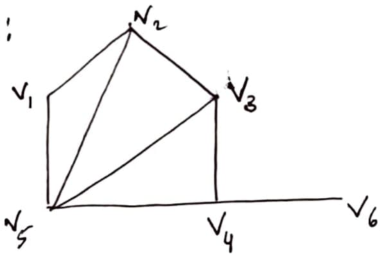
G_3 is not an induced subgraph of G as $v_1 v_5$ are not adjacent in G_3 whereas they are adjacent in G.

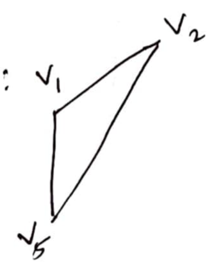
Edge and Vertex Disjoint:

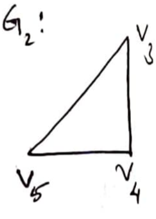
let G be a graph & G_1, G_2 be two subgraphs of G. Then,

1) G_1 & G_2 are said to be edge disjoint if they do not have any edge in common.

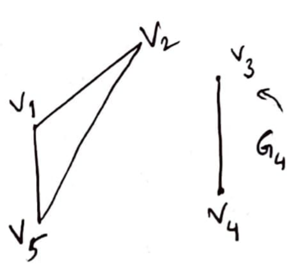
2) G_1 & G_2 are said to be vertex disjoint if they do not have any common edge or common vertex.

G:


G_1:


G_2:


edge disjoint subgraphs

G_3:


Note: Edge disjoint subgraph may have common vertices. (v_5 in above example)

Subgraphs that have no vertex in common cannot possibly have edge in common.

$\because$ Vertex disjoint subgraph must be edge disjoint but converse need not be true

Walk and its Classificat^{n}'s :

Walk : let G be a graph having atleast 1 edge In G consider a finite alternating sequence of vertices & edges of the form $v_i\, e_j\, v_{i+1}\, e_{j+1}\, v_{i+2} \cdots \cdots e_k v_m$, which begins & ends with vertices and each edge in the sequence is incident on the vertices preceding & following it in sequence. Such a ~~ve~~ sequence is called walk in G.

Note : ① In a walk vertex or edge or both can appear more than once.

② The no. of edges present in a walk is called its length.

③ Vertex with which walk begins is called initial vertex & ends, is called terminal vertex.

④ Vertices which are not initial or terminal vertices, are called internal vertices.

⑤ If initial vertex is u & terminal is v, such a walk is denoted by u-v walk.

⑥ A walk which begins & ends at same vertex is called a closed walk.

⑦ A walk which is not closed is called open walk.

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Classifications of Walk :

In a walk, vertices or edges may appear more than once.

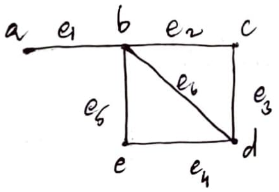
Trail is an open walk in which no edge appears more than once.

Circuit is a closed walk in which no edge appear more than once

Path : A trail in which no vertex appear more than once.

Cycle : A circuit in which terminal vertex doesn't appear as internal vertex & no internal vertex repeat.

Example :



Trail : $w_1 : ae_1b$

$w_2 : ae_1be_2c$

$w_3 : be_2ce_3de_4e$

$w_6 : ae_1be_2ce_3de_6b$

Circuit :

$w_4 : be_2ce_3d.e_6b$

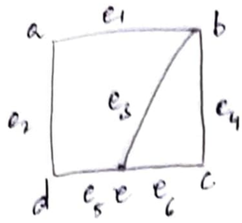
$w_5 : be_2ce_3de_4ee_5b$

Path : w_1, w_2, w_3

~~w7~~ :

cycle : w_4, w_5

Q) Consider the graph given below. Find all paths from a to c also indicate their lengths



→ Path :

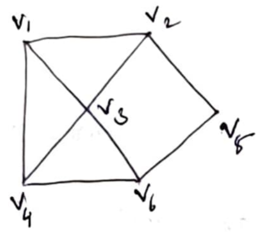
P_1 : a e_2 d e_5 e e_6 c length·3

P_2 : a e_1 b e_4 c length·2

P_3 : a e_1 b e_3 e e_6 c length·3

P_4 : a e_2 d e_5 e e_3 b e_4 c length·4.

Q) Determine no. of different paths of length 2 in the given graph.



8+

1 + 3 + 3

+ 3 + 3 +

4

→ No. of paths of length 2 from a vertex v_1 is no. of pairs of edges incident on v_1.

Since 3 edges are incident on v_1, no. of paths $= {}^3C_2 = 3$

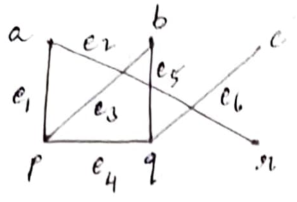
IIIly no. of paths of length 2 that passes through v_2, v_3, v_4, v_5, v_6 are

${}^3C_2$, ${}^4C_2$, ${}^3C_2$, ${}^2C_2$, ${}^3C_2$ resp.

No. of paths of length 2 is ${}^3C_2 + {}^3C_2 + {}^3C_2 + {}^3C_2 + {}^2C_2 + {}^4C_2$

$= \underline{\underline{19}}$

③ Find all cycles present in graph



→ $b e_3 p e_4 q e_5 b$

<u>Connected Graph</u> :

Two vertices in G are said to be connected if there is atleast one path from one vertex to other.

Graph G is a connected graph if every pair of distinct vertices in G are connected, otherwise G is a disconnected graph.

Note: ① A digraph D is said to be connected or disconnected accord. to its underlying graph [a graph obtained from digraph by neglecting directions] is connected or disconnected.

② In graph G all walks, all are connected subgraphs of G.

③ Every graph G consists of one or more connected graphs.
Each such connected graph is a subgraph of G & is called component of G.

4) A connected graph has only one component and a disconnected graph has 2 or more components.

5) A path with n vertices has n-1 edges.

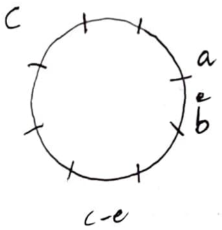
6) A cycle with n vertices has n edges.

7) Degree of any vertex in a cycle is always 2.

---

8) P.T a connected graph G remains connected after removing an edge 'e' from G iff e is a part of some cycle in G.

→ Suppose e is part of some cycle C of G then the end vertices of 'e' say a and b



are joined by atleast two paths one of which is e and the other is C-e.

Hence removal of e from G will not affect connectivity of G. Because even after removal of 'e', end vertices of e remain connected.

Conversely suppose e is not a part of cycle in G then end vertices of 'e' are connected by atmost one path. Hence removal of 'e' from G disconnects these end vertices ⇒ G-e is disconnected Graph

---

iff ⇒ $p \leftrightarrow q$ p = 'e' is part of cycle
q = G-e is connected | $p \leftrightarrow q = p \to q$ & $q \to p$

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Q) If a graph has exactly 2 vertices of odd degree then there must be a path connecting these vertices.

→ Let V_1 and V_2 be 2 vertices of odd degree. Suppose there is no path connecting them, then the graph is disconnected. So V_1 & V_2 belongs to 2 different components of G say H_1 & H_2.

Consequently each of H_1 & H_2 contains only one vertex of odd degree. This is not possible as no. of odd degree vertices in H_1 & H_2 should be even. Therefore there must be a path connecting 2 odd degree vertices.

Q) If G_1, is a simple graph with n vertices in which the degree of every vertex is atleast $\frac{(n-1)}{2}$, P.T G is connected graph.

→ Take any 2 vertices u & v of G. They are either adjacent or not adjacent. If u and v are adjacent, then G is connected.

Suppose u & v are not adjacent. Since degree of each vertex is atleast $\frac{n-1}{2}$

u & v together will have atleast $n-1$ neighbours.

$\left(\deg u + \deg v \geq \frac{n-1}{2} + \frac{n-1}{2} \right.$

$\left. \geq n-1 \right)$

Since G has n vertices, total no. of neighbours which u & v can have is only $n-2$.

$\therefore$ Atleast one vertex say 'x' is neighbour of u & v. Hence there is an edge between u & x & x & v.

$\therefore$ There is a path b/w u & v. So G is connected.

---

⑨ Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

→ Consider a simple graph with n vertices. Choose $(n-1)$ vertices say $v_1, v_2, \ldots, v_{(n-1)}$ of G.

We have maximum no. of edges drawn between these vertices $= {}^{n-1}C_2 = \frac{(n-1)(n-2)}{2}$.

Thus if we have more than $\frac{(n-1)(n-2)}{2}$ edges, atleast one edge should be drawn b/w the n^{th} vertex v_n to some vertex v_i.

$\therefore G$ must be connected.

---

⑩

Eulerian Circuit and Eulerian Trail :

Consider a connected graph G, If there is a circuit in G, that contains all edges in G, then that circuit is called eulerian circuit in G.

If there is a trail in G that contains all edges of G, then that trail is called eulerian trail in G.

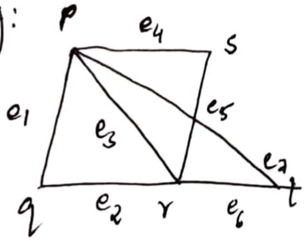
Note: 1) In trail and a circuit no edge can appear more than once but a vertex can appear more than once. This property is carried to Euler trail & Euler circuit also.

2) Since Euler circuit, euler trail includes all edges, they automatically include all vertices.

3) A connected graph that contains Euler circuit is called Euler / Eulerian graph

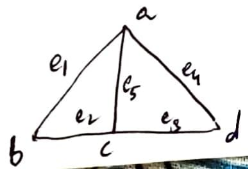
4) A connected graph that contains Euler trail is called semi - Euler graph

eg: 1)



Euler ckt: $p\, e_1\, q\, e_2\, r\, e_5\, s\, e_4\, p\, e_3\, r\, e_6\, t\, e_7\, p$

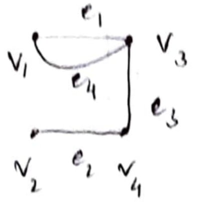
2)



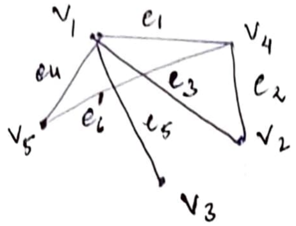
Euler trail: $a\, e_1\, b\, e_2\, c\, e_5\, a\, e_4\, d\, e_3\, c$.

(Q) Draw graph G whose incidence matrix is given b[y]

i) $\begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ v_1 & 1 & 0 & 0 & 1 \\ v_2 & 0 & 1 & 0 & 0 \\ v_3 & 1 & 0 & 1 & 1 \\ v_4 & 0 & 1 & 1 & 0 \end{array}$



(ii) $\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ V_1 & 1 & 0 & 1 & 1 & 1 & 0 \\ V_2 & 0 & 1 & 1 & 0 & 0 & 0 \\ V_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ V_4 & 1 & 1 & 0 & 0 & 0 & 1 \\ V_5 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}$



(Q) Without actually constructing the graph, show that there exist no connected graph whose incidence matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

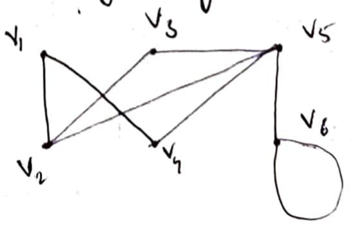
$c_3 \leftrightarrow c_6$

$$\to \begin{bmatrix} \overbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}}^{A(G_1)} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A(G_2)} \end{bmatrix}$$ is reduced to $\begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$

Such a matrix represents disconnected graph

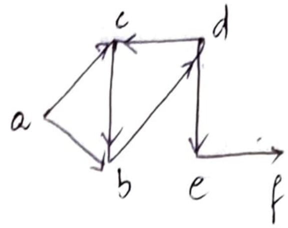
(Q) Construct a graph whose adjacency matrix is given

$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ v_1 & 0 & 1 & 0 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 0 & 1 & 0 \\ v_4 & 1 & 0 & 0 & 0 & 1 & 0 \\ v_5 & 0 & 1 & 1 & 1 & 0 & 1 \\ v_6 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$



4) Construct adjacency matrix for given digraph

Q :

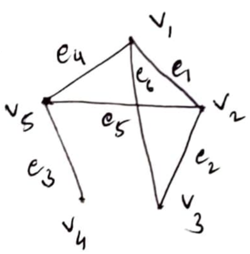


→
$$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 1 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 1 & 0 & 0 \\ c & 0 & 1 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & 0 & 1 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 1 \\ f & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

5) For the given adjacency matrix, construct incidence matrix

$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 1 \\ v_3 & 1 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 1 \\ v_5 & 1 & 1 & 0 & 1 & 0 \end{array}$$

→ G :



$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 0 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 1 & 0 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 0 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}$$

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Eulerian Circuit :

Note : A connected graph G has Euler circuit iff all vertices of G are of even deg.

8) Find all +ve integer $n \geq 2$ for which the complete graph K_n contains Euler circuit. For what value of n does K_n have Euler trail but not a Euler circuit. ?

→ For $n \geq 2$ K_n contains Euler circuit iff n is odd number (deg. of any vertex in K_n is $n-1$). If G has Euler circuit then $(n-1)$ must be even)

For $n=2$ K_2 has 1 edge with 2 end vertices. This edge together with its end vertices constitutes a Euler trail.

---

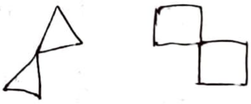
2) a) Does there exist a Euler Graph with even no. of vertices and odd no. of edges?

b) Does there exist a Euler Graph with odd no. of vertices & even no. of edges?

→ a) Yes. Suppose C is a circuit with even no. of vertices and v be any vertex on C. Consider a circuit C' with odd no. of vertices passing through v & x'. C & C' have no edge in common. The circuit Q that consists of edges of C & C' is Euler graph of desired type.

b) Yes. In a) suppose C & C' are circuit with odd (or even) no. of vertices, then Q is Euler graph of desired type.

a) 

b) 

(8) Show that a connected graph with exactly 2 vertices of odd deg. has Euler trail.

→

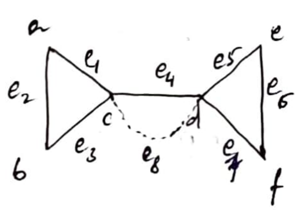
Let a and b be only 2 vertices of odd degree in a connected graph G. Join these vertices by an edge e (even if there is already an edge b/w them).

Then a and b become vertices of even deg.

Since all vertices in G are of even deg., graph $G_1 = G \cup e$ is connected whose vertices are of even degree.

$\therefore$ G_1 has Euler circuit which must include e. The trail got by deleting e from this euler circuit is Euler trail in G.

eg: G_1 :



Euler circuit : $c e_1 a e_2 b e_3 c e_4 d e_5 e e_6 f e_7 d e_8 c$

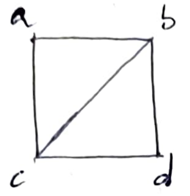
Euler trail : $c e_1 a e_2 b e_3 c e_4 d e_5 e e_6 f e_7 d$

Hamiltonian Graph :

Let G be connected graph if there is a cycle in G that contains all vertices of G then that cycle is called Hamiltonian cycle in G.

Note: 1) A Hamilton cycle in G of n vertices consists exactly of n edges.

2) Hamiltonian cycle includes all vertices of G does not imply it includes all edges of G.

eg: 

Hamiltonian cycle ; abdca

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Note:

1) **Ore's theorem**: If in a simple connected graph with n vertices where $n \geq 3$, the sum of degrees of every pair of non-adjacent vertices is greater than or equal to n then graph is Hamiltonian

(Graph is said to be Hamiltonian if it has Hamiltonian cycle).

2) **Dirac's theorem**: If in a simple connected graph with n vertices $n \geq 3$, degree of every vertex is greater than or equal to $n/2$, is hamiltonian.

1) Prove that complete graph K_n where $n \geq 3$ is hamiltonian graph.

→ In K_n, degree of every vertex is $n-1$.

If $n \geq 3$,

$\Rightarrow$ $n > 2$

$n - 2 > 0$

adding n both sides, $2n - 2 > n$

$2(n-1) > n$ (Dirac's theorem)

$n-1 > n/2$

$\deg v \geq n/2$ K_n is Hamiltonian
by Dirac's theorem

2) Let G be a simple graph with n vertices & m edges where m is atleast 3. If $m \geq \frac{(n-1)(n-2)}{2} + 2$

P.T G is Hamiltonian graph and is the converse true, $m \geq 3$

→ Let u and v be two non-adjacent vertices in G. Let deg of $u = x$ and $\deg v = y$.
If we delete u and v from G, we get subgraph with $(n-2)$ vertices and q edges.

Then $q \leq \frac{(n-2)(n-3)}{2}$ $\quad (\because {}^{n-2}C_2)$

Since u and v are non-adjacent

$m = q + x + y$

$x + y = m - q$

(~~Given~~) $q \leq \frac{(n-2)(n-3)}{2}$

$\Rightarrow -q \geq -\frac{(n-2)(n-3)}{2}$

$\therefore x+y \geq \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2}$

$x+y \geq \frac{(n-2)(n-1-n+3)}{2} + 2$

$x+y \geq \frac{(n-2)(2)}{2} + 2$

$x+y \geq n$

$\deg u + \deg v \geq n$

By ore's theorem G is Hamiltonian

Let G be a graph with 5 vertices and deg of every vertex $= 2$

$m = 5 \quad n = 5$

$m \geq \frac{(n-1)(n-2)}{2} + 2$

$m \geq \frac{4(3)}{2} + 2$

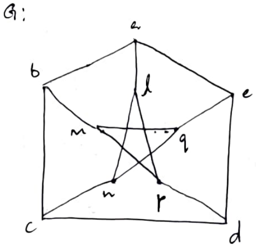
$m \geq 8$ (not true)

$\therefore$ Converse is not true

---

3) Show that Peterson graph has no Hamiltonian cycle but has Hamiltonian path

G:



Peterson graph has 10 vertices and 15 edges. If there is Hamiltonian cycle, it should pass through all 10 vertices, and should include 10 edges.

3 edges are incident on every vertex. Out of these we should include only two in Hamil-tonian cycle (if it exists). So at each of 10 vertices one edge should be excluded say al, bm, cn, de, pm, qe.

No. of edges remaining in the graph is $15-6=9$ edges. These edges are not sufficient to form a cycle. $\therefore$ Peterson graph has no hamiltonian cycle.

The hamiltonian path is a b c d p l n q m.

Note: In complete graph K_n where n is odd no. there are $\frac{(n-1)}{2}$ edge disjoint Hamiltonian cycle.

5/11/21

Q Suppose a new club has odd no of members say $n = 2k+1$ where 'k' is +ve integer. These members meet each day for lunch at round table. They decide to sit in such a way that every member have different neighbour at each lunch. How many days can this arrangement last?

→ Let us consider a graph 'G' in which a member 'x' is represented by a vertex & possibility of his, sitting next to another member 'y' is represented by edge between 'x' and 'y'. Since every member is allowed to sit next to any other member, 'G' is a complete graph.

Since there are 'n' members, there are 'n' vertices, every sitting arrangement around table is hamiltonian cycle.

On first day, they can sit in any order, this will be Hamiltonian cycle (say 'C_1')

On second day, if they are to sit such that every member has different neighbours, we must find hamiltonian cycle C_2, which is edge-disjoint with 'C_1'.

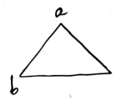
If same arrangement has to be there on subsequent days then for each day, we have to find Hamiltonian cycle which is edge-disjoint with hamiltonian cycles formed earlier.

From above note, number of such cycles is equal to $\frac{n-1}{2} = \frac{2k+1-1}{2} = k$

Therefore sitting arrangement of desired type can last for 'k' days.

eg: a) when $n = 3 = 2.1 + 1$, $k = 1$

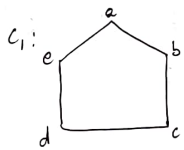
let a, b, c be members, the Hamiltonian cycle is



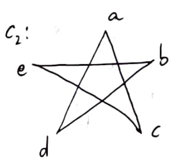
b) When $n = 5 = 2.2 + 1 \Rightarrow k = 2$

let members be a, b, c, d, e

C_1:



C_2:

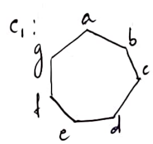


By combining these two graphs we get K_5 (i.e complete graph)

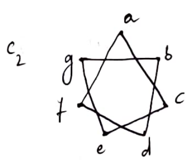
c) $n = 7 = 2.3 + 1 \Rightarrow k = 3$

let members be a, b, c, d, e, f, g

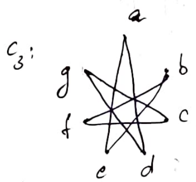
C_1:



C_2



C_3:



TREES :

A tree is a connected acyclic graph denoted by T.

Note : 1) Tree has to be simple graph because loops and parallel edges form cycles.

2) A tree has atleast two pendent vertices which are also called leaves.

3) A disconnected graph where each component is a tree is called forest.

4) In a tree, there is only one path between every pair of vertices.

5) A tree with 'n' vertices will have 'n-1' edges.

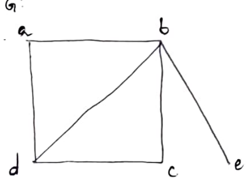
6) A connected graph 'G' is a tree iff adding an edge b/w any two vertices in 'G' creates exactly one cycle in 'G'.

Spanning Tree :

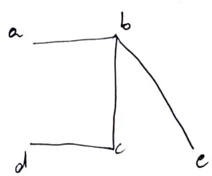
Let 'G' be a connected graph. A subgraph 'T' of G is called spanning tree of 'G' if 'T' is a tree & contains all vertices of 'G'.

G:

eg:



T_1 :

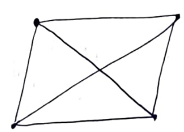


Minimal Spanning Tree :

Let 'G' be a graph and suppose there is a positive real no. associated with each edge of 'G' then 'G' is called a weighted graph and 'T' be its spanning tree. Every branch of 'T' is an edge of 'G' which has some weight, the 'sum' of weights of all branches of tree T is called weight of T.

If G is a connected weighted graph then weight of an edge 'e' is denoted by '$w(e)$' and weight of spanning tree 'T' of 'G' is denoted by $w(T)$.

Suppose we consider all spanning trees of connected weighted graph and find weight of every spanning tree then a spanning tree with least weight is minimal spanning tree and this tree is not unique.



(Matrix Representation of Graph)

Incidence Matrix: Given a graph G without self loops we define incidence matrix $A = [a_{ij}]$ of order $n \times m$ as

$a_{ij} = \begin{cases} 1 & \text{if an edge } e_j \text{ is incident on vertex } v_i \\ 0 & \text{otherwise} \end{cases}$

Observations:

→ Since each edge contains exactly 2 vertices, each column contains 2 1's.

→ 1 in each row represents edge incident from the vertex corresponding to row. ∴ sum of 1's in each row represents deg. of the vertex.

→ Two identical columns ⇒ parallel edges

→ If a row contains 1's in exactly one place ⇒ pendant vertex.

→ Row with zero 1's ⇒ isolated vertex

→ Incidence matrix of every disconnected graph is,

$\begin{bmatrix} A(G_1) & 0 & 0 \\ 0 & A(G_2) & 0 \\ 0 & 0 & A(G_3) \end{bmatrix}$ where $A(G_i)$ represent incident matrix of i^{th} component of G.

Adjacency matrix: let G be a connected graph with no parallel edges. Then, $X = [a_{ij}]$ of order $n \times n$ is,

$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$

Observatn's:

→ Its a symmetric binary matrix

→ Diagonal elements of X(G) are zero iff G has no loop.

→ No info is obtained about ∥le edges, so we avoid ∥le edges

→ 2 rows interchanged ⇒ corresponding columns are also interchanged

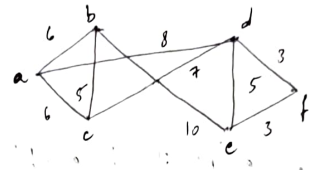
→ No. of 1's in a row (or column) gives deg. of vertex corresponding to the row (or column), counting diagonal elements twice.

KRUSKAL's ALGORITHM

8/11/21

Q) Use Kruskal's algorithm to find minimal spanning tree of weighted graph.

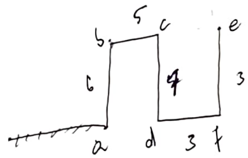
4) a)



→ Graph G has 6 vertices, MST will have 5 edges.

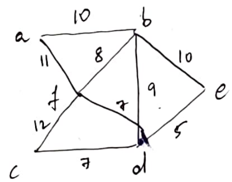
edges	df	ef	de	bc	ab	ac	cd	ad	be
w.g.t	3	3	5	5	6	6	7	8	10
selectn	✓	✓	×	✓	✓	×	✓	—	—

T:

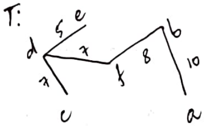


$\Rightarrow w(T) = 24$

5 Q)



T:

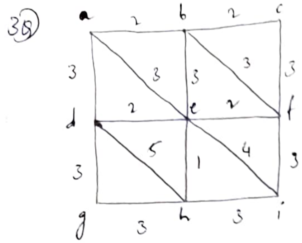


→ Graph G has 6 vertices, MST will have 5 edges.

edges	de	cd	df	bf	bd	be	ab	af	cf
w.g.t	5	7	7	8	9	10	10	11	12
selectn	✓	✓	✓	✓	×	×	✓	—	—

$w(T) = 37$

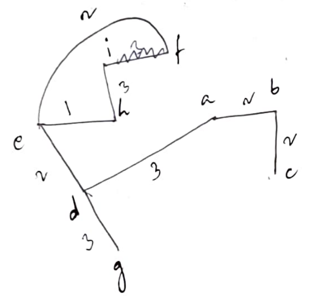
38)



→ G has 9 vertices, MST will have 8 edges

edges	eh	ab	de	bc	ad	dg	gh	hi	if	cf	ae	bf	be	dh	ei
wgt	1	2	2	2	3	3	3	3	3	3	3	3	3	5	4
selectⁿ	✓	✓	✓	✓	✓	✓	×	✓	×	×	×	✓	×	×	×

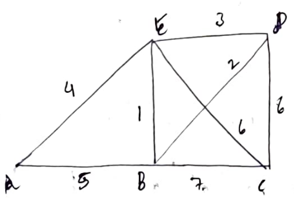
T:



$w(T) = 18$

PRIM'S ALGORITHM

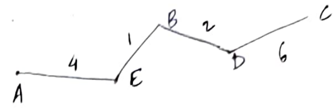
Q) Use prim's algorithm to find MST.



→

	A	B	C	D	E
✓ A	-	5	∞	∞	4
✓ B	5	-	7	2	
C	∞	7	-	6	6
✓ D	∞	2	6	-	3
✓ E	4	1	6	3	-

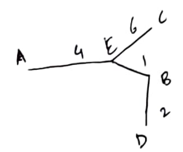
T_1:



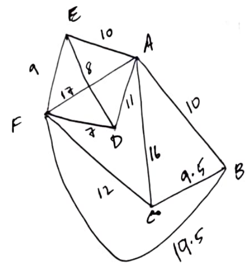
$w(T_1) = 13$

$w(T_2) = 13$

T_2:

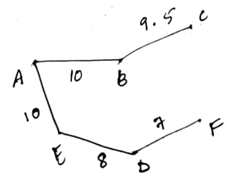


Q) Use Prim's algo to find MST



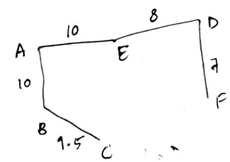
	A	B	C	D	E	F
✓ A	-	10	16	11	10	17
✓ B	10	-	9.5	∞	∞	19.5
✓ C	16	9.5	-	∞	∞	12
✓ D	11	∞	∞	-	8	7
✓ E	10	∞	∞	8	-	9
✓ F	17	19.5	12	7	9	-

T_1:



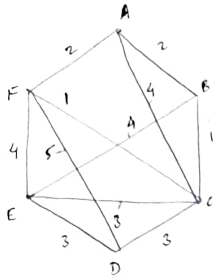
$w(T_1) = 44.5$

T_2:



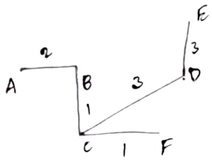
$w(T_2) = 44.5$

(5)



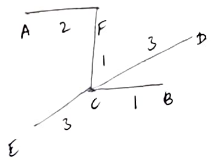
	A	B	C	D	E	F
✓ A	-	2	4	∞	∞	2
✓ B	2	-	1	∞	4	∞
✓ C	4	1	-	3	3	1
✓ D	∞	∞	3	-	3	5
✓ E	∞	4	3	3	-	4
✓ F	2	∞	1	5	4	-

T_1:



$w(T_1) = 10$

T_2:

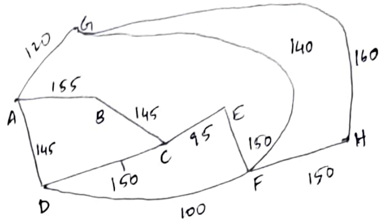


$w(T_2) = 10$

(8) Eight cities A, B, C, D, E, F, G, H are required to be connected by new railway network. The possible track & the cost involved to lay them (in crores) are given.

Track b/w	Cost
A & B	155
A & D	145
A & G	120
B & C	145
C & D	150
C & E	95
D & F	100
E & F	150
F & G	140
F & H	150
G & H	160

→



Using Prim's

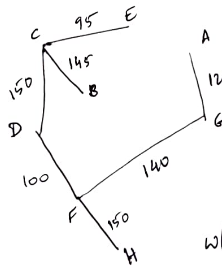
	A	B	C	D	E	F	G	H
✓ A	-	155	∞	145	∞	∞	120	∞
✓ B	155	-	145	∞	∞	∞	∞	∞
✓ C	∞	145	-	150	95	∞	∞	∞
✓ D	145	∞	150	-	∞	100	∞	∞
✓ E	∞	∞	95	∞	-	150	∞	∞
✓ F	∞	∞	∞	100	150	-	140	150
✓ G	120	∞	∞	∞	∞	140	-	160
✓ H	∞	∞	∞	∞	∞	150	160	-

There are 8 vertices, so MST will have 7 edges

Using Kruskal's Algo:

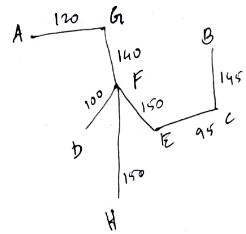
edges	CE	DF	AG	GF	AD	BC	DC	FH	EF ~~AB~~	AB	GH
wgt	95	100	120	140	145	145	150	150	150	155	160
selectⁿ	✓	✓	✓	✓	×	✓	✓	✓	-	-	-

T_K:



$w(T) = 900$ crores

T_P:



$w(T_P) = 900$ crores

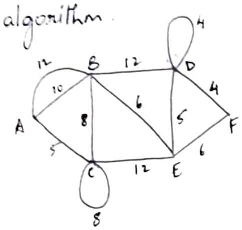
300
285
220
95
900

10/11/21

DIJKSTRA'S ALGORITHM

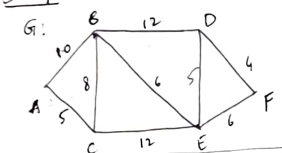
min (temp label ; (permanent label + edge wt))

5) Find shortest path from A to F using Dijkstra's algorithm



→ I step :

G :



(1st Iteration)

min (temp ; permanent + edge wt)

(for B) min (∞ ; 0 + 10) = 10

(for C) min (∞ ; 0 + 5) = 5

(for D) min (∞ ; 0 + ∞) = ∞

(for E) min (∞ ; 0 + ∞) = ∞

IIIly for F

(II iteration)

(for B) min (10 ; ~~0~~ 5+8) = 10

(for D) min (∞ ; 5 + ∞) = ∞

(for E) min (∞ ; 5+ 12) = 17

for F = ∞

III iteration

~~min~~ (for D) min (∞ ; 10 + 12) = 22

⋮

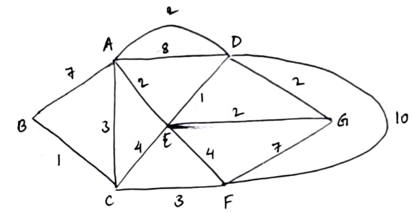
II Step:

	A	B	C	D	E	F
A	$\boxed{0}$	∞	∞	∞	∞	∞
C →	$\boxed{0}$	10	$\boxed{5}$	∞	∞	∞
B	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	∞	17	∞
E	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	22	$\boxed{16}$	∞
D	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	$\boxed{21}$	$\boxed{16}$	22
F	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	$\boxed{21}$	$\boxed{16}$	$\boxed{22}$

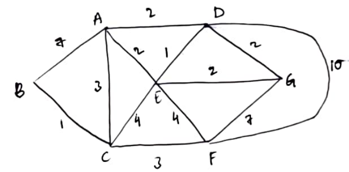
8

B/W	Shortest path	shortest distance
A - F	A - B - E - F	22
A - E	A - B - E	16
A - D	A - B - E - D	21
A - C	A - C	5
A - B	A - B	10

6) Find shortest path from B to G



→ I step:

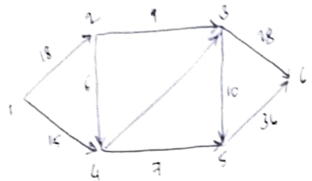


II step :

	B	A	C	D	E	F	G
B	$\boxed{0}$	∞	∞	∞	∞	∞	∞
C	$\boxed{0}$	7	$\boxed{1}$	∞	∞	∞	∞
A	$\boxed{0}$	$\boxed{4}$	$\boxed{1}$	∞	5	4	∞
F	$\boxed{0}$	$\boxed{4}$	$\boxed{1}$	6	5	$\boxed{4}$	∞
E	$\boxed{0}$	$\boxed{4}$	$\boxed{1}$	6	$\boxed{5}$	$\boxed{4}$	11
D	$\boxed{0}$	$\boxed{4}$	$\boxed{1}$	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	7
G	$\boxed{0}$	$\boxed{4}$	$\boxed{1}$	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	$\boxed{7}$

	Shortest distance	Shortest path
B - G	B - C - E - G	7
B - F	B - C - F	4
B - E	B - C - E	5
B - D	B - C - A - D	6
B - C	B - C	1
B - A	B - C - A	4

Q



→

	1	2	3	4	5	6
1	$\boxed{0}$	∞	∞	∞	∞	∞
4	$\boxed{0}$	18	∞	$\boxed{15}$	∞	∞
2	$\boxed{0}$	$\boxed{18}$	29	$\boxed{15}$	22	∞
5	$\boxed{0}$	$\boxed{18}$	27	$\boxed{15}$	$\boxed{22}$	∞
3	$\boxed{0}$	$\boxed{18}$	$\boxed{27}$	$\boxed{15}$	$\boxed{22}$	58
6	$\boxed{0}$	$\boxed{18}$	$\boxed{27}$	$\boxed{15}$	$\boxed{22}$	$\boxed{55}$

$\min(\infty, 0+18) = 18$

$\min(\infty, 0+\infty) = \infty$

$\min(\infty, 0+15) = 15$

$\min(18, 15+\infty) = 18$

$\min(\infty, 15+14) = 29$

$\min(\infty, 15+7) = 22$

	SP	SD
1 to 6	1-2-3-6	55
1 to 5	1-4-5	22
1 to 4	1-4	15
1 to 3	1-2-3	27
1 to 2	1-2	18

Q) Let G be a disconnected graph of even order 'n', with 2 components each of which is complete. Prove that G has minimum of $\frac{N(N-2)}{4}$ edges.

→ ~~(Order = no. of vertices)~~ Let G have 2 components say G_1 & G_2 with x & $n-x$ vertices. As G_1 & G_2 are complete graphs. G_1 has $\frac{x(x-1)}{2}$ edges

G_2 has $\frac{(n-x)(n-x-1)}{2}$ edges

∴ Total no of edges in $G = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$

$$F(x) = \frac{x^2-x}{2} + \frac{(N-x)^2-(N-x)}{2}$$

$$= \frac{x^2-x}{2} + \frac{(N^2+x^2-2Nx-N+x)}{2}$$

$$= \frac{2x^2+N^2-2Nx-N}{2}$$

$$= (\cancel{4x}\ \cancel{+0x})\,x^2 + \frac{N^2-N}{2} - Nx$$

~~$F'(x) = \frac{4x+0-N}{2}$~~

$f'(x) = 2x - N = 0$

$x = \frac{N}{2}$

$f''(x^2) = 2$

$f''\left(\frac{n}{2}\right) = 2$

$f(x)$ attains min at $x = n/2$

Min. of $f(x)$ is $\frac{n^2}{2} - \frac{n^2}{2} + \frac{n^2-n}{2}$

$$= \left(\frac{n}{2}\right)^2 - \frac{n}{2} = \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)$$

COMBINATORICS

Its a branch of discrete maths

12/11/21

Q) 4 different mathematics books, 5 different computer science books and two diff. control theory books are to be arranged in a shelf. a) How many diff. arrangements are possible if books of particular subject must be together?

b) Mathematics books are to be together.

→ a) Math books can be arranged in $4!$ ways, CS book in $5!$ ways and CT books in $2!$ ways.

3 groups of books can be arranged in $3!$ ways.

$\therefore$ Total no. of ways $= 3! \times 4! \times 5! \times 2!$

b) Consider 4 maths books as 1 book

No. of books to be arranged is $1+5+2 = 8$. which can be arranged in $8!$ ways. Maths books can be arranged in $4!$ ways.

$\therefore$ Total no. of ways $= 8! \times 4!$

Q) i) How many arrangements are there for all letters in the word 'Sociological'

ii) In how many of these arrangements

a) a & g are adjacent?

b) All vowels are adjacent?

→ i) No of ~~of~~ arrangements for the word $= \frac{12!}{2! \times 3! \times 2! \times 2!}$

ii) a) No. of arrangements in which a, g are adjacent is $\frac{11!}{2! \times 3! \times 2! \, 2!} \times 2!$

b) All vowels adjacent, $= \frac{7! \times 6!}{2! \times 3! \times 2! \times 2!}$

(consider group of vowels as 1 alphabet)

Q) A question paper contains 10 questions of which 7 are to be answered. In how many ways a student can select 7 que.,

a) if he can choose any 7

b) if he should select 3 que. from first 5, and 4 que. from last 5

c) if he should select atleast 3 from first 5

→ a) $^{10}C_7 = \frac{10!}{7!\,3!} = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{\cancel{3} \times \cancel{2}} = 120$

b) $^5C_3 \times ~~+~~ {}^5C_4 = \frac{5 \times \cancel{4}^{2} \times \cancel{3}}{\cancel{3} \times \cancel{2}} \times \frac{5 \times \cancel{4} \times \cancel{3} \times \cancel{2}}{\cancel{4}\,\cancel{2}} = 10 \times 5 = 50$

c)

5	5
3	4
4	3
5	2

$= {}^5C_3 \times {}^5C_4 = 50$

$= {}^5C_4 \times {}^5C_3 = 50$

$= {}^5C_5 \times {}^5C_2 = 10$

$\left.\right\}$ add $\Rightarrow 110$

Q) How many 4 digit no.s can be formed with 10 digits 0 - 9, if

a) Repetitions are allowed

b) Repetitions are not allowed

c) last digit must be 0 & repetitions are not allowed.

→

a) ~~(9×9 4 ×)~~ ~~8×~~9 × 10 × 10 × 10 = 9000

b) ~~10 + (9×9!)~~ 9 × 9 × 8 × 7 = 4536

c) ~~(9!)~~ 9 × 8 × 7 = 504

81 × 56
486
405
4536

56 × 9
504

Q) In how many ways can 3 men and 3 women be seated round a table if

a) No restriction is imposed

b) Two particular women must not sit together

c) Each women must be seated b/w 2 men.

→

a) ⇒ $(6-1)! = 120$

b) $5! - 4!2! = 72$

c) $3 \times 2 + 3 \times 2 = 12$

men fixed in odd places, women 3! ✓

fixed in even women 3! ✓

[men → in 2 ways
women can be seated in gaps - 3!
3! × 2 = 12]

a) 5!

b) ~~5! - 4!~~ 2!

c) 3!3! × 2! × 2!

Multinomial Theorem :

w.k.t $(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y^r$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} x^{n-r} y^r$$

suppose $n-r = n_1$ & $r = n_2$,

$$= \sum \frac{n!}{n_1!\,n_2!} x^{n_1} y^{n_2}$$

$$= \sum \left(\frac{n!}{n_1!\,n_2!}\right) x^{n_1} y^{n_2} = \sum \binom{n}{n_1\ n_2} x^{n_1} y^{n_2}.$$

Generalisatⁿ of Binomial theorem is multinomial theorem.

$$(x_1 + x_2 + \dots + x_k)^n = \sum \binom{n}{n_1 n_2 \dots n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

where, $\binom{n}{n_1\ n_2\ \dots\ n_k} = \frac{n!}{n_1!\,n_2!\dots n_k!}$

24/11/12

Q) Find coeff. ~~expansion~~ of $x^9 y^3$ in expansion of $(x+2y)^{12}$

→ $(x+2y)^{12} = \sum_{r=0}^{12} \binom{12}{r} x^{12-r} (2y)^r$

$$= \sum_{r=0}^{12} \binom{12}{r} x^{12-r} 2^r y^r$$

To find coeff of $x^9 y^3$ ∴ $r = 3$

coeff of $x^9 y^3 = \binom{12}{3} 2^3$

$$\frac{12!}{9! \times 3!} \times 8 = \frac{12 \times 11 \times 10}{6} \times 8$$

$= 220 \times 8 = 1760$

5) Coeff of x^5y^2 in the expansion of $(2x-3y)^7$

$$(2x-3y)^7 = \sum_{r=0}^{7}\binom{7}{r}(2x)^{7-r}(-3y)^r$$

$$= \sum_{r=0}^{7}\binom{7}{r}2^{7-r}x^{7-r}y^r\cdot(-3)^r$$

For coeff of x^5y^2, $r=2$

$$\Rightarrow \binom{7}{2}2^5(-3)^2 = \frac{7!}{5!\,2!}2^5(-3)^2$$

$$= \frac{7\times6}{2}\times 9\times 32$$

$$= 21\times9\times32 = 189\times32$$

$$= 6048$$

8) Coeff of xyz^{-2} in expansion of $(x-2y+3z^{-1})^4$

→ $$(x-2y+3z^{-1})^4 = \sum\binom{4}{n_1\ n_2\ n_3}(x)^{n_1}(-2y)^{n_2}(3z^{-1})^{n_3}$$

$$= \sum\binom{4}{n_1\ n_2\ n_3}-2^{n_2}3^{n_3}x^{n_1}y^{n_2}z^{-n_3}$$

To find coeff of xyz^{-2}, put $n_1=1$, & $n_2=1$, $n_3=2$

coeff is $$\binom{4}{1\ 1\ 2}-2^1 3^2 = -\frac{4!}{1!\,1!\,\cancel{2!}}\times\cancel{2}\times9$$

$$= -24\times9$$

$$= -216$$

8) Coeff of $w^3x^2yz^2$ in expansion of $(2w-x+3y-2z)^8$

→ $$(2w-x+3y-2z)^8 = \sum\binom{8}{n_1\ n_2\ n_3\ n_4}(2w)^{n_1}(-x)^{n_2}(3y)^{n_3}\cdot(-2z)^{n_4}$$

$$= \sum\binom{8}{n_1\ n_2\ n_3\ n_4}2^{n_1}(-1)^{n_2}3^{n_3}(-2)^{n_4}\cdot w^{n_1}x^{n_2}y^{n_3}z^{n_4}$$

To find coeff of $w^3x^2yz^2$, $n_1=3$, $n_2=2$, $n_3=1$, $n_4=2$

$$\text{coeff} = \frac{8!}{3!\,\cancel{2!}\,1!\,\cancel{2!}}\times2^3\times3\times\cancel{2^2} = \frac{8!\times\cancel{7\times6}}{\cancel{3\times2}}\times2$$

$$= 8!\times\cancel{7\times6}\times4$$

$$= 8!\times4$$

8) P.T if n is a non-negative integer

$$\frac{1}{2}\left[(1+x)^n+(1-x)^n\right] = \binom{n}{0}+\binom{n}{2}x^2+\dots+\binom{n}{k}x^k$$

where $k = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd}\end{cases}$

→ w.k.t

$$(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}(1)^{n-r}x^r = \sum_{r=0}^{n}\binom{n}{r}x^r$$

|||ly $$(1-x)^n = \sum_{r=0}^{n}\binom{n}{r}(-1)^r x^r$$

$$(1+x)^n + (1-x)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r x^r \left[1+(-1)^r\right]$$

$n \to n$ nos

$n \to n-1 \quad n$

when r = even $\quad 1+(-1)^r = 2$

r = odd $\quad 1+(-1)^r = 0$

$$= 2\binom{n}{0} + 2\binom{n}{2}x^2 + 2\binom{n}{4}x^4 \cdots 2\binom{n}{k}x^k$$

---

5) Find the sum of all coefficients in expansion of $(x+y+z)^{12}$

$\to$ $$(x+y+z)^{12} = \sum \binom{12}{n_1, n_2, n_3} x^{n_1} y^{n_2} z^{n_3}$$

$x = 1 = y = z$

$$3^{12} = \sum \binom{12}{n_1, n_2, n_3}$$

---

6) sum, of coeff

$$(2s - 3t + 5u + 6v - 11w + 3x + 2y)^{10}$$

$$= \sum \binom{10}{n_1 \cdots n_7} (2s)^{n_1} (-3t)^{n_2} (5u)^{n_3} (6v)^{n_4} (-11w)^{n_5} (3x)^{n_6} (2y)^{n_7}$$

put $s = t = u = v = w = x = y = 1$

$$(2-3+5+6-11+3+2)^{10} = \sum \binom{10}{n_1 \cdots n_7} \cdots$$

$$4^{10} = \sum \binom{10}{n_1 \cdots n_7} 2^{n_1} (-3)^{n_2} (5)^{n_3} (6)^{n_4} (-11)^{n_5} (3)^{n_6} (2)^{n_7}$$

Principle of Inclusion - Exclusion

If S is a finite set then order of S denoted by $|S|$ is no. of elements in S.

If A and B are subsets of S then

$|A \cup B| = |A| + |B| - |A \cap B|$ —(1) This means no. of elements of $A \cup B$ includes all elements of A and B and excludes elements common to A and B.

* $|\overline{A \cup B}| = |S| - |A \cup B|$, also $|\bar{A} \cap \bar{B}| = |S| - |A \cup B|$

$= |S| - |A| - |B| + |A \cap B|$ —(2)

(1) and (2) are referred to as principle of inclusion - exclusion of two sets.

$$|\bar{A} \cap \bar{B}| = |S| - (|A| + |B|) + |A \cap B|$$

Principle of Inclusion - Exclusion for n sets.

let S be a finite set. $A_1 \cdots A_n$ be subsets of S. Then principle of inclusion - exclusion for $A_1, A_2 \cdots A_n$ states that

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = \left[|A_1| + |A_2| + \cdots + |A_n|\right]$$
$$- \left[|A_1 \cap A_2| + |A_2 \cap A_3| + \cdots + |A_i \cap A_j|\right]$$
$$+ \sum |A_i \cap A_j \cap A_k| - \cdots$$
$$(-1)^{n-1} |A_1 \cap A_2 \cdots \cap A_n|$$

③ $|\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n| = |S| - |A_1 \cup A_2 \cup \dots \cup A_n|$

Suppose A_1 represent set of all those elements of S which satisfy a certain condition c_1 & A_2 represent set of all those elements satisfying condition c_2.

let $N = |S|$, $N(c_i) = |A_i|$, $N(c_i c_j) = |A_i \cap A_j|$

$N(c_i c_j c_k) = |A_i \cap A_j \cap A_k|$, $N(c_i \text{ or } c_j) = |A_i \cup A_j|$

$N(c_i \text{ or } c_j \text{ or } c_k \dots \text{ or } c_n) = S_1 - S_2 + S_3 - \dots + (-1)^{n-1} S_n$

$\bar{N} = S_0 - S_1 + S_2 - S_3 \dots + (-1)^{n-1} S_n$

Generalisat^n; No of elements in S that satisfy exactly m out of n conditions is given by $\binom{n}{n-m}$

$E_m = S_m - \binom{m+1}{1} \sum S_{m+1} + \binom{m+2}{2} \sum S_{m+2} \dots + (-1)^{n-m} \sum S_n$

(continued after $1\frac{1}{2}$ page)

⑤ Among students in a hostel 12 study maths (A) 20 study physics (B) 20 study Chemistry (C) 8 study Bio. (D)

There are 5 students for A & B, 7 for A & C, 4 for A & D, 16 for B & C, 4 for B & D,

3 for C & D, 3 for A, B, C, 2 for A, B, D,

2 for B, C, D, 3 for A, C, D, 2 for A, B, C, D.

Further there are 71 students who do not study any of these subjects. Find total no. of students in hostel.

→ $|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = 71$

$|A \cup B \cup C \cup D| = [|A| + |B| + |C| + |D|] - [\dots$

$= [12 + 20 + 20 + 8] - [5 + 7 + 4 + 16 + 4 + 3] + [3 + 2 + 2 + 3] - [2]$

$= 60 - 39 + 10 - 2$

$= 70 - 41 = \underline{29}$

$S - 29 = 71 \Rightarrow S = 71 + 29$

$= 100$

25/11/21

⑥ How many integers b/w 1 & 300 are divisible by ⓐ atleast one of 5, 6, 8

ⓑ None of 5, 6, 8.

→ let A, B, C be set of (~~factors~~) numbers divisible by 5, 6, 8.

$|A| = \frac{300}{5} = 60$

$|B| = \frac{300}{6} = 50$

$|C| = \left\lfloor \frac{300}{8} \right\rfloor = 37$

$|A \cap B| = \frac{300}{5 \times 6} = 10$

$|B \cap C| = \left\lfloor \frac{300}{\cancel{6 \times 8}\, 24} \right\rfloor = 12$

$|A \cap C| = \left\lfloor \frac{300}{5 \times 8} \right\rfloor = 7$

$|A \cap B \cap C| = \frac{300}{5 \times 24} = \left\lfloor \frac{300}{120} \right\rfloor = \left\lfloor \frac{10}{4} \right\rfloor = 2$

① Atleast $\Rightarrow \cup$

$|A \cup B \cup C| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k|$

$= 60 + 50 + 37 - 10 - 12 - 7 + 2$

$= 120$

2) $|\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - |A \cup B \cup C|$

$= 300 - 120 = 180$

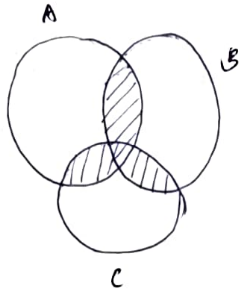
No. of elements in S that satisfy atleast m of n conditions,

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} \mp \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n$$

---

3) Determine no. of integers b/w 1 & 300 which are 1) divisible by exactly 2 of 5, 6, & 8.
2) divisible by atleast two of 5, 6, & 8.

→ 1)



$\Rightarrow |A \cap B| + |B \cap C| + |C \cap A| - 3|A \cap B \cap C|$

$= 10 + 12 + 7 - 3(2)$

$= 23$

2) $|A \cap B| + |B \cap C| + |C \cap A| - 2|A \cap B \cap C|$

$= 10 + 12 + 7 - 2(2) = \underline{\underline{25}}$

Using formula ;

i) $E_m = E_2 = S_2 - \binom{3}{1} S_3$

ii) $L_n = L_2 = S_2 - \binom{2}{1} S_3$

3) Find no. of permutations of English letters which contain 1) exactly 2, 2) atleast 2, 3) exactly 3, 4) atleast 3, of the patterns, CAR, DOG, PUN, BITE

→

$|A_1|$ = no. of words out of 26 letters in which CAR appears.

$\Rightarrow 26-3 = 23,$

$23+1 = 24!$

$|S| = 26!$

$|A_2| = 24!$

$|A_3| = 24!$

$|A_4| = 23!$

$|A_1 \cap A_2| = 26-6+1+1 = 22!$ ~~×2~~

$|A_2 \cap A_3| = 26-6+1+1 = 22!$ ~~×2~~

$|A_3 \cap A_4| = 26-7+2 = (\cancel{20! \times 2})\ 21!$

$|A_4 \cap A_1| = 26-7+2 = \cancel{20! \times 2}\ 21!$

$|A_1 \cap A_3| = 26-6+2 = (\cancel{21! \times 2})\ 22!$

$|A_2 \cap A_4| = 26-7+1 = (\cancel{20! \times 2})\ 21!$

i) E_m

$|A_1 \cap A_2 \cap A_3| = 26-3-3-3+3 = 20!$

$|A_2 \cap A_3 \cap A_4| = 26-3-3-4+3 = 19!$

$|A_3 \cap A_4 \cap A_1| = 19!$, $|A_1 \cap A_2 \cap A_4| = 19!$ ~~$|A_1 \cap A_3$~~

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 26-3-3-3-4+4 = 17!$

1) Exactly 2:

$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$

$= 3 \times 22! + 3 \times 21! - 3\left[1 \times 20! + 3 \times 19!\right] + 6 \times 17!$

2) Atleast two

$L_2 = S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4$

$= 3 \times 22! + 2 \times 21! - 2\left[20! + 3 + 19!\right] + 3 \times 17!$

3) Exactly three

$E_3 = S_3 - \binom{4}{1} S_4$

$= 20! + 3 \times 19! - 4 \times 17!$

4) Atleast three

$L_3 = S_3 - \binom{3}{2} S_4$

$L_3 = 20! + 3 \times 19! - 3 \times 19!$

Q) In how many ways can the integers 1, 2, 3, ... 10 be arranged in a line so that no even integer is in its natural place?

→ $A_1 \to$ 2 is in 2^{nd} posit^n

$A_2 \to$ 4 is in 4^{th} posit^n

$A_3 \to$ 6 is in 6^{th} posit^n

$\vdots$

$A_5 \to$ 10 is in 10^{th} posit^n

$|A_1| = 9!$ (one no. is fixed)

$= |A_2| = |A_3| = |A_4| = |A_5|$

$|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_4| = |A_4 \cap A_5| = |A_5 \cap A_1|$

$= |A_5 \cap A_2| = |A_5 \cap A_3| = |A_1 \cap A_3| = |A_2 \cap A_4| = |A_1 \cap A_4| = 8!$

${}^5C_3 \cdot |A_i \cap A_j \cap A_k| = 7! \times \frac{5 \times 4}{2} = 10 \times 7!$

${}^5C_4 \cdot |A_i \cap A_j \cap A_k \cap A_l| = 6! \times 5$

${}^5C_5 \cdot |A_i \cap A_j \cap A_k \cap A_l \cap A_m| = 5! \times 1$

let $(\bar{A}_z) = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5 = |S| - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$

$= \cancel{2170680}$

29/11/21

Q) In how many ways can 26 letters of English alphabet be permuted, so that none of the patterns CAR, DOG, PUN, or BYTE occurs?

→ $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |S| - |A_1 \cup A_2 \cup A_3 \cup A_4|$

$= |S| - \Big[\Sigma |A_i| - \Sigma |A_i \cap A_j| + \Sigma |A_i \cap A_j \cap A_k| - \Sigma |A_i \cap A_j \cap A_k \cap A_l|\Big]$

$= 26! - \Big[[3 \times 24! + 23!] - [3 \times 22! + 3 \times 21!] + [3 \times 19! + 1 \times 20!] - 17!\Big]$

Q) In how many ways 5 no.s of A's, 4 no.s of b's, and 3 no.s of c's can be arranged so that all the identical letters are not in a single block?

→ $|S| = \frac{12!}{5!\,4!\,3!}$

let A be no. of permutations where a's are together, $|A| = \frac{8!}{4!\,3!}$

IIIly $|B| = \frac{9!}{5!\,3!}$, $|C| = \frac{10!}{5!\,4!}$

$|A \cap B| = \frac{5!}{3!}$, $|B \cap C| = \frac{7!}{5!}$, $|A \cap C| = \frac{6!}{4!}$

$|A \cap B \cap C| = 3!$

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - |A \cup B \cup C|$$

$$= |S| - \left[\frac{12!}{5!\,4!\,3!} - \left[\frac{8!}{4!\,3!} + \frac{9!}{5!\,3!} + \frac{10!}{5!\,4!} \right] - \left[\frac{5!}{3!} + \frac{7!}{5!} + \frac{6!}{4!} \right] + 3! \right]$$

Catalan Number

Consider the sequence b_0, b_1, b_2 of +ve integer defined by $b_0 = 1$ and $b_n = \frac{C(2n, n)}{n+1} = \frac{{}^{2n}C_n}{n+1}$

$$b_n = \frac{(2n!)}{(n!)^2 (n+1)}$$

$$b_1 = \frac{2!}{1! \times 2} = 1$$

$$b_2 = \frac{4!}{(2!)^2 \times 3} = \frac{4 \times 3 \times 2}{2 \times 2 \times 3} = 2$$

$$b_3 = \frac{6!}{(3!)^2 \times 4} = \frac{6 \times 5 \times 4 \times 3 \times 2}{3 \times 3! \times 4} = 5$$

$$b_4 = \frac{8!}{(4!)^2 \times 5} = \frac{8 \times 7 \times 6 \times 5 \times 4!}{4! \times 4! \times 5} = \frac{8 \times 7 \times 6}{4 \times 3 \times 2} = 14$$

$$\boxed{b_n = \binom{2n}{n} - \binom{2n}{n-1}} \quad \text{or} \quad \binom{2n}{n} - \binom{2n}{n+1}$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

Consider x-y plane with O as origin and P as a point with n, n as its coordinates where n is a +ve integer. In this plane suppose we wish to reach point P by starting from O by making only following kinds of moves R and U where ~~R is~~ $R: (x, y) \to (x+1, y)$ and

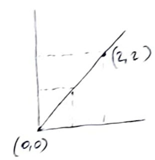
$U: (x, y) \to (x, y+1)$.

Now there is a restriction that we may touch the line $Y = X$, but never rise above it.

A path in which we move from O to P under stated restrictions is called a good path from O to P.

No. of such paths is the catalan number.

eg:



has 2 good paths RRUU & RURU

total paths : 6 : RRUU, UURR, RURU, URUR, RUUR, URRU

No. of good paths = total no. of paths − no. of paths that crosses $y = x$

$$= \binom{4}{2} - \binom{4}{1}$$

$$= \binom{2n}{n} - \binom{2n}{n-1} = b_n$$

Note: For $n \geq 1$, b_n (catalan no.) represent no. of good paths from origin O to point P (n,n) where n is a positive integer and $b_n = \frac{(2n)!}{(n!)^2 (n+1)}$

---

5) Using moves $R(x,y) \to (x+1,y)$ and $U(x,y) \to (x,y+1)$ find no. of ways can one go from

i) (0,0) to (3,3) and not rise above line $y = x$

ii) (0,0) to (6,6) and not " " "

iii) (2,1) to (7,6) and not rise above line $y = x-1$

iv) (3,8) to (10,15) and " " " $y = x+5$

→ i) No. of good paths from (0,0) to (3,3) is

$$b_3 = \frac{(3\times 2)!}{(3!)^2 \cdot 4} = \frac{6\times 5\times 4\times 3!}{3!\times 3!\times 4} = \frac{6\times 5}{3\times 2} = 5$$

ii) $$b_6 = \frac{(2\times 6)!}{(6!)^2\, 7} = \frac{12\times 11\times 10\times 9\times 8\times 7\times 6!}{6!\times 6!\times 7} = \frac{2\times 11\times 2\times 3\times 4}{4\times 3\times 2} = 132$$

iii) Shift origin (0,0) to (2,1)

$X = x-2$ & $Y = y-1$.

Then (7,6) ⇒ also gets shifted to (5,5)
(7−2, 6−1)

and the line $y = x-1$ becomes $Y+1 = X+2-1$

$\Rightarrow Y = X$.

Thus no. of good paths from (2,1) to (7,6) is no. of good paths from (0,0) to (5,5), which is

$$b_5 = \frac{(5\times 2)!}{(5!)^2 \times 6} = \frac{10\times 9\times 8\times 7\times 6\times 5!}{5\times 4\times 3\times 2\times 5!\times 6} = 42$$

iv) $X = x-3$, & $Y = y-8$

then (10,15) also gets shifted to (7,7).

line $y = x+5$ becomes $Y = X$.

No. of good paths from (0,0) to (7,7) is b_7

$$b_7 = \frac{14\times 13\times 12\times 11\times 10\times 9\times 8\times 7!}{7!\times 8\times 7\times 6\times 5\times 4\times 3\times 2} = 13\times 11\times 3 = 429$$

---

6) **Derrangement** :

Permutation of n distinct objects in which none of the object is in its natural place is called derrangement.

eg: Permutation of no.s 1, 2, 3,.. etc in which 1 is not in 1^{st} place, 2 is not in 2^{nd} place, and so on is derrangement.

The no. of possible derrangements of n distinct objects 1, 2, 3.... n is denoted by d_n

→ If there is only one object, it continues to be in its natural posit^n in every arrangement.

$\therefore d_1 = 0$.

→ If there are 2 objects a derrangement can be done in only one way by interchanging their places.

$\therefore d_2 = 1$.

→ For 3 objects 1, 2, 3, no. of derrangements is

$\therefore d_3 = 2$

3,2 | 1,3 | 1,2

3 1 2
2 3 1

→ For n objects,

$$d_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right] \text{ for } n \geq 1$$

$$\Rightarrow d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Note: $e^{-1} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} = 0.36788$

Using Mc lauren series

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$

$\Rightarrow d_n = n!(0.36788)$

Q) Find no. of derrangements of 1, 2, 3, 4.

$$\rightarrow d_4 = 4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = \frac{4!}{2!} - \frac{4!}{3!} + 1$$

$= 12 - 4 + 1$

$= 9$

Q) There are n pair of children gloves in a box. Each pair is of a different colour. Suppose the right gloves are distributed at random to n children and thereafter left gloves are also distributed to them at random find probability that

a) No child gets a matching pair

b) Every child gets a matching pair

c) Exactly one child gets a matching pair.

d) Atleast 2 children get a matching pair.

→

a) No child gets a matching pair ⇒ distribution of left gloves is derrangement.

No. of derrangement $= d_n = n!\,e^{-1}$

whose Probability is $\frac{d_n}{n!} = e^{-1}$

b) (~~⇒ Complement of a)~~)

⇒ Probability that every child gets a matching pair $= \frac{1}{n!}$

c) Exactly one child gets a matching pair ⇒ d_{n-1}

Probability, $\frac{d_{n-1}}{n!} = \frac{(n-1)!\,e^{-1}}{n!} = \frac{(n-1)!\left[1 - \frac{1}{1!} \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right]}{n!}$

(One left glove is in its natural place, remaining n-1 are derranged)

d) Probability of Atleast 2 children getting matching pair

$\Rightarrow P(x \geq 2) = 1 - P(x < 2)$

$= 1 - P(x = 0) - P(x = 1)$

$\Rightarrow$ No child or one child getting matching pair does not occur.

$$P = 1 - \left(\frac{d_n}{n!}\right) - n\left(\frac{d_{n-1}}{n!}\right)$$

$$= 1 - e^{-1} - \frac{e^{-1}}{n}$$

9) 8 letters to 8 people, no. of ways atleast one person gets right letter

$\rightarrow$ No. of ways of placing 8 letters in 8 envelopes $\rightarrow 8!$

$\Rightarrow 8! - d_8$

1/12/11

UNIT 3 : PROBABILITY

Poisson's Distribution :

It is regarded as limiting form of binomial distribution when $n \to \infty$, and $p \to 0$, $np \to$ fixed value λ.

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$p(x) \geq 0$ and $\sum p(x) = e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \dots\right) = e^{-\lambda}e^{\lambda} = 1$

2/12/11

Mean and Variance of Poisson's Distribution

$$\text{Mean} = \frac{\sum p(x)\, x}{\sum P(x)}$$

$$= \sum_{x=0}^{\infty} x\, P(x)$$

$$= \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} \frac{x\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{x\lambda^x}{x(x-1)!}$$

$$= e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!} = e^{-\lambda}\lambda\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \dots\right] = \lambda e^{-\lambda}e^{\lambda}$$

$$\boxed{\text{Mean} = \lambda}$$

Variance :

$$\sigma^2 = \sum_{x=0}^{\infty} P(x)\, x^2 - \lambda^2$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} - \lambda^2$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{x^2\lambda^x}{x!} - \lambda^2$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{[x(x-1) + x]\lambda^x}{x!} - \lambda^2$$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty}\frac{x(x-1)\lambda^x}{x!} + \sum_{x=0}^{\infty}\frac{x\lambda^x}{x!}\right] - \lambda^2$$

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\text{mean} = \frac{\sum P_i x_i}{\sum P_i}$$

$$\text{variance} = \sigma^2 = \frac{\sum P_i (x - \mu)^2}{\sum P_i}$$

$$\sigma^2 = \frac{\sum P_i (x_i - \lambda)^2}{\sum P_i}$$

$$= \sum P_i\left[x_i^2 + \lambda^2 - 2\lambda x_i\right]$$

$$= \sum P_i x_i^2 + \lambda^2 \sum P_i - 2\lambda\sum x_i P_i$$

$$= \sum P_i x_i^2 + \lambda^2 - 2\lambda^2$$

$$= \sum P_i x_i^2 - \lambda^2$$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty} \frac{x(x-1)\lambda^x}{x(x-1)(x-2)!} + \sum_{x=0}^{\infty} \frac{x\lambda^x}{x(x-1)!}\right] - \lambda^2$$

$$= e^{-\lambda}\left[\sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}\right] - \lambda^2$$

$$= e^{-\lambda}\left[\lambda^2\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}\right] - \lambda^2$$

$$= e^{-\lambda}\left[\lambda^2 e^{\lambda} + \lambda e^{+\lambda}\right] - \lambda^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\Rightarrow \boxed{\sigma^2 = \lambda} \quad , \quad \boxed{S.D = \sqrt{\lambda}}$$

---

⑤ The no. of accidents in a year to a taxi in a city follows poisson's distribution with mean 3. Out of 1000 taxi drivers, find approx. no. of drivers with a) no accident in a year, b) more than 3 accidents in a year.

$\rightarrow$ $$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3}3^x}{x!}$$

$\lambda = 3$

$x \rightarrow$ no. of accidents in 1year

a) $$p(x=0) = \frac{e^{-3}3^0}{0!} = e^{-3} = 0.049$$

No. of drivers with no accidents $= 0.049 \times 1000$
$= 49$.

b) $$p(x>3) = 1 - P(x \leq 3)$$

$$= 1 - \left[P(x=0) + P(x=1) + P(x=2) + P(x=3)\right]$$

$$= 1 - \left[e^{-3} + e^{-3}.3 + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!}\right]$$

$$= 1 - e^{-3}\left[1+3+\frac{9}{2}+\frac{9}{2}\right] = 1 - 13e^{-3}$$

$$= 0.3527$$

No. of drivers with more than 3 accidents in a year $\simeq 353$

---

⑥ 2% of fuses manufactured by a firm are found to be defected. Find probability that a box containing 200 fuses contains a) no defective fuses b) 3 or more defective fuses

$\rightarrow$ $x \rightarrow$ no. of defective fuses, $p = 2\%$, $n = 200$

$\lambda = np = 0.02 \times 200 = 4$

a) $$p(x=0) = \frac{e^{-4}\lambda^x}{x!} = e^{-4} = 0.01831$$

b) $$p(x \geq 3) = 1 - \left[P(x<3)\right]$$

$$= 1 - \left[P(x=0) + P(x=1) + P(x=2)\right]$$

$$= 1 - \left(e^{-4} + 4e^{-4} + \frac{4^2}{2}.e^{-4}\right)$$

$$= 1 - e^{-4}\left[1+4+8\right] = 1 - 13e^{-4}$$

$$= 0.7618$$

9) A distributor of bean seeds determines from extensive test that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 98% germinat^n. Determine probability that a particular packet will violate the guarantee.

$n = 200$ | $200 - 98\%$
$p = 5\%$ | $2\% = 4$ seeds packet
$\lambda = np = 0.05 \times 200 = 10$

$P(x=4) = \dfrac{e^{-10}\lambda^4}{x!} = \dfrac{e^{-10} \cdot 10^4}{4!}$

→ $x \to$ No. of seeds which do not germinate per packet.

$p = 5\%$

$\lambda = np = 200 \times 5 = 10$

$P(x > 4) = 1 - P(x \le 4)$

(2% of 200 = 4)

$= 1 - [P(x=0) + P(x=1) + P(x=2) + \cdots P(x=4)]$

$= 1 - e^{-10}\left[1 + 10 + \dfrac{10^2}{2!} + \dfrac{10^3}{3!} + \dfrac{10^4}{4!}\right]$

$= \underline{0.9708}$

---

9) In a certain factory producing razor blades there is a small chance of 0.002 for a blade to be defective. The blades are supplied in packs of 10. Use poisson distribution to calculate approx. no. of packets containing a) no defective blades b) 1 defective c) 2 defective respectively in a consignment of 10,000 packets.

→ $x \to$ no. of defective blades

$p = 0.002$; $n = 10$

$\lambda = np = 10 \times 0.002 = 0.02$

a) $P(x=0) = e^{-0.02} = 0.9801$

No. of packet with no defective blade $= 10^4 \times 0.9801$
$= 9801$

b) $P(x=1) = e^{-0.02}(0.02) = 0.0196$

No. of packets $= 196$

c) $P(x=2) = \dfrac{e^{-0.02}(0.02)^2}{2!} = 1.96 \times 10^{-4}$

⇒ No. of packets $= 2$

0.9801
0.019602
2
0.9801 ×
0.00019602

---

5) A car-hire firm has 2 cars, which it hires out day by day. The no. of demands for a car on each day is distributed as poisson distribution with mean 1.5. Calculate probability that on a certain day,

a) there is no demand
b) demands are refused

→ $\lambda = 1.5$

$x \to$ no. of demands for car on each day.

a) $P(x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-1.5}(1.5)^x}{x!}$

$P(x=0) = e^{-1.5} \cdot \frac{}{} = 0.2231$

b) Demands are refused

$\Rightarrow$ Demand > supply

$\Rightarrow$ $P(x > 2) = 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - \left[e^{-1.5} + e^{-1.5} 1.5 + \frac{(1.5)^2 e^{-1.5}}{2}\right]$

$= 1 - \left[0.2231 + e^{-1.5} 1.5 + \frac{(1.5)^2 e^{-1.5}}{2}\right]$

$= 1 - 0.8605 = \cancel{0.13945} = 0.19115$

---

5) A communication channel receives independent pulses at the rate of 12 pulses / ~~ms~~ μs. The probability of transmission error is 0.001 for each micro second. Compute probabilities of a) no error during a μs

b) one error per μs

c) atleast 1 error per μs

d) Two errors per μs

e) at most 2 error per μs.

$\rightarrow$ $p = 0.001$

$n = 12$

$\lambda = np = 0.012$

$P(x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.012} (0.012)^x}{x!}$

a) $P(x=0) = e^{-0.012} = 0.988$

b) $P(x=1) = e^{-0.012} \times 0.012 = 0.01185$

c) $P(x \geq 1) = 1 - [P(x=0)] = 1 - 0.988 = 0.0119$

d) $P(x=2) = \frac{e^{-0.012} (0.012)^2}{2!} = \cancel{0.0054} \; 7.11 \times 10^{-5}$

e) $P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$

$= 0.9880 + 0.01185 + 7.11 \times 10^{-5}$

0.01185

$= 0.9999$

0.99985

---

6) The frequency of accidents per shift in a factory is as shown in the following table.

accidents per shift	0	1	2	3	4
frequency	180	92	24	3	1

Calculate mean no. of accidents per shift and the corresponding poisson distributn & compare with actual observatn.

$\rightarrow$ $X \rightarrow$ no. of accidents per shift.

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0 \times 180 + 1 \times 92 + 2 \times 24 + 3 \times 3 + 4}{180 + 92 + 24 + 3 + 1}$$

$$= \frac{92 + 48 + 9 + 4}{300} = \frac{153}{300} = 0.51$$

$= \lambda$

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.51}(0.51)^x}{x!}$$

x	$f(x)$	$P(x)$	expected frequency	(rounding off)
0	180	0.6004	$0.6004 \times 300 = 180.12$	= 180
1	92	0.3062	$0.3062 \times 300 = 91.86$	= 92
2	24	0.0780	$0.0780 \times 300 = 23.42$	= 23
3	3	0.0132	$0.0132 \times 300 = 3.96$	= 4
4	1	1.69×10^{-3}	$1.69 \times 10^{-3} \times 300 = 0.5096$	= 1
	300			300

5) Fit a poisson distributⁿ for the following data & calculate theoretical frequencies.

x	0	1	2	3	4
f	111	63	22	3	1

→

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0 \times 111 + 63 \times 1 + 22 \times 2 + 3 \times 3 + 4 \times 1}{200}$$

$$= \frac{63 + 44 + 13}{200} = \frac{120}{200} = 0.6 = \lambda$$

$$P(x) = \frac{e^{-0.6}(0.6)^x}{x!}$$

x	$f(x)$	$P(x)$	expected frequency		
0	111	0.5488	0.5488×200	= 109.76	= 110
1	63	0.3292	0.3292×200	= 65.84	= 66
2	22	0.0987	0.0987×200	= 19.74	= 20
3	3	0.01975	0.01975×200	= 3.94	= 4
4	1	2.963×10^{-3}	$2.963 \times 10^{-3} \times 200$	= 0.5927	= ~~0.6~~ 1 - 1
					200

CONTINUOUS PROBABILITY DISTRIBUTION :

Q. If a random variable takes non-countable infinite no. of values, it is called continuous random variable.

If a variant can take any value in an interval, it will give rise to continuous distributⁿ.

Defⁿ: If for every x belonging to range of continuous random variable 'X', we assign a real no. $f(x)$ satisfying conditⁿ, $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Then $f(x)$ is called continuous probability funcⁿ.

Note: 1) If (a, b) is a subinterval of range of X, then probability that x lies in (a, b) is

$$P(a < x < b) = \int_a^b f(x)\,dx.$$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ geometrically mean that area bounded by the curve $f(x)$ and x-axis is 1.

Mean and Variance :

If x is continuous random variable with continuous probability distribut^n fun^n (CPF), $f(x)$ where x lies b/w $-\infty$ & $+\infty$,

mean $\mu = \int_{-\infty}^{\infty} x f(x)\, dx$

variance, $\sigma^2 = \int_{-\infty}^{\infty} f(x)(x-\mu)^2\, dx$

Exponential Distribution :

The continuous prob. distribu^n having CPF

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$, is known as exponential distribut^n.

Clearly $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = \cancel{\int_{-\infty}^{0} f(x) dx}^{\,0} + \int_{0}^{\infty} f(x)\, dx$

$= \int_0^{\infty} \alpha e^{-\alpha x}\, dx$

$= \left.\frac{\alpha e^{-\alpha x}}{-\alpha}\right]_0^{\infty} = -\left[\cancel{e^{-\infty}} - e^{0}\right]$

$= 1$

$f(x < \;) = 1 - e^{-x/\mu}$

(cumulative)

Mean :

$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_0^{\infty} x \alpha e^{-\alpha x}\, dx$

$= \alpha \int_0^{\infty} x e^{-\alpha x}\, dx$

$\mu = \alpha \left[\frac{x e^{-\alpha x}}{-\alpha} - \frac{e^{-\alpha x}}{\alpha^2}\right]_0^{\infty} = +\alpha\left[\frac{+1}{\alpha^2}\right] = \frac{1}{\alpha}$

$\boxed{\Rightarrow \mu = \frac{1}{\alpha}}$

Variance :

$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_0^{\infty} (x-\mu)^2 \alpha e^{-\alpha x}\, dx$

$= \alpha \int_0^{\infty} (x-\mu)^2 e^{-\alpha x}\, dx = \alpha \left((x-\mu)^2 \frac{e^{-\alpha x}}{-\alpha} - 2(x-\mu)\frac{e^{-\alpha x}}{\alpha^2} + 2\frac{e^{-\alpha x}}{-\alpha^3}\right]_0^{\infty}$

$= -\alpha \left[\mu^2\left(\frac{1}{-\alpha}\right) + \frac{2\mu}{\alpha^2} - \frac{2}{\alpha^3}\right]$

$= -\alpha\left[\frac{-1}{\alpha^3} + \frac{2}{\alpha^3} - \frac{2}{\alpha^3}\right] \Rightarrow \boxed{\sigma^2 = \frac{1}{\alpha^2}}$

$\Rightarrow \boxed{\text{S.D}, \; \sigma = \frac{1}{\alpha}}$

Q) The duration of telephone conversation has been found to have an exponential distribution with mean 3 minutes. Find probability that the conversatn may last for a) more than a minute

b) less than 3 minutes

→ $\mu = \frac{1}{\alpha} = 3$ min $\Rightarrow \alpha = \frac{1}{3}$

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) $P(x > 1) = \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{3} e^{-x/3}\,dx = \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_1^\infty$

$= -\left[e^{-\infty} - e^{-1/3}\right] = e^{-1/3} = 0.71653$

b) $P(x < 3) = \int_0^3 \frac{1}{3} e^{-x/3}\,dx$

$= \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_0^3 = -\left[e^{-1} - 1\right] = 1 - e^{-1}$

$= 0.63212$

Q) In a certain town the duration of a shower is exponentially distribution with mean 5 min. What is the prob. that shower will last for a) 10 min. ~~for~~ or more

b) less than 10 min

c) Between 10-12 min

→ $\mu = \frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) $P(x \geq 10) = \int_{10}^\infty \alpha e^{-\alpha x}\,dx = \int_{10}^\infty \frac{1}{5} e^{-x/5}\,dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_{10}^\infty$

$= -\left[-e^{-2}\right] = e^{-2} = 0.1353$

b) $P(x < 10) = \int_0^{10} \frac{1}{5} e^{-x/5}\,dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_0^{10}$

$= -\left[e^{-2} - 1\right] = 1 - e^{-2}$

$= 0.8647$

c) $P(10 < x < 12) = \int_{10}^{12} \frac{1}{5} e^{-x/5}\,dx = -\left[e^{-x/5}\right]_{10}^{12}$

$= -\left(e^{-12/5} - e^{-2}\right) = 0.04461$

Q) After the appointment of new sales manager, the sales in 2-wheeler showroom is exponentially distributed with $\mu = 4$. If 2 days are selected at random, what is the prob. that

a) On both the days the sale is over 5 units

b) sale is over 5 units on atleast 1 of the 2 days.

$\rightarrow \mu = 4 \Rightarrow \alpha = \frac{1}{4}$

(a) $P(x > 5) = \int_5^{\infty} \frac{1}{4} e^{-x/4} dx = -\left[e^{-5/4}\right] = e^{-5/4}$

$= 0.28650$

(a) (Product rule

Both days - events - independent)

$\Rightarrow P(x>5) \cdot P(x>5) = 0.08208$

(b)

$P\bar{P} + \bar{P}P + P^2$

$2P\bar{P} + P^2 = 2(0.2865)(1-0.2865) + (0.2865)^2$

$= 0.4088327 + 0.08208 = 0.490914$

Q The life of a compressor manufactured by a company is known to be 200 months on an avg. following exponential distribut^n. Find prob. that life of a compressor of that company is

(a) less than 200 months

(b) Between 100 months & 25 yrs.

$\rightarrow \mu = 200, \alpha = \frac{1}{200}$

(a) $P(X < 200) = 1 - e^{-200/\mu} = 0.63212$

(b) $25 \times 12 = 300$ months

$\Rightarrow P(100 < x < 300) = -e^{-x/200} \Big|_{100}^{300}$

$= 0.77686 - 0.393469 = 0.3833$

7/12/21

Normal Distribution:

If $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}}$ where $-\infty < x < \infty$, $-\infty < \mu < \infty$

and $\sigma > 0$ is called normal Distribut^n.

Mean of Normal Distribut^n:

Mean $= \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$

Put $\frac{x-\mu}{\sqrt{2\pi}} = t$, $dx = \sqrt{2\pi}\, dt$

$\int_{-\infty}^{\infty} x f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty}$

2 types of problems;

1) λ & S.D given

2) Calculate mean & variance

you must know ~~to~~ about graph & signs.

Q) In a test on 2000 electric bulbs, it was found that life of a particular bulb was normally distributed with an average light of 2040 hours & SD of 60 hrs. Estimate no. of bulbs likely to burn for
i) more than 2150 hrs ii) less than 1950 hrs
iii) more than 1920 hrs & less than 2160 hrs.

→ $\mu = 2040$ hrs

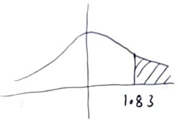
$\sigma = 60$ hrs

$x \to$ life of an electric bulb

let $Z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$

i) $P(x > 2150)$

$Z = \frac{2150-2040}{60} = \frac{110}{60} = 1.83$



$P(x > 2150) = P(Z > 1.83)$

$= 0.5 - \phi(1.83)$

$= 0.5 - 0.4664 = 0.0336$

No. of bulbs which will burn for more than 2150 hrs $= 0.0336 \times 2000 = 67.2 \approx 67$

ii) $P(x < 1950)$

$Z = \frac{x-2040}{60}$; $Z = \frac{1950-2040}{60} = -1.5$

$P(x < 1950) = P(Z < -1.5) = 0.5 - \phi(1.5)$

$= 0.5 - 0.4332 = 0.066$

No. of bulbs which will burn for less than 1950 hrs is 2000×0.066

≈ 132

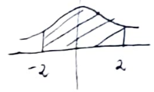
iii) $P(1920 < x < 2160)$ 0.66040

$Z = \frac{x-2040}{60}$

when $x = 1920$, $Z_1 = \frac{1920-2040}{60} = -2$

when $x = 2160$, $Z_2 = \frac{2160-2040}{60} = 2$

$P(1920 < x < 2160) = P(-2 < Z < 2)$



$= 2 \times \phi(2)$

$= 2 \times 0.4772$

$= 0.9544$

No. of bulbs that will burn for more than 1920 hrs and less than 2160 hrs $= 2000 \times 0.9544$

$= 1908.8 \approx 1909$

Q) The mean height of 500 students is 151 cm and SD is 15 cm. Assuming that heights are normally distributed find no. of students whose height lie b/w 120 & 155 cm.

→ $\mu = \cancel{1400}$ 151

$\sigma = 15$

$Z = \frac{x-\mu}{\sigma} = \frac{x-151}{15}$ $x \to$ height of student

$P(120 < x < 155)$ ⇒ $x = 120$, $Z_1 = \frac{120-151}{15} = 2.07$

$x = 155$, $Z_2 = \frac{155-151}{15} = 0.27$

$P(120 < x < 155) = P(-2.07 < x < 0.27)$

$= \phi(2.07) + \phi(0.27)$

$= 0.4808 + 0.1064 = 0.5872$

No. of students whose height is b/w 120 & 155 cm is

$500 \times 0.5872 = 293.6 \simeq 294$

---

(5) A manufacturer of air-mail envelopes knows from past experience that weight of envelope is normally distributed with mean 1.95 g & S.D 0.05 g

$\mu = 1.95$

$\sigma = 0.05$

About how many envelopes weighs &

(i) 2 g or more

(ii) 2.05 g or more can be expected in a given pack of 100 envelopes

$\rightarrow$ $Z = \frac{x-\mu}{\sigma} = \frac{x - 1.95}{0.05}$

(i) $P(x \geq 2)$, $\Rightarrow$ for $x = 2$, $Z = \frac{2 - 1.95}{0.05} = 1$

$P(x \geq 2) \Rightarrow P(Z \geq 1) = 0.5 - \phi(1)$

$= 0.5 - 0.3413 = 0.1527$

No. of envelopes whose weight is 2g or more in packet $= 100 \times 0.1587 \approx 16$

(ii) $P(x \geq 2.05)$

For $x = 2.05$, $Z = \frac{2.05 - 1.95}{0.05} = 2$

$P(x \geq 2.05) = P(Z \geq 2) = 0.5 - \phi(2)$

$= 0.5 - 0.4772 = 0.0228$

No. of envelopes whose wt. is 2.05 or more $= 100 \times 0.0228$

≈ 2

---

(5) In a normal dist^n, ~~35~~ 31% items are under 45, 8% are over 64. Find mean and variance.

$\rightarrow$ $P(x < 45) = 0.31$

$P(x > 64) = 0.08$

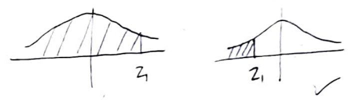
$Z = \frac{x - \mu}{\sigma}$

when $x = 45$, $Z_1 = \frac{45 - \mu}{\sigma}$

when $x = 64$, $Z_2 = \frac{64 - \mu}{\sigma}$

* $P(x < 45) = 0.31$

$P(Z < Z_1) = 0.31$



≥ 0.5 not possible

matches data 0.31

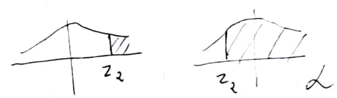
$0.5 - \phi(Z_1) = 0.31$

$\phi(Z_1) = 0.19$

$Z_1 = -0.5$ (from table and sign using graph)

* $P(x > 64) = 0.08$

$P(Z > Z_2) = 0.08$



$0.5 - \phi(Z_2) = 0.08$

$\phi(Z_2) = 0.42$

$Z_2 = 1.41$

$\frac{45 - \mu}{\sigma} = -0.5$

$\Rightarrow 45 = \mu - 0.5\sigma$ — (1)

$\frac{64 - \mu}{\sigma} = 1.41$ $\Rightarrow 64 = 1.41\sigma + \mu$ — (2)

$\mu = 49.97$ ≈ 50 $\sigma = 9.9476$ ≈ 10

Q) In a normal distributⁿ 5% are under 60 and 40% are b/w 60 & 65. Find mean & SD.

→ $P(<60) = 0.05$

$Z = \frac{x-\mu}{\sigma}$

$P(60 < x < 65) = 0.4$

i) when $x = 60$, $Z_1 = \frac{60-\mu}{\sigma}$

when $x = 65$ $Z_2 = \frac{65-\mu}{\sigma}$

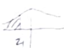
$P(x<60) = 0.05$

$P(z < z_1) = 0.05$

$= 0.5 - \phi(z_1)$

$= 0.05$

$\Rightarrow \phi(z_1) = 0.45$



$z_1 = -1.65$

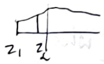
$P(60 < x < 65) = 0.4$

$P(z_1 < z < z_2) = 0.4$

$\phi(z_1) - \phi(z_2) = 0.4$

$\phi(z_2) = 0.05$

$z_2 = 0.13$



$\phi(z_1) - \phi(z_2) = 0.4$

$0.45 - \phi(z_2) = 0.4$

$\phi(z_2) = 0.05$

$\phi(z_1) + \phi(z_2) = 0.4$

$0.45 + \phi(z_2) = 0.4$

$\phi(z_2) = 0.05$

$\frac{60-\mu}{\sigma} = 1.65$

$60 - \mu = 1.65\sigma$

$\sigma = 3.2894$

≈ 3

$\frac{65-\mu}{\sigma} = 0.13$

$65 - \mu = 0.13\sigma$

$\mu = 65.4$

≈ 65

Q) In examinatⁿ taken by 500 students, the avg & S.D of marks obtained are 40% & 10%. Find approx.

i) No. of students who will pass if 50% is fixed as minimum

ii) How many have scored marks above 60%.

iii) What should be the minimum if 350 candidates are to pass.

→ $\mu = 40$

$\sigma = 10$

$x \to$ Marks of student in %.

$Z = \frac{x-\mu}{\sigma}$

$= \frac{x-40}{10}$

i) $P(x \geq 50)$

For $x = 50$, $Z_1 = \frac{50-40}{10} = 1$

$P(z > 1) = 0.5 - \phi(1)$

$= 0.5 - 0.3413 = 0.1587$

No. of students who will pass $= 500 \times 0.1587 = 79.35$

$\simeq 79.$

ii) $P(x > 60)$

$x = 60 \Rightarrow Z = 2$

$P(z > 2) = 0.5 - \phi(2)$

$= 0.5 - 0.4772 = 0.0228$

No. of students scoring $> 60\% \Rightarrow 500 \times 0.0228 = 11.4$

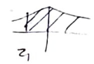
$\simeq 11$

iii) Let M be minimum marks in %.

Probability that ..

$P(x \geq M) = 0.7$

$P(z \geq z_1) = 0.7$ where $z_1 = \frac{M-40}{10}$



$\Rightarrow 0.5 + \phi(z_1) = 0.7$

$\phi(z_1) = 0.2$

$z_1 = 0.52$

$\frac{M-40}{10} = -0.52 \Rightarrow M \approx 35$

5) The income of 10,000 people was found to be normally distributed with μ = Rs 750 p.m

S.D = Rs 50 p.m

S.T of this group, about 95% has income exceeding ₹ 668 and only 5% has income exceeding ₹ 832. Also find lowest income among richest 100.

→ $\mu = 750$ $\quad x \to$ income p.m. $\quad Z = \frac{x-\mu}{\sigma} = \frac{x-750}{50}$

$\sigma = 50$

T.P $p(x > 668) = 0.95$, $\quad p(x > 832) = 0.05$

$\frac{100}{10000} \times 100 = 1\%$

i) $p(x > 668)$

For $x = 668$, $z = -1.64$

$p(z > -1.64) = 0.5 + \phi(1.64)$

$= 0.9495$

No. of people ---- $= 10000 \times .9495 = 9495 \approx 95\%$

ii) $p(x > 832)$

For $x = 832$, $z = 1.64$

$p(z > 1.64) = 0.5 - \phi(1.64)$

$= 0.0505$

No. of people ---- $\approx 5\%$

iii) $p(x > M) = 0.01$

let M be lowest income among richest 100

$p(x > M) = 0.01$

$p(z > z_1) = 0.01$ where $z_1 = \frac{M-750}{50}$

$0.5 - \phi(z_1) = 0.01$

$\phi(z_1) = 0.49$ $\Rightarrow M = 2.33 \times 50 + 750$

$z_1 = 2.33$ $\quad M = 866.5$ p.m

15/12/21

JOINT PROBABILITY:

Expectation ~~(Mean)~~:

$E(x) = \mu_x = \sum_{i=1}^{n} x_i f(x_i)$

Variance:

$V(x) = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) = E(x-\mu)^2$

where μ is mean of x.

$\mu_x = E(x) = \sum_{i=1}^{m} x_i f(x_i)$ and $\mu_y = E(Y) = \sum_{i=1}^{n} y_j g(y_j)$

$E(XY) = \sum_{i,j} x_i y_j J_{ij}$

covariance, $\text{cov}(x,y) = E(XY) - \mu_x \mu_y$

correlat^n of x and Y is $\rho(x,y) = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$

$\sigma_x^2 = E(x^2) - \mu_x^2$

$\sigma_y^2 = E(y^2) - \mu_y^2$

Note: Don't check in $p(x=x_i, y=y_j) = f(x_i) g(y_j)$ for independence. You need to check for each, use instead, $\text{cov}(x,y) = 0$

Q) A fair coin is tossed thrice the random variables X and Y are defined as follows.

X = 0 or 1 accrd. head or tail occurs on first toss.

Y = no. of heads.

i) Determine distribution of X & Y

ii) Joint distribution of X & Y

iii) $E(X)$, $E(Y)$, $E(XY)$, $cov(X,Y)$, $\cancel{cop}(X,Y)$

→ $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$X = \{0,1\}$, $Y = \{0,1,2,3\}$

ii) →

x \ y	0	1	2	3	f(x)
0	0	1/8	2/8	1/8	4/8
1	1/8	2/8	1/8	0	4/8
g(y)	1/8	3/8	3/8	1/8	1

$p(0,0) = 0$
$p(0,1) = 1/8$
$p(0,2) = 2/8$
$p(0,3) = 1/8$
$p(1,0) = 1/8$
$p(1,1) = 2/8$
$p(1,2) = 1/8$
$p(1,3) = 0$

i) Distribution of X

x	0	1
f(x)	4/8	4/8

Distribution of Y

y	0	1	2	3
g(y)	1/8	3/8	3/8	1/8

ii) Joint distribution (1st table)

iii) $E(x) = \sum x_i f(x_i) = 0 + 4/8 = 1/2$

$E(Y) = \sum y_j g(y_j) = 3/8 + 3/8 + 6/8 = 12/8 = 3/2$

$E(XY) = \sum x_i y_j J_{ij} = 2/8 + 2/8 = 1/2$

$cov(X,Y) = E(XY) - \mu_x \mu_y = 1/2 - (1/2)(3/2)$
$= -1/4$

$\rho(X,Y) = \dfrac{cov(X,Y)}{\sigma_x \sigma_y}$

$\sigma_x^2 = E(x^2) - \mu_x^2$

$E(x^2) = \sum x^2 f(x_i) = 4/8$

$\sigma_x^2 = 4/8 - 1/4 = 1/4$

$\sigma_x = 1/2$

$\sigma_y^2 = E(y^2) - \mu_y^2$

$E(y^2) = \sum y^2 g(y_j)$
$= 0 \times 1/8 + 1 \times 3/8 + 4 \times 3/8 + 9 \times 1/8$
$= \dfrac{3+12+9}{8} = \dfrac{24}{8} = \dfrac{6}{2} = 3$

$\sigma_y^2 = 3 - 9/4 = 3/4$

$\sigma_y = \dfrac{\sqrt{3}}{2}$

$\rho(X,Y) = \dfrac{-1/4}{\frac{1}{2}\frac{\sqrt{3}}{2}} = \dfrac{-1}{\sqrt{3}}$

16/12/21

Q) The joint probability distribution of two random variable X and Y is given below.

x \ y	2	3	4	f(x)
1	0.06	0.15	0.09	0.3
2	0.14	0.35	0.21	0.7
g(x)	0.2	0.5	0.3	1

Find the marginal distⁿ of X & Y. Also verify whether X and Y are stochastically independent.

i) Marginal dist^n of x

x	1	2
f(x)	0.3	0.7

" of y:

y	2	3	4
g(y)	0.2	0.5	0.3

covariance:

$cov(x,y) = E(XY) - E(x)E(y)$

$E(x) = \sum x f(x) = 0.3 + 1.4 = 1.7$

$E(y) = \sum y g(y) = 0.4 + 1.5 + 1.2 = 3.1$

$E(xy) = 0.12 + 0.45 + 0.36 + 0.56 + \cancel{1.44}\ 2.1 + 1.68$

$= 5.27$

$cov(x,y) = E(XY) - E(X)E(Y)$

$= 5.27 - 5.27 = \underline{0}$

Q The joint prob. dist^n of 2 random variable X & Y are

x \ y	-4	2	7	f(x)
1	1/8	1/4	1/8	1/2
5	1/4	1/8	1/8	~~3/8~~ 1/2
g(y)	3/8	3/8	2/8	1

Find i) Marginal dist^n of X & Y

ii) E(x), E(Y)

iii) σ_x, σ_y

iv) cov(x, y)

→ i) Marginal dist^n of x:

x	1	5
f(x)	1/2	~~3/8~~ 1/2

" of y:

y	-4	2	7
g(y)	3/8	3/8	2/8

ii) $E(X) = \sum x f(x)$

$\hookrightarrow \frac{1}{2} + \frac{5}{2} = 3$

$E(Y) = -\frac{12}{8} + \frac{6}{8} + \frac{14}{8} = 1$

iii) $\sigma_x^2 = E(x^2) - \mu_x^2 = 13 - 9 = 4 \Rightarrow \sigma_x = 2$

$\sigma_y^2 = E(Y^2) - \mu_y^2 = [-6 + 1.5 + 12.25] - 1 = 6.75$

iv) $cov(x,y) = E(XY) - E(X)E(Y)$

$= \left[-\frac{1}{2} + \frac{1}{2} + \frac{7}{8} + -\frac{20}{4} + \frac{10}{8} + \frac{35}{8}\right] - 3$

$= \frac{3}{2} - 3 = -\frac{3}{2}$

Q The joint probability fun^n of 2 random variables X & Y is given by $f(x,y) = c(2x+y)$ where x and y can assume all integral values such that $0<x<2$, $0<y<3$ and $f(x,y) = 0$, otherwise.

Find i) value of constant c

ii) $P(X \geq 1, Y \leq 2)$

iii) Marginal Prob. dis^n of X & Y

Check whether X & Y are independent.

→ $X = \{0,1,2\}$, $Y = \{0,1,2,3\}$

$f(x,y) = c(2x+y)$

x \ y	0	1	2	3	f(x)
0	0	c	2c	3c	6c
1	2c	3c	4c	5c	14c
2	4c	5c	6c	7c	22c
g(y)	6c	9c	12c	15c	1

i) $42c = 1$

$c = \frac{1}{42}$

ii) $P(X \geq 1, Y \leq 2)$

$= P(x=1, y=0) + P(2,0) + P(1,1) + P(2,1) + P(1,2) + P(2,2)$

$= 2c + 4c + 3c + 5c + 4c + 6c = 24c = \frac{24}{42} = \frac{4}{7}$

iii) Marginal dis^n of x :

x	0	1	2
f(x)	6/42	14/42	22/42

of Y :

y	0	1	2	3
g(y)	6/42	9/42	12/42	15/42

iv) $E(X) = \sum x f(x)$

$= \frac{29}{21}$

$E(Y) = \sum y g(y)$

$= 13/7$

$E(XY) = 3c + 8c + 15c + 10c + 24c + 42c = 102c = \frac{102}{42}$

$\mathrm{cov}(x,y) = E(XY) - E(X)E(Y)$

$\frac{102}{42} - \frac{377}{147} = \frac{20}{147}$

Q) Two fruits are selected at random from a bag containing 3 Apples, 2 Oranges and 4 Mangoes, If X and Y are respectively the no. of Apples and no. of oranges included among the two fruits drawn from the bag, find probability associated with all possible pair of values (x,y). Also find the correlation b/w X & Y.

→ $X = \{0,1,2\}$ X → no. of apples   3A, 2O, 4M → 9 fruits

$Y = \{0,1,2\}$ Y → no. of oranges

X \ Y	0	1	2	f(x)
0	6/36	8/36	1/36	15/36
1	12/36	6/36	0	18/36
2	3/36	0	0	3/36
g(y)	21/36	14/36	1/36	1

$P(x=0, y=0) = \frac{^4C_2}{^9C_2} = \frac{6}{36} = \frac{1}{6}$

$P(x=0, y=1) = \frac{^2C_1 \times {^4C_1}}{^9C_2} = \frac{2\times 4}{36} = \frac{2}{9}$

Marginal, of X :

x	0	1	2
f(x)	15/36	18/36	3/36

of Y:

y	0	1	2
g(y)	21/36	14/36	1/36

$E(X) = \sum x f(x) = 18/36 + 6/36 = 24/36$

$E(Y) = \sum y g(y) = 14/36 + 2/36 = 16/36$

$E(XY) = \sum x_i y_j J_{ij} = 6/36$

$E(x^2) = \sum x_i^2 f(x_i) = 18/36 + 12/36 = \frac{30}{36}$

$E(Y^2) = \sum y_j^2 g(y_j) = \frac{14}{36} + \frac{4}{36} = \frac{18}{36}$

$\sigma_x^2 = E(x^2) - (E(x))^2 = \frac{30}{36} - \left(\frac{24}{36}\right)^2 = \frac{7}{18} \Rightarrow \sigma_x =$

$\sigma_y^2 = E(y^2) - (E(y))^2 = \frac{18}{36} - \left(\frac{16}{36}\right)^2 = \frac{49}{162}$

$\text{cov}(x,y) = E(XY) - E(X)E(Y) = \frac{6}{36} - \frac{24}{36} \times \frac{16}{36}$

$= -\frac{7}{54}$

$\rho(x,y) = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} = \frac{-7/54}{\cancel{0.6499} \times 0.6236 \times 0.5499}$

$= \frac{-7/54}{0.34296} = -0.3779$

8) Two cards are selected at random from a box which contains 5 cards numbered 1, 1, 2, 2, & 3. Find the JPD of X & Y where x denotes the sum and y denotes maximum of 2 no.s drawn. Also find correlation of X & Y.

→ ~~X = {(1,1)~~

$X = \{2, 3, 4, 5\}$ $\quad Y = \{1, 2, 3\}$

• Sample Space : {(1,1), (1,2), (1,3), (1,1), (1,2), (1,3), (1,2), (1,2), (2,1) ~~(2,1)~~ (2,2) (2,3) ~~(2,1)~~ (2,1) (2,1) (2,2) (2,3) ~~(2,2)~~ (3,1) (3,1) (3,2) (3,2) }

X \ Y	1	2	3	f(x)
2	2/20	0	0	2/20
3	0	8/20	0	8/20
4	0	2/20	4/20	6/20
5	0	0	4/20	4/20
g(y)	2/20	10/20	8/20	1

$P(x=2, y=1) = 2/20$

$P(x=3, y=1) = 0/20$

$E(X) = \sum x f(x) = \frac{4}{20} + \frac{24}{20}$

9) X and Y are independent random variable. X takes value 2, 5, 7, with prob. $1/2$, $1/4$, $1/4$ resp. and Y takes value wit 3, 4, 5 with prob. $1/3$, $1/3$, $1/3$

i) Find JPD of X & Y.

ii) Show that $cov(X,Y) = 0$

iii) Find prob. distⁿ. of $Z = X + Y$.

→ Given X and Y are independent

$$\Rightarrow P(X = x, Y = y) = f(x)\, g(y)$$

$x \backslash y$	3	4	5	$f(x)$
2	$1/6$	$1/6$	$1/6$	$1/2$
5	$1/12$	$1/12$	$1/12$	$1/4$
7	$1/12$	$1/12$	$1/12$	$1/4$
$g(y)$	$1/3$	$1/3$	$1/3$	1

$$E(X) = \Sigma x f(x)$$
$$= 1 + \frac{5}{4} + \frac{7}{4}$$
$$= 4$$

$$E(Y) = \Sigma y g(y)$$
$$= 1 + 4/3 + 5/3$$
$$= \cancel{3}4$$

$$E(XY) = \Sigma xy J = 6/6 + 8/6 + 10/6 + 15/12 + 20/12 + 25/12$$
$$+ 21/12 + 28/12 + \frac{35}{12}$$
$$= \cancel{\frac{163}{12}} \; 16$$

$$cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= 16 - 16 = 0$$

iii)

$X+Y$	5	6	7	8	9	10	11	12
P	$1/9$	$1/9$	$1/9$	$1/9$	$1/9$	$2/9$	$1/9$	$1/9$