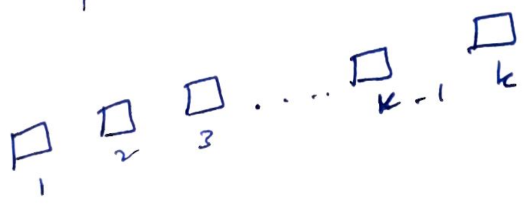
r spaces

also, $\boxed{^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n}$

Let S be a multiset with objects of k - diff. types with finite repetitions no. $n_1, n_2 \ldots n_k$ respectively. Let the size of S be $n = n_1 + n_2 \ldots + n_k$. Then the no. of permutations of S.

$$= \frac{n!}{n_1! \, n_2! \ldots n_k!}$$

Proof. $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots n_k \cdot a_k\}$



$$= {}^{n}C_{n_1} \times {}^{n-n_1}C_{n_2} \times \ldots {}^{n-n_1-n_2-\ldots-n_{k-1}}C_{n_k}$$

H.P

$\cancel{P(n,r) = r! \times {}^{n}C_r = \frac{n!}{r!(n-}}$

Theorem :

Let $n = n_1 + n_2 + \ldots + n_k$

The no. of ways to partition a set of n objects into k labeled boxes $B_1, B_2, \ldots B_k$ in which B_1 contain n_1 objects and so on..

no. of ways to patition $= \frac{n!}{n_1! \, n_2! \ldots n_k!}$

if boxes are not labled and $n_1 = n_2 = \ldots = n_k$

then the no. $= \frac{n!}{k! \, n_1! \, n_2! \ldots n_k!}$

Let S be a multiset with objects of k-diff types with finite repetitions no. $n_1, n_2 \ldots n_k$ respectively. Let the size of S be $n = n_1 + n_2 \ldots + n_k$

then the no. of permutations of S.

$$= \frac{n!}{n_1!\, n_2! \ldots n_k!}$$

Proof. $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots n_k \cdot a_k\}$

$\square_1 \; \square_2 \; \square_3 \; \ldots \; \square_{k-1} \; \square_k$

$$= {}^{n}C_{n_1} \times {}^{n-n_1}C_{n_2} \times \ldots {}^{n-n_1-n_2-\ldots-n_{k-1}}C_{n_k}$$

H.P

~~$P(n,r) = r! \times {}^{n}C_r = \frac{n!}{r!(n-}$~~

Theorem :

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The no. of ways to partition a set of n objects into k labeled boxes $B_1, B_2, \ldots B_k$ in which B_1 contain n_1 objects, and so on..

no. of ways to partition $= \frac{n!}{n_1!\, n_2! \ldots n_k!}$

if boxes are not labeled and $n_1 = n_2 = \ldots = n_k$

then the no. $= \frac{n!}{k!\, n_1!\, n_2! \ldots n_k!}$

MISSISSIPPI

$S = \{1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P\}$

no. of permutations $= \dfrac{11!}{1!\,4!\,4!\,2!}$

Combinations of multisets :

$$S = [\infty \cdot x_1, \infty \cdot x_2, \ldots \infty \cdot x_n]$$

with $k_1, k_2, \ldots k_n$ written under $x_1, x_2, \ldots x_n$.

$k_1 + k_2 + \cdots + k_n = k$

$k_1 \geq 0, k_2 \geq 0, \ldots k_n \geq 0$:

same as the k ball, $(n-1)$ bar, problem

o o | o o o | o o ...

balls are indistinguishable and bars are indistinguishable.

∴ Total no. of permutations =

$$\frac{(k+n-1)!}{k!\,(n-1)!} = \boxed{{}^{k+n-1}C_k \text{ Ans}}$$

Ans.) ${}^{12+8-1}C_{12}$

: ${}^{19}C_{12}$ (Ans).

A bakery boasts of 8 varieties of doughnuts. If a box of doughnuts contains one dozen, how many different options are there for a box of doughnuts?

$$\boxed{\begin{array}{l} n = 8 \\ k = 12 \end{array}}$$

How many ways are there to select 5 bills from a cash box containing : \$1, \$2, \$5, \$10, \$20, \$50, \$100 7 bills?

[Order does NOT matter, bills indistinguishable, & there are atleast 5 bills of each type].

⟶ equal to 00|00|0|||||

6 bars, 5 balls.

7 types of dollar bill — 7 partition for
5 bills = total no. of bills

∴ $\boxed{{}^{11}C_5}$ (Ans)

Q) no. of solutions to the eqn. $x_1 + x_2 + x_3 + x_4 = 20$; $x_1 \geq 3, x_2 \geq 1, x_3 \geq 0, x_4 \geq 5$

$${}^{11+4-1}C_{4-1} = {}^{14}C_3$$

Permutations with indistinguishable objects :

Q) SUCCESS?

\# of S = 3
\# of C = 2

∴ $\dfrac{7!}{3!\,2!}$ - Ans'

Th) The no. of diff. permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 of type 2, ... n_k of type k is :

$$C(n, n_1) \times C(n-n_1, n_2) \times \ldots \times C(n-n_1 - \ldots - n_{k-1}, n_k)$$

$$= \frac{n!}{n_1!\,n_2! \ldots n_k!}$$

How many ways are there to select 5 bills from a cash box containing . \$1, \$2, \$5, \$10, \$20, \$50, \$100 bills?

[order does NOT matter, bills indistinguishable, & there are atleast 5 bills of each type].

→ equal to $00|00|0|||||$

6 bars, 5 balls.

7 types of dollar bills, 7 partition for

5 bills = total no. of bills

∴ $\boxed{{}^{11}C_5}$ (Ans)

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$${}^{11+4-1}C_{4-1} = {}^{14}C_3$$

Q) Permutations with indistinguishable objects:

SUCCESS?

\# of S = 3

\# of C = 2

∴ $\dfrac{7!}{3!\,2!}$ - Ans

Th) The no. of diff. permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 of type 2, ... n_k of type k is:

$$C(n, n_1) \times C(n-n_1, n_2) \times \ldots \times C(n-n_1-\ldots-n_{k-1}, n_k)$$

$$= \frac{n!}{n_1!\,n_2!\ldots n_k!}$$

Q) No. of ways of distributing hands of 5 cards to 4 different people : from a deck of 52 cards.

$$\underline{{}^{52}C_5 \cdot {}^{47}C_5 \times {}^{42}C_5 \times {}^{37}C_5 \times {}^{32}C_5}$$

Q) 4 different employees in 3 indistinguishable offices.

↳ case 1) all emp in 1 office → 1

3 in 1 office, 1 in other → 4

2 in 1 office, 2 in other → 3

2 in 1 office, 1 in other, 1 in third → 6

= 14 (Ans)

Q) 10 indistinguishable balls, 8 distinguishable bins

→ ${}^{17}C_7$ $({}^{n+r-1}C_{r-1})$

$\boxed{r = 8, n = 10}$

Q)

6 copies (identical) ; 4 boxes (identical)

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→ $^{17}C_7$ $\left({}^{n+r-1}C_{r-1}\right)$

$\boxed{r = 8, \ n = 10}$

Q) 6 copies (identical) ; 4 boxes (identical)

Inclusion - Exclusion principle

a set : S

a subset of S = A;

A - satisfies some property P

$$\boxed{\begin{aligned} \therefore |A| &= |S| - |\bar{A}| \\ \text{or } |\bar{A}| &= |S| - |A| \end{aligned}}$$

Count the number of integers b/w 1 & 600 which are not divisible by 6

no.s divisible by 6 : $\frac{600}{6} = 100$

$\therefore$ not divisible = $600 - 100 = 500$

Ex: count the permutation $i_1, i_2, \ldots i_n$ of $(1, 2, 3 \ldots n)$ in which 1 is not at first position.

Indirect: $n! - (n-1)!$

direct : let first position not be 1.

$\therefore (n-1)$ ways

next $(n-1)$ positions can be filled in $(n-1)!$

$\therefore (n-1) \times (n-1)!$ ways.

Ex: The no. of objects of S that have none of the properties $P_1, P_2, \ldots P_n$ is given by.

$$|\bar{A_1} \cap \bar{A_2} \cap \ldots \cap \bar{A_m}| = |S| - \Sigma |A_i| + \Sigma (A_i \cap A_j)$$
$$- \Sigma (|A_i \cap A_j \cap A_k|)$$
$$+ (-1)^m |A_1 \cap A_2 \cap \ldots \cap A_m|$$

where the first sum. over all 1 combination $\{i\}$ of $\{1, 2, \ldots m\}$

" " 2nd " " " 2 combinations $\{i, j\}$ of $\{1, 2, \ldots m\}$

S, $A_1 - P_1$

$A_2 - P_2$

$A_3 - P_3$

$$|(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3)| = |S| - \Sigma A_i + \Sigma(A_i \cap A_j).$$

$$= |S| - (|A_1| + |A_2| + |A_3|)$$
$$+ (|A_1 \cap A_2| + A_2 \cap A_3| + |A_1 \cap A_3|)$$
$$- (|A_1 \cap A_2 \cap A_3|)$$

$\therefore$ $\boxed{({}^mC_0) - ({}^mC_1) + ({}^mC_2) - ({}^mC_3)}$ and so on ...

* gives no. of terms

for $m = 3$ we have verified

Corollary:

the no. of objects which have atleast one of the properties $P_1, P_2, \ldots P_n$

$$|A_1 \cup A_2 \cup \ldots \cup A_m|$$

$$= \Sigma |A_i| - \Sigma |A_i \cap A_j| + \Sigma | A_i \cap A_j \cap A_k| \ldots$$

$$(-1)^{m+1} |A_1 \cap A_2 \ldots \cap A_m|$$

Q How many permutations of the letters M, A, T, H, I, S, F, U, N are there such that none of the words 'MATH', 'IS', 'FUN' occur as consecutive letters.

~~MAT, H, I, S, F, U, N~~

math & IS   is & fun   math & fun

$$n = (6! + 8! + 7!) - (5! + 6! + 4!) + (3!)$$

MATH or IS or FUN is together   math & is & fun

$\therefore$ ans $= 9! - n$

find the no. of integers b/w 1 & 1000 (inclusive) that are not divisible by 5, 6, 8.

$$|S| = 1000$$

$$|A_1| = \left\lfloor \frac{1000}{5} \right\rfloor$$

$$|A_2| = \left\lfloor \frac{1000}{6} \right\rfloor$$

$$|A_3| = \left\lfloor \frac{1000}{8} \right\rfloor$$

$$|A_1 \cap A_2| = \left\lfloor \frac{1000}{30} \right\rfloor$$

$$|A_1 \cap A_3| = \left\lfloor \frac{1000}{40} \right\rfloor$$

$$|A_2 \cap A_3| = \left\lfloor \frac{1000}{24} \right\rfloor$$

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{1000}{\operatorname{lcm}(5, 6, 8)} \right\rfloor$$

periodic : $\dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T f(x)e^{-xs}\,dx$

$\dfrac{1}{t} f(t) \rightarrow \displaystyle\int_x^{\infty} F(x)\,dx$

$f(at) = \dfrac{1}{a} F\left(\dfrac{s}{a}\right)$

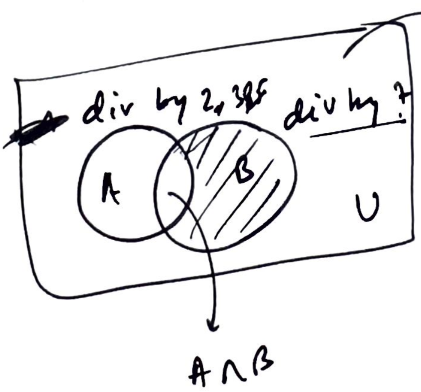
Tutorial 21/10/18

1)

a) $2000 - (1000 + 666 + 400)$

$+ (\text{div by } 6 + \text{div by } 15 + \text{div by } 10)$

$- (\text{div by } 30)$

b) 

$\boxed{210 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{1}{7} = 8}$

$\boxed{2000 \times \frac{8}{210} = 76}$

30 , 7

210

$\dfrac{2000}{210} = 9$

76

~~2000~~

30 , 60 , 90 , 120

Q.2)

$x_1 + x_2 + x_3 + \dots + x_7 = 31$

$0 \le x_1 \le 9$

$0 \le x_2 \le 9$

$\vdots$

$$\boxed{\begin{array}{l}\text{check for} \\ x_i \geq 10 \\ x_i \cap x_j \geq 10 \\ x_i \cap x_j \cap x_k \geq 10\end{array}}$$

remove from total cases.

$x_1 \geq 10$

$x_2 \geq 10$

$u_1 + 10 \geq 10$

$u_1 \geq 0$

$u_2 \geq 0$

$u_3 \geq 0$

$\vdots$

$u_9 \geq 0$

Q.3) $\boxed{\dfrac{12!}{4!\,3!\,5!}}$ simple arrangement with identical objects

Q.4) a) ${}^{n+k-1}C_k$

b) ${}^{n+k-1}C_k \times n!$

Q.5) do yourself

Q.6) no. of ways of arranging.

only (c_i) occurs $\therefore (n-1)!$

only (c_i, c_j) occurs $\therefore (n-2)!$

only $(c_i, c_j \dots c_n)$ occur $\therefore 1!$

no. of ways of selecting ~~Selecting~~:

$$= n! - {}^{n-1}C_1 (n-1)! + {}^{n-1}C_2 (n-2)! - {}^{n-1}C_3 (n-3)! \ldots$$

---

generating function :

Let $h_0, h_1, h_2 \ldots h_n$ is a finite sequence of no-s. $h_0, h_1, h_2 \ldots h_n, 0, 0, \ldots$ is any infinite sequence of no-s.

letting the no-s as the coefficients, from a polynomial

$g(x) = h_0 + h_1 x + \ldots + h_n x^n + \ldots$ $g(x)$ is called the generating function

x^n is placeholder, so that the associated coefficient h_n gives the soln of counting problems.

generating function — Power series / Taylor series

Example : g.f. of infinite sequence $1, 1, 1, \ldots 1, \ldots$

$$g(x) = 1 + x + x^2 + \ldots + x^n + \ldots$$

$$= \frac{1}{1-x}$$

g.f. of binomial coefficients

${}^mC_0, {}^mC_1, \ldots {}^mC_m$

$$g(x) = {}^mC_0 + {}^mC_1 x + \ldots {}^mC_m x^m$$

$$= (1+x)^m$$

Example : If $h_0, h_1, \ldots h_n$ is an infinite sequence, n is a '+'ve integer, then h_n gives the no. of integral soln of $e_1 + e_2 + e_3 + \ldots + e_k = n$,

$$h_n = {}^{n+k-1}C_n$$

$$g(x) = \sum_{k=0}^{\infty} {}^{n+k-1}C_n x^k$$

$$f(x) = \frac{1}{(1-x)^k}$$

$$= \frac{1}{(1-x)} \times \frac{1}{(1-x)} \quad \frac{1}{(1-x)}$$ k-factors, k - type category

$(1 + x + x^2 + \cdots + x^n)(1 + x + x^2 \cdots) \times (\cdots)$ k factors

$h_n = $ coeff of x^n

$$f(x) = \frac{1}{(1-x)^{k_1}(1-x)^{k_2}}$$

Q) Determine the G.F for the no. of n-combinations of apples, bananas, oranges & pears such that the no. of apples, even, no. of bananas - odd, the no's of oranges b/w 0 & 4, and at least one pear.

$f(x) = (1 + x^2 + x^4 + \cdots) \times (x + x^3 + \cdots)(1 + x + x^2 + x^3 + x^4)$
$(x + x^2 + x^3 + \cdots$

$$= \frac{1}{(1-x^2)} \frac{x}{(1-x^2)} \times \frac{(x)}{(1-x)} \frac{(1-x^5)}{(1-x)}$$

$$= \frac{x^2(1-x^5)}{(1-x^2)^2(1-x)^2}$$

Q) Find the no h_n of bags of fruits, A, B, O, P. A - even, B - multiple of 5, O - at least 4, P ...

$\rightarrow$

$\cancel{\frac{1}{1-x^2}}$
$(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)(x^4 + x^5 + \cdots)$
$(1 + x)$

$$= \frac{1}{1-x^2} \frac{1}{1-x^5} \times \frac{x^4}{1-x}(1+x)$$ $n-4$

$$= \frac{1}{(1-x)^2} \cdot \frac{x^4}{(1-x^5)}$$

$$\sum {}^{2+(n-5r)-1}C_{n-5r} \times {}^{1+(n-4-r)-1}C_r$$

$${}^{1+i}C_i \cdot x$$

---

cf

at most 4 oranges

$$\sum_{k=0}^{n} (n+1)x^n \quad ??$$

Example : Let h_n denote the no. of non-negative integral sol^n

$$3e_1 + 4e_2 + 2e_3 + 5e_4 = n$$

$$f_1 + f_2 + f_3 + f_4 = n, \quad f_1 = 3e_1, f_2 = 4e_2, f_3 = 2e_3$$

$$f_4 = 5e_4$$

$$g(x) = (1+x^3+x^6 \ldots)(1+x^4 \ldots)$$

$$(1+x^5+x^{10} \ldots)(1+x^2+x^4+\ldots)$$

$$= \frac{1}{(1-x^3)} \frac{1}{(1-x^4)} \frac{1}{(1-x^5)} \frac{1}{(1-x^2)}$$

$$= \boxed{\sum_{k=0}^{\infty} \bigcirc \, x^n}$$

Example : Recurrences & G.F

$$(1-rx)^{-n} = \sum_{k=0}^{\infty} {}^{-n}C_k (-rx)^k$$

n - +'ve
r - non-negative real

$$\frac{1}{(1-rx)^n} = \sum_{k=0}^{\infty} (-1)^k \cdot {}^{-n}C_k r^k x^k$$

$$ {}^{-n}C_k = (-1)^k \; {}^{n+k-1}C_k$$

$$\frac{1}{(1-rx)^n} = \sum_{k=0}^{\infty} {}^{n+k-1}C_k \; r^k x^k$$

<u>Exp.</u> Determine the G.F. for the sequence of square
$0, 1, 4, \ldots n^2, \ldots$; $n = 2, r = 1$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} {}^{k+1}C_k \; x^k$$

$$= 1 + 2x + 3x^2 + \ldots + nx^{n-1}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \ldots + nx^n$$

diff. $$\frac{1+x}{(1-x)^3} = 1 + 2^2 x + 3^2 x^2 + \ldots$$

$$\boxed{\frac{x(1+x)}{(1-x)^3}} = 1^2 \cdot x + 2^2 \cdot x^2 + 3^2 x^3 \ldots$$

Ans

<u>Generating functions :</u>

<u>Solve the linear homogeneous recurrence relation</u>

1.) $h_n = 5 \cdot h_{n-1} - 6h_{n-2}$ $(n \geq 2)$

$h_0 = 1$; $h_1 = -2$

$g(x) = h_0 + h_1 x + h_2 x^2 + \dots + h_n x^n + \dots$

$h_n - 5h_{n-1} + 6h_{n-2} = 0, \quad n \geq 2$

Soln

$g(x) = h_0 + h_1 x + h_2 x^2 + \dots h_n x^n + \dots$

$-5xg(x) = -5h_0 x - 5h_1 x^2 - 5h_2 x^3 + \dots - 5h_{n-1} x^n \dots$

$6x^2 g(x) = \quad + 6h_0 x^2 + 6h_1 x^3 + \dots + 6h_{n-2} x^n + \dots$

$= 0$

$(1 - 5x + 6x^2) g(x) = h_0 + (h_1 - 5h_0) + (h_2 - 5h_1 + 6h_0) x^2$

$\dots + \underbrace{(h_n - 5h_{n-1} + 6h_{n-2})}_{= 0} x^n \dots$

$= 0$ (from recurrence relation)

clearly $g(x)$

$(1 - 5x + 6x^2) g(x) = (h_0 - h_1 - 5h_0 x)$

$$g(x) = \frac{h_0 - h_1 - 5h_0 x}{1 - 5x + 6x^2}$$

$$= \frac{1 - 7x}{(1 - 2x)(1 - 3x)} = \frac{5}{1 - 2x} - \frac{4}{1 - 3x}$$

$$\frac{1}{1 - 2x} = 1 + 2x + 2^2 x^2 + 2^3 x^3 + \dots 2^n x^n + \dots$$

$$\frac{1}{1 - 3x} = 1 + 3x + 3^2 x^2 + \dots 3^n x^n + \dots$$

$$\therefore g(x) = \sum_{r=0}^{\infty} (5 \cdot 2^r - 4 \cdot 3^r) x^r$$

$$\boxed{5 \cdot 2^n - 4 \cdot 3^n}$$

find h_n :

$$\left.\begin{aligned} h_n + h_{n-1} - 16 h_{n-2} + 20 h_{n-3} = 0 \\ h_0 = 0, \; h_1 = 1, \; h_2 = -1, \; n \geq 3 \end{aligned}\right\}$$

Exponential Generating function :

Let $h_0, h_1, h_2, \ldots h_n, \ldots$ are infinite sequences.

considering the monomial..

$$1, \frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots, \frac{x^n}{n!}, \ldots$$

we get the generating function,

$$g^e(x) = h_0 + h_1 \cdot \frac{x}{1!} + h_2 \frac{x^2}{2!} + h_3 \frac{x^3}{3!} + \ldots + \frac{h_n x^n}{n!} + \ldots$$

$$g^e(x) = \frac{f(x)}{p(x)}, \ldots$$ degree of $p(x)$ is k and that of $f(x) < k$

Sequences of $(1, 1, 1 \ldots 1)$

$$h_0, h_1, h_2, \ldots h_n, \ldots = 1, 1, 1, \ldots 1 \ldots$$

$$g^e(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$= e^x$$

Exponential Generating Function :

(1) Let h_n denote the no. of n-digit no.s with 1, 2, or 3. where no. of 1's are even, no. of 2s are at least three, no. of 3's are at most four.

Determine the exponential generating function

$a+b+c = n$

$0 \le a \le 2k \quad k \le \frac{n}{2} \quad k = \frac{n}{2}$

$3 \le b \le n$

$0 \le c \le 4$

$$g^e(x) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \frac{x^n}{n!} + \cdots\right)\left(\frac{x^3}{3!} + \frac{x^4}{4!} \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)\left(e^x - 1 - x - \frac{x^2}{2!}\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)$$

(Ans)

(2) Determine the no. of ways to color the squares of 1 by n chessboard using the colors, red, white blue if an even no. of squares are to be colored red. $h_0 = 1$

$$g^e(x) = \left(\frac{e^x + e^{-x}}{2}\right)\left(e^{2x}\right)$$

$$= \frac{e^{3x} + e^x}{2}$$

$$= \sum_{r=0}^{\infty} \left(\frac{3^r + 1}{2(r!)}\right) x^r$$

$$\left(\frac{3^r + 1}{\ }\right)$$

Determine the exponential generating function

$a + b + c = n$

$0 \le a \le 2k \quad k = \frac{n}{2} \quad k = \frac{n}{2}$

$3 \le b \le n$

$0 \le c \le 4$

$$g^e(x) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \frac{x^n}{n!} + \cdots\right)\left(\frac{x^3}{3!} + \frac{x^4}{4!} \cdots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)\left(e^x - 1 - x - \frac{x^2}{2!}\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

(Ans)

(2) Determine the no. of ways to color the squares of 1 by n chessboard using the colors, red, white blue if an even no. of squares are to be colored red. $h_0 = 1$

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$$= \frac{e^{3x} + e^x}{2}$$

$$= \sum_{r=0}^{\infty}\left(\frac{3^r + 1}{2(r!)}\right)x^r$$

$$h_r(x) = \left(\frac{3^r + 1}{2}\right)$$

x^n

$a+b+c = 1$

$a \geq 0$
$b \geq 0$
$c \geq 0$

3

$(1+x)(1+x)(1+x)$

$= 3$

21

Q.

no. of square red — even no.
no restriction on white
at least one blue.

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}\cdots\right)\left(1+x+\frac{x^2}{2}+\cdots\right)$$

$$\left(x+\frac{x^2}{2!}+\cdots\right)$$

$$= \left(\frac{e^x+e^{-x}}{2}\right)(e^x)(e^x-1)$$

$$= \frac{1}{2}\left(e^{3x}-e^{-2x}+e^{x}-1\right)$$

$$h_n = \left.\frac{3^n-2^n+1}{2}\right|_{n\geq 0}$$

$$\left.\frac{3^n-2^n+1}{2}-\frac{1}{2}\right|_{n=0}$$

Determine the no. of n digit numbers with each digit odd, and 1 and 3 occur even no. of times.

$1, 3, 5, 7, 9.$

$$g^e(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 (e^x)^3$$

$$g^e(x) = \frac{e^{2x} + e^{-2x} + 2}{4} (e^{3x})$$

$$g^e(x) = \frac{e^{5x} + 2e^{3x} + e^x}{4}$$

$$h_n = (5^n + 2 \cdot 3^n + 1) \cdot \frac{1}{4}$$

Exponential generating function.

Theorem: Let S be the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots n_k \cdot a_k\}$ where $n_1, n_2, \ldots n_k$ are non negative integers. Let h_n be the no. of permutations of S. Then the exponential generating function $g^e(x)$ for the sequence $h_0, h_1, \ldots h_n$ is given by.

$$g^e x = f_1(x) \cdot f_2(x) \cdots f_{n_k}(x) \ldots$$

$$f_{n_i}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{x^{n_i}}{n_i!}$$

. using generating fun, find the no. of quaternary sequence of length n having an even no. of 0's

$$\underline{\underline{a}} + b + c + d = n$$

no. of zero, no. of one, no. of 2, no of 3

$$= \left(x^0 + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} \cdots\right)^3$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)\left(e^{3x}\right)$$

$$= \left(\frac{e^{4x} + e^{2x}}{2}\right)$$

n^{th} term

$$\sum_{n=1}^{\infty} a_n x^n - 2x \sum_{m=0}^{\infty} a_m x^m - x \sum_{m=0}^{\infty} 4^m x^m = 0$$

$$a_n = 4^{n-1}.$$

In how many ways can 4 of the letters ENGINE be arranged.

$$E \rightarrow \left(1 + x + \frac{x^2}{\lfloor 2}\right)$$

N

G, I $\rightarrow$ $(1+x)$

$$f(x) = \left(1 + x + \frac{x^2}{\lfloor 2}\right)^2 (1+x)^2$$

find coeff. of $\frac{x^4}{\lfloor 4}$

A company hires 11 new employees and each of whom is to be assigned. one of 4 division, each subdivision will get at least one new employee. how many ways assignments can be made?

$$f(x) = \left(x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots \right)^4$$

$$= (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1$$

$$= \sum_{i=0}^{4} (-1)^i \left({}^4C_i \right) (4-i)^{11}$$

$\therefore$ ans

$$= 4^{11} - 4(3)^{11} + 6(2^{11}) - 4(1)^{11}$$

Determine the sequences generated by each of the following exponential functions.

a) $f(x) = 7e^{8x} - 4e^{3x} = 7 \sum_{n=0}^{\infty} \frac{(8x)^n}{\lfloor n} - 4 \sum_{n=0}^{\infty} \frac{(3x)^n}{\lfloor n}$

$f(x) = 2e^x + 3x^2$