Classical way to solve constant co-efficient linear Difference equation – A brief introduction.

$$x(n) \rightarrow \boxed{\text{System}} \rightarrow y(n)$$

For a system, $y(n)$ and $x(n)$ may be related by a difference equ^n. (n is integer).

$$\boxed{y(n) + a_1\, y(n-1) + a_2\, y(n-2) = x(n)}$$

This is 2nd order dif. equ^n.

<u>Natural Response:</u> $x(n) = 0$ — system output may then be present by some previous value of say $y(-2)$.

Equ^n to be solved is

$$y(n) + a_1\, y(n-1) + a_2\, y(n-2) = 0$$

We look for $y(n)$ which satisfies this equ^n for all n.

A logical guess is $y(n) = c\, m^n$ $\boxed{\text{General exponential } f^n \text{ in DT}}$

Why? $y(n-1) = c\, m^{n-1} = \frac{c}{m} m^n$ & $y(n-2) = \frac{c}{m^2} m^n$

Thus for any difference we take ~~for~~ nature of y remains same $\rightarrow m^n$. How to know m?

Putting $y(n) = c\, m^n$, we get

$$c\, m^n + a_1 c\, m^{n-1} + a_2 c\, m^{n-2} = 0$$

or $$c\, m^n \left[1 + \frac{a_1}{m} + \frac{a_2}{m^2}\right] = 0$$

Assuming sol^n exist $c\, m^n \neq 0$.

$$\therefore\quad 1 + \frac{a_1}{m} + \frac{a_2}{m^2} = 0$$

or $$\boxed{m^2 + a_1 m + a_2 = 0}$$ Called ch. equ^n.

Suppose roots are m_1 & m_2

$$\therefore\quad y_{nat.}(n) = c_1 m_1^n + c_2 m_2^n$$

c_1, c_2 to be determined from B.C. (to be applied to the total solution)

~~E- Don't apply~~

Before we proceed further to obtain solution due to forcing (or excitation) $x(n)$, let us introduce the forward shift operator E and backward shift operator E^{-1}

E operating on $y(n)$ gives $y(n+1)$ i,e., $E\{y(n)\} = y(n+1)$

& E^{-1} " on $y(n)$ gives $y(n-1)$ i,e., $E^{-1}\{y(n-1)\} = y(n-2)$

[obviously $E^2\{y(n)\} = \cancel{y+2}$ $\cancel{y(n)}$ $y(n+2)$ and so on }.

The given equ^n. $y(n) + a_1 y(n-1) + a_2 y(n-2) = x(n)$

can be written using E operator as:

$$\left[1 + a_1 E^{-1} + a_2 E^{-2}\right] y(n) = x(n)$$

Another way of writing is

Replace n by $n+2$ on both sides to get

$$y(n+2) + a_1 y(n+1) + a_2 y(n) = x(n+2)$$

use E operator

$$\left[E^2 + a_1 E + a_2\right] y(n) = E\{x(n)\}$$

Comming back to the problem, if $x[n] = 0$

for getting natural response:

$$\left[E^2 + a_1 E + a_2\right] y(n) = 0 \qquad \text{assumed sol}^n.\ y(n) = c\,m^n$$

Replace E by m & ch. equ^n is

$$\boxed{m^2 + a_1 m + a_2 = 0} \rightarrow \text{same as obtained earlier without using } E \text{ operator}$$

Note backward shift operator will also lead to the same ch. equ^n.

So solution due to no excitation i,e., $x[n) = 0$

$$\boxed{y_{nat}(n) = c_1 m_1^n + c_2 m_2^n}$$

Solution due to forcing f.^{n} $\boxed{x(n) = K b^n u(n)}$

i, e.,

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = K b^n u(n)$$

$\because$ The RHS is exponential, solution should be of the same form in order that RHS = LHS for all n. in this case for all $(n \geq 0)$.

Let the soln. be $y(n) = p b^n u(n)$ (guessed p to be determined)

putting this in the above eqn.

$$\left(p b^n + a_1 p b^{n-1} + a_2 p b^{n-2}\right) u(n) = K b^n u(n).$$

or $$p b^n \left(1 + \frac{a_1}{b} + \frac{a_2}{b^2}\right) u(n) = K b^n u(n)$$

$\therefore$ output $$y_f(n) = p b^n u(n) = \frac{K b^n u(n)}{1 + \frac{a_1}{b} + \frac{a_2}{b^2}}$$

$$= \frac{b^2 K b^n u(n)}{(b^2 + a_1 b + a_2)}$$

$$y_f(n) = \frac{b^2}{(b^2 + a_1 b + a_2)} \times \underbrace{\boxed{K b^n u(n)}}_{= x(n)}$$ input

-o-

The above result can be interpreted using shift operator as well. (first use backward op. E^{-1}).

$$\left(1 + a_1 E^{-1} + a_2 E^{-2}\right) y_f(n) = K b^n$$

multiply both sides of E^2

$$\left(E^2 + a_1 E + a_2\right) y(n) = E^2 \; K b^n u(n)$$

or $$y_f(n) = \left.\frac{E^2}{E^2 + a_1 E + a_2}\right|_{E=b} K b^n u(n)$$

$$= \frac{b^2}{(b^2 + a_1 b + a_2)} K b^n u(n) \longrightarrow \text{Same Answer.}$$

using forward shift operator

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = K b^n u(n)$$

Replace :- $n \to n+2$ on both sides.

$$y(n+2) + a_1 y(n+1) + a_2 y(n) = K b^{n+2} u(n+2)$$

$$\therefore (E^2 + a_1 E + a_2)\, y(n) = K E^2 \{ b^n u(n) \}.$$

$$\therefore y_f(n) = \frac{E^2}{(E^2 + a_1 E + a_2)}\Bigg|_{E=b} K b^n u(n)$$

$$= \frac{b^2}{b^2 + a_1 b + a_2} K b^n u(n)$$ → same as obtained earlier.

Choose any one of the shift operator to get the forced response. Point to be noted that if $b^2 + a_1 b + a_2 = 0$, This method fails.

So Total soln. is

$$y(n) = y_{nat}(n) + y_f(n)$$

$$y(n) = c_1 m_1^n + c_2 m_2^n + \frac{b^2}{(b^2 + a_1 b + a_2)} K b^n u(n)$$

You now need two B.C to get c_1 & c_2.

Let us solve some problems.

For The causal linear system it is given that

$\cancel{y(n) + 2y(n-1) = x(n)}$

$$y(n) + 2y(n-1) = x(n)$$

Find $y(n)$ (i) if $x(n) = u(n)$ $\cancel{\text{if } y(-1) = 0}$

(ii) if $x(n) = \delta(n)$.

Sol^n

(i) Given equ^n :- $y(n) + 2y(n-1) = x(n) = u(n)$

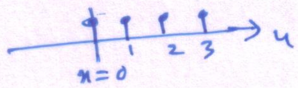
$n \to n+1$: $y(n+1) + 2y(n) = u(n+1)$

$$(E+2)\, y(n) = E\{u(n)\}.$$

For any causal system, The output will be zero if There is no input. i, e.,

$\ldots\; y(-3) = y(-2) = y(-1) = 0$ or $y(n) = 0$ for $n<0$

as input is applied at $n=0$ onwards $(u(n))$.



So for $n<0$ RHS $= 0$

$\therefore\; y(n) + 2y(n-1) = 0$

$\boxed{\text{at } n=0}$ $y(n) + 2y(n-1) = u(n)$

or $y(0) + 2y(-1) = u(0) = 1$

$\therefore\; y(0) = 1 - 2y(-1) = 1$

$\therefore\; \boxed{y(0) = 1}$ is The B.C.

for $n \geq 0$

$$(E+2)\, y(n) = E\{u(n)\} = E\{1^n u(n)\}. \quad \therefore b = 1$$

ch. root $m+2=0$ $\therefore m=-2$

$$\therefore\; y_{total}(n) = c(-2)^n u(n) + \left.\frac{E}{E+2}\right|_{E=1}$$

$\left.\begin{aligned} y(n) &= c(-2)^n u(n) \\ &+ \tfrac{1}{3} u(n) \end{aligned}\right\} n \geq 0$

now $y(0) = 1$

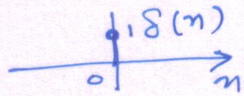
$1 = c + \frac{1}{3}$ or $c = \frac{2}{3}$

$$\boxed{y(n) = \frac{2}{3}(-2)^n u(n) + \frac{1}{3} u(n)}$$

(ii) when $x(n) = \delta(n)$

B.C
$y(-1) = 0$
Causal.

$$y(n) + 2y(n-1) = \delta(n)$$



for $n < 0$ $\quad y(n) + 2y(n-1) = 0$

ch eqn.

$(m+2) = 0$ root $m = -2$

$x(n) = \delta(n)$ is not of the form $b^n u(n)$.

what to do?

at $n = 0$ $\quad y(0) + 2y(-1) = \delta(0) = 1$.

$\therefore y(0) = 1 \quad \because y(-1) = 0 \quad \because$ causal.

for $n > 0$ $\quad \delta(n) = 0$ and sys. eqn.

$$y(n) + 2y(n-1) = 0$$

root $m = -2$
& no forcing fn.

$$\therefore y(n) = c(-2)^n$$

$y(0) = 1$ gives $c = 1$

$$\therefore \boxed{y(n) = (-2)^n u(n)}$$

Go Through it carefully.

I shall try to upload a problem on 2nd order system in a couple of days.

Watch out.