Discrete uniform distribution

Random experiment: Choice of x, where $1 \leq x \leq 100$.

$$X \sim \text{discrete uniform } [a, b]$$

pmf $\boxed{f(x) = \frac{1}{b-a+1}}$

Cumulative distⁿ fⁿ

cdf $\boxed{F(x) = P(X \leq x) = \frac{x-a+1}{b-a+1}}$.

Mean

$$E(X) = \sum_x x f(x) = \boxed{\frac{a+b}{2} \leftarrow \text{Mean}}$$

Variance

$$E(X^2) = \sum_x x^2 f(x) = \frac{1}{(b-a)+1} \sum_x x^2$$

$$V(X) = E(X^2) - \{E(X)\}^2 = \boxed{\frac{(b-a+1)^2 - 1}{12}}.$$

Var

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Bernoulli trial

A Random Exp $\longrightarrow$ only two possible outcomes 'success' & 'Failure'

Bernoulli RV $\longrightarrow$ $X = \begin{cases} 1 & \text{if outcome is } S \\ 0 & \text{if outcome is } F \end{cases}$

If probability of success is p.

$\Rightarrow P(X=1) = p$ and $P(X=0) = 1-p = q$ (say)

Binomial Distribution

- Bernoulli trials are conducted n times
- Trials are independent
- Prob. of success p does not change betⁿ trials.

If X : number of success in n independent trials

PMF of X $\rightarrow$ $f_X(x) = {}^nC_x \, p^x (1-p)^{n-x}$

$x = 0, 1, 2 \cdots n$.

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	Draw with Replacement (Prob of success const)	Draw without Replacement (Success Prob Chang)
Fixed no. of trials	Binomial (Bernoulli – special case $n = 1$)	Hypergeometric
Draw until r^{th} success	Negative Binomial (Geometric – special case $r = 1$)	

$X \sim \text{Binomial}(n, p)$

Properties :

1. $$\sum_x f(x) = \sum_x {}^nc_x\, p^x q^{n-x} = 1.$$

2. $$E(X) = \sum_{x=0}^{n} x\, {}^nc_x\, p^x q^{n-x}$$
$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x}.$$
$$= np \sum_{x-1=0}^{n-1} {}^{(n-1)}c_{(x-1)}\, p^{x-1} q^{(n-1)-(x-1)}$$
$$= np.$$

3. $$E(X^2) = n(n-1)p^2 + np$$

or

$$E(X(X-1)) = n(n-1)p^2$$

4. $$\text{Variance}(X) = npq.$$

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$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1)\, {}^nc_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} p^x q^{n-x}.$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!} p^x q^{n-x}.$$

$$= n(n-1)p^2 \left[\sum_{(x-2)=0}^{n} {}^{n-2}c_{x-2}\, p^{x-2} q^{n-x} \right]$$

$$= n(n-1)p^2.$$

$$E(X^2) - \{E(X)\}^2 = E\, X(X-1) + E(X) - \{E(X)\}^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= npq.$$

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5. Moment Generating f. (mgf)

$$M(t) = \sum_{x=0}^{n} e^{tx} \, {}^{n}c_{x} \, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} {}^{n}c_{x} \, (pe^{t})^{x} q^{n-x}$$

$$= (pe^{t} + q)^{n}$$

A balanced coin is tossed 4 times. Find the probability of getting (1) exactly one head (2) atleast 2 heads (3) At most 3 heads.

Ans. (1) ${}^{4}c_{1} \left(\frac{1}{2}\right)^{1} \left(1-\frac{1}{2}\right)^{3} = \frac{1}{4}$

(2) $P(X=2) + P(X=3) + P(X=4) = \frac{11}{16}$

(3) $P(X=0) + P(X=1) + P(X=2) + P(X=3) = 1 - P(X=4)$

$= \frac{15}{16}$

A fair die is rolled n times. Find the number of trials such that probability of atleast one 6 to be greater than equal to $\cdot 5$.

$\Rightarrow 1 - \left(\frac{5}{6}\right)^{n} \geq \cdot 5$

or, $n \cong 3.8$

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A communication system consists of n components, each of which will independently function with probability p. The total system will be able to operate effectively if atleast one-half of its components function.

(a) For what value of p is a 5-component system more likely to operate effectively than a 3-compont system?

(b) In general, when is a $(2k+1)$ component system better than a $(2k-1)$ component system?

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(a) 5 component system will work with prob.

$$^5C_3\, p^3(1-p)^2 + {}^5C_4\, p^4(1-p) + p^5 \quad \text{— (A)}$$

3 component system :—

$$^3C_2\, p^2(1-p) + p^3 \quad \text{— (B)}$$

5 comp system is better than 3 comp. sys.

$\Rightarrow$ (A) > (B) $\Rightarrow 3(p-1)^2(2p-1) \geq 0 \Rightarrow p \geq \frac{1}{2}$.

(b) X : The number first $(2k-1)$ that function.

$$P_{2k+1}(\text{effective}) = P(X \geq k+1)$$
$$+ P(X=k)\{P(\text{atleast one function from remaining two})$$
$$+ P(X=k-1)P(\text{both function})$$
$$= P(X \geq k+1) + P(X=k)(1-q^2) + P(X=k-1)\cdot p^2 \quad \text{— (A)}$$

$$P_{2k-1}(\text{effective}) = P(X \geq k)$$
$$= P(X=k) + P(X \geq k+1) \quad \text{— (B)}$$

(A) > (B) $\Rightarrow P(X=k-1)\cdot p^2 - P(X=k)(1-p)^2 > 0$

or ${}^{2k-1}C_{k-1}\, p^{k-1} q^k \cdot p^2 - {}^{2k-1}C_k\, p^k q^{k-1} q^2 > 0$

$\Rightarrow p \geq \frac{1}{2}$.

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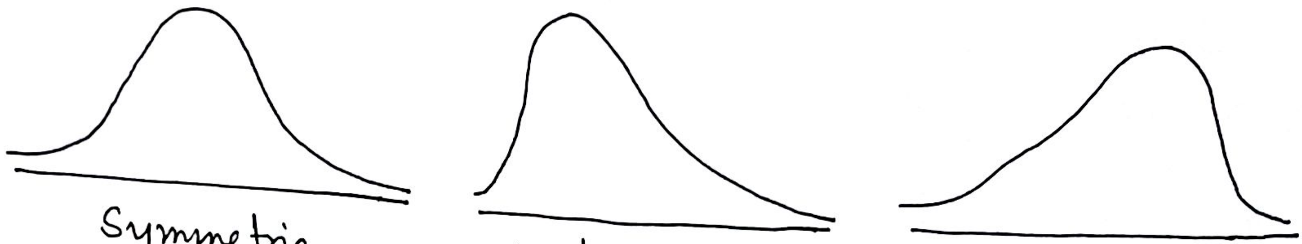
Skewness

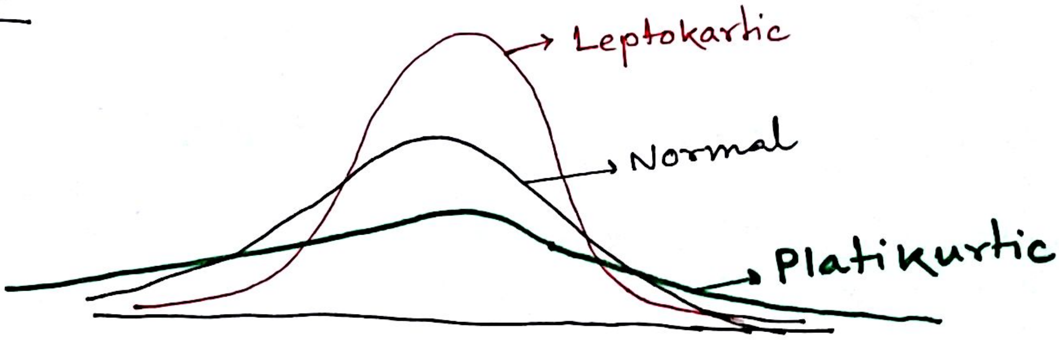


Symmetric   +vely skewed   −vely skewed

Measure

$$\boxed{\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3}{\sigma^3}}$$

Kurtosis



Measure

$$\boxed{\beta_2 = \frac{\mu_4}{\mu_2^2} - 3}$$

$\beta_2 = 0$ Normal

> 0 Leptokartic

< 0 Platikurtic

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Quantiles

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Skewness of Binomial Distribution

$$\beta_1 = \frac{\mu_3}{\sigma^3} = \frac{np(1-p)(1-2p)}{(npq)^{3/2}}$$

$$= \frac{1-2p}{(npq)^{1/2}}$$

$\Rightarrow$ When $p = \frac{1}{2}$, distribution is symmetric

i.e. r^{th} term $= (n-r)^{th}$ term

& $p < \frac{1}{2}$, $\mu_3 > 0$ $\qquad p > \frac{1}{2}$, $\mu_3 < 0$

Kurtosis of Binomial Distribution

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3$$

$$= \frac{3(npq)^2 + npq(1-6pq)}{(npq)^2} - 3$$

$$= \frac{1-6pq}{npq}$$

$\Rightarrow \beta_2 = 0$ if $pq = \frac{1}{6}$

> 0 if $pq < \frac{1}{6}$

< 0 if $pq > \frac{1}{6}$.

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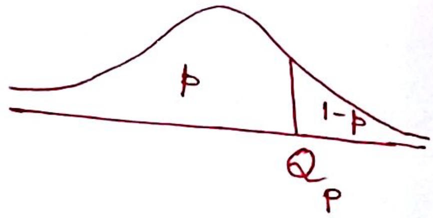
Quantiles

A number Q_p satisfying

$$P(X \leq Q_p) \geq p \quad , \; 0<p<1$$

and $P(X \geq Q_p) \geq 1-p$

is called <u>quantile of order p</u>



$Q_{1/2} \to$ Median of X

$Q_{1/4}, Q_{1/2}, Q_{3/4} \longrightarrow$ Quartiles of X

$Q_{1/10}, Q_{2/10} \cdots Q_{10/10} \to$ Deciles of X

$Q_{1/100}, Q_{2/100} \cdots Q_{100/100} \to$ Percentiles.

Let CDF $F(x)$ of a RV X such that $F(x)=0$ for $x \leq 0$, $F(x) = \frac{x}{2}$ for $0<x\leq 2$ and $F(x) = 1$ for $x > 2$.

Find 1^{st}, 2^{nd} and 3^{rd} Quartiles.

$$F(Q_{1/4}) = .25 \quad , \quad F(Q_{1/2}) = .5 \quad \& \quad F(Q_{3/4}) = .75$$

$$\Rightarrow Q_{1/4} = .5 \quad , \quad Q_{1/2} = 1 \quad , \quad Q_{3/4} = 1.5$$

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Geometric Distribution

Suppose independent Bernoulli trials are conducted, till a success is achieved

$X \to$ No. of trials needed for first success

$$P(X = j) = q^{j-1} \cdot p \quad , j = 1, 2, \ldots \ldots$$

Properties

1. $\sum_{j=1}^{\infty} P(X=j) = p[1 + q + q^2 + q^3 \ldots] = \frac{p}{1-q} = 1.$

2. $E(X) = \sum_{j=1}^{\infty} j\, P(X=j) = \frac{p}{(1-q)^2} = \frac{1}{p}.$

$\Rightarrow$ We need average 6 trials to get a six.

3. $\mu_2' = \frac{q+1}{p^2}$

$\Rightarrow \operatorname{var}(X) = \mu_2' - \mu_1'^2 = \frac{q}{p^2}$

Moment generating fn

4. $M_X(t) = \sum_{j=1}^{\infty} e^{tj} q^{j-1} p$

$$= pe^t \sum_{j=1}^{\infty} (qe^t)^{j-1}$$

$$= \frac{pe^t}{1 - qe^t}.$$

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Suppose independent tests are conducted on patients to test a medicine. If the probability of success is $\frac{1}{3}$ in each trial then what is the probability that atleast 5 trials are needed to get the first success.

pmf $\Rightarrow$ $P(X=j) = \left(\frac{2}{3}\right)^{j-1}\left(\frac{1}{3}\right)$, $j = 1, 2 \ldots$

Required Probability

$$P(X \geq 5) = \sum_{j=5}^{\infty} \left(\frac{2}{3}\right)^{j-1}\left(\frac{1}{3}\right)$$

$$= \left(\frac{2}{3}\right)^4 \cdot \frac{1}{3} \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m$$

$$= \left(\frac{2}{3}\right)^4 \cdot \frac{1}{3} \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right]$$

$$= \frac{\left(\frac{2}{3}\right)^4 \cdot \frac{1}{3}}{1 - 2/3} = \left(\frac{2}{3}\right)^4.$$

Geometric Distribution has Memoryless Property

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Memoryles Property of Geometric Distribution

$$P(X > m) = \sum_{j=m+1}^{\infty} q^{j-1} p$$

$$= q^m \cdot p (1 + q + q^2 + \cdots)$$

$$= \frac{q^m \cdot p}{1-q} = q^m$$

$$P\{\underbrace{(X > m+n)}_{A} / \underbrace{(X > n)}_{B}\}$$

$$= \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{q^{m+n}}{q^n} = q^m$$

$\Rightarrow$ Starting point is immeterial.

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Negative Binomial Distribution

Now let us observe the number of trials needed until r^{th} success occurs

$$P(X = n) = {}^{n-1}C_{r-1}\, p^{r-1}\, q^{n-r} \cdot p$$

$$= {}^{n-1}C_{r-1}\, p^{r}\, q^{n-r}.$$

Moment Generating function

$$M_X(t) = E(e^{tX}) = \sum_{x=r}^{\infty} e^{tx}\, {}^{x-1}C_{r-1}\, p^{r}\, q^{x-r}$$

$$= \sum_{x=r}^{\infty} {}^{x-1}C_{r-1}\, (pe^{t})^{r}\, (qe^{t})^{x-r}$$

$$= \frac{(pe^{t})^{r}}{(1-qe^{t})^{r}}$$

$$E(X) = \frac{r}{p} \qquad \text{Variance}(X) = \frac{rq}{p^2}.$$

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An oil company conducts a geological study that an exploratory oil well should have a 20% chance of striking oil.

(a) What is the probability that the first strike comes on the third well drill?

(b) What is the probability that the third strike comes on the seventh well drilled?

(c) What is the mean and variance of the number of wells that must be drilled if the oil company wants to ~~get~~ set up three producing wells?

(a) $P(X=3) = {}^{3-1}C_{1-1}\, p^1 q^2 = \cdot 2 \times (\cdot 8)^2 = \cdot 128$

(b) $P(X=7) = {}^{7-1}C_{3-1}\, p^3 q^{7-3} = \cdot 049$.

(c) $\text{Mean} = \dfrac{r}{p} = \dfrac{3}{\cdot 20} = 15$

$\text{var}(x) = \dfrac{rq}{p^2} = 60$.

Poisson Model

Limiting form of a Binomial distribution under following conditions

- → The number of trials very large
- → The probability of success p be very small
- → $np = \lambda$, a finite quantity when $n \to \infty, p \to 0$

$$P(X = x) = {}^nC_x \, p^x q^{n-x}$$

$$= \frac{n(n-1)\cdots(n-x+1)}{x!} p^x \cdot (1-p)^n (1-p)^{-x}$$

$$= \frac{n}{n} \cdot \frac{(n-1)}{n} \cdots \frac{(n-x+1)}{n} (np)^x \cdot \frac{1}{x!} (1-p)^n (1-p)^{-x}$$

$$= 1 \cdot \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{x-1}{n}\right) \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$$

As $n \to \infty$

$$\left(1 - \frac{1}{n}\right) \to 1, \quad \left(1 - \frac{x-1}{n}\right) \to 1 \quad \left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda}.$$

$$\boxed{f(x) = \frac{\lambda^x e^{-\lambda}}{x!}} \leftarrow \text{Poisson Distribution.}$$

Moments of Poisson Distribution

1. $\boxed{\text{Mean } (\mu) = \lambda}$

$$E(X) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!} = \lambda e^{-\lambda} \sum_{x=}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots\right)$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda.$$

2. $\boxed{\text{Variance}(X) = \lambda}$

3. $\boxed{\text{Skewness} = \frac{1}{\lambda}}$ $\boxed{\text{Kurtosis} = 3 + \frac{1}{\lambda}}$

4. Moment Generating Function

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot (\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^{t}} = e^{\lambda(e^{t}-1)} \longrightarrow \text{mgf}$$

Alpha particles are emitted by a radio active source at an average rate of 5 in a 20 min interval. What is the probability that (I) there will be exactly 2 emissions in a particular 20 min interval? (II) there will be atleast 3 emissions in 20 min interval.

(I) Probability that exactly 2 emissions in 20 minutes

$$= \frac{5^2}{2!} e^{-5} = .0843$$

(II) Probability of atleast 3 emissions

$$P(x=3) + P(x=4) + \cdots$$

$$= \sum_{x=3}^{\infty} \frac{5^x}{x!} e^{-5}$$

$$= 1 - \sum_{x=0}^{2} \frac{5^x}{x!} e^{-5}$$

$$= .8753.$$

Hyper-Geometric Distribution

An urn contains N balls — M of which are white and $(N-M)$ are red.

Draw a sample of n balls without replacement

Prob(getting k white balls)

$$= \frac{({}^{M}C_{k}) \times ({}^{N-M}C_{n-k})}{{}^{N}C_{n}}$$

Then, X will follow Hypergeometric distn.

$$\boxed{P(X=k) = \frac{({}^{M}C_{k}) \times ({}^{N-M}C_{n-k})}{{}^{N}C_{n}}}$$

Here, $0 \leq k \leq M$, $0 \leq (n-k) \leq (N-M)$

$$\Rightarrow \max(0, n-N+M) \leq k \leq \min(M, n)$$

$$E(X) = \frac{nM}{N}$$

$$Var(X) = \frac{NM(N-M)(N-n)}{N^2(N-1)}.$$

Hypergeometric converges to Binomial distn.

if $N \to \infty$, $\frac{M}{N} \to p$.

Products produced by a machine has a 3% defective rate. What is the probability that the first defective occurs in the fifth item inspected?

Geometric dis^n:

$$P(X=5) = P(4 \text{ non-defectives})\, P(5^{th} \text{ defective})$$

$$= (.97)^4 (.03) = .02655$$

What is the average number of inspections to obtain the first defective

$$E(x) = \sum_{x=1}^{\infty} x\, q^{x-1} p = \frac{1}{p} = \frac{1}{.03}.$$

An item is produced in large numbers. The machine is known to produce 5% defectives. A quality control inspector is examining the items by taking them at random.

What is the probability that atleast 4 items are to be examined in order to get 2 defectives?

Negative binomial with $r=2$, $p = .05$

$$P(X=4) + P(X=5) + \cdots = \sum_{x=4}^{\infty} {}^{x-1}C_{2-1} (.05)^2 (.95)^{x-2}$$

$$= .9928.$$

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