Graph Algorithms-IV

SINGLE-SOURCE SHORTEST PATHS

Introduction

Generalization of BFS to handle weighted graphs

- Direct Graph ***G*** = (***V***, ***E***), edge weight *fn* ; w : ***E*** $\rightarrow$ ***R***
- In BFS w(e)=1 for all e $\in$ E

Weight of path p = $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$$

Shortest Path

Shortest Path = Path of minimum weight

$$\delta(u,v) = \begin{cases} \min\{\omega(p) : u \overset{p}{\leadsto} v\}; & \text{if there is a path from u to v,} \\ \infty & \text{otherwise.} \end{cases}$$

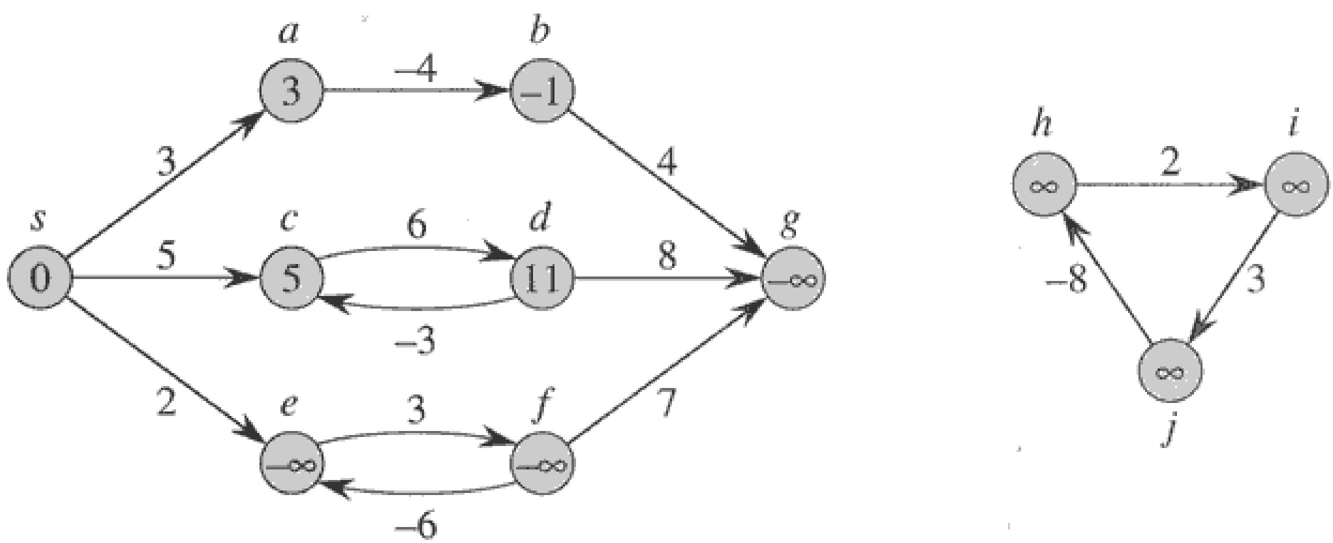
Shortest-Path Variants

- Shortest-Path problems
  - Single-source shortest-paths problem: Find the shortest path from *s* to each vertex *v*. (e.g. BFS)
  - Single-destination shortest-paths problem: Find a shortest path to a given *destination* vertex *t* from each vertex *v*.
  - Single-pair shortest-path problem: Find a shortest path from *u* to *v* for given vertices *u* and *v*.
  - All-pairs shortest-paths problem: Find a shortest path from *u* to *v* for every pair of vertices *u* and *v*.

Negative-weight edges

- No problem, as long as no negative-weight cycles are reachable from the source
- Otherwise, we can just keep going around it, and

get w(s, v) = −∞ for all v on the cycle.



Relaxation

- Maintain d[v] for each $v \in V$
- d[v] is called *shortest-path weight estimate*

```
INIT(G, s)
    for each v ∈ V do
        d[v] ← ∞
        π[v] ← NIL
    d[s] ← 0
```

Relaxation

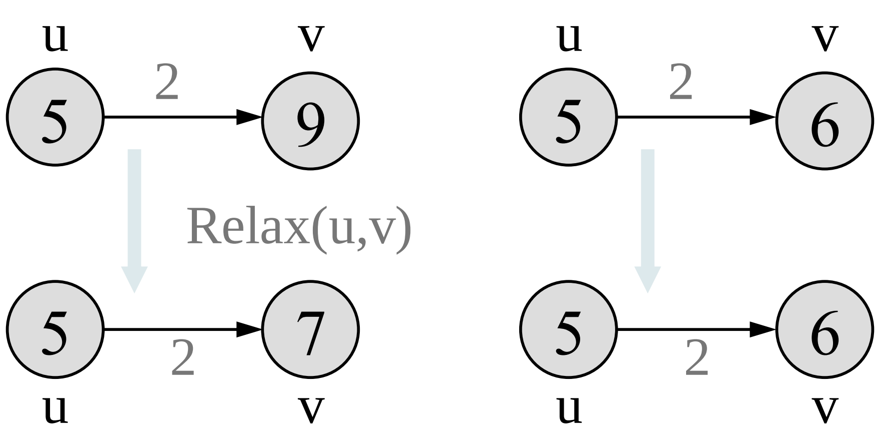
RELAX(u, v)

if d[v] > d[u]+w(u,v) then

d[v] ← d[u]+w(u,v)

π[v] ← u

The process of relaxing an edge(u,v) consists of testing whether we can improve the shortest path to v found so far by going through u,If so, update v.d and v. π



Dijkstra's Algorithm For Shortest Paths

- Non-negative edge weight
- Like BFS: If all edge weights are equal, then use BFS, otherwise use this algorithm
- Use Q = priority queue keyed on d[v] values (note: BFS uses FIFO)

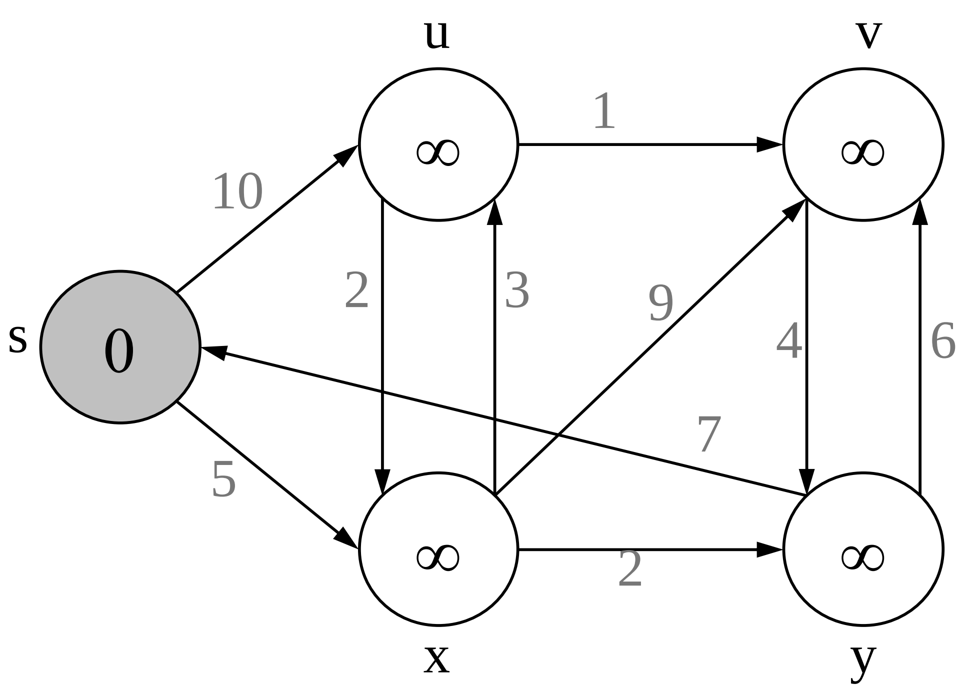
Dijkstra's Algorithm For Shortest Paths

Total time: $O(V+V\lg V + E\lg V) = O(E\lg V)$

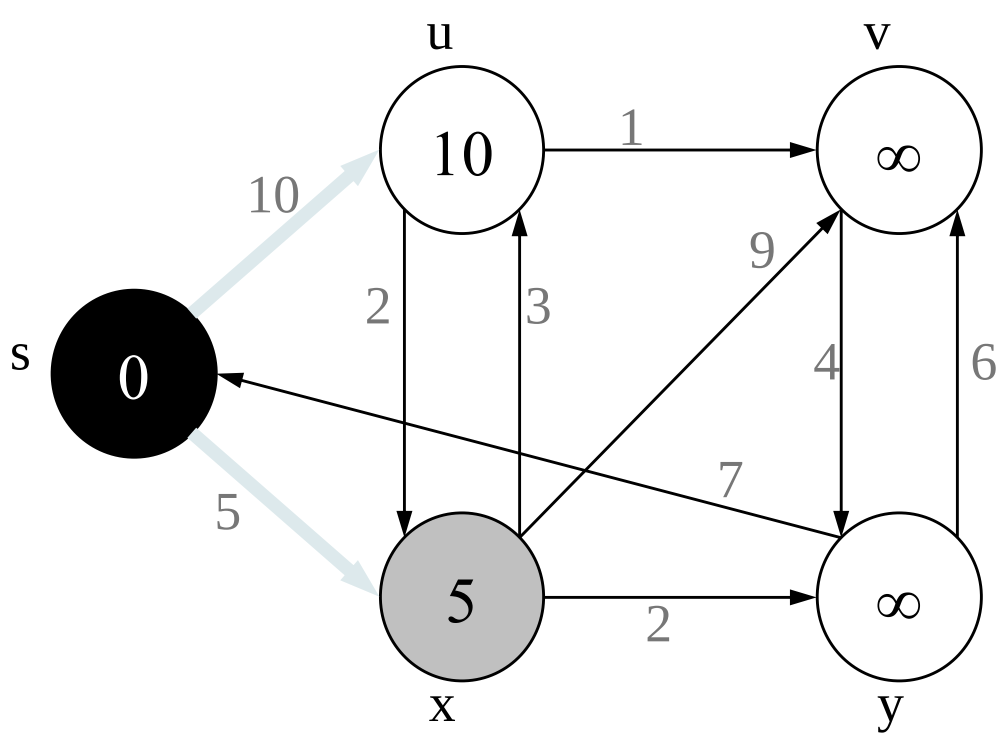
```
DIJKSTRA(G, s)
    INIT(G, s)
    S←Ø
    Q←V[G]
    while Q ≠Ø do
        u←EXTRACT-MIN(Q)
        S←S U {u}
        for each v ∈ Adj[u] do
            RELAX(u, v,w)
```

- INIT(G, s): $O(V)$
- Q←V[G]: $O(V)$ if Q is implemented as a min-heap
- u←EXTRACT-MIN(Q): Takes $O(\lg V)$
- Min-heap operations: $O(V\lg V)$
- for each v ∈ Adj[u] do: Executed O(E) times
- RELAX(u, v,w): Takes $O(\lg V)$
- $O(E\lg V)$

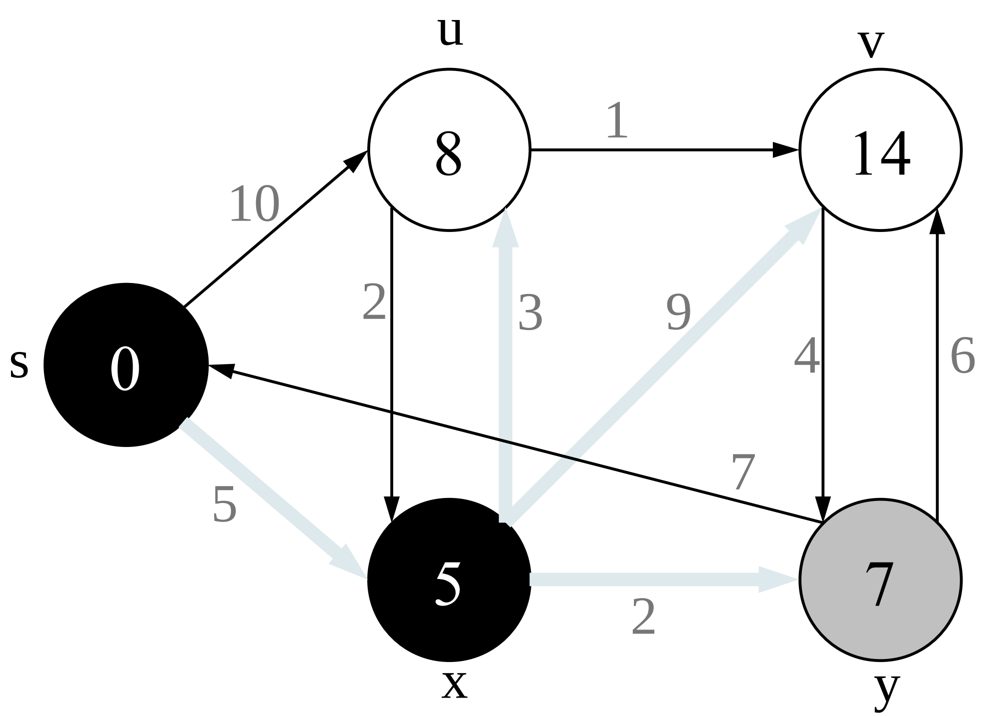
Example



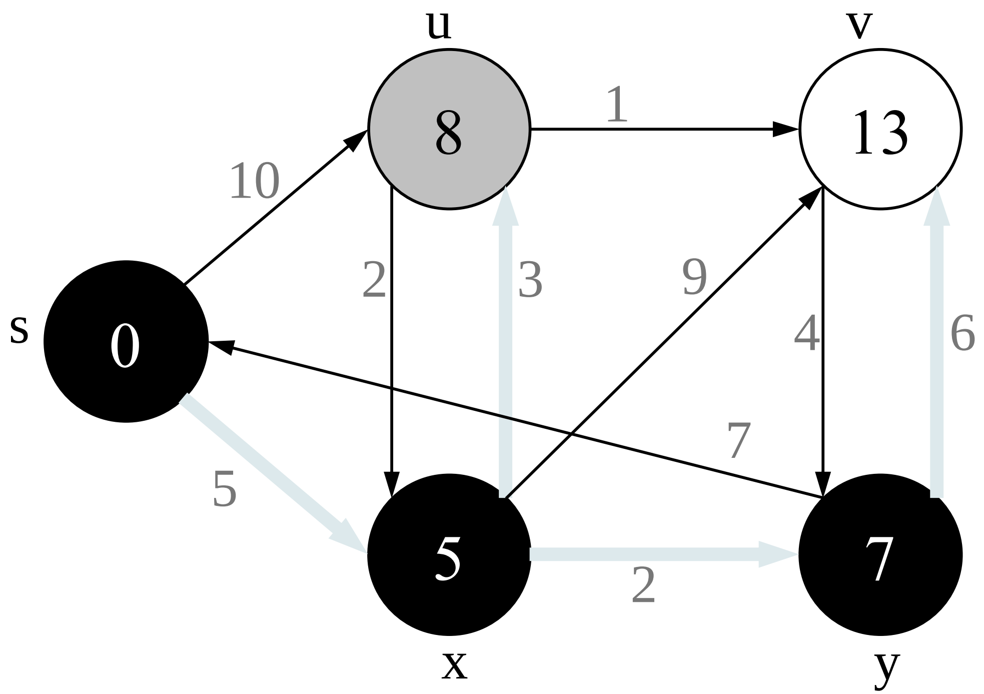
Example



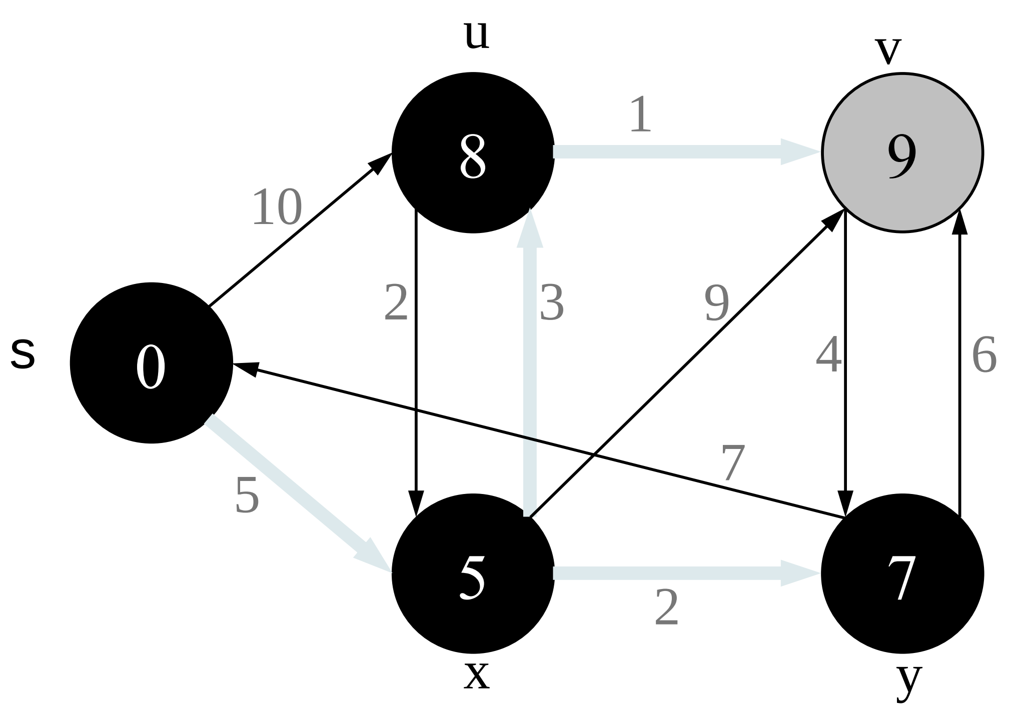
Example



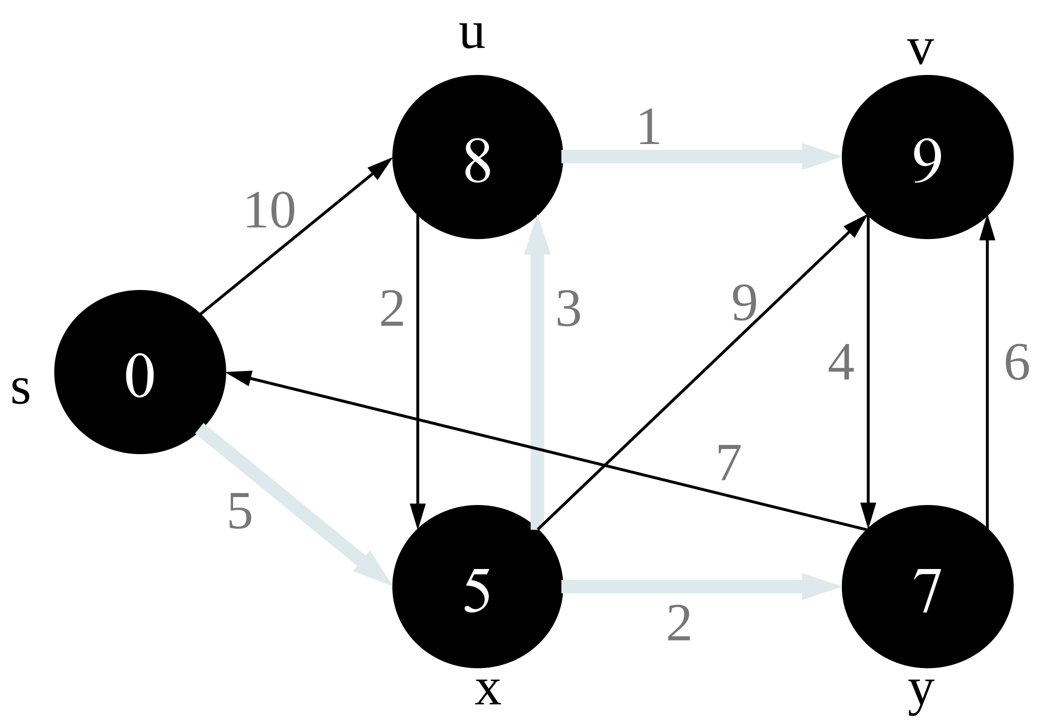
Example



Example



Example



Dijkstra's Algorithm For Shortest Paths

Observe :

- Each vertex is extracted from Q and inserted into S exactly once
- Each edge is relaxed exactly once
- S = set of vertices whose final shortest paths have already been determined
  - i.e. , $S = \{v \in V: d[v] = \delta(s, v) \neq \infty\}$

Dijkstra's Algorithm For Shortest Paths

- *Similar to BFS algorithm:* S corresponds to the set of black vertices in BFS which have their correct breadth-first distances already computed

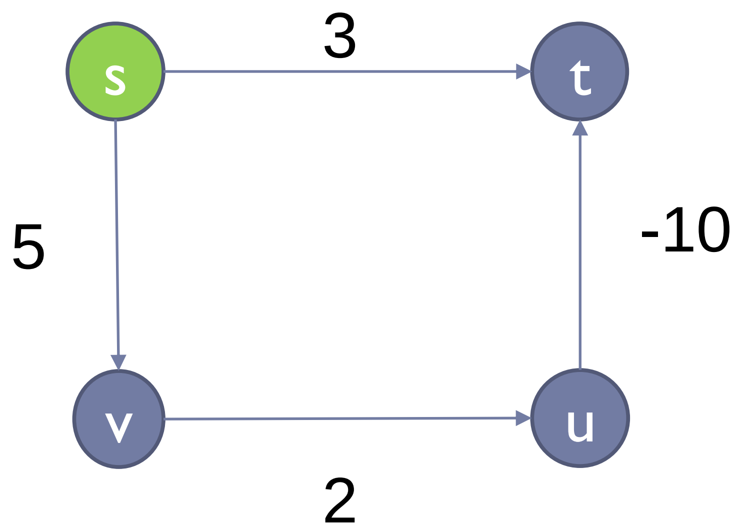
- *Greedy strategy:* Always chooses the closest(lightest) vertex in Q = V-S to insert into S

- Relaxation may reset d[v] values

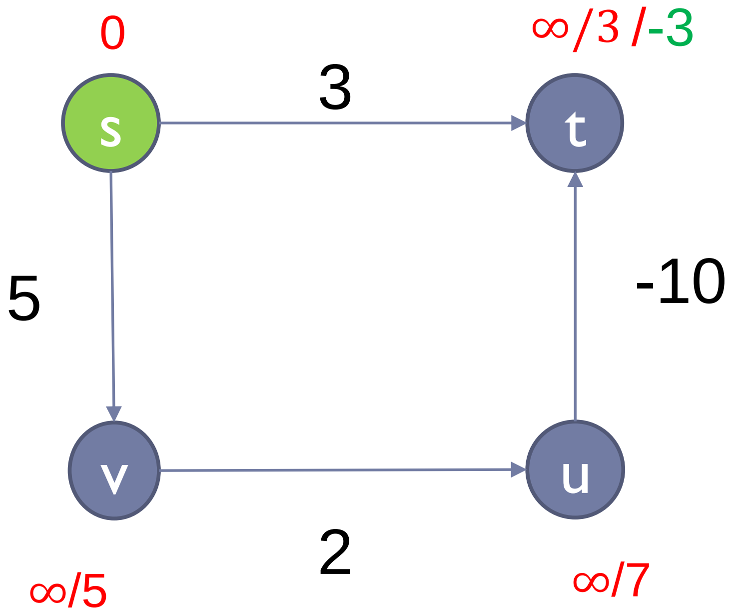
Dijkstra’s Algorithm For Shortest Paths

- Similar to Prim’s MST algorithm: Both algorithms use a priority queue to find the lightest vertex outside a given set S
- Insert this vertex into the set
- Adjust weights of remaining adjacent vertices outside the set accordingly

Dijkstra’s Algorithm-Disadvantages

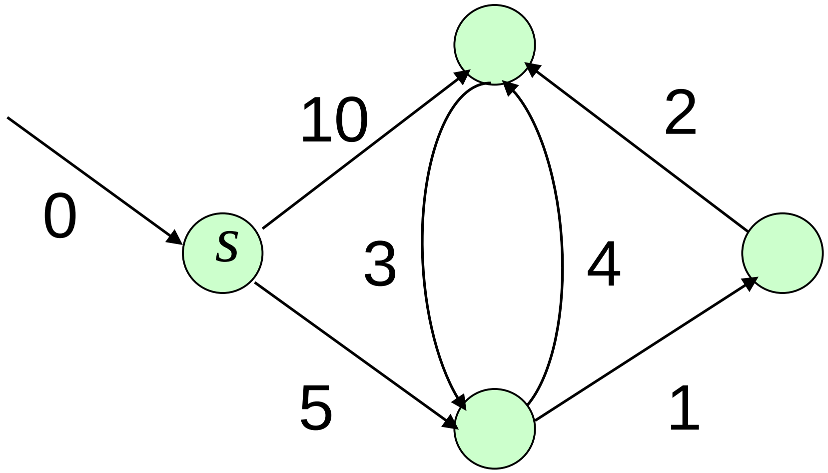


Dijkstra's Algorithm-Disadvantages



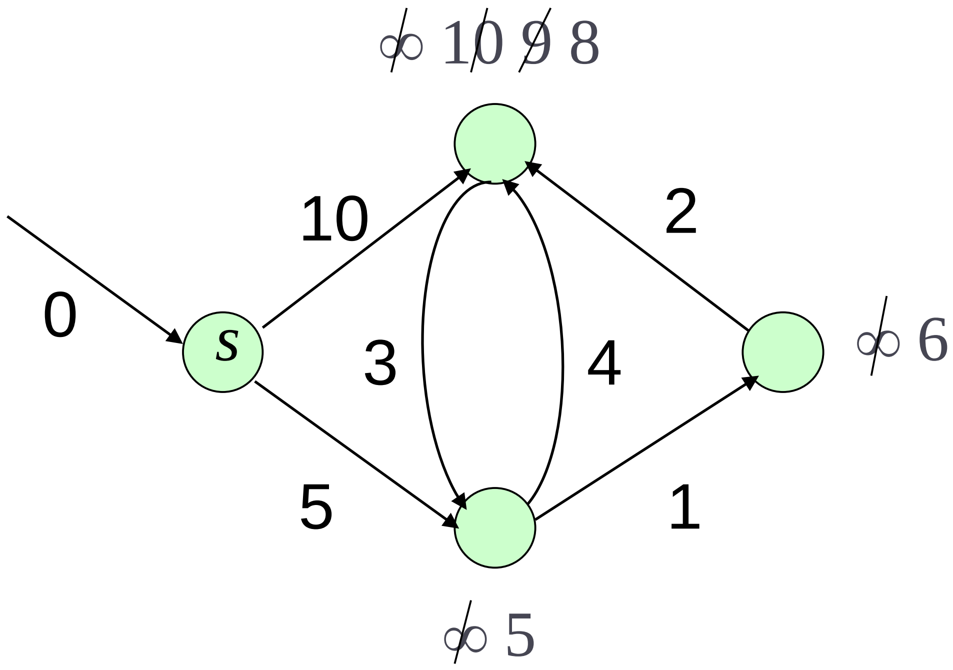
Dijkstra's Algorithm For Shortest Paths

Example: Run algorithm on a sample graph

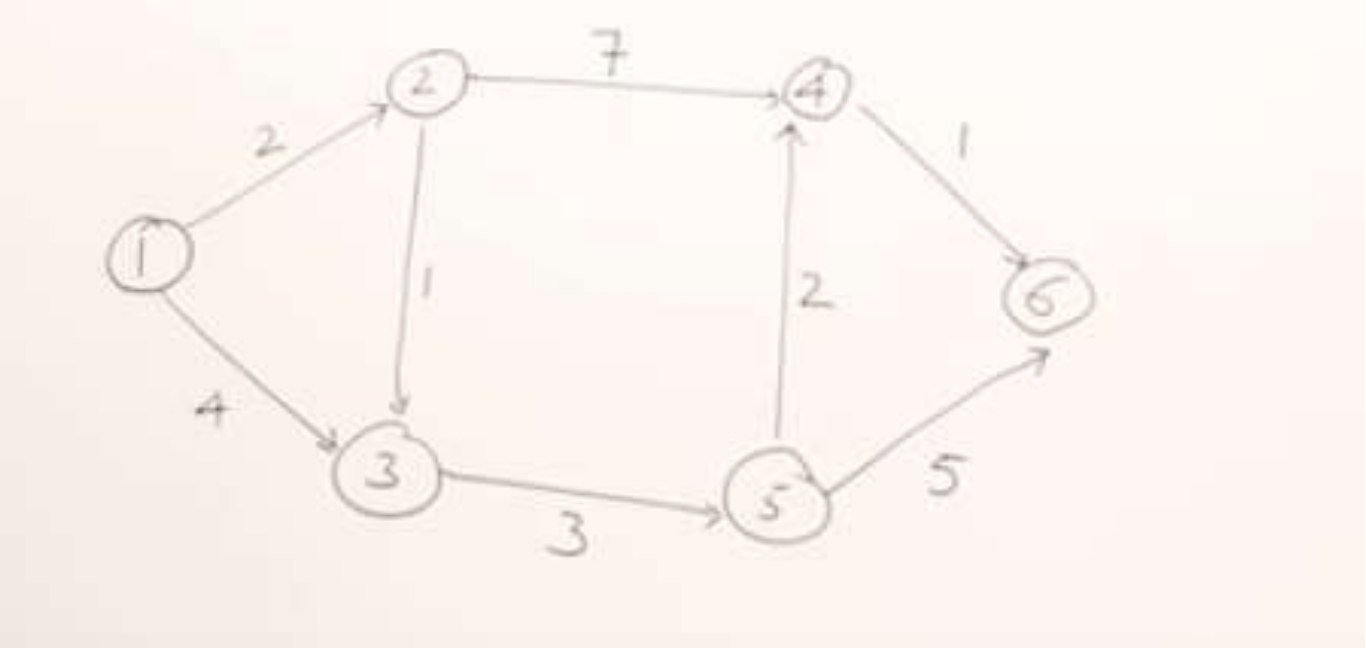


Dijkstra's Algorithm For Shortest Paths

Example: Run algorithm on a sample graph



Problem



All-Pair Shortest Path

The Floyd-Warshall Algorithm

A recursive solution to the all-pairs shortest paths problem:

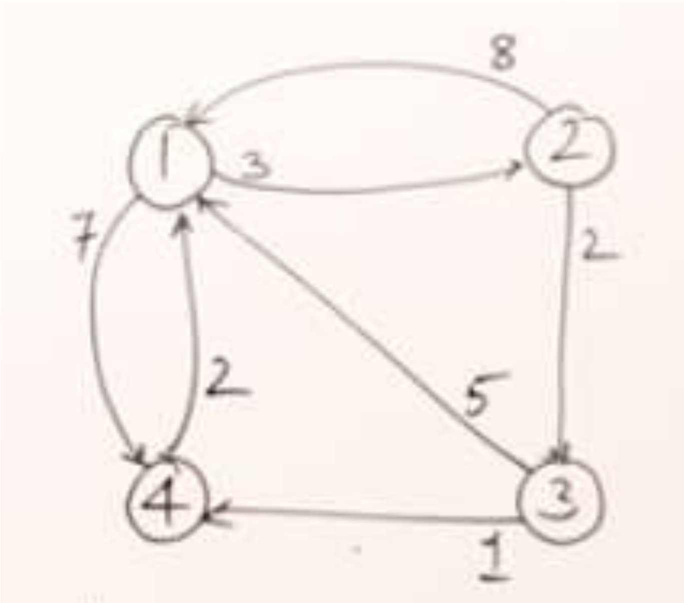
- Let $d_{ij}^{(k)}$ be the weight of a shortest path from vertex i to vertex j with all intermediate vertices in the set {1,2,…,k}.A recursive definition is given by
- $d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k=0, \\ \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}) & \text{if } k \geq 1. \end{cases}$

- The matrix $D^{(n)}=(d_{ij}^{(n)})$ gives the final answer-$d_{ij}^{(n)}=\delta(i, j)$ for all $i,j \in V$-because all intermediate vertices are in the set {1,2,…,n}.

Computing the shortest-path weights bottom up:

- FLOYD-WARSHALL(W)
- n $\leftarrow$ rows[W]
- $D^{(0)} \leftarrow W$
- for k $\leftarrow$ 1 to n
-     do
-         for i $\leftarrow$ 1 to n
-             do for j $\leftarrow$ 1 to n
-                 $d_{ij}^{(k)} \leftarrow \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$
- return $D^{(n)}$

Example:



$$
A^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & \infty \\ 5 & \infty & 0 & 1 \\ 2 & \infty & \infty & 0 \end{bmatrix}
\end{array}
$$

$$A^1 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & \infty & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3 & 5 & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{bmatrix} \end{array}$$

$$A^3 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3 & 5 & 6 \\ 7 & 0 & 2 & 3 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{bmatrix} \end{array}$$

$$A^4 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3 & 5 & 6 \\ 5 & 0 & 2 & 3 \\ 3 & 6 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{bmatrix} \end{array}$$