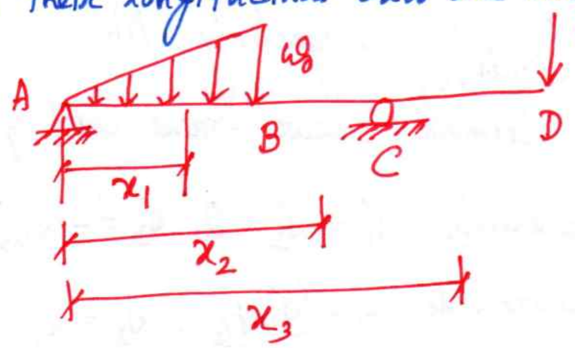
Bending of Beams

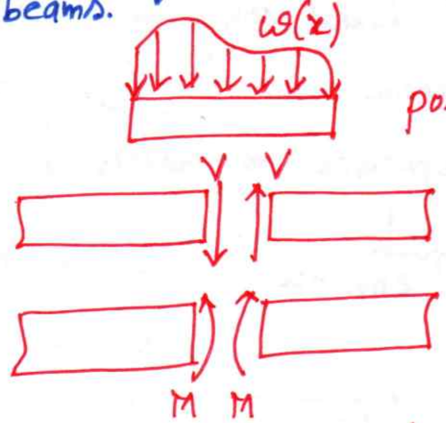
Simply supported beam

Cantilever beam

Overhanging beam

Members that are slender and support loading that are applied perpendicular to their longitudinal axis are called beams.





positive distributed load

positive internal shear

positive internal moment

Beam sign convention

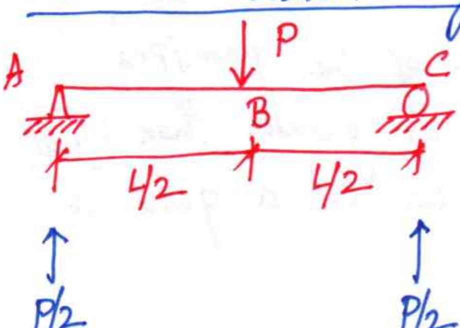
In order to properly design a beam, it is first neessary to determine the maximum shear and moment in the beam. One way to do so this is to express V and M as functions of the arbitrary position x along the beam's axis. These shear and moment functions can then be plotted and represented by graphs called shear and moment diagrams. The maximum values of V and M can then be obtained from these graphs.

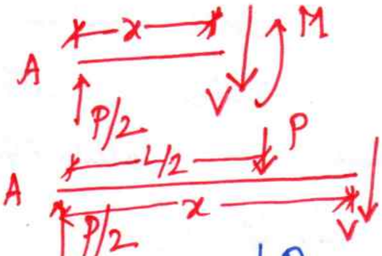
Procedure for analysis :

Support reaction - Determine all the reactive forces and couple moments acting on the beam, and resolve all the forces into components acting perpendicular and parallel to the beam's axis.

Shear and Moment functions •

Shear and Moment diagrams



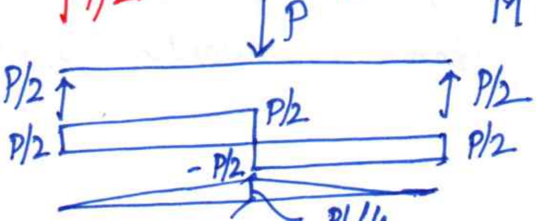


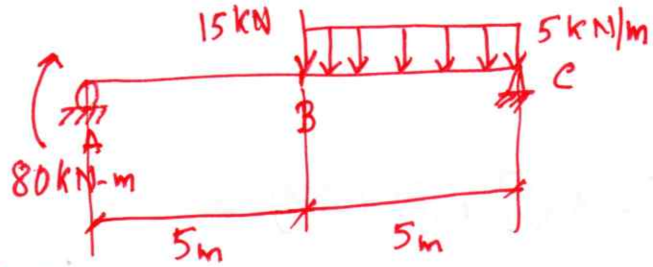
$V = P/2$

$M = P/2\, x$

$V = -P/2$

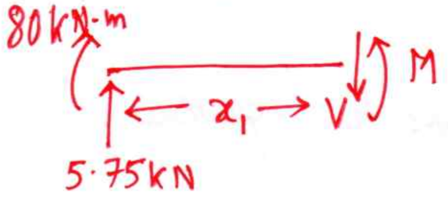
$M = P/2\,(L - x)$





Support reaction, $R_A = 5.75$ kN

$R_B = 34.25$ kN



$0 \le x_1 \le 5\text{m}, \quad V = 5.75 \text{ kN}.$

$$-80 - 5.75x_1 + M = 0$$

$$\Rightarrow M = (5.75x_1 + 80) \text{ kN-m}.$$

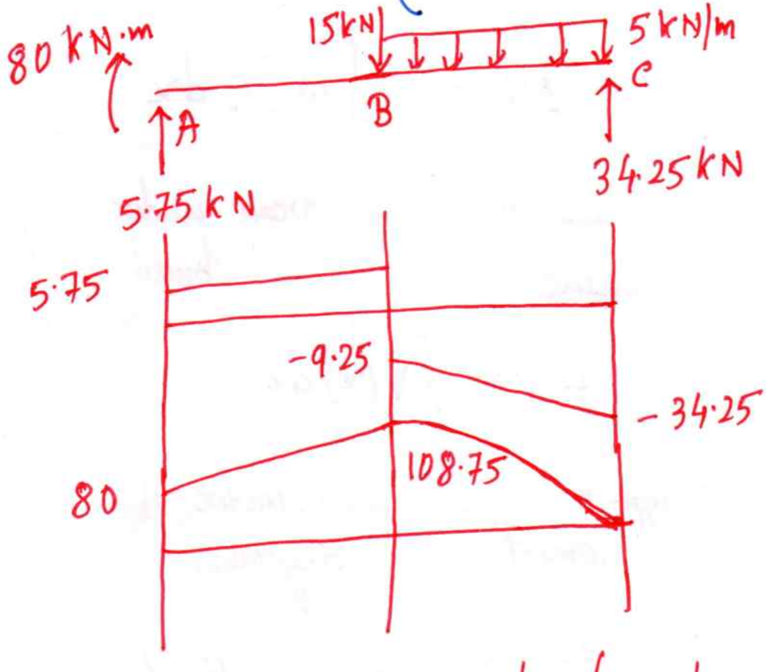
$5\text{m} < x_2 \le 10\text{m}, \quad 5.75\text{ kN} - 15\text{ kN} - 5\text{ kN/m}\,(x_2 - 5) - V = 0$

$$\Rightarrow V = (15.75 - 5x_2) \text{ kN}.$$

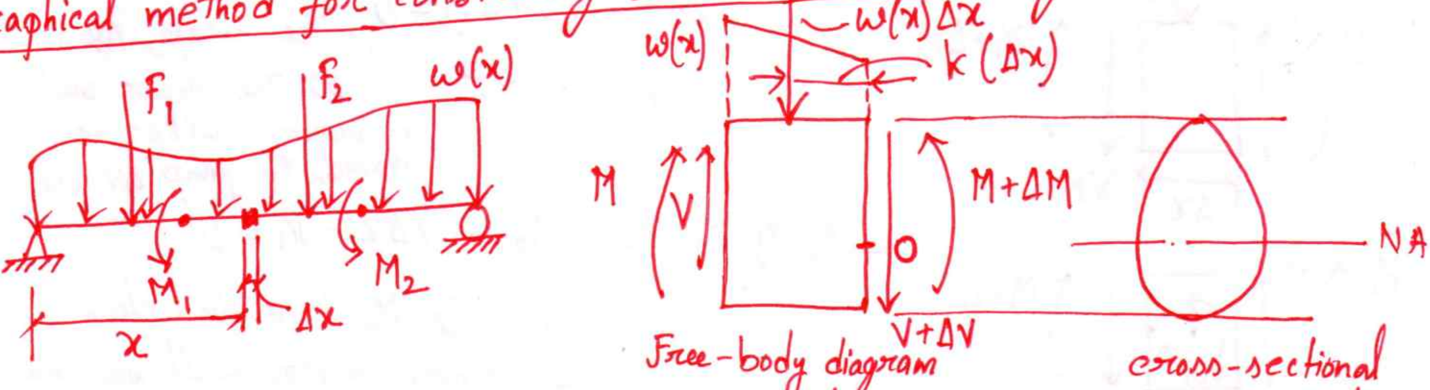
$$-80 - 5.75x_2 + 15(x_2 - 5) \cancel{= 0}$$

$$+ 5(x_2 - 5)\left(\frac{x_2 - 5}{2}\right) + M = 0$$

$$\Rightarrow M = \left(-2.5x_2^2 + 15.75x_2 + 92.5\right) \text{ kN-m}$$



Graphical method for constructing shear and moment diagram:



Free-body diagram of segment Δx

cross-sectional area of segment

$$\Sigma F_y = 0 \Rightarrow V - w(x)\,\Delta x - (V + \Delta V) = 0$$
$$\Rightarrow \Delta V = -w(x)\,\Delta x$$
$$\Sigma M_0 = 0 \Rightarrow -V\Delta x - M + w(x)\,\Delta x\,[k(\Delta x)] + (M + \Delta M) = 0$$
$$\Rightarrow \Delta M = V\Delta x - w(x)\,k(\Delta x)^2$$

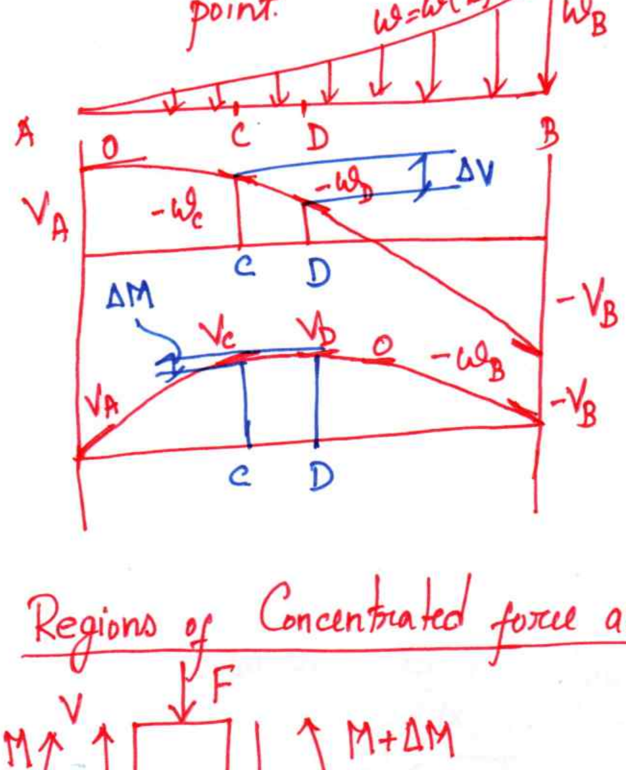
Dividing by Δx and taking the limit as $\Delta x \to 0$, the above two equations become

$$\frac{dV}{dx} = -w(x)$$

slope of shear diagram at each point. = − distributed load intensity at each point.

$$\frac{dM}{dx} = V$$

slope of moment diagram at each point. = shear at each point.



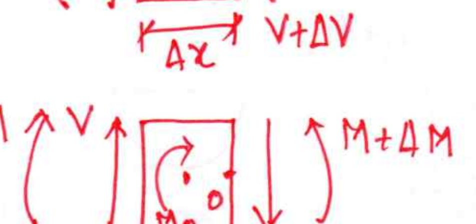
$$\Delta V = -\int w(x)\,dx$$

change in shear = − area under distributed loading

$$\Delta M = \int V(x)\,dx$$

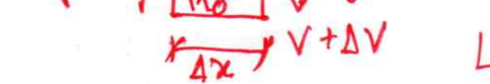
change in moment = area under shear diagram.

Regions of Concentrated force and moment:



$$V - F - (V + \Delta V) = 0$$
$$\Rightarrow V = -F$$

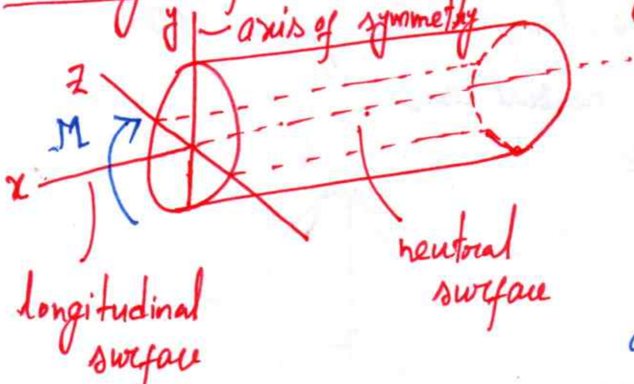
When F acts downward on the beam, ΔV is negative so the shear will "jump" downward. Likewise if F is upward, the jump ΔV will be upward.



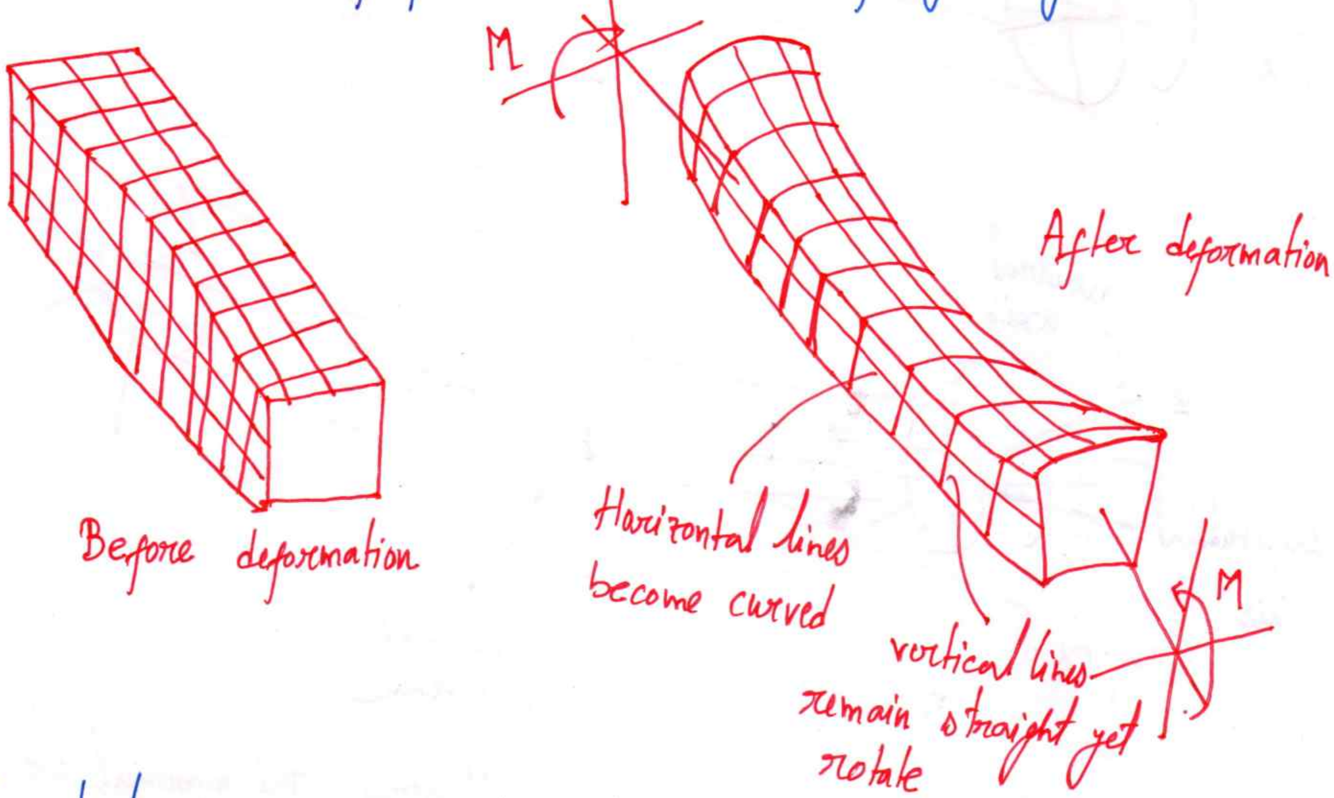
$$\Sigma M_0 = 0,\ M + M - M_0 - V\Delta x - M = 0$$
$$\Rightarrow \Delta M = M_0$$

If M_0 is applied clockwise, ΔM is positive so the moment diagram will jump upward. Likewise, M_0 acts counterclockwise the jump (ΔM) will be downward.

Bending deformation of a straight member:



We will discuss the deformations that occur when a straight prismatic beam, made of homogeneous material, is subjected to bending. The discussion will be limited to beams having a cross-sectional area that is symmetrical with respect to an axis and the bending moment is applied about an axis perpendicular to this axis of symmetry.

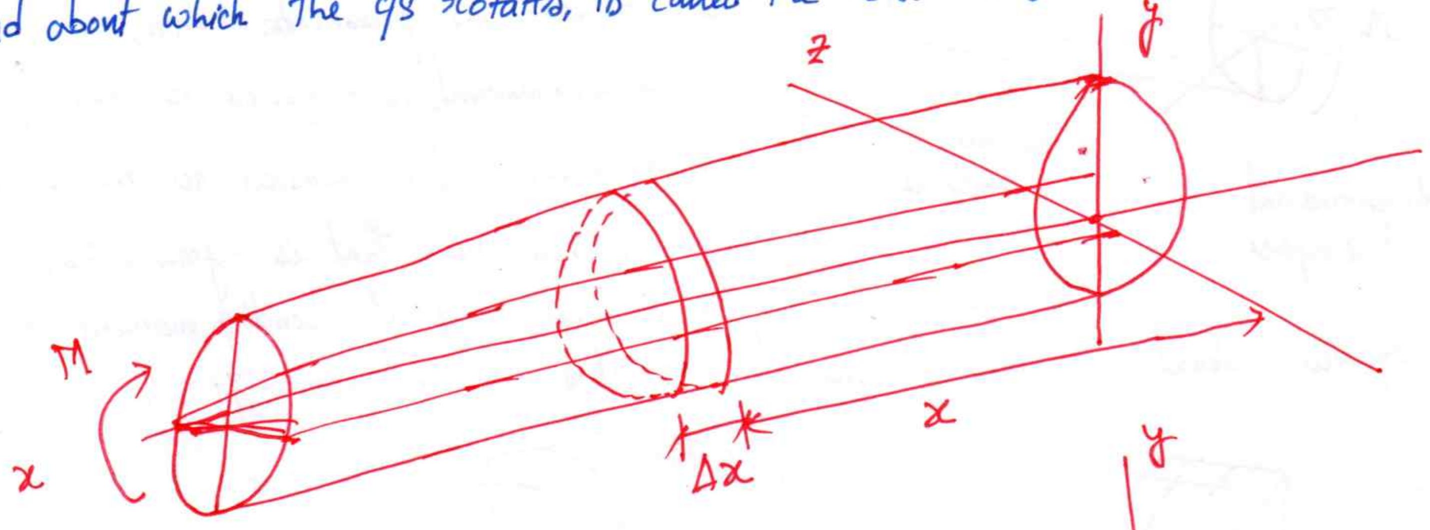


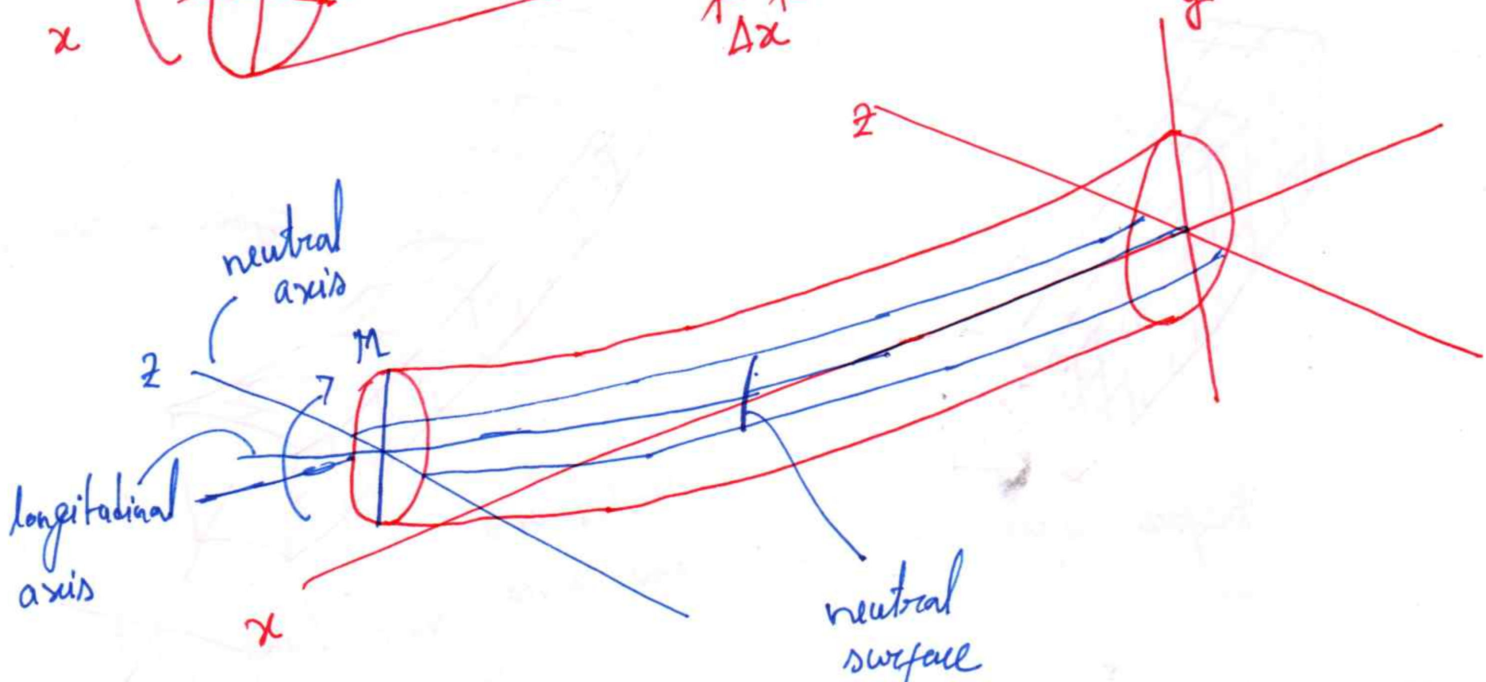
The behaviour of any deformation bar subjected to a bending moment causes the material within the bottom portion of the bar to stretch and the material within the top portion to compress. Consequently, between these two lines/regions there must be a surface, called the neutral surface, in which the longitudinal fibers of the material will not undergo a change in length.

Assumptions: ① The longitudinal axis x, which lies within the natural surface (next page figure) doesnot experience any change in length. Rather the moment will tend to deform the beam so that this line becomes a curve that lies in the x-y plane of symmetry.

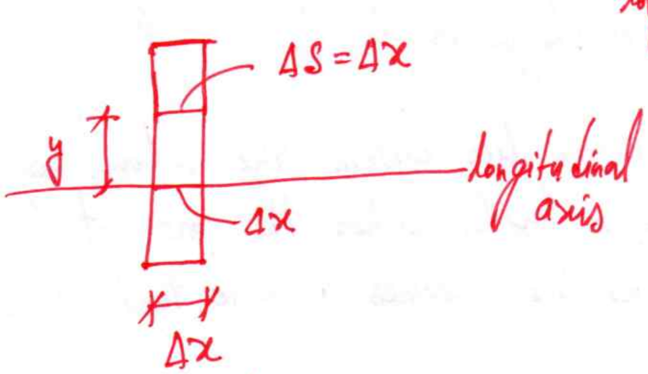
② All c/s of the beam ~~rep~~ remains plane and perpendicular to the longitudinal axis during deformation.

(3) Any deformation of the c/s within its own plane will be neglected. In particular, the z-axis, lying in the plane of the c/s and about which the c/s rotates, is called the neutral axis.

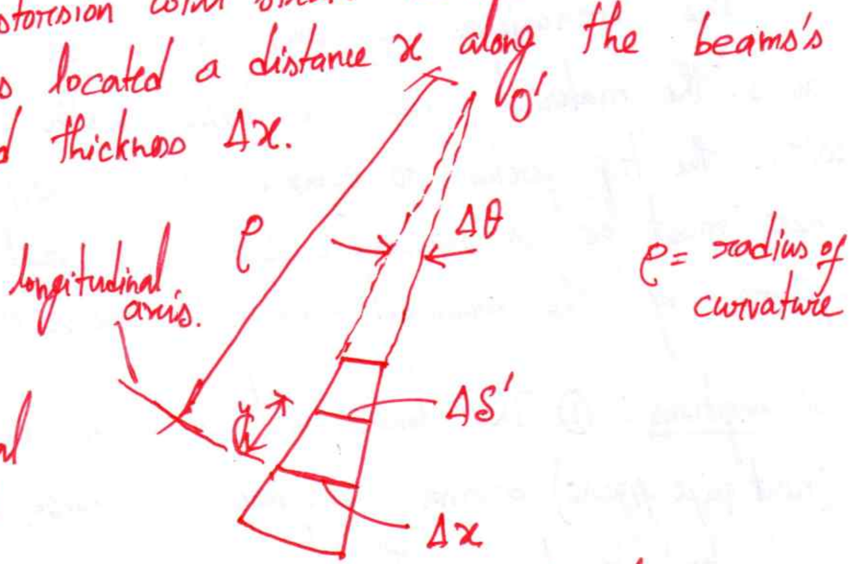




In order to show how this distorsion will strain the material, we will isolate a segment of the beam that is located a distance x along the beams's length and has an undeformed thickness Δx.



undeformed element



deformed element.

Notice that any line segment Δx, located on the neutral surface does not change in length, whereas any line segment ΔS, located at the arbitrary distance y above the neutral surface, will contract and become $\Delta S'$ after deformation.

By definition, the normal strain along ΔS is $\epsilon = \lim\limits_{\Delta S \to 0} \dfrac{\Delta S' - \Delta S}{\Delta S}$

Before deformation: $\Delta S = \Delta x$

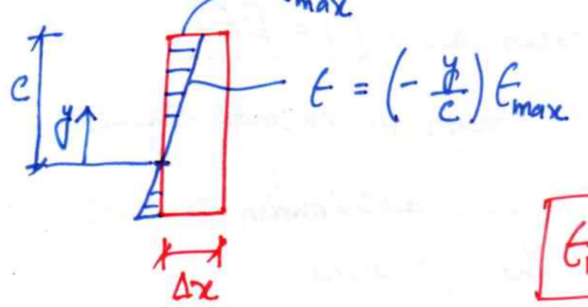
After deformation Δx has a radius of curvature ρ, with center of curvature at point O'.

$\Delta x = \Delta S = \rho \Delta\theta$ Since $\Delta\theta$ defines the angle between the c/s sides of the element

In the same manner, the deformed length of ΔS becomes $\Delta S' = (\rho - y)\Delta\theta$.

$$\epsilon = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} \Rightarrow \boxed{\epsilon = -\frac{y}{\rho}}.$$

This indicates that the longitudinal normal strain of any element within the beam depends on its location y on the cross-section and the radius of curvature of the beam's longitudinal axis. For any specific cross-section, the longitudinal normal strain will vary linearly with y from the neutral axis.

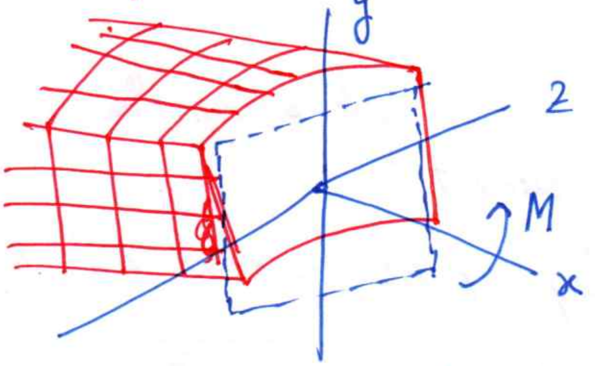


A contraction $(-\epsilon)$ will occur in fibers located above the neutral axis $(+y)$ whereas elongation $(+\epsilon)$ will occur in fibers located below the axis $(-y)$.

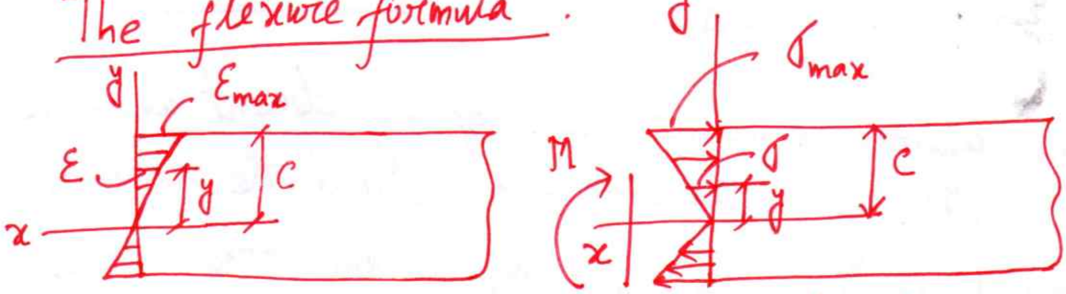
$$\boxed{\epsilon_{max} = c/\rho} \qquad \boxed{\epsilon = -\frac{y}{c}\,\epsilon_{max}}$$

This normal strain depends only on the assumptions made with regards to the deformation. Provided only a moment is applied to the beam, then it is reasonable to further assume that this moment causes a normal stress only in the longitudinal or x direction. All other components of normal and shear stresses are zero, since the beam's surface is free from any other loads.

It is this uniaxial state of stress that causes the material to have the longitudinal normal strain component ϵ or ϵ_x ($\sigma_x = E\epsilon_x$). Furthermore, by Poisson's ratio there must also be associated strain components $\epsilon_y = -\nu\epsilon_x$ and $\epsilon_z = -\nu\epsilon_x$, which deform the planes of the c/s area, although here we have neglected these deformation. Such deformations will, however, cause the c/s dimensions to become smaller below the neutral axis and larger above the neural axis.



The flexure formula:



Normal strain variation (profile view)

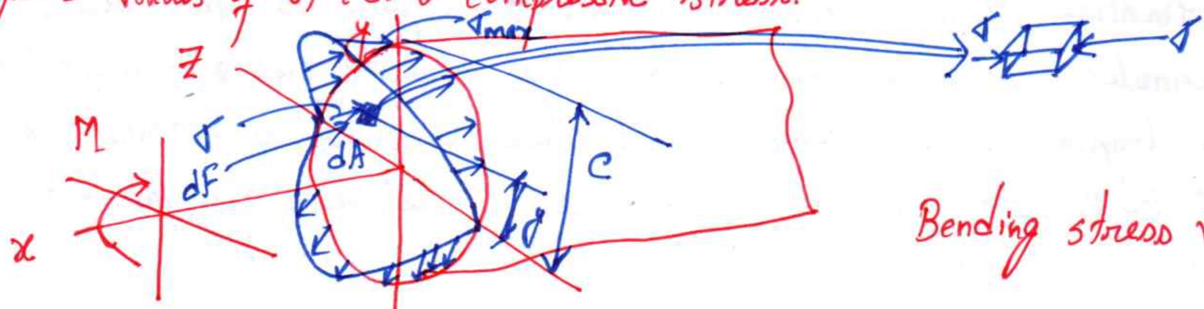
Bending stress variation (profile view).

Material behaves in a linear-elastic manner - Hooke's law - $\boxed{\sigma = E\epsilon}$.

A linear variation of normal strain $\Rightarrow$ a linear variation in normal stress.

$$\boxed{\sigma = -\frac{y}{c}\sigma_{max}}$$ Stress distribution accross the c/s area.

For positive M, which acts in the +z direction, positive values of y give negative values of σ, i.e. a compressive stress.



Bending stress variation

Location of neutral axis — By satisfying the condition that the resultant force produced by the stress distribution over the c/s area must be equal to zero.

$dF = \sigma dA$ acts on the arbitrary element dA.

$$F_R = \Sigma F_x ; \quad 0 = \int_A dF = \int_A \sigma dA = \int_A -\left(\frac{y}{c}\right) \sigma_{max} dA$$

$$= -\frac{\sigma_{max}}{c} \int_A y dA$$

Since $\frac{\sigma_{max}}{c} \neq 0$, $\boxed{\int_A y dA = 0}$ — The first moment of the member's c/s area about the neutral axis must be zero.

This condition can only be satisfied if the neutral axis is also the horizontal centroidal axis for the c/s.*

* The location $\bar{y}$ for the centroid of the c/s area is defined from the equation $\bar{y} = \frac{\int y dA}{\int dA}$ if $\int y dA = 0$, then $\bar{y} = 0$. So the centroid lies on the reference (neutral) axis.

Resultant internal moment M = moment produced by the stress distribution about the neutral axis.

$dM = y dF$ [moment of dF about neutral axis]

the moment is +ve since by right hand rule, the thumb is directed along (+ve) z direction.

$dF = \sigma dA$

$$(M_R)_z = \Sigma M_z ; \quad M = \int_A y dF = \int_A y(\sigma dA) = +\int_A y \left(\frac{y}{c} \sigma_{max}\right) dA$$

$$\Rightarrow \boxed{M = +\frac{\sigma_{max}}{c} \int_A y^2 dA} \Rightarrow \boxed{\sigma_{max} = +\frac{Mc}{I}}$$

σ_{max} = the max^m normal stress in the member, which occurs at a point on the c/s area farthest away from the neutral axis.

M = the resultant internal moment, determined from the method of sections and the equations of equilibrium and computed about the neutral axis of the c/s.

I = the moment of inertia of the c/s area computed about the neutral axis.

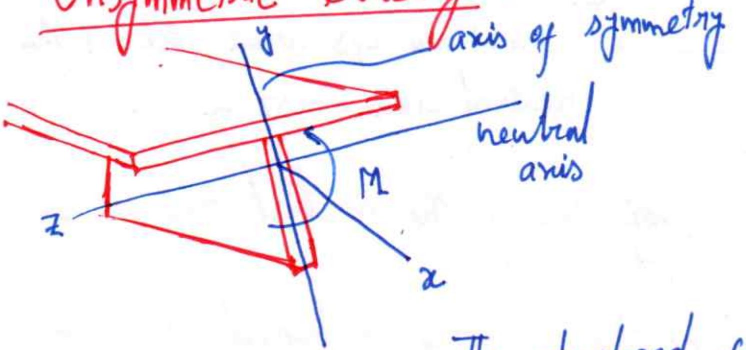
c = the perpendicular distance from the neutral axis to a point farthest away from the neutral axis where σ_{max} acts.

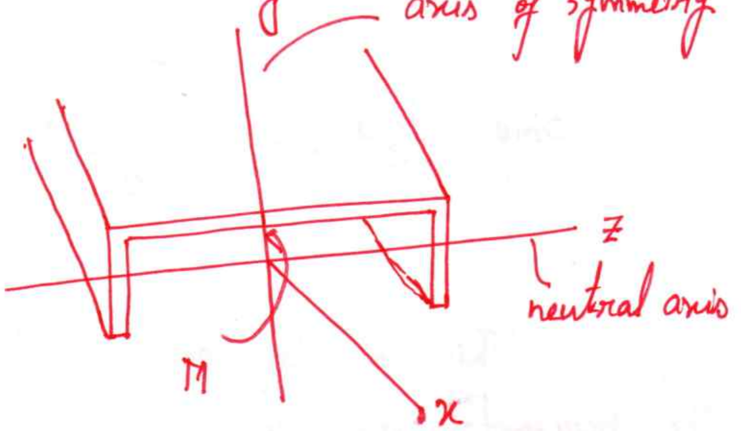
Since $\sigma_{max}/c = -\sigma/y$

$$\boxed{\sigma = -\frac{My}{I}}$$ flexure formula.

This is used to determine the normal stress in a straight member, having a c/s that is symmetrical with respect to an axis and the moment is applied perpendicular to this axis.

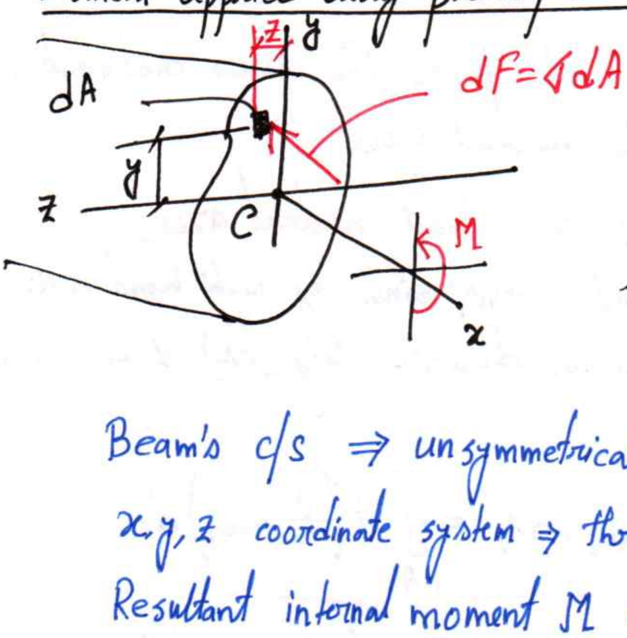
Unsymmetric Bending:

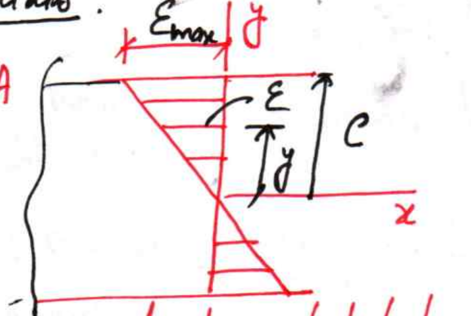




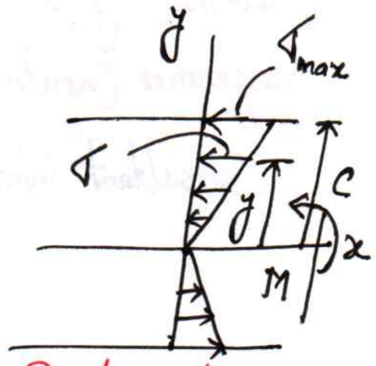
The developed flexure formula is valid for these cases.

Moment applied along principal axis:





normal strain distribution (profile view)



Bending stress distribution (profile view)

Beam's c/s ⇒ unsymmetrical.

x, y, z coordinate system ⇒ through the centroid C (origin).

Resultant internal moment M acts along the +z axis.

$dF = \sigma\, dA$

$F_R = \Sigma F_x; \quad 0 = \int_A \sigma\, dA$

$(M_R)_y = \Sigma M_y; \quad 0 = \int_A z\,\sigma\, dA$

$(M_R)_z = \Sigma M_z; \quad M = \int_A -y\,\sigma\, dA$

Stress distribution acting over the entire cross-sectional area to have a zero force resultant, the resultant internal moment about y axis to be zero, and the resultant internal moment about the z axis to equal M.

location of dA (0, y, z).

$$0 = \int_A z\,dA = -\frac{\sigma_{max}}{c}\int_A y z\,dA \quad \text{i.e.,} \quad \boxed{\int_A y z\,dA = 0}$$

product of inertia for the area

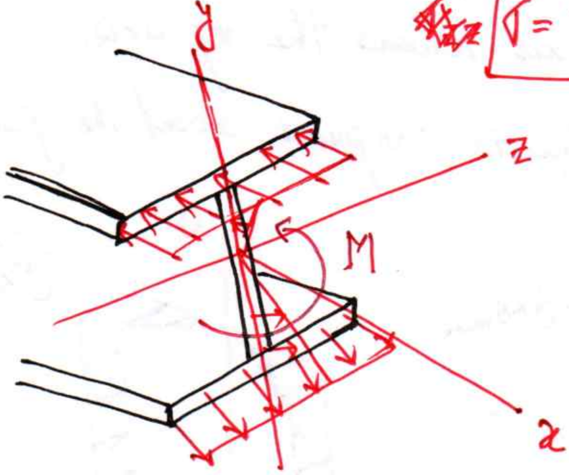
this will be zero provided the y and z axes are chosen as principal axes of inertia for the area.

For arbitrarily shaped area – principal axes can be determined by inertia transformation equation or Mohr's circle of inertia.

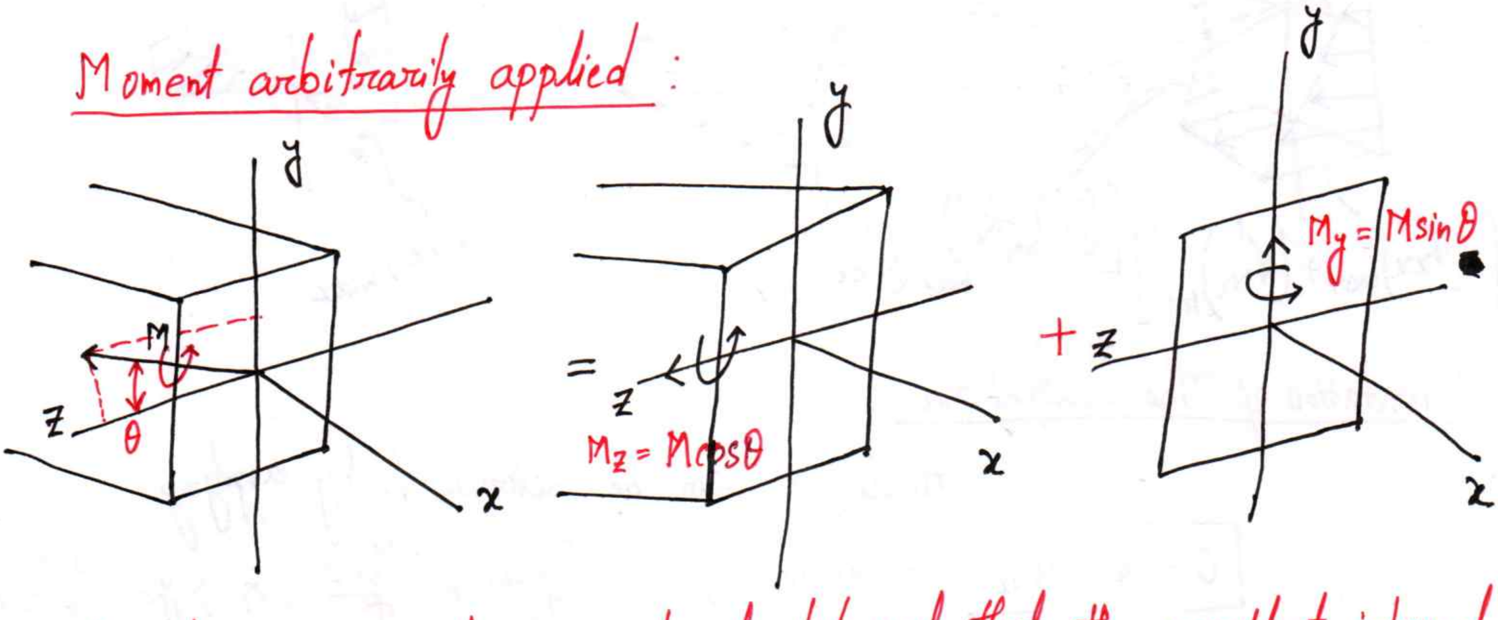
If the area has an axis of symmetry – principal axes will always be oriented along the axis of symmetry and perpendicular to it.

$$\boxed{\sigma = \sigma_{xx} = -\frac{M y}{I_z}}$$

Since M is applied about one of the principal axes (z axis).



Moment arbitrarily applied :



Sometimes a member may be loaded such that the resultant internal moment does not act about one of the principal axes of the c/s.

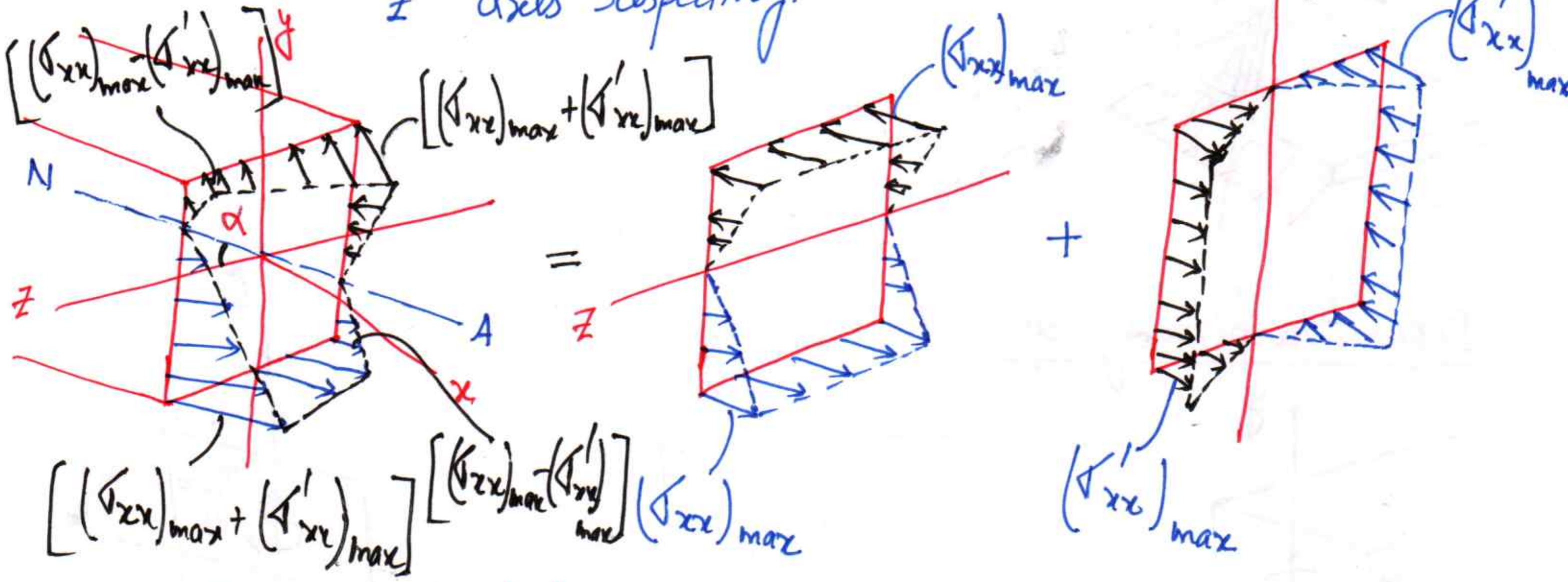
$$\boxed{\sigma = \sigma_{xx} = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}} \; *$$

$\sigma = \sigma_{xx}$ = the normal stress at the point.

y, z = the coordinates of the point measured from x, y, z axes having their origin at the centroid of the c/s area and forming a right-handed coordinate system. The x-axis is directed outward from the c/s and the y and z axes represent respectively the principal axes of min^m & max^m moment of inertia.

M_y, M_z = the resultant internal moment components directed along the principal y and z axes. They are (+ve) if directed along the $+y$ and $+z$ axes, otherwise they are negative, or stated another way, $M_y = M\sin\theta$ and $M_z = M\cos\theta$, where θ is measured +ve from the $+z$ axis toward the $+y$ axis.

I_y, I_z = the principal moments of inertia computed about the y and z axes respectively.



Orientation of the neutral axis:

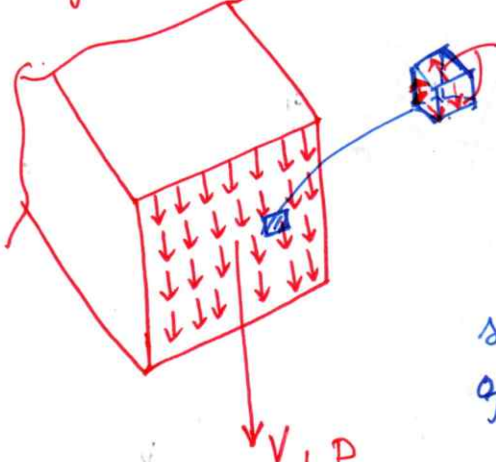
Angle α can be determined by applying $\boxed{0 = \sigma = \sigma_{xx}}$ condition. $\Rightarrow -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0 \Rightarrow y = \frac{M_y I_z}{M_z I_y} z$

if $M_z = M\cos\theta$ & $M_y = M\sin\theta \Rightarrow$ $y = \left(\frac{I_z}{I_y}\tan\theta\right) z$

equation of line that defines Neutral axis (N.A)

$\boxed{\tan\alpha = \frac{I_z}{I_y}\tan\theta}$ since, $\boxed{\alpha = y/z}$. $\boxed{\text{Note } \theta \neq \alpha}$

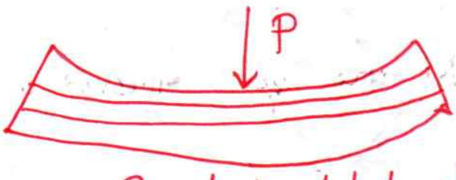
~~Composite beam~~ : Transverse Shear



Associated longitudinal shear stress will also act along longitudinal planes of the beam. (complementary property of shear).

If the top and bottom surfaces of each board are smooth, and the boards are not bonded together, then application of P will cause the boards to slide relative to one another.

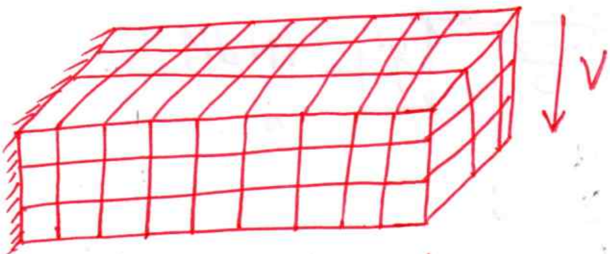
Boards not bonded together.



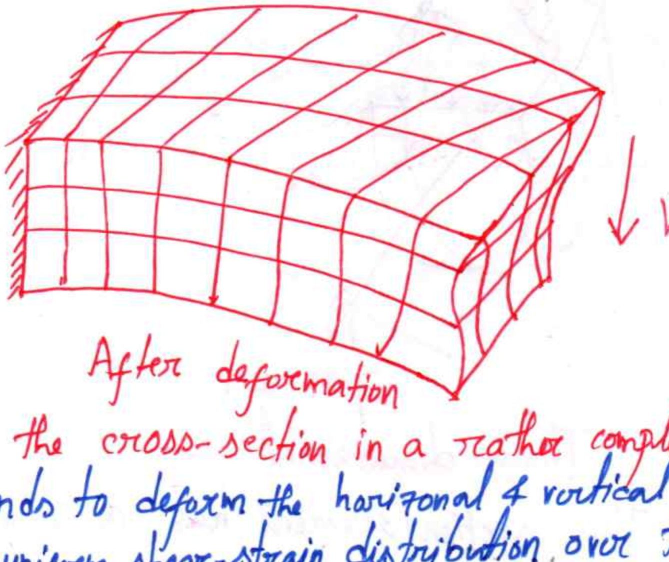
Boards bonded together

If the boards are bonded together, then the longitudinal shear stresses between the boards will prevent the relative sliding of the boards and consequently the beam will act as a single unit.

Shear stress will lead to shear strains.



Before deformation



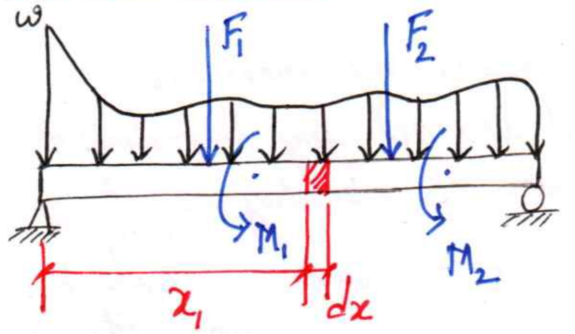
After deformation

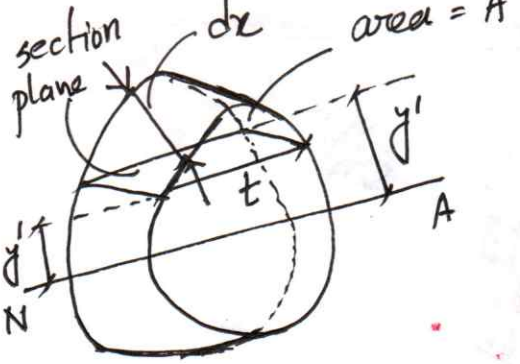
The shear strains tend to distort the cross-section in a rather complex manner. When a shear V is applied, it tends to deform the horizonal & vertical lines into the pattern shown above. This nonuniform shear-strain distribution over the cross-section will cause the cross-section to wrap, that is not to remain plane.

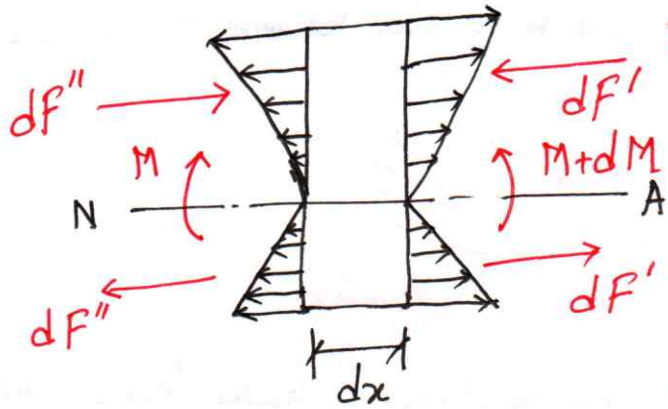
In the development of the flexure formula, we assumed that cross sections must remain plane and perpendicular to the longitudinal axis of the beam after deformation.

Although this is violated when the beam is subjected to both bending and shear, we can generally assume the cross-sectional warping described above is small enough so that it can be neglected. This assumption is particularly true for the most common case of a slender beam, i.e., one that has a small depth compared with its length.

The Shear Formula:



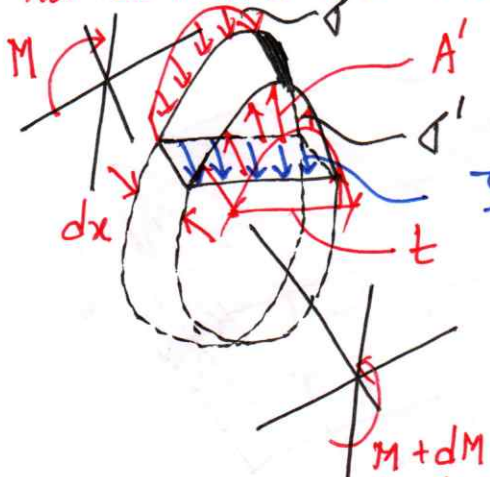




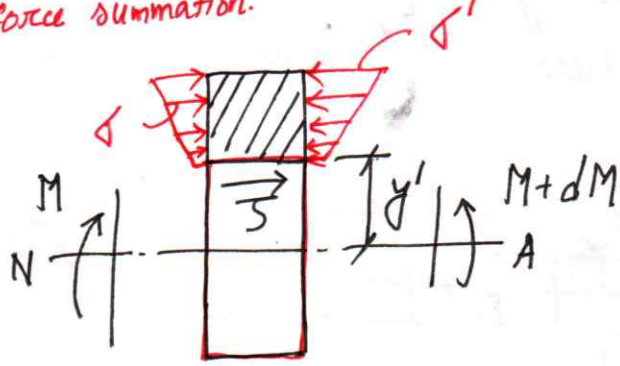
$\Sigma F_x = 0$ satisfied

$$V = \frac{dM}{dx}$$

A free body diagram of the element shows only the normal- ~~shear~~ stress distribution acting on it. This distribution is due to the bending moments M and M+dM. We have excluded the effects of V and V+dV, and $w(x)$ on the FBD since these forces are vertical and will therefore not be involved in a horizontal force summation.



Three-dimensional view.



profile view

This section/segment has a width t at the section and the c/s sides have an area A'. Because the resultant moments on each side of the element differ by dM, $\Sigma F_x = 0$ will not be satisfied unless a longitudinal shear stress τ acts over the bottom face of the segment. Here, we are going to assume that this shear stress is constant accross the width t of the bottom face. It acts on area $t\,dx$.

$$\Sigma F_x = 0 \; ; \quad \int_{A'} \sigma' dA - \int_{A'} \sigma dA - \tau (t\,dx) = 0$$

$$\Rightarrow \int_{A'} \left(\frac{M+dM}{I}\right) y\,dA - \int_{A'} \left(\frac{M}{I}\right) y\,dA - \tau(t\,dx) = 0$$

$$\Rightarrow \left(\frac{dM}{I}\right) \int_{A'} y\,dA = \tau(t\,dx) \Rightarrow \boxed{\tau = \frac{1}{It}\left(\frac{dM}{dx}\right)\int_{A'} y\,dA}$$

Now, $V = \frac{dM}{dx} \Rightarrow \boxed{\tau = \frac{V}{It}\int_{A'} y\,dA}$ this represents the first moment of area A' about the NA or Neutral Axis.

Let us assume $\int_{A'} y\,dA = Q$

The location of the centroid of the area A' is $\bar{y}' = \int_{A'} y\,dA / A'$

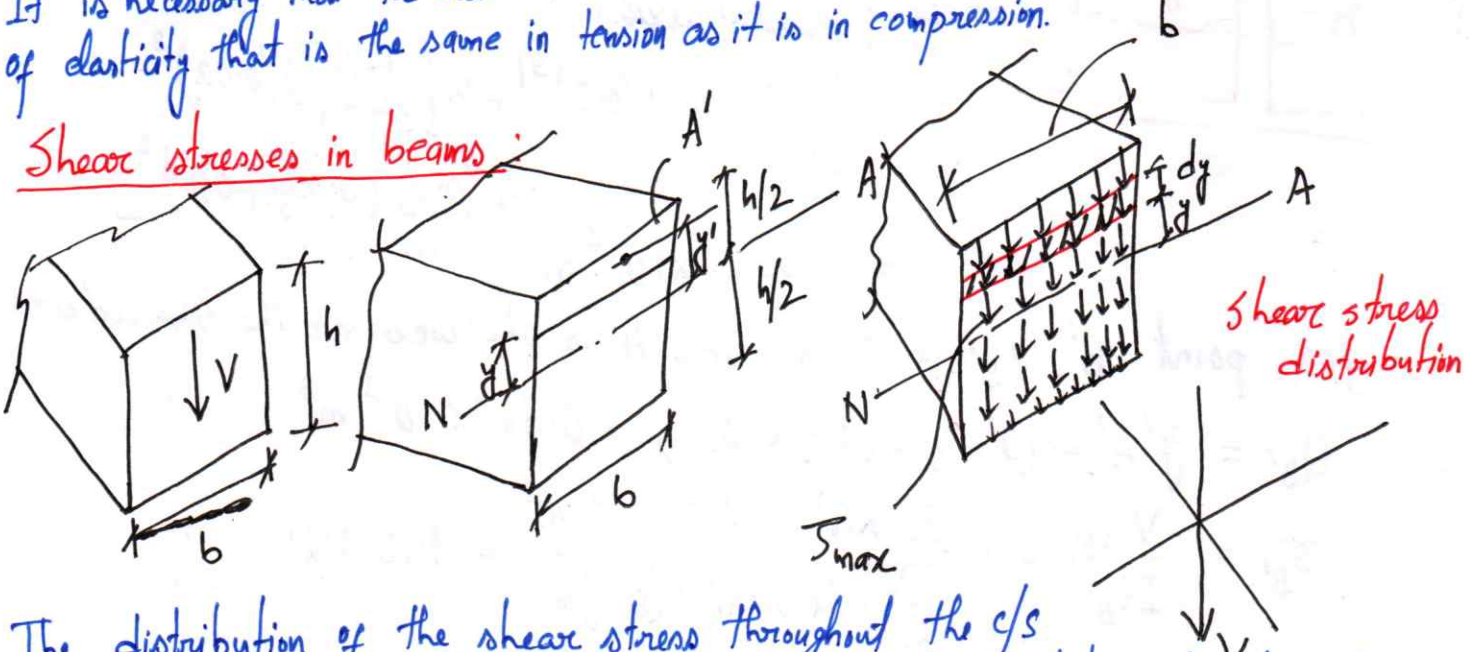
so, $Q = \int_{A'} y\,dA = \bar{y}' A'$

$$\boxed{\tau = \frac{VQ}{It}}$$ Shear formula

In the derivation, we considered only the shear stresses acting on the beam's longitudinal plane, the formula applies as well for finding the transverse shear stress on the beam's cross-sectional area. This, of course, is because the transverse and longitudinal shear stresses are complementary are numerically equal.

It is necessary that the material behave in a linear-elastic manner and have a modulus of elasticity that is the same in tension as it is in compression.

Shear stresses in beams :



The distribution of the shear stress throughout the c/s can be determined by computing the shear stress at an arbitrary height y from the neutral axis.

$$Q = \bar{y}'A' = \left[y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right]\left(\frac{h}{2} - y\right) b = \frac{1}{2}\left(\frac{h^2}{4} - y^2\right) b$$

$$\tau = \frac{VQ}{It} = \frac{V\left(\frac{1}{2}\right)\left[h^2/4 - y^2\right] b}{\left(\frac{1}{12} b h^3\right) b} \Rightarrow \boxed{\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)}$$

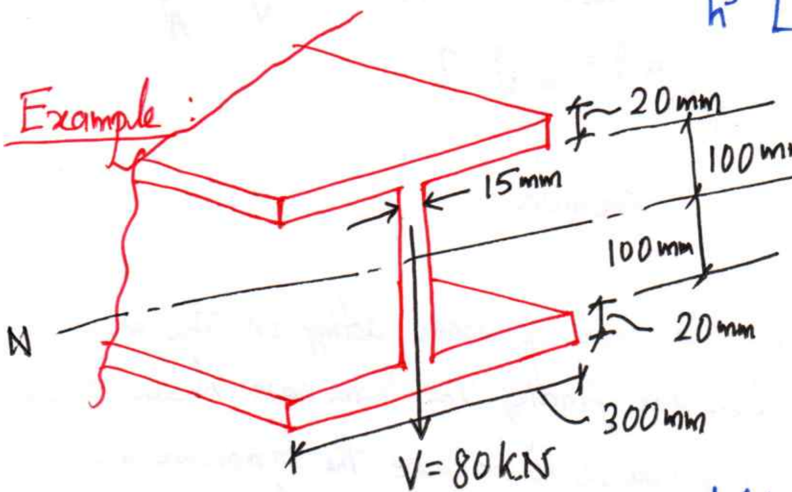
This result indicates that the shear-stress distribution over the c/s is parabolic. The intensity varies from zero at the top and bottom, $y = \pm h/2$, to a maximum value at the neutral axis, $y = 0$. Specifically, since the area of the cross-section is $A = bh$, then at $y = 0$, we have

$$\boxed{\tau_{max} = 1.5\frac{V}{A}}$$

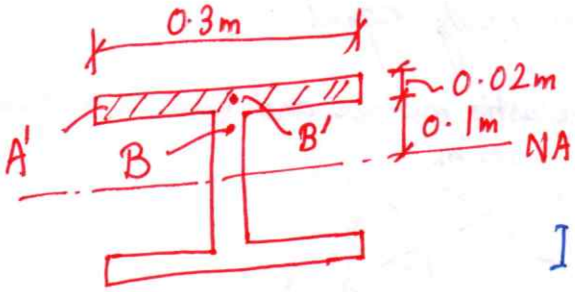
$$\int_A \tau \, dA = \int_{-h/2}^{h/2} \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) b \, dy = \frac{6V}{h^3}\left[\frac{h^2}{4}y - \frac{1}{3}y^3\right]_{-h/2}^{h/2}$$

$$= \frac{6V}{h^3}\left[\frac{h^2}{4}\left(\frac{h}{2} + \frac{h}{2}\right) - \frac{1}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right)\right] = V$$

Example :



A steel wide-flange beam has the dimensions as shown. If it is subjected to a shear of $V = 80\,kN$, (a) plot the shear-stress distribution acting over the beam's c/s area. (b) determine the shear force resisted by the web.



part (a) We must first determine the moment of inertia of the cross-sectional area about the neutral axis.

$$I = \left[\frac{1}{12}(0.015)(0.2)^3\right] + 2\left[\frac{1}{12}(0.3)(0.02)^3 + (0.3)(0.02)(0.110)^2\right]$$

$$= 155.6 \times 10^{-6}\, m^4$$

for point B', $t_{B'} = 0.3\,m$ and A' is the web or the shaded area.

$$Q_{B'} = \bar{y}'A' = (0.110) \times (0.3) \times (0.02) = 0.660 \times 10^{-3}\, m^3$$

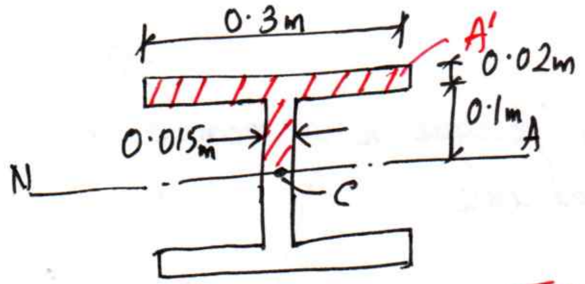
$$\tau_{B'} = \frac{VQ_{B'}}{It_{B'}} = \frac{(80\,kN)(0.660 \times 10^{-3})\,m^3}{(155.6 \times 10^{-6}\,m^4)(0.3)\,m} = 1.13\,MPa$$

For point B, $t_B = 0.015\,m$ and $Q_B = Q_{B'}$, Hence,

$$\tau_B = \frac{VQ_B}{It_B} = \frac{80(kN)(0.660) \times 10^{-3}\,m^3)}{(155.6 \times 10^{-6}\,m^4)(0.015\,m)} = 22.6\,MPa$$

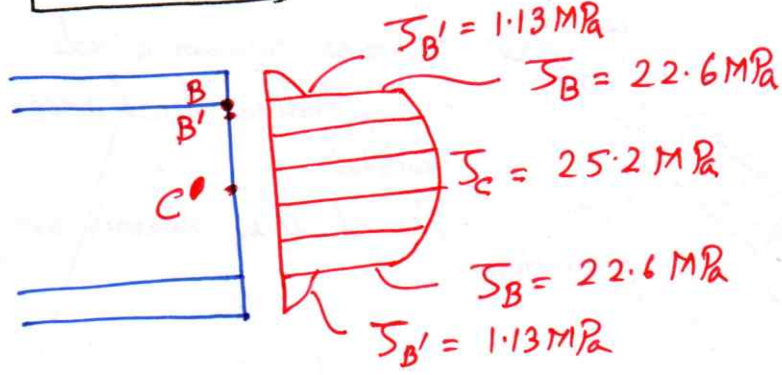
For point C, $t_c = 0.015\,m$, and A' is the shaded area (next page figure)

$$Q_c = \Sigma \bar{y}'A' = (0.110)(0.3)(0.02) + (0.05)(0.015)(0.1) = 0.735 \times 10^{-3}\, m^3$$

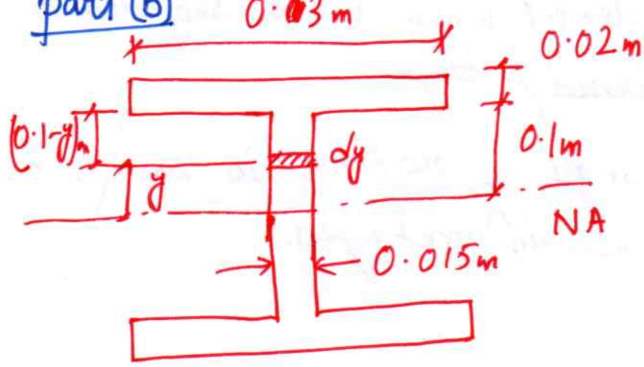


$$\tau_c = \tau_{max} = \frac{VQ_c}{It_c} = \frac{(80\,kN)[0.735\times10^{-3}\,m^3]}{(155.6\times10^{-6}\,m^4)(0.015\,m)}$$

$$= 25.2\ MPa$$



part (b)



The shear force in the web will be determined by first formulating the shear stress at the arbitrary location y within the web. Using units of meters, we have

$I = 155.6\times10^{-6}\ m^4$

$t = 0.015\ m$

$$A' = 0.3\times0.2 + 0.015\times(0.1-y)\ m^2$$

$$Q = \Sigma \bar{y}'A' = 0.11\times0.3\times0.02 + \left[y + \tfrac{1}{2}(0.1-y)\right]0.015(0.1-y)$$

$$= (0.735 - 7.5y^2)\times10^{-3}\ m^3$$

so that, $$\tau = \frac{VQ}{It} = \frac{80\,kN\,(0.735-7.5y^2)\,10^{-3}\ m^3}{155.6\times10^{-6}\,m^4 \times 0.015\,m}$$

$$= (25.192 - 257.07\,y^2)\ MPa$$

This stress acts on the area strip $dA = 0.015dy$ and therefore the ~~there~~ shear force resisted by the web is

$$V_w = \int_{A_w} \tau\, dA = \int_{-0.1m}^{0.1m} (25.192 - 257.07\,y^2)\times10^6 \times 0.015\ dy$$

$$= 73.0\ kN.$$