DT – LTI

$$x[n] = z^n \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \underbrace{z^n}_{I/p} \underbrace{\left[\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}\right]}_{\text{scale} = H(z)}$$

$$z^n \longrightarrow z^n H(z)$$

Don't put in verticle bracket. It is const f^n.

$$z = e^{j\omega}$$

$I/p \rightarrow$ Sustained sinusoid O/p.

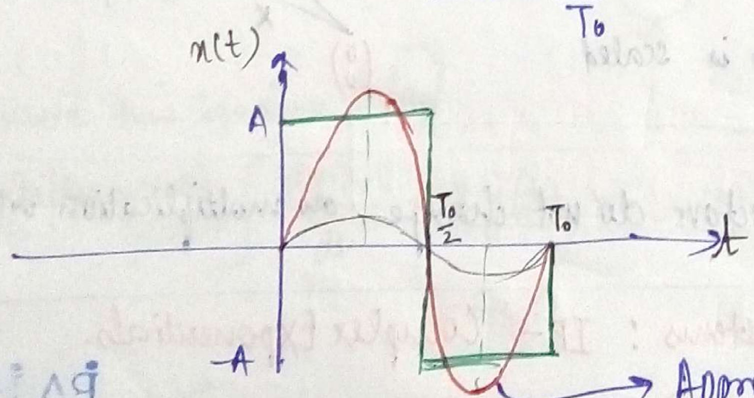
Fourier Series $\Rightarrow$ Only applicable for periodic signals with finite discontinuities.

* Dirichle Conditions

FOURIER SERIES

$$x(t) = x(t + T_0) \quad \forall t$$

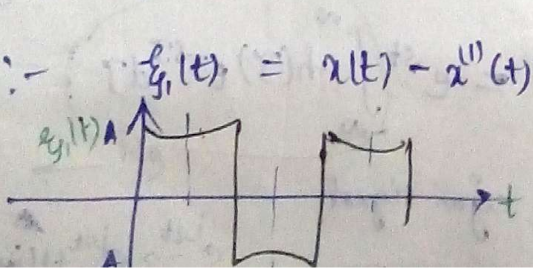
$$\Omega_0 = \frac{2\pi}{T_0}$$



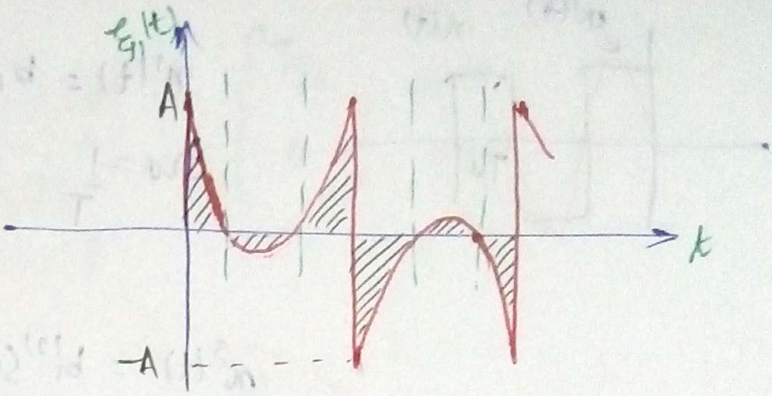
$\rightarrow$ Approximation of signal

$$x^{(1)}(t) = b_1^{(1)} \sin(\Omega_0 t)$$

Approx error :- $\varepsilon_1(t) = x(t) - x^{(1)}(t) = x(t) - b_1^{(1)} \sin(\Omega_0 t)$



$$P_{\varepsilon_1} = \frac{1}{T_0}\int_0^{T_0} |\varepsilon_1^*(t)|^2 dt$$



$$P_{\varepsilon_1} = \frac{1}{T_0}\left[\int_0^{T_0} |\varepsilon_1^*(t)|^2 dt\right]$$

Averaging operator

$$= \frac{J_1}{T_0}$$

MSE (Mean Square Error)

* $J \to$ function of $b_1^{(1)}$. ($\because$ freq, T = fixed, only Amp. of approx. signal can be changed)

$$\frac{\partial J}{\partial b_1^{(1)}} = 0 \quad \text{(Minimise Power)}$$

$$b_1^{(1)} = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(\Omega_0 t)\,dt = \frac{2}{T_0}\left\{\int_0^{T_0/2} A\sin(\Omega_0 t)\,dt + \int_{T_0/2}^{T_0} (-A)\sin(\Omega_0 t)\,dt\right\}$$

$$= \frac{4A}{\pi}$$

$$x^{(2)}(t) = b_1^{(2)}\sin(\Omega_0 t) + b_2^{(2)}\sin(2\Omega_0 t)$$

AE:- $\varepsilon_2(t) = x(t) - \left\{b_1^{(2)}\sin(\Omega_0 t) + b_2^{(2)}\sin(2\Omega_0 t)\right\}$

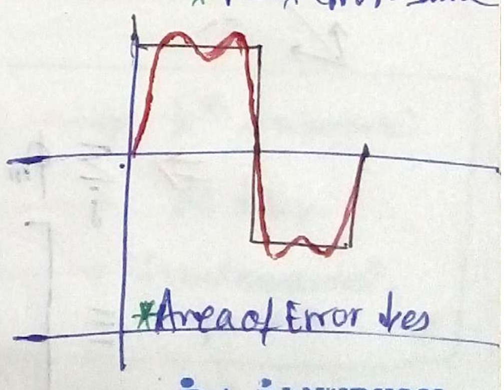
$$P_{\varepsilon_2} = \frac{1}{T_0}\int_0^{T_0} |\varepsilon_2(t)|^2 dt = \frac{J_2}{T_0}$$

$$\frac{\partial J_2}{\partial b_1^{(2)}} = 0 \qquad \frac{\partial J_2}{\partial b_2^{(2)}} = 0$$

$$b_1^{(2)} = \frac{4A}{\pi} = b_1^{(1)} \qquad b_2^{(2)} = 0.$$

$$\frac{2}{T_0}\int_0^{T_0} x(t)\sin(\Omega_0 t)\,dt$$

* Max. error = Same



* Area of Error dec

$$x^3(t) = b_1^{(3)}\sin(\Omega_0 t) + b_2^{(3)}\sin(2\Omega_0 t) + b_3^{(3)}\sin(3\Omega_0 t)$$

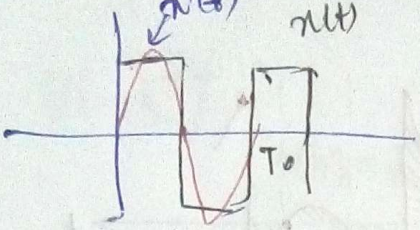
AE:-

$$P_{\varepsilon_3} = \text{—} = \frac{J_3}{T_0}$$

$$b_1^{(3)} = \frac{4A}{\pi} \qquad b_2^{(3)} = 0$$

$$b_3^{(3)} = \frac{4A}{3\pi}$$

22/08/16



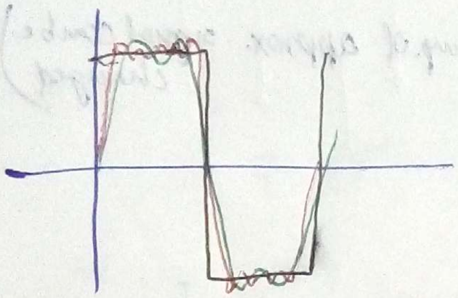
$$x^1(t) = b_1^{(1)} \sin(\Omega_0 t)$$

$$\Omega_0 = \frac{1}{T}$$

$$x^2(t) = b_1^{(2)} \sin(\Omega_0 t) + b_2^{(2)} \sin(2\Omega_0 t)$$

$$x^3(t) = b_1^{(3)} \sin(\Omega_0 t) + b_2^{(3)} \sin(2\Omega_0 t) + b_3^{(3)} \sin(3\Omega_0 t)$$

$$b_1^{(3)} = \frac{4A}{\pi} = b_1^{(1)} \qquad b_2^{(3)} = b_2^{(2)} = 0$$



$$b_3^{(3)} = \frac{4A}{3\pi} = \frac{b_1^{(3)}}{3}$$

TRIGONOMETRIC FOURIER SERIES

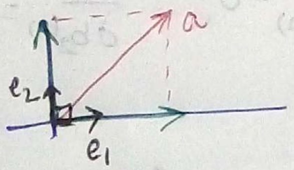
$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\Omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\Omega_0 t)$$

$$= a_0 + \sum_{k=1}^{\infty} a_k \phi(t) + \sum_{k=1}^{\infty} b_k \psi_k(t)$$

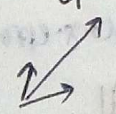
Synthesis eq^n

ϕ_k, ψ_k : basic func^n.

$$x(t) = x(t + T_0) \qquad T_0 = 1/\Omega_0$$



(Sinusoids) Orthogonal fun^n :-



$$\int_{t_0}^{t_0+T_0} \underbrace{\cos(m\Omega_0 t)}_{\phi_m(t)} \underbrace{\cos(k\Omega_0 t)}_{\phi_k(t)} dt = \begin{cases} T_0/2, & m = k \\ 0, & m \neq k \end{cases}$$

$$\Rightarrow \sum_j \phi_m(t_j)\, \phi_k(t_j) = 0$$

$$\equiv \begin{bmatrix} \phi_m(t_1) \\ \phi_m(t_2) \\ \vdots \\ \phi_m(t_j) \end{bmatrix}^T \begin{bmatrix} \phi_k(t_1) & \phi_k(t_2) & \cdots \end{bmatrix}^T = 0$$

$$\int_{t_0}^{t_0+T_0} \underbrace{\sin(m\Omega_0 t)}_{\psi_m(t)} \underbrace{\sin(k\Omega_0 t)}_{\psi_k(t)}\, dt = \begin{cases} T_0/2, & m=k \\ 0, & m\neq k \end{cases}$$

* Weighted linear combination of Sine & Cosine f^n.

$$\int_{t_0}^{t_0+T_0} \cos(m\Omega_0 t)\, \sin(k\Omega_0 t)\, dt = 0$$

$$\int_{t_0}^{t_0+T_0} \underbrace{x(t)}_{\text{Refer Synthesis eq}^n} \cos(k\Omega_0 t)\, dt = \cancel{a_0 \int_{t_0}^{t_0+T_0} \cos(k\Omega_0 t)\, dt} + \int_{t_0}^{t_0+T_0} \sum_{l=1}^{\infty} a_l \cos(l\Omega_0 t)\cos(k\Omega_0 t)$$

$$+ \cancel{\sum_{l=1}^{\infty} b_l \int_{t_0}^{t_0+T_0} \sin(l\Omega_0 t)\cos(k\Omega_0 t)\, dt}$$

$$\boxed{\frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)\, dt = a_k \quad A3}$$

(when $l=k$, $a_l \int - = (T_0/2)\, a_k$)

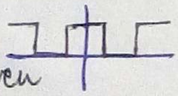
Analysis equations

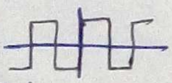
IIly

$$\boxed{\frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)\, dt = b_k. \quad A2}$$

Also,

$$\boxed{\int_{t_0}^{t_0+T_0} x(t)\, dt = a_0 T_0} \Rightarrow \boxed{\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\, dt = a_0 \quad A1}$$

$x(t)$ 
= even

$x(t)$ 
= odd

Read opp:-
How to minimize calculations in such cases

$\cos(k\Omega_0 t), \sin(k\Omega_0 t)$	$\longrightarrow$	k^{th} harmonic
a_0	$\longrightarrow$	dc term
$K=1$	$\longrightarrow$	Fundamental components

EXPONENTIAL FOURIER SERIES (EFS)

24/08/16

Synthesis eqn $\quad x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{jk\Omega_0 t}$

Real valued sig.

$x^*(t) = x(t) \quad | \; x(t) \in R.$

$x(t) = \sum_{k=-\infty}^{\infty} C_k^* e^{-jk\Omega_0 t} = x^*(t)$

$C_{-k} = C_k^*$

$x(t) = C_0 + \sum_{k=1}^{\infty} \left[C_k e^{jk\Omega_0 t} + C_k^* e^{-jk\Omega_0 t} \right]$

$(C_k \cos(k\Omega_0 t) + j C_k \sin(k\Omega_0 t)) + (C_k^* \cos(k\Omega_0 t) - j C_k^* \sin(k\Omega_0 t))$

$= (C_k + C_k^*)\cos(k\Omega_0 t) + (C_k - C_k^*) j \sin(k\Omega_0 t)$

$x(t) = C_0 + 2\sum_{k=1}^{\infty} Re\left\{ C_k e^{jk\Omega_0 t} \right\}$

It is Real valued.

Trigonometric Fourier Series (TFS) $\leftrightarrow$ EFS

$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\Omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\Omega_0 t)$

$C_0 = a_0$

$(C_k + C_k^*)\cos(k\Omega_0 t) + j(C_k - C_k^*)\sin(k\Omega_0 t)$

$= a_k \cos(k\Omega_0 t) + b_k \sin(k\Omega_0 t)$

$C_k + C_k^* = a_k \qquad jC_k - jC_k^* = b_k$

$\therefore \quad \boxed{C_k = \frac{1}{2}(a_k - jb_k) \qquad C_k^* = \frac{1}{2}(a_k + jb_k)}$

k → small (not capital)

$\int_{t_0}^{t_0+T_0} x(t) e^{-jn\Omega_0 t} dt = \int_{t_0}^{t_0+T_0} \sum_{k=-\infty}^{+\infty} C_k e^{jk\Omega_0 t} e^{-jn\Omega_0 t} dt \quad - (1)$

$\int_{t_0}^{t_0+T_0} e^{j(k-n)\Omega_0 t} dt = \begin{cases} T_0, & k=n \\ 0, & k \neq n \end{cases}$ (Orthogonal Function)

* x(t) should be represented by weighted linear combination of orthogonal basis function (for easy calculations).

① continue . . .

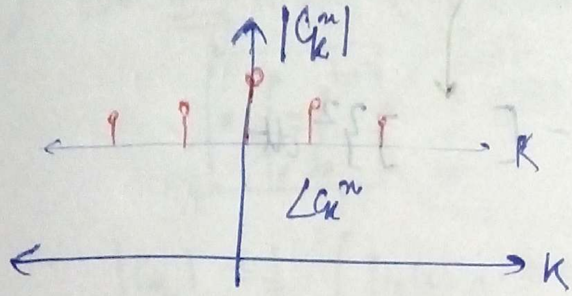
$$= \sum_{k=-\infty}^{\infty} c_k \int_{t_0}^{t_0+T_0} e^{-j(k-n)\Omega_0 t}\, dt.$$

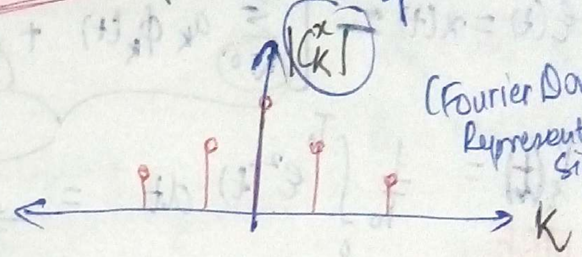
$$\Rightarrow \frac{1}{T} \int_{t_0}^{t_0+T_0} x(t)\, e^{-jn\Omega_0 t}\, dt = c_n.$$

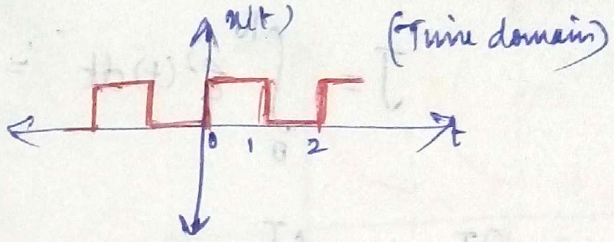
Analysis eq^n.

$\ldots \; |C_{-1}^x|, \; |C_0^x|, \; |C_1^x| \; \ldots$

$\frac{1}{2} \quad 1 \quad 2 \quad 1 \quad \frac{1}{2}$





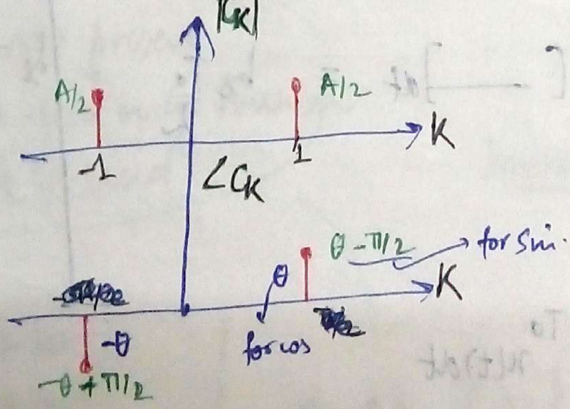


$$x(t) = A\cos(\Omega_0 t + \theta)$$

$$= \frac{A}{2} e^{j\theta} e^{j\Omega_0 t} + \frac{A}{2} e^{-j\theta} e^{-j\Omega_0 t}$$

$$c_1 = \frac{A}{2} e^{j\theta} \qquad c_2 = 0,\; c_3 = 0 \ldots$$

$$c_{-1} = \frac{A}{2} e^{-j\theta} \qquad c_{-2} = 0,\; c_{-3} = 0 \ldots$$



$$x(t) = A\sin(\Omega_0 t + \theta)$$

$$= \frac{A}{2j} e^{j\theta} e^{j\Omega_0 t} - \frac{A}{2j} e^{-j\theta} e^{-j\Omega_0 t}$$

$$= \frac{A}{2} e^{j(\theta - \pi/2)} e^{j\Omega_0 t} + \frac{A}{2} e^{-j(\theta - \pi/2)} e^{-j\Omega_0 t}$$

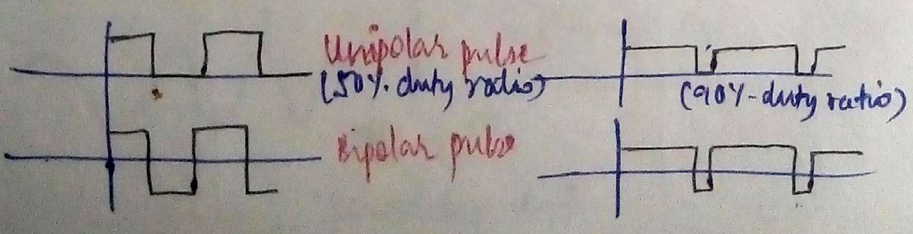
(Draw both magnitude & phase spectra)

$$c_1 = \frac{A}{2} e^{j(\theta - \pi/2)} \Rightarrow |c_1| = \frac{A}{2}$$

$$\angle c_1 = \theta - \pi/2$$

$$c_2 = \frac{A}{2} e^{-j(\theta - \pi/2)} \Rightarrow |c_1| = A/2$$

$$\angle c_2 = -(\theta - \pi/2)$$



TFS

$$S \rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} \cos(k\Omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\Omega_0 t)$$

$$= \sum_{k=0}^{\infty} a_k \phi_k(t) + \sum_{l=1}^{\infty} b_l \psi_l(t)$$

$$\varepsilon(t) = x(t) - \underbrace{\left[\sum_{k=0}^{\infty} a_k \phi_k(t) + \sum_{l=1}^{\infty} b_l \psi_l(t)\right]}$$

$$P_\varepsilon(t) = \frac{1}{T_0}\int_0^{T_0} \varepsilon^2(t)\, dt = \frac{J}{T_0}$$

MSE

$$J = \int_0^{T_0} \varepsilon^2(t)\, dt = \int_0^{T_0} \{x(t) - [\quad]\}^2 dt$$

$$\frac{\partial J}{\partial a_k} = 2\int^{T} \varepsilon(t)\phi_k(t)\, dt = 0 \quad -(1)$$

$$\frac{\partial J}{\partial a_0} = 2\int_0^{T_0} \varepsilon(t) \cdot 1 \cdot dt = 0 \quad -(3)$$

$$\frac{\partial J}{\partial b_l} = 2\int_0^{T} \varepsilon(t)\psi_l(t)\, dt = 0 \quad -(2)$$

$$\Rightarrow \int_0^{T_0} x(t)\phi_k(t)\, dt = \int_0^{T_0} \phi_k(t)\left[\sum_{m=0}^{\infty} a_m \phi_m(t)\right] dt$$

$(k \neq 0)$

$$= \frac{T_0}{2} a_k.$$

$$\Rightarrow \int_0^{T_0} x(t)\psi_l(t)\, dt = \int \psi_l(t)[\underline{\qquad}]dt = b_l \frac{T_0}{2}$$

$$\Rightarrow \int_0^{T_0} x(t)\, dt - a T_0 = 0$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt$$

$$a_k = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(k\Omega_0 t)$$

$$b_l = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(k\Omega_0 t)\, dt$$

25/08/16

TFS (Compact Form)

$$x(t) = d_0 + \sum_{k=1}^{\infty} d_k \cos(k\Omega_0 t + \theta_k)$$

$$= d_0 + \sum_{k=1}^{\infty} \{[d_k \cos(\theta_k)] \cos(k\Omega_0 t)\} + \sum_{k=1}^{\infty} \{[-d_k \sin(\theta_k)] \sin(k\Omega_0 t)\}$$

$$\left.\begin{aligned} d_0 &= a_0 \\ a_k &= d_k \cos(\theta_k) \end{aligned}\right\} \quad b_k = -d_k \sin(\theta_k)$$

$$d_k = \sqrt{a_k^2 + b_k^2}$$

$$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right)$$

Pg-195-201 Opp

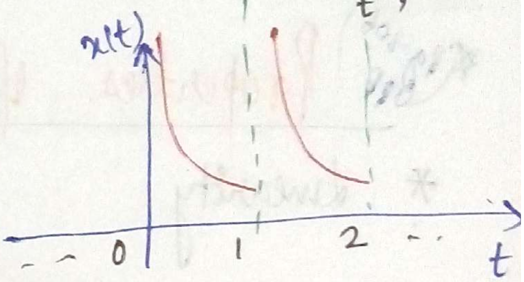
Dirichlet Cond^n

① Over any period, $x(t)$ should be ~~exactly~~ absolutely integratable.

$$\boxed{* \int_{T_0} \equiv \int_{t_0}^{t_0+T_0}}$$

$$\int_{t_0}^{t_0+T_0} |x(t)|\, dt < \infty$$

$x(t) = \frac{1}{t}$, $0 < t \le 1$

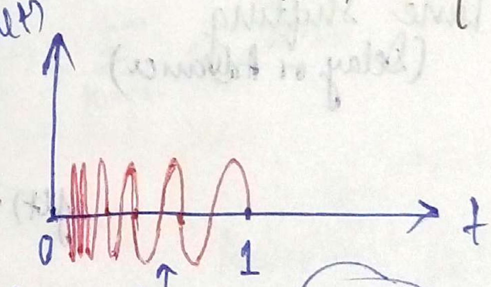


$$|c_k| \le \frac{1}{T} \int_T |x(t) e^{-jk\Omega_0 t}|\, dt$$

$$= \frac{1}{T} \int_T |x(t)|\, dt.$$

② For any given span of time, variation of signal $\boxed{x(t)}$ is bounded.
(Any finite interval)

$x(t) = \sin(2\pi/t)$, $0 < t \le 1$.



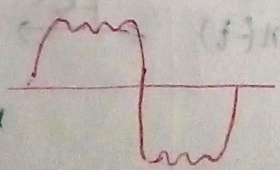
Instantaneous frequency changes very fast.
(decreases)

Cannot use Fourier Series

When all freq are present all the time, then only Fourier Series has to be used.

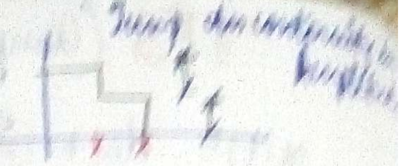
Ω_0	0-30
$3\Omega_0$	30-45
$5\Omega_0$	45-60

$\Omega_0, 3\Omega_0, 5\Omega_0$ | 0-60s.
(present all time)

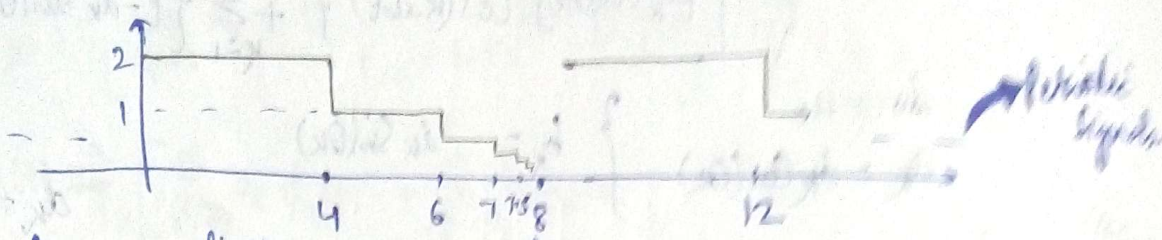


Completely diff

(3) No. of discontinuities and amount (jump) of discontinuities must be finite



(eg)



Amount of discontinuity = finite

No. of ——— = infinite

$\Rightarrow$ Cannot use fourier series

(Pg-206 BP) Properties of FS :-

* Linearity

$$x(t) \xleftrightarrow{FS} C_k^x$$

Syn. eqn — Ana. eqn — exp. F.S

$$y(t) \xleftrightarrow{FS} C_k^y$$

$$z(t) = a\,x(t) + b\,y(t) \xleftrightarrow{FS} a C_k^x + b C_k^y = C_k^z$$

(a, b → Finite) Complex/Real

* Time Shifting (Delay or Advance)

$$x(t) \xleftrightarrow{FS} C_k^x$$

$$y(t) = x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} C_k^x = C_k^y$$

Only phase changes / Phase Spectra changes

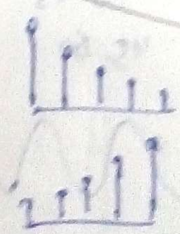
Mag. = same

but Mag. spectra remains unchanged.

* Time Reversal

$$x(t) \xleftrightarrow{FS} C_k^x$$

$$x(-t) \xleftrightarrow{FS} C_{-k}^x$$



F.S → $x(t) = \sum$ (infinite / not) Weighted linear combination of signal independent components / basis functions

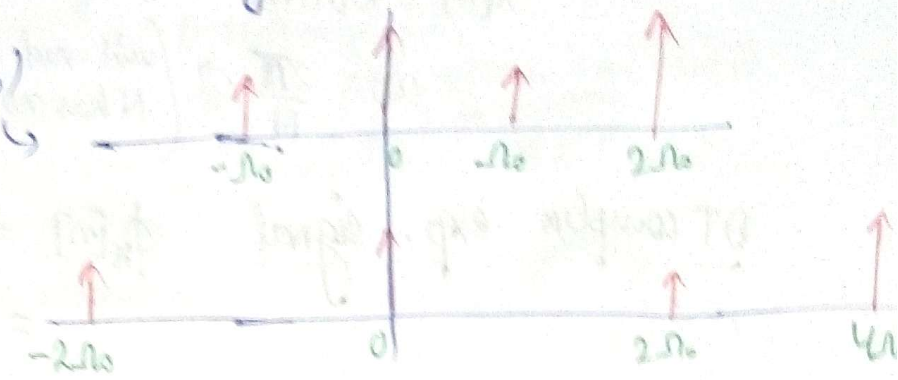
$$x(\alpha t) = \sum_{k=-\infty} C_k e^{jk\alpha\Omega_0 t}$$

Complex → $k = -\infty$ to ∞

Real valued f^n → $k = 0$ to ∞

Fourier coeff. remain unchanged.

(location changes)



Properties of FS

29/08/16

* Multiplication

$$x(t) \xleftrightarrow{FS} C_k^x$$

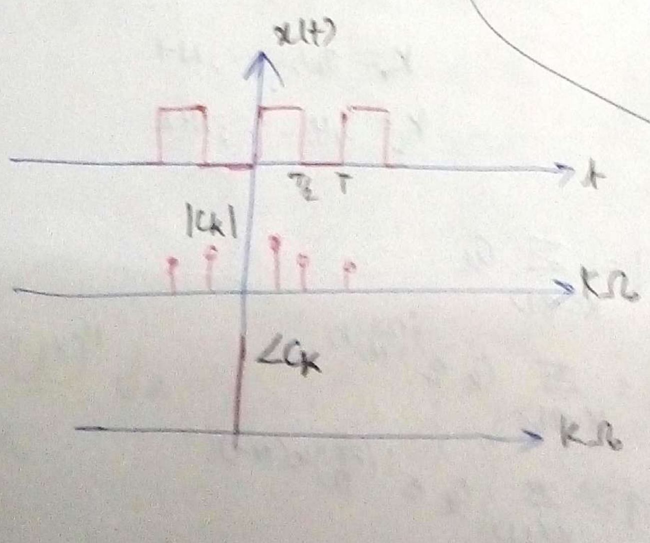
$$y(t) \xleftrightarrow{FS} C_k^y$$

$$z(t) = x(t)y(t) \xleftrightarrow{FS} h_k^z = \sum_{l=-\infty}^{+\infty} C_l^x C_{k-l}^y$$

Parseval's Rel^n:-

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |C_k|^2$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t}$$



$C_k e^{jk\Omega_0 t}$

Power → $|C_k|^2$

Power in both domains are constant

Time & freq domain

Av. power of a periodic signal (Time domain) = sum of av. power of all subdomains

Cont-Time Signal Fourier Series (CTSFS)

DTFS

- Periodic signal

$x[n] = x[n+N]$ Periodicity $= N$.

$\omega_0 = \frac{2\pi}{N}$ (unit = rad as N has no units → It is digital frequency)

DT complex exp. signal $\phi_k[n] = e^{jk\omega_0 n}$ $\{k \in Z\}$

$= e^{jk(\frac{2\pi}{N})n}$

$= \cos(k\_) + j\sin(k\_)$

(Harmonically Related)

$\phi_{(k+rN)}[n] = \phi_k[n]$

$e^{j(\frac{2\pi}{N})n\{k+rN\}}$

$= e^{j(\frac{2\pi}{N})nk} * e^{j(\frac{2\pi}{N})nrN}$ $\rightarrow 1$

Synthesis equation $x[n] = \sum_{k=\langle N\rangle} c_k \phi_k[n]$ ⇒ Finite Linear Combination

↳ finite no. of N consecutive terms/numbers.

$= \sum_{k=\langle N\rangle} c_k e^{jk(\frac{2\pi}{N})n}$

$k = 0, 1, \dots, N-1$

$k = 3, 4, \dots, N+2$

$x[0] = x[N]$

$x[1] = x[N+1]$

N equations
$$\begin{cases} x[0] = \sum_{k=\langle N\rangle} c_k \\ x[1] = \sum_{k=\langle N\rangle} c_k e^{j(\frac{2\pi}{N})k} \\ \vdots \\ x[N-1] = \sum_{k=\langle N\rangle} c_k e^{j(\frac{2\pi}{N})k(N-1)} \end{cases}$$

x Bar ← $\underline{x}$ = column vector

$$\underline{x} = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{array}{c|c} & k \rightarrow \\ \hline n \downarrow & \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & \dots & \vdots \\ \vdots & & \phi_k[n] & \vdots \\ 1 & \dots & \dots & e^{j\frac{2\pi}{N}(N-1)} \end{bmatrix} \end{array} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

$$\underline{x} = \underline{\underline{D}}\ \underline{C}$$

$\hookrightarrow$ double bar

$\Rightarrow \underline{C} = \underline{\underline{D}}^{-1}\underline{x}$ $\rightarrow$ Assuming inverse exists

$D \rightarrow$ Orthogonal

$\Rightarrow \cancel{\underline{C} = D^H x}$ $\rightarrow$ Hermition Matrix $\rightarrow$ see bottom of page

$$\sum_{n=\langle N\rangle} \underbrace{e^{j\left(\frac{2\pi}{N}\right)kn}}_{\phi_k[n]} = \begin{cases} N, & k=0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$D^H D = \begin{bmatrix} x & & \\ & x & \\ & & x \\ & & & x \end{bmatrix}$$

$$\begin{bmatrix} d_1 & d_2 & d_k & d_N \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Check whether $\underline{\underline{D}}$ is harmition or not.

$$D^H: \begin{bmatrix} \underline{d}_1^H \\ d_2^H \\ d_3^H \\ \vdots \\ d_N^H \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

$$D^H D = \begin{bmatrix} d_1^H d_1 & d_1^H d_2 & d_1^H d_3 & d_1^H d_4 \\ d_2^H d_1 & d_2^H d_2 & d_2^H d_3 & d_2^H d_4 \\ d_3^H d_1 & d_3^H d_2 & d_3^H d_3 & d_3^H d_4 \\ d_4^H d_1 & d_4^H d_2 & d_4^H d_3 & d_4^H d_4 \end{bmatrix}$$

$$(\underline{d}_k)^H \underline{d}_l = \sum_{n=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)nk} \cdot e^{j\left(\frac{2\pi}{N}\right)nl} = \sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)(-k+l)n}$$

$$= \begin{cases} N, & l=k \\ 0, & \text{otherwise} \end{cases}$$

eg $\rightarrow d_1^H d_2$

$$4\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\underline{\underline{D}}^H \underline{\underline{D}} = \begin{bmatrix} N & & \\ & N & \\ & & N \end{bmatrix} = N\underline{\underline{I}}$$

$$\frac{1}{N}\underline{\underline{D}}^H = \underline{\underline{D}}^{-1} \qquad \underline{C} = \frac{1}{N}\underline{\underline{D}}^H \underline{x}$$

ANUBHAV JAIN NOTES

Analysis eqn.

$$\boxed{c_k = \sum_{n=\langle N\rangle} x[n]\, e^{-j(\frac{2\pi}{N})kn}}$$

31/08/16

D.T.F.S.

Synthesis Eqn: $x[n] = \sum_{k=\langle N\rangle} c_k e^{j(\frac{2\pi}{N})kn}$

Analysis Eqn: $c_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j(\frac{2\pi}{N})nk}$

Linearity Prop.

$$\begin{cases} x[n] \xleftrightarrow{FS} c_k^x \\ y[n] \xleftrightarrow{FS} c_k^y \\ ax[n]+by[n] \xleftrightarrow{FS} a c_k^x + b c_k^x \end{cases}$$

* $z[n] = x[n]y[n] \xleftrightarrow{FS} \sum_{l=\langle N\rangle} c_l^x c_{k-l}^y = c_k^z$

$c_k = \{\frac{1}{2}, 1, 3\}$

$c_{-k} = \{3, 1, \frac{1}{2}\}$

Parseval's Reln

$$\boxed{\frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |c_k^x|^2}$$

* eigen signal → o/p is signal multiplied by a scalar.

z → eigen ⇒ z^n = eigen.

FS & LTI sys.

$x(t) = e^{st} \longrightarrow y(t) = H(s)e^{st}$

$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$

System function

Impulse Response of system

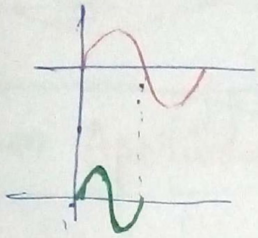
→ I/p = $x(t) = \delta(t)$ → Impulse Input

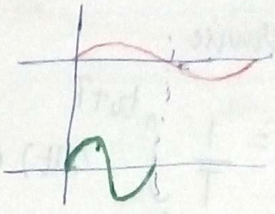
O/p = $y(t) = h(t)$ → Impulse Response

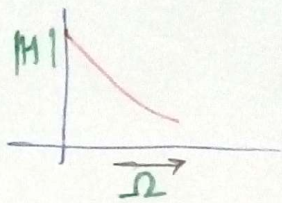
$s = j\Omega$

$$\underbrace{H(j\Omega)}_{\text{Frequency Response of the LTI system}} = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\Omega\tau}\, d\tau$$

$$e^{j2\pi\omega t} \longrightarrow \boxed{LTI} \longrightarrow H(j2\pi)\, e^{j2\pi t}$$





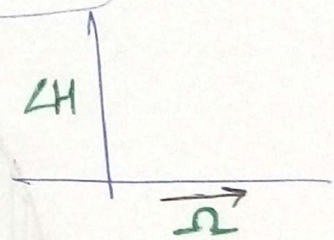


Characteristics of I/p & ch. of system decides o/p.

$$x[n] = z^n \longrightarrow y[n] = H(z)\, x[n]$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

System function



$|z| = 1$

$z = e^{j\omega}$

$$\underbrace{H(e^{j\omega})}_{\text{Freq. Response of the sys.}} = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t} \xrightarrow[\text{system}]{\text{LTI}}$$

$$x(t) \longrightarrow \boxed{} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(j\Omega_0 k)\, e^{jk\Omega_0 t}$$

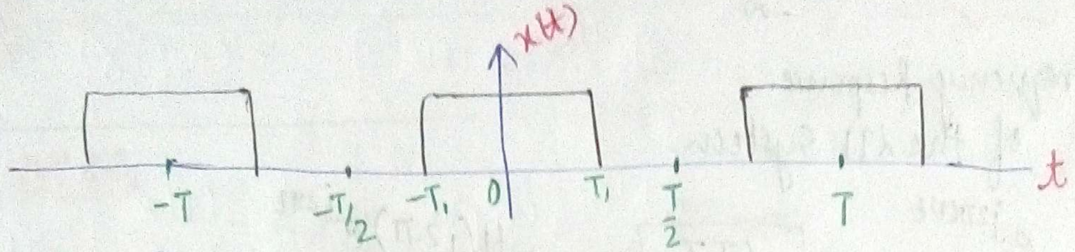
if H is complex, there will simply be a "phase shift" as it will add to power of $e^{jk\Omega_0 t}$

$H = |H| e^{j\phi}$

Freq = unchanged for LTI system

Amp. & phase → May change.

Rep^n of Aperiodic Sig

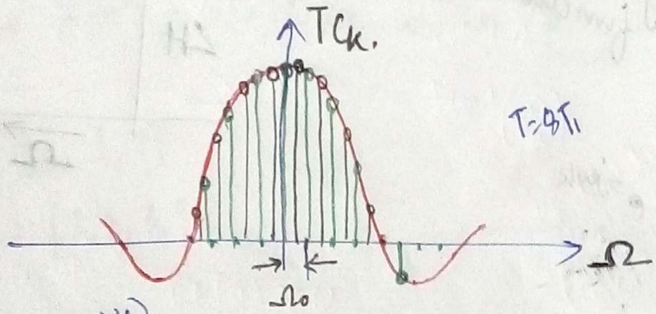


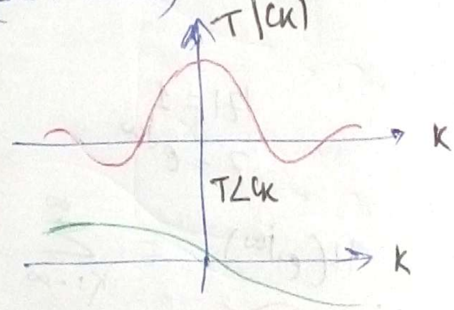
$$x(t) = \begin{cases} 1 ; & |t| < T_1 \\ 0 ; & \text{otherwise} \end{cases}$$

$$C_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, e^{-jk\Omega_0 t}\, dt$$

$$C_k = \frac{2 \sin(k\Omega_0 T_1)}{k\Omega_0 T}, \quad \Omega_0 = \frac{2\pi}{T}$$

$$T C_k = \frac{2 \sin(\Omega T_1)}{\Omega} \qquad (\Omega = k\Omega_0)$$

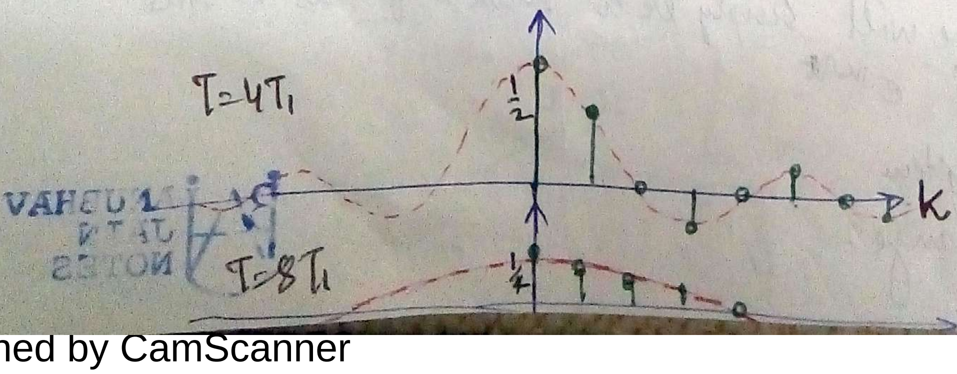


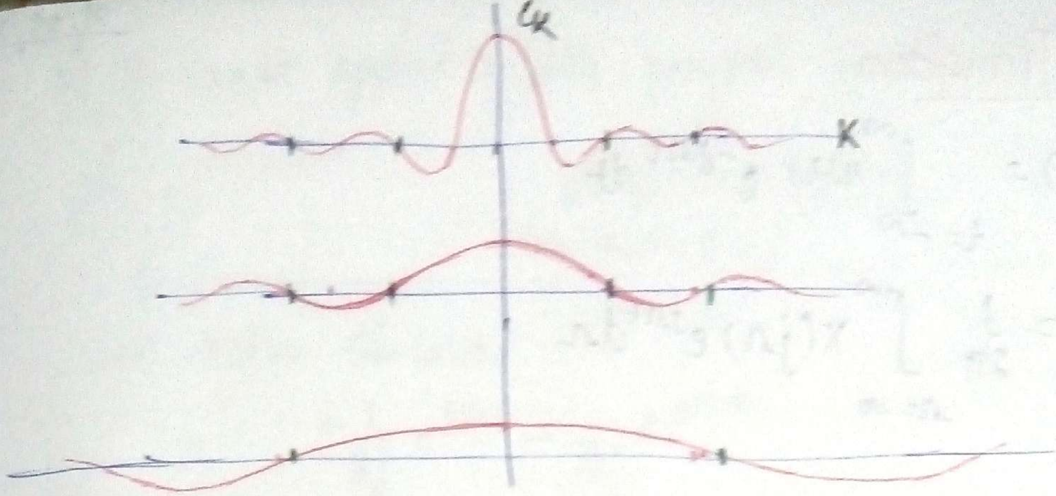


$T = 4T_1$, $\Omega_0 = \frac{\pi}{2T_1}$ (rad/s), $C_k = \frac{\sin(k\pi/2)}{k\pi}$, $C_0 = 1/2$

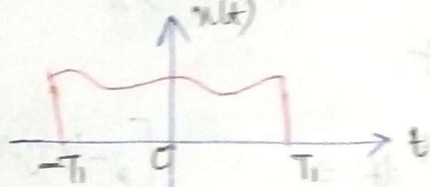
$T = 8T_1$, $\Omega_0 = \frac{\pi}{4T_1}$, $C_k = \frac{\sin(k\pi/4)}{k\pi}$, $C_0 = 1/4$

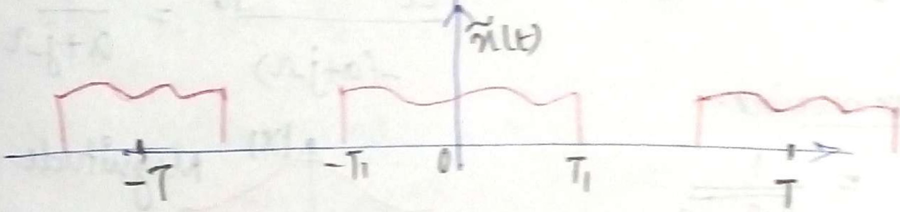
$T = 16T_1$, $\Omega_0 = \frac{\pi}{8T_1}$, $C_k = \frac{\sin(k\pi/8)}{k\pi}$, $C_0 = 1/8$





$T \to \infty$ Aperiodic





S $\quad \tilde{x}(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\Omega_0 t}$ Synthesis eq^n.

λ $\quad c_k = \frac{1}{T}\int_{-T/2}^{T/2} \overset{=x(t)}{\tilde{x}(t)}\, e^{-jk\Omega_0 t}\, dt$ Analysis eq^n., $\quad \Omega_0 = 2\pi/T$

$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\Omega_0 t}\, dt$

Define :- * $\boxed{X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt}$

* $\boxed{c_k = \frac{1}{T} X(j\Omega)\Big|_{\Omega = k\Omega_0}}$ Analysis Eq^n.

Fourier Transform Pair

$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\Omega_0) e^{jk\Omega_0 t}$

$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\Omega_0)\, e^{jk\Omega_0 t}\, \Omega_0$

$T \to \infty \quad \Omega_0 \to 0$

* $\boxed{x(t) = \tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega) e^{j\Omega t}\, d\Omega}$ Synthesis eq^n.

ANUBHAV JAIN NOTES

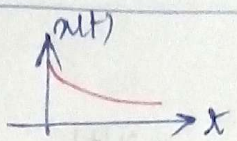
Scanned by CamScanner

Fourier Transform

t to Ω — Forward Transform: $X(j\Omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$

Inverse/Reverse Transform (Ω to t): $x(t) = \frac{1}{2\pi} \int_{\Omega=-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$

$x(t) = e^{-at}\, u(t)$



If a = Real → a > 0

$X(j\Omega) = \int_0^\infty e^{-at}\, e^{-j\Omega t}\, dt$

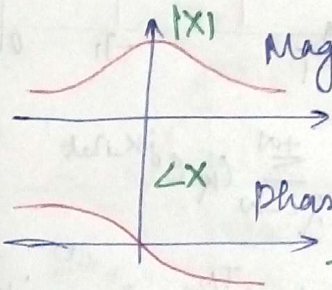
$= \int_0^\infty e^{-(a+j\Omega)t}\, dt = \frac{\left[e^{-(a+j\Omega)t}\right]_0^\infty}{-(a+j\Omega)} = \frac{1}{a+j\Omega}, a>0.$

⇒ 'X' can be a complex number.

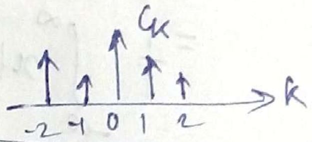
$|X(j\Omega)| = \frac{1}{\sqrt{a^2+\Omega^2}}$

$\angle X(j\Omega) = -\tan^{-1}\left(\frac{\Omega}{a}\right)$

Characteristics of a low pass filter (low freq. signal)



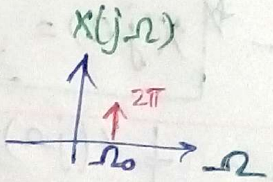
If a = Complex: $Re(a) > 0$.



F.T. to periodic signal.

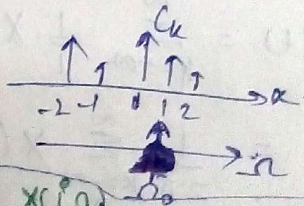
periodic $x(t)$

$X(j\Omega) = 2\pi\, \delta(\Omega - \Omega_0)$



$x(t) = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \cancel{2\pi}\, \delta(\Omega-\Omega_0)\, e^{j\Omega t}\, d\Omega.$

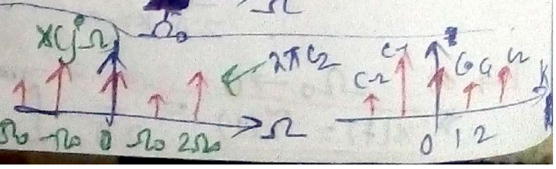
$x(t) = e^{j\Omega_0 t}$



$X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k\, \delta(\Omega - k\Omega_0)$

$x(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\Omega_0 t}$

(line spectra)



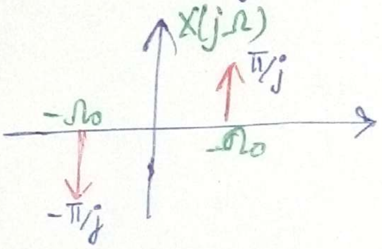
F.T. can be applied to both periodic & aperiodic signals
↓
we get line spectra
(Similar to F.S.)

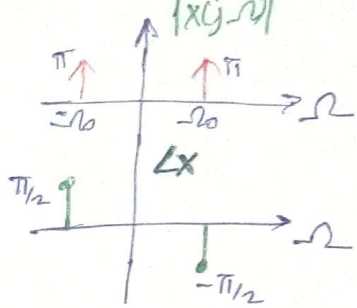
---

* $x(t) = \sin(\Omega_0 t)$

$$= \frac{1}{2j} e^{j\Omega_0 t} - \frac{1}{2j} e^{-j\Omega_0 t}$$

$\rightarrow c_1 = \frac{1}{2j}$, $c_{-1} = \frac{-1}{2j}$ all other c_k's are zeroes.

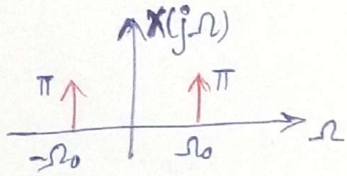




* $x(t) = \cos(\Omega_0 t)$

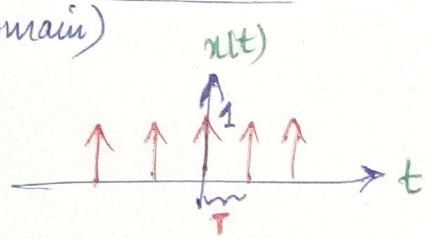
$$= \frac{1}{2} e^{j\Omega_0 t} + \frac{1}{2} e^{-j\Omega_0 t}$$

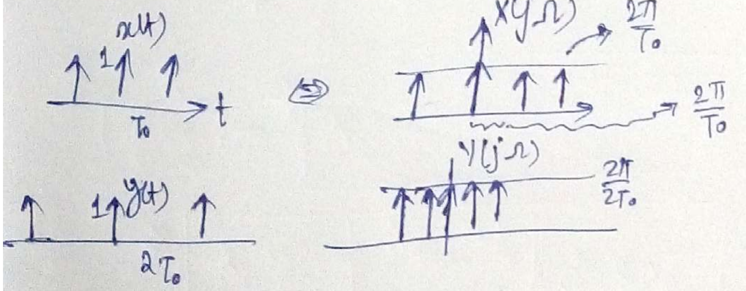
$\rightarrow c_1 = c_{-1} = 1/2$



Impulse Train (in Time domain or freq domain)

$x(t) =$





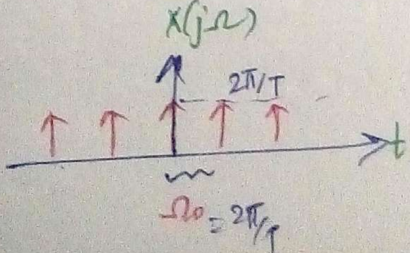
$$c_k = \frac{1}{T} \int_{t_0}^{t_0+T_0} \delta(t)\, e^{-jk\Omega_0 t}\, dt$$

$t_0 \to \int_{-T/2}^{T/2}$, $\delta(t) \to x(t)$

$$c_k = \frac{1}{T} \delta(0)\, e^{-(0)} = \frac{1}{T}$$

$$X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi\, c_k\, \delta(\Omega - k\Omega_0)$$

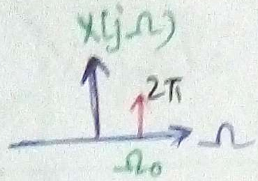
$$\sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\Omega - k\Omega_0)$$

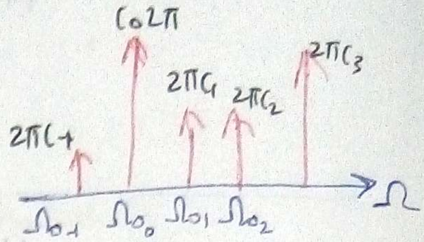


ANUBHAV JAIN NOTES

05/09/16

FT for periodic signal





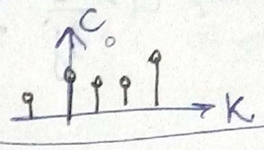
$$X(j\Omega) = 2\pi\,\delta(\Omega-\Omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\Omega-\Omega_0)\, e^{j\Omega t}\, d\Omega$$

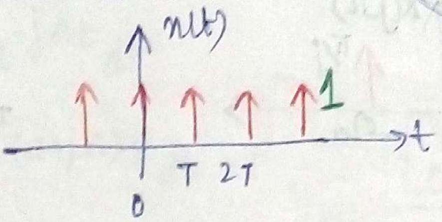
$$= e^{j\Omega_0 t}$$

$$\Rightarrow X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k\, \delta(\Omega-\Omega_0)$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\Omega_0 t}$$



Impulse Train

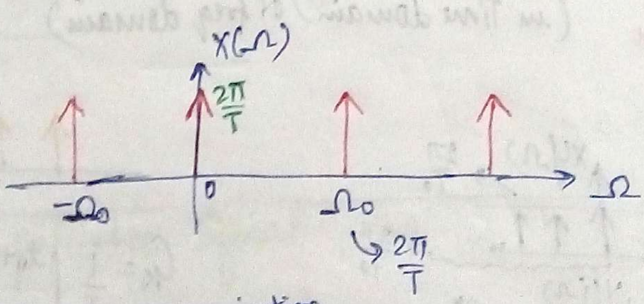


$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

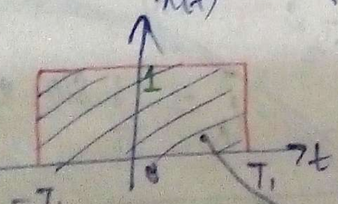
Remember.
$\boxed{X(\Omega) \equiv X(j\Omega)}$

CTFS : $c_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\Omega_0 t}\, dt = \frac{1}{T}$

CTFT $\rightarrow$ $X(j\Omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\,\delta(\Omega - k\Omega_0)$, $\quad \Omega_0 = \frac{2\pi}{T}$



(T → smaller
$\frac{2\pi}{T}$ → bigger)



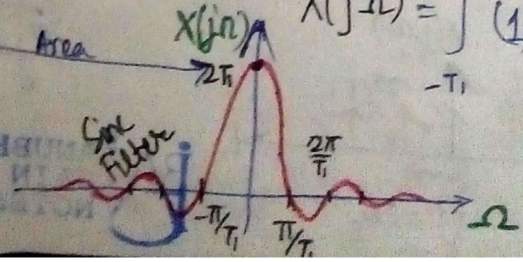
no characterisation in Time domain

~~Brick wall type filter~~

freq domain

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & \text{otherwise} \end{cases}$$

Area

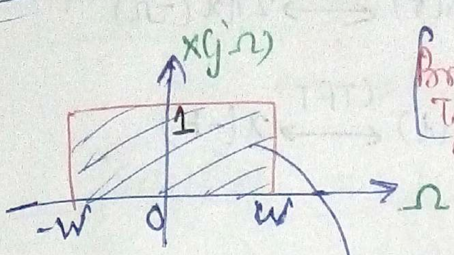


$$X(j\Omega) = \int_{-T_1}^{T_1} (1)\, e^{-j\Omega t}\, dt = \frac{2\pi \sin(\Omega T_1)}{\Omega T_1}$$

$$= 2\pi\, \mathrm{sinc}\left(\frac{\Omega T_1}{\pi}\right)$$

$$\mathrm{Sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

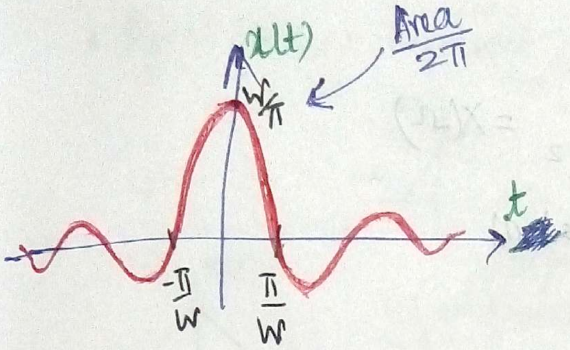
Filter → Frequency domain spectrum



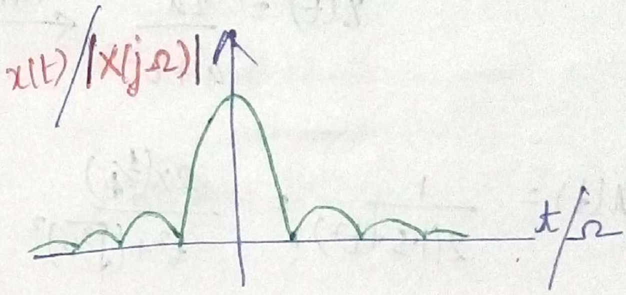
Brick wall Type Filter

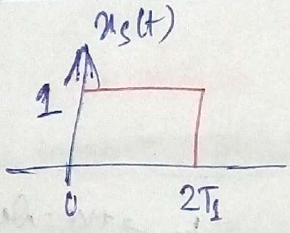
$$X(j\Omega) = \begin{cases} 1, & |\Omega| < W \\ 0, & \text{Otherwise} \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega t}\, d\Omega = \frac{\sin(Wt)}{\pi t}$$

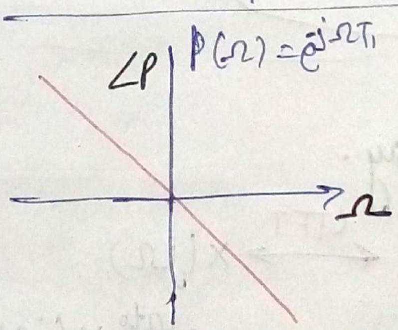


$$x(t) = \frac{W}{\pi}\,\text{Sinc}\left(\frac{Wt}{\pi}\right)$$





$$X_s(j\Omega) = \quad |X_s(j\Omega)| = |X(j\Omega)|$$

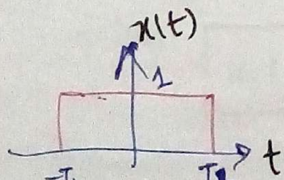


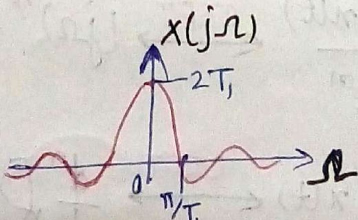
$$\angle P(j\Omega) = -T_1\Omega.$$

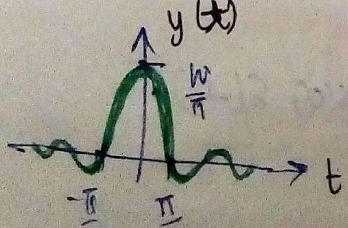
Phase Delay or Phase Lag

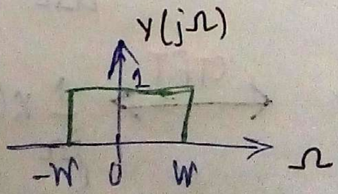
07/04/16

TIME FREQUENCY DUALITY









Duality

$$x(t) \xleftrightarrow{CTFT} X(\Omega) \iff X(t) \xleftrightarrow{CTFT} 2\pi x(-\Omega)$$

$$x(t) \xleftrightarrow{CTFT} X(F) \iff X(t) \xleftrightarrow{CTFT} x(-F)$$

$$x(t) = \frac{1}{3+2t^2} \xrightarrow{CTFT} X(\Omega) = ?$$

known → $$x(t) = e^{-a|t|} \xleftrightarrow{CTFT} \frac{2a}{a^2+\Omega^2} = X(\Omega)$$

$$X(t) = \frac{2a}{a^2+t^2} \xleftrightarrow{CTFT} 2\pi\, e^{-a|-\Omega|}$$

$$x(t) = \frac{1}{2(1.5+t^2)} = \frac{2\times(\frac{1}{4})}{t^2+(\sqrt{3/2})^2} = \frac{1}{4\sqrt{\frac{3}{2}}} \frac{2\sqrt{3/2}}{t^2+(\sqrt{\frac{3}{2}})^2} = \frac{1}{(2\sqrt{6})}\left\{\frac{2\sqrt{\frac{3}{2}}}{t^2+(\sqrt{\frac{3}{2}})^2}\right\}$$

$$\Rightarrow X(-\Omega) = \frac{1}{(2\sqrt{6})} \cdot 2\pi\, e^{-\sqrt{3/2}\,|\Omega|}$$

~~Duality~~ Delay.

t_0 → +ve - Delay
→ -ve - Advance

$$x(t) \xleftrightarrow{CTFT} X(j\Omega)$$

(Linear Phase System)

$$x(t-t_0) \longleftrightarrow e^{-j\Omega t_0} X(j\Omega)$$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{CTFT} (j\Omega)^n X(j\Omega)$$

$$(-jt)^n x(t) \xleftrightarrow{CTFT} \frac{d^n}{d\Omega^n}[X(j\Omega)]$$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{CTFT} \frac{1}{j\Omega} X(j\Omega) + \pi X(0)\, \delta(\Omega)$$

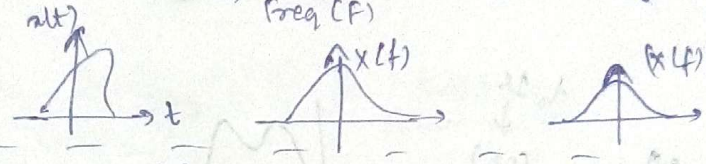
Parseval's Rel^n (Energy = Const in diff domains).

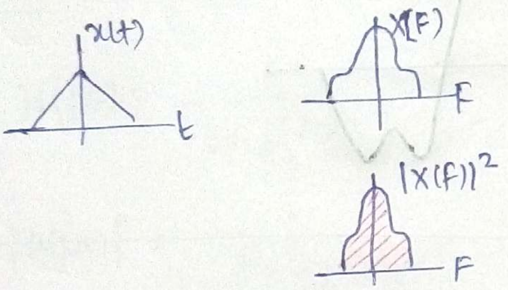
$$x(t) \xleftrightarrow{CTFT} X(\Omega)$$

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\infty} |X(\Omega)|^2 d\Omega = \int_{F=-\infty}^{\infty} |X(F)|^2 dF$$

$|X(F)|^2$ → Energy Spectral Density ESD → J/Hz = Energy per unit frequency

- Signal same, Represent → Time (t), omega (Ω), Freq (F). But energy remains same.





$$\boxed{ESD = |X(F)|^2 = \frac{|X(j\Omega)|^2}{2\pi}}$$

Convolution

O/p in Time domain: $y(t) = h(t) * x(t)$ ← convolution

O/p in Freq domain: $Y(j\Omega) = H(j\Omega) \cdot X(j\Omega)$ ← multiplication

$X(j\Omega) \rightarrow \boxed{H_1(j\Omega)} \rightarrow \boxed{H_2(j\Omega)} \rightarrow Y(j\Omega)$ } Write everything in Freq-domain only (in Block diagram)

$X(j\Omega) \rightarrow \boxed{H_1(j\Omega) \cdot H_2(j\Omega)} \rightarrow Y(j\Omega)$

$X(j\Omega) \rightarrow \boxed{H_2(j\Omega)} \rightarrow \boxed{H_1(j\Omega)} \rightarrow Y(j\Omega)$

Time Delay System

No gain or attenuation.

$$x(t) \longrightarrow y(t) = x(t - t_0)$$

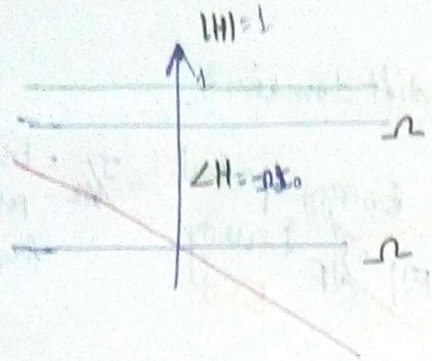
$$Y(j\Omega) = e^{-j\Omega t_0} X(j\Omega)$$

$$= \boxed{H(j\Omega)} X(j\Omega)$$

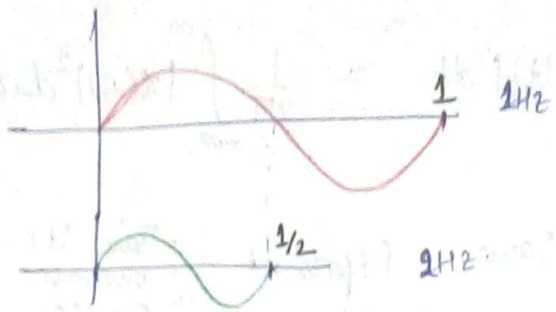
$H(j\Omega) = 1\, e^{-j\Omega t_0}$

$|H(j\Omega)| = 1$ (Graphs next page)

$\angle H(j\Omega) = -\Omega t_0$



2Hz wave Delay will be double from delay caused by 1Hz wave



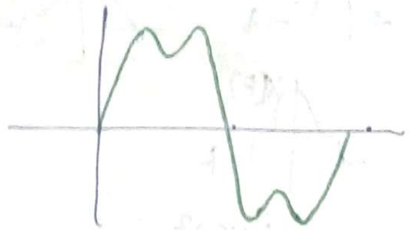
If Time ~~delay~~ fixed = Same

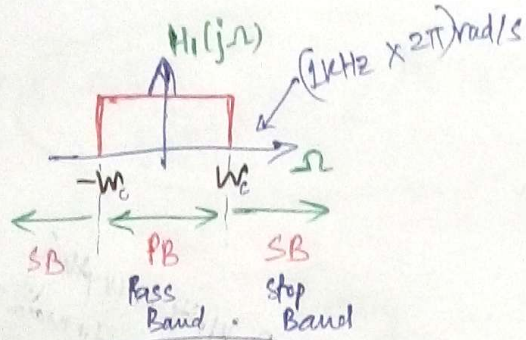
$t_0 = 1s$

Phase delay → double ~~of~~ 1Hz (2π)

~~Hmm~~ for 2Hz (4π)

(phase delay $\propto \Omega$)





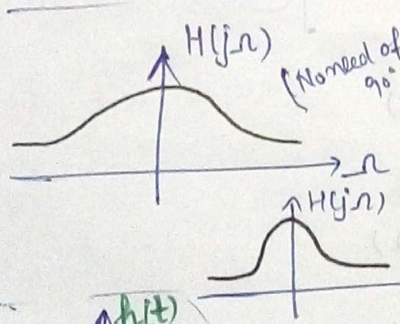
$W = 2\pi \times 10^3$ rad/s

$X(j\Omega) \rightarrow \boxed{H(j\Omega)} \rightarrow Y(j\Omega)$

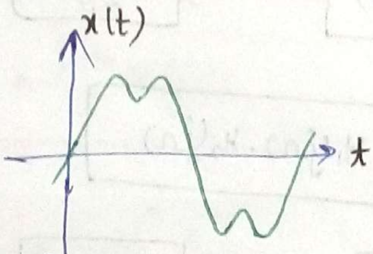
$x(t) = \sin(\Omega_0 t) + \frac{1}{3}\sin(3\Omega_0 t)$

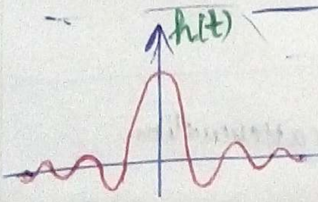
$\Omega_0 = 2\pi \times 500$ r/s

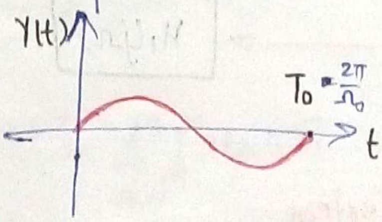
$3\Omega_0 = 2\pi \times 1500$ r/s



(No need of exact 90°)







$T_0 = \frac{2\pi}{\Omega_0}$

(3rd Harmonic is suppressed. only Fundamental remains)

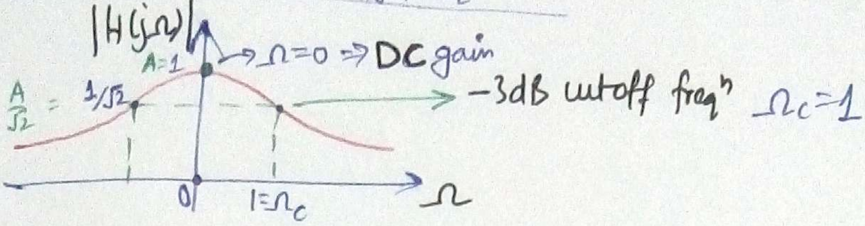
$h(t) \neq 0$, $t < 0$

★ System is causal if system's impulse response $h(t) = 0$ for $t < 0$

DTFT — Not in Midsem

$$H(j\Omega) = \frac{1}{1+j\Omega}$$

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\Omega^2}}$$



$$20\log\left(\frac{1}{\sqrt{2}}\right) = 20\log_{10}(2^{-1/2}) = -3.01$$

$$H(j\Omega) = \frac{1}{1+j(\Omega/\Omega_c)}$$

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^2}}$$

$$20\log_{10}\{|H()|\} = 20\log_{10}\left\{\frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^2}}\right\}$$

gain $= \frac{1}{\sqrt{2}}$ at $\Omega = \Omega_c$
↓
-3 dB cutoff freq.