MA 20104 Probability and Statistics

1. Probability

Classical, relative frequency and axiomatic definition of probability. Addition rule and conditional probability, Multiplication rule, Total probability, Bayes' Theorem and Independence, Problems.

— 6 Lectures.

Introduction to Probability

Origin → Prediction of gamblers in gambling

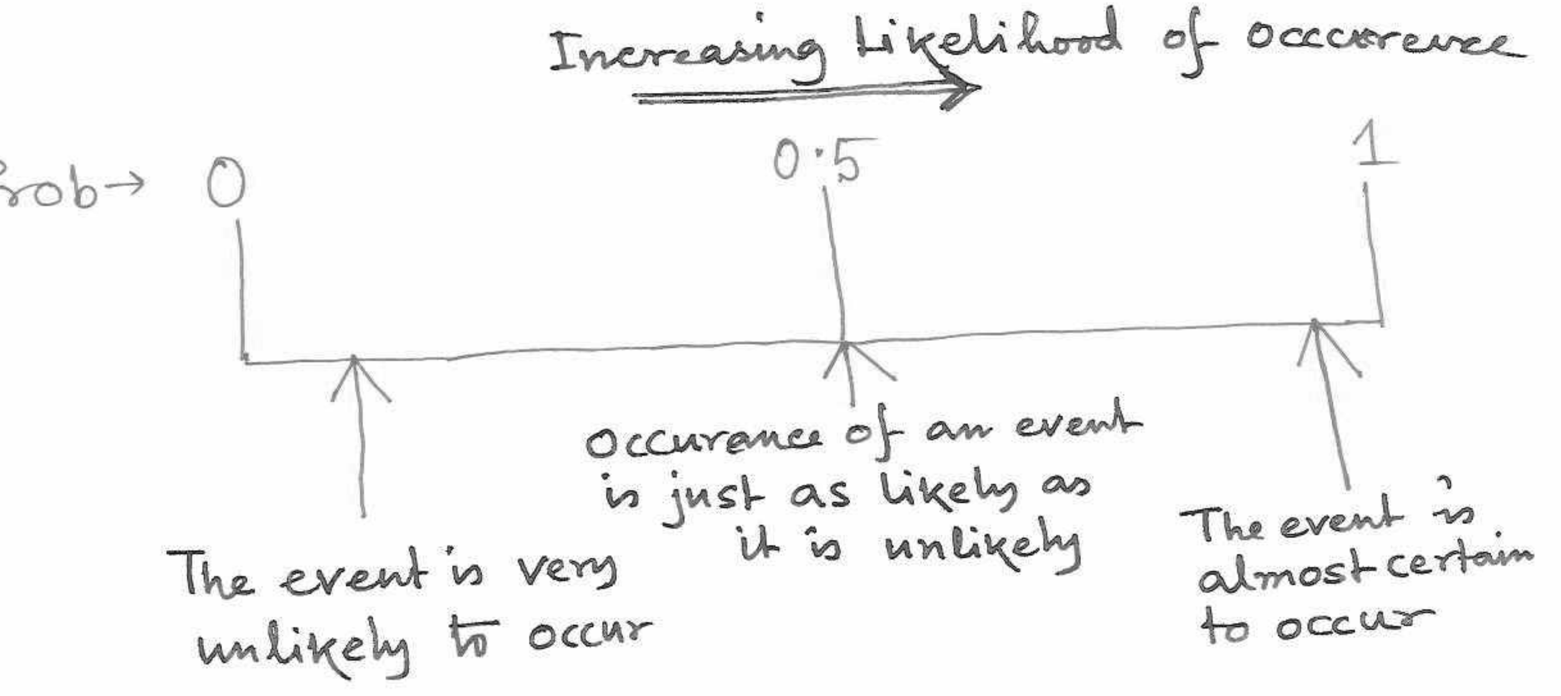
Interpretation → A measure of how likely an event will occur

Meaning → Likelihood, Chance, Degree of belief etc.

Happening of an event

- Sure or Certain
- Never or Impossible
- May happen or probable.

Probability as a numerical measure of the Likelihood of occurance



Imagine you were living in 17th Century. One day your friend was visiting and challenged you to game of chance. You agreed to play the game with him. He said —

"I can get a ~~s~~ sum of 8 and a sum of 6 rolling two dice before you can get two sums of 7's."

Would you continue to play the game?

so, there are ten possible ways to get the favourable result for him but there are six ways for you.

Strategy 1

Roll a die. At least one 6 would appear during a total of four rolls.

Strategy 2

A double 6 on twenty four rolls of two dice.

Chance of winning

Strateg 1 → 51·8 %

Strategy 2 → 49·1 %.

1654 → Pascal & Fermat create the mathematical theory of probability

1812 → Laplace gave classical def^{n}.

1933 → A. N. Kolmogorov gave axiomatic def^{n}.

Strategy 1

P(Atleast one 6)

$\Rightarrow$ P(getting one, two, three, four [illegible] six)

or 1 − P(getting no six)

$$= 1 - \left(\frac{5}{6}\right)^4 = 51.8\%$$

Stretegy 2

36 possible rolls of two dice

required probability

$$= 1 - \left(\frac{35}{36}\right)^{24}$$

$$= 49.1\%$$

Four Perspectives on Probability

- Classical
- Empirical
- Subjective
- Axiomatic

Classical ("A priori" or "Theoretical")

e.g. tossing a fair ~~coin~~ die

$\Rightarrow$ six possible outcomes are equally likely.

$\Rightarrow$ probability of each $\frac{1}{6}$.

Assume situation

A random experiment has N possible outcomes which are mutually exclusive, exhaustive and equally likely.

Def^n $\rightarrow$ Let M of these outcomes be favourable to the happening of event A.

Probability of A

$$= P(A)$$

$$= \frac{M}{N}.$$

Experiment An act of conducting controlled test or investigation

Results may be one or more.

Based on number of possible resuts —

→ Deterministic

→ Probabilistic

Deterministic

→ Only one possible result

→ Predictable

→ Known prior to its conduct

Example To verify laws of science

e.g. Newton's laws of motion etc.

Probabilistic or Indeterministic or Unpredictable

→ More than one possible results or outcomes

→ Can not be predicted prior to conduct

→ Outcome of experiment must be independent or identically distributed

e.g.

Advantage:

Conceptually simple

Disadvantage:

1. If do not have finitely many equally likely outcomes
   i.e. N is not finite.

2. Definition is circular in nature.
   'equally likely' means 'equally probable'!

e.g. studying people's income over time

– infinitely many possible outcomes.

2. Emperical ("A posteriori" or "Frequentist")

This perspective defines probability via a thought process.

e.g. Given a die which is weighted
Find probability of each outcome.

e.g. If the die is fair. And toss it 100 times, 500 times, 1000 times ------ ..

Number of toses	Proportion of one.
100	0·19
200	0·19
300	0·18333
400	0·1825
500	0·182
600	0·18
700	0·177142
800	0·17125
900	0·1744
1000	0·175

Let us formalise <u>frequency def^n. of probability</u>

<u>Assume</u>

A random experiment is conducted a large number of times independently under identical condition, say N times

<u>Def^n:-</u>

Let M be the number of times of event A occurs in N trials of experiments

Then,

$$P(A) = \lim_{N \to \infty} \frac{M}{N}$$

provided the limit exists.

Find P(A) where A = {number of heads occured}

H H T H H T H H T ....

Is the coin biased?

☒ Example $A \equiv \{\text{number of head occured}\}$

H H T H H T H H T ......

$$\frac{M}{N} \rightarrow \frac{1}{1}, \frac{2}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{4}{6}, \ldots\ldots$$

$$= \begin{cases} \dfrac{2N-1}{3N-1} & N = 1, 2, \ldots\ldots \\[2ex] \dfrac{2N}{3N-1} & N = 1, 2, \ldots\ldots \\[2ex] \dfrac{2N}{3N} & N = 1, 2, \ldots\ldots \end{cases}$$

$$\Rightarrow \lim_{N \to \infty} \frac{M}{N} = \frac{2}{3}$$

So, biased coin in favour of H.

- The empirical view is used in most statistical inference procedure which are based on sampling.

Disadvantages ......

Drawbacks

Difficult to arrange large number of experiments. because of high cost etc.

- failure rate of launched satelite is very difficult to measure, so difficult to calculate success rate.

- It leaves open problem — how large N has to be to get good approximation

- if $\lim\limits_{N \to \infty} \dfrac{\sqrt{N}}{N} = 0$, then $P(A) = 0$

  $\not\Rightarrow$ impossible event

- $\lim\limits_{N \to \infty} \dfrac{N - \sqrt{N}}{N} = 1 \Rightarrow P(A) = 1$

  $\not\Rightarrow$ Sure event

so, Axiomatic def^{n} ......
based on set theory

3. Subjective

- Individual person's measure of belief that an event will occur.
- Basis for Bayesian statistics
- Must obey the coherence (consistency) condition in order to be workable.

4. Axiomatic (Kolmogorov 1933)

Axiomatic perspective says that probability is a function from events to numbers satisfying three conditions (Axioms).

Axiom 1: $0 \leq P(A) \leq 1$

for every allowable event A.
0 is smallest allowable probability
1 is largest allowable probability.

Axiom 2: The certain event has probability 1.

e.g. in rolling a die "One of 1, 2, 3, 4, 5, 6 come up"

<u>Axiom 3</u> : Probability of union of mutually exclusive events is sum of probabilities of individual event.

Two events are mutually exclusive if they both cannot occur simultaneously.

e.g.

$A \equiv$ a 1 comes up on the die

$B \equiv$ an even number comes up on the die.

Sample Space – the set of all possible outcomes. Denoted by Ω or S.

Event – A subset of sample space

Probability fn – A fn giving the prob of event.

Examples

- Toss a fair coin. Observe up face

$$\Omega = \{H, T\}$$

- Toss a coin three times observe the seq. of heads and tails

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

- An automobile door is assembled with a large number of spot welds. After assembly each weld is inspected, and the total no. of defectives is counted.

$$\Omega = \{0, 1, 2, 3, \cdots K\}$$ $K =$ total no. of welds in the door.

▨ A cathode ray tube is manufactured, put on life test and aged to failure. The elapsed time (in hours) at failure is recorded.

$$\Omega = \{t : t \in \mathbb{R},\ t \geq 0\}$$

The set is uncountable.

▨ A monitor records the emission count from a radioactive source in one minute.

$$\Omega = \{0, 1, 2, \ldots\}$$

The set is countably infinite

▨ In a missile launch, the three components of velocity are mentioned from the ground as a function of time. At 1 minute after launch these are printed for a control unit.

$$\Omega = \{(v_x, v_y, v_z) \mid v_x, v_y, v_z \text{ are real numbers}\}$$

▨ The velocity components are continuously recorded for 5 minutes.

Write the sample space for the following random experiments.

(i) A coin is tossed three times and the results are noted.

(ii) From five players A, B, C, D, E two players are selected for a match

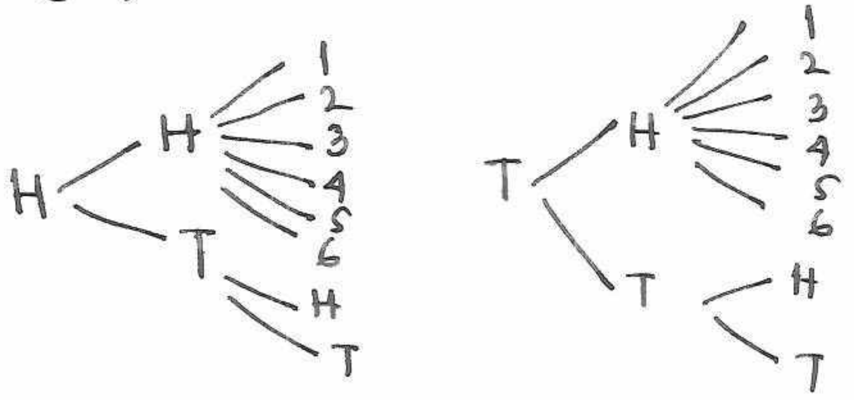
(iii) A coin is tossed twice. If the second throw results in a head, a die is thrown, otherwise a coin is tossed.

Ans. (i) $\Omega = \{TTT, TTH, THT, HTT, HHT, HTH, THH, HHH\}$

$n(\Omega) = 8$

(ii) $n(\Omega) = 10$

(iii)



$n(\Omega) = 16.$

Event

Tossing three coins at a time

Event A : Number of heads exceeds the number of tails

{HHH, HHT, HTH, THH}

Event B : Getting two heads only

{HHT, HTH, THH}

Equally Likely Event

Outcomes of an experiment is said to be equally likely if taking into consideration all the relevant evidences that there is no reason to expect one in preference to ther Other.

Mutually exclusive event

Two events are such that they can not occur simultaneously.

Exhaustive Events

A die is rolled. An event getting of an even no and another event getting odd no are exhaustive events

Independent Event

Happening of one event does not depend on the happening of other.

$\Omega \rightarrow$ Sample space

$B \rightarrow$ Subset of Ω which is σ-field

union, intersection operations give a pt in B.
i.e. closed under set operations

$(\Omega, B) \rightarrow$ Measurable space

Axiomatic definition of Probability (Kolmogorov 1933)

Let (Ω, B) be a measurable space. A set fn.
$P: B \rightarrow R$ is said to be probability function
if it satisfies the following three axioms.

P_1: Non-negativity axiom

$$P(A) \geq 0 \quad \forall \; A \in B$$

P_2: Axiom of completeness

$$P(\Omega) = 1$$

P_3: Axiom of countable-additivity

For pairwise disjoint subsets $E_i \in B$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

(Ω, B, P) is called Probability space.

X : Set

B : σ-field or σ-algebra

P is f^n from B to real number

P is called measure if (for $A \in B$)

- Non-negativity $P(A) \geqslant 0$
- Null empty set $P(\phi) = 0$
- Countable additivity

  for countable collection pairwise disjoints

  $\{A_i\}_{i \in \mathbb{N}}$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Pair (X, B) is called measurable space.

Union of two events

$A \cup B \rightarrow$ occurence of atleast one of A & B

$\bigcup_{i=1}^{n} A_i \rightarrow$ occurence of atleast one A_i, $i = 1, 2 \ldots n$

$\bigcup_{i=1}^{\infty} A_i \rightarrow$ occurence of atleast one A_i, $i = 1, 2 \ldots$

Intersection of two events

$A \cap B \rightarrow$ simultaneous occurence of A & B

$\bigcap_{i=1}^{n} A_i \rightarrow$ simultaneous occurence of $A_1, A_2 \ldots A_n$

$\boxed{\bigcup_{i=1}^{n} A_i = \Omega} \rightarrow$ $A_1, A_2 \ldots A_n$ are exhaustive events

$\boxed{A \cap B = \phi} \rightarrow$ A and B are mutually exclusive events

$A^C \rightarrow$ Not happening of A

$A - B \rightarrow$ Happening of A but not B

$= A \cap B^C$

- Ω is discrete and contains n pts.

$$P(\omega_j) = \frac{1}{n}, \quad \Omega = \{\omega_j \mid j = 1, 2 \dots n\}$$

$$\Rightarrow P(A) = \frac{m}{n} \quad \text{if } A \text{ contains } m \text{ events.}$$

- Ω is discrete and contains countable number of pts but not equally likely probabilities.

$$P(A) = \sum_{\omega \in A} P(\omega)$$

- Ω contains uncountably many pts.

$$\Omega = (0, \infty)$$

$$P(I) = \int_I e^{-x}\, dx \quad \forall\, I \subseteq \Omega$$

$$\Rightarrow P(I) \geqslant 0, \quad P(\Omega) = 1.$$

Consequences of Axiomatic defn. (Proof !!)

$\boxed{1}$ $P(\phi) = 0$

$\boxed{2}$ For any finite collections $A_1, A_2 \dots A_n$ of pairwise disjoint sets in B

$$P\left(\bigcup_{i=1}^{n}\right) A_i = \sum_{i=1}^{n} P(A_i)$$

$\boxed{3}$ Probability fn. P is monotone & subtractive

i.e. $A \subseteq B \Rightarrow \begin{cases} \text{(i) } P(A) \leq P(B) \\ \text{(ii) } P(B-A) = P(B) - P(A) \end{cases}$

$\boxed{4}$ For any $A \in B$, $0 \leq P(A) \leq 1$

$\boxed{5}$ $P(A^c) = 1 - P(A)$

$\boxed{1}$ Take $A_1 = \Omega$, $A_2 = A_3 = A_4 = \cdots = \phi$

By P_3 $$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

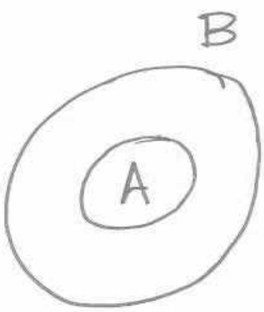
$$\Rightarrow P(\Omega) = P(\Omega) + P(\phi) + \cdots\cdots$$

By P_1 $$P(\phi) \geq 0$$
$$\Rightarrow \boxed{P(\phi) = 0}$$

$\boxed{2}$ Take $A_{n+1} = A_{n+2} = \cdots = \phi$ in P_3

Then follows.

$\boxed{3}$



$$B = (A \cap B) + (B - A)$$
$$= A + (B - A)$$

From $\boxed{2}$ (i) & (ii) follow.

$\boxed{4}$ As $A \subset \Omega \Rightarrow P(A) \leq P(\Omega) = 1$ from $\boxed{3}$

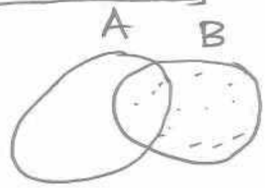
By P_1 and above.

$\boxed{5}$ $A \uplus A^c = \Omega$

$$\Rightarrow P(A) + P(A^c) = P(\Omega)$$

Addition Rule

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



As, $A \cup B = A \cup (B - (A \cap B))$

$$\Rightarrow P(A \cup B) = P(A) + P(B - (A \cap B))$$

$$= P(A) + P(B) - P(A \cap B) \text{ by } \boxed{3}$$

$$\boxed{\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) - P(A \cap B) \\ & - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \end{aligned}}$$

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ &= P(A) + P(B \cup C) - P((A \cap B) \cup (A \cap C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(A \cap B \cap A \cap C \end{aligned}$$

Follows as ~~A∩∩~~

$$\begin{aligned} A \cap B \cap A \cap C &= A \cap A \cap B \cap C \\ &= A \cap B \cap C \end{aligned}$$

General Addition Rule or Principle of Inclusion - Exclusion follows.......

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

Using method of induction

Assume it ~~for~~ k and prove for (k+1)

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)$$

$$\Rightarrow LHS = P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\bigcup_{i=1}^{k} A_i \cap A_{k+1}\right)$$

$$= \left\{\sum_{i=1}^{k} P(A_i) - \sum_{i<j}^{k} P(A_i \cap A_j) + \sum_{i<j<k'}^{k} P(A_i \cap A_j \cap A_{k'}) - \cdots + (-1)^{k+1} P\left(\bigcap_{i=1}^{k} A_i\right)\right\} + P(A_{k+1})$$

$$- P\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$ using dist. law.

$$= \cdots - \left\{\sum_{i=1}^{k} P(A_i \cap A_{k+1}) - \sum_{i<j}^{k} P((A_i \cap A_{k+1}) \cap (A_j \cap A_{k+1}))\right.$$

$$+ \sum_{i<j<k'}^{k} P((A_i \cap A_{k+1}) \cap (A_j \cap A_{k+1}) \cap (A_{k'} \cap A_{k+1})) -$$

$$\left.\cdots + (-1)^{k+1} P\left(\bigcap_{i=1}^{k} (A_i \cap A_{k+1})\right)\right)$$

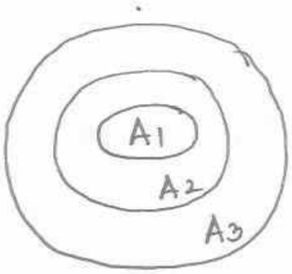
Monotonic Sequence of Events.

If $\{A_n\}$ is a monotonic sequence of sets in B then

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

Let $\{A_n\}$ be monotonic increasing sequence

$$\Rightarrow \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n .$$



Let $B_1 = A_1$, $B_2 = A_2 - A_1$, $B_3 = A_3 - A_2 \dots$

$\dots \quad B_n = A_n - A_{n-1}$

$\Rightarrow \{B_n\}$ is disjoint sequence of sets

Again, $A_n = \bigcup_{i=1}^{n} B_i$

or, $P(A_n) = \sum_{i=1}^{n} P(B_i)$

$$P\left(\lim_{n\to\infty} A_n\right) =$$

$$= P\left(\lim_{n\to\infty} \bigcup_{i=1}^{n} B_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right)$$

$$= \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n\to\infty} P(A_n) .$$

Monotonic decreasing sequence the case is reverse.

Subadditivity property of Probability fn.

- For A, B, $P(A \cup B) \leqslant P(A) + P(B)$

- For any $A_1, A_2 \cdots A_n \in \mathcal{B}$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leqslant \sum_{i=1}^{n} P(A_i)$$

By induction process we can prove it.

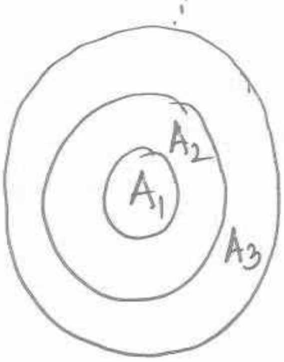
$$P\left(\bigcup_{i=1}^{k+1} A_i\right) \leqslant P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1})$$

$$\leqslant \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$$

Generalisation

For any countable sequence

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} P(A_i)$$



Let $B_1 = A_1$, $B_2 = A_2 - A_1$, $B_3 = A_3 - (A_1 \cup A_2)$

$\cdots$ $B_n = A_n - \left(\bigcup_{i=1}^{n-1} A_i\right)$

$\Rightarrow \{B_i\}$ disjoint sequence & $B_i \subset A_i$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$$

$$\leqslant \sum_{i=1}^{\infty} P(A_i)$$

Boole's Inequality

$$P(A \cap B) \geq 1 - P(A^c) - P(B^c)$$

For countable sequence $\{A_i\}$

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{\infty} P(A_i^c)$$

Proof.

$$P(A \cap B) + P(B^c) \geq P(A)$$

$$\Rightarrow P(A \cap B) + P(B^c) \geq 1 - P(A^c) \qquad \square$$

Again,

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - P\left(\bigcap_{i=1}^{\infty} A_i\right)^c$$

$$= 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right)$$

$$\geq 1 - \sum_{i=1}^{\infty} P(A_i^c).$$

Benferroni's Inequality

For any events $A_1, A_2, \ldots$.

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j) \leq P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$\leq \sum_{i=1}^{n} P(A_i).$$

Proof. RHS bound is sub-additivity property.
LHS bound can be proved by induction.

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)$$

$$= P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right)$$

Assuming that it is true for index k

$$\geq \sum_{i=1}^{k} P(A_i) - \sum_{i<j}^{k} P(A_i \cap A_j) + P(A_{k+1})$$

$$- P\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

$$= \sum_{i=1}^{k+1} P(A_i) - \sum_{i<j}^{k+1} P(A_i \cap A_j). \quad \square$$

Problem 1 Suppose there are n persons in a party. Assume $n \leq 365$ and no person has birth day on 29^{th} February.
What is the probability that atleast two persons share the same birthday.
What will be the value of 'n' so that the probabili is significant.

Problem 2 An interval of length 1, say $(0, 1)$ is divided into three intervals by choosing at random. What is the probability that the three line segments form a triangle?

Problem 3 The integers x and y are choosen at random with replacement from the set of natural numbers $\{1, 2 \ldots 9\}$. Find the probability that $|x^2 - y^2|$ is divisible by 2

Proble 4 Show that probability of obtaining six atleast once in 4 throws with a die is slightly greater than $\frac{1}{2}$, and that of obtaining double six at the least once in 24 throws with two dice is slightly less than $\frac{1}{2}$.

Problem 5 Find the minimum number of times a die has to be thrown such that the probability of no six is less than $\frac{1}{2}$.

Ans 1

A ≡ Event that two persons share the same birthday.

A^c ≡ No two persons have same birthday.

$$P(A^c) = \frac{{}^{365}P_n}{(365)^n} = \frac{365 \cdot 364 \cdot 363 \cdots (365-n+1)}{(365)^n}$$

$$= 1 \cdot \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)$$

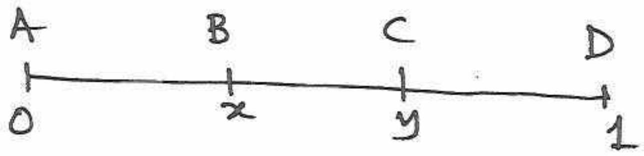
$$\Rightarrow P(A)$$

$$= 1 - \left\{\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)\right\}$$

For $0 \leq P(A) \leq 1$ $n \cong 60$.

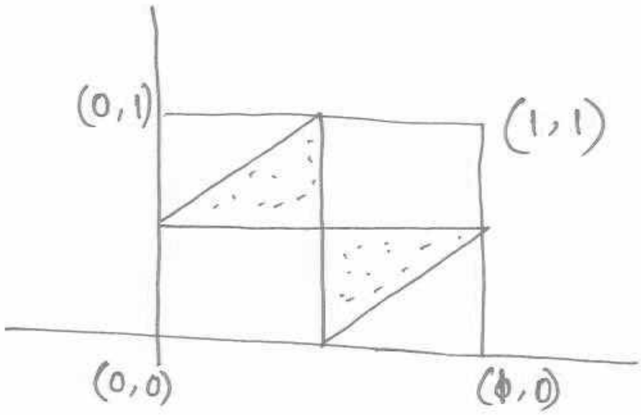
n	$P(A)$
10	0.129
20	0.411
23	0.507
30	0.706
50	0.970
60	0.994

Ans 2.



$AB + BC > CD \Rightarrow x + (y - x) > 1 - y \Rightarrow y > \frac{1}{2}.$

$AB + CD > BC \Rightarrow x + (1 - y) > y - x \Rightarrow y - x < \frac{1}{2}.$



$\text{Prob} = \frac{1}{4}.$

$$\boxed{\begin{array}{l} 0 < x < \frac{1}{2} < y < 1 \quad \& \quad y - x < \frac{1}{2} \\ \text{or} \\ 0 < y < \frac{1}{2} < x < 1 \quad \& \quad x - y < \frac{1}{2} \end{array}}$$

Ans 3

$A \equiv$ Event where x and y are both even

$B \equiv$ Event where x and y are both odd

(x, y) are pts for $x, y = 1, 2 \ldots 9$.

A contains 4^2 points

B containts 5^2 points

A and B are mutually exclusive.

Total number $\rightarrow 9^2$.

So. $P(A + B) = P(A) + P(B) = \frac{4^2}{9^2} + \frac{5^2}{9^2} = \frac{41}{81}.$

Problem 4

Required probability $= 1 - \left(\frac{5}{6}\right)^4 = \cdot 518$

and $1 - \left(\frac{35}{36}\right)^4 = \cdot 49$

Problem 5

$\left(\frac{5}{6}\right)^k < \frac{1}{2} \Rightarrow k \geqslant 4.$

Conditional Probability

A die is roll.

B $\longrightarrow$ Event that an even number occurs $\{2, 4, 6\}$

C $\longrightarrow$ Event that 2 occurs.

Then, $P(C) = \frac{1}{6}$ but $P(C/B) = \frac{1}{3}$.

Defn:

Let $(\Omega, \mathcal{B}, P)$ be a probability space and let $E \in \mathcal{B}$ be an event $P(E) > 0$. Then for any $A \in \mathcal{B}$ the conditional probability of A given that E has already occurred is

$$P(A/E) = \frac{P(A \cap E)}{P(E)}.$$

Prove that conditional probability fn. is well defined or conditional probability is a valid probability measure.

It satisfies probability ~~axioms~~ axioms.

Axiom 1 : $P(A/E) \geq 0 \quad \forall\ A \in B$

Axiom 2 : $P(\Omega/E) = 1.$

Axiom 3 : Let $A_1, A_2 \ldots$ are pairwise disjoint events,

$$P\left(\bigcup_{i=1}^{\infty} A_i / E\right) = \sum_{i=1}^{\infty} P(A_i/E).$$

Proof:

$$P(A/E) = \frac{P(A \cap E)}{P(E)} \geq 0 \text{ --- } P_1$$

$$P(\Omega/E) = \frac{P(\Omega \cap E)}{P(E)} = \frac{P(E)}{P(E)} = 1 \text{ --- } P_2$$

$$P\left(\bigcup_{i=1}^{\infty} A_i / E\right)$$

$$= \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap E\right)}{P(E)}$$

$$= P\left(\bigcup_{i=1}^{\infty} (A_i \cap E)\right) / P(E)$$

$$= \sum_{i=1}^{\infty} P(A_i \cap E) / P(E) \text{ --- } P_3$$

Roll a fair die twice. $X_1 \equiv$ result in the first roll $X_2 \equiv$ result of the second roll.
Given that $X_1 + X_2 = 7$. What is the probability that $X_1 = 4$ or $X_2 = 4$?

$A \equiv$ event that $X_1 = 4$ or $X_2 = 4$

$$\{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (1,4), (2,4), (3,4), (4,4), (5,4), (6,4)\}$$

$B \equiv$ event that $X_1 + X_2 = 7$

$$\{(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)\}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = ?$$

$$P(A/B) = \frac{2/36}{6/36} = \frac{1}{3}.$$

Example 1 A certain product has been purchased. The manual states that the lifetime T of the product, defined as the amount of time (in years) the product works properly until it breaks down, satisfies

$$P(T \geq t) = e^{-\frac{t}{5}}, \text{ for all } t \geq 0.$$

For example, the probability that the product lasts more than (or equal) to 2 years is $P(T \geq 2) = e^{-\frac{2}{5}} = \cdot 6703$. If the product is being used for 2 years without any problem, then what is the probability that it breaks down in the third year?

$A \equiv$ the event that the purchased product breaks down in third year.

$B \equiv$ the event that the purchased product does not break down in first two years

$P(A/B) = ?$

$$P(B) = e^{-2/5}$$

$$\begin{aligned} P(A) &= P(2 \leq T \leq 3) \\ &= P(T \geq 2) - P(T \geq 3) \\ &= e^{-2/5} - e^{-3/5} \end{aligned}$$

Again $A \cap B = A$ (as $A \subset B$)

$$\Rightarrow P(A/B) = \frac{P(A \cap B)}{P(B)} = \cdot 1813$$

$$\boxed{\begin{aligned} P(A \cap B) &= P(B)\, P(A/B) \\ &= P(A)\, P(B/A) \end{aligned}}$$

$$\boxed{P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \cdots P\left(A_n \Big/ \bigcap_{i=1}^{n-1} A_i\right)}$$

By method of inductión

$$P\left(\bigcap_{i=1}^{k+1} A_i\right) = P\left(\underline{(A_1 \cap A_2)} \bigcap_{i=3}^{k+1} A_i\right)$$

$$= P(\underline{A_1 \cap A_2})\, P(A_3/\underline{A_1 \cap A_2})\, P(A_4/A_1 \cap A_2 \cap A_3) \cdots P(A_{k+1}/A_1 \cap A_2 \cdots \cap A_k)$$

$$= \underline{P(A_1)\, P(A_2/A_1)}\, P(A_3/A_1 \cap A_2) \cdots P(A_{k+1}/A_1 \cap A_2 \cap A_3 \cdots \cap A_k)$$

In a town, it's rainy one third of the days. Given that it is rainy, there will be ~~heav~~ heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $1/4$.

If it is rainy and there is heavy traffic, I arrive late for work with probability $\frac{1}{2}$.

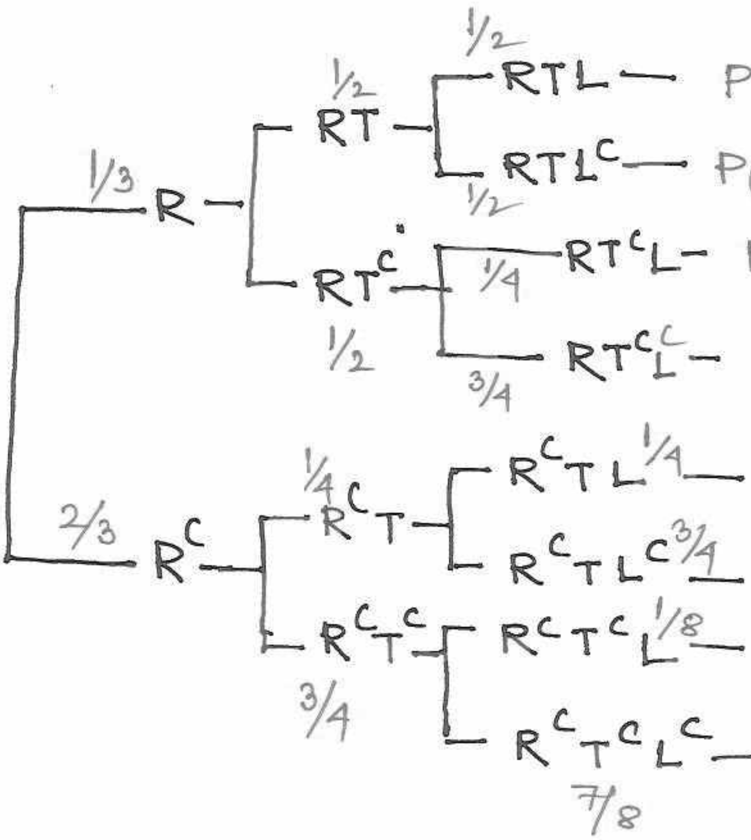
On the other hand, probability of being late is reduced to $1/8$ if it is not rainy but no heavy traffic.

In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is 0.25. Pick a random day

(i) What is the prob that it's not rainy but there is heavy traffic and I am not late?

(ii) What is the probability that I am late?

(iii) Given that I arrived late at work, what is the probility that it rained that day?



$P(RLT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$

$P(RTL^c) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$

$P(RT^cL) = \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{24}$

$P(RT^cL^c) = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{3}{24}$

$P(R^cTL) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{24}$

$P(R^cTL^c) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{8}$ (i)

$P(R^cT^cL) = \frac{1}{8} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{16}$

$P(R^cT^cL^c) = \frac{7}{8} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{7}{16}$

(ii) $P(L) = P(RTL) + P(RT^cL) + P(R^cTL) + P(R^cT^cL)$

$= \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16}$

$= \frac{11}{48}.$

(iii) $P(R/L) = \dfrac{P(R \cap L)}{P(L)} = \dfrac{1/8}{11/48} = \dfrac{6}{11}$

$P(R \cap L) = P(R \cap L \cap T) + P(R \cap L \cap T^c)$

$= \frac{1}{12} + \frac{1}{24} = \frac{1}{8}$

A box contains 5000 IC, of which 1000 are manufactured by company X and the rest by company Y. 10% chips made by X and 5% made by Y are defective. If randomly choosen chip ~~f~~ is found to be defective, find the probability that it came from company X.

A : Chip is made by company X

B : chip is defective.

$$P(A) = \frac{1000}{5000} = \cdot 2$$

$$P(B) = \frac{300}{5000} = \cdot 06$$

$$P(A \cap B) = \frac{100}{5000} = \cdot 02$$

$$P(A/B) = \frac{\cdot 02}{\cdot 06} = \frac{1}{3}.$$

Independence of events

Two events A and B are independent if

$$P(A \cap B) = P(A)\, P(B)$$

$$\Rightarrow P(A/B) = P(A \cap B)/P(B) = P(A)$$

and $P(B/A) = P(B)$

Disjoint $\Longrightarrow$ $A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$

Independence $\Longrightarrow P(A \cap B) = P(A) \cdot P(B)$

Two events A and B with $P(A) \neq 0$ and $P(B) \neq 0$ if A and B are disjoint then they are not independent.

A and B are disjoint

$\Rightarrow A \cap B = \phi$

$\Rightarrow P(A \cap B) = 0 \neq P(A)\, P(B)$

$\Rightarrow$ A and B are not independent.

- A & B independent $\Rightarrow$ A & B^c independent
- A & B independent $\Rightarrow$ A^c & B independent
- A & B independent $\Rightarrow$ A^c & B^c independent

$$
\begin{aligned}
P(A \cap B^c) = P(A - B) &= P(A) - P(A \cap B) \\
&= P(A) - P(A) \cdot P(B) \\
&= P(A) \cdot P(B^c)
\end{aligned}
$$

$$
\begin{aligned}
P(A^c \cap B^c) = P(A \cup B)^c &= 1 - P(A \cup B) \\
&= (1 - P(A))(1 - P(B))
\end{aligned}
$$

If $A_1, A_2 \ldots A_n$ are independent then

$$
\begin{aligned}
&P(A_1 \cup A_2 \cup \ldots \cup A_n) \\
&\quad = 1 - (1 - P(A_1))(1 - P(A_2)) \cdots (1 - P(A_n))
\end{aligned}
$$

Suppose that the probability of being killed in a single flight is $p = \frac{1}{4 \times 10^5}$ based on available statistics. Assume that different flights are independent.

If a businessman takes 20 flights per year, what is the probability that he killed in a plane crash within next 20 years.

Total number of flights he takes in 20 years

$$= 20 \times 20.$$

P(he survives in a single flight)

$$= 1 - p.$$

P(he is killed in plane crash in next 20 yrs)

$$= 1 - (1-p)^{400}$$

$$\approx \frac{1}{10000}.$$

Assume he will not die because of any other reason.

Total Probability

For any two events A and B,

$$P(A) = P(A \cap B) + P(A \cap B^C)$$

$$= P(B)\, P(A/B) + P(B^C)\, P(A/B^C)$$

A more general version considering the partions of the sample space Ω

$$\boxed{P(A) = \sum_{i=1}^{\infty} P(B_i)\, P(A/B_i)}$$

$B_1, B_2, \ldots$ are palitions of sample space.

Three bags each contains 100 marbles

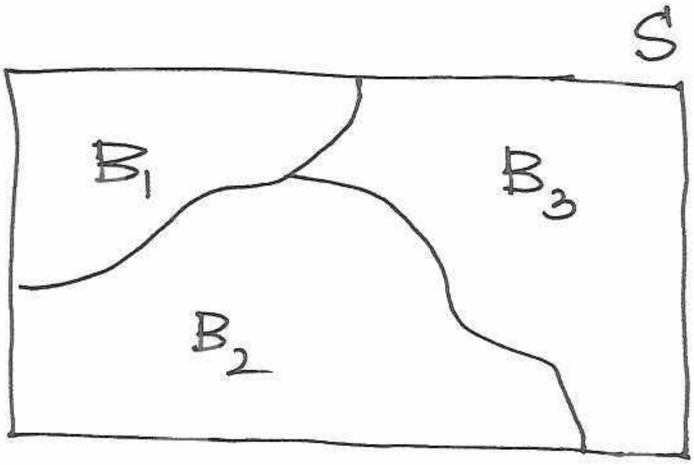
Bag 1 has 75 red and 25 blue

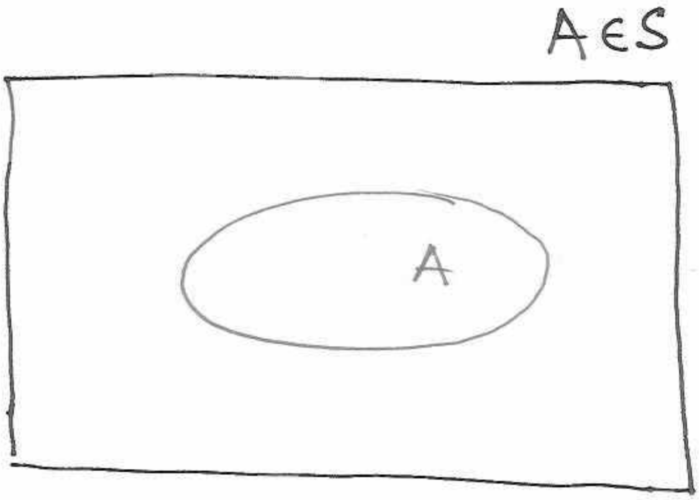
Bag 2 has 60 red and 40 blue

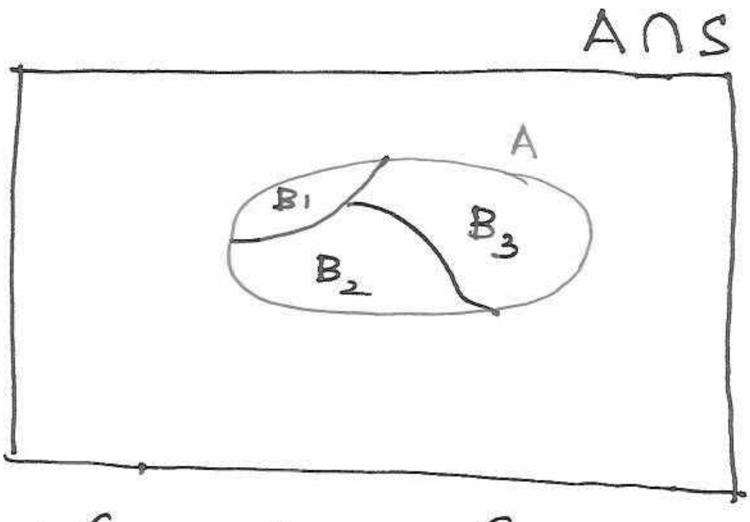
Bag 3 has 45 red and 55 blue.

A bag is choosen at random and then pick a marble at random.

What is the prob that choosen marble is red.







$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$

<u>Theorem</u> Let $B_1, B_2 \ldots$ be pairwise disjoint events with $B = \bigcup_{i=1}^{\infty} B_i$.

Then for any event A,

$$P(A \cap B) = \sum_{i=1}^{\infty} P(B_i)\, P(A/B_i)$$

Further if, $P(B) = 1$ or $B = \Omega$ then

$$P(A) = \sum_{i=1}^{\infty} P(B_i)\, P(A/B_i).$$

Proof

$$A \cap B = A \cap \left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} (A \cap B_i)$$

As B_is are pairwise disjoint so $(A \cap B_i)$s

Applying countable additivity axiom

$$P(A \cap B) = \sum_{i=1}^{\infty} P(A \cap B_i)$$

$$= \sum_{i=1}^{\infty} P(B_i)\, P(A/B_i)$$

If $B = \Omega \Rightarrow P(B) = 1$

$\Rightarrow P(B^c) = 0$.

Hence,

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

$$= P(A \cap B) + 0.$$

$$= \sum_{i=1}^{\infty} P(B_i)\, P(A/B_i).$$

Conditionally Independent

A box contains two coins — a regular & a fake, two headed coin.

I choose ~~two~~ a. coins sequentially at random and toss it twice.

Events are

A : First coin results H

B : Second coin results H

C : Regular coin has been selected.

Find $P(A/C)$, $P(B/C)$, $P(A \cap B/C)$, $P(A)$, $P(B)$ and $P(A \cap B)$

Note that, A and B are not independent, but they are conditionally independent given C.

Two events A and B are conditionally indeped given an event C with $P(C) > 0$ if

$$P(A \cap B / C) = P(A/C)\, P(B/C)$$

We know, $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ if $P(B) > 0$.

Conditioning with C,

$$P(A/B, C) = \frac{P((A \cap B)/C)}{P(B/C)}$$

$$= \frac{P(A/C)\, P(B/C)}{P(B/C)}$$

$$= P(A/C)$$

Thus if A and B are conditionally ind. to C.

$$P(A/B, C) = P(A/C).$$

$$P(A/c) = P(B/c) = \frac{1}{2}$$

$$\Rightarrow P(A \cap B/c) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

$$P(A) = P(A/c)\,P(C) + P(A/c^c)\,P(c^c)$$
$$= \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4}.$$

$$P(B) = \frac{3}{4}$$

$$P(A \cap B) = P(A \cap B/c)\,P(c) + P(A \cap B/c^c)\,P(c^c)$$
$$= \frac{1}{4} \cdot \frac{1}{2} + P(A/c^c)\,P(B/c^c)\,P(c^c)$$
$$= \frac{1}{8} + 1 \cdot 1 \cdot \frac{1}{2}$$
$$= \frac{5}{8}.$$

Check: $P(A \cap B) \neq P(A) \cdot P(B)$

Bayes' Theorem

For any two events A and B, $P(A) \neq 0$

$$P(B/A) = \frac{P(A/B)\,P(B)}{P(A)}.$$

If $B_1, B_2, B_3 \ldots$ form a partition of sample space Ω, A is any event with $P(A) \neq 0$

$$P(B_i/A) = \frac{P(A/B_i)\,P(B_i)}{\sum_i P(A/B_i)\,P(B_i)}$$

For the previous example we got $P(R) = .6$
Now, suppose that chosen marble is Red, what is the probability Bag 1 is chosen?

$$P(B_1/R) = \frac{P(R/B_1)\,P(B_1)}{P(R)}$$

$$= \frac{.75 \times \frac{1}{2}}{.6} = \frac{5}{12}.$$