$\Rightarrow a_{m',n'} = m' + \frac{n'(n'+1)}{2}$

if $n = 0$

$$a_{m,n} = a_{m-1,n} + 1$$
$$= (m-1) + \frac{n(n+1)}{2} + 1$$
$$= m + \frac{n(n+1)}{2} \quad \checkmark \text{ True}$$

if $n > 0$

$$a_{m,n} = a_{m,n-1} + n$$
$$= m + \frac{(n-1)n}{2} + n$$
$$= m + \frac{n(n+1)}{2} \quad \text{True}$$

Set - Function - Relation

* **Set** :- unordered collection of objects.

$A = \{1, 2, 3\}$

Subset :- If X & Y are two sets and if each element of X is element of Y, then X is a subset of Y

$X \subseteq Y$

If X is a subset of Y & $X \neq Y$, then X is a proper subset of Y, $X \subset Y$

Number of elements of $X = |X| \rightarrow$ cardinality of 'X'.

* $X = Y$

$\Rightarrow \forall x \,(x \in X \rightarrow x \in Y) \wedge (x \in Y \rightarrow x \in X)$

* empty set $\rightarrow \emptyset, \{\ \}$
* Universal set $\rightarrow U$

$A = \{a, b, c\}$, $|A| = 3$

subsets $\rightarrow \{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

P(A) is the power set of A, which ~~cont~~ is the collection of all subsets of A.

$|P(A)| = 2^3 = 8$

Subsets containing element 'a'	subsets not containing 'a'.
$\{a\}$	$\emptyset$
$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{c\}$
$\{a, b, c\}$	$\{b, c\}$
$4 = 2^{n-1}$	$4 = 2^{n-1}$

<u>Theorem</u>: If $|X| = n$, then $|P(X)| = 2^n$, where X is a set.

<u>proof</u>:-

<u>Basis step</u>:-

$n = 0 \Rightarrow X = \emptyset$

$P(x) = \{\emptyset\} \Rightarrow 2^n = 2^0 = 1$

$|P(X)| = 1$ ← True

<u>Inductive step</u>:-

Let the result is true for 'n'.

Let X be a set of cardinality 'n+1'.

We have to show that $|P(X)| = 2^{n+1}$

Let 'y' be a set obtained from X by dropping one element 'a' from X.

$\Rightarrow |P(Y)| = 2^n$

Y is a set having elements of X except 'a'.

From the property we have seen above,

$$|P(Y)| = \frac{|P(X)|}{2}$$

$$|P(X)| = 2|P(Y)|$$

$$\boxed{\therefore |P(X)| = 2^{n+1}}$$

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* Operations on a set :-

Union, Intersection, Difference.

A & B are two sets

$A \cup B \rightarrow$ Collection of all distinct elements that exist in A or B.

$A \cap B \rightarrow$ Intersection is the collection of the elements that exist in both A & B.

$A - B \neq B - A$

Let U be the universal set and A, B, C are the subsets of U then the following properties hold for U, A, B, C and $\cup, \cap, -$ (Difference)

(a) Associative : $(A \cup B) \cup C = A \cup (B \cup C)$,
$(A \cap B) \cap C = A \cap (B \cap C)$

(b) commutative : $A \cup B = B \cup A$, $A \cap B = B \cap A$

(c) Distributive : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(d) Identity : $A \cup \phi = A$, $A \cap U = A$

(e) complementation : $A \cup \bar{A} = U$, $A \cap \bar{A} = \phi$

(f) Idempotent : $A \cup A = A$, $A \cap A = A$

(g) Bound : $A \cup U = U$, $A \cap \phi = \phi$

(h) Absorption : $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

(i) Involution : $\bar{\bar{A}} = A$

(j) 0/1 laws : $\bar{\phi} = U$, $\bar{U} = \phi$

(k) Demorgan : $\overline{A \cup B} = \bar{A} \cap \bar{B}$
$\overline{A \cap B} = \bar{A} \cup \bar{B}$

* (c) Distributive

prove $\underbrace{A \cap (B \cup C)}_{X} = \underbrace{(A \cap B) \cup (A \cap C)}_{Y}$

S Let x be an element of X. $\Rightarrow x \in X$

$x \in A \cap (B \cup C)$

$\Rightarrow$ x exists both in A and $(B \cup C)$

i.e $x \in A$ and $x \in B \cup C$

$x \in B$ or $x \in C$

$\therefore$ i.e $(x \in A$ and $x \in B)$ (or) $(x \in A$ and $x \in C)$

$x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

Now, Let $x \in Y$

i.e $x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow x \in A \cap B$ (or) $x \in A \cap C$

$(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$x \in A$ and $(x \in B$ or $x \in C)$

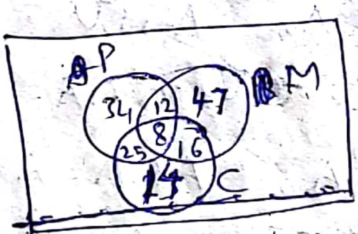
$x \in A$ and $x \in B \cup C$

$x \in A \cap (B \cup C)$

* Venn diagram

→ pictorial representation of sets

U $A, B \subseteq U$



Out of 165 students of a class ~~say 8 students~~

~~take P, M, C.~~ take M, P, C

8 students take M, P, C

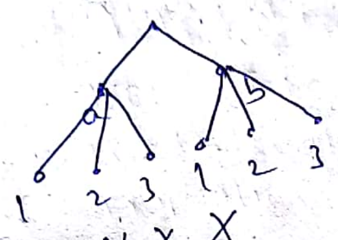
20	"	"	M & P
24	"	"	M & C
33	"	"	P & C
79	"	"	P
83	"	"	M
63	"	"	C

How Many students didn't take any! Ans: 9

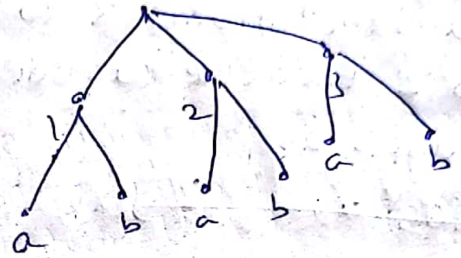
* If X and Y are two sets, $x \in X$, $y \in Y$
Cartesian product is defined as
$X \times Y$, which is the set of all ordered pairs (x, y) such that $x \in X$ & $y \in Y$

Ex:- $X = \{1, 2, 3\}$, $Y = \{a, b\}$

$X \times Y = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$



$Y \times X$



$X \times Y$

$Y \times X = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

$X \times Y \neq Y \times X$

$|X \times Y| = |Y \times X|$

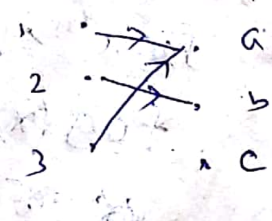
Function:-

* Let X & Y be two sets and a function 'f' is defined as ~~the~~ a subset of the cartesian product $X \times Y$ such that for all $x \in X$ there is exactly one $y \in Y$ s.t. $(x, y) \in f$

$f : X \longrightarrow Y$

$X = \{1, 2, 3\}$ a subset of $X \times Y = \{(1,a), (3,a), (2,b)\}$

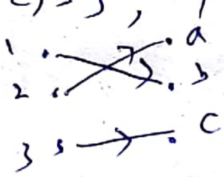
$Y = \{a, b, c\}$



* Injective:-

'f' is injective func^n, if for every $y \in Y$, there is atmost one $x \in X$ such that $(x, y) \in f$

$X = \{1, 2, 3\}$, $Y = \{a, b, c\}$



* **Surjective :-**

If the range of funcⁿ 'f' is Y, then it is onto funcⁿ or surjection

* **Bijection :-**

One-one & onto is a Bijection.

* **Composition :-**

Let 'g' be a funcⁿ from X to Y and f be a funcⁿ from Y to Z then we can define composition of two funcⁿs as

~~$(f\circ g)(x) = f(g(x))$~~

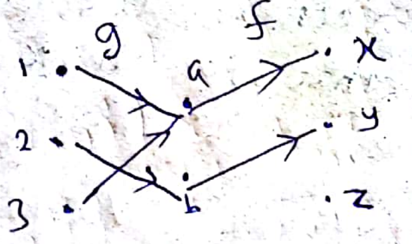
$(f\circ g) : X \longrightarrow Z$

$g : X \longrightarrow Y = \{(x,y) : (x,y) \in g\}$

$f : Y \rightarrow Z = \{(y,z) \mid (y,z) \in f\}$

$X = \{1,2,3\}, \quad Y = \{a,b\}, \quad Z = \{x,y,z\}$

$g = \{(1,a),(2,b),(3,a)\}, \quad f = \{(a,x),(b,y)\}$



$f\circ g = \{(1,x),(2,y),(3,x)\}$

Ex: $f(x) = x^4, \quad g(x) = \log_3 x$

$f\circ g(x) = f(\log_3 x) = (\log_3 x)^4$

$g\circ f(x) = \log_3 x^4 = 4\log_3 x$

* **Sequence and string :-**

Sequence is a special type of funcⁿ where the domain consists of consecutive integers.

$a_0, a_1, a_2, \ldots$

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S = subset of $X \times Y$ = $\{(1,a), (2,b), (3,d), (4,e), \ldots\}$

$S_1 \subseteq S$ → domain consists of consecutive values
↓
Subsequence

A string over X is a finite sequence of elements of X, X is a finite set.

$X = \{a, b, c, d\}$

$\beta_1 = a, \beta_2 = b, \beta_3 = c$

$\beta_2 \beta_1 \beta_3 = \{bac\}$ → string

$b^2 a^4 c$ = bb aaaa c

Ex:- Given a sequence $S_n = 2^n + 4 \cdot 3^n$, $n \geq 0$

Find S_0, S_1, Find a formula for S_i, S_{n-1}, S_{n-2}. Prove that S_n satisfies $S_n = 5S_{n-1} - 6S_{n-2}, n \geq 2$

$S_0 = 1 + 4 = 5$

$S_1 = 2 + 4 \times 3 = 14$

$S_i = 2^i + 4 \cdot 3^i$

$S_{n-1} = 2^{n-1} + 4 \cdot 3^{n-1}$

$S_{n-2} = 2^{n-2} + 4 \cdot 3^{n-2}$

$$\begin{aligned}
RHS &= 5S_{n-1} - 6S_{n-2} \\
&= 5(2^{n-1} + 4 \cdot 3^{n-1}) - 6(2^{n-2} + 4 \cdot 3^{n-2}) \\
&= 2^{n-2}[10 - 6] + 4 \cdot 3^{n-2}(15 - 6) \\
&= 2^n + 4 \cdot 3^n = S_n
\end{aligned}$$

$S_1, S_2, \ldots, S_i, \ldots S_n, S_{n+1}, \ldots$

If $S_n < S_{n+1}$ $\forall n$, it is increasing sequence

If $S_n \leq S_{n+1}$ $\forall n$, it is nondecreasing sequence

If $S_n > S_{n+1}$ $\forall n$, it is decreasing sequence

If $S_n \geq S_{n+1}$ $\forall n$, it is non-increasing sequence

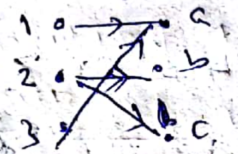
* Relation

Defⁿ:- A relation (binary) R from X to Y is a subset of the cartesian product of X × Y, if $(x,y) \in R$, we call $x R y$ or 'x' is related to y. If $X = Y$, R is on X.

$\{x \in X \mid (x,y) \in R$ for $y \in Y\} \rightarrow$ domain of R

$\{y \in Y \mid (x,y) \in R$ for $x \in X\} \rightarrow$ Range of R

$X = \{1,2,3\}$, $Y = \{a, b, c\}$



Special type of

Fⁿ: A funcⁿ is a relation with the following constraints

(i) The domain should be X

(ii) for all $x \in X$, there is exactly one $y \in Y$

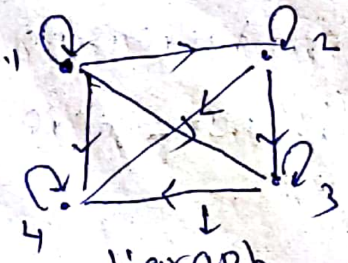
Ex:- $R = \{(x,y) \mid x \leq y\}$

$X = \{2,3,4\}$, $Y = \{3,4,5,6\}$

$R = \{(2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)$
$(4,4), (4,5), (4,6)\}$

Ex 2:- $X = \{1,2,3,4\}$ Above R on X

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4)$
$(3,3), (3,4), (4,4)\}$



called a digraph

* If a Relation R contains $\{(x,x) \mid x \in X\}$, then it is reflexive relation.

* If $(x,y) \in R$, then $(y,x) \in R$. Such a relation is called symmetric relation.

Ex:- $R = \{(a,b), (b,c), (c,b), (d,d)\}$
$X = \{a, b, c, d\}$. Is R reflexive, symmetric.

Not reflexive since (b,b), (c,c) are not present & symmetric because (b,c) is there & (c,b) is there.

Antisymmetric:-

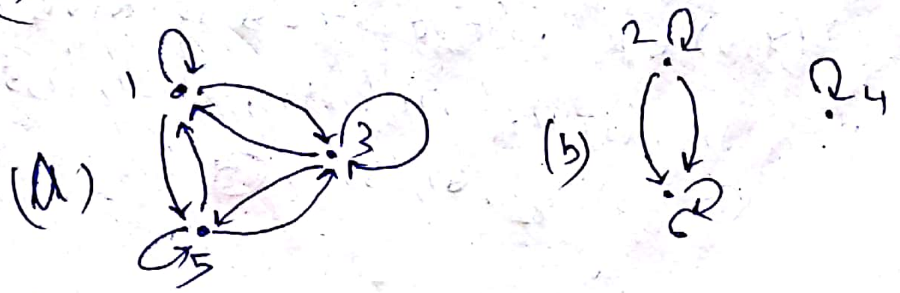
If $(x,y) \in R$ and $x \neq y$ then $(y,x) \notin R$, then R is antisymmetric.

$X = \{2,3,4\}$, $Y = \{3,5,6\}$
xRy: x divides y
$R = \{(2,6), (3,6), (3,3)\}$

* $R = \{(a,a), (b,b), (d,d), (c,c)\}$
↓
both symmetric & Anti-symmetric
i.e if (x,y) s.t. $x \neq y \notin R$, then ~~sy relation~~ if $(x,y) \in R$ and $x \neq y$, then ~~R is~~ the condition that $(y,x) \in R$ is vacuously true. Relation is symmetric as well as anti-symmetric.

~~*Relation~~
* **Transitive :-**
Relation 'R' is transitive if $(x,y) \in R$ & $(y,z) \in R$ then $(x,z) \in R$

Ex: $R = \{(x,y) \mid x \leq y\}$, $R = \{(x,y) \mid x \text{ divides } y\}$



~~R= {(1,1)~~

(a) $R = \{(1,1), (3,3), (5,5), (1,3), (3,1), (1,5), (5,1), (3,5), (5,3)\}$

$\Rightarrow$ R is reflexive, symmetric, transitive

* ~~Partition~~
* Partial Order:

If a relation is reflexive, antisymmetric & transitive, then the relation is called a partial order.

* Equivalence:

If a relation is reflexive, symmetric and transitive, then the relation is an equivalence relation.

* ~~Total order~~ ~~Conside~~ Let R is on X. consider a subset 'S' on 'X'

say $x \in X$ & $x \in S$

$x R a$ for some $a \in X$

if a also $\in S$, for all x, a in S.

the R is on 'S' also.

If this property holds for all subsets of 'X', then it is total order

antisymmetric, transitive, connex property (any two elements of the set are related $\Leftarrow$ $(n + {}^nC_2)$ pairs)

* Partition:

R is an equivalence relation on X, $a \in X$,

$[a] = \{x \in X \mid x R a\}$ → all the elements related to 'a'

$S = \{[a] \mid a \in X\}$ is a partion on X

All sets in 'S' are called the equivalence classes

$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$(x, y) \in R$ where $x - y$ is divisible '3'.

$R = \{(1,1), (1,4), (1,7), (1,10), (2,2), (2,5), (2,8), (3,3), (3,6), (3,9), \ldots\}$

$[1] = \{1, 4, 7, 10\} \rightarrow$ equivalence class of '1' (also of 4, 7, 10)

$[2] = \{2, 5, 8\} \rightarrow$ " " '2' (also of 5, 8)

$[3] = \{3, 6, 9\} \rightarrow$ " " '3' (also, 6, 9)

⋮

* Recurrence Relation:

Defn: A recurrence relation of a sequence is an equation that relates the n^{th} term of the sequence to certain predecessors.

$a_1, a_2 \ldots a_n, \ldots$

Ex: $a_n = a_{n-1} + 3$

Ex: Some one invests ₹. 10,000 at 15% compound interest

$a_0 = 10000$

$a_n = a_{n-1} + 15\%. a_{n-1} = (1.15) a_{n-1}$

$a_1 = 11500 = (1.15)\, 10{,}000$

$a_2 = (1.15)^2 \times 10000$

⋮

$a_n = (1.15)^n \times 10{,}000$

Ex: The No. of subsets (S_n) of a set of 'n' elements.

$S_n = 2S_{n-1}$, $S_0 = 1$

No. of subsets containing n^{th} element = those not containing it

$= 2 \cdot 2 \cdot S_{n-2}$

$= 2^2 S_{n-2}$

$\Rightarrow S_n = 2^n \cdot S_0 \Rightarrow \boxed{S_n = 2^n}$

Ex: Find the number of n-bit strings, which does not contain '111'

Sol

Sol: Strings that don't contain 111, begin with
1. 0
2. 10
3. 110

$\therefore S_n = S_{n-1} + S_{n-2} + S_{n-3}$

$S_1 = 2, \; S_2 = 4, \; S_3 = 7$

Ex: Find the no. of n-bit strings which has no two consecutive 0's.

Sol. strings begin with
1. 1
2. 01

$S_n = S_{n-1} + S_{n-2}$

$S_1 = 2, \; S_2 = 3$

Ex: Code - Enumeration:

n-bit code words are formed from decimal digits, 0, 1, 2, ----, 9.
A valid code word contains even no. of 0's.
Find the number of n-bit valid codewords.

Sol strings begin with

$\left.\begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}\boxed{}\right\} S_{n-1}$

$0 \; \underbrace{\boxed{}}_{\text{invalid words of 'n-1' length.}}$

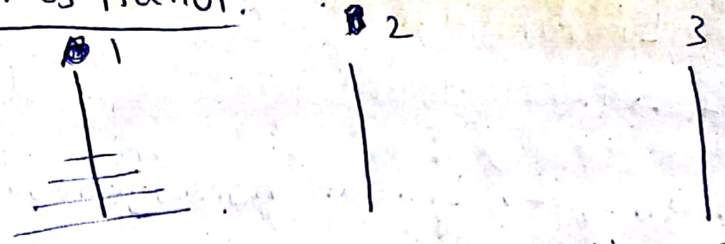
$\therefore S_n = 9S_{n-1} + (10^{n-1} - S_{n-1})$

$S_n = 10^{n-1} + 8 . S_{n-1}$

$S_1 = 9$

$\Rightarrow S_2 = 82$

* Tower of Hanoi :-



No. of moves to transfer n-discs from 1 to 2.

Let C_n be the No. of moves

$\Rightarrow C_1 = 1$

Move top n-1 discs from 1 to 3 & then n^{th} disc from 1 to 2 & then (n-1) discs from 3 to 2

$$\Rightarrow C_n = 2C_{n-1} + 1$$
$$= 2(2C_{n-2} + 1) + 1$$
$$= 2^2 C_{n-2} + 2 + 1$$
$$\vdots$$
$$= 2^{n-1} C_1 + 2^{n-2} + \cdots + 1$$
$$= 2^{n-1} + 2^{n-2} + \cdots + 1$$
$$= 2^n - 1$$

prove that the sol^n is optimal. $\Rightarrow$ we have to prove $C_n = d_n$

Proof: let d_n be the optimal sol^n.

B.S :-

if $n=1$, $C_n = 1 = d_n$ $\Rightarrow$ true

- 1 move is the optimal.

I.S :-

Let the result is true for (n-1) discs

i.e $C_{n-1} = d_{n-1}$

$$d_n \geq 2d_{n-1} + 1$$
$$\geq 2C_{n-1} + 1$$
$$d_n \geq C_n$$

But from assumption, $C_n \geq d_n$ [$\because d_n$ is optimal]

* ~~$y = ax+b$~~ → ~~Linear Homogeneous equation~~/Recurre
~~$y = ax^2+bx+c$ → L.~~

* $a_n = a_{n-1} + a_{n-2}$ → Linear homogeneous recurre relation

$a_n = 2a_{n-1} + 5a_{n-2}$ → with constant coefficients

$a_n = 5n\,a_{n-1}$ → Linear homogeneous with variable coefficient.

$a_n^2 = 2a_{n-1}+5$ → Non-linear non-homogeneous eq^n

$a_n = 2a_{n-1}a_{n-2}$ → Non-homogeneous

* Def^n :-

A linear homogeneous recurrence relation of ~~order~~ order 'k' with constant coefficients is of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, $c_k \neq 0$ for order 'k'

with 'k' initial conditions
$a_0 = P_0,\ a_1 = P_1, \ldots\ldots, a_k = P_k$

Ex:- Solve $a_n = 5a_{n-1} - 6a_{n-2}$; $a_0 = 7,\ a_1 = 16$

~~1st order recurrence r/o~~

{ let sol^n be of the form t^n.

$\Rightarrow t^n = 5t^{n-1} - 6t^{n-2}$

$t^{n-2}(t^2 - 5t + 6) = 0$

$t = 2, 3, \cancel{0}$

$a_n = 2^n$ $\qquad a_n = 3^n$ → 2 sol^ns → Order '2'

Let S and T be the ~~two~~ Sol^ns

& $S_n = 2^n$, $T_n = 3^n$

$U = bS + dT$ is one of the sol^ns of a_n

$\therefore a_n = bS_n + dT_n$

$a_0 = 7 \Rightarrow b \times 2^0 + d \times 3^0 = 7$

$b + d = 7$

$a_1 = 16 \Rightarrow b \times 2 + d \times 3 = 16$

$2b + 3d = 16$

$2(7) + d = 16$

$\boxed{d = 2} \Rightarrow \boxed{b = 5}$

$\therefore a_n = 5 \cdot 2^n + 2 \cdot 3^n$

Theorem:-

1. Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a 2nd order linear homogeneous recurrence relation with constant coefficients. If S & T are solutions then $U = bS + dT$ is also a solution of the recurrence relation.

2. If r is a root of $t^2 - c_1 t - c_2 = 0$, then the sequence r^n, $n = 0, 1, 2, \ldots$ is a soln. If a is sequence defined initial conditions $a_0 = P_0, a_1 = P_1, \ldots$ and r_1, r_2 are roots with $r_1 \neq r_2$ then there exists b & d such that $a_n = b r_1^n + d r_2^n$, $n = 0, 1, \ldots$

proof: 1. If S & T are two solns of the recurrence relation of 2nd order

$S_n = c_1 S_{n-1} + c_2 S_{n-2}$

$T_n = c_1 T_{n-1} + c_2 T_{n-2}$

$\therefore U_n = bS_n + dT_n$

$= b(c_1 S_{n-1} + c_2 S_{n-2}) + d(c_1 T_{n-1} + c_2 T_{n-2})$

$= c_1 (bS_{n-1} + dT_{n-1}) + c_2 (bS_{n-2} + dT_{n-2})$

$U_n = c_1 U_{n-1} + c_2 U_{n-2}$

2. Since r_1 is a root,

$r^2 - c_1 r - c_2 = 0$

$r^2 = c_1 r + c_2$

$\Rightarrow r^2 \times r^{n-2} = c_1 r \times r^{n-2} + c_2 r^{n-2}$

$r^n = c_1 r^{n-1} + c_2 r^{n-2}$

Ex. Assume that the deer population ~~increases by 1% each year~~

at time $n=0$, it is 200

$n=1$, 220

and increase from $(n-1)$ to n is twice that of from $(n-2)$ to $(n-1)$

$\Rightarrow S_n - S_{n-1} = 2(S_{n-1} - S_{n-2})$

$S_n = 3S_{n-1} - 2S_{n-2}$, $S_0 = 200$

$S_1 = 220$

$S_n = t^n$

$\Rightarrow t^2 - 3t + 2 = 0$

$t = 1, 2$

$\therefore S_n = a \cdot 2^n + b$

$S_0 = 200 \Rightarrow a + b = 200$

$S_1 = 220 \Rightarrow 2a + b = 220$

$\boxed{a = 20} \Rightarrow \boxed{b = 180}$

$\therefore S_n = 20 \cdot 2^n + 180$

* Fibonacci

$f_n = f_{n-1} + f_{n-2}$

$t^2 - t - 1 = 0$

$t = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

$\therefore f_n = b \cdot \left(\frac{\sqrt{5}+1}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$ $f_0 = 0$, $f_1 = 1$

$\Rightarrow b + d = 0 \Rightarrow \boxed{d = -b}$

$b\left(\frac{\sqrt{5}+1}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$b\left(\frac{\sqrt{5}+\cancel{1}-\cancel{1}+\sqrt{5}}{2}\right) = 1$

$\boxed{b = \frac{1}{\sqrt{5}}}$, $\boxed{d = -\frac{1}{\sqrt{5}}}$

$\therefore f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Theorem:

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ be a 2nd order linear homogeneous recurrence relation with initial conditions $a_0 = P_0$, $a_1 = P_1$. If both the roots are equal then the soln is $a_n = br^n + dnr^n$.

Proof: $t^n - c_1 t^{n-1} - c_2 t^{n-2} = 0$

$\Rightarrow t^2 - c_1 t - c_2 = 0$

$\Rightarrow (t-r)^2 = 0$ [$\because$ both roots are 'r']

$t^2 - 2rt + r^2 = 0$

$\therefore c_1 = 2r,\ c_2 = -r^2$

Now consider $a_n = nr^n$

$\Rightarrow c_1 a_{n-1} + c_2 a_{n-2}$

$= 2r((n-1)r^{n-1}) - r^2((n-2)r^{n-2})$

$= (2n-2)r^n - (n-2)r^{n-2}$

$= nr^n$

$= a_n$ ✓

$\therefore U_n = br^n + dnr^n$

Ex: $d_n = 4(d_{n-1} - d_{n-2})$, $d_0 = 1$, $d_1 = 1$

& $t^2 - 4t + 4 = 0 \Rightarrow t = 2, 2$

$\therefore$ ~~$d_n = \ldots$~~

$d_n = 2^n(a + bn)$

$d_0 = 1 \Rightarrow \boxed{a = 1}$

$d_1 = 1 \Rightarrow 2(1 + b) = 1$

$\boxed{b = -\frac{1}{2}}$

~~$d_n = \ldots$~~

$d_n = 2^n - n 2^{n-1}$

* Soln:-

A linear homogeneous recurrence relation of order k; $t^k - c_1 t^{k-1} - c_2 t^{k-2} - \ldots - c_k = 0$

order k; If r is a root of multiplicity m, then the soln contains $r^n, nr^n, \ldots n^{m-1} r^n$

* Linear Non-homogeneous Recurrence relation

eg:→ $a_n = 3a_{n-1} + 2n$

$a_n = 3a_{n-2} + n^2 + n + 1$

$a_n = 3a_{n-1} + 4a_{n-2} + 3^n$

$a_n = 3a_{n-1} + 5a_{n-2} + 6a_{n-3} + 7^n + 5$

~~$a_n = c_1 + a_n$~~

In general,

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$

→ Solution is the sum of two separate part of recurrence relation

↓

2 parts

homogeneous part

particular ~~form~~ ~~sol~~ part

$a_n = \{ a_n^{(p)} + a_n^{(h)} \}$

$a_n^{(p)}$ → particular sol^n

$a_n^{(h)}$ → sol^n for homogeneous part

Theorem:-

If $\{a_n^{(p)}\}$ is a particular sol^n of non-homogeneous linear recurrence relation with constant coefficients,

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$

then every sol^n is of the form $\{a_n^{(p)} + a_n^{(h)}\}$

$a_n^{(p)}$ → particular sol^n

$a_n^{(h)}$ → sol^n to homogeneous part

$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n)$

let ~~the 2nd solution~~ b_n be a sol^n of a_n

$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n)$

$b_n - a_n^{(p)} = c_1 (b_{n-1} - a_{n-1}^{(p)}) + c_2 (b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k (b_{n-k} - a_{n-k}^{(p)})$

$\Rightarrow b_n - a_n^{(p)} = a_n^{(h)}$.

$\therefore \boxed{b_n = a_n^{(h)} + a_n^{(p)}}$

Ex:- $a_n = 3a_{n-1} + 2n$, $a_1 = 3$

& soln of homogeneous part $a_n^{(h)} = \alpha 3^n$.

$a_n^{(p)} = cn + d$, since the highest degree of 'n' is the eqn is '1'.

$\Rightarrow a_n = \alpha 3^n + cn + d$

$3a_{n-1} + 2n = \alpha 3^n + 3c(n-1) + 3d + 2n$

$= \alpha 3^n + (3c+2)n + (3d - 3c)$

$\Rightarrow 3c + 2 = c \Rightarrow \boxed{c = -1}$

$3d - 3c = d \Rightarrow 2d = 3c$

$\boxed{d = -3/2}$

$a_1 = 3 \Rightarrow \alpha \times 3 - 1 - \frac{3}{2} = 3$.

$3\alpha = \frac{11}{2}$

$\boxed{\alpha = 11/6}$

$\therefore a_n = \frac{11}{6} 3^n - n - \frac{3}{2}$

Ex Find all the solns of recurrence relation

$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

& $a_n^{(h)}$

$t^2 - 5t + 6 = 0$

$t = 2, 3$

$a_n^{(h)} = \alpha 2^n + \beta 3^n$.

let $a_n^{(p)} = c \cdot 7^n$

$\therefore a_n = \alpha 2^n + \beta 3^n + c \cdot 7^n$

$5 a_{n-1} - 6a_{n-2} + 7^n = \alpha 2^n + \beta 3^n + 5c 7^{n-1} - 6c 7^{n-2} + 7^n$.

$\therefore c \cdot 7^n = 5c 7^{n-1} - 6c 7^{n-2} + 7^n$

$49c = 35c - 6c + 49$

$20c = 49$

$\boxed{c = 49/20}$

*Solve by substitution:-

Non-linear recurrence relation

$$a_n^2 - 2a_{n-1}^2 = 1, \quad n \geq 1, \quad a_0 = 1$$

let. $a_n^2 = b_n \Rightarrow b_0 = a_0^2 = 1$

$$b_n - 2b_{n-1} = 1$$

$$b_n^{(h)} = \alpha 2^n, \quad b_n^{(p)} = c$$

$$\therefore b_n = \alpha 2^n + c$$

$$b_n = 2b_{n-1} + 1$$

$$= 2(\alpha 2^{n-1}) + 2c + 1$$

$$= \alpha 2^n + 2c + 1$$

$$c = 2c + 1 \Rightarrow \boxed{c = -1}$$

$$\therefore b_n = \alpha 2^n - 1 \qquad b_0 = 1 \Rightarrow \boxed{\alpha = 2}$$

$$\therefore b_n = 2^{n+1} - 1$$

$$a_n = \sqrt{2^{n+1} - 1}$$