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UNIT - I

SYSTEM OF EQUATIONS AND VECTOR SPACES

Consider equations, $2x + 3y = 4$
$3x + 5y = 6$

$$[AB] = \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 3 & 5 & 6 \end{array}\right]$$

$$[AB] \xrightarrow{R_2 = 2R_1 - R_2} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$$

On column transform

$$[AB] \sim \left[\begin{array}{cc|c} 3 & 2 & 4 \\ 5 & 4 & 6 \end{array}\right]$$

corresponds to y   corresponds to x

[∴ avoid column transform]

Elementary row transformation :

* $R_i \leftrightarrow R_j$ interchange i^{th} and j^{th} row
* $R_i \rightarrow kR_i$ multiply a row with a scalar
* $R_i \rightarrow \underbrace{R_i + kR_j}$ add scalar multiple of a row R_j to R_i
  $\rightarrow aR_i + bR_j$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 2 & 2 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_1 \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 2 & 2 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 2 & 2 \end{bmatrix} \rightarrow A'$$

$$R_3 - 2R_2 \Rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$E_2 A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & -6 & -12 \end{bmatrix}$$

E_1 , E_2 are elementary matrices.

Similar matrices :

Two matrices A and B are said to be similar if there exists a matrix S λ'

$S^{-1} A S = B$ [or $AS = SB$]

Row equivalent matrices

Two matrices A and B are said to be equivalent if one matrix can be obtained by row transformation of other.

Echelon form of a matrix

A matrix is said to be in the echelon form if

- All zero rows (if it exists), should appear at the bottom of the matrix.
- All entries below the leading entry (1st non-zero entry) of a row are zeroes.
- All the leading entries after the 1st row appear to the right of the leading entries in the previous row.

$$A = \begin{bmatrix} 4 & -2 & 0 & 3 \\ 0 & 0 & 5 & -7 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{is in the echelon form}$$

(usually an upper triangular matrix)

$$A = \begin{bmatrix} 3 & -1 & 2 & 4 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 2 & 0 \end{bmatrix} \rightarrow \text{in echelon form.}$$

$$A = \begin{bmatrix} -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 2 & 0 & -1 \end{bmatrix} \rightarrow \text{is not in echelon form.}$$

$$R_2 \leftrightarrow R_3 \Rightarrow A \sim \begin{bmatrix} -2 & 0 & 2 & 3 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \rightarrow \text{in echelon form.}$$

Given a matrix, it can be made to echelon form using row transformation.

Rank of a matrix, $\rho(A)$

$\rho(A) = r$ if

- there exist atleast one non-zero minor of order r.
- all minors of order greater than r vanish.

(minor = determinant of square sub-matrix)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 4 & -1 \\ 1 & 6 & 13 & 11 \end{bmatrix}$$

3rd order, 2nd order, 1st order } minor can be constructed

$\rho(A) \leq 3$

$\rho(A) \leq \min(m,n)$ where A is of order $m \times n$

3x3 minors,

$$M_1 = \begin{vmatrix} 2 & 3 & 4 \\ 0 & 4 & -1 \\ 6 & 13 & 11 \end{vmatrix} = 0, \quad M_2 = 0, \quad M_3 = 0, \quad M_4 = 0$$

$\Rightarrow \rho(A) < 3$

2x2 minors,

$$M_5 = \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4 \Rightarrow \rho(A) = 2$$

- Determinant of upper / lower Δ matrix is product of its diagonal elements
- No. of non-zero rows in the echelon form of the matrix is the rank of the matrix.

Example: Determine rank of

$$A = \begin{bmatrix} 2 & 5 & -4 & 6 \\ 1 & 2 & -2 & 3 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$$

$R_4 \to 2R_2 - R_4 \Rightarrow A \sim \begin{bmatrix} 2 & 5 & -4 & 6 \\ 1 & 2 & -2 & 3 \\ -1 & -3 & 2 & -2 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

$R_3 \to R_3 + R_2 \Rightarrow A \sim \begin{bmatrix} 2 & 5 & -4 & 6 \\ 1 & 2 & -2 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix}$

If we use def^n, we need to check if 4^{th} order minor is zero or non-zero.

$R_2 \to R_2 + R_3$, $R_3 \to 2R_3 + R_1$

$$A \sim \begin{bmatrix} 2 & 5 & -4 & 6 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$$

$R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 2 & 5 & -4 & 6 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & -1 & 3 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_2 - R_4$

$$A \sim \begin{bmatrix} 2 & 5 & -4 & 6 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$R_4 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 2 & 5 & -4 & 6 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rho(A) = 4$

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(class sol^n)

→ $R_2 \leftrightarrow R_1 \Rightarrow A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

Step 2 → Bring zeroes below a_{11}

$$\left.\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 2R_1 \end{array}\right\} \Rightarrow A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \Rightarrow A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \Rightarrow A \sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No. of non-zeroes in echelon form is the rank of the matrix.

$\Rightarrow \rho(A) = 4$

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(3) $$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_2 \leftrightarrow R_1 \Rightarrow A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$\left.\begin{array}{l} R_2 \to 2R_1 - R_2 \\ R_3 \to 3R_1 - R_3 \\ R_4 \to 6R_1 - R_4 \end{array}\right\} \Rightarrow A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & -5 & -3 & -7 \\ 0 & -4 & -9 & -10 \\ 0 & -9 & -12 & -17 \end{bmatrix}$$

$$\left.\begin{array}{l} R_3 \to R_3 - \frac{4}{5}R_2 \\ R_4 \to R_4 - \frac{9}{5}R_2 \end{array}\right\} \Rightarrow A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & -5 & -3 & -7 \\ 0 & 0 & -\frac{33}{5} & -\frac{22}{5} \\ 0 & 0 & -\frac{33}{5} & -\frac{22}{5} \end{bmatrix}$$

$$R_4 \to R_3 - R_4 \Rightarrow A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & -5 & -3 & -7 \\ 0 & 0 & -\frac{33}{5} & -\frac{22}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 3$

Linear system of equations,

$AX = B$

$B = 0$ → Homogenous ($AX = 0$)

Non-homogeneous

Homogenous ($AX = 0$): $x = 0$ is always a solution [Trivial solⁿ]

$\rho(A) = n$

Always consistent [solⁿ exists]

Non-trivial solⁿ exists [non-zero solⁿs]

$\rho(A) \le n$, n = no. of unknowns

If $\rho(A) = r$, then r leading entries and their coefficients (leading variables) would exist. The remaining $(n-r)$ variables are free variables which can be assigned arbitrary values.

Q Discuss the solⁿ of

$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

→ $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$

$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$

$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

$\rho(A) = 3 = r$

n = no. of unknowns $= 3$

$n = r \Rightarrow$ trivial solⁿ exists

(Gauss elimination method : Step by step elimination of variables)

Equivalent system :

$$x + 2y + 3z = 0$$
$$-2y - 5z = 0$$
$$z = 0$$

② $- 3 \times$ ① → ④

$-2y - 5z = 0$

③ $- 7 \times$ ① → ⑤

$-4y - 9z = 0$

⑤ $- 2 \times$ ④ $\Rightarrow z = 0$

$z = 0$ in ④ $\Rightarrow y = 0$

From ①,

$x = 0$

Q Discuss the solution of

$$2x_1 + 3x_2 - 4x_3 + x_4 = 0$$
$$x_1 - x_2 + x_3 + 2x_4 = 0$$
$$5x_1 - x_3 + 7x_4 = 0$$
$$7x_1 + 8x_2 - 11x_3 + 5x_4 = 0$$

$A = \begin{bmatrix} 2 & 3 & -4 & 1 \\ 1 & -1 & 1 & 2 \\ 5 & 0 & -1 & 7 \\ 7 & 8 & -11 & 5 \end{bmatrix}$

$R_2 \leftrightarrow R_1$

$A \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & 3 & -4 & 1 \\ 5 & 0 & -1 & 7 \\ 7 & 8 & -11 & 5 \end{bmatrix}$

$\left.\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \\ R_4 \to R_4 - 7R_1 \end{array}\right\} \Rightarrow A \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 5 & -6 & -3 \\ 0 & 5 & -6 & -3 \\ 0 & 15 & -18 & -9 \end{bmatrix}$

$\left.\begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 3R_2 \end{array}\right\} \Rightarrow A \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 5 & -6 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2 = r$, $n = 4$ $\Rightarrow n - r = 2$ free variables.

Equivalent system

$$x_1 - x_2 + x_3 + 2x_4 = 0$$
$$5x_2 - 6x_3 - 3x_4 = 0$$

leading variables

free variables

Let $x_1 = a$, $x_2 = b$

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Gauss elimination method:

$$\left.\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 0 \\ 5x_2 - 6x_3 - 3x_4 &= 0 \end{aligned}\right] \text{backward substitution}$$

leading variables as they are coefficients of leading entries (r). Remaining $n-r$ variables are called free variables.

Free variables take arbitrary values.

$x_3 = 5a \quad x_4 = 5b$

$5x_2 - 6(5a) - 3(5b) = 0$

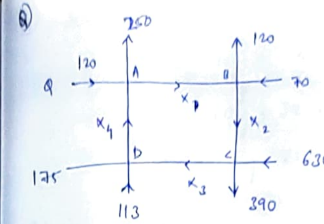
$x_2 = 6a + 3b$

$x_1 - (6a+3b) + 5a + 2(5b) = 0 \Rightarrow x_1 = a - 7b.$

$AX = B$

- $B = 0$: $AX = 0$ Homogenous
  - always consistent as $x = 0$ (trivial) ~~to~~ is always a solution $[\rho(A) = r = n]$
  - If $\rho(A) = r < n$, then non-trivial solⁿ exists & $(n-r)$ free variables are ~~designed~~ assigned arbitrary values
- $B \neq 0$: Non homogeneous linear system of eq
  - $\rho(A) \neq \rho(A:B)$: Inconsistent
  - $\rho(A) = \rho(A:B) = r$: Consistent
    - $r = n$: ~~(Inconsistent)~~ unique
    - $r \neq n$: infinitely many solutions

Q2



$120 + x_4 = x_1 + 250 \qquad x_1 - x_4 = -130$ —(1)

$x_1 + 70 = x_2 + 120 \qquad x_1 - x_2 = 50$ —(2)

$630 + x_2 = 390 + x_3 \qquad x_2 - x_3 = -240$ —(3)

$115 + x_3 = x_4 + 175 \qquad x_3 - x_4 = 60$ —(4)

(1) − (2) $\Rightarrow x_2 - x_4 = -180$ —(5)

(5) − (3) $\rightarrow x_3 - x_4 = 60$ —(6)

(4) = (6) $\Rightarrow |A| = 0$ in AXB

$x_1 - x_4 = -130$

$x_2 - x_4 = -180$

$x_3 - x_4 = 60$

$$[A:B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -130 \\ 1 & -1 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & -240 \\ 0 & 0 & 1 & -1 & 60 \end{array}\right]$$

** For infinitely many solⁿ, $(n-r)$ free variables are assigned arbitrary values.

from $[A:B]$

$$R_2 \rightarrow R_2 - R_1 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -130 \\ 0 & -1 & 0 & 1 & 180 \\ 0 & 1 & -1 & 0 & -240 \\ 0 & 0 & 1 & -1 & 60 \end{array}\right]$$

$$R_3 \rightarrow R_3 + R_2 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -130 \\ 0 & -1 & 0 & 1 & 180 \\ 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 1 & -1 & 60 \end{array}\right], \quad R_4 \rightarrow R_4 + R_3 \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -130 \\ 0 & -1 & 0 & 1 & 180 \\ 0 & 0 & -1 & 1 & -60 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 3$ and $\rho(A:B) = 3 \Rightarrow r = 3$

$\rho(A) = \rho(A:B) \Rightarrow AX = B$ is consistent.

$r = 3$ & $n = 4$, $r < n \Rightarrow$ infinitely many solutions exist. (3 planes intersecting on a line)

Equivalent system,

$x_1 - x_4 = -130$

$-x_2 + x_4 = 180$

$-x_3 + x_4 = -60$

Free variable x_4

Let $x_4 = a$

$x_1 - a = -130 \Rightarrow x_1 = a - 130$

$-x_2 + a = 180 \Rightarrow x_2 = a - 180$

$-x_3 + a = -60 \Rightarrow x_3 = a + 60$

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Q Discuss solution of

$$\left.\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ 3x_1 - 4x_2 + 3x_3 &= -1 \\ 2x_1 - x_2 + 2x_3 &= -3 \\ 3x_1 + x_2 - 2x_3 &= 5 \end{aligned}\right\}$$

more eq^ns than unknowns ⇒ overdetermined systems

→ $[A:B] = \begin{bmatrix} 2 & 3 & -1 & : & 1 \\ 3 & -4 & 3 & : & -1 \\ 2 & -1 & 2 & : & -3 \\ 3 & 1 & -2 & : & 5 \end{bmatrix}$

$R_3 \to R_3 - R_1,\ R_2 \to R_2 - \frac{3}{2}R_1,\ R_4 \to R_4 - 3/2\, R_1$

$[A:B] \sim \begin{bmatrix} 2 & 3 & -1 & : & 1 \\ 0 & -17/2 & 9/2 & : & -5/2 \\ 0 & -4 & 3 & : & -4 \\ 0 & -7/2 & -1/2 & : & 7/2 \end{bmatrix}$

$R_3 \to R_3 - \frac{8}{17}R_2,\ R_4 \to R_4 - \frac{7}{17}R_2$

$[A:B] \sim \begin{bmatrix} 2 & 3 & -1 & : & 1 \\ 0 & -17/2 & 9/2 & : & -5/2 \\ 0 & 0 & 15/17 & : & -48/17 \\ 0 & 0 & -40/17 & : & 77/17 \end{bmatrix}$

$R_4 \to R_4 + \frac{8}{3}R_3$

$\Rightarrow [A:B] \sim \begin{bmatrix} 2 & 3 & -1 & : & 1 \\ 0 & -17/2 & 9/2 & : & -5/2 \\ 0 & 0 & 15/17 & : & -48/17 \\ 0 & 0 & 0 & : & -51/17 \end{bmatrix}$

$\rho(A) = 3 \qquad \rho(A:B) = 4$

$\rho(A) \neq \rho(A:B)$

⇒ The given system is inconsistent.

Q Solve

$$\begin{aligned} 3x + 2y + 2z &= 0 \\ x + 2y &= 4 \\ 10y + 3z &= -2 \\ 2x - 3y - z &= 5 \end{aligned}$$

by Gauss elimination method.

→ $[A:B] = \begin{bmatrix} 3 & 2 & 2 & : & 0 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix}$ ..(overdetermined system)

~~$(R_2 \to 3R_2 - R_1,\ R_4 \to R_4 - \frac{2}{3}R_1)$~~

$R_1 \leftrightarrow R_2$

$[A:B] \sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 3 & 2 & 2 & : & 0 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix}$

$R_2 \to R_2 - 3R_1,\ R_4 \to R_4 - 2R_1$

$[A:B] \sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -4 & 2 & : & -12 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix}$

$R_3 \to R_3 + \frac{10}{4}R_2,\ R_4 \to R_4 - 7/4\, R_2$

$[A:B] \sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -4 & 2 & : & -12 \\ 0 & 0 & 8 & : & -32 \\ 0 & 0 & -9/2 & : & 18 \end{bmatrix}$

$R_4 \to R_4 + 9/16\, R_3$

$[A:B] \sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -4 & 2 & : & -12 \\ 0 & 0 & 8 & : & -32 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$

$\rho(A) = \rho(A:B) = 3,\ n = 3$

$r = n$

⇒ System has unique solution

Equivalent system is

$$\begin{aligned} x + 2y &= 4 \\ -4y + 2z &= -12 \\ 8z &= -32 \end{aligned}$$

(start with bottom most eq^n & move up)

$\Rightarrow z = -4$

$-4y - 8 = -12 \Rightarrow y = 1$

$x + 2 = 4 \Rightarrow x = 2$

Q) Discuss the solution of

$$3x_2 - 6x_3 + 6x_4 + 4x_5 = -5$$
$$3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9$$
$$3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15$$

No. of eq^n is less than no. of unknown ⇒ under determined sol^n

→ Under determined systems cannot have unique solution.

$$[A:B] = \left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$[A:B] \sim \left[\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$[A:B] \sim \left[\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{array}\right]$$

$R_3 \to R_3 + \frac{2}{3}R_2$

$$[A:B] \sim \left[\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{8}{3} \end{array}\right]$$

$\rho(A) = \rho(A:B) = 3 = r$ &

$n = 5$

$n - r = 2$ free variables

Equivalent system,

$$3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9$$
$$3x_2 - 6x_3 + 6x_4 - 4x_5 = -5$$
$$\frac{2}{3}x_5 = \frac{8}{3}$$

$\Rightarrow x_5 = 4$

let $x_3 = a$, $x_4 = b$

$3x_2 - 6(a) + 6(b) + 4(4) = -5$

$\Rightarrow x_2 = 2a - 2b - 7$

$3x_1 - 7(2a - 2b - 7) + 8(a) - 5(b) + 8(4) = 9$

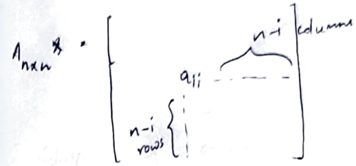
$3x_1 = 9 - 32 + 5b - 8a - 49 + 14a + 14b$

$3x_1 = -72 - 6a + 9b$

$x_1 = +2a - 3b - 24$

$x = [2a-3b-24 \quad 2a-2b-7 \quad a \quad b \quad 4]$

Flops → floating point operations per second



To bring zero below a_{ii}

No. of divisions → $(n-i)$ to replace $a_{ii} = 1$

	for A	for [A:B]
No. of multiplications →	$(n-i)(n-i)$	$(n-i)(n-i+1)$
No. of additions →	$(n-i)(n-i)$	$(n-i)(n-i+1)$

Total no. of operations $= \sum_{i=1}^{n-1}(n-i) + 2\sum_{i=1}^{n-1}(n-i)^2$

$= \{(n-1) + (n-2) + \dots + 3 + 2 + 1\}$

$+ 2\{(n-1)^2 + (n-2)^2 + \dots + 3^2 + 2^2 + 1\}$

$= \sum_{k=1}^{n-1} k + 2\sum_{k=1}^{n-1} k^2 = \frac{(n-1)(n)}{2} + \frac{(n-1)(n)(2n-1)}{3}$

For large n the no. of flops $\approx \frac{2n^3}{3}$

Algorithm	n = 1000	n = 10000
Elimination	0.7s	11 min
$O(n^2)$ Backward substitution	0.001s	0.1 s

LU factorization method:

$A_{n\times n} = LU$ → upper triangular matrix; L ↓ lower triangular matrix

Eg:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$a_{11} = l_{11}u_{11}$ $a_{12} = l_{11}u_{12}$ $a_{13} = l_{11}u_{13}$

$a_{21} = l_{21}u_{11}$ $a_{22} = l_{21}u_{12} + l_{22}u_{22}$ $a_{23} = l_{21}u_{13} + l_{22}u_{23}$

$a_{31} = l_{31}u_{11}$ $a_{32} = l_{31}u_{12} + l_{32}u_{22}$ $a_{33} = l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33}$

$u_{11} = 1 = u_{22} = u_{33}$ → crout's method

$l_{11} = 1 = l_{22} = l_{33}$ → Doolittle's method

$(E_n \cdots E_3E_2E_1)A = U$ (Gauss eliminatⁿ)

$A = LU$ $L = (E_nE_{n-1}\cdots E_3E_2E_1)^{-1}$

Methods to factorise A as $L.U$:

1) Obtain the echelon form of $A_{n\times n}$, and name it U using $R_i - l_{ij}R_j$ only. (No row swapping)

2) Write $L = (l_{ij})$, a lower △lar matrix with $l_{ii} = 1$.

Then we get $A^* = LU$

* Principle minor of A of all orders should be non-zero

$a_{11} \neq 0$ $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$

Solution of $AX = B$ by LU factorisatⁿ

$AX = B \rightarrow (LU)X = B$

$L(UX) = B$ (Associativity)

let $UX = Y \Rightarrow LY = B$] → solve by forward substitution.

Now solve $UX = Y$] → solve by backward substitution.

Q) Solve
$3x + 2y + 7z = 4$
$2x + 3y + z = 5$
$3x + 4y + z = 7$
by LU decomposition.

→ $A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

Step 1: Echelon form of A, (Use $R_i - l_{ij}R_j$ ONLY!)

$R_3 \rightarrow R_3 - R_1$, $R_2 \rightarrow R_2 - \frac{2}{3}R_1$

$A \sim \begin{bmatrix} 3 & 2 & 7 \\ 0 & +5/3 & -11/3 \\ 0 & 2 & -6 \end{bmatrix}$ and $l_{21} = 2/3$, $l_{31} = 1$

$R_3 \rightarrow R_3 - \frac{6}{5}R_2 \Rightarrow A \sim \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} = U$ and $l_{32} = \frac{6}{5}$

$\frac{2}{(5/3)} = \frac{6}{5}$

$-\frac{11}{3} \cdot \frac{6}{5}$, $-30, \frac{22}{5}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix}$

$AX = B \Rightarrow (LU)X = B \Rightarrow L(UX) = B$

$UX = Y \Rightarrow LY = B$

let $Y = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ $LY = B \Rightarrow$ $x_1 = 4$

$\frac{2}{3}x_1 + y_1 = 5$

$x_1 + \frac{6}{5}y_1 + z_1 = 7$

$\frac{2}{3} \cdot 4 + y_1 = 5 \Rightarrow 5 - \frac{8}{3} = \frac{7}{3}$

$4 + \frac{6}{5} \cdot \frac{7}{3} + z_1 = 7$

$z_1 = 7 - 4 - \frac{14}{5}$

$= 3 - \frac{14}{5} = \frac{1}{5}$

$UX = Y \Rightarrow 3x + 2y + 7z = 4$

$\frac{5}{3}y - \frac{11}{3}z = \frac{7}{3}$

$-\frac{8}{5}z = \frac{1}{5} \Rightarrow z = -\frac{1}{8}$

$5y - 11z = 7$

$5y + \frac{11}{8} = 7 \Rightarrow y = \frac{1}{5}\left(7 - \frac{11}{8}\right) = \frac{9}{8}$

$3x + \frac{9}{4} + -\frac{7}{8} = 4 \Rightarrow x = \left(4 - \frac{18+7}{8}\right)\frac{1}{3}$

$= \left(4 - \frac{11}{8}\right)\frac{1}{3}$

$= \frac{7}{8}$

Q) Solve $2x_1 + 4x_2 - 6x_3 = b_1$ 3/6/22

$x_1 + 5x_2 + 3x_3 = b_2$

$x_1 + 3x_2 + 2x_3 = b_3$ by LAU factorization method

when $B = \begin{bmatrix} -4 \\ 10 \\ 5 \end{bmatrix}$ and $B = \begin{bmatrix} 20 \\ 49 \\ 32 \end{bmatrix}$

$\rightarrow$ $A = \begin{bmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix}$

obtain echelon form using $R_i \to R_i - l_{ij}R_j$

$\left.\begin{array}{l} R_2 \to R_2 - \frac{1}{2}R_1 \\ R_3 \to R_3 - \frac{1}{2}R_1 \end{array}\right\} \Rightarrow A \sim \begin{bmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 1 & 5 \end{bmatrix}$ & $l_{21} = \frac{1}{2}$, $l_{31} = \frac{1}{2}$

$R_3 \to R_3 - \frac{1}{3}R_2 \Rightarrow A \sim \begin{bmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix} = U$ & $l_{32} = \frac{1}{3}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}$

$AX = B \Rightarrow (LU)X = B \Rightarrow LY = B$

$A = LU$

let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ ~~then~~ ~~LY=B~~,

$LY = \begin{bmatrix} -4 \\ 10 \\ 5 \end{bmatrix} \Rightarrow$ $y_1 = -4$

$\frac{1}{2}y_1 + y_2 = 10$

$\frac{1}{2}y_1 + \frac{1}{3}y_2 + y_3 = 5$

$\frac{1}{2}(-4) + y_2 = 10$

$y_2 = 12$

$-2 + 4 + y_3 = 5$

$y_3 = 3$

$UX = Y \Rightarrow 2x_1 + 4x_2 - 6x_3 = -4$

$3x_2 + 6x_3 = 12$

$3x_3 = 3$

$x_3 = 1, \quad 3x_2 + 6 = 12 \Rightarrow x_2 = 2, \quad 2x_1 + 8 - 6 = -4$

$x_1 = \frac{-4-2}{2} = -3$

$X = [-3 \quad 2 \quad 1]$

b) $LY = \begin{bmatrix} 20 \\ 49 \\ 32 \end{bmatrix}$

$\Rightarrow y_1 = 20$

$\frac{1}{2}y_1 + y_2 = 49 \qquad y_2 = 39$

$\frac{1}{2}y_1 + \frac{1}{3}y_2 + y_3 = 32 \qquad 10 + 13 + y_3 = 32$

$y_3 = 9$

$UX = Y \Rightarrow 2x_1 + 4x_2 - 6x_3 = 20$

$3x_2 + 6x_3 = 39$

$3x_3 = 9$

$\Rightarrow x_3 = 3 \qquad 3x_2 + 18 = 39 \Rightarrow x_2 = 7$

$2x_1 + 28 - 18 = 20$

$x_1 = 5$

$X = [5 \quad 7 \quad 3]$

8) Solve
$3x_1 - 6x_2 - 3x_3 = -3$
$2x_1 + 6x_3 = -22$
$-4x_1 + 7x_2 + 4x_3 = 3$ by LU decomposition.

$\rightarrow A = \begin{bmatrix} 3 & -6 & -3 \\ 2 & 0 & 6 \\ -4 & 7 & 4 \end{bmatrix}$

Obtain echelon form using $R_i - l_{ij}R_j$

$\Rightarrow R_2 \rightarrow R_2 - \frac{2}{3}R_1, \quad R_3 \rightarrow R_3 + \frac{4}{3}R_1$

$7 + \frac{4}{3}(7)$
$= 7(1 + 4/3)$
$= 4^1/_3$

$\Rightarrow A \sim \begin{bmatrix} 3 & -6 & -3 \\ 0 & 4 & +8 \\ 0 & \frac{49}{3}-1 & 0 \end{bmatrix} \qquad l_{21} = \frac{2}{3}, \quad l_{31} = \frac{-4}{3}$

Simplify eq^n 1 & 2 & proceed.

$A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & 3 \\ -4 & 7 & 4 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} -1 \\ -11 \\ 3 \end{bmatrix}$

Obtain echelon form,

$\left.\begin{matrix} \Rightarrow R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + 4R_1 \end{matrix}\right\} \Rightarrow A \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 4 \\ 0 & -1 & 0 \end{bmatrix} \qquad l_{21} = 1, \quad l_{31} = -4$

$R_3 \rightarrow R_3 + \frac{1}{2}R_2 \Rightarrow A \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} = U \quad \& \quad l_{32} = \frac{-1}{2}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & -1/2 & 1 \end{bmatrix}$

Let y be $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$AX = B$ and $A = LU$

$UX = Y \Rightarrow LY = B$.

$y_1 = -1$

$y_1 + y_2 = -11$ $\quad y_2 = -10$

$-4y_1 - \frac{1}{2}y_2 + y_3 = 3$ $\quad 4 + 5 + y_3 = 3$ $\quad y_3 = -6$

$UX = Y \Rightarrow$ $x_1 - 2x_2 - x_3 = -1$

$2x_2 + 4x_3 = -10$

$2x_3 = -6 \Rightarrow x_3 = -3$

$2x_2 - 12 = -10$

$x_2 = 1$

$x_1 - 2 + 3 = -1$

$x_1 = -2$

$X = \begin{bmatrix} -2 & 1 & -3 \end{bmatrix}$

Problems on consistency

5) Find a and b for which the system $x + ay + z = 3$

$x + 2y + 2z = b$

$x + 5y + 3z = 9$ has

i) Unique solution $\quad$ ii) Infinitely many solⁿs $\quad$ iii) No solution

$\underbrace{\text{i) and ii)}}_{\rho(A) = \rho(A:B)}$

iii) $\rho(A) \neq \rho(A:B)$

$\rho(A) < \rho(A:B)$

~~(i) Unique solutⁿ)~~

$|A| = 0 \Rightarrow \rho(A) = n$

$[A:B] = \begin{bmatrix} 1 & a & 1 & : & 3 \\ 1 & 2 & 2 & : & b \\ 1 & 5 & 3 & : & 9 \end{bmatrix}$

~~$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$~~

~~$A = \begin{bmatrix} 1 & a & 1 \\ 0 & 2-a & 1 \\ 0 & 5-a & 2 \end{bmatrix}$~~

$R_3 \leftrightarrow R_1$

$[A:B] \sim \begin{bmatrix} 1 & 5 & 3 & : & 9 \\ 1 & 2 & 2 & : & b \\ 1 & a & 1 & : & 3 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$[A:B] \sim \begin{bmatrix} 1 & 5 & 3 & : & 9 \\ 0 & -3 & -1 & : & b-9 \\ 0 & a-5 & -2 & : & -6 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ (only for missing coeff problem)

$[A:B] \sim \begin{bmatrix} 1 & 5 & 3 & : & 9 \\ 0 & -3 & -1 & : & b-9 \\ 0 & a+1 & 0 & : & -2b+12 \end{bmatrix}$

i) No solution $\Leftrightarrow \rho(A) \neq \rho(A:B)$

This is possible only if $\rho(A) = 2$ and $\rho(A:B) = 3$

$\rho(A) = 2 \Leftrightarrow a+1 = 0$

i.e $a = -1$

$a+1 = 0$ but $\rho(A:B) = 3$ only if $12 - 2b \neq 0$

$\therefore$ The system is inconsistent if $a = -1$ and $b \neq 6$

ii) Unique solution,

$\rho(A) = \rho(A:B) = r$

and $r = n$

Since $A_{3\times3} X_{3\times1} = B_{3\times1}$

if $\rho(A) = 3$ then $\rho(A:B) = 3$

$\Rightarrow \rho(A) = 3$ only if $a+1 \neq 0$

i.e $a \neq -1$

Unique solution exists if $a \neq -1$ & for any value of b

Note: In general $A_{m\times n} X_{n\times 1} = B_{n\times 1}$ $[m>n]$.

$\rho(A) = n$ does not guarantee unique solution as $\rho(A:B)$ could be greater than n.

(iii) Infinitely many solutions,

$\{ \rho(A) = \rho(A:B)$ & $r < n \}$

$\rho(A) = \rho(A:B) = 2$ only if $\{a+1=0$ and $12-2b=0\}$

Infinitely many solutions exist if $a=-1$ & $b=6$.

Infinitely many

5) Determine for which the system,

$3x - y + 4z = 3$

$x + 2y - 3z = -2$

$6x + 5y + \lambda z = -3$ is consistent. Solve them completely in each case.

→ $[A:B] = \left[\begin{array}{ccc:c} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{array}\right]$

$R_1 \leftrightarrow R_2$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & -2 \\ 3 & -1 & 4 & 3 \\ 6 & 5 & \lambda & -3 \end{array}\right]$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 6R_1$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & -7 & \lambda+18 & \cancel{-15}\ 9 \end{array}\right]$

$R_3 \to R_3 - R_2$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & -2 \\ 0 & -7 & 13 & 9 \\ 0 & 0 & \lambda+5 & 0 \end{array}\right]$

i) Unique solution,

$\rho(A) = \rho(A:B) = r$ & $r = n$

$\Rightarrow \lambda \neq -5$

ii) Infinitely many solutions,

$\rho(A) = \rho(A:B)$ & $r < n$

$\Rightarrow \rho(A) = \rho(A:B) = 2$

$\Rightarrow \lambda = -5$

$x - 2y - 3z = -2$

$-7y + 13z = 9$

Suppose $(\lambda+5) = a$

$az = 0$

Since $a \neq 0$, $z = 0$

$-7y + 0 = 9 \Rightarrow y = \frac{-9}{7}$

$x + 2\left(\frac{-9}{7}\right) - z(0) = -2 \Rightarrow x = \frac{4}{7}$

If $\lambda \neq -5$ then $x = \left[\frac{4}{7} \ \ \frac{-9}{7} \ \ 0\right]$ is the unique solution.

If $\lambda = -5$ then $r = 2$ and hence the system will have infinitely many solutions.

$x + 2y - 3z = -2$

$-7y + 13z = 9$

$z = k \Rightarrow -7y + 13k = 9 \Rightarrow y = \frac{13k-9}{7}$

$\Rightarrow x + 2\left(\frac{13k-9}{7}\right) - 3k = -2 \Rightarrow x = -2 + 3k - \frac{26k+18}{7}$

$= \frac{-14 + 21k - 26k + 18}{7}$

$= \frac{4-5k}{7}$

$X = \left[\frac{4-5k}{7} \quad \frac{13k-9}{7} \quad k \right]$ is the solution when $\lambda = -5$

For what values of k will the equation

$x + y + z = 1$

$2x + y + 4z = k$

$4x + y + 10z = k^2$

have a solution. Solve them completely in each case

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{array}\right] = [A:B]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

Vector Spaces :

Binary operation : set S and $S \neq \{\}$, $a, b \in S$.
$*$ is a binary operation on S, $a * b \in S$ for all $a, b \in S$.

set G, $G \neq \{\}$ and a binary operator $*$

a) Closure property : If $*$ is defined on G, then closure property will be satisfied.

b) Associative property: $(a * b) * c = a * (b * c)$

c) Existence of identity 'e' :
If $e \in G$ such that $a * e = a = e * a$ for all $a \in G$.

d) Existence of inverse :
If $\forall$ $a \in G$, there exists an element b $\ni$ $a * b = e = b * a$

e) If a1, a2, a3 and a4 are satisfied then G is a group w.r.t $*$, and we write $(G, *)$

f) If $a * b = b * a$ for all $a, b \in G$ (commutative property) then $(G, *)$ is an abelion group.

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Fields

$(F, *_1, *_2)$ is a field if $(F, *_1)$ and $(F, *_2)$ are abelian groups

$(\mathbb{R}, +, \cdot)$

$(\mathbb{R}, +)$ is an abelian group

$(\mathbb{R} - \{0\}, \cdot)$ is an abelian group

Vector space

Consider a non-empty V and a field $(F, +, \cdot)$

Define vector addition $u+v \in V$ for all $u, v \in V$

and define scalar multiplication $k.u \in V$ for all $u \in V$ and $k \in F$

If i) $(V, +)$ is an abelian group

ii) $k.(u+v) = k.u + k.v$ $\forall$ $u, v \in V$ and $k \in F$

iii) $(k_1 + k_2).u = k_1.u + k_2.u$ $\forall$ $u \in V$ & $k_1, k_2 \in F$

iv) $(k_1.k_2).u = k_1.(k_2.u)$

v) $1.u = u$

then V is said to be a vector space and we write $(V, +, \cdot)$ (operator ① & ② are not same)

① $P_n(t)$ be the set of all polynomials of degree less than or equal to n along with the zero polynomial. with real coefficients.

If $P_1(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$

$P_2(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_n t^n$

$P_1 + P_2 = (a_0+b_0) + (a_1+b_1)t + (a_2+b_2)t^2 + \dots + (a_n+b_n)t^n$

Let $(\mathbb{R}, +, \cdot)$ be the field of reals.

For $k \in \mathbb{R}$ $kP_1(t) = k(a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n)$

Verify if $(P_n(t), +, \cdot)$ is a vector space.

By definition of vector addition closure property is satisfied as $a_i + b_i \in \mathbb{R}$ for $i = 0$ to n

$$(P_1+P_2)+P_3 = \{(a_0+b_0) + (a_1+b_1)t + \dots + (a_n+b_n)t^n\} + c_0 + c_1 t + c_2 t^2 + \dots + c_n t^n$$

$$= \{(a_0+b_0)+c_0\} + \{(a_1+b_1)+c_1\}t + \dots + \{(a_n+b_n)+c_n\}t^n$$

$$= \{a_0 + (b_0+c_0)\} + \{a_1 + (b_1+c_1)\}t + \dots + \{a_n + (b_n+c_n)\}t^n$$

$$= (a_0 + a_1 t + \dots + a_n t^n) + \{(b_0+c_0) + (b_1+c_1)t + \dots + (b_n+c_n)t^n\}$$

$$= P_1 + (P_2 + P_3)$$

let $o(t)$ be the identity w.r.t vector addition

λ' $p_1(t) + o(t) = p_1(t) \Rightarrow o(t) = 0 + 0t + 0t^2 + \dots + 0t^n$

Consider $p_1(t) + p_2(t) = o(t)$

$\Rightarrow (a_0+b_0) + (a_1+b_1)t + \dots + (a_n+b_n)t^n = 0 + 0t + \dots + 0t^n$

$a_0 + b_0 = 0 \qquad a_1 + b_1 = 0 \quad \dots \quad a_n + b_n = 0$

$\Rightarrow b_0 = -a_0 \qquad b_1 = -a_1 \quad \dots \quad b_n = -a_n$

$\Rightarrow p_2 = -a_0 + (-a_1)t + (-a_2)t^2 + \dots + (-a_n)t^n = -p_1(t)$

$\forall\ p_1(t) \in V, \ -p_1(t) \in V$

$p_1(t) + p_2(t) = (a_0+b_0) + (a_1+b_1)t + \dots + (a_n+b_n)t^n$

$= (b_0+a_0) + (b_1+a_1)t + \dots + (b_n+a_n)t^n$

$= p_2(t) + p_1(t)$

$\Rightarrow (V,+)$ is an abelian group

ii) To prove $k.(p_1+p_2) = k.p_1 + k.p_2 \quad \forall\ p_1, p_2 \in V \ \&\ k \in \mathbb{R}$

Consider $k \in \mathbb{R}$ and $p_1, p_2 \in P_n(t)$

$k(p_1+p_2) = k\{(a_0+b_0) + (a_1+b_1)t + \dots + (a_n+b_n)t^n\}$

$= k(a_0+b_0) + k(a_1+b_1)t + \dots + k(a_n+b_n)t^n$

$= (ka_0+kb_0) + (ka_1+kb_1)t + \dots + (ka_n+kb_n)t^n$

$= (ka_0 + ka_1t + \dots + ka_nt^n) + (kb_0 + kb_1t + \dots + kb_nt^n)$

$= k.p_1 + k.p_2$

iii) To prove $(k_1+k_2)p_1 = k_1p_1 + k_2p_1 \quad \forall\ p_1 \in P_n(t)$

scalar addition (usual addition as $k_1, k_2 \in \mathbb{R}$) — vector addition

$(k_1+k_2)p_1 = (k_1+k_2)(a_0 + a_1t + \dots + a_nt^n)$

$= (k_1+k_2)a_0 + (k_1+k_2)a_1t + \dots + (k_1+k_2)a_nt^n$

$= (k_1a_0 + k_2a_0) + (k_1a_1 + k_2a_1)t + \dots + (k_1a_n + k_2a_n)t^n$

$= (k_1a_0 + k_1a_1t + \dots + k_1a_nt^n) + (k_2a_0 + k_2a_1t + \dots + k_2a_nt^n)$

$= k_1(a_0 + a_1t + \dots + a_nt^n) + k_2(a_0 + a_1t + \dots + a_nt^n)$

$= k_1.p_1 + k_2.p_1$

iv) To prove $(k_1.k_2)p_1 = k_1.(k_2.p_1)$

$(k_1.k_2)p_1 = (k_1k_2)a_0 + (k_1k_2)a_1t + \dots + (k_1k_2)a_nt^n$

$= k_1(k_2a_0) + k_1(k_2a_1)t + \dots + k_1(k_2a_n)t^n$

$= k_1\{k_2a_0 + k_2a_1t + \dots + k_2a_nt^n\}$

$= k_1\{k_2(a_0 + a_1t + \dots + a_nt^n)\}$

$= k_1.\{k_2.p_1\}$

v) $1.p_1 = p_1$

(not value but symbol)

$1.p_1 = 1(a_0 + a_1t + a_2t^2 + \dots + a_nt^n)$

$= a_0 + a_1t + \dots + a_nt^n$

$= p_1$

$a + e = a$, e is zero element

$a.e = a$, e is unit element

Since all the properties of a vector space are satisfied, $(P_n(t), +, \cdot)$ is a vector space over over the field of reals.

HW
Show that the set of n-tuples in $\mathbb{R}^n$ is a vector space over the field $(\mathbb{R}, +, \cdot)$ under usual vector addition and scalar multiplicat^n λ' $u \in \mathbb{R}^n$

$\Rightarrow u = (u_1, u_2, \ldots, u_n)$

[only Transformat^n req
$1.P^n \longrightarrow \mathbb{R}^{n+1}$]

9/6/22

(8) Consider set of $\mathbb{R}^+$ and field of reals $(\mathbb{R}, +, \cdot)$

Vector addition is defined by $u+v = uv$ for $u, v \in \mathbb{R}^+$ and the scalar multiplication is defined by $k.u = u^k$ for $u \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

Show that $(\mathbb{R}^+, +, \cdot)$ is a vector space over $\mathbb{R}$.

→ i) To prove $(\mathbb{R}^+, +)$ is abelian,
→ $u, v \in \mathbb{R}^+ \Rightarrow u+v = uv \in \mathbb{R}^+ \Rightarrow$ vector addition is closed.

ii) → $\underbrace{(u+v)+w}_{\text{vector addition}} = (uv)+w = (uv)w = u(vw)$ [∵ Associativity of usual $\times^{\ln}$]

$= u+(vw)$

$= \underbrace{u+(v+w)}_{\text{vector addition}}$

Associative property is satisfied

→ Existence of identity wrt '+'

let $u+e = u = e+u$

$u+e = u \Rightarrow \underbrace{ue = u}_{\text{the usual multiplicat}^n} \Rightarrow e = 1 \in \mathbb{R}^+$

→ Existence of inverse

Consider $u+v = 1 = v+u$

$u+v = 1 \Rightarrow uv = 1 \Rightarrow v = \frac{1}{u} \in \mathbb{R}^+$

∴ for any $u \in \mathbb{R}^+$, $u^{-1} = \frac{1}{u} \in \mathbb{R}^+$

∴ $(\mathbb{R}^+, +)$ is a group [4 properties sufficient to prove to be a group]

→ Commutative property

$u+v = uv = vu = v+u$

∴ $(\mathbb{R}^+, +)$ is an abelian group

ii) To prove $k.(u+v) = k.u + k.v$

→ Consider $k.(u+v) = k.(uv)$ [∵ vector addition]

$= (uv)^k$ (∵ scalar multiplicat^n)

$= u^k v^k$

$= u^k + v^k$ [∵ vector addition]

$= k.u + k.v$ [scalar multiplicat^n]

iii) To prove $\underbrace{(k_1+k_2)}_{\text{scalar addition}} u = k_1.u + k_2.u$ (usual addit^n of real no.s)

$(k_1+k_2)u = u^{k_1+k_2}$

$= u^{k_1}.u^{k_2}$ [∵ theory of indices]

$= u^{k_1} + u^{k_2}$ [∵ vector addition]

$= k_1.u + k_2.u$ ($\because$ scalar multiplicatⁿ)

iv) To prove $(k_1 k_2).u = k_1.(k_2.u)$

$(k_1 k_2).u = u^{k_1 k_2} = u^{k_2 k_1} = (u^{k_2})^{k_1} = k_1.(u^{k_2})$

$= \cancel{(k_2.u)}^{k_1}$ [scalar multiplicati.

$= k_1.(k_2.u)$.

v) To prove $1.u = u$

$1.u = u^1 = u$

Since all the properties of vector space are satisfied $(\mathbb{R}^+, +, \cdot)$ is a vector space over the field $(\mathbb{R}, +, \cdot)$

HW

8) Consider the set of all continuous functions $c[0,1]$, with point wise addition and scalar multiplicatⁿ ie $(f+g)(x) = f(x) + g(x)$ and $k.f$ is defined by $(k.f)(x) = k f(x)$. If $c[0,1]$ a vector space over the field of reals.

9) Show that $M_{2\times 2} = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}, a, b \in \mathbb{R} \right\}$ is a vector space over the field of reals under matrix addition and scalar multiplicatⁿ.

→

i) To prove $(M_{2\times2}, +)$ ~~$= \left\{\begin{bmatrix} 0 & a \\ b & 0\end{bmatrix} a,b\in\mathbb{R}\right\}$~~ is an ~~vect~~ abelian group.

→ By definition of ~~vector~~ matrix addition,

for matrices $M_1 = \begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}$ or A and $M_2 = \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix}$ or B, closure property

$$\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_1 + a_2 \\ b_1 + b_2 & 0 \end{bmatrix} \in M_{2\times2}.$$

as $a_1 + a_2 \in \mathbb{R}$ & $b_1 + b_2 \in \mathbb{R}$

→ Associative property,

$$(A+B)+C = \left(\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix}\right) + \begin{bmatrix} 0 & a_3 \\ b_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_1 + a_2 \\ b_1 + b_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_3 \\ b_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & (a_1 + a_2) + a_3 \\ (b_1 + b_2) + b_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a_1 + (a_2 + a_3) \\ b_1 + (b_2 + b_3) & 0 \end{bmatrix} = A + (B + C)$$

→ Zero vector, for $M_{2\times2}$,

$A + E = A$ $\Rightarrow$ Let $E = \begin{bmatrix} 0 & e_1 \\ e_2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e_1 \\ e_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} \Rightarrow \begin{matrix} a_1 + e_1 = a_1 \Rightarrow e_1 = 0 \\ b_1 + e_2 = b_1 \Rightarrow e_2 = 0 \end{matrix}$$

$$\Rightarrow E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{2\times2}$$

→ Additive Inverse,

$A+B=E$

$$\Rightarrow \begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} b_1 = -a_1 \\ b_2 = -a_2 \end{matrix}$$

$$\Rightarrow B = \begin{bmatrix} 0 & -a_1 \\ -a_2 & 0 \end{bmatrix} \in M_{2\times 2}$$

→ Commutative property,

$$A+B = \begin{bmatrix} 0 & a_1+a_2 \\ b_1+b_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_2+a_1 \\ b_2+b_1 & 0 \end{bmatrix} = B+A$$

$\Rightarrow (M_{2\times 2}, +)$ is an abelian group.

ii) To prove $k.(A+B) = k.A + k.B$

$$= k.\left(\begin{bmatrix} 0 & a_1+a_2 \\ b_1+b_2 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & k(a_1+a_2) \\ k(b_1+b_2) & 0 \end{bmatrix} = \begin{bmatrix} 0 & ka_1+ka_2 \\ kb_1+kb_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & ka_1 \\ kb_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & ka_2 \\ kb_2 & 0 \end{bmatrix} = k.A + k.B$$

iii) To prove $(k_1+k_2)A = k_1.A + k_2.A$

$$= (k_1+k_2)\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & (k_1+k_2)a_1 \\ (k_1+k_2)b_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & k_1a_1+k_2a_1 \\ k_1b_1+k_2b_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & k_1a_1 \\ k_1b_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_2a_1 \\ k_2b_1 & 0 \end{bmatrix} = k_1\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}$$

$$= k_1.A + k_2.A$$

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Vector space $(V, +, \cdot)$ over a field F,

Subspaces,

A non-empty subset U of V is said to be a subspace under the same vector addition and scalar multiplication.

Theorem: let U be a non-empty subset of a vector space V. Then U is a subspace of V if $u_1 + u_2 \in U$ $\forall$ $u_1, u_2 \in U$ and $k.u \in U$ $\forall$ $u \in U$ and $k \in F$.

Q) Is W, a set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ $x=0$ a subspace of R^2?

$\rightarrow$ $W = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix}, y \in \mathbb{R} \right\} \subset \mathbb{R}^2$

Consider $u_1 = \begin{bmatrix} 0 \\ y_1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ y_2 \end{bmatrix}$ then $u_1 + u_2 = \begin{bmatrix} 0 \\ y_1 + y_2 \end{bmatrix} \in W$

$k.u_1 = k \begin{bmatrix} 0 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ ky_1 \end{bmatrix} \in W$

$\rightarrow$ W is a subspace of R^2

Q) Is W, a set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, $x \geq 0$, $y \geq 0$ a subspace of $\mathbb{R}^2$?

$\rightarrow$ $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x \geq 0, y \geq 0 \right\}$

$u_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $u_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ $\Rightarrow$ $u_1 + u_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} \in W$

as $x_1, x_2, y_1, y_2 \geq 0$

let $k = -1$, then $k.u_1 = -1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -y_1 \end{bmatrix} \notin W$

$\Rightarrow$ W is not a subspace as scalar multiplication is not closed.

Q) W consists of the points (a,b,c) such that $a = 2b = 3c$. Is W a subspace of $\mathbb{R}^3$?

$\rightarrow$ $a = 2b = 3c$

$\left[\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \right]$ $\Rightarrow$ line

$\Rightarrow$ $\frac{a}{6} = \frac{b}{3} = \frac{c}{2}$ (line through origin)

$(\vec{r} - \vec{a}) = \lambda \vec{b}$

$W = \left\{ (a,b,c) \ / \ \frac{a}{6} = \frac{b}{3} = \frac{c}{2} \right\} \subset \mathbb{R}^3$

$u_1 = (a_1, b_1, c_1)$ and $u_2 = (a_2, b_2, c_2)$

$\Rightarrow a_1 = 2b_1 = 3c_1$ — 1a and $a_2 = 2b_2 = 3c_2$ — 1b

$u_1 + u_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

$a_1 + a_2 = 2b_1 + 2b_2 = 3c_1 + 3c_2$

From 1a + 1b $a_1 + a_2 = 2(b_1 + b_2) = 3(c_1 + c_2)$ — 3

$\Rightarrow$ $u_1 + u_2 \in W$

$k.u_1 = (ka_1, kb_1, kc_1)$

$k \times$ 1a $\Rightarrow$ $ka_1 = k(2b_1) = k(3c_1)$

$\Rightarrow$ $ka_1 = 2(kb_1) = 3(kc_1)$

$\Rightarrow$ $k.u_1 \in W$ $\rightarrow$ 4

From ③ and ④ W is a subspace of $\mathbb{R}^3$.

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⑤ Is W a subspace of $\mathbb{R}^3$ if W is the set of all points (a,b,c) such that $a+2b-c=0$

→ $W = \{ a+2b-c=0 \,/\, (a,b,c) \} \subset \mathbb{R}^3$

↳ plane passing through origin

$u = (a_1, b_1, c_1)$ $\quad v = (a_2, b_2, c_2)$

$\Rightarrow a_1 + 2b_1 - c_1 = 0$ — (1a) and

$a_2 + 2b_2 - c_2 = 0$ — (1b)

$u+v = (a_1+a_2,\ b_1+b_2,\ c_1+c_2)$

(1a) + (1b) $\Rightarrow a_1 + 2b_1 - c_1 + a_2 + 2b_2 - c_2 = 0$

$(a_1+a_2) + 2(b_1+b_2) - (c_1+c_2) = 0$ — ②

From ② $u+v \in W$ — ③

$k.u = (ka_1, kb_1, kc_1)$

$k \times$ (1a) $\Rightarrow k(a_1 + 2b_1 - c_1) = 0 \Rightarrow ka_1 + 2(kb_1) - kc_1 = 0$ — ④

From ④ $k.u \in W$ — ⑤

③ and ⑤

$\Rightarrow W$ is a subspace of $\mathbb{R}^3$

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⑨ Is $W = \{(a,b,c) \,/\, a+2b-3c=1\}$ a subspace of $\mathbb{R}^3$?

$(0,0,0) \notin W \Rightarrow W$ is not a subspace.

(

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⑧ Is $W = \{(a,b,c) \,/\, a^2+b^2+c^2=1\}$ a subspace of $\mathbb{R}^3$?

→ ~~let u = (a,b~~

Consider $(1,0,0)$ and $k=2$

$k.u = (2,0,0) \notin W$

[For disproving use any one example scalar $\times^{tn}$, vector additn or using zero vector conditn]

HW Let $W \subset \mathbb{R}^3$ with

i) Is W a subspace of $\mathbb{R}^3$ if $a \le b \le c$ (No)

ii) Is W a subspace of $\mathbb{R}^3$ if $a = b^2$ (No)

Linear Combination :

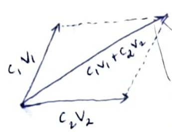
Consider a set $S = \{v_1, v_2, v_3 \ldots v_n\} \subset V$, a vector space over a field F. A vector $v \in V$ is said to be a linear combination of vectors in S if there exists scalars, $c_1, c_2 \ldots c_n$ ∍'

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_n v_n$$

Consider
$$3x + 4y + z = 1$$
$$2x + 3y - z = 2$$
$$x - y + 3z = -2$$

$$\Rightarrow x\begin{bmatrix}3\\2\\1\end{bmatrix} + y\begin{bmatrix}4\\3\\-1\end{bmatrix} + z\begin{bmatrix}1\\-1\\3\end{bmatrix} = \begin{bmatrix}1\\2\\-2\end{bmatrix}$$

Let c_1, c_2



if $=V$ there exists a linear combination

can be $=V$ or $\neq V$

HW soln

i) $(a,b,c) \in W \Rightarrow a \leq b \leq c$

$u = (1,2,3) \in W$

$k = -1 \Rightarrow k.u = (-1)(1,2,3) = (-1,-2,-3) \notin W$

$\Rightarrow k.u \notin W$

$\therefore$ W is not a subspace.

---

ii) $u = (a,b,c) \in W \Rightarrow a = b^2$

$v = (x,y,z) \in W \Rightarrow x = y^2$

$u+v = (a+x, b+y, c+z)$

$(b+y)^2 = b^2 + 2by + y^2$

$= a + 2by + x$

$\Rightarrow (b+y)^2 \neq (a+x) \Rightarrow u+v \notin W$

$\therefore$ W is not a subspace.

Q) Is $v = (4,-9,2)$ a linear combinatn of $(1,2,-1)$, $(1,4,2)$ and $(1,-3,2)$?

→ let $u_1 = (1,2,-1)$, $u_2 = (1,4,2)$, $u_3 = (1,-3,2)$

Consider $x_1u_1 + x_2u_2 + x_3u_3 = V$

$$x_1\begin{bmatrix}1\\2\\-1\end{bmatrix} + x_2\begin{bmatrix}1\\4\\2\end{bmatrix} + x_3\begin{bmatrix}1\\-3\\2\end{bmatrix} = \begin{bmatrix}4\\-9\\2\end{bmatrix}$$

$$A = \begin{bmatrix}1&1&1\\2&4&-3\\-1&2&2\end{bmatrix} \quad B = \begin{bmatrix}4\\-9\\2\end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc:c}1&1&1&4\\2&4&-3&-9\\-1&2&2&2\end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c}1&1&1&4\\0&2&-5&-17\\0&3&3&6\end{array}\right]$$

$R_3 \to R_3 - \frac{3}{2}R_2$

$$\Rightarrow [A:B] \sim \left[\begin{array}{ccc:c}1&1&1&4\\0&2&-5&-17\\0&0&\frac{21}{2}&\frac{63}{2}\end{array}\right]$$

$\rho(A) = \rho(A:B) = 3$

$\Rightarrow$ System is consistent

No. of unknowns, $n = 3 = r$

= Unique solutn.

$\Rightarrow x_1 + x_2 + x_3 = 4$

$2x_2 - 5x_3 = -17$

$\frac{21}{2}x_3 = \frac{63}{2}$

$x_3 = 3$

$2x_2 - 15 = -17$

$2x_2 = -2 \Rightarrow x_2 = -1$

$x_1 - 1 + 3 = 4$

$x_1 = 2$

$\Rightarrow 2u_1 - u_2 + 3u_3 = V$

Q) Is $V = (1,4,6)$ a linear combination of $V_1 = (1,1,2)$, $V_2 = (2,3,5)$, $V_3 = (3,5,8)$? If yes find the scalars

→ Consider $x_1V_1 + x_2V_2 + x_3V_3 = V$

$$x_1\begin{bmatrix}1\\1\\2\end{bmatrix} + x_2\begin{bmatrix}2\\3\\5\end{bmatrix} + x_3\begin{bmatrix}3\\5\\8\end{bmatrix} = V$$

$$A = \begin{bmatrix}1&2&3\\1&3&5\\2&5&8\end{bmatrix} \quad B = \begin{bmatrix}1\\4\\6\end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & 4 \\ 2 & 5 & 8 & 6 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\rho(A) = 2$, $\rho(A:B) = 3$

$\rho(A) \neq \rho(A:B)$

$\Rightarrow$ System is inconsistent

$\Rightarrow x_1v_1 + x_2v_2 + x_3v_3 \neq V$

for any $x_1, x_2, x_3 \in \mathbb{R}$

(Here the one of the vectors is coplanar with other 2 vectors thus we cannot form a point in 3D)

③ Express $V = t^2 + 4t - 3$ in $P(t)$ as a linear combination if $p_1 = t^2 - 2t + 5$; $p_2 = 2t^2 - 3t$, $p_3 = t + 1$

$\to$ Let $x_1p_1 + x_2p_2 + x_3p_3 = V$

$\Rightarrow x_1(t^2 - 2t + 5) + x_2(2t^2 - 3t) + x_3(t+1) = (t^2 + 4t - 3)$

Coeff of t^2 : $x_1 + 2x_2 + \cancel{x_3} = 1$

Coeff of t : $-2x_1 - 3x_2 + x_3 = +4$

Const : $5x_1 + x_3 = -3$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & +4 \\ 5 & 0 & 1 & -3 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 5R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & -10 & 1 & -8 \end{array}\right]$$

$R_3 \to R_3 + 10R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 11 & 52 \end{array}\right]$$

$\rho(A) = \rho(A:B) = 3$

$n = r = 3$

$\Rightarrow$ System is consistent & unique solⁿ exists

$x_1 + 2x_2 = 1$

$x_2 + x_3 = 6$

$11x_3 = 52$

$x_3 = \frac{52}{11}$

$x_2 + \frac{52}{11} = 6$

$x_2 = 6 - \frac{52}{11}$

$= \frac{14}{11}$

$x_1 + \frac{28}{11} = 1$

$x_1 = 1 - \frac{28}{11} = \frac{-17}{11}$

$$\Rightarrow \frac{-17}{11}p_1 + \frac{14}{11}p_2 + \frac{52}{11}p_3 = V$$

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Subspace :

Let U be a non-empty subset of a vector space V. Then U is a subspace of V if $\boxed{\text{zero vector } 0 \in U}$

$u + v \in U \quad \forall \; u, v \in U$ and $k.u \in U, \; \forall \; k \in F, u \in U$

Spanning set :

Let $S = \{v_1, v_2, v_3, \dots v_n\} \subset V$, a vector space over the field F. The span of S denoted span (S) is obtained by all possible linear combinat's of the vectors in S.

i.e $\text{span}(S) = \{c_1v_1 + c_2v_2 + c_3v_3 + \dots + c_nv_n,\ v_i \in S$

and $c_i \in F$ for $i = 1$ to $n\}$

$\text{span}(S)$ is a subspace of V.

Q) Verify whether $p = t^3 + t^2 + t$ is in the span of $\{P_1, P_2, P_3\}$

where $P_1 = t^3 + t^2$ $\quad P_2 = t^3 + t$ $\quad P_3 = t^2 + t$

→ $p \in \text{span}\{P_1, P_2, P_3\}$ if and if

$p = x_1p_1 + x_2p_2 + x_3p_3$ $\Big\} \Rightarrow$ (verify whether p is a linear combⁿ of P_1, P_2, P_3 & this Qn is same)

$t^3 + t^2 + t = x_1(t^3 + t^2) + x_2(t^3 + t) + x_3(t^2 + t)$

$x_1 + x_2 = 1$ $\quad$ ($\because$ coeff of t^3)

$x_1 + x_3 = 1$ $\quad$ ($\because$ coeff of t^2)

$x_2 + x_3 = 1$ $\quad$ ($\because$ coeff of t)

$AX = B$ where $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$

$R_2 \to R_2 - R_1 \Rightarrow [A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$

$R_3 \to R_3 + R_1 \Rightarrow [A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$

$\rho(A) = 3 = \rho(A:B) = r \Rightarrow$ solⁿ exists [$\because$ system is consistent]

no. of unknowns $n = 3$

$r = n \Rightarrow$ unique solⁿ

$x_1 + x_2 = 1$

$-x_2 + x_3 = 0$

$2x_3 = 1 \Rightarrow x_3 = \frac{1}{2}$

$x_2 = \frac{1}{2}$

$x_1 = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore p \in \text{span}\{P_1, P_2, P_3\}$ as $p = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3$

Q) Find the conditions n' any vector $v = (a, b, c) \in \mathbb{R}^3$ belongs to $W = \text{span}(S)$ where $S = \{u_1 = (1, 2, 0),$ $u_2 = (-1, 1, 2)$, $u_3 = (3, 0, -4)\}$

→ Suppose $v \in \text{span}(S)$

$\Rightarrow v = x_1u_1 + x_2u_2 + x_3u_3$

$[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 2 & 1 & 0 & b \\ 0 & 2 & -4 & c \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 2 & -4 & c \end{array}\right]$

$R_3 \to R_3 - \frac{2}{3}R_2$

$[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 0 & 0 & c - \frac{2}{3}(b-2a) \end{array}\right]$

$\rho(A) = 2$

Solⁿ of $AX = B$ exists only if $\rho(A:B) = 2$

$\rho(A:B) = 2 \Rightarrow c - \frac{2}{3}(b - 2a) = 0$

i.e $4a - 2b + 3c = 0$

[a plane passing through origin]

Linear dependence and independence

A set $S = \{v_1, v_2, v_3 \dots v_n\}$ of vectors in a vector space V over a field F is said to be linearly dependent if there exists scalars $c_1, c_2, c_3 \dots c_n$ in F ~~and~~ not all zeroes such that

$$c_1v_1 + c_2v_2 + c_3v_3 + \dots + c_nv_n = 0 \longrightarrow ①$$

(0 = zero vector)

ie if equation ① has non-trivial sol^{n}s, then S is linearly dependent.

S is linearly independent if $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$ only if $c_1 = 0, c_2 = 0 \dots c_n = 0$

ie S is linearly independent if eq^n ① has only trivial sol^{n}s.

Q) Determine whether $u_1 = (1,2,5)$ $u_2 = (2,5,1)$ and $u_3 = (1,5,2)$ are linearly dependent or independent.

→ Consider $x_1u_1 + x_2u_2 + x_3u_3 = 0$

$\Rightarrow AX = 0$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -9 & -3 \end{bmatrix}$$

$R_3 \to R_3 + 9R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{bmatrix}$$

$\rho(A) = 3$

$n = 3$

$r = n$

$\Rightarrow$ trivial sol^n exists

$\Rightarrow x_1u_1 + x_2u_2 + x_3u_3 = 0$ only if $x_1 = 0, x_2 = 0, x_3 = 0$

$\Rightarrow \{u_1, u_2, u_3\}$ is a linearly independent set.

For 2 vectors,

$\{u_1, u_2\}$ is linearly dependent if $u_2 = ku_1$.

$[\because x_1u_1 + x_2u_2 = 0 \Rightarrow u_2 = -\frac{x_1}{x_2}u_1]$

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Q) Determine whether $t^3 - 4t^2 + 3t + 3$; $t^3 + 2t^2 + 4t - 1$ and $2t^3 - t^2 - 3t + 5$ are linearly dependent or independent.

→ Consider $x_1p_1 + x_2p_2 + x_3p_3 = 0$

$\Rightarrow AX = 0$

$$A = \begin{bmatrix} 1 & -4 & 3 & 3 \\ 1 & 2 & 4 & -1 \\ 2 & -1 & -3 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -4 & 2 & -1 \\ 3 & 4 & -3 \\ 3 & -1 & 5 \end{bmatrix}$$

$R_2 \to R_2 + 4R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 7 \\ 0 & 1 & -9 \\ 0 & -4 & -1 \end{bmatrix},$$

~~$(R_3 \to R_3 - \frac{R_2}{6})$~~

$R_3 \leftrightarrow R_2 \Rightarrow$

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \\ 0 & 6 & 7 \\ 0 & -4 & -1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \\ 0 & 6 & 7 \\ 0 & -4 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 6R_2$, $R_4 \to R_4 + 4R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & 61 \\ 0 & 0 & -37 \end{bmatrix}$$

$R_4 \to R_4 + \frac{37}{61} R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & 61 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3 = r$ and no. of unknowns, $n = 3$, $r = n$

$\Rightarrow$ only trivial sol^n exists

$\Rightarrow$ $x_1 p_1 + x_2 p_2 + x_3 p_3 = 0$

only if $x_1 = 0 = x_2 = x_3$

$\Rightarrow$ $\{p_1, p_2, p_3\}$ are linearly independent set

Q) Is $(1,2,5), (1,3,1), (2,5,7), (3,1,4)$ linearly dependent or independent ?

→ Consider $x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4 = 0$

$\Rightarrow AX = 0$

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 1 \\ 5 & 1 & 7 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 5R_1$, $R_2 \to R_2 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & -4 & -3 & -11 \end{bmatrix}$$

$R_3 \to R_3 + 4R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & 31 \end{bmatrix}$$

$\rho(A) = 3$

$n = 4$

$r < 4$

$\Rightarrow$ non-trivial sol^n exists

$\Rightarrow$ linearly dependent

Note : → Consider $S = \{0, v_1, v_2, \dots v_n\} \subset V$ where V is a vector space over the field F.

Is S linearly dependent or independent ?

Even Consider $x_0 0 + x_1 v_1 + \dots + x_n v_n = 0$

Even if $x_1, x_2 \dots x_3$ are zeroes, x_0 need not be zero $\Rightarrow S$ is linearly dependent {even if one scalar is zero it is sufficient}

$\therefore$ Any set of vectors containing the zero vector will be linearly dependent.

Basis (plu: Bases) { and dimension of a vector space)

A subset $S = \{u_1, u_2, \dots u_n\}$ of a vector space V over a field F is a basis of V if

i) S is linearly independent

ii) S spans V {i.e $V = span(S)$}

The no. of vectors in the basis of V is called dimension of V and is denoted by $\dim(V)$.

$B = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is a basis of $\mathbb{R}^3$, Hence $\dim(\mathbb{R}^3) = 3$

$p(t) \in P_n(t) \Rightarrow p = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$, $a_i \in \mathbb{R}$

$B = \{1, t, t^2, t^3, \dots t^n\} \Rightarrow \dim(P_n(t)) = n+1$

Theorem: let V be a vector space of dimension n, then i) Any set of 'n' linearly independent vectors will be a basis of V.

ii) Any set containing more than n vectors will be linearly dependent.

iii) Any linearly independent set containing less than n vectors cannot span V.

Q) Determine whether $B = \{(1,1,1), (1,2,3)$ & $(2,-1,1)\}$ is a basis of $\mathbb{R}^3$?

→ Step 1: Prove B is linearly independent
Step 2: Prove that $\text{span}\{B\} = \mathbb{R}^3$
} Assuming above theorem is not given (not used)

→ Consider $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0 \Rightarrow AX = 0$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

$\rho(A) = 3$

$r = n$

$\Rightarrow$ only trivial sol^n exists

$\Rightarrow B$ is linearly independent. — ①

Consider (an arbitrary vector) V

$x_1 v_1 + x_2 v_2 + x_3 v_3 = V$, where $V = (a,b,c) \in \mathbb{R}^3$

$AX = B$

Note: Let B be basis of V, $B = \{v_1, v_2, \dots, v_n\}$
i) B is linearly independent
ii) For any $v \in V$,
$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ is unique

$$[A:B] = \begin{bmatrix} 1 & 1 & 2 & : & a \\ 1 & 2 & -1 & : & b \\ 1 & 3 & 1 & : & c \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$[A:B] = \begin{bmatrix} 1 & 1 & 2 & : & a \\ 0 & 1 & -3 & : & b-a \\ 0 & 2 & -1 & : & c-a \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$[A:B] = \begin{bmatrix} 1 & 1 & 2 & : & a \\ 0 & 1 & -3 & : & b-a \\ 0 & 0 & 5 & : & c+a-2b\cancel{+2a} \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + 2x_3 = a$

$x_2 - 3x_3 = b - a$

$5x_3 = c + a - 2b$

$x_3 = \dfrac{a - 2b + c}{5}$

$x_2 = b - a + 3\left(\dfrac{a - 2b + c}{5}\right)$

$= \dfrac{5b - 5a + 3a - 6b + 3c}{5}$

$x_2 = \dfrac{-2a - b + 3c}{5}$

$x_1 = a - 2x_3 - x_2$

$= a - \cancel{2}\left(\dfrac{-2a - b + 3c}{5}\right) - 2\left(\dfrac{a - 2b + c}{5}\right)$

$= (5a + 2a + b - 3c - 2a + 4b - 2c)/5 = (5a + 5b - 5c)/5$

$x_1 = a + b - c$

Since solut^n exist for any vector (a,b,c) we can state that $\text{span}(B) = \mathbb{R}^3$ — ②

From ① and ② B is a basis of $\mathbb{R}^3$.

Alternate sol^n,
→ Verify B is linearly independent.
→ If B is linearly independent & no. of vectors in $B = \dim(V)$, then B is basis of V.
[i.e state theorem (i)]

For this que., Solve upto ①, Since $\dim(\mathbb{R}^3) = 3$ & only linearly independent set of 3 vectors is a basis of $\mathbb{R}^3$, B is basis of $\mathbb{R}^3$.

Row space & Column Space of Matrix

Consider $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

$$\left.\begin{aligned} r_1 &= (a_{11}, a_{12}, a_{13} \cdots a_{1n}) \\ r_2 &= (a_{21}, a_{22}, \cdots a_{2n}) \\ &\vdots \\ r_m &= (a_{m1}, a_{m2} \cdots a_{mn}) \end{aligned}\right\} \Rightarrow R = \{r_1, r_2 \cdots r_m\}$$

span (R) will be a subspace of $\mathbb{R}^n$ and this is called row space of A and is denoted by rowsp (A) = rowspace (A) = span (R)

|||ly.

$$\left.\begin{aligned} c_1 &= (a_{11}, a_{21}, a_{31} \cdots a_{m1}) \\ c_2 &= (a_{12}, a_{22}, a_{32} \cdots a_{m2}) \\ &\vdots \\ c_n &= (a_{1n}, a_{2n}, \cdots a_{mn}) \end{aligned}\right\} \Rightarrow C = \{c_1, c_2, \cdots c_n\}$$

span (c) will be a subspace of $\mathbb{R}^m$ and this is the column space of A and is denoted by col sp (A) = colspace (A) = span (C)

(dim of col space & row space will be same)

Rank - Nullity Theorem :

$$\text{rank}(A) + \text{nullity}(A) = \underbrace{n}_{\text{no. of columns}}$$

nullity (A) = dim of null space of A.

Finding basis of a subspace from a spanning set.

W = span(S) is subspace of V
S is called spanning set of W.

i) Row space algorithm :

Step 1: Write the vectors along the rows of the matrix. A.

Step 2: Obtain echelon form of the matrix A.

Step 3: Dimension of the sub space is the rank of A ie $\rho(A)$ and output the non-zero rows as the basis of this subspace, W = span(S).

ii) Column space algorithm : (Casting out algorithm)

Step 1: Write the vectors of S along columns of matrix A.

(Step 2 ~~& 3~~ from above ~~with columns instead of rows~~)

Step 3: Cast out the original vectors corresponding to leading entries as the vectors of the basis.

Difference b/w the 2 algorithms :

→ Row space algorithm helps in finding a basis of span (S)

→ Column space algorithm gives a subset of S which forms a basis of span (S)

→ Find a basis {either of 2 algos}

→ Find subset of S which is basis of span(S) : only use casting algo

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Q Determine whether (1,1,1,1) (1,2,3,2) (2,5,6,4) and (2,6,8,5) forms a basis of $\mathbb{R}^4$. If not find the dimension of the subspace they span.

→ $\dim(\mathbb{R}^4) = 4$

Accrd. to theorem, any linearly independent set of 4 vectors in $\mathbb{R}^4$ forms a basis of $\mathbb{R}^4$

[∵ It is sufficient to check if the set is linearly independent]. We can apply row-space or col-space algorithm.

~~Consider~~ $\cancel{x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4}$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - 4R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

⇒ only 3 independent vectors in given set.

⇒ $S = \{(1,1,1,1)\ (1,2,3,2)\ (2,5,6,4)\ (2,6,8,5)\}$ cannot be a basis of $\mathbb{R}^4$

$\dim(W) = 3$ [∵ $W = \text{span}(S)$]

and basis of W,

$B = \{(1,1,1,1), (0,1,2,1)\ (0,0,-2,-1)\}$

(We do not extract vectors from row space algo, as row exchange operat^n would affect posit^n of vectors. In col-space, row exchange doesn't change position of vectors)

Q Find a subset of $S = \{u_1, u_2, u_3, u_4\}$ where $u_1 = (1,1,1,2,3)$ $u_2 = (1,2,-1,-2,1)$ $u_3 = (3,5,-1,-2,5)$ $u_4 = (1,2,1,-1,4)$

→ We need subset of S which is a basis of $W = \text{span}(S)$.

Hence we use column-space algorithm

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 2 & 5 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -2 & -2 & -1 \\ 3 & 1 & 5 & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 2R_1$, $R_5 \to R_5 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & 0 \\ 0 & -4 & -8 & -3 \\ 0 & -2 & -4 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$, $R_4 \to R_4 + 4R_2$, $R_5 \to R_5 + 2R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$\cancel{R_4 \to R_4 - \frac{1}{2}R_3,\ R_5}$

$R_4 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$, $R_5 \to R_5 - 3R_3$

$$A \sim \begin{array}{c} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \ \ \downarrow\ \ \ \downarrow\ \ \ \downarrow\ \ \ \downarrow \\ \ \ u_1\ \ u_2\ \ u_3\ \ u_4 \end{array}$$

$\rho(A) = 3$

⇒ Only 3 independent vectors are there in set i.e

$S = \{(1,1,1,2,3)$ $\cancel{(0,1,2,}$ $(1,2,-1,-2,1)$, $(1,2,1,-1,4)\}$ is the basis of span(S)

(u_3 is cast out)

Note: Column ③ does not have a leading entry.

⇒ u_3 is a linear combinat^n of u_1 and u_2 (∵ $u_1 + 2u_2 = u_3$)

$\left[\begin{array}{l} x_2 = 2 \\ x_1 + x_2 = 3 \\ x_1 = 1 \end{array}\right.$ $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right]$

Q) Find a basis and dimension of the subspace spanned by $u_1 = t^3 + t^2 - 3t + 2$, $u_2 = 2t^3 + t^2 + t - 4$ and $u_3 = 4t^3 + 3t^2 - 5t + 2$

→ We apply row-space algorithm,

$$A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & 1 & 1 & -4 \\ 4 & 3 & -5 & 2 \end{bmatrix}$$

[cubic polynomial maps to $\mathbb{R}^4$
$\Rightarrow \dim(\mathbb{R}^4) = 4$
we have 3 vectors
$\therefore$ $\text{span}(S) \neq P_3(t)$]

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -1 & 7 & -8 \\ 0 & -1 & 7 & -6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -1 & 7 & -8 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\rho(A) = 3$

$\Rightarrow S = \{u_1, u_2, u_3\}$ is linearly independent and hence a basis of $W = \text{span}(S)$ and $\dim(W) = 3$

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Q) Is $S = \{(1,2,-1,3,4), (2,4,-2,6,8), (1,3,2,2,6), (1,4,5,1,8), (2,7,3,3,9)\}$ a basis of $\mathbb{R}^5$?
If no, then find a subset of S which forms a basis of $W = \text{span}\{S\}$ and also find the dimension of this subspace.

→ (In row-space we obtain basis from echelon form
In column-space, using leading entries, we extract basis from original vector set)

→ Since we need a subset of S which is a basis of $\text{span}(S)$ we use col-space algorithm

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_4 \to R_4 - 3R_1$
$R_3 \to R_3 + R_1$, $R_5 \to R_5 - 4R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$, $R_4 \to R_4 + R_2$
$R_5 \to R_5 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

$R_5 \to R_5 - \frac{5}{4} R_3$

$$A \sim \begin{bmatrix} \boxed{1} & 2 & 1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & \boxed{-4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow u_1$ $\quad \downarrow u_3$ $\quad \downarrow u_5$

$B = \{(1,2,-1,3,4), (1,3,2,2,6), (2,7,3,3,9)\}$ is a basis of $W = \text{span}(S)$ and $\dim(W) = 3$

Q) Find the basis and dimension of sol^n - space of system of equations,

$x_1 + 2x_2 - 3x_3 + 2x_4 - 4x_5 = 0$

$2x_1 + 4x_2 - 5x_3 + x_4 - 6x_5 = 0$ and

$5x_1 + 10x_2 - 13x_3 + 4x_4 - 16x_5 = 0$

→ $AX = 0$ where

$$A = \begin{bmatrix} 1 & 2 & -3 & 2 & -4 \\ 2 & 4 & -5 & 1 & -6 \\ 5 & 10 & -13 & 4 & -16 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -3 & 2 & -4 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 2 & -6 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 2 & -3 & 2 & -4 \\ 0 & 0 & \underline{1} & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

no. of unknowns, $n = 5$

$n - r = 5 - 2 = 3$ free variables

let $x_2 = k_1$, $x_4 = k_2$, $x_5 = k_3$

(Assign arbitrary values to free variables)

equivalent system is,

$x_1 + 2x_2 - 3x_3 + 2x_4 - 4x_5 = 0$

$x_3 - 3x_4 + 2x_5 = 0$

$x_1 + 2k_1 - 3x_3 + 2k_2 - 4k_3 = 0$

$x_3 - 3k_2 + 2k_3 = 0$

$x_3 = 3k_2 - 2k_3$

$x_1 - 3(3k_2 - 2k_3) = 4k_3 - 2k_2 - 2k_1$

$x_1 = -2k_3 - 2k_1 + 7k_2$

$$X = \begin{bmatrix} -2k_3 - 2k_1 + 7k_2 \\ k_1 \\ 3k_2 - 2k_3 \\ k_2 \\ k_3 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Basis of $solut^n$ space of $AX = 0$ is

$B = \{(-2,1,0,0,0), (7,0,3,1,0), (-2,0,-2,0,1)\}$ and

$\dim(sol^n \text{ space}) = 3$

Note: i) Sol^n space of $AX = 0$ is called null space of A.

ii) $\dim\{sol^n \text{ space of } A_{m\times n}X = 0\}$

$= \dim\{\text{null space}(A_{m\times n})\} = n - r$

where $r = \rho(A)$ whenever $r < n$

Q) Find the basis and dimension of the row-space, col^{mn} space and null space of $\begin{bmatrix} 0 & 0 & 3 & 1 & 4 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \end{bmatrix} = A$

→ $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 3 & 1 & 4 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 4R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 4 & 0 & 3 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$\cancel{R_3 \to R_3 - \frac{1}{3}R_2, \; R_4 \to R_4 - \frac{4}{3}R_2}$

$$A \sim \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 1 & 4 \\ 0 & 0 & 4 & 0 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - 4R_2$

$$A \sim \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}$$

$\rho(A) = 3$

Basis of row space (A),

$B_1 = \{(1,3,1,2,1), (0,0,1,-1,-1), (0,0,0,4,7)\}$

$\dim(\text{row sp}(A)) = 3$

Basis of col sp(A), $B_2 = \{(0, 0, 3, 4), (3, 1, 4, 8), (1, 2, 5, 8)\}$

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$\dim(\text{col sp}(A)) = 3$

Null spa (A) = solⁿ space of $AX = 0$

Equivalent system is
$$x_1 + \boxed{3x_2} + x_3 + 2x_4 + x_5 = 0$$
$$x_3 - x_4 - x_5 = 0$$
$$4x_4 + \boxed{7x_5} = 0$$

x_2 and x_5 are free variables

$\dim(\text{null space}(A)) = n - r$

Let $x_2 = k_1$ and $x_5 = k_2$

$\Rightarrow 4x_4 + 7k_2 = 0 \Rightarrow x_4 = -\frac{7}{4}k_2$

& $x_3 + \frac{7}{4}k_2 - k_2 = 0 \Rightarrow x_3 = -\frac{3}{4}k_2$

& $x_1 + 3k_1 - \frac{3}{4}k_2 + 2\left(-\frac{7}{4}\right)k_2 + k_2 = 0$

$\Rightarrow x_1 + 3k_1 \frac{-3-14+4}{4} k_2 = 0$

$x_1 = -3k_1 + \frac{13}{4}k_2$

~~Equivalent system~~ →

$$X = \begin{bmatrix} -3k_1 + \frac{13}{4}k_2 \\ k_1 \\ -\frac{3}{4}k_2 \\ -\frac{7}{4}k_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} 13 \\ 0 \\ -3 \\ -7 \\ 4 \end{bmatrix} \frac{k_2}{4}$$

Basis of the null space (A), $B_N = \{(-3, 1, 0, 0, 0), (+13, 0, -3, -7, 4)\}$

Note: $\dim(\text{null space}(A)_{m\times n}) = n - r$, $r = \rho(A)$

Q) Obtain basis & dimension of row space, col space & null space of
$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ 2 & 4 & -1 & -1 & 0 \\ 3 & 6 & -1 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix} \\ \begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \end{array}$$

(Set $S = \{c_1, c_2, c_3, c_4, c_5\}$ is linearly dependent

$0.c_1 + 0.c_2 + 0.c_3 + 0.c_4 + kc_5 = 0, k \neq 0$

∵ Any set containing zero vector is linearly dependent.

→ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$,

$$A \sim \begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 10 & 0 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$R_4 \to R_4 - \frac{1}{2}R_3$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

Basis of row space (A)

$B_1 = \{(1, 2, -1, -2, 0), (0, 0, 1, 3, 0), (0, 0, 0, 4, 0)\}$

Basis of col space (A)

$B_2 = \{(1, 2, 3, 0), (-1, -1, -1, 1), (-2, -1, 4, 5)\}$

$\dim(\text{row sp}(A)) = \dim(\text{col spa}(A)) = 3$

Null spa(A) = solⁿ space of $AX = 0$ $[A_{4\times5} X_{5\times1} = 0]$

Equivalent system is
$$x_1 + 2x_2 - x_3 - 2x_4 = 0$$
$$x_3 + 3x_4 = 0$$
$$4x_4 = 0$$
$n = 5$

Since $x = [x_1 \ x_2 \ x_3 \ x_4 \ x_5]^T$,

the free variables are x_2 & x_5

$x_2 = a$ & $x_5 = b$,

$\Rightarrow$ $x_4 = 0$, $x_3 = 0$, $x_1 + 2x_2 = 0$

$x_1 + 2a = 0$

$x_1 = -2a$

$$x = [-2a \ \ a \ \ 0 \ \ 0 \ \ b]^T$$

$$= a[-2 \ \ 1 \ \ 0 \ \ 0 \ \ 0]^T + b[0 \ \ 0 \ \ 0 \ \ 0 \ \ 1]^T$$

Basis of null sp(A) $= \{(-2,1,0,0,0), (0,0,0,0,1)\}$

$\Rightarrow$ dim (null sp(A)) $= 2$

---

5) Extend $S = \{(1,1,1,1), (2,2,3,4)\}$ to a basis of $\mathbb{R}^4$.

→ dim $(\mathbb{R}^4) = 4$

S contains 2 linearly independent vectors.

We need to add 2 more vectors

$B = \{(1,0,0,0), (0,1,0,0)\ (0,0,1,0)\ (0,0,0,1)\}$ is a standard basis

We need row space algo. (to choose vector from above)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \Rightarrow A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

If we consider $P = \{(1,1,1,1), (2,2,3,4)\}$

(to make $\rho(A) = 4$, we need leading entries in 2^{nd} & 4^{th} col.) $\Rightarrow$

we consider $P = \{(1,1,1,1)\ (2,2,3,4)\ (0,1,0,0)\ (0,0,0,1)\}$

then P is a basis of $\mathbb{R}^4$.

$$\left[\because M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right]$$

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6) Find a homogeneous system whose solution space is spanned by $S = \{u_1, u_2, u_3\}$ where $u_1 = (1,-2,0,3,-1)$ $u_2 = (2,-3,2,5,-3)$ and $u_3 = (1,-2,1,2,-2)$

→ Now $S \subset \mathbb{R}^5$

$\Rightarrow$ Any vector in sol^n space will be of the form $(x_1, x_2, x_3, x_4, x_5)$

Since S spans the sol^n space, v is a linear combination of u_1, u_2, u_3

$\therefore c_1 u_1 + c_2 u_2 + c_3 u_3 = v$ [using c_1, c_2, c_3, as x_1, x_2, x_3 is already used in v]

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 2 \\ 5 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \Rightarrow AX = B$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ -2 & -3 & -2 & x_2 \\ 0 & 2 & 1 & x_3 \\ 3 & 5 & 2 & x_4 \\ -1 & -3 & -2 & x_5 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$, $R_4 \to R_4 - 3R_1$, $R_5 \to R_5 + R_1$

$$\Rightarrow [A:B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 0 & 1 & 0 & x_2+2x_1 \\ 0 & 2 & 1 & x_3 \\ 0 & -1 & -1 & x_4-3x_1 \\ 0 & -1 & -1 & x_5+x_1 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$, $R_4 \to R_4 + R_2$,
$R_5 \to R_5 + R_2$

$$\Rightarrow [A:B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 0 & 1 & 0 & x_2+2x_1 \\ 0 & 0 & 1 & x_3-2x_2-4x_1 \\ 0 & 0 & -1 & x_4-3x_1+x_2+2x_1 \\ 0 & 0 & -1 & x_5+x_1+2x_1+x_2 \end{array}\right]$$

$R_4 \to R_4 + R_3$, $R_5 \to R_5 + R_3$

$$\Rightarrow [A:B] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & x_1 \\ 0 & 1 & 0 & x_2+2x_1 \\ 0 & 0 & 1 & x_3-2x_2-4x_1 \\ 0 & 0 & 0 & x_4-x_1+x_2+x_3-2x_2-4x_1 \\ 0 & 0 & 0 & x_5-x_1-x_2+x_3 \end{array}\right]$$

As V is a linear combination of u_1, u_2, u_3, $AX=B$ should be consistent.

Now $\rho(A) = 3$

Condition for consistency is $\rho(A) = \rho(A:B)$

$\rho(A:B) = 3$ only if $x_4 - 5x_1 - x_2 + x_3 = 0$ &

$x_5 - x_1 - x_2 + x_3 = 0$

Q Find the homogenous system whose solⁿ space is spanned by $(1,-2,0,3)$ $(1,-1,-1,4)$ $(1,0,-2,5)$

$\rightarrow$ $S = \{u_1, u_2, u_3\}$, where $u_1 = (1,-2,0,3)$ $u_2 = (1,-1,-1,4)$
$u_3 = (1,0,-2,5)$

$S \subset \mathbb{R}^4$

Any vector in solⁿ space will be of the form (x_1, x_2, x_3, x_4).

Since S spans solⁿ space, V is a linear combinatⁿ of u_1, u_2, u_3

$\therefore$ $c_1u_1 + c_2u_2 + c_3u_3 = v$

$$c_1\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \\ -1 \\ 4 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow AX = B$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ -2 & -1 & 0 & x_2 \\ 0 & -1 & -2 & x_3 \\ 3 & 4 & 5 & x_4 \end{array}\right]$$

As V is a linear combinatⁿ of u_1, u_2, u_3 $AX=B$ must be consistent.

$R_2 \to R_2 + 2R_1$, $R_4 \to R_4 - 3R_1$

$$\Rightarrow [A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 0 & 1 & 2 & x_2+2x_1 \\ 0 & -1 & -2 & x_3 \\ 0 & 1 & 2 & x_4-3x_1 \end{array}\right]$$

$R_3 \to R_3 + R_2$, $R_4 \to R_4 - R_2$

$$\Rightarrow [A:B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 0 & 1 & 2 & x_2+2x_1 \\ 0 & 0 & 0 & x_3+x_2+2x_1 \\ 0 & 0 & 0 & x_4-5x_1-x_2 \end{array}\right]$$

Now $\rho(A) = 2$

Conditⁿ for consistency is $\rho(A) = \rho(A:B)$

$\rho(A:B) = 2$ only if
$x_3 + x_2 + 2x_1 = 0$ &
$x_4 - 5x_1 - x_2 = 0$

Note $\rho(A) = 2$, but we have 3 vectors u_1, u_2, u_3
$\Rightarrow$ S spans solⁿ space but is not a basis.

Coordinate Vector:

Consider an ordered basis $B = \{u_1, u_2, u_3 \dots u_n\}$ of a vector space V over a field F. Let $v \in V$ be any vector then

$$v = c_1u_1 + c_2u_2 + c_3u_3 + \dots + c_nu_n$$

[ie any vector can be expressed as linear combinat^n of basis vectors]

The n-tuple of these scalars is in $\mathbb{R}^n$ and is called the coordinate vector of v relative to the basis B and is denoted as

$$[v]_B = [c_1, c_2 \dots c_n] \in \mathbb{R}^n$$

[[] → used to represent coordinate vectors related to a basis]

[∵ coordinate vector of zero-vector = 0]

$[u_1]_B = [1, 0, 0 \dots 0]$

[∵ $u_1 = c_1u_1 + c_2u_2 + \dots + c_nu_n$ $\Rightarrow c_1 = 1, c_2 = 0, c_3 = 0, c_4 = 0$]

$[u_i]_B = [0, 0, 0 \dots, 1, 0, 0]$ ↳ i^{th} component

Q $S = \{p_1, p_2, p_3\}$ where $p_1(t) = t+1$, $p_2(t) = t-1$, $p_3(t) = (t-1)^2$ is a basis of $P_2(t)$. Find the coordinate vector of $v = 2t^2 - 5t + 9$ relative to S.

→ let $v = c_1p_1 + c_2p_2 + c_3p_3$

$$2t^2 - 5t + 9 = c_1(t+1) + c_2(t-1) + c_3(t^2 + 1 - 2t)$$

coeff of t^2: $2 = c_3$

coeff of t: $-5 = c_1 + c_2 - 2c_3$ $\Rightarrow$ $-5 = c_1 + c_2 - 4$

constant: $9 = c_1 - c_2 + c_3$ $\Rightarrow$ $9 = c_1 - c_2 + 2$

$\Rightarrow$ $c_1 + c_2 = -1$
$c_1 - c_2 = 7$

$2c_1 = 6 \Rightarrow c_1 = 3$

$3 + c_2 = -1 \Rightarrow c_2 = -4$

$[v]_S = [3, -4, 2]$

Q Find the coordinate vector of $(2, 7, -4)$ relative to basis $S = \{(1,2,0)(1,3,2)(0,1,3)\}$ of $\mathbb{R}^3$ by LU decomposition.

→ Let $v = c_1u_1 + c_2u_2 + c_3u_3$

$$c_1\begin{bmatrix}1\\2\\0\end{bmatrix} + c_2\begin{bmatrix}1\\3\\2\end{bmatrix} + c_3\begin{bmatrix}0\\1\\3\end{bmatrix} = \begin{bmatrix}2\\7\\-4\end{bmatrix} \Rightarrow AX = B$$

$$A = \begin{bmatrix}1 & 1 & 0\\2 & 3 & 1\\0 & 2 & 3\end{bmatrix} \quad B = \begin{bmatrix}2\\7\\-4\end{bmatrix}$$

Reduce to echelon form using $R_i \to R_i - l_{ij}R_j$

$R_2 \to R_2 - 2R_1 \Rightarrow A \sim \begin{bmatrix}1 & 1 & 0\\0 & \cancel{3}1 & 1\\0 & 2 & 3\end{bmatrix}$ and $l_{21} = 2$, $l_{31} = 0$ (no operat^n performed)

$R_3 \to R_3 - 2R_2 \Rightarrow A \sim \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} = U$ and $l_{32} = 2$

$$L = \begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\0 & 2 & 1\end{bmatrix}$$

$AX = B \Rightarrow (LU)X = B$ as $A = LU$

$L(UX) = B \Rightarrow LY = B$ [ie $UX = Y$]

let $Y = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \Rightarrow LY = B \Rightarrow$

$x_1 = 2$

$2x_1 + x_2 = 7 \Rightarrow x_2 = 7 - 4 = 3$

$2x_2 + x_3 = -4 \Rightarrow -(6+4) = x_3$, $-10 = x_3$

Now $UX = Y \Rightarrow$ $c_1 + c_2 = 2$, $c_2 + c_3 = 3$, $c_3 = -10$ $\Rightarrow$ $c_2 = 13$, $c_1 = -11$ $\Rightarrow [v]_S = [-11, 13, -10]$

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Unit 1:

→ Rank of a matrix

$$\rho(A_{m\times n}) = \min\{m, n\}$$

→ Solution of system of eq^n $AX = B$

i) $A_{m\times n} X_{n\times 1} = 0 \rightarrow \rho(A) = r$

$r = n \rightarrow$ only trivial sol^n

$r < n \rightarrow$ non-trivial sol^n also exists (infinitely many)

ii) $A_{m\times n} X_{n\times 1} = B_{m\times 1}$ consistent if $\rho(A) = \rho(A:B) = r$

else inconsistent.

$r = n \Rightarrow$ Unique sol^n

$r < n \Rightarrow$ Infinitely many solutions

Vector spaces

$(V, +)$ is an abelian group

$k.(u+v) = k.u + k.v$

$(k_1 + k_2).u = k_1 u + k_2 u$

$(k_1 k_2) u = k_1 (k_2 u)$ and $1.u = u$

Subspaces

W is a subspace of V if

i) $0 \in W$

ii) $u+v \in W \ \forall \ u, v \in W$

iii) $k.u \in W \ \forall \ k \in F$ and $k \in W$

→ linear combination $c_1u_1 + c_2u_2 + c_3u_3 + \dots + c_nu_n = v$

$\Rightarrow AX = B$

$[A:B] = \left[\begin{array}{l} u_1, u_2, u_3 \dots u_n \text{ are entered as columns} \\ \text{and } v \text{ is entered as last column} \end{array} \right]$

→ spanning set S = set of all possible linear combinations of vectors in S = $\{ c_1u_1 + c_2u_2 + \dots + c_nu_n \mid u_i \in S, c_i \in F \}$

This S generates a subspace = span(S).

→ linear dependence - independence,

$c_1u_1 + c_2u_2 + \dots + c_nu_n = 0$ for non-zero scalars $c_1, c_2, \dots c_n$ then $S = \{u_1, u_2, \dots, u_n\}$ are linearly dependent.

If $c_1u_1 + \dots + c_nu_n = 0$ only if $c_1 = 0, c_2 = 0, \dots c_n = 0$ then S is linearly independent.

→ Basis : linearly independent spanning sets.

Every basis is a spanning set. Converse need not be true

Row space (A), sol space (A), Null space (A)

Row space (A) ↓ rows of A ↓ when subset not req ↓ Extending a subset to basis of W

sol space (A) ↓ columns of A ↓ when need linearly independent subset of S

Null space (A): Sol^n space of $AX = 0$

→ Finding $AX=0$ when spanning set of sol^n space is given

→ Coordinate vector of V relative to a basis.

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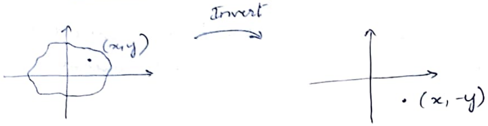
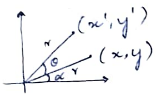
UNIT : 2 : LINEAR TRANSFORMATION

i)



$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

ii)



$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cancel{r\cos(\theta+\alpha)} & \cancel{0} \\ \cancel{0} & \cancel{r\sin(\theta+\alpha)} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$x' = r\cos(\theta+\alpha) = r\{\cos\alpha\cos\theta \mp \sin\alpha\sin\theta\} = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$y' = r\sin(\theta+\alpha) = r\{\sin\alpha\cos\theta + \cos\alpha\sin\theta\} = r\sin\alpha\cos\theta + r\cos\alpha.\sin\theta.$

$r\cos(\alpha) x \qquad r\sin(\alpha) = y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta & -y\sin\theta \\ y\cos\theta & + x\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Linear transformatⁿ = linear mapping.

Consider U and V, the vector spaces over the same field F. A mapping $T : U \to V$ is said to be linear if

i) Vector addition is preserved $(T(u+v) = T(u) + T(v))$

ii) $T(k.u) = k.T(u)$ (scalar multiplicatⁿ is presered).

Mapping : A rule that assigns to every element in U a unique element in V.

Q) Verify if $T : M_{2\times2} \to M_{2\times2}$, $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & c-d \end{bmatrix}$ is a linear transformation.

→ [Note : $T(u+v) = T(u) + T(v)$]

$T(k.u) = k.T(u)$ → are defined on V.

are defined on U

Let $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $v = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ => $u+v = \begin{bmatrix} a+x & b+y \\ c+z & d+t \end{bmatrix}$

$$T(u+v) = T.\begin{bmatrix} a+x & b+y \\ c+z & d+t \end{bmatrix} = \begin{bmatrix} a+x+b+y & 0 \\ 0 & c+z-(d+t) \end{bmatrix}$$

$$= \begin{bmatrix} a+b & 0 \\ 0 & c-d \end{bmatrix} + \begin{bmatrix} x+y & 0 \\ 0 & z-t \end{bmatrix}$$

$T(u+v) = T(u) + T(v)$ ——— ①

$$T(k.u) = T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = \begin{bmatrix} ka+kb & 0 \\ 0 & kc-kd \end{bmatrix}$$

$$= \begin{bmatrix} k(a+b) & 0 \\ 0 & k(c-d) \end{bmatrix} = k\begin{bmatrix} a+b & 0 \\ 0 & c-d \end{bmatrix}$$

$T(k.u) = kT(u)$ ——— ②

From ① and ②, we can say that the mapping is linear.

Q) Verify if $L: P_2(t) \to P_3(t)$ defined by $L[f(t)] = t f(t)$ is a linear mapping or not.

→ Consider $u, v \in P_2(t)$

$\Rightarrow u = a_0t^2 + a_1t + a_2$, $a_0, a_1, a_2 \in R$ | $P_n(t)$ = vector space of polynomials of degree $\leq n$

III^{ly}

$v = b_0t^2 + b_1t + b_2$

$$u + v = (a_0 + b_0)t^2 + (a_1 + b_1)t + (a_2 + b_2)$$

$$L(u+v) = t[(a_0+b_0)t^2] + t[(a_1+b_1)t] + t[a_2+b_2]$$

$$= (a_0+b_0)t^3 + (a_1+b_1)t^2 + (a_2+b_2)t$$

$$= a_0t^3 + b_0t^3 + a_1t^2 + b_1t^2 + a_2t + b_2t$$

$$= (a_0t^3 + a_1t^2 + a_2t) + (b_0t^3 + b_1t^2 + b_2t)$$

$$= t(a_0t^2 + a_1t + a_2) + t(b_0t^2 + b_1t + b_2)$$

$$L(u+v) = L(u) + L(v) \quad \text{——— ①}$$

$$k.u = k(a_0t^2 + a_1t + a_2)$$

$$= ka_0t^2 + ka_1t + ka_2$$

$$L(k.u) = t.ka_0t^2 + tka_1t + tka_2$$

$$= ka_0t^3 + ka_1t^2 + ka_2t$$

$$= k\{t(a_0t^2 + a_1t + a_2)\} = k.L(u)$$

$$L(k.u) = k.L(u) \quad \text{——— ②}$$

From ① and ② we conclude that $L: P_2(t) \to P_3(t)$ is a linear transformation

---

Q) Let $T: V_1(R) \to V_3(R^3)$ given by $T(x) = (x, x^2, x^3)$. Is T a linear transformation?

$V_1(R) = R$

$V_3(R) = R^3$

→ (~~Consider $u, v \in V_1(R)$~~)

(~~Let u~~)

Let $x, y \in R$

$\Rightarrow T(x+y) = (x+y, (x+y)^2, (x+y)^3)$

$T(x+y) = (x+y,\ x^2 + 2xy + y^2,\ x^3 + 3xy(x+y) + y^3)$

$T(x) + T(y) = (x, x^2, x^3) + (y, y^2, y^3)$

$= (x+y, x^2+y^2, x^3+y^3)$

$\Rightarrow T(x+y) \neq T(x) + T(y)$

$\therefore T: V_1(R) \to V_3(R)$ is not a linear transformation

---

<u>Theorem</u>: Let $T: U \to V$ be a linear transformation then $T(0_U) = 0_V$

Example: $T: R^3 \to P_2(t)$ $\quad T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = at^2 + bt + c$

$T: R^2 \to M$ $\quad T(x,y) = \begin{bmatrix} x+y & 0 \\ 0 & x-y \end{bmatrix}$ $\quad T(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

In both cases, above, zero vectors of R^3 and R^2 map to zero vectors of $P_2(t)$ & M respectively. [i.e zero vector of that vector space maps to the zero vector of mapped vector space]

Range of Linear transformation:

Consider a linear transformatⁿ $T: U \to V$.

The subset of V which contains the images of every vector in U is called the range of a linear transformation and is denoted by range (T).

$$\text{range}(T) = \{T(u), \forall u \in U\}$$

Note: range(T) is a subspace of V.

Q) Is $u = (1,-1,0)$ in the image space of $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L(x,y,z) = (x+z, y+z, x+2y+2z)$? If yes find its pre-image.

→ $u \in$ Image sp (T) = Range (T), only if there is a pre-image $v = (x,y,z)$ such that $T(v) = u$

Suppose $T(v) = T(x,y,z) = u$

$\Rightarrow (x+z, y+z, x+2y+2z) = (1,-1,0)$

$$\Rightarrow \left.\begin{aligned} x+z &= 1 \\ y+z &= -1 \\ x+2y+2z &= 0 \end{aligned}\right\} \Rightarrow AX = B \text{ where}$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$, $[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & -1 \end{array}\right]$,

$R_3 \to R_3 - 2R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$\rho(A) = 3 = \rho(A:B) \Rightarrow$ The system $AX = B$ is consistent

$\Rightarrow u \in$ range (L)

$x + z = +1$

$y + z = -1$

$-z = 1 \Rightarrow z = -1 \Rightarrow y = 0 \Rightarrow$, $x = 2$

$\Rightarrow L(2,0,-1) = (1,-1,0)$ $\left[\because (2,0,-1) \text{ is the pre-image of } (1,-1,0)\right]$

Q) Find the condition for which (a,b,c) belongs to range(T) if $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T(x,y,z) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

→ $(a,b,c) \in$ Image sp(T) only if (there is a ~~pre-image~~) ~~$\ni T(v) = u$~~)

$\forall$

$$L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ for some } \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow AX = B$$

$$[A:B] = \left[\begin{array}{ccc|c} -1 & 2 & 0 & a \\ 1 & 1 & 1 & b \\ 2 & -1 & 1 & c \end{array}\right]$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + 2R_1$

$$[A:B] \sim \left[\begin{array}{ccc|c} -1 & 2 & 0 & a \\ 0 & 3 & 1 & b+a \\ 0 & 3 & 1 & c+2a \end{array}\right]$$

$R_3 \to R_3 - R_2 \Rightarrow [A:B] \sim \left[\begin{array}{ccc|c} -1 & 2 & 0 & a \\ 0 & 3 & 1 & b+a \\ 0 & 0 & 0 & c+a-b \end{array}\right]$

Since $(a,b,c) \in$ range (T), $AX = B$ should be consistent.

Now $\rho(A) = 2 = \rho(A:B)$ only if $c+a-b = 0$

Range $(T) = \{(a,b,c) \,/\, a-b+c=0\} \subset \mathbb{R}^3$ proper sub-space

plane

30/6/22

5) $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(u) = Au$

$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ for any $u \in \mathbb{R}^2$. let $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ $\quad c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

i) Find the image of u in $\mathbb{R}^3$

ii) Does $b \in$ Range (T)? Is there more than one x in $\mathbb{R}^2$ such that $T(x) = b$?

iii) Does $c \in$ Range (T)?

→ i) $T\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad (\because T(u) = Au)$

$T\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

ii) Consider $x \in \mathbb{R}^2$ [ie $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$] such that

$T(x) = b$

$\Rightarrow Ax = b \Rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$\Rightarrow [A:B] = \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right]$

$\left.\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}\right\} \Rightarrow [A:B] = \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right]$

$R_3 \to R_3 - \frac{4}{14} R_2 \Rightarrow [A:B] \sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 - 3x_2 = 3$

$14x_2 = -7 \Rightarrow x_2 = \frac{-7}{14} = -0.5$

$x_1 - 3(-0.5) = 3 \Rightarrow x_1 = 1.5$

$T\begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \Rightarrow (3, 2, -5) \in$ Range (T)

Since $T(x) = b$ has a unique solⁿ there is only one vector x for which $T(x) = b$

$Ax = C \Rightarrow [A : C] = \begin{bmatrix} 1 & -3 & : & 3 \\ 3 & 5 & : & 2 \\ -1 & 7 & : & 5 \end{bmatrix}$

$\left.\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}\right\} \Rightarrow (A : B) \sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 4 & : & 8 \end{bmatrix}$

$R_3 \to R_3 - \frac{4}{14} R_2 \Rightarrow [A : B] \sim \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 14 & : & -7 \\ 0 & 0 & : & 10 \end{bmatrix}$

$Ax = C$ is inconsistent as $\rho(A) \neq \rho(A : B)$

$[\because \rho(A) = 2 \quad \rho(A : B) = 3]$

$\Rightarrow$ ~~g~~ C does not have a preimage in $\mathbb{R}^2$

and hence $C \notin$ Range (T).

IE!

<u>Theorem</u> : Let $T : U \to V$ be a linear transformation

Let $B = \{u_1, u_2, u_3, \dots u_n\}$ be a basis of U

and $S = \{v_1, v_2, v_3 \dots v_n\}$ be any subset of V

then there exists a unique linear transformation

such that $T(u_1) = v_1, \ T(u_2) = v_2 \dots T(u_n) = v_n$

<u>Note</u>: i) We need a basis of domain and their images to differine a linear transformation.

ii) $u \in U \Rightarrow u = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$

$\Rightarrow T(u) = T(c_1 u_1) + T(c_2 u_2) + \dots + T(c_n u_n)$

$= c_1 T(u_1) + c_2 T(u_2) + \dots + c_n (T(u_n))$

Q) Find the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $\underset{\hookrightarrow \text{Domain}}{(1,2)}$ to $(3,0)$ and $\underset{\hookrightarrow \text{Domain}}{(2,1)}$ to $(1,2)$

$\rightarrow$ Reqd : i) Basis of Domain

ii) Image of this basis

$S = \{(1,2), (2,1)\}$ is a basis of the $\mathbb{R}^2$

Now $(1,2) = 1(1,0) + 2(0,1) = 1e_1 + 2e_2$ [Note: $\mathbb{R} = \{e_1, e_2\}$ is called the standard basis]

$[\because 1i + 2j]$

$\Rightarrow T(1,2) = T(e_1 + 2e_2)$

$(3,0) = T(e_1) + 2T(e_2) \longrightarrow ①$

$(2,1) = 2e_1 + e_2$

$T(2,1) = 2T(e_1) + T(e_2)$

$(1,2) = 2T(e_1) + T(e_2) \longrightarrow ②$

Solve ① and ②

$T(e_1) + 2T(e_2) = (3,0)$

$2 \times ② \Rightarrow 4T(e_1) + 2T(e_2) = (2,4)$

$-3T(e_1) = (1,-4) \Rightarrow T(e_1) = \frac{-1}{3}(1,4) = \left(-\frac{1}{3}, \frac{4}{3}\right)$

$② \Rightarrow T(e_2) = (1,2) - 2T(e_1) = (1,2) - 2\left(\frac{-1}{3}, \frac{4}{3}\right)$

$= \left(\frac{5}{3}, -\frac{2}{3}\right)$

$(x,y) \in \mathbb{R}^2 \to (x,y) = xe_1 + ye_2$

$T(x,y) = T(xe_1 + ye_2) = xT(e_1) + yT(e_2)$

$T(x,y) = x\left(\frac{-1}{3}, \frac{4}{3}\right) + y\left(\frac{5}{3}, \frac{-2}{3}\right)$

$= \left(\frac{-x+5y}{3}, \frac{4x-2y}{3}\right)$

Q $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(2,-4) = (1,1)$ and $T(1,2) = (1,3)$

→ Reqd: i) Basis of Domain
ii) Image of basis

$S = \{(2,-4)\ (1,2)\}$ is a basis of $\mathbb{R}_2$

Now $(2,-4) = 2e_1 - 4e_2$

$T(2,-4) = 2T(e_1) - 4T(e_2)$

$(1,1) = 2T(e_1) - 4T(e_2) \longrightarrow ①$

$(1,2) = e_1 + 2e_2$

$T(1,2) = T(e_1) + 2T(e_2)$

$(1,3) = T(e_1) + 2T(e_2) \longrightarrow ②$

Solve ① and ②

$2T(e_1) - 4T(e_2) = (1,1)$

$2T(e_1) + 4T(e_2) = (2,6)$

$4T(e_1) = (3,7)$

$\Rightarrow T(e_1) = \left(\frac{3}{4}, \frac{7}{4}\right), T(e_2) = \left(\frac{1}{8}, \frac{5}{8}\right)$

$(x,y) = xe_1 + ye_2 \Rightarrow T(x,y) = xT(e_1) + yT(e_2)$

$T(x,y) = x\left(\frac{3}{4}, \frac{7}{4}\right) + y\left(\frac{1}{8}, \frac{5}{8}\right)$

$= \left(\frac{6x+y}{8}, \frac{14x+5y}{8}\right)$

Q Find the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(-1,0) = (-1,0,2)$ and $T(2,1) = (1,2,1)$ 1/7/22

→ $S = \{(-1,0), (2,1)\}$ is a basis of $\mathbb{R}^2$ as S is linearly independent set of two vectors

$(-1,0) = (-1,0) = -e_1$

$T(-1,0) = T(-e_1) = -T(e_1)$

$-T(e_1) = (-1,0,2) \Rightarrow T(e_1) = -(-1,0,2)$

$T(e_1) = (1,0,-2) \longrightarrow ①$

$(2,1) = 2e_1 + e_2$

$T(2,1) = 2T(e_1) + T(e_2)$

$(1,2,1) = 2(1,0,-2) + T(e_2)$

$T(e_2) = (1,2,1) - 2(1,0,-2) = (-1,2,5)$

$(x,y) = xe_1 + ye_2$

$T(x,y) = xT(e_1) + yT(e_2)$

$= x(1,0,-2) + y(-1,2,5) = (x-y, 2y, -2x+5y)$

$T(x,y) = (x-y, 2y, -2x+5y)$

Q Find $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(1,1,1) = (1,1,1)$ and $T(1,2,3) = (-1,-2,-3)$

→ $S = \{(1,1,1), (1,2,3)\}$ is linearly independent, but not a basis of $\mathbb{R}^3$ [∵ $\mathbb{R}^3$ requires 3 independent vectors ~~?~~ in basis]

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$B = \{(1,1,1), (1,2,3), (0,0,1)\}$ is a basis of $\mathbb{R}^3$

The linear transformation depends on the image of $(0,0,1)$ $\Rightarrow$

Suppose $T(0,0,1) = (0,0,0)$

$(1,1,1) = 1e_1 + 1e_2 + 1e_3$

$T(1,1,1) = T(e_1) + T(e_2) + T(e_3)$

$(1,1,1) = T(e_1) + T(e_2) + T(e_3)$ —— ①

$(1,2,3) = 1e_1 + 2e_2 + 3e_3$

$T(1,2,3) = T(e_1) + 2T(e_2) + 3T(e_3)$

$(-1,-2,-3) = T(e_1) + 2T(e_2) + 3T(e_3)$ —— ②

Note: $(0,0,1) = e_3$ & $T(e_3) = (0,0,0)$ [~~we could have chosen it to be non-zero as well~~] —— ③

③ in ① $\Rightarrow T(e_1) + T(e_2) = (1,1,1)$

$T(e_1) + 2T(e_2) = (-1,-2,-3)$

$= T(e_2) = (2,3,4)$

$T(e_2) = (-2,-3,-4)$

$T(e_1) + (-2,-3,-4) = (1,1,1)$

$T(e_1) = (1,1,1) - (-2,-3,-4) = (3,4,5)$

$(x,y,z) = xe_1 + ye_2 + ze_3$

$T(x,y,z) = x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$

$= x(3,4,5) + y(-2,-3,-4) + z(0,0,0)$

$= (3x - 2y,\ 4x - 3y,\ 5x - 4y)$

③ Find the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^4$ whose range space is spanned by $(1,2,0,-4)$ $(2,0,-1,-3)$

$\rightarrow$ Consider any basis of $\mathbb{R}^3$ with $u_1 = (1,2,0,-4)$ and $u_2 = (2,0,-1,-3)$ as the images of any 2 vectors in the basis.

For convenience consider $S = \{e_1, e_2, e_3\}$ the standard basis of $\mathbb{R}^3$.

Let $T(e_1) = u_1 = (1,2,0,-4)$

$T(e_2) = u_2 = (2,0,-1,-3)$

Now $T(e_3) = (0,0,0,0)$ [This cannot be independent of u_1 & u_2 as $\{u_1, u_2\}$ is a spanning set of Range(T) $\therefore$ choose either zero vector or multiple of u_1, u_2]

$(x,y,z) = xe_1 + ye_2 + ze_3$

$T(x,y,z) = x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$

$= x(1,2,0,-4) + y(2,0,-1,-3) + z(0,0,0,0)$

$= (x + 2y,\ 2x,\ -y,\ -4x - 3y)$

Matrix of a linear transformation (mapping):

Consider a linear transformation $T: U \to V$ where $S = \{u_1, u_2, u_3, \ldots, u_m\}$ and $B = \{v_1, v_2, \ldots, v_n\}$ are the bases of U and V respectively.

$T(u_1) = c_{11}v_1 + c_{12}v_2 + c_{13}v_3 + \cdots + c_{1n}v_n$ (as $T(u_1) \in V$) —— ①

$T(u_2) = c_{21}v_1 + c_{22}v_2 + \cdots + c_{2n}v_n$ —— ②

$\vdots$

$T(u_m) = c_{m1}v_1 + c_{m2}v_2 + \cdots + c_{mn}v_n$ → (m)

(1) ⇒ $c_1 = [c_{11}, c_{12}, c_{13} \cdots c_{1n}]$

(2) ⇒ $c_2 = [c_{21}, c_{22}, c_{23} \cdots c_{2n}]$

(3) ⇒ $c_3 = [c_{31}, c_{32}, c_{33} \cdots c_{3n}]$

⋮

(m) ⇒ $c_m = [c_{m1}, c_{m2}, c_{m3} \cdots c_{mn}]$

} coordinate vector of $T(u_1) + T(u_2) + \cdots T(u_m)$

Matrix of a linear transformation is given by $[T]_{SB} = [c_1 \ c_2 \ c_3 \ \cdots c_m]_{n\times m}$ with c_i's written along columns.

If $A = [T]_{SB}^T$

$$T\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} c_{11}v_1 + c_{12}v_2 + \cdots c_{1n}v_n \\ c_{21}v_1 + c_{22}v_2 + \cdots c_{2n}v_n \\ \vdots \\ c_{m1}v_1 + c_{m2}v_2 + \cdots c_{mn}v_n \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$T\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = A\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

ie To write matrix of linear transformation,

i) Find image of basis vectors of domain

ii) Find coordinate vectors of images relative to basis of codomain

iii) Enter the coordinate vectors along columns to get matrix of linear transformⁿ.

11/2/22

(1) $T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(a,b) = (a+b, a-b, 2b)$.

wrt ~~$S = \{(1,0),(1,1)\}$ and the~~ ~~st~~ standard basis of $\mathbb{R}^3$.

→ Req, i) bases s and s' of Domain and co-domain

ii) Image of basis of s

iii) coordinate vectors of the image relative to s'

$s = \{e_1 = (1,0), e_2 = (0,1)\}$ is the basis of $\mathbb{R}^2$

$s' = \{e_1' = (1,0,0), e_2' = (0,1,0), e_3' = (0,0,1)\}$ is basis of $\mathbb{R}^3$.

$T(e_1) = T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} = e_1' + e_2' + 0e_3'$

$[T(e_1)] = [1, 1, 0] = c_1$

$T(e_2) = T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix} = e_1' - e_2' + 2e_3'$

$[T(e_2)] = [1, -1, 2] = c_2$

Matrix of linear transformation is

$[T]_{ss'} = [c_1 \ c_2]$ where c_1 and c_2 are along column

$$[T]_{ss'} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}$$

Note: $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ y\cos\theta + x\sin\theta \end{bmatrix}$ is geometrically rotation in space

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$T_1(x) = Au$

Reflection ~~of~~ about y-axis

$T_2(x) = Ax \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

~~stretching~~ Scaling

$T_3(x) = Ax \qquad A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \quad$ i.e $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$

$$T_1(T_2(T_3(x))) = A_1\{A_2(A_3 x)\} = A_1 A_2 A_3 X$$

* A matrix $A_{m\times n}$ can be viewed as a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined as

$T(x) = Ax \quad x \in \mathbb{R}^n$

$T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a-b \\ 2b \end{bmatrix} = \begin{bmatrix} 1 & +1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \qquad T(X) = AX \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}$

Now $[T]_{SS'} = A$ if S and S' are std. bases

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③ Find the matrix of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+3y \\ 4x-5y \end{bmatrix}$

w.r.t $S = \{(1,2), (2,5)\}$ (same for domain and codomain).

$\rightarrow \quad T(u_1) = T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix} = c_{11}u_1 + c_{12}u_2 = c_{11}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_{12}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

~~$T(u_2) = T\begin{bmatrix} 2 \\ 5 \end{bmatrix} =$~~

$AX = B \Rightarrow [A:B] = \left[\begin{array}{cc:c} 1 & 2 & 8 \\ 2 & 5 & -6 \end{array}\right]$ (can use calculator to do this part)

$R_2 \to R_2 - 2R_1$

$[A:B] \sim \left[\begin{array}{cc:c} 1 & 2 & 8 \\ 0 & 1 & -22 \end{array}\right] \Rightarrow c_{11} + 2c_{12} = 8$

$c_{12} = -22$

$\Rightarrow c_{11} = 8 - 2(-22)$
$= 8 + 44$
$= 52$

$[T(u_1)]_S = [52, -22] = c_1$

$T(u_2) = T\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 19 \\ -17 \end{bmatrix}$

$T(u_2) = c_{21}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_{22}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$AX = B \Rightarrow [A:B] \sim \left[\begin{array}{cc:c} 1 & 2 & 19 \\ 0 & 1 & -55 \end{array}\right]$

$c_{22} = -55$
$c_{21} = 19 - 2(-55)$
$= 19 + 110$
$= 129$

$\Rightarrow [T]_{SS} = \begin{bmatrix} 52 & 129 \\ -22 & -55 \end{bmatrix}$

---

⑤ $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 0 \\ 1 & 4 & -2 \end{bmatrix}$ defines a linear transformation on $\mathbb{R}^3$ and $S = \{(1,1,1)\ (0,1,1), (1,2,3)\}$ is a basis of $\mathbb{R}^3$. Find a matrix of linear transformation relative to the basis

Note: A = matrix of LT wrt std basis

col sp (A) = range (T)

13/7/22 To find coordinate vector of image of basis vector,

$$T\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1 & -2 & 1\\3 & -1 & 0\\1 & 4 & -2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\2\\3\end{bmatrix}$$

$T(v_1) = c_{11}v_1 + c_{12}v_2 + c_{13}v_3$

$$\begin{bmatrix}0\\2\\3\end{bmatrix} = c_{11}\begin{bmatrix}1\\1\\1\end{bmatrix} + c_{12}\begin{bmatrix}0\\1\\1\end{bmatrix} + c_{13}\begin{bmatrix}1\\2\\3\end{bmatrix} \Rightarrow AX = B$$

$$A = \left[\begin{array}{ccc:c}1 & 0 & 1 & 0\\1 & 1 & 2 & 2\\1 & 1 & 3 & 3\end{array}\right]$$

$c_{11} = -1$

$c_{12} = 1 \Rightarrow \begin{bmatrix}-1\\1\\1\end{bmatrix} = c_1$

$c_{13} = 1$

$$T\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1 & -2 & 1\\3 & -1 & 0\\1 & 4 & -2\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}-1\\\cancel{0}-1\\2\end{bmatrix}$$

$c_{21}v_1 + c_{22}v_2 + c_{23}v_3 = T(v_2)$

$$\Rightarrow c_{21}\begin{bmatrix}1\\1\\1\end{bmatrix} + c_{22}\begin{bmatrix}0\\1\\1\end{bmatrix} + c_{23}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$\Rightarrow c_{21} = -4,\quad c_{22} = -3,\quad c_{23} = 3 \Rightarrow \begin{bmatrix}-4\\-3\\3\end{bmatrix} = c_2$$

$$T_3\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1 & -2 & 1\\3 & -1 & 0\\1 & 4 & 2\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\1\\\cancel{13}3\end{bmatrix}$$

$c_{31}v_1 + c_{32}v_2 + c_{33}v_3 = T(v_3)$

$$c_{31}\begin{bmatrix}1\\1\\1\end{bmatrix} + c_{32}\begin{bmatrix}0\\1\\1\end{bmatrix} + c_{33}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\1\\\cancel{8}3\end{bmatrix}$$

$c_{31} = \cancel{-4} -2,\ c_{32} = -1\cancel{3} \qquad c_{33} = \cancel{17}\ 2$

$$c_3 = \begin{bmatrix}\cancel{-14}\\\cancel{-13}\\\cancel{44}\end{bmatrix} \qquad c_3 = \begin{bmatrix}-2\\-1\\2\end{bmatrix}$$

$$\Rightarrow [T]_S = \begin{bmatrix}-1 & -4 & -2\\1 & -3 & -1\\1 & 3 & 2\end{bmatrix}$$

---

(3) $T: P_1(t) \to P_2(t)$ is defined by $T(f(t)) = t f(t)$. Find the ~~nature~~ matrix of the linear transformation relative to the bases

$A = \{1, t\}$ $B = \{1+t, 1-t, t^2\}$

→ Step 1: Find image of vectors in A

Step 2: Find the coordinate vectors of images relative to B.

$A = \{u_1 = 1,\ u_2 = t\}$

$B = \{v_1 = 1+t,\ v_2 = 1-t,\ v_3 = t^2\}$

$T(u_1) = T(1) = t$

$T(1) = c_{11}v_1 + c_{12}v_2 + c_{13}v_3$

$c_{11}(1+t) + c_{12}(1-t) + c_{13}(t^2) = t$

$c_{11} + c_{12} + \cancel{0}(c_{11} - c_{12})t + c_{13}t^2 = t$

$c_{11} + c_{12} = 0 \qquad c_{13} = 0 \Rightarrow c_1 = \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}$

$c_{11} - c_{12} = 1$

$T(u_2) = T(t) = t^2$

$T(u_2) = c_{21}v_1 + c_{22}v_2 + c_{23}v_3$

$t^2 = c_{21}(1+t) + c_{22}(1-t) + c_{23}t^2$

$\Rightarrow c_{21} = 0 \quad c_{22} = 0 \quad c_{23} = 1 \Rightarrow c_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[T]_{AB} = [c_1 \; c_2] = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 0 \\ 0 & 1 \end{bmatrix}$

---

5) Let V be vector space of function $f: \mathbb{R} \to \mathbb{R}$ with the basis $S = \{\sin(2t), \cos(2t), e^{3t}\}$.

$D: V \to V$ is the derivative map ie $D\{f(t)\} = \frac{df}{dt}$.

Find the matrix of the linear transformation.

$\to$

$S = \{u_1 = \sin(2t), \; u_2 = \cos(2t), \; u_3 = e^{3t}\}$

$D(u_1) = \frac{d}{dt}\{u_1\} = 2\cos(2t) = c_{11}u_1 + c_{12}u_2 + c_{13}u_3$

$= 0\sin(2t) + 2\cos(2t) + 0e^{3t}$

$c_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$D(u_2) = \frac{d}{dt}(u_2) = -2\sin(2t)$

$-2\sin(2t) = c_{21}u_1 + c_{22}u_2 + c_{23}u_3$

$= (-2)\sin(2t) + 0\cos(2t) + 0e^{3t}$

$c_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$

$D(u_3) = \frac{d}{dt}(u_3) = 3e^{3t} = c_{31}u_1 + c_{32}u_2 + c_{33}u_3$

$3e^{3t} = 0\sin(2t) + 0\cos(2t) + 3e^{3t}$

$\cancel{3e^{3t}} \quad c_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

$[D]_{SS} = [c_1 \; c_2 \; c_3] = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Note: If $T: U \to V$ is a L.T with S and S' as bases of U & V respectively, then

$[T]_{SS'} \cdot [v]_S = [T(v)]_{S'}$

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Q) The linear transformation $F: \mathbb{R}^3 \to \mathbb{R}^2$ is given by $F(x,y,z) = (3x + 2y - 4z,\ x - 5y + 3z)$

$S = \{(1,1,1), (1,1,0), (1,0,0)\}$ and $S' = \{(1,3), (2,5)\}$ are bases of $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

i) Find $[F]_{ss'}$

ii) Verify $[F]_{ss'} [v]_s = [F(v)]_{s'}$ for $v = (1,-3,2) \in \mathbb{R}^3$

→ $S = \{v_1 = (1,1,1),\ v_2 = (1,1,0),\ v_3 = (1,0,0)\}$

$S' = \{w_1 = (1,3),\ w_2 = (2,5)\}$

i) Step 1: Find images of vectors in S

Step 2: Find coordinate vectors of these images w.r.t S'

i) $F(v_1) = F(1,1,1) = (1,-1) = c_{11} w_1 + c_{12} w_2$

$= c_{11}(1,3) + c_{12}(2,5)$

$\Rightarrow 3(c_{11} + 2c_{12} = 1) \quad \cdot \quad 3c_{11} + 6c_{12} = 3$

$3c_{11} + 5c_{12} = -1 \quad = \quad 3c_{11} + 5c_{12} = -1$

$c_{12} = 4$

$c_{11} = 1 - 8$

$= -7$

$c_1 = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$

$F(v_2) = F(1,1,0) = (5,-4)$

$= c_{21} w_1 + c_{22} w_2$

$= c_{21}(1,3) + c_{22}(2,5)$

$\Rightarrow 3(c_{21} + 2c_{22} = 5) = 3c_{21} + 6c_{22} = 15$

$3c_{21} + 5c_{22} = -4$

$c_{22} = 19$

$c_{21} = -33$

$c_2 = \begin{bmatrix} -33 \\ 19 \end{bmatrix}$

$F(v_3) = F(1,0,0) = (3,1)$

$= c_{31} w_1 + c_{32} w_2$

$= c_{31}(1,3) + c_{32}(2,5)$

$\Rightarrow 3(c_{31} + 2c_{32} = 3) = 3c_{31} + 6c_{32} = 9$

$3c_{31} + 5c_{32} = 1$

$c_{32} = 8$

$c_{31} = 3 - 16$

$= -13$

$c_3 = \begin{bmatrix} -13 \\ 8 \end{bmatrix}$

Matrix of linear transformation $= [c_1\ c_2\ c_3]$

$= \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

ii) ~~F(v)~~

Step 1: Find coordinate vector of v

Step 2: Find coordinate vector of F(v)

Step 3: Check condition

$v = (1,-3,2) = av_1 + bv_2 + cv_3$

$(1,-3,2) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$a = 2,\ b = -5,\ c = 4$

$[v] = [2, -5, 4] \cong \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$

$\cancel{[F(v)]}\ [F]_{SS'} \cdot [v]_S = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 99 \\ -55 \end{bmatrix}$ —— (1)

$F(v) = \cancel{\begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}} = F(1,-3,2)$

$= (-11, 22) = a w_1 + b w_2$

$a\begin{bmatrix} 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -11 \\ 22 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -11 \\ 22 \end{bmatrix}$

$a = 99, \ b = -55$ —— (2)

From (1) and (2),

$[F]_{SS'} [v]_S = [F(v)]_{S'}$

---

(9) $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation whose matrix of the L.T is $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ w.r.t the natural basis. Find $L\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

→ Natural basis = std. basis

$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$

then $L : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $L(x) = AX$.

Note: $v = [v]_S \in \mathbb{R}^n$ if $S \to$ std basis

if v = polynomial then $[v]_S$ = n-tuple of coefficients of v when S is std basis

(9) $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ is the matrix of L.T w.r.t. $S = \{v_1, v_2, v_3\}$ and $T = \{w_1, w_2\}$ where

$v_1 = (-1,1,0), \ v_2 = (0,1,1), \ v_3 = (1,0,0)$

$w_1 = (1,2), \ w_2 = (1,-1)$

Compute $[L(v_1)]_T, [L(v_2)]_T, [L(v_3)]_T, L(v_1), L(v_2), L(v_3)$.

→ Reqd. (i) Defn of matrix of LT

(ii) " " coordinate vectors

By definition, each column of A is the coordinate vector image of vector in S ie

$A = \left[[L(v_1)]_T \ \ [L(v_2)] \ \ [L(v_3)]_T \right]$

$\Rightarrow [L(v_1)]_T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ [L(v_2)]_T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ [L(v_3)]_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$[v]_B = [c_1, c_2 \dots c_n]$

$B = \{v_1, v_2 \dots v_n\}$ then

$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

$[L(v_i)]_T = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow L(v_i) = a w_1 + b w_2$

$L(v_1) = (1) w_1 + (-1) w_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$L(v_2) = 2 w_1 + w_2 = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$L(v_3) = 1 \cancel{0} w_1 + 0 w_2 = w_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(a) Matrix of the linear transformat^n $L: \mathbb{R}^2 \to \mathbb{R}^2$ is

$A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$ w.r.t $S = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Compute $[L(v_1)]_S$, $[L(v_2)]_S$, $L(v_1)$, $L(v_2)$, $L\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\to$ $[L(v_1)]_S = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad [L(v_2)]_S = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$L(v_1) = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \cancel{\begin{bmatrix} -1 \\ 7 \end{bmatrix}} = \cancel{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$L(v_2) = -3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -10 \end{bmatrix}$

(iii) $L\begin{bmatrix} -2 \\ 3 \end{bmatrix}$,

$[v]_S = ?$ $\quad av_1 + bv_2 = v$

$\Rightarrow a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 1 & +1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \Rightarrow a = 1/3, \quad b = -7/3$

$2x + 2y = -4$
$2x - y = 3$
$3y = -7$
$y = -\frac{7}{3}$
$x = -2 + \frac{7}{3}$
$= \frac{1}{3}$

$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/3 \\ -7/3 \end{bmatrix}$

$[L(v)] = [L]_S [v]_S$

$= \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1/3 \\ -7/3 \end{bmatrix} = \begin{bmatrix} 23/3 \\ -29/3 \end{bmatrix} = [L(v)]_S$

$L(v) = \frac{23}{3} v_1 - \frac{29}{3} v_2$

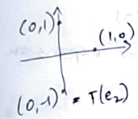
$= \frac{23}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{29}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -6/3 \\ \cancel{17/3}\ 25 \end{bmatrix}$

$23 \times 2 = 46$
$+29$
25

Geometric linear transformation on $\mathbb{R}^2$

Reflection about x-axis



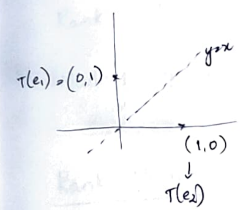
$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Reflection about y-axis

$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

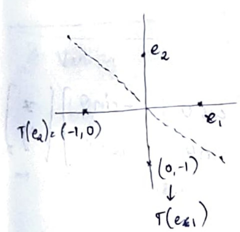
$\downarrow T(e_1) \quad \downarrow T(e_2)$

Reflection about $x=y$ line



$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Reflection about $y = -x$



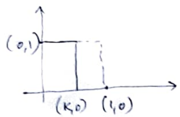
$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Contraction and Expansion

→ Horizontal $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}k & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$0<k<1$ for contraction

$k>1$ For expansion



$0<k<1$ (contraction)

→ Vertical $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & k\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

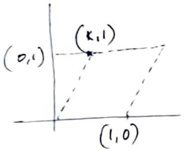
$0<k<1$ for contraction

$k>1$ for expansion

Shear

→ Horizontal :

$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & k\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

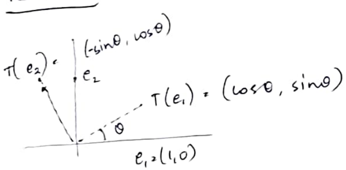


$k>0$

→ Vertical : $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & 0\\k & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Rotation :

$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$



Kernel of a linear transformation :

If $T: U \to V$ is a linear transformation,

$$\ker(T) = \{ u \in U \;/\; T(u) = 0 \}$$

* $\ker(T)$ is a subspace of U.
* $0_u \in \ker(T)$ as $T(0_u) = 0_v$
* $u, v \in \ker(T) \Rightarrow T(u+v) = T(u) + T(v) = 0_u + 0_v = 0_v$
  $\Rightarrow u+v \in \ker(T)$

$T(k.u) = k.T(u) = k(0) = 0 \Rightarrow k.u \in \ker(T)$

Rank of L.T , $\text{rank}(T) = \dim\{\text{Range}(T)\}$

Nullity of a linear transformation, $\text{nullity}(T) = \dim\{\ker(T)\}$

Rank - Nullity theorem :

If $T: U \to V$

then $\text{rank}(T) + \text{nullity}(T) = \dim(U)$

Q) Verify rank-nullity theorem for $T: \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(x,y,z,t) = (x-y+z+t,\; x+2z+t,\; x+y+3z-3t)$

→ $T(x,y,z,t) = \begin{bmatrix}1 & -1 & 1 & 1\\1 & 0 & 2 & 1\\1 & 1 & 3 & -3\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\end{bmatrix}$

$T(x) = AX$

$A \to$ matrix of linear transformatⁿ w.r.t std. basis.

Since columns are images of std basis vectors.

$\therefore$ col. space (A) = range sp (A) [when s' is std basis]

$\Rightarrow$ apply col space algo on A

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$$

$R_3 \to R_3 - R_1$, $R_2 \to R_2 - R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} \boxed{1} & -1 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{-4} \end{bmatrix}$$

Basis of range space =

$\{(1,1,1), (-1,0,1), (1,1,-3)\}$

rank (T) = dim $\{$Range $(T)\}$ = 3,

Ker (T) = $\{u \in U \,/\, T(u) = 0\}$

Ker (T), $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $[A:B]$ $\left[\begin{array}{cccc:c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right]$

$-4t = 0 \Rightarrow t = 0$

$y + z = 0$

$x - y + z + t = 0$

Free variable is $z = k$

$y = -k$

$x + k + k + 0 = 0 \Rightarrow x = -2k$

$X = k(-2,-1,1,0)$

Basis of Ker (T) =

$\{(-2,-1,1,0)\}$

$\Rightarrow$ Nullity (T)

= dim $\{$Ker $(T)\}$

= 1

Rank (T) + Nullity (T)

= 3 + 1 = 4

= dim $(\mathbb{R}^4)$

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Q) Verify rank-nullity theorem for $T: \mathbb{R}^5 \to \mathbb{R}^4$ defined by

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & 5 & -7 & 0 & 4 \end{bmatrix}$$

$\to$ $T: \mathbb{R}^5 \to \mathbb{R}^4$ is given by $T(x) = AX$.

$\Rightarrow$ A = matrix of L.T w.r.t std basis

$\Rightarrow$ col sp (A) = Range (A)

$\Rightarrow$ We apply column space algorithm on A.

$R_2 \leftrightarrow R_1$ $\Rightarrow$ $A \sim \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 2 & -1 & 1 & -6 & 8 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & 5 & -7 & 0 & 4 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 7R_1$, $R_4 \to R_4 - 4R_1$ $\Rightarrow$ $A \sim \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & -6 & -18 & 24 & -24 \\ 0 & 13 & 9 & -12 & 12 \end{bmatrix}$

$R_3 \to R_3 + 2R_2$, $R_4 \to R_4 - \frac{13}{3}R_2$ $\Rightarrow$ $A \sim \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -30 & 40 & -40 \end{bmatrix}$

$R_4 \leftrightarrow R_3$

Basis of image (T) = $\{(2,1,-7,4), (-1,-2,8,5), (1,-4,+10,-7)\}$

dim (range (T)) = 3

Kernel (T) = $\{x \in \mathbb{R}^5 \mid T(x) = 0\}$

Consider $T(x) = 0$ $\Rightarrow$ $AX = 0$

$x_1 - 2x_2 - 4x_3 + 3x_4 - 2x_5 = 0$

$\Rightarrow$ $3x_2 + 9x_3 - 12x_4 + 12x_5 = 0$

$-30x_3 + 40x_4 - 40x_5 = 0$

$x_4 = a$, $x_5 = b$ $\Rightarrow$ $-3x_3 + 4a - 4b = 0$ $\Rightarrow$ $x_3 = \frac{4a - 4b}{3}$

$x_2 + 3\left(\frac{4a-4b}{3}\right) - 4a + 4b = 0 \Rightarrow x_2 = 0$

$x_1 - 2(0) - 4\left(\frac{4a-4b}{3}\right) + 3a - 2b = 0$

$x_1 = 2b - 3a + \frac{16a}{3} - \frac{16b}{3}$

$= \frac{7a}{3} - \frac{10b}{3}$

$X = \left(\frac{7a}{3} - \frac{10b}{3}, 0, \frac{4a-4b}{3}, a, b\right)$

$= \frac{a}{3}(7, 0, 4, 3, 0) + \frac{b}{3}(-10, 0, -4, 0, 3)$

~~Nullity(T) = 2~~ Basis of $\ker(T) = \{(7,0,4,3,0), (-10,0,-4,0,3)\}$

$\text{rank}(T) = \dim\{\text{Range}(T)\} = 3$

$\text{Nullity}(T) = \dim\{\ker(T)\} = 2$

$\text{rank}(T) + \text{nullity}(T) = 3 + 2 = 5 = \dim(\mathbb{R}^5)$

---

8) Let $L: P_2(t) \to P_2(t)$ be defined by

$L(at^2 + bt + c) = (a+2b)t + (b+c)$

i) Is $-4t^2 + 2t - 2$ in kernel of L?

ii) Is $t^2 + 2t + 1$ in range of L?

iii) Is L one to one?

iv) Is L onto?

v) Verify rank-nullity theorem.

→ i) Consider $L(-4t^2 + 2t - 2) = (-4+4)t + (2-2)$

$= 0t + 0 = 0$

Since $L(-4t^2+2t-2) = 0$, $-4t^2 + 2t - 2 \in \ker(L)$

ii) $L(at^2 + bt + c) = k_1 t + k_2$, $k_1 = a + 2b$, $k_2 = b + c$

$k_1 t + k_2 \neq t^2 + 2t + 1$

for any k_1 and k_2

$\therefore t^2 + 2t + 1 \notin \text{Range}(L)$

iii) $L(0) = 0$ by def^n of linear mapping

$L(-4t^2 + 2t - 2) = 0$ (from (i))

Two different vectors have same image.

$\therefore L$ cannot be one-~~to~~-one.

iv) Quadratic polynomials belong to co-domain but do not have pre-images ie they do not belong to range (L)

$\Rightarrow L$ is not onto.

[Note if codomain was $P_1(t)$, it would be onto]

v) $S = \{1, t, t^2\}$ is a basis of $P_2(t)$

$L(1) = 1$

$L(t) = 2t + 1$

$L(t^2) = t$

These vectors span range sp (L)

Basis of range space $(T) = \{1, t\}$

Consider $L(at^2 + bt + c) = 0$

$(a+2b)t + (b+c) = 0t + 0$

$\Rightarrow (a+2b)t + (b+c)$

$\Rightarrow a + 2b = 0$

$b + c = 0$

$c = k \Rightarrow b = -k,\ a = 2k$

$at^2 + bt + c = 2kt^2 - kt + k = k(2t^2 - t + 1)$

$\ker(L) = \{k(2t^2 - t + 1), k \in \mathbb{R}\}$

ie basis of $\ker(L) = \{2t^2 - t + 1\}$

Nullity $(L) = \dim(\ker(L)) = 1$

Rank $(L) = \dim(\text{range sp}(L)) = 2$

Rank (L) + nullity $(L) = 2 + 1 = 3 = \dim(P_2(t))$

18/7/22 # Singular and Non-singular

Consider a linear transformation $T: U \to V$

T is said to be singular, if the image of any non-zero vector is zero.

ie If $T(u) = 0_v$ for $u \neq 0_u$

$\ker(T) = \{v / T(u) = 0\}$

$\{0_u\} \subset \ker(T)$

If $T: U \to V$ is such that $T(u) = 0$ only if $u = 0$ then we say that T is non-singular.

If T is non-singular, then $\ker(T) = \{0_u\}$

Theorem: A linear mapping $T: U \to V$ is one-to-one iff it is non-singular.

Def^n: A bijective linear transformation is called an isomorphism.

Theorem:
Consider finite dimensional vector spaces U and V with dimension of $U = \dim(V)$. The linear transformation $T: U \to V$ is an isomorphism iff T is non-singular

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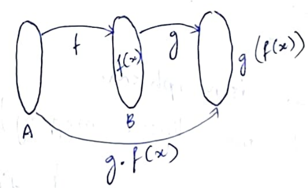
20/7/22 Composition of mappings:

$f: A \to B$ $\quad g: B \to C$

$f \cdot g(x) = f\{g(x)\}$ $\quad g(f(x)) = g \cdot f(x)$



Homomorphism: The vector space of all linear mappings from U to V is called homomorphism and is denoted as Hom (U, V)

$f: U \to V$ & $g: U \to V$, $f + g: U \to V$ is also linear map where $(f+g)(x) = f(x) + g(x)$.

$(k \cdot f)(x) = k\{f(x)\}$

With these definition of vector addition and scalar multiplication Hom (U, V) is a vector space.

Hom $(\cancel{\mathbb{F}}\, V, V) = A(V)$, algebra of linear operations

Invertible linear operators :

Consider a linear mapping $f: V \to V$

[ie $F \in A(V)$].

F is said to be invertible if there exists a linear mapping in $A(V)$, denoted by F^{-1}, λ'

$F^{-1} \circ F = I = F \circ F^{-1}$ where I is the identity map.

Q) let T be a linear transformation on $\mathbb{R}^3$ defined by $T(a,b,c) = (3a, a-b, 2a+b+c)$ $\forall$ $(a,b,c) \in \mathbb{R}^3$. Is T invertible? If yes, find the inverse. Also find T^2.

→ F^{-1} exists if F is an isomorphism.

[$\because$ Isomorphism $\Rightarrow$ bijective linear mapping]

Then: If $T: U \to V$ is non-singular and $\dim(U) = \dim(V)$, then T is an isomorphism

Given : $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\Rightarrow$ dim{ Domain} = dim{ Codomain} — ①

~~Ker(T)~~ : $T(a,b,c) = (0,0,0)$

$\Rightarrow$ $3a = 0$ $\Rightarrow a = 0$

$a - b = 0$ $\Rightarrow b = 0$

$2a+b+c = 0$ $\Rightarrow c = 0$

$\ker(T) = \{0,0,0\}$

$\therefore$ T is non-singular. — ②

From ① and ②, T is an isomorphism.

Hence T is invertible.

$$T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3a \\ a-b \\ 2a+b+c \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = AX$$

Suppose $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\Rightarrow AX = Y \Rightarrow X = A^{-1}Y$

$= T^{-1}(Y)$

$$A^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$X = A^{-1}Y$ $\Rightarrow$ $\frac{1}{3}x = a$

$\frac{1}{3}x - y = b$

$-x + y + z = c$

$$T^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x/3 \\ x/3 - y \\ -x + y + z \end{bmatrix}$$

$T^2 = T(T(x))$

~~$F: \mathbb{R}^2 \to \mathbb{R}^2$~~

$T^2 = T(T(x)) = T(Ax) = A(Ax) = A^2 x$

$$T^2\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 2 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 9a \\ 2a+b \\ 9a+c \end{bmatrix}$$

(6) $F: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $F(a,b) = (2a+b,\ 3a+2b)$

Is T invertible? If yes, find F^{-1}, also find F^2 and F^{-2}.

→

Ker (F) ;

$F(a,b) = (0,0)$

$\Rightarrow \quad 2a+b=0 \qquad \Rightarrow 4a+2b=0$

$\quad 3a+2b=0 \qquad \Rightarrow 3a+2b=0 \qquad \Rightarrow a=0$

$\Rightarrow b=0$

$Ker(F) = \{0,0\}$ — ①

Given : $F: \mathbb{R}^2 \to \mathbb{R}^2$, dim (domain) = dim (codomain) — ②

From ① & ②, F is non-singular.

Hence F is invertible.

$x = \begin{bmatrix} a \\ b \end{bmatrix}, \quad y = \begin{bmatrix} x \\ y \end{bmatrix}$

$F = AX$

$X = A^{-1}y = F^{-1}$

$$F = \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \Rightarrow F^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-y \\ -3x+2 \end{bmatrix}$$

$$F^2 = F(F(x)) = F(AX) = A(AX) = A^2X$$

$$= \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$F^2\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7a+4b \\ 12a+7b \end{bmatrix}$$

$$F^{-2}(x) = F^{-1}(F^{-1}(x))$$

$$F^{-2}\begin{bmatrix} a \\ b \end{bmatrix} = (A^{-1})^2(x) = \begin{bmatrix} 7 & -4 \\ -12 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7a-4b \\ -12a+7b \end{bmatrix}$$

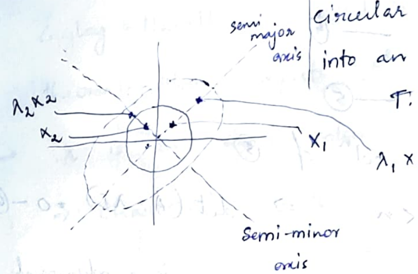
25/7/22

Eigen values and eigenvectors ; UNIT-3

↳ an adjective (something special or characteristic)

∴ eigenvalues → characteristic values

eigen vectors → " vectors



Circular membrane stretched through into an ellipse

$T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(x) = AX$

(All points on semi major axis & semi major axis, continue to remain on the axis)

$T(x_1) = \lambda_1 x_1$

$T(x_2) = \lambda_2 x_2$

Defn: Consider a linear transformn, $T: V \to V$. ~~(A non-zero vector)~~ A scalar λ, is said to be an eigen value of the linear transformn if ∃ a non-zero vector $X \in V$ s.t. $T(x) = \lambda X$, then X is called, the eigen vector corresponding to the eigenvalue, λ.

Polynomials of matrices : If A is a square matrix, and $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ is any polynomial, then $f(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n$ is the polynomial of matrix A.

$T(X*) \; AX$ — ① [∵ By defⁿ of linear mapping]

If λ is an eigenvalue then

$T(X) = \lambda X$ — ② [Defⁿ of eigenvalue]

$A_{n\times n} X_{n\times 1} = \lambda X_{n\times 1}$ — ③

$\Rightarrow AX - \lambda X = 0$ — ④

$\Rightarrow (A - \lambda I)X = 0$ — ⑤

We need non-trivial solⁿ of ⑤

$\therefore \; \text{rank}\, \rho(A - \lambda I) < n \quad \Rightarrow \det(A - \lambda I) = 0$ — ⑥

↓ is a polynomial.

eqⁿ ⑥ is called characteristic eqⁿ, which is an n^{th} degree polynomial in λ.

⑥ $\Rightarrow (-1)^n \{\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n \lambda^n\} = 0$

The roots of eqⁿ ⑥ are called eigenvalues.

For each root of eqⁿ ⑥, we find a non-trivial solⁿ of eqⁿ ⑤ which are termed as eigenvectors.

Theorem: Cayley-Hamilton theorem:

Statement: Every square matrix satisfies its own characteristic eqⁿ.

ie If $\Delta(t)$ is a characteristic polynomial of matrix A then $\Delta(A) = 0$

ie $\Delta(t) = (-1)^n t^n + a_1 t^{n-1} + a_2 t^{n-2} + \dots + a_n$

$\Rightarrow (-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0$

---

⑤ Compute A^{-1}, given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, using Cayley-Hamilton theorem. Also compute A^8.

→ λ is an eigen value if $AX = \lambda X$, $X \neq 0$

$\Rightarrow$ Characteristic eqⁿ : $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(-1-\lambda) - 4 = 0$

$\Rightarrow -1 + \lambda^2 - 4 = 0 \quad \Rightarrow \lambda^2 - 5 = 0$

[Shortcut for solving $|A - \lambda I| = 0 \Rightarrow \lambda^2 - Tr(A)\lambda + |A| = 0$

[∵ Trace (A) ⇒ sum of principal diagonal elements]

Accrd. to C-H theorem,

$A^2 - 5I = 0$

multiply A^{-1} on both sides,

$A - 5A^{-1} = 0 \quad \Rightarrow A^{-1} = \frac{1}{5} A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

$A^2 = 5I \Rightarrow$ ~~$(A^2)^4 = (5I)^4 \Rightarrow A^8 = 625I$~~

$A^4 = 5(5I) = 25I$

$A^8 = 25A^4 = 25(25I) = 625I$

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8) Compute A^{-1} and A^4 when $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

using Cayley - Hamilton theorem.

→ $AX = \lambda X \quad (X \neq 0) \Rightarrow (A - \lambda I) X = 0$

$|A - \lambda I| = 0 \Rightarrow -\lambda^3 + Tr(A)\lambda^2 - \{\text{sum of principal minors}\} \cdot \lambda + |A| = 0$

$Tr(A) = 1 + 4 + 6 = 11$

$M_{11} = \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = -1, \quad M_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} = -3, \quad M_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$|A| = -1$

$-\lambda^3 + 11\lambda^2 + 4\lambda - 1 = 0$

By Cay-Hamilton theorem

$-A^3 + 11A^2 + 4A - I = 0$ — zero matrix — (1)

Multiply (1) by A^{-1}

$-A^2 + 11A + 4I - A^{-1} = 0$

$\Rightarrow A^{-1} = A^2 + 11A + 4I$

$= \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

A^4 / From (1), (x ly A)

$-A^4 + 11A^3 + 4A^2 - A = 0$

$A^4 = 11A^3 + 4A^2 - A$ — (2)

(1) $\Rightarrow A^3 = 11A^2 + 4A - I$ — (3)

(3) in (2), $A^4 = 11(11A^2 + 4A - I) + 4A^2 - A$

$A^4 = 125A^2 + 43A - 11I$

$= \begin{bmatrix} 1782 & 3211 & 4004 \\ 3211 & 5786 & 7215 \\ 4004 & 7215 & 8997 \end{bmatrix}$

Eigen ~~value~~ Spaces

If $T: V \to V$ is a linear transformat° then $T(x) = \lambda x$, $x \neq 0 \Rightarrow \lambda$ = eigenvalue and x = eigen vector.

For each λ which is a root of $|A - \lambda I| = 0$

$E(\lambda) = \{ kx \mid k \in F \ \& \ T(x) = \lambda x \}$ is the eigenspace.

x = eigen vector $\Leftrightarrow x \neq 0$

But zero-vector $\in E(\lambda)$

Algebraic Multiplicity and geometric multiplicity of an eigenvalue :

The no. of times an eigen value λ is repeated is called the algebraic multiplicity of λ. The dimension of the eigen space $E(\lambda)$ is the geometric multiplicity.

(8) Find the eigen space of linear ~~multiplicity~~ mapping

$T: \mathbb{R}^3 \to \mathbb{R}^3 \quad T(x,y,z) = (2x+y,\ y-z,\ 2y+4z)$

$$T(x) = T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+y \\ y-z \\ 2y+4z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$T(x) = AX = \lambda X$

Characteristic equation : $|A - \lambda I| = 0$

$$-\lambda^3 + Tr(A)\lambda^2 - \left\{ \begin{matrix}\text{sum of} \\ \text{principal minors}\end{matrix} \right\} \lambda + |A| = 0$$

$M_{11} = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = 6 \quad M_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8 \quad M_{33} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$

sum $= 6 + 8 + 2 = 16$

$|A| = 12$

$\Rightarrow -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$

$\Rightarrow x_1 = 3 \quad x_2 = 2$

$\gamma_1 + \gamma_2 + \gamma_3 = -\frac{b}{a} \qquad \gamma_1\gamma_2\gamma_3 = \frac{-d}{a}$

$3 + 2 + x_3 = \cancel{\frac{-7}{-1}}\ 7 \Rightarrow x_3 = 2$

$\left[\because \sum_{i=1}^{n} \lambda_i = Tr(A) \qquad \prod_{i=1}^{n} \lambda_i = \lambda_1\lambda_2\lambda_3 \dots \lambda_n = |A| \right]$

Algebraic multiplicity of $\lambda = 3$ is 1.
and that of $\lambda = 2$ is 2.

Now $AX = \lambda X$, $X \neq 0 \Rightarrow (A - \lambda I)X = 0$, $\lambda \neq 0$

$\lambda = 3 \Rightarrow A - \lambda I,\ A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

[since we know the det of $A - 3I = 0$,]

we can skip $R_3 \to R_3 + R_2$.

$$A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A-3I)X = 0 \Rightarrow -x + y = 0$
$-2y - z = 0$

$z = k$, $\Rightarrow 2y + k = 0 \Rightarrow y = \frac{-k}{2}$, $x = y = \frac{-k}{2}$

$$X = \left(-\frac{k}{2}, -\frac{k}{2}, k\right) = +\frac{k}{2}(-1,-1,2)$$

$E(\lambda = 3) = \{ a(-1,-1,2) \mid a \in \mathbb{R} \}$

$\Rightarrow$ Geometric multiplicity of $\lambda = 3$ is one

For $\lambda = 2 \Rightarrow A - \lambda I = A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 \to R_2 + R_1$

$\Rightarrow A \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow (A-2I)X = 0 \Rightarrow \begin{matrix} y = 0 \\ -z = 0 \end{matrix} \Rightarrow z = 0$

$x \to$ free variable $\Rightarrow x = k$

$X = (k,0,0)$

$E(\lambda = 2) = \{ k(1,0,0) \mid k \in \mathbb{R} \}$ is one.

$\Rightarrow$ Geometric multiplicity of $\lambda = 2$

(8) $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T(x,y,z) = (2x+2y+z, x+3y+z, x+2y+2z)$.

Determine the geometric multiplicity of each of the eigenvalues.

$\rightarrow$ $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = AX$

$AX = \lambda X \Rightarrow (A - \lambda I)X = 0 \quad , X \neq 0$

characteristic eq^n : $|A - \lambda I| = 0$

$\Rightarrow -\lambda^3 + Tr(A)\lambda^2 - \left\{ \begin{matrix} \text{sum of principal} \\ \text{minors} \end{matrix} \right\} \lambda + |A| = 0$

$Tr(A) = 7 \qquad , |A| = 5$

$M_{11} = 4 \qquad M_{22} = 3 \qquad M_{33} = 4$

$\Rightarrow \quad -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$

$\lambda_1 = 5 ; \quad \lambda_2 = 1 \qquad \lambda_3 = 1$

Algebraic multiplicity of $\lambda = 5$ is one and of $\lambda = 1$ is two.

Eigen vectors are sol^n of $(A - \lambda I)X = 0 \quad , X \neq 0$. (∵ Non-trivial sol^n)

$\lambda = 5 \Rightarrow A - \lambda I = A - 5I = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$

$A - 5I \sim \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ as $|A - 5I| = 0$

[∵ Last row of echelon form of $A - \lambda I$ is always zero if λ is an eigen value of A. Remaining rows can be chosen such that they are independent (and convenient).]

$R_2 \to R_2 - R_1 \Rightarrow A - 5I \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$(A - 5I)X = 0 \Rightarrow x - 2y + z = 0$
$\qquad 4y - 4z = 0$

$z = k \Rightarrow 4y - 4k = 0 \Rightarrow y = k$

$x - 2k + k = 0 \Rightarrow x = k$

$X = (k, k, k) = k(1,1,1)$

$E(\lambda = 5) = \{ k(1,1,1) \ / \ k \in \mathbb{R} \} \Rightarrow$ Geometric multiplicity of $\lambda = 5$ is one.

$\lambda = 1 \Rightarrow A - \lambda I = A - I$

$A - I = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(A - I)X = 0 \Rightarrow x + 2y + z = 0$

$y = k_1 \quad z = k_2 \Rightarrow x + 2k_1 + k_2 = 0 \Rightarrow x = -2k_1 - k_2$

$X = (-2k_1 - k_2, k_1, k_2) = k_1(-2,1,0) + k_2(-1,0,1)$

$E(\lambda = 1) = \{ k_1(-2,1,0) + k_2(-1,0,1) \ / \ k_1, k_2 \in \mathbb{R} \}$

Geometric multiplicity of $\lambda = 1$ is two

[Note : Geometric multiplicity $\leq$ Algebraic multiplicity]

8) Find the eigen space of $T: P_2(t) \to P_2(t)$ given by $T(at^2+bt+c) = (2a-c)t^2 + (2a+b-2c)t + (-a+2c)$

$\to$ $T(at^2+bt+c) = \lambda(at^2+bt+c)$ [$\because$ By def^n of eigen values]

$(2a-c)t^2 + (2a+b-2c)t + (-a+2c) = \lambda at^2 + \lambda bt + \lambda c$

$$\left.\begin{array}{l} 2a-c = \lambda a \\ 2a+b-2c = \lambda b \\ -a+2c = \lambda c \end{array}\right\} \Rightarrow \begin{array}{l} (2-\lambda)a - c = 0 \\ 2a + (1-\lambda)b - 2c = 0 \\ a + (2-\lambda)c = 0 \end{array}$$

$(A-\lambda I)x = 0$ $\quad A-\lambda I = \begin{bmatrix} 2-\lambda & 0 & -1 \\ 2 & 1-\lambda & -2 \\ -1 & 0 & 2-\lambda \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$

$|A-\lambda I| = 0$

$\Rightarrow -\lambda^3 + Tr(A)\lambda^2 - \left\{\text{sum of principal minors}\right\}\lambda + |A| = 0$

$Tr(A) = 5$

$M_{11} = 2 \quad M_{22} = 3 \quad M_{33} = 2$

$-\lambda^3 + 5x^2 - 7\lambda + 3 = 0$

$\lambda_1 = 3 \quad ; \lambda_2 = 1$

$\lambda_3 = 5-4 = 1$

Algebraic multiplicity of $\lambda = 3$ is one and of $\lambda = 1$ is two.

$\lambda = 3 \Rightarrow A - \lambda I = A - 3I = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -2 & -2 \\ -1 & 0 & -1 \end{bmatrix}$

$A - 3I \sim \begin{bmatrix} -1 & 0 & -1 \\ 2 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$(A-3I)x = 0 \Rightarrow \left.\begin{array}{l} -x - z = 0 \\ -2y - 4z = 0 \end{array}\right\}$ or $\begin{array}{l} -a - c = 0 \\ -2b - 4c = 0 \end{array}$

$c = k \Rightarrow -2b - 4k = 0 \Rightarrow b = -2k, \quad a = -k$

$x = (-k, -2k, k) = k(-1, -2, 1)$

$\Rightarrow P_1(t) = k(-t^2 - 2t + 1)$

as $(a,b,c) \Rightarrow p(t) = at^2 + bt + c$

$E(\lambda=3) = \left\{ k(-t^2 - 2t + 1) \mid k \in \mathbb{R} \right\}$

$\lambda = 1 \Rightarrow A - \lambda I = A - I = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix}$

$(A-I)x = 0 \Rightarrow a - c = 0$

let $b = k_1 \quad c = k_2 \Rightarrow a = k_2$

$x = (k_2, k_1, k_2) = k_1(0,1,0) + k_2(1,0,1)$

$\Rightarrow P_2(t) = k_1 t + k_2(t^2+1)$

$E(\lambda=1) = \left\{ k_1 t + k_2(t^2+1) \mid k_1, k_2 \in \mathbb{R} \right\}$

1/8/22

Monic polynomial

A polynomial whose coefficient of heighest degree term is unity is called a monic polynomial.

If $f(t)$ is a monic polynomial of degree 'n'

then $f(t) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \dots + a_n$

Minimal polynomial:

Consider a set of polynomials which has a square matrix A as the root.

$J(A) = \{ f(t) \in P(t) \mid f(A) = 0 \}$

$f(t) = |A - tI|$ $\qquad g(t) = |tI - A|$ = characteristic polynomial, $\Delta(t)$

[If n is odd $f(t) = g(t)$
 n is even $f(t) = -g(t)$].

$f(t), g(t) \in J(A)$ by Cayley-Hamilton theorem

The monic polynomial of least degree in $J(A)$ is the minimal polynomial of A and is denoted by $m(t)$

Note: $\Delta(A) = 0$, $m(A) = 0$

$\deg(m(t)) \leq \deg\{\Delta(t)\}$

$m(t)$ divides $\Delta(t)$

$m(t)$ contains the same irreducible factors of $\Delta(t)$

---

i) Find the minimal polynomial of $\begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$

$\rightarrow$ $AX = \lambda X$

$(A - \lambda I)X = 0$ $\qquad |A - \lambda I| = 0$

$\Rightarrow -\lambda^3 + Tr(A)\lambda^2 \mp \{\text{sum of principal minors}\}\lambda + \det(A) = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - [2 + (-3) + 8]\lambda + 3 = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - [7]\lambda + 3 = 0$

$\lambda_1 = 3, \quad \lambda_2 = 2$

$\lambda_1 + \lambda_2 + \lambda_3 = Tr(A) = 3 + 1 + \lambda_3 = 5 \Rightarrow \lambda_3 = 1$

If $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$ and $r_1, r_2, r_3, \dots r_n$ are the roots then

$f(x) = (x - r_1)(x - r_2)(x - r_3) \dots (x - r_n)$

$\Rightarrow (\lambda - 3)(\lambda - 1)(\lambda - 1) = 0$

$\Delta(t) = |tI - A| = (t-3)(t-1)(t-1) = (t-3)(t-1)^2$

Consider $f(t) = (t-3)(t-1)$, $g(t) = (t-3)(t-1)^2$

We know that $\Delta(A) = 0$ [$\because$ by Cayley Hamilton theorem]

If $f(A) = 0$, then $m(t) = f(t)$ else $m(t) = g(t)$ or $\Delta(t)$

Consider $f(A) = (A - 3I)(A - I)$

$$= \begin{bmatrix} -1 & 2 & -5 \\ 3 & 4 & -15 \\ 1 & 2 & -7 \end{bmatrix} \begin{bmatrix} 1 & 2 & -5 \\ 3 & 6 & 15 \\ 1 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$f(A) = 0 \Rightarrow m(t) = (t-3)(t-1)$ is the minimal polynomial of A.

Q) Find the characteristic and minimal polynomial of $A = \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$

ch eqⁿ, $|A - \lambda I| = 0$

→ characteristic polynomial ⇒ '' $-\lambda^3 + Tr(A)\lambda^2 - \{$ sum of principal minors $\}$ $+ \det(A) = 0$

$\Rightarrow -\lambda^3 + 4\lambda^2 - [-2 + 11 + -4]\lambda + 2 = 0$

$\Rightarrow -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$

$\Rightarrow \lambda_1 = 2, \lambda_2 = 1$

$\Rightarrow \lambda_3 = 4 - 3 = 1$

$\Delta(t) = |tI - A| = (t-2)(t-1)(t-1)$

let $f(t) = (t-2)(t-1)$ $\quad g(t) = (t-2)(t-1)^2$

If $f(A) = 0$, then $m(t) = f(t)$ else $m(t) = \Delta(t)$

consider,

$f(A) = (A - 2I)(A - I)$

$= \begin{bmatrix} 1 & -2 & 2 \\ 4 & -6 & 6 \\ 2 & -3 & 3 \end{bmatrix}\begin{bmatrix} 2 & -2 & 2 \\ 4 & -5 & 6 \\ 2 & -3 & 4 \end{bmatrix}$

$= \begin{bmatrix} -2 & 2 & -2 \\ -4 & 4 & -4 \\ -2 & 2 & -2 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$f(A) \neq 0 \Rightarrow m(t) = \Delta(t) = (t-2)(t-1)^2$ is the minimal polynomial

H.W

Q) Find the characteristic and minimal polynomial of $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & -2 \\ -1 & -1 & 3 \end{bmatrix}$

3/8/22

Block Matrices :

$A = \left[\begin{array}{cc|cc|c} 2 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ \hline -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$A = \left[\begin{array}{cc|cc|c} 2 & 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ \hline -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$

Square block matrix : let M be a block matrix, M is said to be a sqr. block matrix if the diagonal blocks are also square

eg: $A = \left[\begin{array}{cc|c|cc} 1 & 2 & 2 & 4 & 5 \\ -1 & 1 & -1 & 1 & -1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$

Diagonal block matrix : A square block matrix is a diagonal block matrix if atleast some diagonal blocks are non-zero and all other blocks are zero.

If A is a diagonal block matrix

ie $A = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 \\ 0 & 0 & A_{33} & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & A_{nn} \end{bmatrix} = \text{diag}\{A_{11}, A_{22}, A_{33}, A_{44} \cdots A_{nn}\}$

i) $f(A) = \text{diag}\{f(A_{11}), f(A_{22}), f(A_{33}), \dots f(A_{nn})\}$

ii) $\Delta(t) = \Delta_{A_{11}}(t) \cdot \Delta_{A_{22}}(t) \cdot \Delta_{A_{33}}(t) \cdots \Delta_{A_{nn}}(t)$

iii) $M(t) = \text{L.CM}\{M_1(t), M_2(t) \dots M_n(t)\}$ $M_i(t)$ = minimal polynomial of A_{ii}

<u>Note</u>: The characteristic polynomial of block UTM, and block lower Δ^r Matrix can be found similar to that of the block diagonal matrix.

But minimal polynomial cannot be found similar to that of block diagonal matrix.

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5) $A = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

$A_{11} = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ $A_{22} = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}$ $A_{33} = [7]$

$|A_{11} - \lambda I| = \begin{vmatrix} 2-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 \Rightarrow \Delta_1(t) = (2-t)^2 = (t-2)^2$

$|A_{22} - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 5-\lambda \end{vmatrix} = \lambda^2 - 9\lambda + 14 = (\lambda - 2)(\lambda - 7)$

$\Delta_2(t) = (t-2)(t-7)$

$|A_{33} - \lambda I| = |7 - \lambda| \Rightarrow \Delta_3(t) = t-7$

$\Delta(t) = \Delta_1(t) \cdot \Delta_2(t) \cdot \Delta_3(t) = (t-2)^3 (t-7)^2$

From $\Delta_1(t) \Rightarrow f_1(t) = t-2$ $f_1(A_{11}) = A_1 - 2I = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} \neq 0$

$\Rightarrow$ $t-2$ is not a minimal polynomial.

$\therefore m_1(t) = \Delta_1(t) = (t-2)^2$

$\Delta_2(t) \Rightarrow f_2(t) = (t-2)(t-7) = \Delta_2(t)$ & $\Delta_2(A_{22}) = 0$

$\therefore m_2(t) = (t-2)(t-7)$

$\Delta_3(t) = t-7 \Rightarrow m_3(t) = t-7$

$m(t) = \text{LCM}\{m_1(t), m_2(t), m_3(t)\}$
$= \text{LCM}\{(t-2)^2, (t-2)(t-7), (t-7)\}$
$= (t-2)^2(t-7)$

---

6) Find the characteristic and minimal polynomial of

$A = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \Rightarrow A_{11} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ $A_{22} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

~~$A_{33} = 3$~~

$|A_{11} - \lambda I| = \lambda^2 - 6\lambda + 9$
$= (\lambda - 3)^2 \Rightarrow \Delta(t) = (t-3)^2$

$|A_{22} - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3 \Rightarrow \Delta(t) = (t-3)^3$

$\Delta(t) = \Delta_1(t)\,\Delta_2(t) = (t-3)^2(t-3)^3 = (t-3)^5$

$\Delta(t) = f(t) = t-3 \Rightarrow f(A_{11}) = A_{11} - 3I$
$= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$

$\Rightarrow m_1(t) = (t-3)^2$

$\Delta_2(t) \Rightarrow f_1(t) = (t-3)$ $\qquad f_2(t) = (t-3)^2$

$$f_1(A_{22}) = A_{22} - 3I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

$$f_2(A_{22}) = (A_{22} - 3I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ & & \\ & & \end{bmatrix} \neq 0$$

$$\Rightarrow M_2(t) = \Delta_2(t) = (t-3)^3$$

$$m(t) = LCM\{M_1(t), M_2(t)\}$$
$$= LCM\{(t-3)^2, (t-3)^3\} = (t-3)^3.$$

---

③ Find the characteristic polynomial of

$$A = \left[\begin{array}{cc|cc} 1 & 1 & 2 & 2 \\ 0 & 3 & 3 & 4 \\ \hline 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

$$A_{11} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 5 & 5 \\ 0 & 6 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}.$$

characteristic polynomial $\Delta(t) = \Delta_1(t) \cdot \Delta_2(t)$

$\Delta_1(t) \to A_{11}$

$\Delta_2(t) \to A_{22}$

$$|A_{11} - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) \Rightarrow \cancel{3} \; \Delta(t) = (t-1)(t-3)$$

$$|A_{22} - \lambda I| = \begin{vmatrix} 5-\lambda & 5 \\ 0 & 6-\lambda \end{vmatrix} \Rightarrow \Delta(t) = (t-5)(t-6).$$

$$\Delta(t) = \Delta_1(t)\,\Delta_2(t) = (t-1)(t-3)(t-5)(t-6)$$

Since all factors are linear, $m(t) = \Delta(t)$.

4/8/22 **Companion matrix:**

If $f(t)$ is any monic polynomial

i.e $f(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1} + t^n$

then,

$$c(f) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}_{n \times n}$$

is called the companion matrix and

$\Delta(t) = m(t) = f(t)$

If $f(t) = t^3 - 8t^2 + 5t + 7$ then $c(f) = \begin{bmatrix} 0 & 0 & -7 \\ 1 & 0 & -5 \\ 0 & 1 & 8 \end{bmatrix}$

with $\Delta(t) = m(t) = t^3 - 8t^2 + 5t + 7$

Nilt-potent operator:

A linear operator $T: V \to V$ is termed as a nilt potent operator of index k, if $T^k = 0$ for $k \in \mathbb{N}$ but $T^{k-1} \neq 0$

$\Rightarrow \{ k =$ smallest +ve $\mathbb{Z}$ for which $T^k = 0 \}$

Jordan nil-potent block of index r,

$$N(r) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Jordan block : $J(\lambda_i) = \lambda_i I + N_i$

$$J(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \lambda_i \end{bmatrix}$$

Jordan Canonical Form :

A linear operator $T: V \to V$ whose characteristic polynomial $\Delta(t) = (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}(t-\lambda_3)^{n_3} \ldots (t-\lambda_r)^{n_r}$ and minimal polynomial

$$m(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}(t-\lambda_3)^{m_3} \ldots (t-\lambda_r)^{m_r}$$

where λ_i's are distinct eigen values of the linear operator T, can be represented by a block diagonal matrix J whose diagonal blocks are Jordan block $J_{ij}(\lambda_i)$

Properties:

1) The same eigen value λ_i can appear in multiple blocks if it has several independent eigen vectors.

2) The no. of jordan blocks corresponding to an eigen value λ_i = geometric multiplicity of λ_i

3) For each eigen value λ_i, there exists atleast 1 jordan block of order m_i & all the other jordan blocks of λ_i are of order $\leq m_i$

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8) Find the jordan canonical form of the linear transfn whose ch polynomial is $(t-2)^4(t+5)^3$ and $m(t) = (t-2)^2(t+5)^3$

→ $\Delta(t) = (t-2)^4(t+5)^3 = (t-2)^{n_1}(t+5)^{n_2}$

$m(t) = (t-2)^2(t+5)^3 = (t-2)^{m_1}(t+5)^{m_2}$

case i): $n_1 = 4 = 2+2$ $\quad n_2 = 3 = m_2$ $\quad \Rightarrow$ (Geometric multiplicity of $\lambda = 2$ is two)

$$J_{11}(\lambda=2) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = J_{12}(\lambda=2)$$

$$J_{21}(\lambda=-5) = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

$J = \text{diag}\{J_{11}(\lambda=+2), J_{12}(\lambda=2), J_{21}(\lambda=-5)\}$

case ii): $4 = 2+1+1$ ($\Rightarrow$ Geometric multiplicity of $\lambda=2$ is three)

$$J_{11}(\lambda=2) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad J_{12}(\lambda=2) = [2] = J_{13}(\lambda=2)$$

$$J_{21}(\lambda=-5) = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

$J = \text{diag}\{J_{11}(\lambda=2), J_{12}(\lambda=2), J_{13}(\lambda=2), J_{21}(\lambda=-5)\}$

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9) Find all possible jordan canonical forms of $T: V \to V$ whose ch. polynomial is $(t-3)^5$ also find minimal polynomial in each case

→ case i) : $m(t) = (t-3)^5$ then

$$J = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

case ii) : $5 = 4+1$

$$J_{11}(\lambda = 3) = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad J_{12}(\lambda = 5) = [3]$$

$J = \text{diag}\{J_{11}, J_{12}\}$

$m(t) = (t-3)^4$

case iii) : a) $5 = 3+2$ (Geometric multiplicity of 2)

$$J_{11}(\lambda = 3) = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad J_{12}(\lambda = 3) = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$J = \text{diag}\{J_{11}, J_{12}\}$, $m(t) = (t-3)^3$

b) $5 = 3+1+1$ (Geometric multiplicity of $\lambda = 3$ is 3)

$$J_{11}(\lambda = 3) = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad J_{12}(\lambda = 3) = J_{13} = [3]$$

$J = \text{diag}\{J_{11}, J_{12}, J_{13}\}$, $m(t) = (t-3)^3$

case iv) : $5 = 2+2+1$ (Geometric multiplicity is three)

$$J_{11} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = J_{12} \quad J_{13} = [3] , \quad m(t) = (t-3)^2$$

$J = \text{diag}\{J_{11}, J_{12}, J_{13}\}$

case v) : $5 = 2+1+1+1$ (Geometric multiplicity 4)

$$J_{11} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad \& \quad J_{12} = J_{13} = J_{14} = [3]$$

$J = \text{diag}\{J_{11}, J_{12}, J_{13}, J_{14}\}$, $m(t) = (t-3)^2$

case vi) : $5 = 1+1+1+1+1$

$J_{11} = J_{12} = J_{13} = J_{14} = J_{15} = [3]$

$J = \text{diag}\{J_{11}, J_{12}, J_{13}, J_{14}, J_{15}\} = 3I$

$m(t) = t - 3$

Q) Find the jordan canonical form of the linear operator given by

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{bmatrix}$$

→ characteristic polynomial ⇒ $-\lambda^3 + Tr(A)\lambda^2 - \{\text{sum of principal minors}\}\lambda + \det(A) = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - [6+2+2]\lambda + 4 = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = 0$

$\Delta(t) = t^3 - 6\lambda^2 + 12\lambda - 8 = (t-2)^3$

$$\lambda = 2 \Rightarrow A - \lambda I = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(n - r) = 3 - 1 = 2 = \dim\{\text{sol}^n \text{ space of } (A - \lambda I)X = 0\}$ is two.

⇒ Geometric multiplicity of $\lambda = 2$ is two.

⇒ Jordan canonical form should have 2 jordan blocks.

$\Rightarrow (t-2)^3 = (t-2)^2 (t-2)$ ie $3 = 2+1$

$J_{11} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $J_{12} = [2] \Rightarrow J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Differential equation in its fundamental form

$$\text{IVP}\begin{cases} \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1(t), & x_1(t_0) = x_{10} \\ \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2(t), & x_2(t_0) = x_{20} \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n(t), & x_n(t_0) = x_{n0} \end{cases}$$

IVP $\Rightarrow$ Initial Value problem

$$\frac{dx}{dt} = v, \quad x(0) = x_0$$

$$\frac{dv}{dt} = -\frac{k}{m}x - \frac{\gamma}{m}v + \frac{1}{m}f(t), \quad v(0) = v_0$$

$$X = \begin{bmatrix} x \\ v \end{bmatrix} \Rightarrow \frac{dX}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{bmatrix}$$

$$\frac{dX}{dt} = \begin{bmatrix} 0x + v \\ -\frac{k}{m}x - \frac{\gamma}{m}v \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

$$\frac{dX}{dt} = AX + F(t) \quad X(0) = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = X_0 \longrightarrow ①$$

Eqn ② is called the diff eq^n in its fundamental form

Example 2: 5/8/22

$$X = [x_1 \ x_2 \ \cdots \ x_n]^T$$

$$\frac{dX}{dt} = AX + F(t) \qquad X(t_0) = X_0 \ —— ①$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} \quad X_0 = \begin{bmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{bmatrix}$$

Eqn ① is called the differential eq^n in its fundamental form.

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③ Express $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0$ $\quad x(0) = 4 \quad \left.\frac{dx}{dt}\right|_{t=0} = 5$ in its fundamental form.

$\rightarrow \frac{dx}{dt} = y$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} = 2\frac{dx}{dt} + 3x = 2y + 3x, \quad x(0) = 4$$

$$\frac{dy}{dt} = 3x + 2y \qquad y(0) = \left.\frac{dx}{dt}\right|_{t=0} = 5$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \frac{dX}{dt} = \begin{bmatrix} 0x + y \\ 3x + 2y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad X(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

$$\frac{dX}{dt} = AX \qquad X(0) = X_0$$

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \qquad X_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

(8) Express $\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} - x(t) = 0$ $\quad x(-1) = 2$

$\left.\frac{dx}{dt}\right|_{t=-1} = 1$ and $\left.\frac{d^2x}{dt^2}\right|_{t=-1} = -205$ in its

fundamental form.

→ $\frac{dx}{dt} = y \Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2} = z$

$\frac{dz}{dt} = \frac{d^3x}{dt^3} = x(t) - \frac{d^2x}{dt^2} = x - z$

$\frac{dx}{dt} = y \quad \frac{dy}{dt} = z \quad \frac{dz}{dt} = x - z$ #

$= 0x + y + 0z \quad = 0x + 0y + z$

$x(-1) = 2 \quad y(-1) = \left.\frac{dx}{dt}\right|_{t=-1} = 1 \quad z(-1) = \left.\frac{d^2x}{dt^2}\right|_{t=-1} = -205$

$\frac{dX}{dt} = AX \quad X(-1) = X_0$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad X_0 = \begin{bmatrix} 2 \\ 1 \\ -205 \end{bmatrix}$

Sol^n of $\frac{dx}{dt} = AX + F(t) \quad X(t_0) = X_0$

$\frac{dx}{dt} - AX = F(t)$ → similar to linear 1st order ODE

Sol^n is $e^{-At}X = \int e^{-At}F(t)\,dt + C$

$X = e^{At}\int e^{-At}F(t)\,dt + C$

Diagonalization

1) Similar matrices :

Two matrices A & B are said to be similar if there exists an invertible matrix S

st $A = SBS^{-1}$ [or $S^{-1} = P \Rightarrow P^{-1}AP = B$] $\left(A = SBS^{-1} = P^{-1}BP\right)$

2) Diagonalisation :

A matrix is said to be diagonizable if it is similar to a diagonal matrix D.

ie $A = PDP^{-1}$ where D → diagonal matrix

Note: → If A is a sq^r matrix with eigen values $\lambda_1, \lambda_2 \dots \lambda_n$ and independent eigen vectors $X_1, X_2 \dots X_n$ then $P = [X_1\ X_2 \dots X_n]$ and

$A = PDP^{-1} \quad D = \text{diag}\{\lambda_1, \lambda_2, \dots \lambda_n\}$

→ A is diagonisable if geometric multiplicity of each eigen value is equal to its algebraic multiplicity

$e^{At} = Pe^{Dt}P^{-1} \quad e^{Dt} = \text{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t}, \dots e^{\lambda_n t}\}$

$[\because A = PDP^{-1} \quad f(A) = Pf(D)P^{-1}]$

→ Alternate sol^n:

Consider $\frac{dx}{dt} = AX$ — ①

We assume $X = ve^{\lambda t}$ — ②

② in ① $\Rightarrow \frac{d}{dt}\{ve^{\lambda t}\} = Ave^{+\lambda t}$

$v\lambda e^{\lambda t} = Ave^{\lambda t}$

$\Rightarrow AV = \lambda V$

$\Rightarrow \lambda$ is the eigenvalue of A & V is the eigenvector

Complementary funct

$CF = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ (for distinct eigen values)

$= c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t} + \dots + c_n v_n e^{\lambda_n t}$

If λ_1 and λ_2 are repeated ie $\lambda_1 = \lambda_2 = \lambda$

$CF = (c_1 + c_2 t) v e^{\lambda t}$ + rest.

$PI = e^{At} \int e^{-At} F(t)\, dt$

$e^{At} = Pe^{Dt}P^{-1}$ $[\because A = PDP^{-1}]$

$PI = Pe^{Dt}P^{-1} \int Pe^{-Dt}P^{-1} F(t)\, dt$

$= Pe^{Dt} \int e^{-Dt} P^{-1} F(t)\, dt = Pe^{Dt} \int (Pe^{Dt})^{-1} F(t)\, dt$

$[\because (AB)^{-1} = B^{-1}A^{-1}]$

If $Pe^{Dt} = \phi(t)$ then

$PI = \phi(t) \int [\phi(t)]^{-1} F(t)\, dt$

8/8/22

Q Solve $\frac{dx}{dt} = -2x + 3y$ $\frac{dy}{dt} = -x + 2y$

given $x(2) = 2$ $y(2) = 4$ by reducing it to fundamental form.

$\rightarrow \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2x + 3y \\ -x + 2y \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \end{bmatrix}$ $A = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$ $x(2) = \begin{bmatrix} x(2) \\ y(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\frac{dx}{dt} = AX$ $x(2) = X_0$ — ① This is the fundamental form.

$|A - \lambda I| = 0 \Rightarrow \lambda^2 - Tr(A)\lambda + |A| = 0$

$\Rightarrow \lambda^2 - 0\lambda + (-1) = 0 \Rightarrow \lambda^2 - 1 = 0$

$\lambda = 1, -1$

$\lambda = 1 \Rightarrow A - \lambda I = A - I$

$= \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

$(A - I)x = 0 \Rightarrow -x + y = 0$

$y = k \Rightarrow x = k \Rightarrow X = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = 1 \Rightarrow V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1 \Rightarrow A - \lambda I = A + I = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}$

$(A + I)x = 0 \Rightarrow -x + 3y = 0$

$y = k \Rightarrow x = 3k \Rightarrow X = \begin{bmatrix} 3k \\ k \end{bmatrix} = k\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda_2 = -1 \Rightarrow V_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Soln is $X = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t} = \underbrace{\begin{bmatrix} e^{t} & 3e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}}_{\phi(t)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$X(2) = \begin{bmatrix}2\\4\end{bmatrix} \Rightarrow \begin{bmatrix} e^{2} & 3e^{-2} \\ e^{2} & e^{-2}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix} e^{2} & 3e^{-2}\\ e^{2} & e^{-2}\end{bmatrix}^{-1}\begin{bmatrix}2\\4\end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} e^{-2} & -3e^{-2}\\ -e^{2} & e^{2}\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-10e^{-2}\\ 2e^{2}\end{bmatrix} = \begin{bmatrix}5e^{-2}\\ -e^{2}\end{bmatrix}$$

$$X = \begin{bmatrix} e^{t} & 3e^{-t}\\ e^{t} & e^{-t}\end{bmatrix}\begin{bmatrix}5e^{-2}\\ -e^{2}\end{bmatrix} = \begin{bmatrix}5e^{t-2} - 3e^{-(t-2)} \\ 5e^{t-2} - e^{-(t-2)}\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

---

⑨ Solve $\ddot{x} - 2\dot{x} - 3x = 0$ $\quad x(0) = 4 \quad \dot{x}(0) = 5$

by reducing it to fundamental form $\quad$ [· → represents time derivative

→ $\begin{bmatrix}\dot{x}\\ \ddot{x}\end{bmatrix} = \begin{bmatrix} & \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}\begin{bmatrix}x\\ \dot{x}\end{bmatrix}$

$\begin{bmatrix}\dot{x}\\ \ddot{x}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\end{bmatrix}$ $\qquad \ddot{x} = 3x + 2\dot{x}$

$\dot{x} = y \qquad X = \begin{bmatrix}x\\y\end{bmatrix} \qquad \dot{X} = \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} \qquad [\because \dot{y} = \ddot{x}]$

$\frac{dX}{dt} = AX \qquad X(0) = \begin{bmatrix}4\\5\end{bmatrix} \qquad A = \begin{bmatrix}0 & 1\\ 3 & 2\end{bmatrix}$

Characteristic eqⁿ $|A - \lambda I| = 0 \Rightarrow \lambda^2 - Tr(A)\lambda + |A| = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow \lambda = -1, 3$

$\lambda = -1 \Rightarrow A - \lambda I = A + I$

$= \begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$

$(A + I)X = 0 \Rightarrow x + y = 0 \qquad y = k \Rightarrow x = -k$

$X = \begin{bmatrix}-k\\k\end{bmatrix} = k\begin{bmatrix}-1\\1\end{bmatrix} \qquad \therefore \lambda_1 = -1, \Rightarrow v_1 = \begin{bmatrix}-1\\1\end{bmatrix}$

$\lambda = 3 \Rightarrow A - \lambda I = A - 3I = \begin{bmatrix}-3 & 1\\ 3 & -1\end{bmatrix} \sim \begin{bmatrix}-3 & 1\\ 0 & 0\end{bmatrix}$

$(A - 3I)X = 0 \Rightarrow -3x + y = 0$

$y = 3k \Rightarrow -3x + 3k = 0 \Rightarrow x = k$

$X = \begin{bmatrix}k\\3k\end{bmatrix} = k\begin{bmatrix}1\\3\end{bmatrix} \qquad \therefore \lambda_2 = 3 \Rightarrow v_2 = \begin{bmatrix}1\\3\end{bmatrix}$

Solution is $X = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

$X = c_1\begin{bmatrix}-1\\1\end{bmatrix}e^{-t} + c_2\begin{bmatrix}1\\3\end{bmatrix}e^{+3t}$

$= \begin{bmatrix}-e^{-t} & e^{3t}\\ e^{-t} & 3e^{3t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$

$X(0) = \begin{bmatrix}4\\5\end{bmatrix} \Rightarrow \begin{bmatrix}-1 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$

$-c_1 + c_2 = 4$
$c_1 + 3c_2 = 5$

$4c_2 = 9/4$

~~$c_2 = 4.5$~~
~~$c_1 = 5 - 4.5$~~
~~$= 0.5$~~
$5 - 3(2.25)$
$=$ ~~2.75~~
$= -7/4$

$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-7/4\\ 9/2\end{bmatrix}$

---

⑧ Solve $y''' + 2y'' - y' - 2y = 0$

$y(0) = 3, y'(0) = 2, \quad y''(0) = 0$

by reducing it to fundamental form

→ $y''' = 2y + y' - 2y''$

$$\begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$$

$X = \begin{bmatrix} y \\ u \\ v \end{bmatrix}$ $\quad X' = AX \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \quad x(0) = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$

Sol^n is $X = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t} + C_3 v_3 e^{\lambda_3 t}$

$[\because \lambda_1 \neq \lambda_2 \neq \lambda_3]$

Q) Solve $\dot{x} = -3x - 2y$, $\dot{y} = 5x - y$ by reducing it to fundamental form

$\rightarrow \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \Rightarrow \dot{X} = \begin{bmatrix} -3 & -2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \dot{X} = AX$

$A = \begin{bmatrix} -3 & -2 \\ 5 & -1 \end{bmatrix}$

Characteristic eq^n : $|A - \lambda I| = 0$

$\lambda^2 - Tr(A)\lambda + |A| = 0$

$\lambda^2 - (-4)\lambda + (13) = 0$

$\Rightarrow \lambda = -2 \pm 3i$

$\lambda_1 = -2 + 3i$, $\lambda_2 = -2 - 3i$

$\lambda = -2 + 3i \Rightarrow A - \lambda I = A - (-2 + 3i) I = \begin{bmatrix} -3 - (-2 + 3i) & -2 \\ 5 & -1 - (-2 + 3i) \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -1 - 3i & -2 \\ 5 & 1 - 3i \end{bmatrix} \sim \begin{bmatrix} 5 & 1 - 3i \\ 0 & 0 \end{bmatrix}$

$(A - \lambda I) X = 0 \Rightarrow 5x + (1 - 3i) y = 0$

$y = 5k \Rightarrow 5x + (1 - 3i) 5k = 0 \Rightarrow x = -(1 - 3i) k$

$X = \begin{bmatrix} -(1 - 3i) k \\ 5k \end{bmatrix} = k \begin{bmatrix} -1 + 3i \\ 5 \end{bmatrix}$ $\quad \lambda_1 = -2 + 3i$

$\Rightarrow v_1 = \begin{bmatrix} -1 + 3i \\ 5 \end{bmatrix}$

$X = C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}$

$(\lambda_1, \lambda_2 = \text{complex})$

Then instead of finding v_2 just seperate the real and imaginary parts of 1st term alone

$X = C_1 \begin{bmatrix} -1 + 3i \\ 5 \end{bmatrix} e^{(-2 + 3i)t}$

$= C_1 \begin{bmatrix} -1 + 3i \\ 5 \end{bmatrix} e^{-2t} [\{ \cos(3t) + i \sin(3t) \}]$

$= C_1 e^{-2t} \begin{bmatrix} -\cos(3t) - 3\sin(3t) + i(3\cos(3t) - \sin(3t)) \\ 5\cos(3t) + i\, 5\sin(3t) \end{bmatrix}$

$= C_1 \begin{bmatrix} e^{-2t} \{ -\cos(3t) - 3\sin(3t) \} \\ e^{-2t}\, 5\cos(3t) \end{bmatrix} + C_1 i \begin{bmatrix} e^{-2t} \{ 3\cos(3t) - \sin(3t) \} \\ e^{-2t}\, 5\sin(3t) \end{bmatrix}$

$C_1 i = C_2$

$$X = \underbrace{\begin{bmatrix} e^{-2t} \{ -\cos(3t) - 3\sin(3t) \} & e^{-2t} (3\cos(3t) - \sin(3t)) \\ 5e^{-2t} \cos(3t) & 5e^{-2t} \sin(3t) \end{bmatrix}}_{\phi(t)} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

10/8/22

UNIT : 4 : INNER PRODUCT SPACES

Def^n: Consider a vector space V, a func^n denoted by $\langle , \rangle$ which assigns a unique scalar to vectors u and v denoted by $\langle u, v \rangle$ is said to be an inner product if it satisfies

i) linearity property : i.e

$$\langle au_1 + bu_2 , v \rangle = a \langle u_1 , v \rangle + b \langle u_2 , v \rangle$$

for all $u_1, u_2, v \in V$ and $a, b \in K$, field.

ii) Conjugate symmetry :

$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

iv) Positive definite property

$\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$

The vector space together with the inner product is called inner product space.

If field $K = \mathbb{R}$,

a) linearity property : $\langle au_1 + bu_2 , v \rangle = a \langle u_1, v \rangle + b \langle u_2 , v \rangle$

If $z \in \mathbb{R}$, $\bar{z} = z$

b) Symmetry : $\langle u, v \rangle = \langle v, u \rangle$

c) Positive definite $\langle u, v \rangle \geq 0$ and $\langle u, v \rangle = 0$ if and only if $u = 0$

Consequence of field $K = \mathbb{R}$

$$\langle u, av_1 + bv_2 \rangle = \langle av_1 + bv_2 , u \rangle \quad [\because \text{Symmetric property}]$$

$$= a \langle v_1, u \rangle + b \langle v_2, u \rangle \quad [\because \text{linearity}]$$

$$= a \langle u, v_1 \rangle + b \langle u, v_2 \rangle \quad [\because \text{Symmetric property}]$$

Linearity property can be applied in 2nd operat^n also when the field is $\mathbb{R}$.

Normed vector space :

Consider a vector inner product space V. For any vector $v \in V$ the function $\| . \|$ assigns a unique scalar $\| v \|$. The function $\| . \|$ is said to be a norm

if i) Positive definite property

$\| u \| \geq 0$ and $\| u \| = 0$ iff $u = 0$

ii) $\| ku \| = |k| \, \| u \|$

iii) Triangular law of inequality

$\| u + v \| \leq \| u \| + \| v \|$

The inner product space together with $\| . \|$ is called a normed vector space.

If $V = \mathbb{R}^n$ $K = \mathbb{R}$ $u \in \mathbb{R}^n \rightarrow u = (u_1, u_2, u_3 \dots u_n)$

$\| u \|_\infty = |u_1| + |u_2| + |u_3| + \dots + |u_n|$ → Infinity norm

$\| u \|_1 = \max \{ |u_1|, |u_2|, \dots |u_n| \}$

These two result in normal space but not inner product space.

Usual norm $\| u \| = \sqrt{\langle u, u \rangle}$

With this norm, every inner product space is a normed vector space.

Normalising a vector

A vector whose norm is unity is called a unit vector. i.e $\|u\| = 1$

If $\|u\| \neq 1$ then $\left\| \frac{u}{\|u\|} \right\| = \sqrt{\left\langle \frac{u}{\|u\|}, \frac{u}{\|u\|} \right\rangle}$

$= \sqrt{\frac{1}{\|u\|^2} \langle u, u \rangle} = 1$

$\therefore \frac{u}{\|u\|}$ is a unit vector.

Process of obtaining unit vector corresponding to given vector is called normalisat^n.

Q) Prove that R^n with $\langle \, , \rangle$ defined as the usual dot product is an inner product space. Also evaluate $\langle 5u_1 + 6u_2, 6v_1 - 7v_2 \rangle$ and $\| 6u - 7v \|^2$

→ $u_1, u_2, v \in R^n \Rightarrow u_1 = (u_{11}, u_{12}, u_{13} \dots u_{1n})$

$u_2 = (u_{21}, u_{22}, u_{23} \dots u_{2n})$

$v = (v_1, v_2 \dots v_n)$

$\langle u, v \rangle = u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 + \dots + u_nv_n$

i) To prove $\langle au_1 + bu_2, v \rangle = a\langle u_1, v \rangle + b\langle u_2, v \rangle$

$au_1 + bu_2 = (au_{11} + bu_{21}, au_{12} + bu_{22}, \dots, au_{1n} + bu_{2n})$

$\langle au_1 + bu_2, v \rangle = (au_{11} + bu_{21})v_1 + (au_{12} + bu_{22})v_2 + \dots + (au_{1n} + bu_{2n})v_n$

$= a(u_{11}v_1 + u_{12}v_2 + \dots u_{1n}v_n) + b(u_{21}v_1 + u_{22}v_2 + \dots + u_{2n}v_n)$

$= a\langle u_1, v \rangle + b\langle u_2, v \rangle$

ii) To prove $\langle u, v \rangle = \langle v, u \rangle$

$\langle u, v \rangle = u_1v_1 + u_2v_2 + \dots + u_nv_n$

$= v_1u_1 + v_2u_2 + \dots + v_nu_n = \langle v, u \rangle$

iii) $\langle u, u \rangle = u_1^2 + u_2^2 + \dots + u_n^2$

If $u \neq 0$ then at least one of $u_1, u_2, \dots u_n$ is non-zero.

$\Rightarrow u_1^2 + u_2^2 + u_3^2 + \dots + u_n^2 > 0$

Suppose $u_1^2 + u_2^2 + u_3^2 + \dots + u_n^2 = 0$

Sum of positive numbers $= 0$ only if all terms are zero.

$\therefore u_1 = 0, u_2 = 0 \dots u_n = 0$

$\therefore u = (0, 0, 0, \dots 0) \Leftrightarrow \langle u, u \rangle = 0$

$\therefore \langle \, , \rangle$ is an inner product.

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iv) Expand $\langle 5u_1 + 8u_2, 6v_1 - 7v_2 \rangle$ & $\| 2u - 3v \|^2$

$\langle 5u_1 + 8u_2, 6v_1 - 7v_2 \rangle = 5\langle u_1, 6v_1 - 7v_2 \rangle + 8\langle u_2, 6v_1 - 7v_2 \rangle$ [$\because$ linearity property]

$= 5\{6\langle u_1, v_1 \rangle - 7\langle u_1, v_2 \rangle\} + 8\{6\langle u_2, v_1 \rangle - 7\langle u_2, v_2 \rangle\}$ [$\because$ Field $= \mathbb{R} \Rightarrow$ linearity can be applied to 2nd position]

$= 30\langle u_1, v_1 \rangle - 35\langle u_1, v_2 \rangle + 48\langle u_2, v_1 \rangle - 56\langle u_2, v_2 \rangle$

$\|u\| = \sqrt{\langle u,u\rangle} \Rightarrow \|2u-3v\|^2 = \langle 2u-3v, 2u-3v\rangle$

$\|2u-3v\|^2 = 2\langle u, 2u-3v\rangle - 3\langle v, 2u-3v\rangle$

$= 2\{2\langle u,u\rangle - 3\langle u,v\rangle\} - 3\{2\langle v,u\rangle - 3\langle v,v\rangle\}$

[∵ linearity]

$= 4\langle u,u\rangle - 6\langle u,v\rangle - 6\langle v,u\rangle$
$+ 9\langle v,v\rangle$ [∵ linearity when fied $= \mathbb{R}$]

$= 4\langle u,u\rangle - 12\langle u,v\rangle + 9\langle v,v\rangle$ [∵ $\langle v,u\rangle = \langle u,v\rangle$ when field $= \mathbb{R}$]

MATH LA UNIT-4

11/8/22

9) $f(t) = t+2 \quad g(t) = 3t-2 \quad h(t) = t^2 - 2t$
in $P(t)$

$$\langle P, Q \rangle = \int_0^1 P(t)\,q(t)\,dt$$

Find $\langle f, g \rangle$, $\langle f, h \rangle$. Also normalise f and g.

→ $$\langle f, g \rangle = \int_0^1 f(t)\,g(t)\,dt = \int_0^1 (t+2)(3t-2)\,dt$$

$$= \int_0^1 (3t^2 + 4t - 4)\,dt = \left[t^3 + 2t^2 - 4t \right]_0^1$$

$= -1$

$$\langle f, h \rangle = \int_0^1 f(t)\,h(t)\,dt = \int_0^1 (t+2)(t^2 - 2t)\,dt$$

$\frac{1}{4} - \frac{14}{4} - 6$
$-13-24$
-37

$$= \int_0^1 (t^3 - 7t - 6)\,dt = \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t \right]_0^1$$

$= -\frac{37}{4}$ $\quad = -\frac{7}{4}$

normalise f and g $\Leftrightarrow$ Find $\frac{f}{\|f\|}$ & $\frac{g}{\|g\|}$

$$\|f\| = \sqrt{\langle f, f \rangle}$$

$$\langle f, f \rangle = \int_0^1 (f(t))^2 \, dt = \int_0^1 (t+2)^2 \, dt$$

$$= \frac{(t+2)^3}{3} \Big|_{t=0}^{1} = \frac{3^3 - 2^3}{3} = \frac{19}{3}$$

$$\|f\| = \sqrt{\frac{19}{3}} \qquad \frac{f(t)}{\|f(t)\|} = \frac{t+2}{\sqrt{\frac{19}{3}}}$$

$$= \sqrt{\frac{3}{19}}\,(t+2)$$

$$\langle g, g \rangle = \int_0^1 (g(t))^2 \, dt = \int_0^1 (3t-2)^2 \, dt$$

$$= \frac{(3t-2)^3}{3(3)} \Big|_{t=0}^{1} = \frac{1^3 - (-2)^3}{9} = 1$$

TVS

$\Rightarrow \|g(t)\| = 1 \Rightarrow g(t)$ is a normalised vector.

Angle between vectors u and v

$$\cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \, \|v\|}$$

Find the angle between $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ when $\langle A, B \rangle = Tr\{B^T A\}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$B^T A = \begin{bmatrix} b_{11}a_{11} + b_{21}a_{21} + b_{31}a_{31} & ? & ? \\ ? & b_{12}a_{12} + b_{22}a_{22} + b_{32}a_{32} & ? \\ ? & ? & b_{13}a_{13} + b_{23}a_{23} + b_{33}a_{33} \end{bmatrix}$$

TVS

$\langle A, B \rangle = Tr(B^T A)$

$= b_{11}a_{11} + b_{21}a_{21} + b_{31}a_{31} +$
$b_{12}a_{12} + b_{22}a_{22} + b_{32}a_{32} +$
$b_{13}a_{13} + b_{23}a_{23} + b_{33}a_{33}$

$=$ Sum { Elementary product of A and B }

$A * B$ = usual matrix product

$A .* B$ → elementwise product of A & B

$\langle A, B \rangle = Tr(B^T A) = Sum\{A .* B\}$

Angle b/w A & B is given by

$$\cos(\theta) = \frac{\langle A, B \rangle}{\|A\| \, \|B\|}$$

$\langle A, B \rangle = Sum\{A .* B\}$

$$= Sum\left\{\begin{bmatrix} 9(1) & 8(2) & 7(3) \\ 6(4) & 5(5) & 4(6) \end{bmatrix}\right\}$$

$= 9 + 16 + 21 + 24 + 25 + 24 = 119$

TVS

$\|A\| = \sqrt{\langle A, A \rangle}$

$\langle A, A \rangle = Tr(A^T A) = Sum(A .* A)$

$$= Sum\left\{\begin{bmatrix} 9^2 & 8^2 & 7^2 \\ 6^2 & 5^2 & 4^2 \end{bmatrix}\right\}$$

$= 81 + 64 + 49 + 36 + 25 + 16 = 271$

$\|A\| = \sqrt{271}$

$\langle B, B \rangle = Tr(B^T B) = Sum\{B .* B\}$

$$= Sum\left[\begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 4^2 & 5^2 & 6^2 \end{bmatrix}\right]$$

$= 1 + 4 + 9 + 16 + 25 + 36 = 91$

$\|B\| = \sqrt{91}$

Angle b/w A & B is given by

$$\cos(\theta) = \frac{119}{\sqrt{271}\sqrt{91}} \Rightarrow \theta = \cos^{-1}\left\{\frac{119}{\sqrt{271}\sqrt{91}}\right\}$$

$= 40.73°$

TVS

Q) Find the angle b/w the vectors $(1,3,-5,4)$ and $(2,-3,4,1)$ in $\mathbb{R}^4$ if the inner product is the usual dot product

→ $\theta = \cos^{-1}\left(\frac{-23}{\sqrt{51}\sqrt{30}}\right)$

---

u and v are orthogonal $\Rightarrow \theta = \frac{\pi}{2}$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \langle u, v \rangle = 0$

Suppose $f(t) = \cos(t)$
$g(t) = \sin(t)$

$t \in [0, \pi]$

$\langle f, g \rangle = \int_0^{\pi} f(t)g(t)\,dt$

$= \int_0^{\pi} \cos t \sin t\,dt = \int_0^{\pi} \frac{\sin 2t}{2}\,dt$

$= -\frac{\cos 2t}{4}\Big|_{t=0}^{\pi}$

$= \cancel{-}\frac{1}{4}[1-1] = 0 \Rightarrow$ $f(t)$ & $g(t)$ orthogonal

TVS

Q) Find the non-zero vector(s) which are orthogonal to $(1,2,1)$ and $(2,5,4)$ in $\mathbb{R}^3$

→ Consider $u = (x, y, z) \in \mathbb{R}^3$

let $v = (1,2,1)$, & $w = (2,5,4)$

Task to find u s.t. $\langle u, v \rangle = 0$ &
$\langle u, w \rangle = 0$

$\langle u, v \rangle = 0 \Rightarrow x + 2y + z = 0$ —(1)
$\langle u, w \rangle = 0 \Rightarrow 2x + 5y + 4z = 0$ —(2)

(1) & (2) $\Rightarrow AX = 0$ $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$R_2 \to R_2 - 2R_1 \Rightarrow A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$AX = 0 \Rightarrow x + 2y + z = 0$
$y + 2z = 0$

let z be k

$\Rightarrow y = -2k$

$x = +\cancel{2}k - k = 3k$

$u = (3k, -2k, k) = k(3, -2, 1)$

Orthogonal complements of a vector v

Consider an inner product space V. For any $v (\neq 0) \in V$, the orthogonal complement denoted by $v^{\perp}$ (read as v perp) is the set of all vectors which are orthogonal to vector v.

ie $v^{\perp} = \{u \in V \mid \langle u, v \rangle = 0\}$

If $S = \{u_1, u_2, u_3 \ldots u_n\}$ is any subset of V, then

$$S^{\perp} = \{u \in V, \langle u, u_1 \rangle = 0, \langle u, u_2 \rangle = 0 \ldots \langle u, u_n \rangle = 0$$

If $W = \text{span}(S)$ then $S^{\perp} = W^{\perp}$

$w \in W \Rightarrow w = c_1 u_1 + \cdots + c_n u_n$

$u \in S^{\perp}$

$\langle u, w \rangle = c_1 \langle u, u_1 \rangle + \cdots + c_n \langle u, u_n \rangle$

$\langle u, w \rangle = c_1 0 + c_2 0 + \cdots + c_n 0 = 0$

$S^{\perp} = W^{\perp}$

$v^{\perp}$ & $S^{\perp} = W^{\perp}$ are subspaces of V

~~Find the~~ Orthogonal Sets & Orthogonal Basis :

A subset $S = \{u_1, u_2 \ldots u_n\}$ of an inner product space V is said to be an orthogonal set if

$\langle u_i, u_j \rangle = 0$ for $i \neq j$

(~~If S is an independent set of ortho~~)

If S is a basis of a subspace of V and S is an orthogonal set, then S is said to be an orthogonal basis.

Orthonormal set:

The subset $S = \{u_1, u_2, \dots u_n\}$ of an inner product space V is an orthonomal set if

$$\langle u_i, u_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$S = \{e_1, e_2, \dots e_n\} \in \mathbb{R}^n$ is an orthonormal basis

But not all standard basis are orthonormal

eg: $S = \{1, t, t^2\} \notin \subset P_2(t)$

Q) Find the orthonormal basis of the orthogonal complement of $u = (1, -2, -1, 3)$

Consider $u = (1, -2, -1, 3)$

Consider $v = \{x, y, z, t\}$

By def^n $u^\perp = \{ v \in \mathbb{R}^4 / \langle v, u \rangle = 0 \}$

$\Rightarrow$ $x(1) + (-2)y + (-1)z + t(3) = 0$

$\Rightarrow$ $x - 2y - z + 3t = 0$

$\rho(A) = 1 = r$ $n = 4$ $\Rightarrow$ $\dim(u^\perp) = n - r = 3$

Using 3 free variable we get basis. We need orthogonal basis.

we find the orthogonal vectors individually (orthogonalisat^n $\rightarrow$ an easier step) (would be learnt later)

y, z, t are free variables

Let $y = 0$, $z = 1$ $t = 0$

then $x - 2(0) - 1 + 3(0) = 0$

$\Rightarrow x = 1$

TVS

$u_1 = (1,0,1,0)$

Now we require $\langle v, u \rangle = 0$ $[\because v \in U^{\perp}]$

and $\langle v, u_1 \rangle = 0$ $[\because v \in$ orthogonal basis$]$

$$\left.\begin{aligned}\langle v, u\rangle = 0 &\Rightarrow x - 2y - z + 3t = 0\\ \langle v, u_1\rangle = 0 &\Rightarrow x + z = 0\end{aligned}\right\}$$

$AX = 0$, $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 1 & 0 & -1 & 0 \end{bmatrix}$

$A \sim \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 2 & 2 & -3 \end{bmatrix}$

$AX = 0$, $\Rightarrow$ $x - 2y - z + 3t = 0$

$2y + 2z - 3t = 0$

z & t are free variable

$z = +1$ $t = 0$ $\Rightarrow$ $2y + 2(1) - 3(0) \Rightarrow$ $y = -1$

$x - 2(-1) - 1 + 3(0) = 0$ $\Rightarrow x = -1$

TVS

$v = (-1,-1,1,0) = u_2$

Now we require $\langle v, u \rangle = 0$ $[\because v \in U^{\perp}]$

& $\langle v, u_1 \rangle = 0$,

$\langle v, u_2 \rangle = 0$ $[\because v \in$ orthogonal basis$]$

$\langle v, u \rangle = 0 \Rightarrow x - 2y - z + 3t = 0$

$\langle v, u_1 \rangle = 0 \Rightarrow x + z = 0$

$\langle v, u_2 \rangle = 0 \Rightarrow -x - y + z = 0$

$AX = 0$, $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + R_1$ $\Rightarrow A \sim \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 2 & 2 & -3 \\ 0 & -3 & 0 & 3 \end{bmatrix}$

$R_3 \to R_3 + \frac{3}{2} R_2 \Rightarrow A \sim \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 2 & 2 & -3 \\ 0 & 0 & 3 & -3/2 \end{bmatrix}$

$AX = 0 \Rightarrow \quad x - 2y - z + 3t = 0$

$2y + 2z - 3t = 0$

$3z - \frac{3}{2}t = 0$

t is a free variable

$t = 2 \quad 3z - \frac{3}{2}(2) = 0 \Rightarrow z = 1$

$2y + 2(1) - 3(2) = 0 \Rightarrow y = 2$

$x - 2(2) - (1) + 3(2) = 0 \Rightarrow x = -1$

$v = (-1, 2, 1, 2) = u_3$

$S = \{ u_1 = (1,0,1,0),$
$u_2 = (-1,-1,1,0),$
$u_3 = (-1,2,1,2) \}$ is an

orthogonal basis of $u^{\perp}$

TVS

Theorem : Any orthogonal set of non-~~linear~~ zero vectors will be linearly independent.

If $S = \{u_1, u_2, \ldots, u_n\}$ is an orthogonal basis of V and $v \in V$ then

$[v]_S = [c_1 \ldots c_n]$ where $c_i = \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle}$

Fourier coefficients.

Q) Show that $S = \{u_1, u_2, \ldots, u_n\}$
$u_1 = (1,1,0,-1)$, $u_2 = (1,2,1,3)$
$u_3 = (1,1,-9,2)$, $u_4 = (16,-13,1,3)$ is an orthogonal basis of $\mathbb{R}^4$. Hence find coordinate vector of $v = (a,b,c,d)$

$\rightarrow$ To show that S is an orthogonal basis. Now S is orthogonal if
$\langle u_i, u_j \rangle = 0$ for $i \neq j$

$\langle u_1, u_2 \rangle = 1(1) + 1(2) + 0(1) + (-1)(3) = 0$

$\langle u_1, u_3 \rangle = 1(1) + 1(1) + 0(-9) + -1(2) = 0$

TVS

$\langle u_1, u_4 \rangle = 1(16) + 1(-13) + 0(1) + (-1)(3) = 0$

$\langle u_2, u_3 \rangle = 0 \quad \langle u_2, u_4 \rangle = 0 \quad \langle u_3, u_4 \rangle = 0$

$\therefore$ Since $\langle u_i, u_j \rangle = 0$ for $i \neq j$

$S = \{u_i\}_{i=1}^{4}$ is an orthogonal set.

Theorem: Any orthogonal set of non-zero vectors is linearly independent.
$\therefore$ S is linearly independent.

→ Any set of linearly independent vectors of an n-dimensional vector space V is a basis of V.

Since $\dim(\mathbb{R}^4) = 4$ and S is linearly independent set of four vectors, S is an orthogonal basis of $\mathbb{R}^4$

$v = (a, b, c, d) = c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4$

$\xi_1 \quad c_i = \dfrac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle}$

TVS

for $i = 1$

$\langle v, u_1 \rangle = a(1) + b(1) + c(0) + d(-1) = a + b - d$

$\langle u_1, u_1 \rangle = 1^2 + 1^2 + 0 + (-1)^2 = 3$

$c_1 = \dfrac{a+b-d}{3}$

$i = 2$, $\langle v, u_2 \rangle = a(1) + b(2) + c(1) + d(3)$
$= a + 2b + c + 3d$

$\langle u_2, u_2 \rangle = 1^2 + 2^2 + 1^2 + 3^2 = 15$

$c_2 = (a + 2b + c + 3d)/15$

$i = 3$ $\langle v, u_3 \rangle = a(1) + b(1) + c(-9) + d(2)$
$= a + b - 9c + 2d$

$\langle u_3, u_3 \rangle = 1^2 + 1^2 + (-9)^2 + 2^2 = 87$

$c_3 = (a + b - 9c + 2d)/87$

$i = 4$ ~~$\langle u_4, u_4 \rangle$~~

$\langle v, u_4 \rangle = a(16) + b(-13) + c(1) + d(3)$

$= 16a - 13b + c + 3d$

$\langle u_4, u_4 \rangle = 16^2 + (-13)^2 + 1^2 + 3^2 = 435$

$c_4 = (16a - 13b + c + 3d)/435$

Q Show that $S = \{(1,1,1,1), (1,1,-1,-1), (1,-1,1,-1), (1,-1,-1,1)\}$ is an orthogonal basis of R^4. Hence find coordinate vector of $(1,3,-5,6)$ relative to S.

S is orthogonal if $\langle u_i, u_j \rangle = 0$ for $i \neq j$

$\langle u_1, u_2 \rangle = 1(1) + 1(1) + 1(-1) + 1(-1) = 0$

$\langle u_1, u_3 \rangle = 0 \qquad \langle u_1, u_4 \rangle = 0$

$\langle u_2, u_3 \rangle = 1(1) + 1(-1) + (-1)(1) + (-1)(-1) = 0$

$\langle u_2, u_4 \rangle = 0 \qquad \langle u_3, u_4 \rangle = 0$

$\therefore$ since $\langle u_i, u_j \rangle = 0$ for $i \neq j$

$S = \{u_i\}^4$ is an orthogonal set of non-zero vectors, S is linearly independent.

Since dimension of R^4 is 4, and S is linearly independent set of four vectors, S is an orthogonal basis of R^4.

$v = (1,3,-5,6) = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4$

$$c_i = \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle}$$

$i=1 \quad \langle v, u_1 \rangle = 1(1) + 3(1) \cancel{+} -5(1) + 6(1) = 5$

$\langle u_1, u_1 \rangle = 1^2 + 1^2 + 1^2 + 1^2 = 4$

$c_1 = 5/4$

$i=2 \quad \langle v, u_2 \rangle = 1(1) + 3(1) + (-5)(-1) + 6(-1) = 3$

$\langle u_2, u_2 \rangle = 4 \qquad c_2 = 3/4$

$i=3 \quad \langle v, u_3 \rangle = 1(1) + 3(-1) + (-5)(1) + 6(-1) = -13$

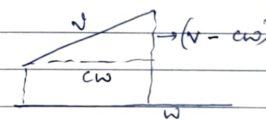
$\langle u_3, u_3 \rangle = 4 \qquad c_3 = \frac{-13}{4}$

$i=4 \quad \langle v, u_4 \rangle = 1(1) + 3(-1) + (-5)(-1) + 6(1) = 9$

$\langle u_4, u_4 \rangle = 4 \qquad c_4 = 9/4$

$$[v]_S = \left[5/4,\ 3/4,\ -13/4,\ 9/4 \right]$$

Projections:



Projection of v on w denoted by $\mathrm{proj}(v,w) = cw$

From diagram $v - cw$ is orthogonal to w

$\langle v - cw, w \rangle = 0 \Rightarrow \langle v, w \rangle - c\langle w, w \rangle = 0$

$$c = \frac{\langle v, w \rangle}{\langle w, w \rangle} \Rightarrow \mathrm{proj}(v,w) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w$$

TVS

$$\text{proj}(\vec{a},\vec{b}) = \frac{\langle \vec{a},\vec{b}\rangle}{\langle \vec{b},\vec{b}\rangle}\vec{b} = \frac{(\vec{a}.\vec{b})}{b^2}\vec{b}$$

$$= \frac{(\vec{a}.\vec{b})}{b}\left(\frac{\vec{b}}{b}\right) = (\vec{a}.\vec{b}).\hat{b}$$

If $w = \text{span}\{s\} \subset V$
$s = \{u_1, u_2, \dots u_n\}$ and $v \in V$ then
projection of v on w i.e
$\text{proj}(v,w) = c_1u_1 + c_2u_2 + \dots + c_nu_n$
where $c_i \frac{\langle v,u_i\rangle}{\langle u_i,u_i\rangle}$

Q) Find the projection of $(1,2,3,4,6)$ on $w = \text{span}\{\underset{u_1}{(1,2,1,2,1)}, \underset{u_2}{(1,-1,2,-1,1)}\}$

→ $v = (1,2,3,4,6)$

$\text{Proj}(v,w) = c_1u_1 + c_2u_2$

$\langle v,u_1\rangle = 1(1) + 2(2) + 3(1) + 4(2) + 6(1) = 22$

$\langle u_1,u_1\rangle = 1^2 + 2^2 + 1^2 + 2^2 + 1^2 = 11 \quad c_1 = \frac{22}{11} = 2$

$\langle v,u_2\rangle = 1(1) + 2(-1) + 3(2) + 4(-1) + 6(1) = 7$

$\langle u_2,u_2\rangle = 1^2 + (-1)^2 + 2^2 + (-1)^2 + 1^2 = 8$

$c_2 = 7/8$

14/8/22

TVS

Q) Projection of
$f(t) = t^2 \quad g(t) = t+2$ in $P(t)$
when $\langle u,v\rangle = \int_0^1 u(t)v(t)\,dt$

→ $\text{Proj}(f(t),g(t)) = \frac{\langle f,g\rangle}{\langle g,g\rangle}g(t)$

$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$

$= \int_0^1 t^2(t+2)\,dt = \int_0^1 (t^3+2t^2)\,dt$

$\langle f,g\rangle = \left[\frac{t^4}{4} + \frac{2t^3}{3}\right]_{t=0}^1 = \frac{1}{4} + \frac{2}{3} = \frac{11}{12}$

$\langle g,g\rangle = \int_0^1 g^2\,dt = \int_0^1 (t+2)^2\,dt = \left.\frac{(t+2)^3}{3}\right|_{t=0}^1$

$= \frac{3^3-2^3}{3} = \frac{19}{3}$

$\text{Proj}(f(t),g(t)) = \frac{(11/12)}{19/3}\cdot(t+2) = \frac{11}{76}(t+2)$

Q Find the projection of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ on $B = \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$ in $M_{2\times 2}$ when

$\langle A, B \rangle = Tr(B^T A)$

$\rightarrow$ $Proj(A, B) = \dfrac{\langle A, B\rangle}{\langle B, B \rangle} \cdot B$

$\langle A, B \rangle = Tr(B^T A) = Sum\{A.*B\}$
where $A.*B$ is the elementwise product of A & B.

$= Sum\left\{\begin{bmatrix} 1(1) & 2(1) \\ 3(5) & 4(5) \end{bmatrix}\right\} = 1+2+15+20$
$= 38$

$\langle B, B \rangle = Sum\{B.*B\} = Sum\left\{\begin{bmatrix} 1^2 & 1^2 \\ 5^2 & 5^2 \end{bmatrix}\right\}$

$= 1+1+25+25 = 52$

$Proj(A, B) = \dfrac{38}{52}\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} = \dfrac{19}{26}\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$

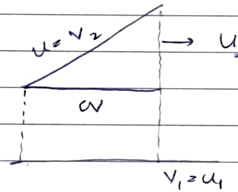
$= \begin{bmatrix} 19/26 & 19/26 \\ 95/26 & 95/26 \end{bmatrix}$

Q Find the projection of $(1, 3, 1, 2)$ on $(1, -2, 7, 4)$ in $\mathbb{R}^4$ w.r.t the usual dot product.

$\rightarrow$ $proj(u, v) = \dfrac{1}{7}(1, -2, 7, 4) = \left(\dfrac{1}{7}, \dfrac{-2}{7}, \dfrac{7}{7}, \dfrac{4}{7}\right)$

Gram - Schmidt orthogonalization process

If $u_1, u_2, u_3 \ldots u_n$ be linearly independent,



$\rightarrow u_2 \neq c v_1$

$v_1 = u_1$

$v_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} \cdot v_1$

$v_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1 - \dfrac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} \cdot v_2$

$v_4 = u_4 - \dfrac{\langle u_4, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \dfrac{\langle u_4, v_2\rangle}{\langle v_2, v_2\rangle} \cdot v_2 - \dfrac{\langle u_4, v_3\rangle}{\langle v_3, v_3\rangle} \cdot v_3$

TVS

$$v_n = u_n - \frac{\langle u_n, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \cdots - \frac{\langle u_n, v_{n-1}\rangle}{\langle v_{n-1}, v_{n-1}\rangle} v_{n-1}$$

<u>Note</u>: If vectors u & v are orthogonal then $k_1 u$ & $k_2 v$ are also orthogonal.

Q) Construct an orthogonal basis and hence an orthonormal basis of the subspace spanned by $(1,1,1,1)$ & $(1,2,4,5)$ and $(1,-3,-4,-2)$ in $\mathbb{R}^4$.

→ $u_1 = (1,1,1,1)$, $u_2 = (1,2,4,5)$, $u_3 = (1,-3,-4,-2)$

$v_1 = u_1 = (1,1,1,1)$

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$

$\langle u_2, v_1\rangle = 1+2+4+5 = 12$

$\langle v_1, v_1\rangle = 1+1+1+1 = 4$

$v_2 = (1,2,4,5) - \frac{12}{4}(1,1,1,1) = (-2,-1,1,2)$

$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$

TVS

$\langle u_3, v_1\rangle = 1-3-4-2 = -8$

$\langle u_3, v_2\rangle = 1(-2) + (-3)(-1) + (-4)(1) + (-2)(2) = -7$

$\langle v_2, v_2\rangle = (-2)^2 + (-1)^2 + 1^2 + 2^2 = 10$

$v_3 = (1,-3,-4,-2) - \left(\frac{-8}{4}\right)(1,1,1,1)$

$- \frac{-7}{10}(-2,-1,1,2)$

$= (1,-3,-4,-2) + (2,2,2,2) + \left(\frac{7}{5}, \frac{-7}{10}, \frac{7}{10}, \frac{7}{5}\right)$

$= \left(\frac{8}{5}, \frac{-17}{10}, \frac{-13}{10}, \frac{7}{5}\right)$

⇒ let $v_3 = (16, -17, -13, 14)$

$S = \{v_1 = (1,1,1,1), v_2 = (-2,-1,1,2),$

$v_3 = (16,-17,-13,14)\}$ is an orthogonal basis of span $\{u_1, u_2, u_3\}$

$w_1 = \frac{v_1}{\|v_1\|}$, $w_2 = \frac{v_2}{\|v_2\|}$, $w_3 = \frac{v_3}{\|v_3\|}$

$\|v_1\| = \sqrt{\langle v_1, v_1\rangle} = \sqrt{4} = 2$

$\|v_2\| = \sqrt{10}$

$\langle v_3, v_3 \rangle = 16^2 + (-17)^2 + (-13)^2 + 14^2$
$= 910$

$\| v_3 \| = \sqrt{910}$

$w_1 = \frac{1}{2}(1,1,1,1)$ $\quad w_2 = \frac{1}{\sqrt{10}}(-2,-1,1,2)$

$w_3 = \frac{1}{\sqrt{910}}(16,-17,-13,14)$

$T = \{w_1, w_2, w_3\}$ is an orthonormal basis.

17/8/22

Q) Obtain the orthonormal basis of subspace W spanned by
$S = \{u_1 = (1,1,1,1), u_2 = (1,-1,2,2), u_3 = (1,2,-3,-4)\}$

$\rightarrow v_1 = u_1 = (1,1,1,1)$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$\langle u_2, v_1 \rangle = 1(1) + 1(-1) + 1(2) + 1(2) = 4$

$\langle v_1, v_1 \rangle = 1^2 + 1^2 + 1^2 + 1^2 = 4$

$v_2 = (1,-1,2,2) - \left(\frac{4}{4}\right)(1,1,1,1) = (0,-2,1,1)$

$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$

TVS

$\langle u_3, v_1 \rangle = 1(1) + 2(1) + (-3)(1) + (-4)(1) = -4$

$\langle u_3, v_2 \rangle = 1(0) + 2(-2) + (-3)(1) + (-4)(1)$
$= -11$

$\langle v_2, v_2 \rangle = 0^2 + (-2)^2 + 1^2 + 1^2 = 6$

$v_3 = (1,2,-3,-4) - \left(\frac{-4}{4}\right)(1,1,1,1)$

$- \left(\frac{-11}{6}\right)(0,-2,1,1)$

$= (2,3,-2,-3) - \frac{11}{6}(0,-2,1,1)$

$\cancel{= \left(2, \frac{20}{3}, -\frac{55}{6}, -\frac{29}{6}\right)}$

$= \left(2, -\frac{2}{3}, -\frac{1}{6}, -\frac{7}{6}\right)$

v_3 is orthogonal to v_1 & v_2

$\Rightarrow kv_3$ is also orthogonal

$\therefore v_3 = (12,-4,-1,-7)$

$T = \{v_1, v_2, v_3\}$ is an orthogonal basis

TVS

$w_1 = \frac{v_1}{\|v_1\|}$, $w_2 = \frac{v_2}{\|v_2\|}$, $w_3 = \frac{v_3}{\|v_3\|}$

$\|v_1\| = \sqrt{\langle v_1, v_1\rangle}$

$w_1 = \frac{(1,1,1,1)}{\sqrt{4}}$ $= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$w_2 = \frac{(0,-2,1,1)}{\sqrt{6}}$ $= \left(0, \frac{-2}{\sqrt{6}}, \frac{+1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$\langle v_3, v_3\rangle = 12^2 + (-4)^2 + (-1)^2 + (-7)^2$
$= 144 + 16 + 1 + 49 = 210$

16
1
49
10

$w_3 = \frac{(12, -4, -1, 7)}{\sqrt{210}}$

$X = \{w_1, w_2, w_3\}$ is an orthonormal basis of $W = span(S)$

Q) Obtain an orthonormal basis of $P_2(t)$ whose basis is $S = \{1, t, t^2\}$ and $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$

TVS

→ $S = \{u_1 = 1, u_2 = t, u_3 = t^2\}$

$v_1 = u_1 = 1$

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$

$\langle u_2, v_1\rangle = \int_0^1 u_2 \cdot v_1 \, dt = \int_0^1 t(1)\, dt$

$= \frac{t^2}{2}\Big|_0^1 = \frac{1-0}{2}$

$= \frac{1}{2}$

$\langle v_1, v_1\rangle = \int_0^1 v_1^2 \, dt = \int_0^1 1 \, dt$

$= t\Big|_0^1 = 1$

$v_2 = t - \frac{1/2}{1}\, 1 = t - \frac{1}{2}$

let $v_2 = LCM \times v_2 = 2t - 1$

$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$

$\langle u_3, v_1\rangle = \int_0^1 u_3 v_1 \, dt = \int_0^1 t^2 \, dt$

$= \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$

$$\langle u_3, v_2 \rangle = \int_0^1 u_3 v_2 \, dt = \int_0^1 t^2 (2t-1) \, dt$$

$$= \int_0^1 (2t^3 - t^2) \, dt = \left[\frac{2t^4}{4} - \frac{t^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\langle v_2, v_2 \rangle = \int_0^1 v_2^2 \, dt = \int_0^1 (2t-1)^2 \, dt$$

$$\left[\because \int (ax+b)^n = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1}, \quad n \neq -1 \right] = \frac{1}{2} \frac{(2t-1)^3}{3} \Big|_0^1$$

$$= \frac{1}{6} \left\{ 1^3 - (-1)^3 \right\} = \frac{2}{6}$$

$$= \frac{1}{3}$$

$$v_3 = t^2 - \frac{\frac{1}{3}}{1} - \frac{\frac{1}{6}}{\frac{1}{3}} (2t-1)$$

$$= t^2 - \frac{1}{3} - \frac{1}{2}(2t-1) = t^2 - t + \frac{1}{6}$$

let $v_3 = \mathrm{LCM} \times v_3 = 6t^2 - 6t + 1$

$S = \{v_1, v_2, v_3\}$ is an orthogonal basis of $P_2(t)$

$$w_1 = \frac{v_1}{\|v_1\|} \qquad w_2 = \frac{v_2}{\|v_2\|} \qquad w_3 = \frac{v_3}{\|v_3\|}$$

$$w_1 = \frac{1}{1} = 1, \quad w_2 = \frac{2t-1}{\sqrt{1/3}} = \cancel{\frac{6t-3}{}} \; \sqrt{3}(2t-1)$$

~~w_3~~

$$= 2\sqrt{3}t - \sqrt{3}$$

$$\langle v_3, v_3 \rangle = \int_0^1 v_3^2 \, dt = \int_0^1 (6t^2 - 6t + 1)^2 \, dt$$

$$= \frac{1}{\cancel{33}\,5}$$

$$36t^4 - 72t^3 + 48t^2 - 12t + 1 \quad \cancel{\frac{6t^3}{3} - \frac{6t^2}{2} + t}$$

$$w_3 = \frac{6t^2 - 6t + 1}{\sqrt{\frac{1}{\cancel{33}\,5}}} = \sqrt{5}\,(6t^2 - 6t + 1)$$

Orthogonal Matrix:

A square matrix A is said to be orthogonal if $A^T A = A A^T = I$

Theorem: If ~~α~~ A is a square matrix then the following statements are equivalent.

TVS

i) A is orthogonal
ii) Rows of A are orthonormal
iii) Columns of A are orthonormal

Q) Construct an orthogonal matrix A whose 1st row is $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

→ (~~Note~~) We have $u_1 = \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$

we need u_2, u_3 which are orthogonal to u_1

$u = (x, y, z)$ and $\langle u, u_1 \rangle = 0$
then $x\left(\frac{1}{3}\right) + y\left(\frac{2}{3}\right) + z\left(\frac{2}{3}\right) = 0$

$x + 2y + 2z = 0$

$y = k_1$ and $z = k_2 \Rightarrow x = -2k_1 - 2k_2$

$u = (-2k_1 - 2k_2, k_1, k_2) = k_1(-2, 1, 0) + k_2(-2, 0, 1)$

$\underbrace{u_2 = (-2, 1, 0) \quad u_3 = (-2, 0, 1)}$ basis of orthogonal complement basis of u_1

$\langle u_1, u_2 \rangle = 0 \quad \langle u_1, u_3 \rangle = 0$ but
$\langle u_2, u_3 \rangle \neq 0$

TVS

$\therefore$ let $v_2 = u_2 = (-2, 1, 0)$

$$v_3 = u_3 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$\langle v_3, u_2 \rangle = (-2)(-2) + 1(0) + 0(1) = 4$

$\langle v_2, v_2 \rangle = (-2)^2 + (1)^2 + 0^2 = 5$

$v_3 = (-2, 0, 1) - \frac{4}{5}(-2, 1, 0) = \left(-\frac{2}{5}, -\frac{4}{5}, 1\right)$

$v_3 = (-2, -4, 5) \quad [\because v_3 = \text{LCM} \times v_3]$

$\{u_1, v_2, v_3\}$ is an orthogonal set

$w_1 = \frac{u_1}{\|u_1\|} \qquad w_2 = \frac{v_2}{\|v_2\|} \qquad w_3 = \frac{v_3}{\|v_3\|}$

$\langle u_1, u_1 \rangle = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = 1$

$\langle v_2, v_2 \rangle = (-2)^2 + (1)^2 + (0)^2 = 5$

$\langle v_3, v_3 \rangle = (-2)^2 + (4)^2 + (5)^2 = 4 + 16 + 25 = 45$

$w_2 = \frac{1}{\sqrt{5}}(-2, 1, 0) = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$

$w_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$w_3 = \frac{(-2, -4, 5)}{\sqrt{45}}$

TVS

$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/\sqrt{45} & \frac{-4}{\sqrt{45}} & \frac{5}{\sqrt{45}} \end{bmatrix} \quad \text{with } A^TA = AA^T = I$$

Q) Find an orthogonal matrix whose 1st two rows are ~~multiples~~ linear combination of $(1,1,1)$ and $(1,-2,3)$

$\Rightarrow$ $u_1 = (1,1,1)$ $\quad u_2 = (1,-2,3)$

we have $v_1 = u_1$ and $v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$

$= (1,-2,3) - \frac{(1-2+3)}{1+1+1}(1,1,1)$

$= (1,-2,3) - \frac{2}{3}(1,1,1) = \left(\frac{1}{3}, \frac{-8}{3}, \frac{7}{3}\right)$

let $v_2 = (1,-8,7)$ $\quad [\because v_2 = \text{LCM} \times v_2]$

We need v_3 s.t. $\langle v_3, v_1\rangle = 0$ & $\langle v_3, v_2\rangle = 0$

let $v_3 = (x,y,z)$

$\langle v_3, v_1\rangle = 0 \Rightarrow x+y+z = 0$

$\langle v_3, v_2\rangle = 0 \Rightarrow x - 8y + 7z = 0$

$AX = 0$ with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 7 \end{bmatrix}$

TVS

$R_2 \to R_2 - R_1 \Rightarrow A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -9 & 6 \end{bmatrix}$

$AX = 0 \Rightarrow x+y+z = 0$

$9y + 6z = 0$

$z = 3k \Rightarrow -9y + 18k = 0 \Rightarrow y = 2k$

$x + 2k + 3k = 0 \Rightarrow x = -5k$

$X = (-5k, 2k, 3k) = k(-5, 2, 3)$

$v_3 = (-5, 2, 3)$

$w_1 = \frac{v_1}{\|v_1\|}$ $\quad w_2 = \frac{v_2}{\|v_2\|}$ $\quad w_3 = \frac{v_3}{\|v_3\|}$

$\langle v_1, v_1\rangle = 1^2+1^2+1^2 = 3 \Rightarrow \|v_1\| = \sqrt{3}$

$w_1 = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\langle v_2, v_2\rangle = 1^2 + (-8)^2 + 7^2 = \frac{114}{8} \Rightarrow \|v_2\| = \sqrt{114}$

$w_2 = \frac{(1,-8,7)}{\sqrt{114}} = \left(\frac{1}{\sqrt{114}}, \frac{-8}{\sqrt{114}}, \frac{7}{\sqrt{114}}\right)$

$\langle v_3, v_3\rangle = (-5)^2 + 2^2 + 3^2 = 38$

$w_3 = \frac{(-5,2,3)}{\sqrt{38}} = \left(\frac{-5}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}\right)$

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{114}} & \frac{-8}{\sqrt{114}} & \frac{7}{\sqrt{114}} \\ \frac{-5}{\sqrt{38}} & \frac{2}{\sqrt{38}} & \frac{3}{\sqrt{38}} \end{bmatrix}$$ is the required orthogonal matrix

QR factorisation

If A is a matrix with independent columns, then it can be factorised as $A = QR$ where Q is a matrix with orthonormal columns;
$R \rightarrow$ upper triangular matrix with diagonals positive. $= Q^TA$

Q) Obtain the QR factorisation of $A =$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

→ We have $u_1 = (1,1,1,0)$,
$u_2 = (-1,0,-1,1)$,
$u_3 = (-1,0,0,-1)$

(Choose columns only and orthonormalise them)

$v_1 = u_1 = (1,1,1,0)$

$v_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$

$\langle u_2, v_1\rangle = (-1)(1) + 0(1) + (-1)(1) + 1(0)$
$= -2$

$\langle v_1, v_1\rangle = 1^2 + 1^2 + 1^2 + 0 = 3$

$v_2 = (-1,0,-1,1) + \left(\dfrac{2}{3}\right)(1,1,1,0)$

$= \left(\dfrac{-1}{3}, \dfrac{2}{3}, \dfrac{-1}{3}, 1\right)$

let $v_2 = \text{LCM} * v_2 \Rightarrow v_2 = (-1, 2, -1, 3)$

$v_3 = u_3 - \dfrac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \dfrac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$

$\langle u_3, v_1\rangle = (-1)(1) + 0(1) + 0(1) + 1(0) = -1$
$\langle u_3, v_2\rangle = (-1)(-1) + 0(2) + 0(-1) + (-1)(3) = -2$
$\langle v_2, v_2\rangle = (-1)^2 + 2^2 + (-1)^2 + 3^2 = 15$

$v_1 = (-1,0,0,-1) - \left(\dfrac{-1}{3}\right)(1,1,1,0) - \left(\dfrac{-2}{15}\right)(-1,2,-1,3)$

$= \left(-\dfrac{4}{5}, \dfrac{3}{5}, \dfrac{1}{5}, \dfrac{-3}{5}\right)$

$v_3 = (-4, 3, 1, -3)$

$w_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{(1,1,1,0)}{\sqrt{3}} = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, 0\right)$

$w_2 = \dfrac{v_2}{\|v_2\|} = \dfrac{(-1, 2, -1, 3)}{\sqrt{15}}$

TVS

$\langle v_3, v_3 \rangle = (-4)^2 + 3^2 + 1^2 + (-3)^2$
$= 35$

$w_3 = \frac{v_3}{\|v_3\|} = \frac{(-4, 3, 1, -3)}{\sqrt{35}}$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{15} & -4/\sqrt{35} \\ 1/\sqrt{3} & 2/\sqrt{15} & 3/\sqrt{35} \\ 1/\sqrt{3} & -1/\sqrt{15} & 1/\sqrt{35} \\ 0 & 3/\sqrt{15} & -3/\sqrt{35} \end{bmatrix}$$

$R = Q^T A = [r_{ij}]$ $\quad r_{ji} = \langle u_i, w_j \rangle$
$\quad$ upper $\Delta \Rightarrow r_{ij} = 0$ for $i > j$

$r_{11} = \langle u_1, w_1 \rangle = 1\left(\frac{1}{\sqrt{3}}\right) + 1\left(\frac{1}{\sqrt{3}}\right) + 1\left(\frac{1}{\sqrt{3}}\right) + 0$
$= \frac{3}{\sqrt{3}} = \sqrt{3}$

$r_{12} = \langle u_2, w_1 \rangle = -1\left(\frac{+1}{\sqrt{3}}\right) + 0 + -1\left(\frac{+1}{\sqrt{3}}\right) + 1\left(\frac{0}{\sqrt{3}}\right)$
$= \frac{-1}{\sqrt{3}} + \frac{(-1)}{\sqrt{3}} = \frac{-2}{\sqrt{3}}$

$r_{13} = \langle u_3, w_1 \rangle = (-1)\frac{1}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$

TVS

$r_{21} = \langle u_1, w_2 \rangle = 0$ $\quad$ obtained from v_2, v_2 is orthonormal to $u_1 \Rightarrow 0$

$r_{22} = \langle u_2, w_2 \rangle = -1\left(\frac{-1}{\sqrt{15}}\right) + 1\left(3/\sqrt{15}\right) + (-1)\left(\frac{1}{\sqrt{15}}\right)$
$= 5/\sqrt{15}$

$r_{23} = \langle u_3, w_2 \rangle = -1\left(-1/\sqrt{15}\right) + -1\left(3/\sqrt{15}\right) = \frac{-2}{\sqrt{15}}$

$r_{31} = 0$, $r_{32} = 0$

$r_{33} = \langle u_3, w_3 \rangle = (-1)\left(\frac{-4}{\sqrt{35}}\right) - 1\left(\frac{-3}{\sqrt{35}}\right)$
$= 7/\sqrt{35}$

$$R = \begin{bmatrix} \sqrt{3} & -2/\sqrt{3} & \frac{-1}{\sqrt{3}} \\ 0 & 5/\sqrt{15} & -2/\sqrt{15} \\ 0 & 0 & 7/\sqrt{15} \end{bmatrix} \qquad A = QR$$

26/8/22

Method of least squares

Consider $AX = B$ is an overdetermined system.

Suppose $AX = B$ is inconsistent
$\Rightarrow \|A\hat{X} - B\| \neq 0$ where $\hat{X}$ is an approximation to the solution of $AX = B$

TVS

$a_1 x_{\cancel{x}} = b_1 \qquad a_2 x_{\cancel{2}} = b_2 \qquad a_3 x = b_3$

with $\frac{a_1}{b_1} \neq \frac{a_2}{b_2} \neq \frac{a_3}{b_3}$

$E_1 = a_1 x - b_1 \qquad E_2 = a_2 x - b_2 \qquad E_3 = a_3 x - b_3$

$$E_{sq} = E_1^2 + E_2^2 + E_3^2$$
$$= (a_1 x - b_1)^2 + (a_2 x - b_2)^2 + (a_3 x - b_3)^2$$

E_{sq} is min/max when $\frac{dE_{sq}}{dx} = 0$

$$\frac{dE_{sq}}{dx} = 2(a_1 x - b_1)a_1 + 2(a_2 x - b_2)a_2 + 2(a_3 x - b_3)a_3$$

$$\frac{dE_{sq}}{dx} = 0 \Rightarrow (a_1 x - b_1)a_1 + (a_2 x - b_2)a_2 + (a_3 x - b_3)a_3 = 0$$

$$a_1^2 x - a_1 b_1 + a_2^2 x - a_2 b_2 + a_3^2 x - a_3 b_3 = 0$$

$$(a_1^2 + a_2^2 + a_3^2) x = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$x = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{a_1^2 + a_2^2 + a_3^2} \quad \text{--- } (2)$$

TVS

$AX = B \qquad A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad X = [x] \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$x = \frac{A^T B}{A^T A}$ [$\because$ From (2)]

(1) $\Rightarrow$ $[(A^T A) X = A^T B]$

The least square error is $\|AX - B\|$

Q) Determine the least square error solution and the least square error for the system of equation $AX = B$ where

$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

$\rightarrow$ $AX = B \Rightarrow (A^T A) X = A^T B$

$$A^T A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 3 \end{bmatrix}$$

TVS

$$A^TB = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$(A^TA)x = A^TB$$

$$\begin{bmatrix} 6 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \qquad X = \begin{bmatrix} \frac{24}{17} \\ \frac{-8}{17} \end{bmatrix}$$

$$AX = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 24/17 \\ -8/17 \end{bmatrix} = \begin{bmatrix} \frac{40}{17} \\ 24/17 \\ 8/17 \\ -32/17 \end{bmatrix}$$

$$AX - B = \begin{bmatrix} \frac{-11}{17} \\ 7/17 \\ -26/17 \\ -15/17 \end{bmatrix}$$

$$\|AX-B\| = \sqrt{\langle AX-B, AX-B\rangle}$$
$$= \sqrt{\frac{63}{17}}$$
$$= 1.92506$$

TVS

⑧ Determine for

$AX = B$ where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ -2 \end{bmatrix}$

$\rightarrow$ $AX = B \Rightarrow (A^TA)X = A^TB$

$$A^TA = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 0 & 1 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 18 & 14 \\ 18 & 39 & 24 \\ 14 & 24 & 16 \end{bmatrix}$$

$$A^TB = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 0 & 1 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$$

$$(A^TA)X = A^TB \Rightarrow X = \begin{bmatrix} -23/15 \\ -28/15 \\ 64/15 \end{bmatrix}$$ is the least square error soln

$$AX = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -23/15 \\ -28/15 \\ 64/15 \end{bmatrix} = \begin{bmatrix} 97/15 \\ 63/5 \\ \frac{286}{15} \\ 6/5 \\ 23/15 \end{bmatrix} \qquad AX - B = \begin{bmatrix} 112/15 \\ 11/5 \\ 286/15 \\ 1/5 \\ 53/15 \end{bmatrix}$$

TVS

~~$\|AX-B\| = 20.896$~~

2/9/22

UNIT-5 (Symmetric Matrices & Quadratic forms)

Q) Is $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ diagonalizable?

If yes, then find the modal matrix of the transformation

→ $A = \frac{1}{3}M \qquad M = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

If λ is an eigenvalue of M, then $\frac{1}{3}\lambda$ is the eigen value of $A = \frac{1}{3}M$

$|M - \lambda I| = 0 \Rightarrow -\lambda^3 + Tr(M)\lambda^2 - \{\text{sum of principle minors}\}\lambda + |M| = 0$

$-\lambda^3 + 3\lambda^2 - \left\{\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\right\}\lambda - 27 = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 + 9\lambda - 27 = 0 \Rightarrow \lambda = 3, -3$

$\lambda_1 + \lambda_2 + \lambda_3 = 3$

TVS

$3 + (-3) + \lambda_3 = 3 \Rightarrow \lambda_3 = 3$

Eigenvalues of $M = 3, 3, -3$

$\therefore$ eigen values of A are $1, 1, -1$

$\lambda = 1 \Rightarrow A - \lambda I = A - I = \begin{bmatrix} -2/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & -2/3 \\ 2/3 & -2/3 & -2/3 \end{bmatrix}$

$(A - I) \sim \begin{bmatrix} -2/3 & 2/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(A - I)X = 0 \Rightarrow -\frac{2}{3}x + \frac{2}{3}y + \frac{2}{3}z = 0$

$y = k_1, \; z = k_2 \Rightarrow -x + k_1 + k_2 = 0$

$\Rightarrow x = k_1 + k_2$

$X = [k_1 + k_2 \quad k_1 \quad k_2]^T = k_1[1 \quad 1 \quad 0]^T + k_2[1 \quad 0 \quad 1]^T$

$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -1 \Rightarrow A - \lambda I = A + I = \begin{bmatrix} 4/3 & 2/3 & 2/3 \\ 2/3 & 4/3 & -2/3 \\ 2/3 & -2/3 & 4/3 \end{bmatrix}$

$(A+I) \sim \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \left[\because R_1 \to \frac{3}{2}R_1,\ R_2 \to \frac{3}{2}R_2\right]$

$(A+I) \sim \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$

$(A+I)X = 0 \Rightarrow 2x + y + z = 0$

$\frac{3}{2}y + \frac{-3}{2}z = 0$

$z = k \Rightarrow y - k = 0 \Rightarrow y = k$

$2x + 2k = 0$

$\Rightarrow x = -k$

$X = \begin{bmatrix} -k & k & k \end{bmatrix}^T = k\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T$

$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

and $P^{-1}AP = D$

5) Orthogonally diagonalise $\begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ 4 & 8 & 17 \end{bmatrix}$

→ Orthogonal diagonalisation → the modal matrix P should be orthogonal

⇒ eigenvectors should be orthonormal.

$|A - \lambda I| = 0 \Rightarrow -\lambda^3 + Tr(A)\lambda^2 - \left\{\begin{matrix}\text{sum of} \\ \text{principle minors}\end{matrix}\right\}\lambda + |A| = 0$

$-\lambda^3 + 24\lambda^2 - 45\lambda + 22 = 0 \Rightarrow \lambda = 22, -1$

$\lambda_1 + \lambda_2 + \lambda_3 = Tr(A) \Rightarrow \lambda_3 = 24 - 1 - 22$

$= 1$

$\lambda = 22 \Rightarrow A - 22I = \begin{bmatrix} -20 & 2 & 4 \\ 2 & -17 & 8 \\ 4 & 8 & -5 \end{bmatrix}$

$(A - 22I) \sim \begin{bmatrix} 2 & -17 & 8 \\ 4 & 8 & -5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & -17 & 8 \\ 0 & 42 & -21 \\ 0 & 0 & 0 \end{bmatrix}$

$(A - 22I)X = 0 \Rightarrow 2x - 17y + 8z = 0$

$42y - 21z = 0$

TVS

$z = 2k \Rightarrow 21\{2y - 2k\} = 0 \Rightarrow y = k$

$2x - 17(k) + 8(2k) = 0$

$\Rightarrow 2x = k \Rightarrow x = \frac{k}{2}$

$X = \begin{bmatrix} k/2 & k & 2k \end{bmatrix}^T = \frac{k}{2}\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$

$\lambda = 1 \Rightarrow A - \lambda I = A - I = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$

$(A - I) \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(A - I)X = 0 \Rightarrow x + 2y + 4z = 0$

$y = k_1, \; z = k_2 \Rightarrow x = -2k_1 - 4k_2$

$X = \begin{bmatrix} -2k_1 - 4k_2 & k_1 & k_2 \end{bmatrix}^T$

$= k_1\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^T + k_2\begin{bmatrix} -4 & 0 & 1 \end{bmatrix}^T$

we have $X_1 = \underset{u_1}{(1, 2, 4)}$ $\quad X_2 = \underset{u_2}{(-2, 1, 0)}$

$X_3 = (-4, 0, 1) = u_3$

TVS

$v_1 = u_1 = (1, 2, 4)$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = (-2, 1, 0) - \frac{0}{21}(1, 2, 4)$

$= (-2, 1, 0)$

$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$

$= (-4, 0, 1) - \frac{0}{21}(1, 2, 4) - \frac{8}{5}(-2, 1, 0)$

$v_3 = \left(-\frac{4}{5}, -\frac{8}{5}, 1\right)$

let $v_3 = (-4, -8, 5)$

$w_1 = \frac{v_1}{\|v_1\|} \qquad w_2 = \frac{v_2}{\|v_2\|} \qquad w_3 = \frac{v_3}{\|v_3\|}$

$\|v\| = \sqrt{\langle v, v \rangle}$

$w_1 = \frac{(1, 2, 4)}{\sqrt{21}} \qquad w_2 = \frac{(-2, 1, 0)}{\sqrt{5}} \qquad w_3 = \frac{(-4, -8, 5)}{\sqrt{105}}$

$P = \begin{bmatrix} \frac{1}{\sqrt{21}} & \frac{-2}{\sqrt{5}} & \frac{-4}{\sqrt{105}} \\ \frac{2}{\sqrt{21}} & \frac{1}{\sqrt{5}} & \frac{-8}{\sqrt{105}} \\ \frac{4}{\sqrt{21}} & 0 & \frac{5}{\sqrt{105}} \end{bmatrix}$ & $D = \begin{bmatrix} 22 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

where $P^{-1} = P^T$ &

TVS

$P^{-1}AP = D \Rightarrow P^{T}AP = D$

8) Orthogonally diagonalize $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \\ 2 & -2 & 3 \end{bmatrix}$

$\Rightarrow A - \lambda I = 0$

$-\lambda^3 + Tr(A)\lambda^2 - \{\text{sum of principle minors}\}\lambda + \det(A) = 0$

$-\lambda^3 + 9\lambda^2 - \{5 + 5 + 5\}\lambda - 25 = 0$

$-\lambda^3 + 9\lambda^2 - 15\lambda - 25 = 0 \Rightarrow \lambda = -1, 5$

(-ve det $\Rightarrow$ one -ve eigen value)

$\lambda_1 + \lambda_2 + \lambda_3 = 9 \Rightarrow \lambda_3 = 9 - 5 + 1 = 5$

For $\lambda = -1$,

$A + I = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & -2 \\ 2 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 4 & 2 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

let $z = k$

$4x + 2y + 2k = 0$

$3y = 3k \Rightarrow y = k$

$4x = -2k - 2k = -4k$

$x = -1$

TVS

$X_1 = k\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T$

for $\lambda = 5$

$A - 5I = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & -2 \\ 2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

let $y = k_1$, $z = k_2$

$-2x = -2k_1 - 2k_2 \Rightarrow x = k_1 + k_2$

$X_2 = k_2\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T \quad X_3 = k_3\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$

We have the eigen vectors, $u_1 = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix}$

$u_2 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$

$v_1 = u_1$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$= (1, 1, 0) - \left(\frac{-1+1}{3}\right) \cdot 3 v_1 = 1 \; 1 \; 0$

$v_3 = u_3 - \frac{\langle u_3 v_1 \rangle}{\langle v_1 v_1 \rangle} v_1 - \frac{\langle u_3 v_2 \rangle}{\langle v_2 v_2 \rangle} v_2$

$= u_3 - 0 - \frac{1}{2} (1\,1\,0) = \left(\frac{1}{2} \; -\frac{1}{2} \; 1\right)$

$$w_1 = \frac{v_1}{\|v_1\|} \qquad w_2 = \frac{v_2}{\|v_2\|} \qquad w_3 = \frac{v_3}{\|v_3\|}$$

$$= \left(-\tfrac{1}{\sqrt3} \quad \tfrac{1}{\sqrt3} \quad \tfrac{1}{\sqrt3}\right) \qquad \left(\tfrac{1}{\sqrt2} \quad \tfrac{1}{\sqrt2} \quad 0\right) \qquad \left(\tfrac{1}{\sqrt2} \quad 0 \quad \tfrac{1}{\sqrt2}\right)$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt3} & 0 & \frac{1}{\sqrt2} \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

14/9/22

U & V → orthogonal

Σ → diagonal

Singular Value Decomposition:

$$A_{m\times n} = \underset{\text{rotate}}{U_{m\times m}} \; \underset{\text{stretch}}{\Sigma_{m\times n}} \; \underset{\text{rotate}}{\left(V^T_{n\times n}\right)}$$

U → matrix with orthonormal eigenvectors of AA^T

Σ = matrix of singular value $\left(\sqrt{\text{positive eigen values}}\right)$ of AA^T or A^TA

V → matrix with orthonormal eigenvectors of $(A^TA)_{n\times n}$

~~Suppose~~ If $n < m$ then we find V first (else U)

~~If~~ ∴ $u_1 = \frac{1}{\sigma_1}(Av_1)$ for $\sigma_1 \neq 0$

$\sigma_i = \sqrt{\lambda_i}$ λ_i = non-zero eigen values of A^TA or AA^T

TVS

— Method used by dept Math

Q) Obtain singular value decomposition of $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{2\times 3}$

$\rightarrow$ $size(A) = 2\times 3 \Rightarrow size(AA^T) = 2\times 2$
and $size(A^TA) = 3\times 3$

$\therefore AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$AA^T - \lambda I = 0$

$\lambda^2 - Tr(AA^T)\lambda + |AA^T| = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\lambda = 3, 1$ (Arrange in desc order)

$\sigma = \sqrt{\lambda} = \sqrt{3}, 1$

$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (appended to make $size(\Sigma) = size(A)$)

$(AA^T - \lambda I)X = 0 \quad X = ?$

$\lambda = 3 \Rightarrow AA^T - \lambda I = (AA^T - 3I)$

TVS

$= \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$

$-x - y = 0$

$y = k \Rightarrow x = -k \Rightarrow X = \begin{bmatrix} -k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 1 \Rightarrow (AA^T - \lambda I) = AA^T - I$

$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x - y = 0$

$y = k \Rightarrow x = k \Rightarrow X = \begin{bmatrix} k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

X_1 & X_2 are eigen vectors of AA^T

$\Rightarrow u_1 = \dfrac{X_1}{\|X_1\|} \qquad u_2 = \dfrac{X_2}{\|X_2\|}$

TVS

$A =$

$U_{2\times2}\ \Sigma_{2\times3}\ V^T_{3\times3}$

$U = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} \end{bmatrix}$

$U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$A = U\Sigma V^T$

$\Rightarrow U^T A = \Sigma V^T$

Take transpose on both sides

$\frac{1}{\sigma_i} U_i^T A = V_i^T$

$V_i = \frac{1}{\sigma_i} A^T u_i$

$i=1 \quad v_1 = \frac{1}{\sigma_1} A^T U_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -\sqrt{2}/3 \\ 1/\sqrt{6} \end{bmatrix}$

$V_2 = \frac{1}{\sigma_2} A^T U_2 = \frac{1}{1} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

TVS

$= \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

Find V_3 s.t. V_3 is orthogonal to v_1 & v_2

let $V_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $\langle v_3, v_1 \rangle = 0$ & $\langle v_3, v_2 \rangle = 0$

$\frac{1}{\sqrt{6}} x - \frac{\sqrt{2}}{3} y + \frac{1}{\sqrt{6}} z = 0$

$\frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} z = 0$

$\Rightarrow \begin{aligned} x - 2y + z &= 0 \\ -x + z &= 0 \end{aligned} \Rightarrow \begin{aligned} x - 2y + z &= 0 \\ -2y + 2z &= 0 \end{aligned}$

[$\because$ eliminate x by adding]

$z = k, \Rightarrow -2y + 2k = 0$

$y = k$

$x - 2k + k = 0 \Rightarrow x = k$

$X = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

TVS

$$v_3 = \frac{X}{\|X\|} = \frac{\begin{bmatrix}1\\1\\1\end{bmatrix}}{\sqrt{1^2+1^2+1^2}} = \begin{bmatrix}1/\sqrt3\\1/\sqrt3\\1/\sqrt3\end{bmatrix}$$

$$V = \begin{bmatrix}1/\sqrt6 & -1/\sqrt2 & 1/\sqrt3\\ -\sqrt{2/3} & 0 & 1/\sqrt3\\ 1/\sqrt6 & 1/\sqrt2 & 1/\sqrt3\end{bmatrix}$$

5) Find SVD of $A = \begin{bmatrix}1 & -1\\ -2 & 2\\ 2 & -2\end{bmatrix} = \underset{3\times3}{U}\ \underset{3\times2}{\Sigma}\ \underset{2\times2}{V^T}$

(Traditional method)

AA^T $\quad$ A^TA

$$A^TA = \begin{bmatrix}9 & -9\\ -9 & 9\end{bmatrix}$$

$$\lambda^2 - Tr(A^TA)\lambda + |A| = 0$$

$$\lambda^2 - 18\lambda + 0 = 0$$

$$\lambda(\lambda-18) = 0$$

$\lambda = 18, 0$ (descending order)

$\sigma = \sqrt{\lambda} = \sqrt{18}$

TVS

$$\Sigma = \begin{bmatrix}\sqrt{18} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

Find V.

$\lambda = 18 \Rightarrow A^TA - \lambda I = \begin{bmatrix}-9 & -9\\ -9 & -9\end{bmatrix}$

$A^TA - 18I \sim \begin{bmatrix}-9 & -9\\ 0 & 0\end{bmatrix}$

$\{(A^TA) - 18I\}X = 0 \Rightarrow -9x - 9y = 0$

$y = k, \Rightarrow -9x - 9k = 0$

$\Rightarrow x = -k$

$X = \begin{bmatrix}-k\\ k\end{bmatrix} = k\begin{bmatrix}-1\\ 1\end{bmatrix} \Rightarrow X_1 = \begin{bmatrix}-1\\ 1\end{bmatrix}$

$\lambda = 0 \Rightarrow A^TA - \lambda I = A^TA = \begin{bmatrix}9 & -9\\ -9 & 9\end{bmatrix} \leftrightarrow \begin{bmatrix}9 & -9\\ 0 & 0\end{bmatrix}$

$(A^TA)X = 0 \Rightarrow 9x - 9y = 0$

$y = k \Rightarrow 9x - 9k = 0 \Rightarrow x = k$

$X = \begin{bmatrix}k\\ k\end{bmatrix} = k\begin{bmatrix}1\\ 1\end{bmatrix} \Rightarrow X_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$

TVS

$V_1 = \frac{X_1}{\|X_1\|}$, $V_2 = \frac{X_2}{\|X_2\|}$

$$V_1 = \frac{\begin{bmatrix}-1\\1\end{bmatrix}}{\sqrt{(-1)^2+1^2}} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$V_2 = \frac{\begin{bmatrix}1\\1\end{bmatrix}}{\sqrt{1^2+1^2}} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \Rightarrow V = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

$A = U\Sigma V^T \Rightarrow AV = U\Sigma$

$u_i = \frac{1}{\sigma_i} AV_i \quad \sigma_i \neq 0$

$$u_1 = \frac{1}{\sigma_1} AV_1 = \frac{1}{\sqrt{18}}\begin{bmatrix}1 & 1\\-2 & 2\\2 & -2\end{bmatrix}\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$= \frac{1}{\sqrt{18}}\begin{bmatrix}-2/\sqrt{2}\\4/\sqrt{2}\\-4/\sqrt{2}\end{bmatrix} = \frac{1}{\sqrt{18}}\begin{bmatrix}-\sqrt{2}\\2\sqrt{2}\\-2\sqrt{2}\end{bmatrix}$$

$$= \begin{bmatrix}-1/3\\2/3\\-2/3\end{bmatrix}$$

TVS

we need u_3 ∵ size(U) = 3x3

let $u = (x, y, z) = \begin{bmatrix}x\\y\\z\end{bmatrix}$ λ' $\langle u, u_1 \rangle = 0$

$\Rightarrow -\frac{x}{3} + 2y/3 - \frac{2z}{3} = 0$

If $y = k_1$ & $z = k_2$ $\Rightarrow -x + 2k_1 - 2k_2 = 0$

$\Rightarrow x = 2k_1 - 2k_2 \Rightarrow X = \begin{bmatrix}2k_1 - 2k_2\\k_1\\k_2\end{bmatrix}$

$$X = k_1\begin{bmatrix}2\\1\\0\end{bmatrix} + k_2\begin{bmatrix}-2\\0\\1\end{bmatrix}$$

$$X_1 = \begin{bmatrix}2\\1\\0\end{bmatrix}, \quad X_2 = \begin{bmatrix}-2\\0\\1\end{bmatrix}$$

Orthogonal matrix ⇒ orthonormal rows & columns.

∴ Apply gram schidart process on X_1 & X_2

$w_1 = x_1 = (2,1,0)$

$$w_2 = x_2 - \frac{\langle x_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (-2,0,1) - \left(\frac{-4}{5}\right)(2,1,0)$$

$$= \left(\frac{-2}{5}, \frac{4}{5}, 1\right)$$

$w_2 = (-2,4,5)$

$$u_2 = \frac{w_1}{\|w_1\|}, \quad u_3 = \frac{w_2}{\|w_2\|}$$

$$u_2 = \frac{(2,1,0)}{\sqrt{2^2+1^2}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$u_3 = \frac{(-2,4,5)}{\sqrt{(-2)^2+4^2+5^2}} = \left(\frac{-2}{\sqrt{45}}, \frac{4}{\sqrt{45}}, \frac{5}{\sqrt{45}}\right)$$

$$\Rightarrow U = \begin{bmatrix} -1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix}$$

Schematic

$$A = U \Sigma V^T$$
$$A^T A = (V \Sigma^T U^T) U \Sigma V^T$$

$$A^T A = V \Sigma^T \Sigma V^T \quad (\text{similar to } PDP^{-1})$$

$\Rightarrow \lambda$ for $A^TA \Rightarrow \sigma^2$ for A

$$A A^T = U \Sigma \Sigma^T U^T$$

Quadratic forms

Funct^n $f: \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = X^T A X = \sum_{i,j=1}^{n} A_{ij} x_i x_j$$

(homogenous eq^n of second degree)

let $X = Py \qquad X^T = (Py)^T = y^T P^T$

$$f(x) = X^T A X \Rightarrow (Py)^T A (Py) = y^T P^T A P y$$
$$= Y^T D Y$$
$$= f^*(Y)$$

($P \to$ orthonormal modal matrix $P^T = P^{-1}$)

canonical form of

$\int \sin x \, dx = -\cos x + C$

$\int \cos x \, dx = \sin x + C$

$\int \sec^2 x \, dx = \tan x + C$

$\int \csc^2 x \, dx = -\cot x + C$

$\int \sec x \tan x \, dx = \sec x + C$

$\int \operatorname{cosec} x \cot x \, dx = -\csc x + C$

$\int \frac{1}{x} \, dx = \ln|x| + C$

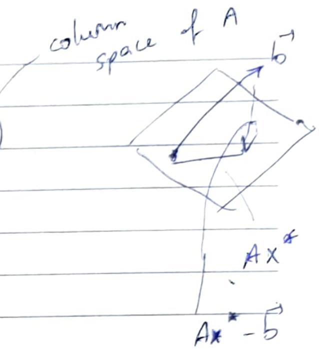
$\int e^x \, dx = e^x + C$

$\int a^x \, dx = \frac{a^x}{\ln a} + C$

$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$

$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$

$\int \frac{1}{|x|\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$



$Ax^* = \text{proj } \vec{b} \text{ on } C(A)$

$Ax^* - b = \text{proj}_{C(A)} \vec{b} - \vec{b}$

$Ax^* - \vec{b} \in C(A)^{\perp}$

$C(A)^{\perp} = N(A^T)$ — Null space of A^T

$\Rightarrow Ax^* - \vec{b} \in N(A^T)$

$\Rightarrow A^T(Ax^* - \vec{b}) = \vec{0}$

$A^T A \vec{x}^* = A^T \vec{b}$

TVS

→ A matrix is orthogonal if $P^{-1} = P^T$
Then its column vectors form orthonormal set

→ ~~Orthogonal~~
Eigenvectors obtained from symmetric matrices would be orthogonal to eachother

QR Decomposition

Q → Orthogonal matrix
R → Upper Δ^r matrix

$$A = QR$$
$$Q^T A = Q^T Q R = IR = R$$

Least Square Sol^n

$$A^T A \hat{x} = A^T B$$
$$(QR)^T (QR) \hat{x} = (QR)^T B$$
$$R^T Q^T QR \hat{x} = R^T Q^T B$$
$$R^T R \hat{x} = R^T Q^T B$$

[cancel out R^T if its invertible]

TVS

Diagonalisation

→ If 2 matrices are similar, they have same eigenvectors

→ We want a matrix that is similar to the given matrix

→ We need to find new basis that will make it diagonal. This comes from the eigenvectors

→ A nxn matrix is diagonalizable iff it has n distinct linearly independent eigenvectors

Symmetric Matrices & Orthogonal Diagonalisation

If $A = A^T \rightarrow A$ is symmetric

If A is nxn symmetric,

1) A is diagonalisable
2) All eigenvalues of A are real
3) If λ is an eigenvalue of A with multiplicity k, then ~~B~~ λ has k linearly independent eigenvectors

Insurance database

ADA . 2^{nd} unit leftover & 3 complete

$$\sin x \cdot \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$$

$$\cos x \cdot \cos y = \frac{\cos(x+y) + \cos(x-y)}{2}$$

$$\sin x \cdot \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}$$

Test -3 Portions

Unit - 4
+
2 concepts of unit 5
ie
orthogonal diagonalisat
+ digonalisat

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

2, 3