① Prob-Stats | Post-MidSem | Anshul

Joint Distribution Function :-

{joint cumulative probability distribution function}

$$F(a,b) = P\{X \le a, Y \le b\} \quad \forall\ -\infty < a,b < \infty$$

$$F_X(a) = P(X \le a) \equiv F(a,\infty)$$
$$F_Y(b) = P(Y \le b) \equiv F(\infty,b)$$

Marginal Distributions of X & Y.

* $P\{X>a, Y>b\} = 1 - F_X(a) - F_Y(b) + F(a,b)$
* $P\{a_1 < x \le a_2,\ b_1 < y \le b_2\} = F(a_2,b_2) + F(a_1,b_1) - F(a_1,b_2) - F(a_2,b_1)$

Joint Probability Mass function :-

{for discrete Random Variables}

$$p(x,y) = P(X=x, Y=y)$$

$$p_X(x) = P(X=x) = \sum_{y:p(x,y)>0} p(x,y)$$
$$p_Y(y) = P(Y=y) = \sum_{x:p(x,y)>0} p(x,y)$$

→ Marginal probability mass function

Note :- X and Y are said to be jointly continuous if there exists $f(x,y)$, for all real x & y, having property that for every set C in 2D plane,

$$P\{(X,Y)\in C\} = \iint_{(x,y)\in C} f(x,y)\,dx\,dy$$ ②

~~# Jointly~~ ~~Continuous~~

Jointly Probability density function:

$$C = \{(x,y) : x\in A, y\in B\}$$

$$P\{X\in A, Y\in B\} = \int_B\int_A f(x,y)\,dx\,dy$$

Note :-

* $F(a,b) = P\{X\in(-\infty,a],\ Y\in(-\infty,b]\}$
$$= \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$$

* $$\boxed{f(a,b) = \frac{\partial^2}{\partial a\,\partial b}F(a,b)}$$

* X & Y jointly continuous ⟹ they are individually continuous

* $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

→ Marginal density functions

Generalisation :- {n Random Var.}

$$F(a_1,a_2,\cdots a_n) = P\{X_1\le a_1, X_2\le a_2, \cdots X_n \le a_n\}$$

$$P\{(X_1,X_2,\cdots X_n)\in C\} = \iint_{(x_1,x_2,\dots,x_n)\in C}\cdots\int f(x_1,\dots,x_n)\,dx_1\,dx_2\cdots dx_n$$

$$P\{X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n\}$$
$$= \int_{A_n}\int_{A_{n-1}} \cdots \int_{A_1} f(x_1, \ldots, x_n)\, dx_1 dx_2 \cdots dx_n$$

The multinomial distribution:-

Sequence of n independent and identical experiments is performed.

$r \rightarrow$ No. of possible outcomes.

$X_i \rightarrow$ No. of experiments result in outcome number i. $(1 \le i \le r)$

$P_1, P_2, \ldots, P_r \rightarrow$ respective probabilities

$$\sum_{i=1}^{r} P_i = 1$$

$$P\{X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r\}$$
$$= \frac{n!}{n_1! n_2! \cdots n_r!} P_1^{n_1} P_2^{n_2} \cdots P_r^{n_r}$$
$$\forall \sum_{i=1}^{r} n_i = n$$

Independent Random Variable:-

$$P\{X \in A, Y \in B\} = P\{X \in A\} P\{Y \in B\}$$
$$P\{X \le a, Y \le b\} = P\{X \le a\} P\{Y \le b\}$$
$$F(a,b) = F_X(a) F_Y(b)$$
$$p(x,y) = p_X(x) p_Y(y)$$
$$f(x,y) = f_X(x) f_Y(y)$$

$\rho \rightarrow$ Correlation Coefficient

$$= \frac{Cov(X,Y)}{\sigma_x \sigma_y}$$

Sums of Independent Random Variables:-

$X+Y \rightarrow$ New Random Variable

$$F_{X+Y}(a) = P\{X+Y \le a\}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x) f_Y(y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)\, dx\, f_Y(y)\, dy$$
$$= \int_{-\infty}^{\infty} F_X(a-y) f_Y(y)\, dy$$

Convolution

$$f_{X+Y}(a) = \frac{d}{da} F_{X+Y}(a)$$
$$= \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$$

$$F_n(x) = P\{\underbrace{X_1 + \cdots + X_n}_{\text{Uniform Random variables }(0,1)} \le x\}$$

for $x \le 1$,

$$F_n(x) = \frac{x^n}{n!}, \quad 0 \le x \le 1$$

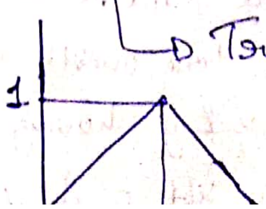
↳ Can be proved by induction

(A) Sum of two independent uniform random variables:-

$$f_X(a) = f_Y(b) = \begin{cases} 1 & ; 0 < a < 1 \\ 0 & ; o.w. \end{cases}$$

$$f_{X+Y}(a) = \begin{cases} a, & 0 \le a \le 1 \\ 2-a, & 1 < a < 2 \\ 0, & ow \end{cases}$$

↳ Triangular Distribution



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(B) Gamma Random Variables :-

$$f(x) = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \; ; \; 0 < x < \infty$$

$X \sim gamma(s, \lambda)$

$Y \sim gamma(t, \lambda)$

$X+Y \sim gamma(s+t, \lambda)$

$$f_{X+Y}(a) = \frac{\lambda^{s+t} a^{s+t-1} e^{-\lambda a}}{\Gamma(s+t)}$$

$X_1 + X_2 \cdots + X_n \sim gamma(\sum_1^n t, \lambda)$

(C) Normal Random Variables :-

$$N\left(\sum_1^n \mu_i, \sum_1^n \sigma_i^2\right)$$

$$Z = \frac{\sum X_i - \sum \mu_i}{\sqrt{\sum \sigma_i^2}}$$

(D) Poisson Random Variables :-

$$Poisson\left(\sum_1^n \lambda_i\right)$$

$$f_{X+Y}(a) = \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^a}{a!}$$

(E) Binomial Random Variables :-

$$Binomial(n+m, p)$$

$$P\{X+Y=k\} = p^k q^{n+m-k} \sum_{i=0}^{n} \binom{n}{i}\binom{m}{k-i}$$

(F) Geometric Random Variables :-

$$\overset{S_n}{\overset{\shortparallel}{P\{X_1+X_2+\cdots+X_n = k\}}} = \binom{k-1}{n-1} p^n (1-p)^{k-n} \; , \; k \geq n$$

P is same for all

(This is basically converted into Negative Binomial) $\sim NB(n, p)$

if P is different for all,

$$P(S_n = k) = \sum_{i=1}^{n} P_i q_i^{k-1} \prod_{j \neq i} \frac{P_j}{P_j - P_i}$$

(G) $Exp(\lambda) \longrightarrow Gamma(n, \lambda)$ $\leftarrow$ No. of...

Conditional Distribution :-

(A) Discrete Case :-

$$P_{X|Y}(x|y) = \frac{P(x,y)}{P_Y(y)}$$

$$F_{X|Y}(x|y) = P\{X \leq x \mid Y = y\}$$

$$= \sum_{a \leq x} P_{X|Y}(a|y)$$

(B) Continuous Case :-

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\, dx$$

$$F_{X|Y}(a|y) = \int_{-\infty}^{a} f_{X|Y}(x|y)\, dx$$

** # The Bivariate Normal Distribution

if $\mu_x, \mu_y, \sigma_x > 0, \sigma_y > 0, -1 < \rho < 1$, then

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}}$$

* Conditional

if $\boxed{Y=y}$ then X is normally distributed

with $\mu = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y)$

$$\sigma^2 = \sigma_x^2 (1-\rho^2)$$

* $f_X(x) \sim N(\mu_x, \sigma_x^2)$

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Properties of Expectation :-

$g(X,Y) \rightarrow$ function of two random variables X & Y.

$$E[g(X,Y)] = \sum_y \sum_x g(x,y)\, p(x,y)$$

(for Discrete Rand. Var.)

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

or

$\{\text{if } g(x,y) \geq 0\} \longrightarrow \iint_{y\ x} g(x,y) f(x,y).\, dx\, dy$

(for Continuous Rand. Var.)

$\rightarrow E(X+Y) = E(X) + E(Y)$

The Sample Mean :-

$X_1, \ldots, X_n \rightarrow$ independent and identically distributed r.v.'s having distribution fn^c F and expected value μ.

$$\boxed{\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}}$$

$$E(X) = E(\bar{X}) = \mu = E[X_i]$$

(A) Binomial random variables :-

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = np$$

$\forall\ X = X_1 + X_2 + \cdots + X_n$

(B) Negative Binomial R.V.,

$X = X_1 + X_2 + \cdots + X_r$

$$E(X) = \frac{r}{p}$$

(C) Hypergeometric R.V.,

$X = X_1 + X_2 + \cdots + X_m \rightarrow$ No. of white balls selected

$$E(X) = \frac{mn}{N}$$

$n \rightarrow$ balls selected
$N \rightarrow$ Total balls

~~Moments of the Number~~ of ~~Events that Occur~~ :-

Method of Transformation :-

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)\ |J|}$$

$$\forall\ J = \frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Product Moments :-

$$\boxed{\mu'_{r,s} = E(X^r Y^s)}$$

Central product Moments :-

$$\boxed{\mu_{r,s} = E[(x-\mu_x)^r (Y-\mu_y)^s]}$$

Covariance :-

$$Cov(X,Y) = E((X-\mu_x)(Y-\mu_y))$$

$$\boxed{Cov.(x,y) = E(XY) - E(x)E(Y)}$$

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov.(X,Y)$$

Note:- if X and Y are independent then,

$$Cov(X,Y) = 0$$

since, $E(XY) = E(X)E(Y)$

Correlation :-

$\rho \rightarrow$ Correlation Coefficient

$$\boxed{\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}}$$

$$-1 \le \rho_{X,Y} \le 1$$

Note:- For transformation,

$$U = \frac{X-a}{h} \qquad V = \frac{Y-b}{k}$$

$$\boxed{Cov(X,Y) = hk\, Cov(U,V)}$$

$$\boxed{\rho_{X,Y} = \rho_{U,V}}$$

↳ independent of change

Note:-

$Var(X_1+X_2+\cdots+X_n)$

$$= Var(X_1)+\cdots+Var(X_n) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} Cov(X_i,X_j)$$

Weibull Distribution :-

$X \sim W(\alpha,\beta)$

$$f_X(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} & , \forall x>0,\ \alpha>0,\ \beta>0 \\ 0 & , \text{o.w} \end{cases}$$

$$F_X(x) = \begin{cases} 1-e^{-\alpha x^\beta} & , x>0 \\ 0 & , x\le 0 \end{cases}$$

$$\text{Mean} = \mu = \frac{\Gamma\left(\frac{\beta+1}{\beta}\right)}{\alpha^{1/\beta}}$$

$$\mu_k = \frac{\Gamma\left(\frac{\beta+k}{\beta}\right)}{\alpha^{k/\beta}}$$

$$\text{Variance} = \sigma^2 = \alpha^{-2/\beta}\left[\Gamma\left(\frac{\beta+2}{2}\right) - \left(\Gamma\left(\frac{\beta+1}{\beta}\right)\right)^2\right]$$

Sampling Distribution

Population → group of objects under st.

Sample → A finite subset of statistical individuals

Sampling → The process of selecting a sample s.t.,
- ↳ to get max info. with minimum inputs;
- ↳ find limits of parameters and degree of confidence

Statistic → Statistical measures computed from sample observation

Sampling Distribution → Probability distribution of statistic. (eg:- $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$)

Central Limit Theorem :- (CLT)

$X_1, X_2, \ldots, X_n \rightarrow$ identical, independent r.v.'s (iid)

↳ each with mean μ and Var σ^2

let, $S_n = (X_1+X_2+\cdots+X_n)/n = \bar{X}$

mean $= \sum_{i=1}^{n} \mu_i = \mu$

Var $= \sigma^2/n$

$$\boxed{\text{if } Z_n = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}, \text{ then } \lim_{n\to\infty} \frac{F_n(z)}{\phi(z)} = 1}$$

(11)

i.e. if we have a large number of iid random variables then the Distribution for the sum of these random variables is approximately the same as Standard Normal distribution

$$Z_n \sim N(0,1)$$

$$\forall \; Z_n = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}, \; n \to \infty$$

→ Generally $n \geq 30$.

Random Sampling :–

$X_1, X_2, X_3, \ldots, X_n \to n$ iid rv's each with pdf $f(x)$

then $(x_1, x_2, \ldots, x_n)$ is a random sample if the joint distribution is,

$$\boxed{f(x_1, x_2, \ldots, x_n) = f(x_1)\, f(x_2) \ldots f(x_n)}$$

Statistic :– a function of random sample.

eg:- $\bar{X} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$ (Sample Mean)

$$\boxed{s^2 = \frac{1}{(n-1)} \Sigma (x_i - \bar{x})^2} \quad \text{(Sample Variance)}$$

$$\left.\begin{array}{l} X_{\left(\frac{n+1}{2}\right)} \text{ if } n \text{ is odd} \\ \dfrac{X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}}{2} \text{ if } n \text{ is even} \end{array}\right\} \text{(Sample Median)}$$

~~Sample Range : $X_{(n)} - X_{(1)}$~~

☆ iid → identically and independently distributed

(12)

Sampling distribution of Mean :–

we assume that the concerned population has a mean μ and variance σ^2.

if $(x_1, x_2, \ldots, x_n)$ is a random sample then,

$$\text{Sample Mean} = \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\boxed{E(\bar{x}) = \frac{1}{n} \Sigma E(x) = \mu}$$

$$\boxed{Var(\bar{x}) = \frac{\sigma^2}{n}}$$

↳ This is known as Standard Error of $\bar{x}$ (Sample Mean).

let $X_{11}, X_{12}, \ldots, X_{1n} \to$ iid Rv's with mean μ_1, and var. σ_1^2 and $X_{21}, X_{22}, \ldots X_{2n} \to$ iid Rv's with mean μ_2 and var. σ_2^2.

if $\bar{x}_1 = \frac{\Sigma x_{1i}}{n}$ & $\bar{x}_2 = \frac{\Sigma x_{2i}}{n}$ then,

$$\boxed{\frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} \to N(0,1) \text{ as } n_1 \to \infty, \; n_2 \to \infty}$$

Distributions arising from Normal Distribution :–

Chi-Square Distribution :–

$Z_1, Z_2, \ldots, Z_K \to$ normally and iid r.v.'s with $\mu = 0, \sigma^2 = 1$ each.

$$\chi^2 = Z_1^2 + Z_2^2 + \ldots + Z_K^2$$

$$\chi_n^2 = \sum_{1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 \quad \text{Chi-Square Variable}$$

$$f(x) = \begin{cases} \dfrac{x^{K/2-1} \cdot e^{-x/2}}{2^{K/2}\,\Gamma(K/2)}, & x > 0 \\ 0, & \text{o.w.} \end{cases}$$

Same as gamma with $\alpha = K/2$, $\lambda = 1/2$

$$\boxed{\frac{1}{2} \times \frac{\left(\frac{x}{2}\right)^{K/2-1} e^{-x/2}}{\Gamma(K/2)}}$$

$K \to$ degree of freedom.

Mean $= \mu = K$.

Variance $= \sigma^2 = 2K$

$$\boxed{P\{\chi_K^2 \geq \chi_{\alpha,K}^2\} = \int_{\chi_{\alpha,K}^2}^{\infty} f(x)\,dx = \alpha}$$

$M_{\chi^2}(t) = \text{MGF} = E(e^{tx}) = (1-2t)^{-K/2}$

Note:- χ^2 is identicle to Gamma$\left(\frac{K}{2}, \frac{1}{2}\right)$

$$f(x) = \frac{\frac{1}{2} e^{-\frac{x}{2}} \left(\frac{x}{2}\right)^{\frac{K}{2}-1}}{\Gamma\left(\frac{K}{2}\right)}, \quad x > 0$$

- Sum of Chi-Sq. random var.,

$$\chi^2_{\Sigma K_i}$$

t-Distribution :-

$Z \sim N(0,1)$

$V \to$ chi-Sq. r.v. with K dg. of freedom. (χ_K^2)

if Z and V are independent then,

$$\boxed{t_K = \frac{Z}{\sqrt{V/K}}}$$

$$\left(\frac{1}{\beta\left(\frac{1}{2}, \frac{K}{2}\right)}\right)$$

$$\boxed{f(t) = \frac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{K\pi}\,\Gamma\left(\frac{K}{2}\right)} \times \frac{1}{\left(\frac{t^2}{K}+1\right)^{\left(\frac{K+1}{2}\right)}}}$$

with K degree of freedom

Marginal pdf

Mean $= \mu = 0$

Var $= \sigma^2 = \frac{K}{K-2}$

Student's t-distribution :-

$X \sim N(0,1)$, $Y \sim \chi_n^2$ } independent then

$T = \dfrac{X}{\sqrt{Y/K}}$ is said to be Student's

transformation,

$T = \dfrac{X}{\sqrt{Y/K}}$, $U = Y \Rightarrow X = \sqrt{\frac{u}{K}}\,T$, $y = u$

$$\boxed{f_{T,U}(t,u) = \frac{1}{2^{\frac{K+1}{2}} \sqrt{K\pi}\,\Gamma\left(\frac{K}{2}\right)} e^{-\frac{u}{2}\left(1+\frac{t^2}{K}\right)} \cdot u^{\frac{K-1}{2}}}$$

$\forall\ -\infty < t < \infty$, $u > 0$

$\to$ T is symmetric at $t=0$

$\to$ odd order moments vanish.

F-Distribution :-

$W, Y \to$ Chi-Square r.v. with u, v degree of freedom

$$F = \frac{W/u}{Y/v}$$

$$\boxed{h(f) = \frac{\Gamma\left(\frac{u+v}{2}\right)\left(\frac{u}{v}\right)^{u/2} f^{\left(\frac{u}{2}\right)-1}}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)\left[\frac{u}{v}f+1\right]^{\frac{u+v}{2}}}}, \text{ o.w.}$$

Mean $= \mu = \dfrac{v}{v-2}$, $v > 2$

Var $= \sigma^2 = \dfrac{2v^2(u+v-2)}{u(v-2)^2(v-4)}$, $v > 4$

$$\boxed{P\{F \geq F_{\alpha,u,v}\} = \int_{F_{\alpha,u,v}}^{\infty} h(f)\,df = \alpha}$$

Parameter Estimation

Point Estimation :-

a single numerical value of a statistic that corresponds to that parameter.

Eg :-

1. Mean μ , $\hat{\mu} = \bar{x}$ ↳ Sample Mean
   ↓ estimator of μ
2. Var. σ^2 , $\hat{\sigma}^2 = s^2$ ↳ Sample variance
3. $\mu_1 - \mu_2$, $\hat{\mu}_1 - \hat{\mu}_2 = \bar{x}_1 - \bar{x}_2$
4. proportion p , $\hat{p} = x/n$

(A) Properties of Estimator :-

$\hat{\theta}$ is unbiased estimator of θ if,

$$E(\hat{\theta}) = \theta$$

→ Mean Square Error of estimator,

$$\boxed{\begin{aligned} MSE(\hat{\theta}) &= E(\hat{\theta} - \theta)^2 \\ &= Var(\hat{\theta}) + \underbrace{(\theta - E(\hat{\theta}))^2}_{\text{bias}} \end{aligned}}$$

↓ important when comparing two estimators,

The relative efficiency of $\hat{\theta}_2$ to $\hat{\theta}_1$ is,

$$= \frac{MSE(\hat{\theta}_1)}{MSE(\hat{\theta}_2)}.$$

→ Minimum variance of unbiased estimator $\hat{\theta}$ is,

$$V(\hat{\theta}) \geq \frac{1}{n E\left(\frac{d}{d\theta} \ln(f(x,\theta))\right)^2}$$

↳ Cramer - Rao Lower Bound

(B) Method of Maximum Likelihood :-

$x_1, x_2, \dots, x_n$ → observed value in a random sample of size n.

$f(x, \theta)$ → Prob. dist. f^n
↳ Unknown parameter.

$$\boxed{L(\theta) = f(x_1, \theta) \cdot f(x_2, \theta) \dots f(x_n, \theta)}$$

The value of θ that maximises $L(\theta)$ is called Maximum likelihood Estimator (MLE) of θ.

Eg :-

For mean μ , MLE = $\bar{x}$

(C) Method of Moments :-

$\theta_1, \theta_2, \dots, \theta_k$ → k Unknown Parameters

$x_1, x_2, \dots, x_n$ → Random Sample of size n from X

$\left.\begin{array}{l} f(x; \theta_1, \theta_2, \dots \theta_k) \\ P(x; \theta_1, \theta_2, \dots, \theta_k) \end{array}\right\}$ Cont. / Disc. Probability functions

$$\boxed{m'_t = \frac{1}{n} \sum_{i=1}^{n} x_i^t} \quad \forall\ t = 1, 2, \dots, k$$

↳ first k sample moments about origin.

$$\boxed{\begin{aligned}\mu'_t = E(X^t) &= \int_{-\infty}^{\infty} x^t f(x;\theta_1,\cdots\theta_k)\,dx \quad \text{Cont.}\\ &= \sum_{x\in R_x} x^t P(x;\theta_1,\cdots\theta_k) \quad \text{Disc.}\end{aligned}}$$

↳ first K population Moments about the origin.

$$\boxed{\mu'_t = m'_t} \quad \forall t = 1,2,\cdots,K$$

↓ Solving these K equations gives,

$\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$

(Moment estimators of $\theta_1, \theta_2, \ldots \theta_k$)

Precesion of Estimation :-

$$\sigma_{\hat{\theta}} = \sqrt{V(\hat{\theta})}$$

↳ Standard error of $\hat{\theta}$ = Standard deviation of $\hat{\theta}$

Single - Sample Confidence interval Estimation :-

$$\boxed{P\{L \le \theta \le U\} = 1-\alpha}$$

$\theta \rightarrow$ Unknown Parameter

L & $U \rightarrow$ Lower and Upper-Confidence limits

$$\boxed{1-\alpha \rightarrow \text{Confidence Coefficient}}$$

→ One-sided $100(1-\alpha)\%$ lower Confidence interval on θ,

$$P(L \le \theta) = 1-\alpha$$

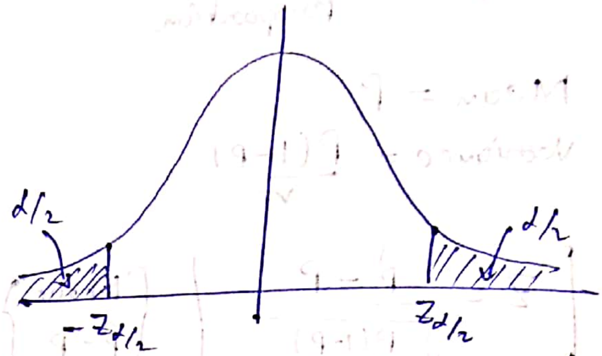
Similary upper.

→ $\theta - L$ or $U - \theta$ → accuracy of the estimator

(A) Confidence Interval on the mean of Normal Distribution :-
(Variance Known)

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \qquad \{\text{error} : \bar{X}-\mu\}$$

$$P\{-z_{\alpha/2} \le Z \le z_{\alpha/2}\} = 1-\alpha$$



So $100(1-\alpha)\%$ two-sided Confidence interval on μ is,

$$\boxed{\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}}$$

if Variance is Unknown,

$$t_{n-1} = \frac{\bar{X}-\mu}{S/\sqrt{n}} \qquad \left\{\frac{S^2(n-1)}{\sigma^2} = \chi^2_{n-1}\right\}$$

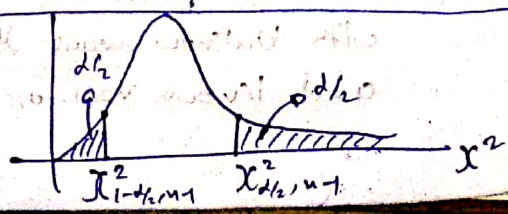
$$P\{-t_{\alpha/2,n-1} \le t \le t_{\alpha/2,n-1}\} = 1-\alpha$$

$$\boxed{\bar{X} - t_{\alpha/2,n-1}\, S/\sqrt{n} \le \mu \le \bar{X} + t_{\alpha/2,n-1}\, S/\sqrt{n}}$$

(B) Confidence interval on Variance of Normal distribution :-

$$\chi^2_{n-1} = \frac{(n-1)S^2}{\sigma^2}$$

$$\boxed{P\{\chi^2_{1-\alpha/2,n-1} \le \chi^2 \le \chi^2_{\alpha/2,n-1}\} = 1-\alpha}$$



$$\boxed{\frac{(n-1)S^2}{\chi^2_{\alpha/2,\,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,\,n-1}}}$$

(C) Confidence Interval on a Proportion :-

$\hat{p} = \frac{X}{n}$ → Point estimator of Proportion

Mean = P

Variance = $\frac{P(1-P)}{n}$

$$\boxed{Z = \frac{\hat{P} - P}{\sqrt{\frac{P(1-P)}{n}}}}$$

$\left[\begin{array}{c}\text{Error} \\ \hat{P} - P\end{array}\right]$

→ approximately standard Normal ∀ n → relatively large

$$P\left\{-Z_{\alpha/2} \le Z \le Z_{\alpha/2}\right\} = 1-\alpha$$

$100(1-\alpha)\%$ two-sided Confidence interval on P is :-

$$\boxed{\hat{p} - Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le P \le \hat{p} + Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

Two-Sample Confidence Interval Estimation :-

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

X_1, X_2 → two independent r.v's with Unknown means μ_1, μ_2 and known Var. σ_1^2, σ_2^2.

$$P\{-Z_{\alpha/2} \le Z \le Z_{\alpha/2}\} = 1-\alpha$$

$$\boxed{\bar{X}_1 - \bar{X}_2 - Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{X}_1 - \bar{X}_2 + Z_{\alpha/2}\sqrt{\quad}}$$

(A) Confidence interval on the difference b/w means of two normal distributions, Variance Unknown :-

$\{\sigma_1^2 = \sigma_2^2 = \sigma\}$

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$$

$$t = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$P\left\{-t_{\frac{\alpha}{2},\,n_1+n_2-2} \le t \le t_{\alpha/2,\,n_1+n_2-2}\right\} = 1-\alpha$$

$$\boxed{\begin{array}{c}\bar{X}_1 - \bar{X}_2 - t_{\alpha/2,\,n_1+n_2-2}\, S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} \\ \le \mu_1 - \mu_2 \le \;\_\_ + \_\_\_\_\end{array}}$$

(B) Confidence Interval on the ratio of Variance of two Normal distributions :-

$$F = \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2}$$

$$\boxed{\frac{S_1^2}{S_2^2} F_{1-\frac{\alpha}{2},\,n_2-1,\,n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{S_1^2}{S_2^2} F_{\frac{\alpha}{2},\,n_2-1,\,n_1-1}}$$

Tests of Hypotheses :-

Statistical Hypotheses :-

eg:-

$H_0 : \mu = 2500$ psi

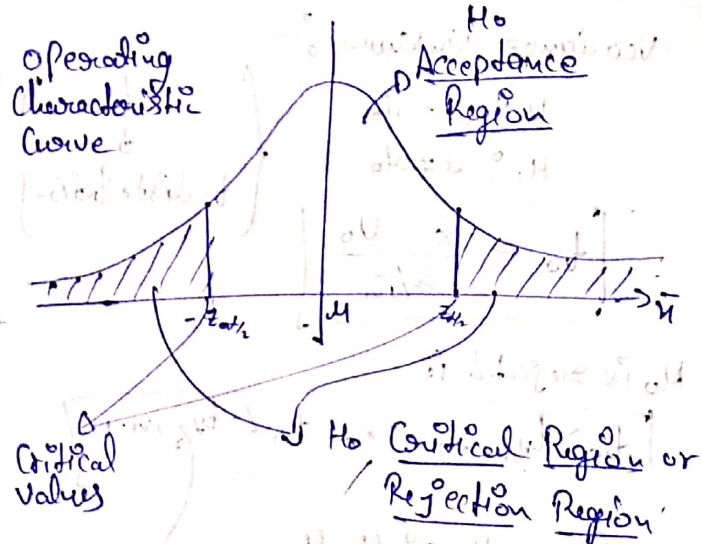
$H_1 : \mu \neq 2500$ psi

$H_0 \rightarrow$ Null Hypotheses.

$H_1 \rightarrow$ alternative hypothesis.

$H_1 : \mu \neq 2500$ psi $\rightarrow$ two-sided alt. hypo.
($\mu > 2500$ or $\mu < 2500$)

$H_1 : \mu > 2500$ psi $\rightarrow$ One-sided alt. hypo.



Type I and II errors :-

$\alpha = P(\text{type I error}) = P\{\text{reject } H_0 / H_0 \text{ is true}\}$

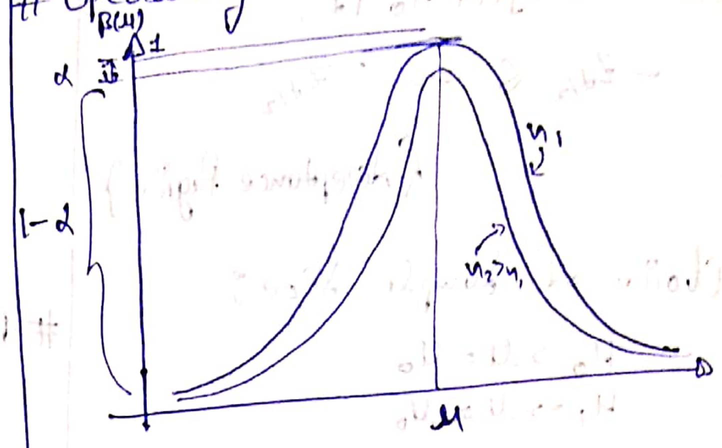
$\beta = P(\text{type II error}) = P\{\text{accept } H_0 / H_0 \text{ is false}\}$

Power of the test $= 1-\beta = P\{\text{reject } H_0 / H_0 \text{ is false}\}$

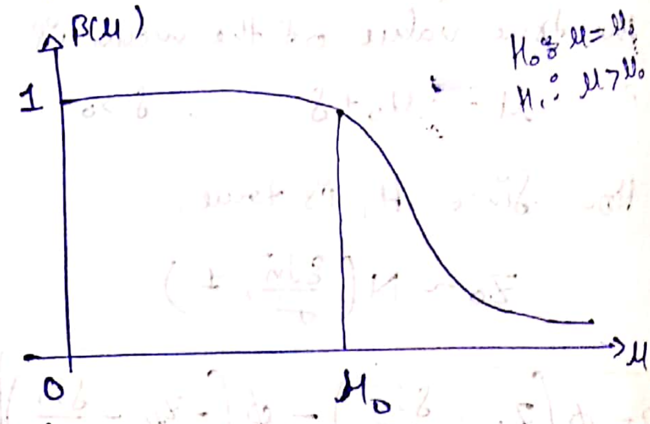
★ $\boxed{\beta(\mu) = 1-\alpha(\mu)}$ → for mean

Note:- The probability of Type I error, α is often called the significance level or size of the test.

Operating Characteristic Curve :-



↳ Two-Sided test.



↳ One-sided test.

Test of Hypotheses on a Single Sample

(A) On mean of a Normal Dist., Variance Known :-

$H_0 : \mu = \mu_0$

$H_1 : \mu \neq \mu_0$

now for a random sample of size n,

$$Z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

→ Reject H_0 if either,

$$Z_0 > Z_{\alpha/2} \quad \text{or} \quad Z_0 < -Z_{\alpha/2}$$

{Critical or Rejection Region}

→ Fail to reject H_0 if,

$$-Z_{\alpha/2} \leq Z_0 \leq Z_{\alpha/2}$$

{Acceptance Region}

Choice of Sample Size :-

$H_0 \Rightarrow \mu = \mu_0$
$H_1 \Rightarrow \mu \neq \mu_0$

if null hypothesis is false and the true value of the mean is

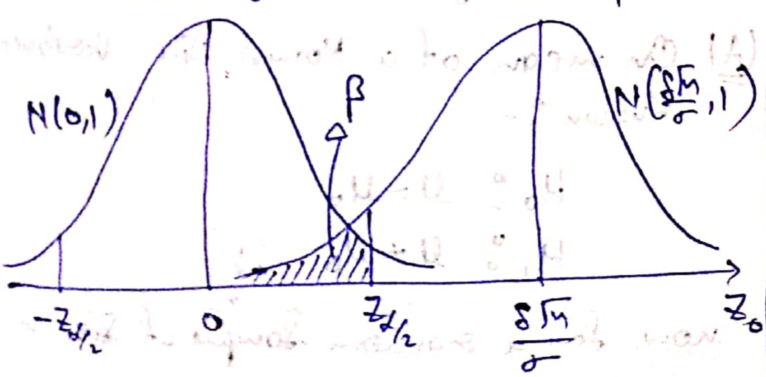
$$\mu = \mu_0 + \delta \quad , \ \delta > 0$$

Now, since H_1 is true,

$$Z_0 \sim N\left(\frac{\delta\sqrt{n}}{\sigma}, 1\right)$$

$$\boxed{\beta = \phi\left(Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \phi\left(-Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)}$$

Under H_0 Under H_1



$$\boxed{\text{Parameter } d = \frac{|\mu - \mu_0|}{\sigma} = \frac{|\delta|}{\sigma}}$$

if $\delta > 0$,

$$\phi\left(-Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) \approx 0$$

So $\beta \approx \phi\left(Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$

$$-Z_\beta \approx Z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}$$

$$\boxed{n \approx \frac{(Z_{\alpha/2} + Z_\beta)^2 \sigma^2}{\delta^2}}$$

P-value → The smallest value of significance that would lead to rejection of H_0.

(B) On mean of Normal Distribution, Variance Unknown :-

$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$

{t → t-distribution}

$$\boxed{t_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}}$$

H_0 is rejected if,

$$\boxed{t_0 > t_{\alpha/2, n-1} \ \text{or} \ t_0 < -t_{\alpha/2, n-1}}$$

$\alpha/2$

→ for, $H_0 : \mu = \mu_0$
$H_1 : \mu > \mu_0$

H_0 is rejected if,

$$\boxed{t_0 > t_{\alpha, n-1}}$$

α

(C) On Variance of Normal Dist. :-
{for Normal Population}

$H_0 : \sigma^2 = \sigma_0^2$
$H_1 : \sigma^2 \neq \sigma_1^2$

(25)

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

H_0 is rejected if,

$\chi_0^2 > \chi^2_{\alpha/2, n-1}$ "or"

$\chi_0^2 < \chi^2_{1-\alpha/2, n-1}$

→ for large sample :-

$$\boxed{Z_0 = \frac{s-\sigma}{\sigma/\sqrt{2n}}}$$

(D) On a Proportion :-

$H_0 : P = P_0$

$H_1 : P \neq P_0$

$$\boxed{Z_0 = \frac{X - nP_0}{\sqrt{nP_0(1-P_0)}}} = \frac{\frac{X}{n} - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}}$$

$\hat{P} \leftarrow$

H_0 rejected if,

$$\boxed{Z_0 > Z_{\alpha/2} \quad \text{or} \quad Z < -Z_{\alpha/2}}$$

Tests of Hypotheses on Two Samples :-

(A) On Means of two Normal dist's, Variance known :-

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

$$\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

~~Since actual means are same for both the samples~~

$$\boxed{Z_0 = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}}$$

~~Since it is assumed true H_0 i.e.~~

→ Curve is always made assuming H_0 to be true.

H_0 is rejected if,

$$\boxed{Z_0 > Z_{\alpha/2} \quad \text{or} \quad Z_0 < Z_{\alpha/2}}$$

$$\boxed{d = \frac{|\delta|}{\sqrt{\sigma_1^2 + \sigma_2^2}}}$$

$$\boxed{n = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2/n_1 + \sigma^2/n_2}} \rightarrow \text{Choice of Sample Size}$$

(B) Variance Unknown in (A) :-

$$\left.\begin{array}{l} S_p^2 = \dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} \\ \boxed{t_0 = \dfrac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{1/n_1 + 1/n_2}}} \end{array}\right\} \sigma_1^2 = \sigma_2^2$$

H_0 is rejected if,

$t_0 > t_{\alpha/2, n_1+n_2-2}$ or $t_0 < -t_{\alpha/2, n_1+n_2-2}$

$$\boxed{t_0^* = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}} \rightarrow \{\sigma_1^2 \neq \sigma_2^2$$

$$\boxed{d = \frac{|\delta|}{2\sigma}}$$