Fourier Series expansion of a periodic f^n.

$$x(t) = x(t \pm T) \qquad \left.\begin{array}{l} T = \text{fund. period} \\ \omega = \text{fund. frequency} \end{array}\right\} T = \frac{2\pi}{\omega}.$$

$$x(t) = c_0 + \sum_{K=1}^{\infty} a_K \cos K\omega t + \sum_{K=1}^{\infty} b_K \sin K\omega t$$

- To get c_0

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} c_0\,dt + \frac{1}{T}\sum_{K=1}^{\infty}\int_{-T/2}^{T/2} a_K \cos K\omega t\,dt + \sum_{K=1}^{\infty}\int_{-T/2}^{T/2} b_K \sin K\omega t\,dt$$

$$= c_0 \frac{1}{T}\int_{-T/2}^{T/2} dt + 0 + 0 = c_0$$

$$\therefore \boxed{c_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt}$$ → nothing but the average value of $x(t)$ over a period.

- To get a_K

$$\int_{-T/2}^{T/2} x(t)\cos m\omega t\,dt = c_0\int_{-T/2}^{T/2}\cos m\omega t\,dt + \sum_{K=1}^{\infty}\int_{-T/2}^{T/2} a_K \cos m\omega t \cos K\omega t\,dt$$

$$+ \sum_{K=1}^{\infty}\int_{-T/2}^{T/2} b_K \cos m\omega t \sin K\omega t\,dt$$

$$= c_0 \times 0 + \int_{-T/2}^{T/2} a_m \cos^2 m\omega t\,dt + 0$$

$$\int_{-T/2}^{T/2} x(t)\cos m\omega t\,dt = \frac{a_m}{2}\int_{-T/2}^{T/2}(1 - \cos 2m\omega t)\,dt = \frac{a_m}{2}\cdot T$$

$$\therefore a_m = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos m\omega t\,dt$$

$$\therefore \boxed{a_K = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos K\omega t\,dt}$$

To get b_K :

$$\int_{-T/2}^{T/2} x(t)\sin m\omega t\,dt = c_0\int_{-T/2}^{T/2}\sin m\omega t\,dt + \sum_{K=1}^{\infty}\int_{-T/2}^{T/2} a_K \sin m\omega t\cos K\omega t\,dt$$

$$+\sum_{K=1}^{\infty}\int_{-T/2}^{T/2} b_K \sin m\omega t\sin K\omega t\,dt$$

or $$\int_{-T/2}^{T/2} x(t)\sin m\omega t\,dt = c_0\times 0 + 0 + \int_{-T/2}^{T/2} b_m \sin^2 m\omega t\,dt$$

$$= \frac{b_m}{2}\int_{-T/2}^{T/2}(1-\cos 2m\omega t)\,dt$$

$$= \frac{b_m}{2}\cdot T$$

$$\therefore\ b_m = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin m\omega t\,dt$$

i.e., $$\boxed{b_K = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin K\omega t\,dt}$$

—•—

To summarise :- For periodic $x(t)$

$$x(t) = c_0 + \sum_{K=1}^{\infty} a_K\cos K\omega t + \sum_{K=1}^{\infty} b_K\sin K\omega t$$

where

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$a_K = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos K\omega t\,dt$$

and $$b_K = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin K\omega t\,dt$$

Fourier Series in complex form:-

We Know:- $$x(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos k\omega t + \sum_{k=1}^{\infty} b_k \sin k\omega t$$

or $$x(t) = c_0 + \sum_{k=1}^{\infty} \left(a_k \cos k\omega t + b_k \sin k\omega t\right)$$

or $$x(t) = c_0 + \sum_{k=1}^{\infty} \left[a_k \frac{\left(e^{jk\omega t} + e^{-jk\omega t}\right)}{2} + b_k \frac{\left(e^{jk\omega t} - e^{-jk\omega t}\right)}{2j}\right]$$

or $$x(t) = c_0 + \sum_{k=1}^{\infty} \left[a_k \frac{\left(e^{jk\omega t} + e^{-jk\omega t}\right)}{2} - jb_k \frac{\left(e^{jk\omega t} - e^{-jk\omega t}\right)}{2}\right]$$

or $$x(t) = c_0 + \sum_{k=1}^{\infty} \left[\frac{(a_k - jb_k)}{2} e^{jk\omega t} + \frac{(a_k + jb_k)}{2} e^{-jk\omega t}\right]$$

or $$\boxed{x(t) = c_0 + \sum_{k=1}^{\infty} \frac{(a_k - jb_k)}{2} e^{jk\omega t} + \sum_{k=1}^{\infty} \frac{(a_k + jb_k)}{2} e^{-jk\omega t}}$$

Now let $c_k = \frac{a_k - jb_k}{2}$ then $c_k^* = \frac{a_k + jb_k}{2}$

$$\therefore x(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega t} + \sum_{k=1}^{\infty} c_k^* e^{-jk\omega t}$$

• Let us now call $c_k^* = c_{-k}$ then

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega t} + \sum_{k=1}^{\infty} c_{-k} e^{-jk\omega t}$$

or $$x(t) = \sum_{k=-\infty}^{-1} c_k e^{jk\omega t} + c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega t}$$

or $$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}}$$

Complex Fourier Series:-

we got

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

How to get c_k ?

$$\int_{-T/2}^{T/2} x(t)\, e^{-jm\omega t}\, dt = \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} c_k\, e^{j(k-m)\omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \int_{-T/2}^{T/2} c_k\, e^{j(k-m)\omega t}\, dt$$

Now $\int_{-T/2}^{T/2} c_k\, e^{j(k-m)\omega t}\, dt = 0$ if $k \neq m$

and for $k=m$ $\int_{-T/2}^{T/2} c_m\, e^{j(m-m)\omega t}\, dt = c_m \int_{-T/2}^{T/2} dt = c_m T$ for $k=m$

$$\therefore \int_{-T/2}^{T/2} x(t)\, e^{-jm\omega t}\, dt = c_m T$$

or $$c_m = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jm\omega t}\, dt$$

or $$\boxed{c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt}$$

Alternatively:

$$c_k = \frac{1}{2}\left(a_k - jb_k\right) = \frac{1}{2}\frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos k\omega t\, dt - \frac{2}{T}\frac{j}{2}\int_{-T/2}^{T/2} x(t)\sin k\omega t\, dt$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$ same as above.

(ii) If K is odd

$$a_K = \frac{1}{T}\int_0^{T/2} x(t)\cos K\omega t\, dt - \frac{1}{T}\int_0^{T/2} x(\tau)\cos(K\omega\tau + K\pi)\, d\tau$$

or $$a_K = \frac{1}{T}\int_0^{T/2} x(t)\cos K\omega t\, dt + \frac{1}{T}\int_0^{T/2} x(\tau)\cos K\omega\tau\, d\tau$$

or $$a_K = \frac{1}{T}\int_0^{T/2} x(t)\cos K\omega t\, dt \qquad K = \text{odd}$$

only odd harmonics & $C_0 = 0$ if $x(t) \rightarrow$ H.W. Symm.

Similarly :-

$$b_K = \frac{1}{T}\int_0^{T} x(t)\sin K\omega t\, dt = \frac{1}{T}\int_0^{T/2} x(t)\sin K\omega t\, dt + \frac{1}{T}\int_{T/2}^{T} x(t)\sin K\omega t\, dt$$

or $$b_K = \frac{1}{T}\int_0^{T/2} x(t)\sin K\omega t\, dt - \frac{1}{T}\int_{T/2}^{T} x\left(t-\frac{T}{2}\right)\sin K\omega t\, dt$$

$\tau = t - \frac{T}{2}$

$$\therefore\ b_K = \frac{1}{T}\int_0^{T/2} x(t)\sin K\omega t\, dt - \frac{1}{T}\int_0^{T/2} x(\tau)\sin(K\omega\tau + K\pi)$$

(i) If K is <u>even</u> $$b_K = \frac{1}{T}\int_0^{T/2} x(t)\sin K\omega t\, dt - \frac{1}{T}\int_0^{T/2} x(\tau)\sin K\omega\tau\, d\tau$$

or $b_K = 0$

(ii) If K is <u>odd</u> $$b_K = \frac{1}{T}\int_0^{T/2} x(t)\sin K\omega t\, dt + \frac{1}{T}\int_0^{T/2} x(\tau)\sin K\omega\tau\, d\tau$$

or $$\boxed{b_K = \frac{2}{T}\int_0^{T/2} x(t)\sin K\omega t\, dt} \qquad b_K \neq 0,\ K = \underline{\text{odd}}.$$

$\therefore$ If $x(t)$ has H.W Symmetry

$C_0 = 0$

$a_K \neq 0$ & $b_K \neq 0$ only for $K = \text{odd}$

all even harmonics will be present.

Finally if $x(t)$ is Q. wave symmetric.

i, e., $x(t) \rightarrow$ H.W Symmetry + either even or odd

Even Q.wave symm :- $c_0 = 0$ only cosine terms $a_k \neq 0$
$x(t)$ $K = \text{odd}$

Odd Qwave symm:- $c_0 = 0$ only sine terms $b_k \neq 0$
$K = \text{odd}$.

A periodic f^n $x(t)$ can be even or odd

$$x(t) = c_0 + \sum_{K=1}^{\infty} a_K \cos K\omega t + \sum_{K=1}^{\infty} b_K \sin K\omega t = \sum_{K=-\infty}^{\infty} c_K e^{jK\omega t}$$

- If $x(t)$ is even then $\boxed{b_K = 0 \text{ must be zero.}}$
  only c_0 and a_K exist.
- If $x(t)$ is odd then then $\boxed{c_0 = a_K = 0 \text{ (must be)}}$.
  only b_K exist.

- If $x(t)$ is half-wave symmetric then $\boxed{c_0 = 0}$ (must be) and only odd harmonics

$x(t)$ is halfwave symmetric
if $x(t) = -x(t \pm T/2)$

Quarter Wave Symmetry: $x(t)$ is QW.Sym.
if (i) $x(t)$ is either even or odd
(ii) $x(t)$ has halfwave symmetry $[x(t) = -x(t \pm T/2)]$.

(i) $x(t) \rightarrow$ even $\longrightarrow$ $c_0 = 0$, $b_K = 0$ and a_K exist for $K =$ odd

(ii) $x(t) \rightarrow$ odd $\longrightarrow$ $c_0 = 0$, $a_K = 0$ and b_K exist for $K =$ odd.

① If $x(t)$ is even $b_k = 0$; only cosine & c_0 present.

② If $x(t)$ is odd $a_k = c_0 = 0$; only sine terms

③ If $x(t)$ has H.W sym. $c_0 = 0$

$$x(t) = -x(t - T/2)$$

$$c_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\int_0^{T/2} x(t)\,dt + \frac{1}{T}\int_{T/2}^{T} x(t)\,dt$$

$$\text{or } c_0 = \frac{1}{T}\int_0^{T/2} x(t)\,dt - \frac{1}{T}\int_{T/2}^{T} x(t - T/2)\,dt \longrightarrow \tau = t - T/2,\quad d\tau = dt$$

$$\text{or } c_0 = \frac{1}{T}\int_0^{T/2} x(t)\,dt - \frac{1}{T}\int_0^{T/2} x(\tau)\,d\tau$$

or $\boxed{c_0 = 0}$ proved.

~~Suppose x(t) has Q. wave Symmetry~~

~~which means~~ $x(t) \rightarrow$ H.W symmetry ~~+ either even or odd~~.

$\boxed{c_0 = 0}$ $\quad x(t) = -x(t - T/2)$

Let us examine a_K & b_K

Now

$$a_K = \frac{1}{T}\int_0^T x(t)\cos k\omega t\,dt = \frac{1}{T}\int_0^{T/2} x(t)\cos k\omega t\,dt + \frac{1}{T}\int_{T/2}^{T} x(t)\cos k\omega t\,dt$$

~~put~~

$$\text{or } a_K = \frac{1}{T}\int_0^{T/2} x(t)\cos k\omega t\,dt - \frac{1}{T}\int_{T/2}^{T} x(t - T/2)\cos k\omega t\,dt$$

put $\tau = t - T/2 \quad d\tau = dt$

$$\therefore a_K = \frac{1}{T}\int_0^{T/2} x(t)\cos k\omega t\,dt - \frac{1}{T}\int_0^{T/2} x(\tau)\cos k\omega(\tau + T/2)$$

$\boxed{\omega T = 2\pi}$

$$\text{or } a_K = \frac{1}{T}\int_0^{T/2} x(t)\cos k\omega t\,dt - \frac{1}{T}\int_0^{T/2} x(\tau)\cos(k\omega\tau + k\pi)\,dt$$

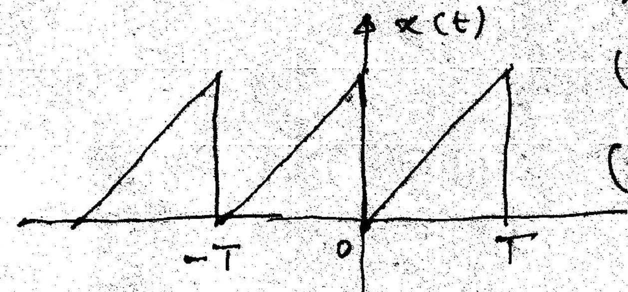
(i) If k is even

$$a_K = \frac{1}{T}\int_0^{T/2} x(t)\cos k\omega t\,dt - \frac{1}{T}\int_0^{T/2} x(\tau)\cos k\omega\tau\,dt = 0$$

$\therefore a_K = 0$ for k even $\rightarrow$ no even harmonics

Some Periodic signals and comments on the terms which will be present and type of harmonics

①



(i) $x(t) \rightarrow$ neither even nor odd

(ii) $x(t) \rightarrow$ not half wave symmetric

Conclusions: All terms (c_0, a_K, b_K are expected) with both even & odd harmonics.

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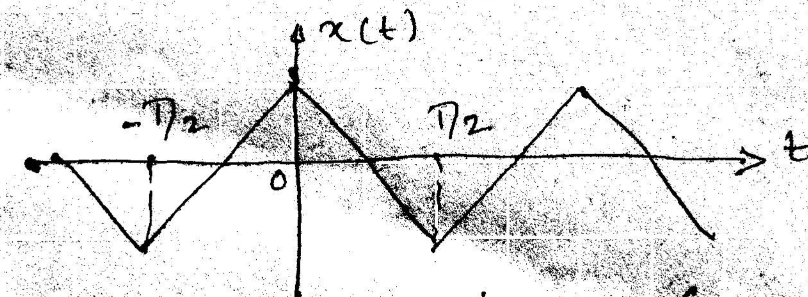
②



(i) $x(t)$ is even $\longrightarrow$ cosine terms. (no sine terms).

(ii) $x(t)$ is H.W sym. $\longrightarrow$ $c_0 = 0$, only odd harmonics.

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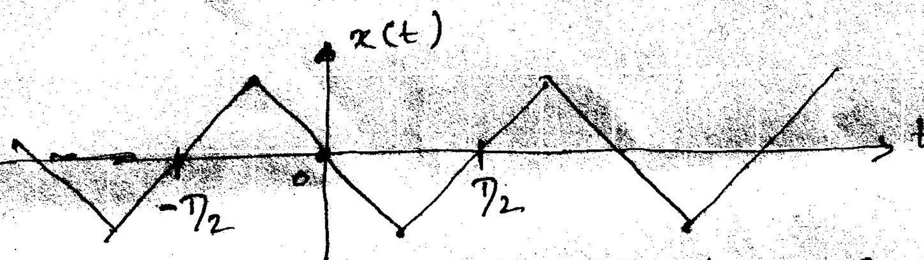
③



(i) $x(t)$ is half wave symmetry $c_0 = 0$, odd harmonics.

(ii) $x(t)$ is odd f.n $\rightarrow$ only sine terms (no cosine terms)

(iii) $x(t)$ has Q.W. sym.

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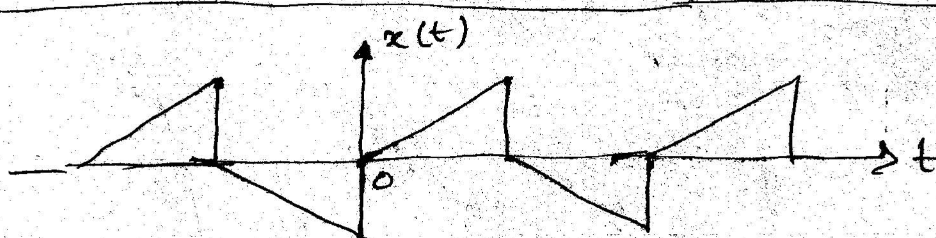
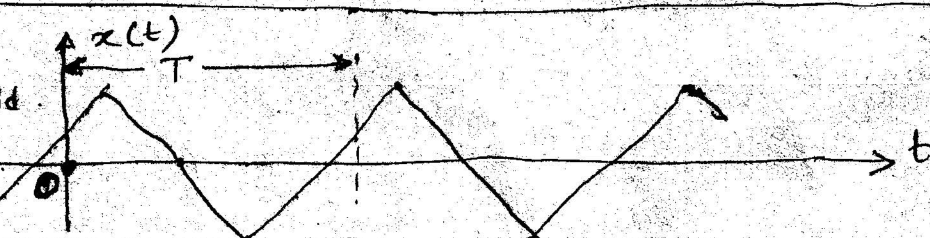
④



$x(t) \rightarrow$ neither even nor odd.

$x(t) \rightarrow$ H.W. Sym.
$c_0 = 0$
only odd harmonics.

$x(t) \rightarrow$ not Q.wave sym. $\rightarrow$ both sine & cosine terms with odd harmonics will be present.



(i) $x(t) \rightarrow$ neither even nor odd.

(ii) H.W Sym ✓
$c_0 = 0$
& odd harmonics.

(iii) not Q. wave sym. only sine & cosine terms with ~~w~~ with odd harmonics will be present.