# Elementary Graph Algorithms

Breadth First Search, Depth First Search

# Graphs

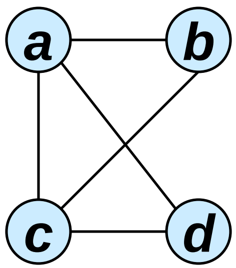
- *Graph* $G = (V, E)$
  - $V$ = set of vertices
  - $E$ = set of edges $\subseteq (V \times V)$
- Types of graphs
  - Undirected: edge $(u, v) = (v, u)$; for all $v$, $(v, v) \notin E$ (No self loops.)
  - Directed: $(u, v)$ is edge from $u$ to $v$, denoted as $u \rightarrow v$. Self loops are allowed.
  - Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
  - Dense: $|E| \approx |V|^2$.
  - Sparse: $|E| << |V|^2$.
- $|E| = O(|V|^2)$

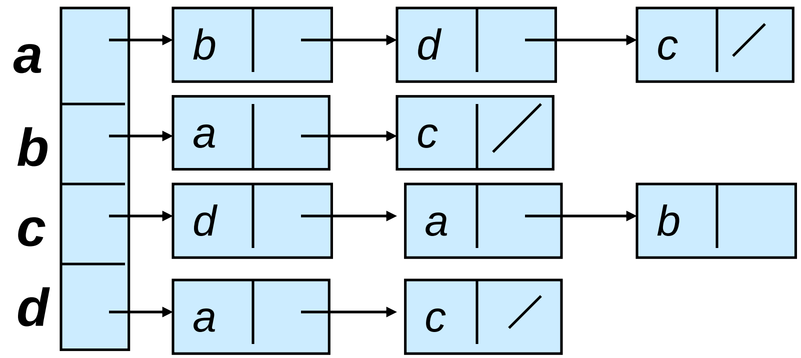
# Graphs

- If $(u, v) \in E$, then vertex $v$ is adjacent to vertex $u$.
- Adjacency relationship is:
  - Symmetric if $G$ is undirected.
  - Not necessarily so if $G$ is directed.
- If $G$ is connected:
  - There is a path between every pair of vertices.
  - $|E| \geq |V| - 1$.
  - Furthermore, if $|E| = |V| - 1$, then $G$ is a tree.

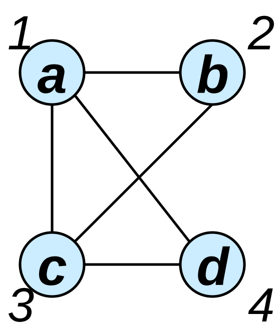
# Representation of Graphs

- Two standard ways.
  - Adjacency Lists.



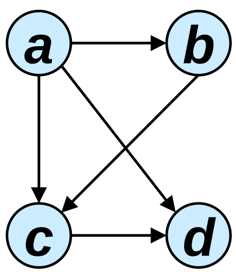

  - Adjacency Matrix.

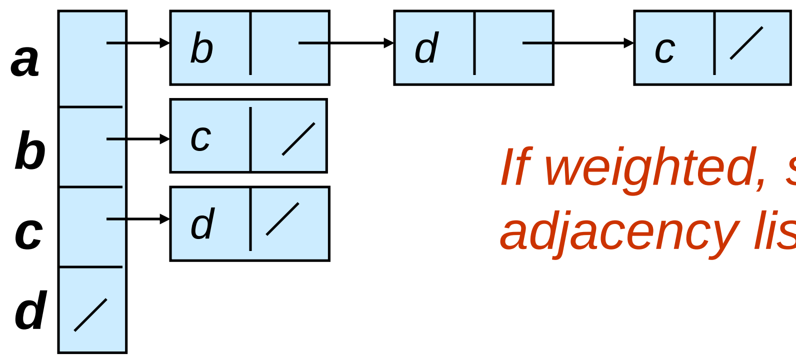



| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 0 | 1 |
| 4 | 1 | 0 | 1 | 0 |

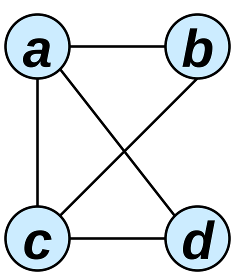
# Adjacency Lists

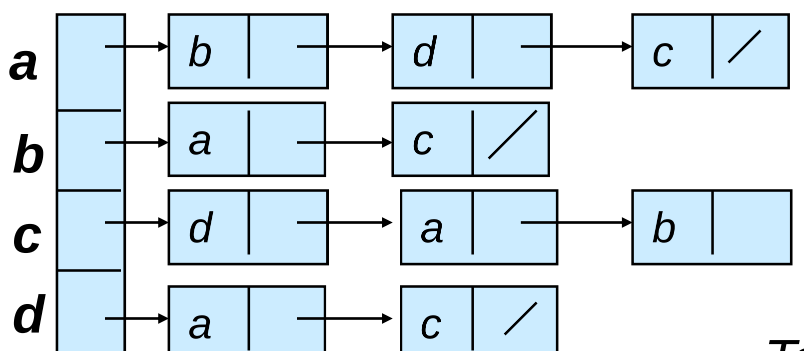
- Consists of an array *Adj* of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to $u$.




*If weighted, store weights also in adjacency lists.*




*Total storage:* $\Theta(V+E)$

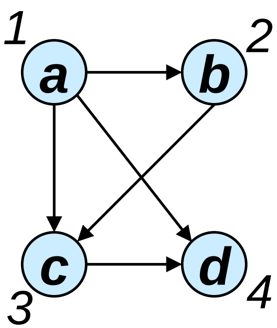
# Pros and Cons: adj list

- Pros
  - Space-efficient, when a graph is sparse.
  - Can be modified to support many graph variants.
- Cons
  - Determining if an edge $(u,v) \in G$ is not efficient.
    - Have to search in $u$'s adjacency list. $\Theta(\text{degree}(u))$ time.
    - $\Theta(V)$ in the worst case.

# Adjacency Matrix

- $|V| \times |V|$ matrix $A$.
- Number vertices from 1 to $|V|$ in some arbitrary manner.
- $A$ is then given by:

$$A[i,j] = a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
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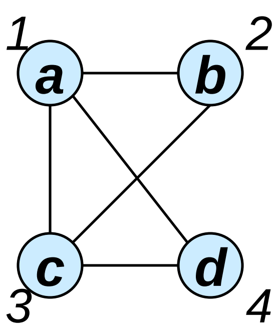
| 3 | 0 | 0 | 0 | 1 |
| 4 | 0 | 0 | 0 | 0 |




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 1 |
| 2 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 0 | 1 |
| 4 | 1 | 0 | 1 | 0 |

*$A = A^T$ for undirected graphs.*

# Space and Time

- **Space:** $\Theta(V^2)$.
  - Not memory efficient for large graphs.
- **Time:** to list all vertices adjacent to $u$: $\Theta(V)$.
- **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.
- Can store weights instead of bits for weighted graph.
- Advantages:
  - Simpler, preferred for graphs that are reasonably small.
  - Only one bit per entry for unweighted graphs

# Graph Searching

- Given: a graph $G = (V, E)$, directed or undirected
- Goal: methodically explore every vertex and every edge
- Ultimately: build a tree on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a tree

# Breadth-First Search

- “Explore” a graph, turning it into a tree
  - One vertex at a time
  - Expand frontier of explored vertices across the *breadth* of the frontier
- Builds a tree over the graph
  - Pick a *source vertex* to be the root
  - Find (“discover”) its children, then their children, etc.

# Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - $d[v]$ = distance (smallest # of edges, or shortest path) from $s$ to $v$, for all $v \in V$. $d[v] = \infty$ if $v$ is not reachable from $s$.
  - $\pi[v] = u$ such that $(u, v)$ is last edge on shortest path $s \rightsquigarrow v$.
    - $u$ is $v$'s predecessor.
  - Builds breadth-first tree with root $s$ that contains all reachable vertices.

*Definitions:*
*Path between vertices u and v: Sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $u=v_1$ and $v=v_k$, and $(v_i,v_{i+1}) \in E$, for all $1 \le i \le k-1$.*
*Length of the path: Number of edges in the path.*
*Path is simple if no vertex is repeated.*

# Breadth-first Search

- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
  - A vertex is "discovered" the first time it is encountered during the search.
  - A vertex is "finished" if all vertices adjacent to it have been discovered.
- Colors the vertices to keep track of progress.
  - White – Undiscovered.
    - All vertices start out white
  - Gray – Discovered but not finished/fully explored.
    - Adjacent to white vertices
  - Black – Discovered and Finished/fully explored.
    - Colors are required only to reason about the algorithm. Can be implemented without colors.
- Explore vertices by scanning adjacency list of grey vertices

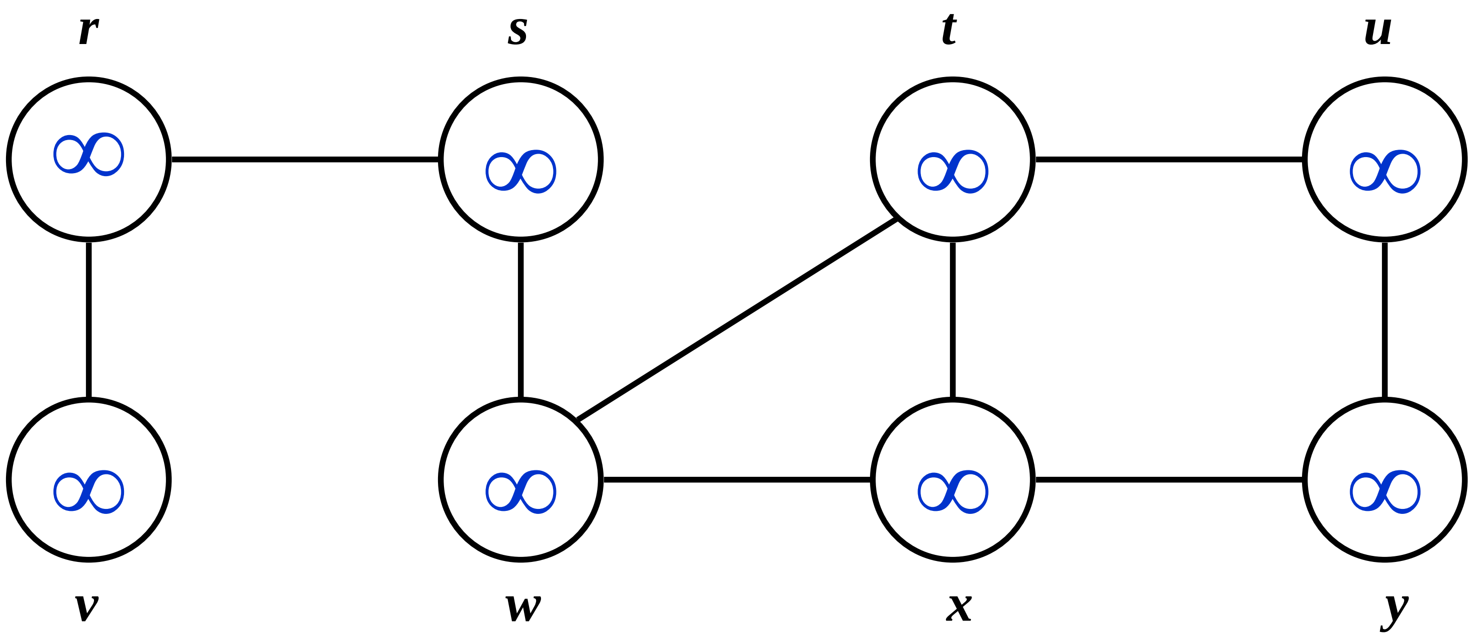
**BFS(G,s)**

```
for each vertex u in V[G] – {s}
        do color[u] ← white
            d[u] ← ∝
            π[u] ← nil
color[s] ← gray
d[s] ← 0
 π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
        do u ← dequeue(Q)
                for each v in Adj[u]
                        do if color[v] = white
                                then color[v] ← gray
                                    d[v] ← d[u] + 1
                                    π[v] ← u
                                    enqueue(Q,v)
                color[u] ← black
```

*white: undiscovered*
*gray: discovered*
*black: finished*

*Q: a queue of discovered vertices*
*color[v]: color of v*
*d[v]: distance from s to v*
*π[u]: predecessor of v*

# Breadth-First Search: Example



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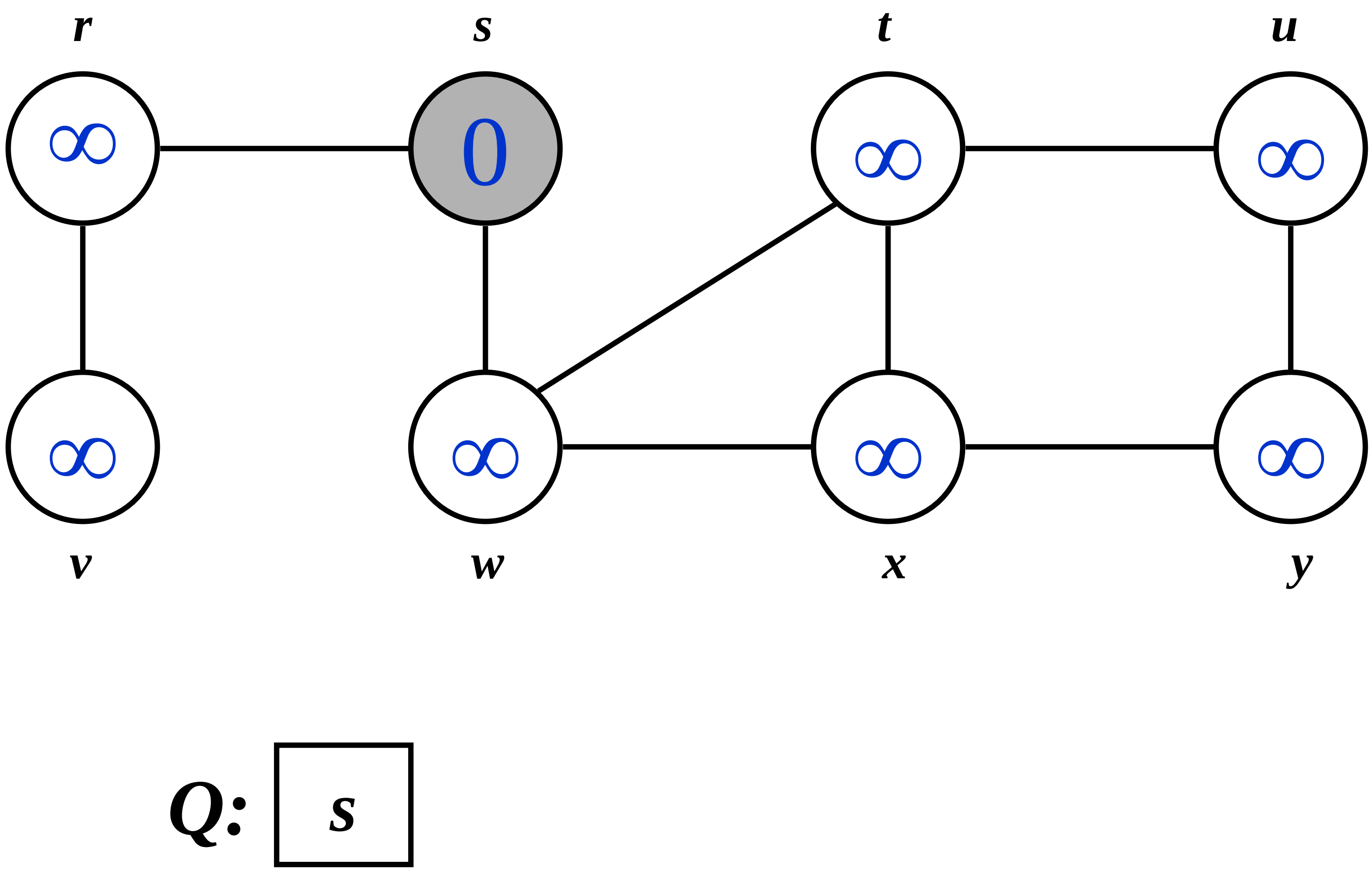

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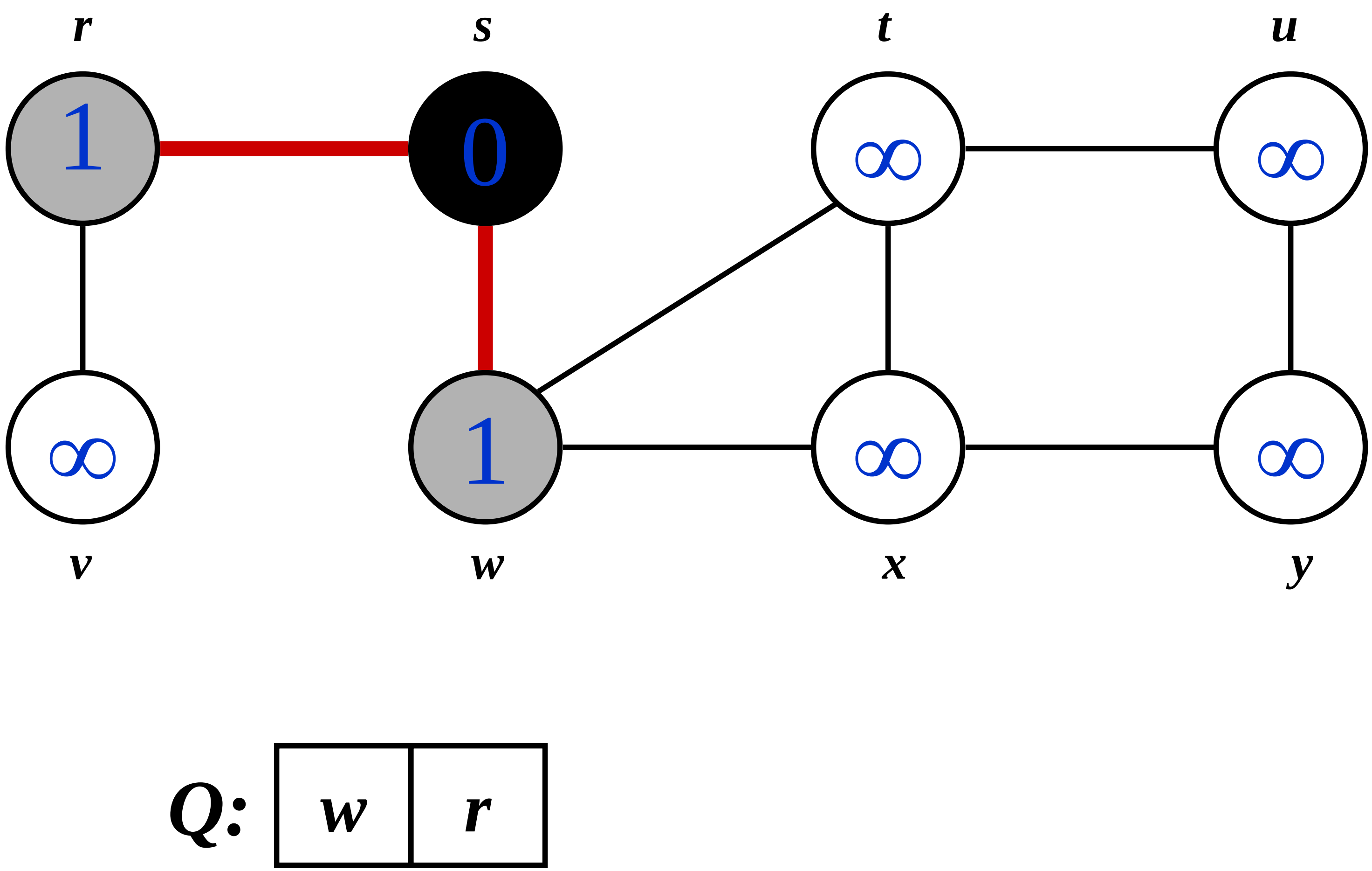

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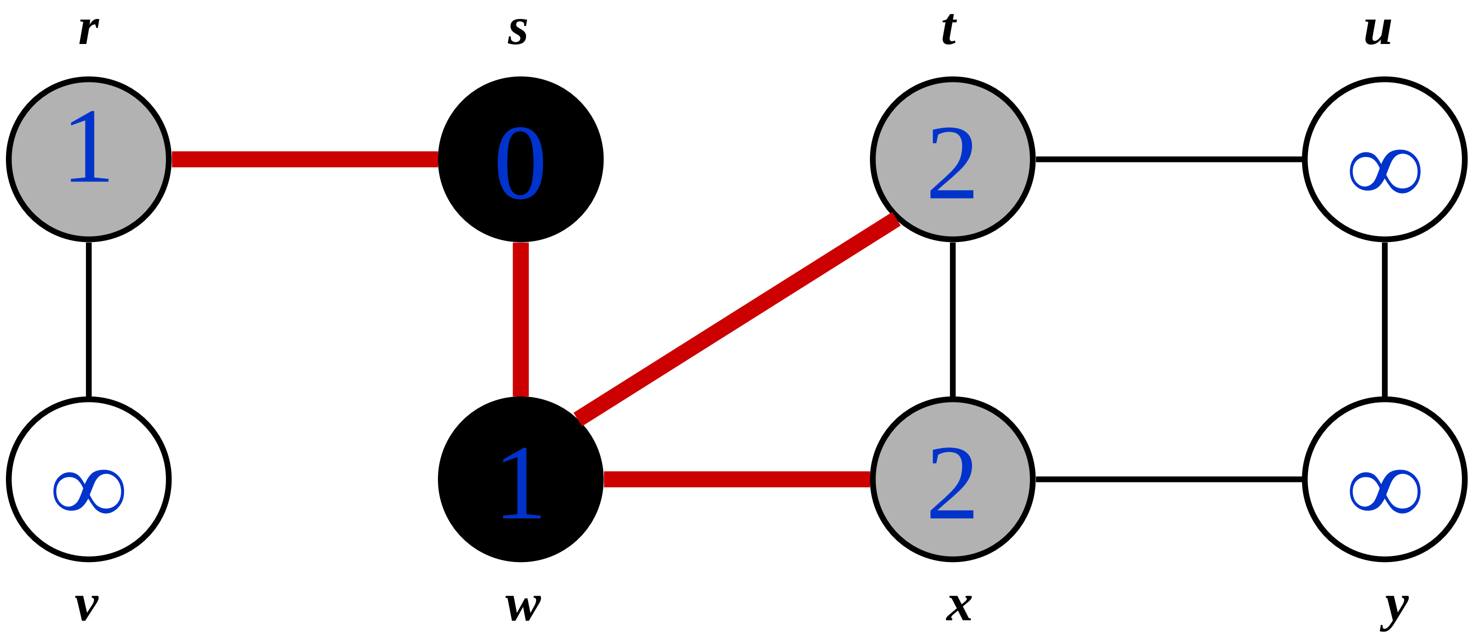
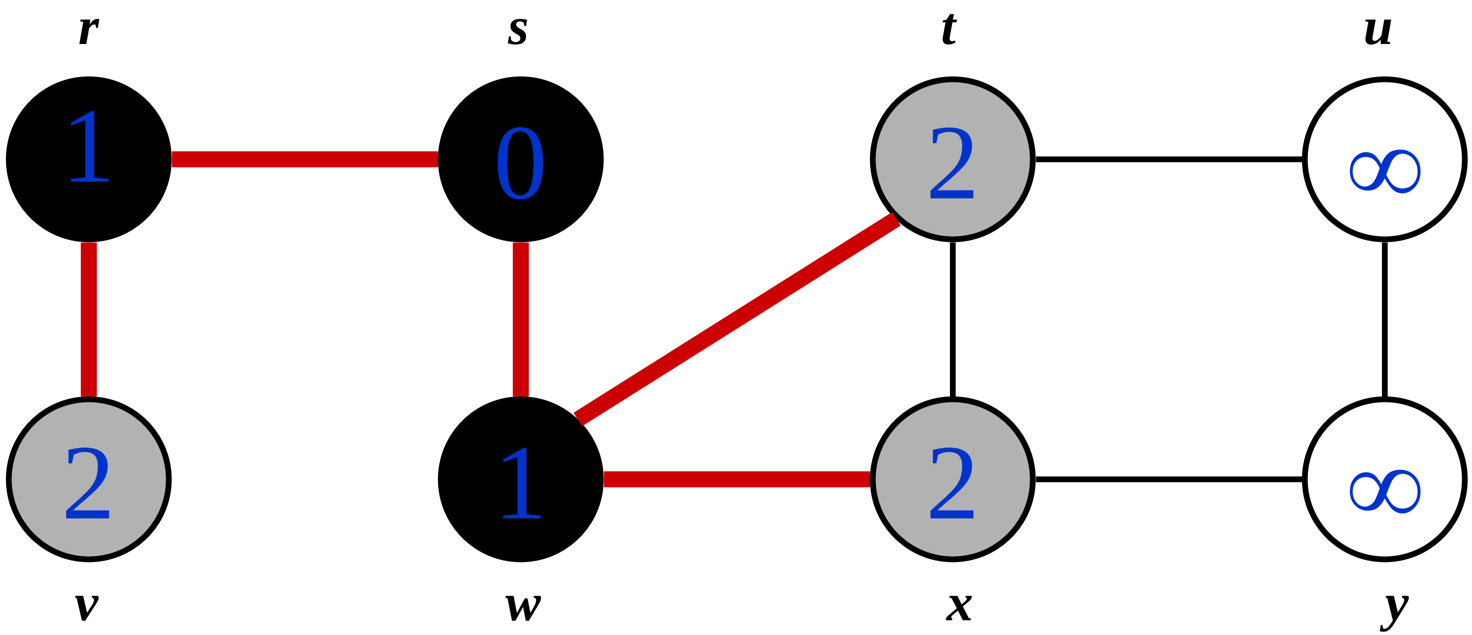


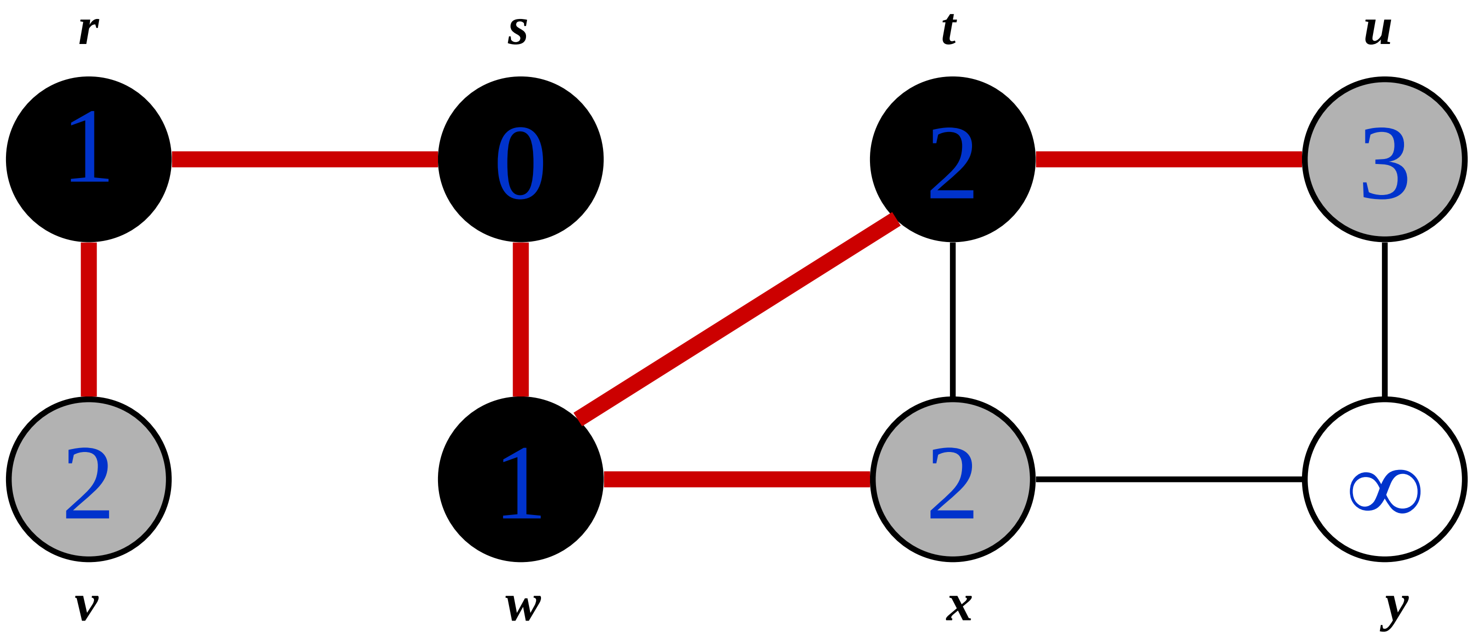

**Q:** | *r* | *t* | *x* |

# Breadth-First Search: Example



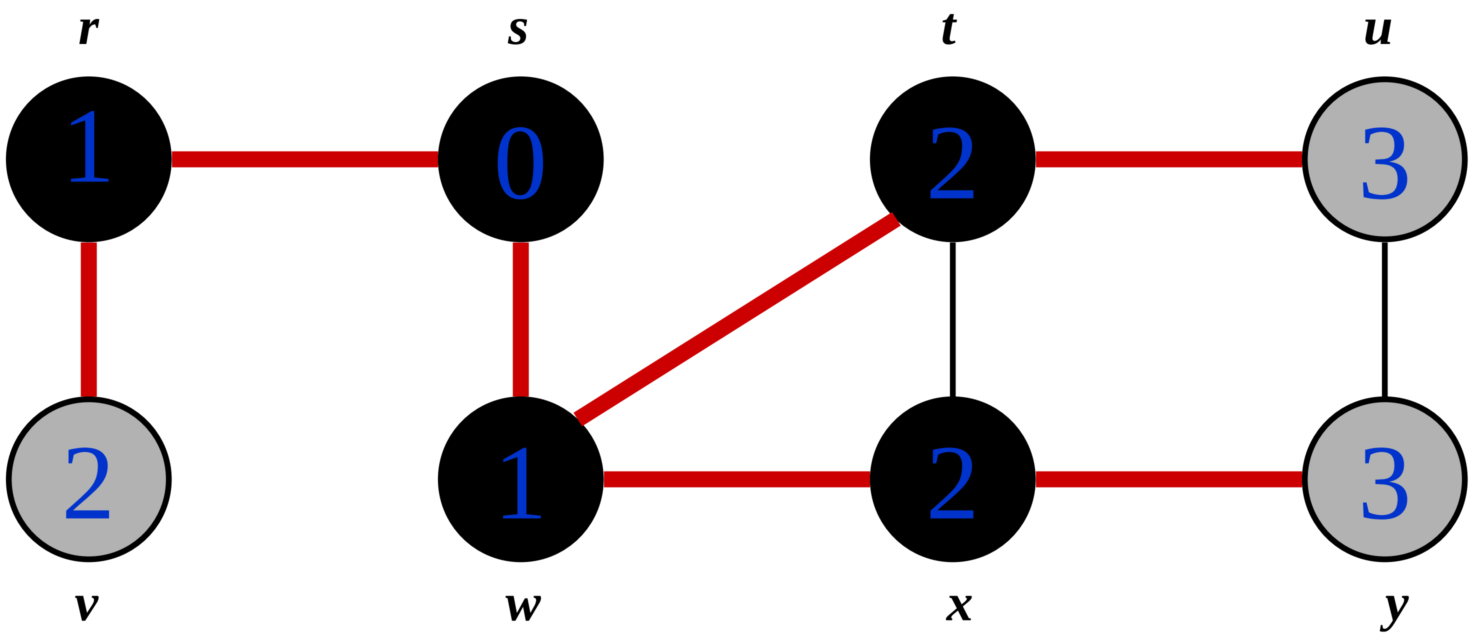

| Q: | t | x | v |
|---|---|---|---|

# Breadth-First Search: Example




Q: | x | v | u |

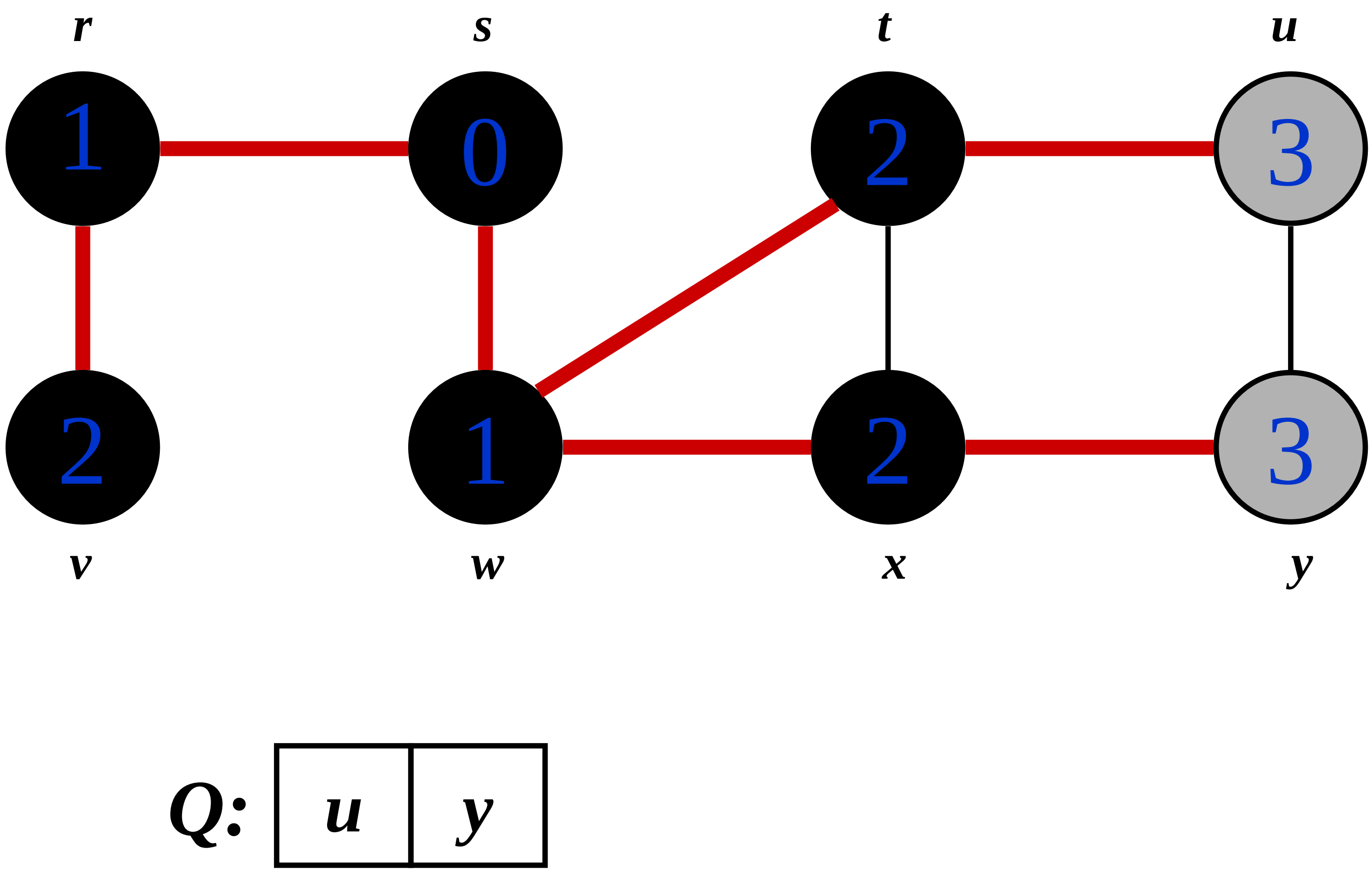
# Breadth-First Search: Example




**Q:**

| | | |
|---|---|---|
| *v* | *u* | *y* |

# Breadth-First Search: Example



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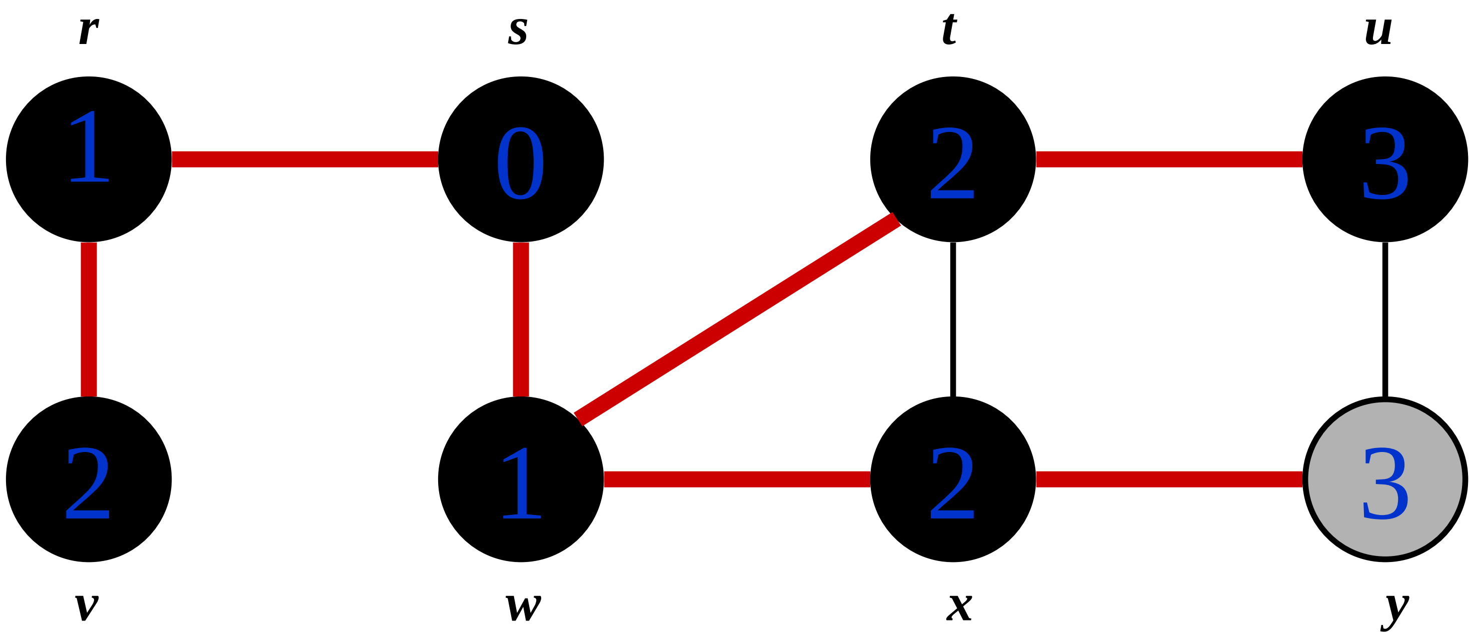
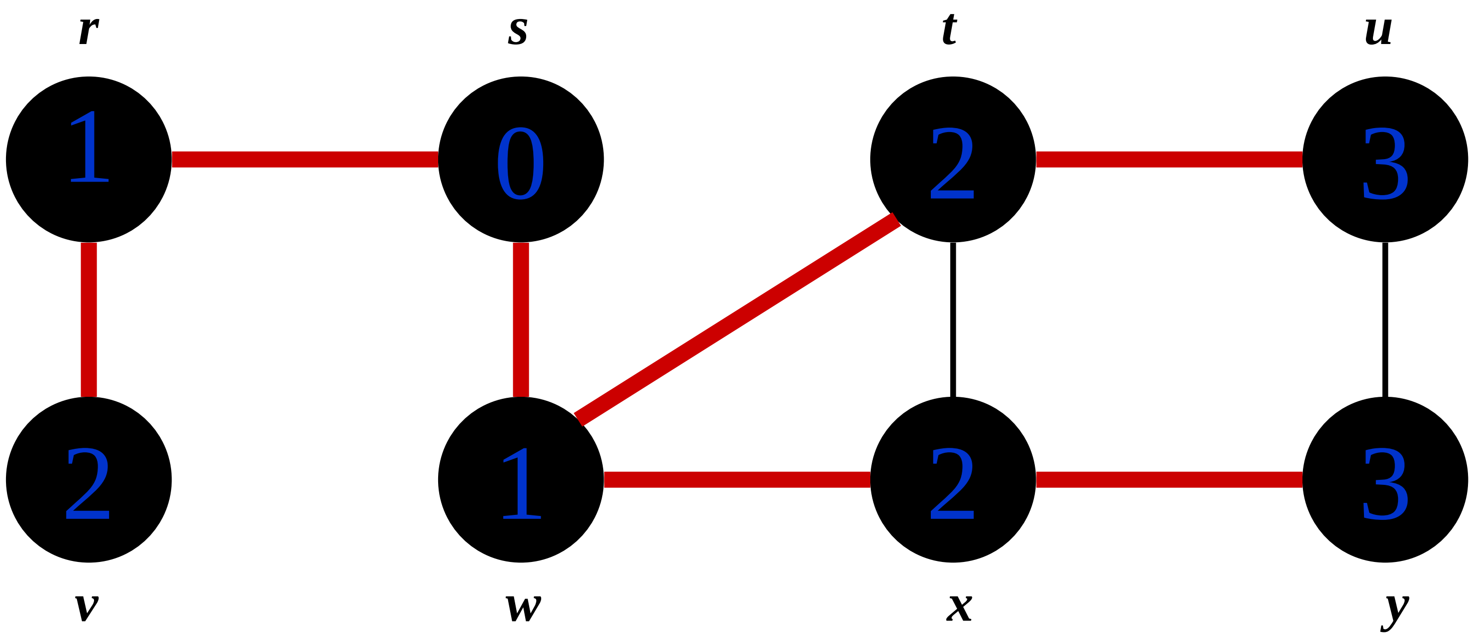



**Q:** $y$

# Breadth-First Search: Example




***Q:*** Ø

# Analysis of BFS

- Initialization takes $O(V)$.
- Traversal Loop
  - After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(V)$.
  - The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(E)$.
- Summing up over all vertices => total running time of BFS is $O(V+E)$, linear in the size of the adjacency list representation of graph.

# Breadth-First Search: Properties

- BFS calculates the *shortest-path distance* to the source node
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or $\infty$ if v not reachable from s
- BFS builds *breadth-first tree*, in which paths to root represent shortest paths in G
  - Thus can use BFS to calculate shortest path from one vertex to another in $O(V+E)$ time

# Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore "deeper" in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex *v* that still has unexplored edges
  - When all of *v*'s edges have been explored, backtrack to the vertex from which *v* was discovered

# Depth-First Search

- Vertices initially colored white
- Then colored gray when discovered
- Then black when finished

# Pseudo-code

**DFS(G)**

```
for each vertex u ∈ V[G]
    do color[u] ← white
       π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
    do if color[u] = white
          then DFS-Visit(u)
```

***DFS-Visit(u)***

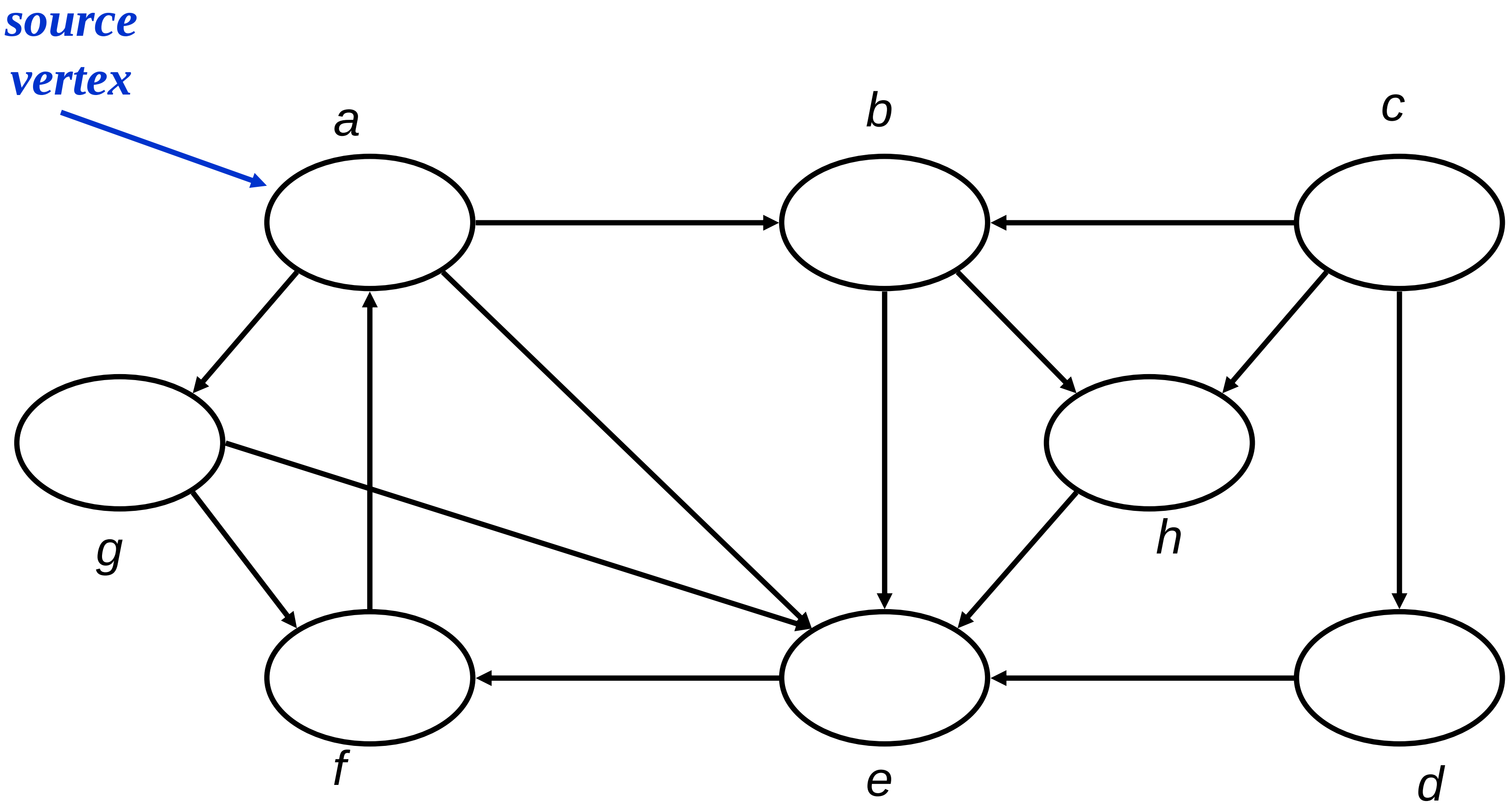
```
color[u] ← GRAY    ▽ White vertex u has been discovered
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]
    do if color[v] = WHITE
          then π[v] ← u
               DFS-Visit(v)
color[u] ← BLACK    ▽ Blacken u; it is finished.
f[u] ← time ← time + 1
```

*Uses a global timestamp **time**.*

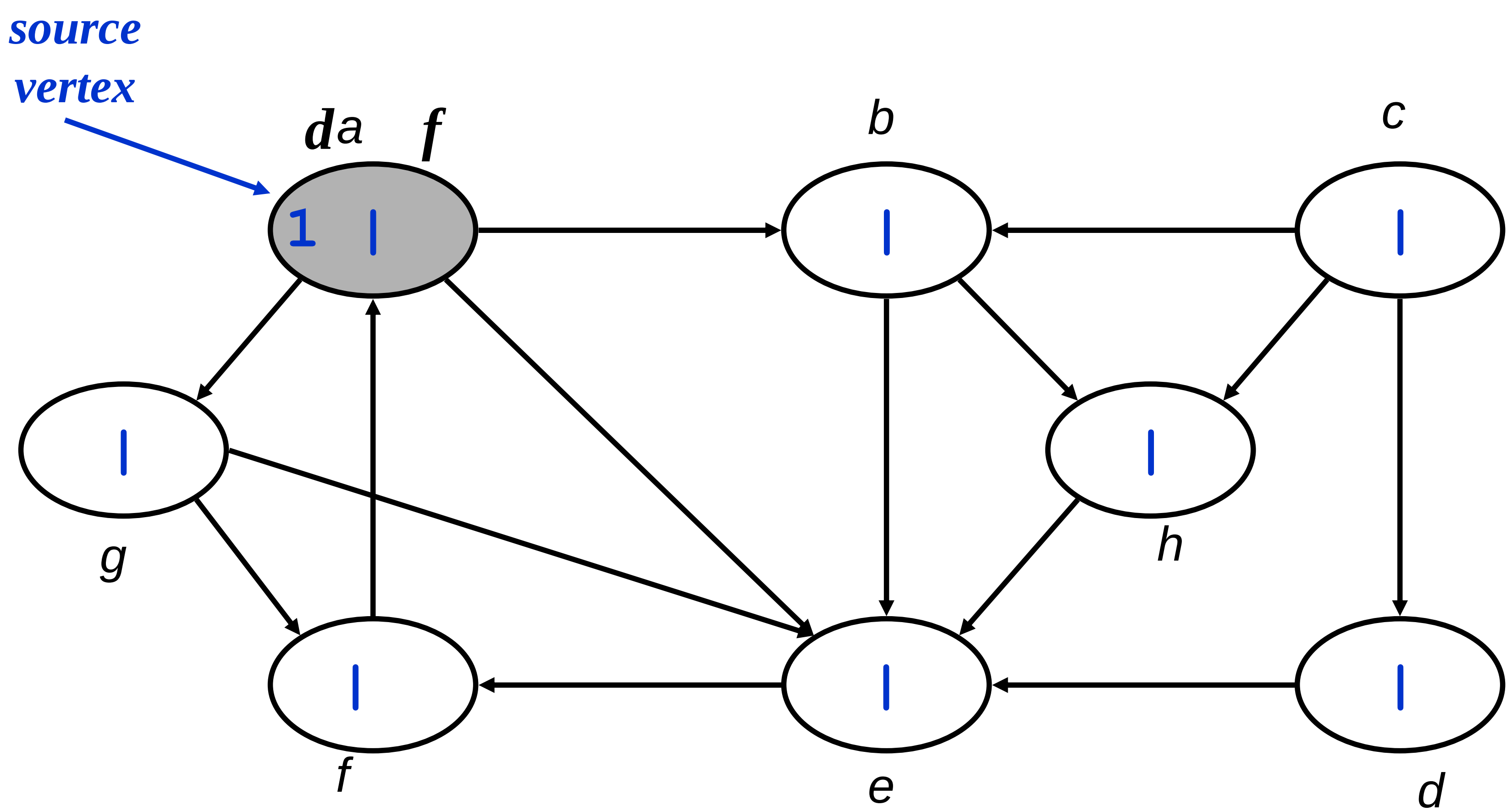
# Depth-First Sort Analysis

- "Charge" the exploration of edge to the edge:
  - Each loop in DFS_Visit can be attributed to an edge in the graph
  - Thus loop will run in O(E) time, algorithm O(V+E)

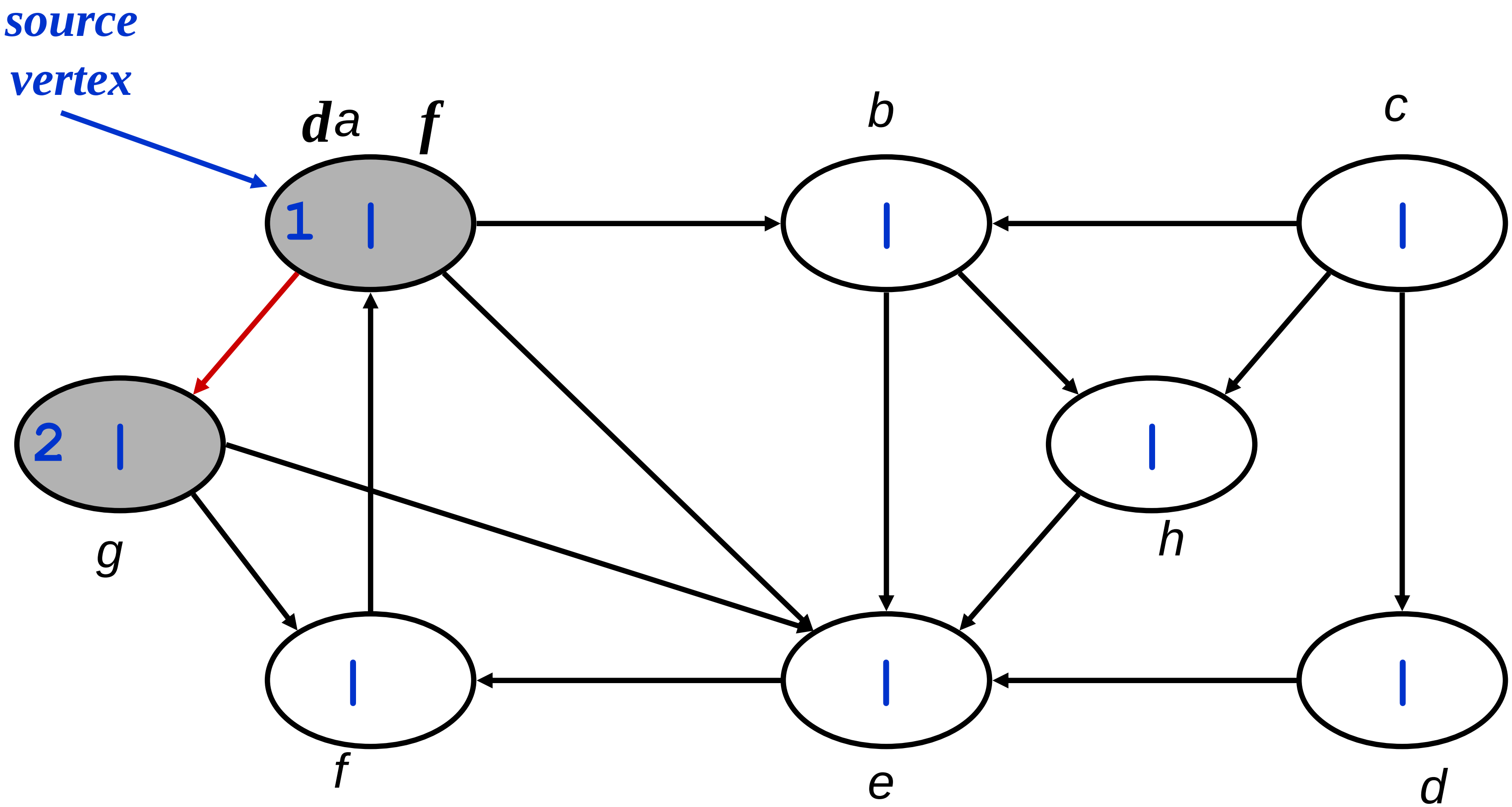
# DFS Example



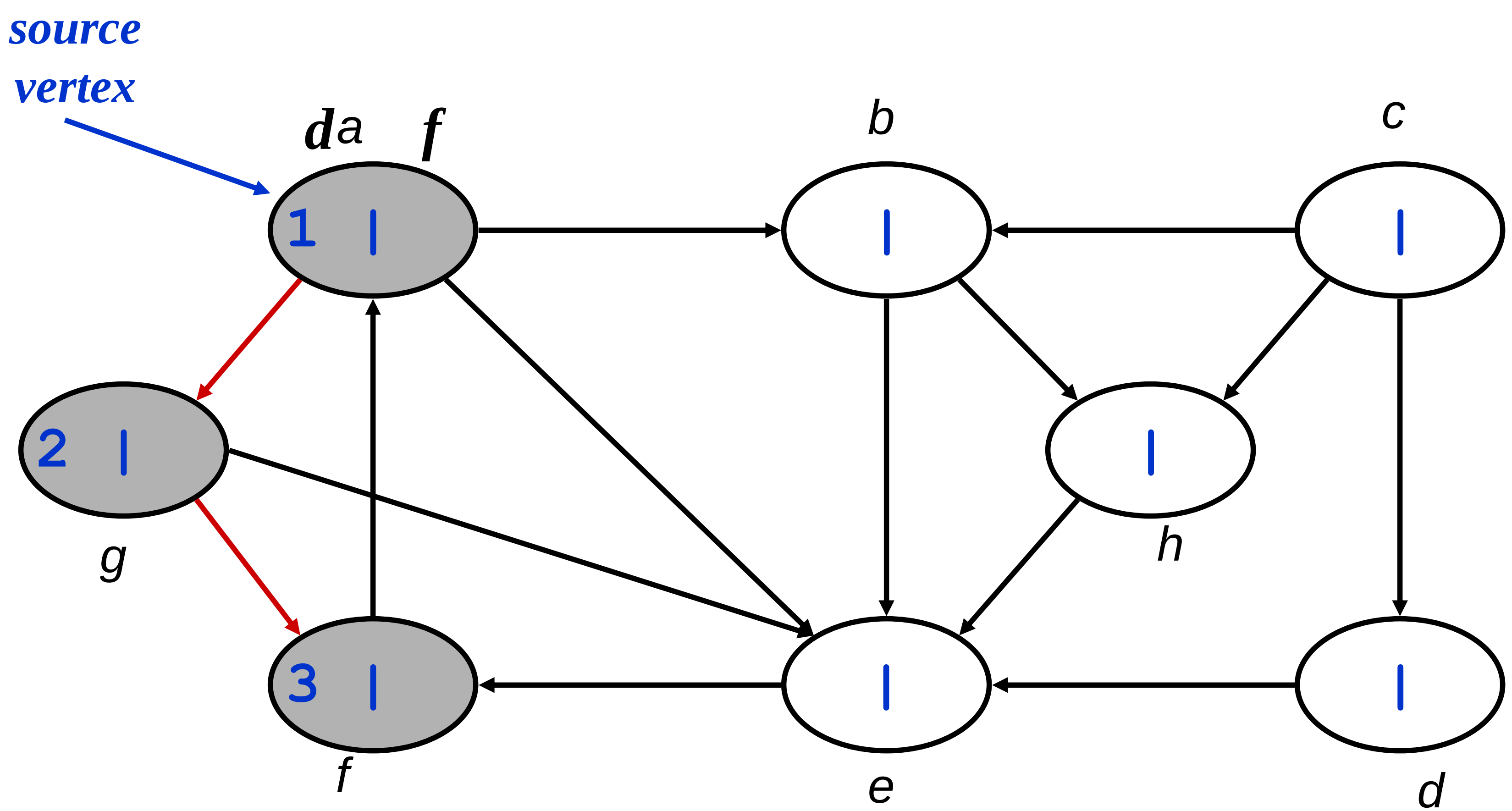
# DFS Example



# DFS Example



# DFS Example



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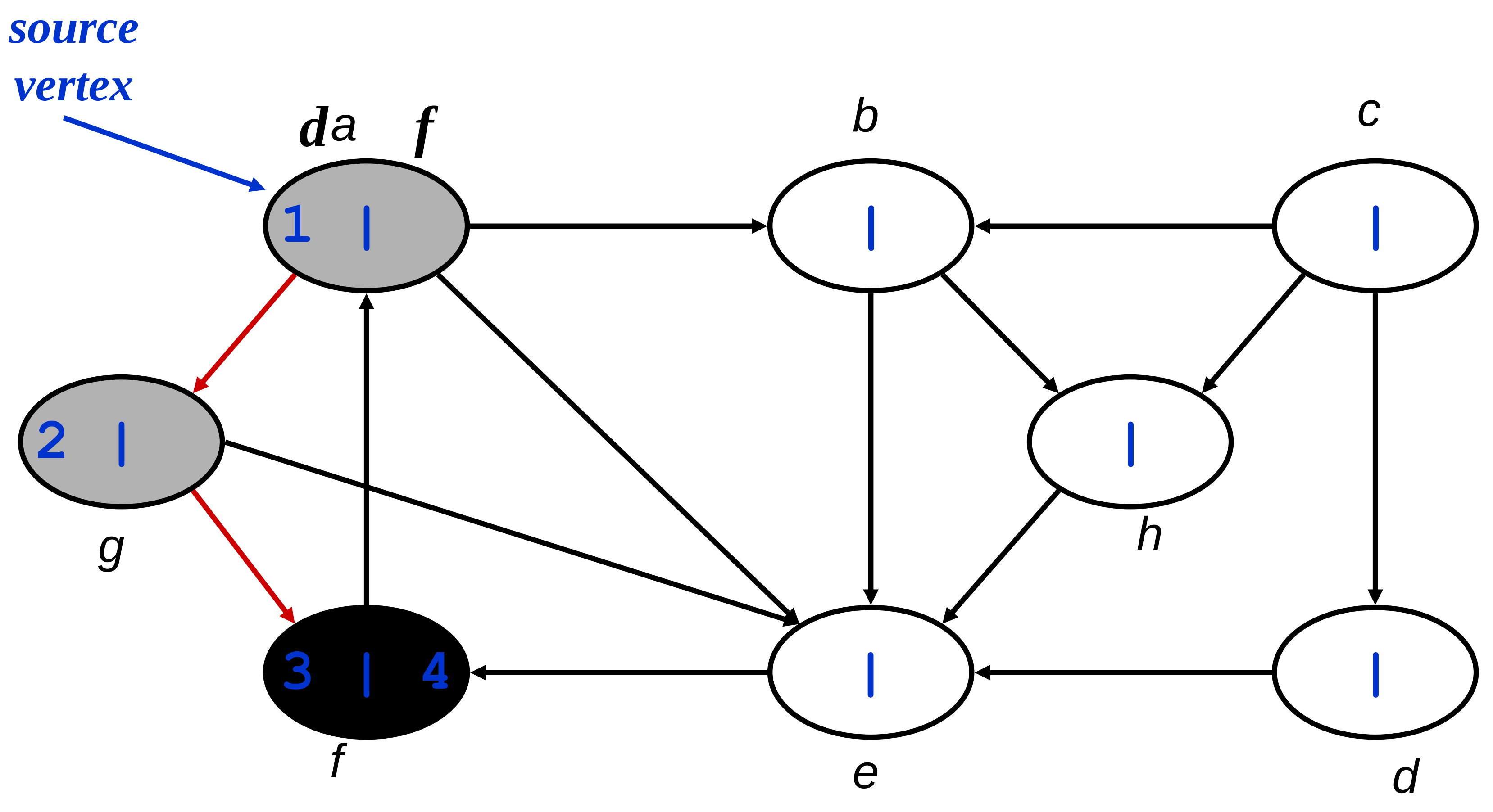

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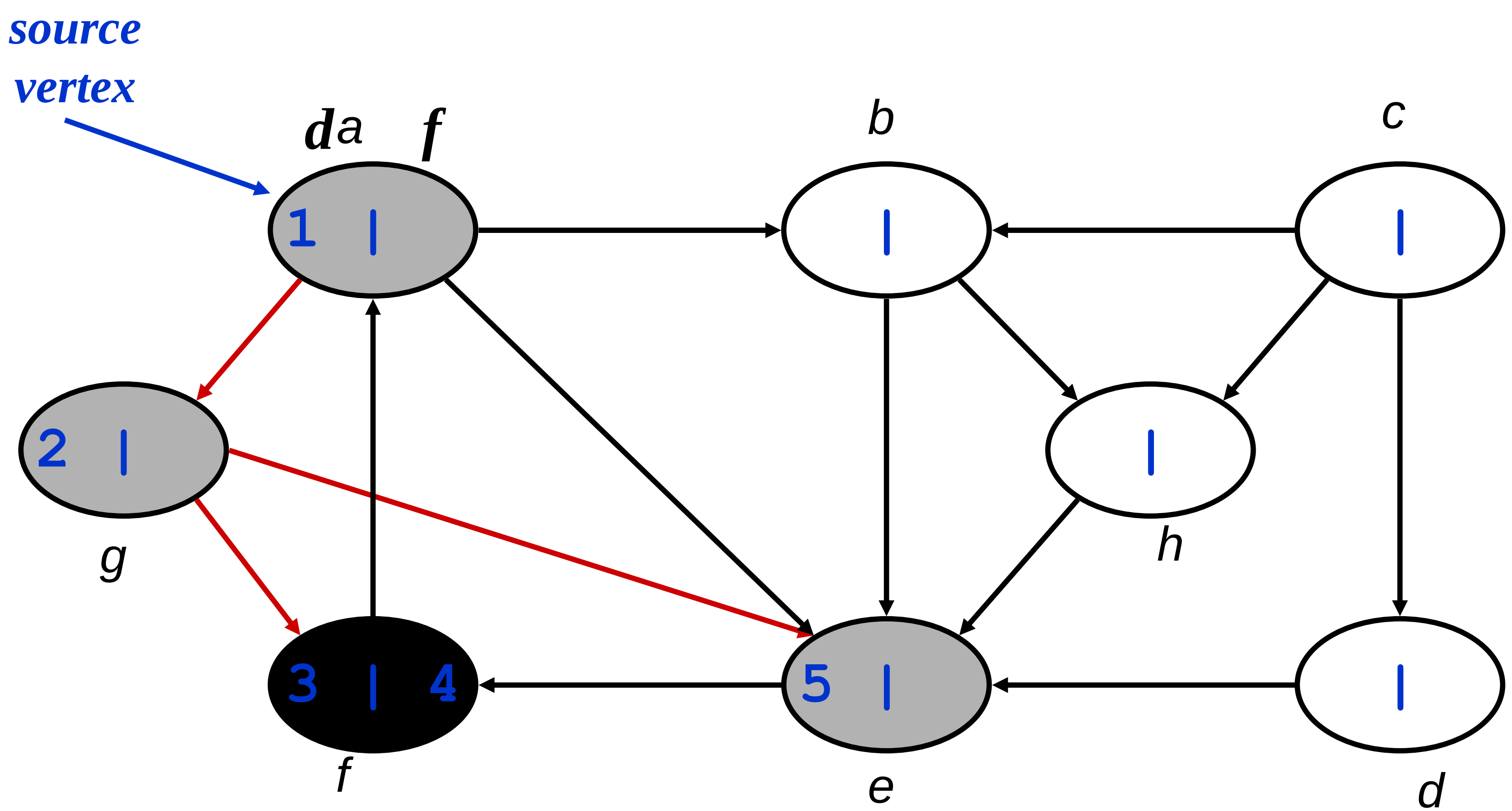

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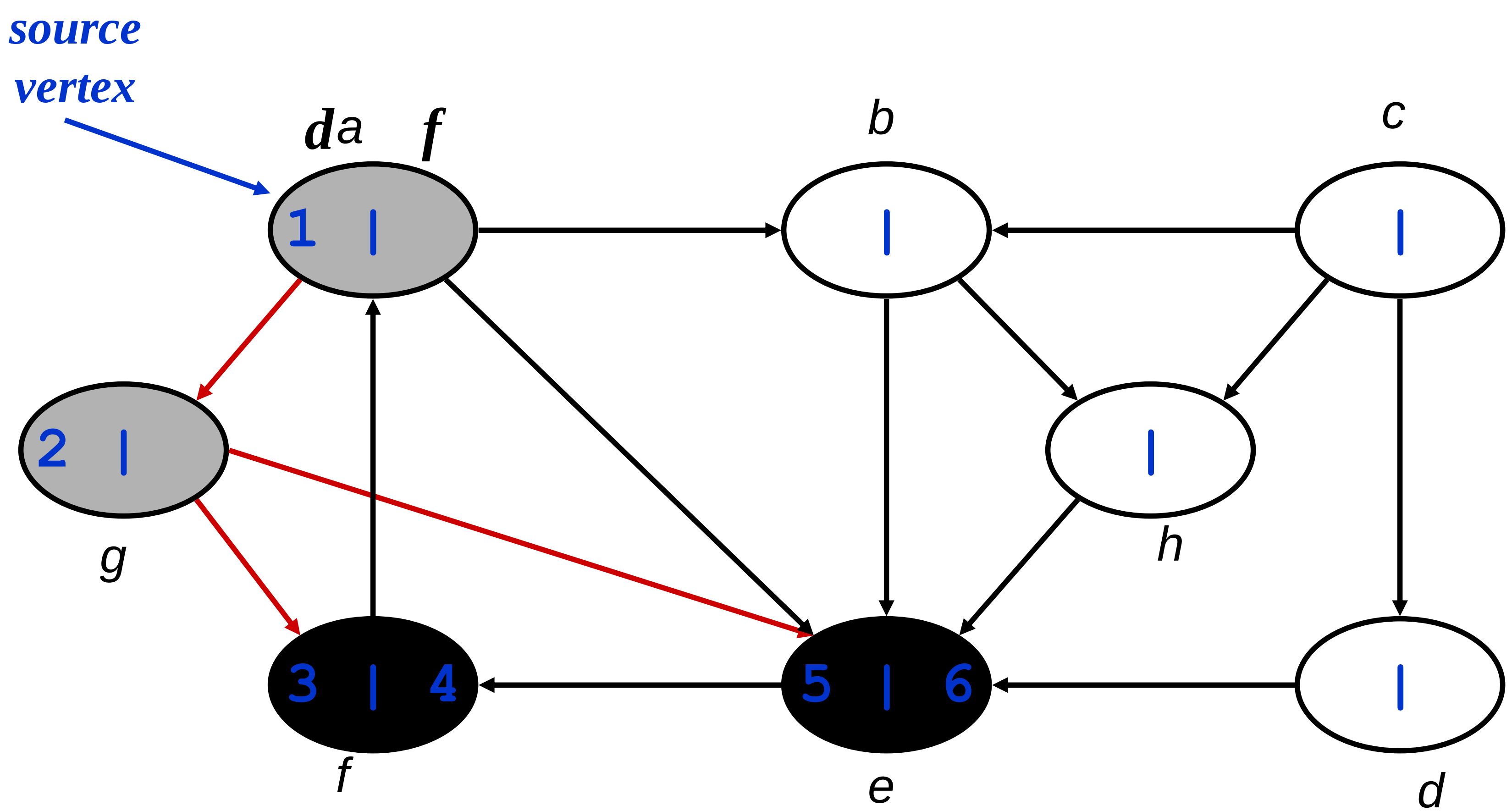

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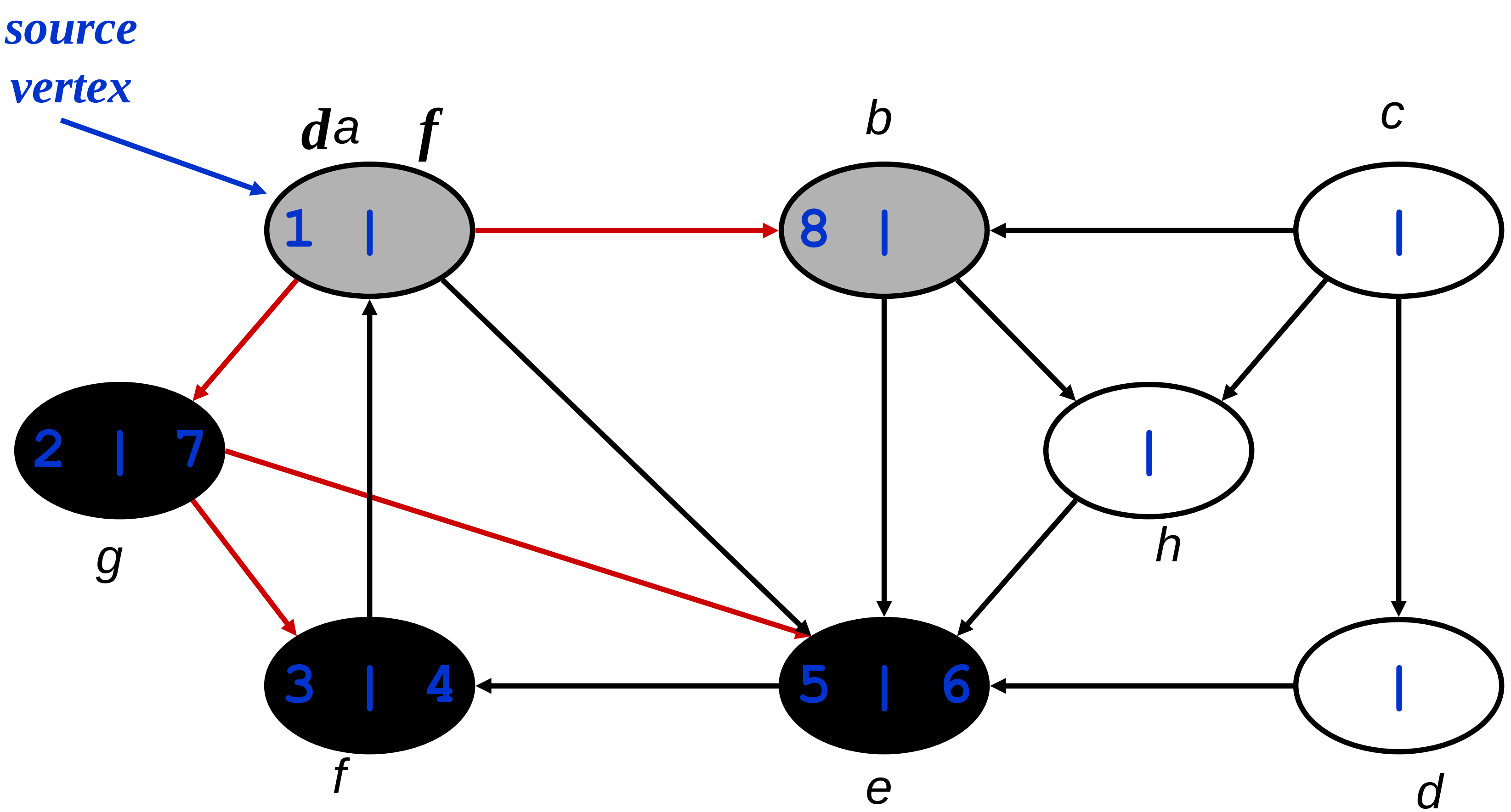

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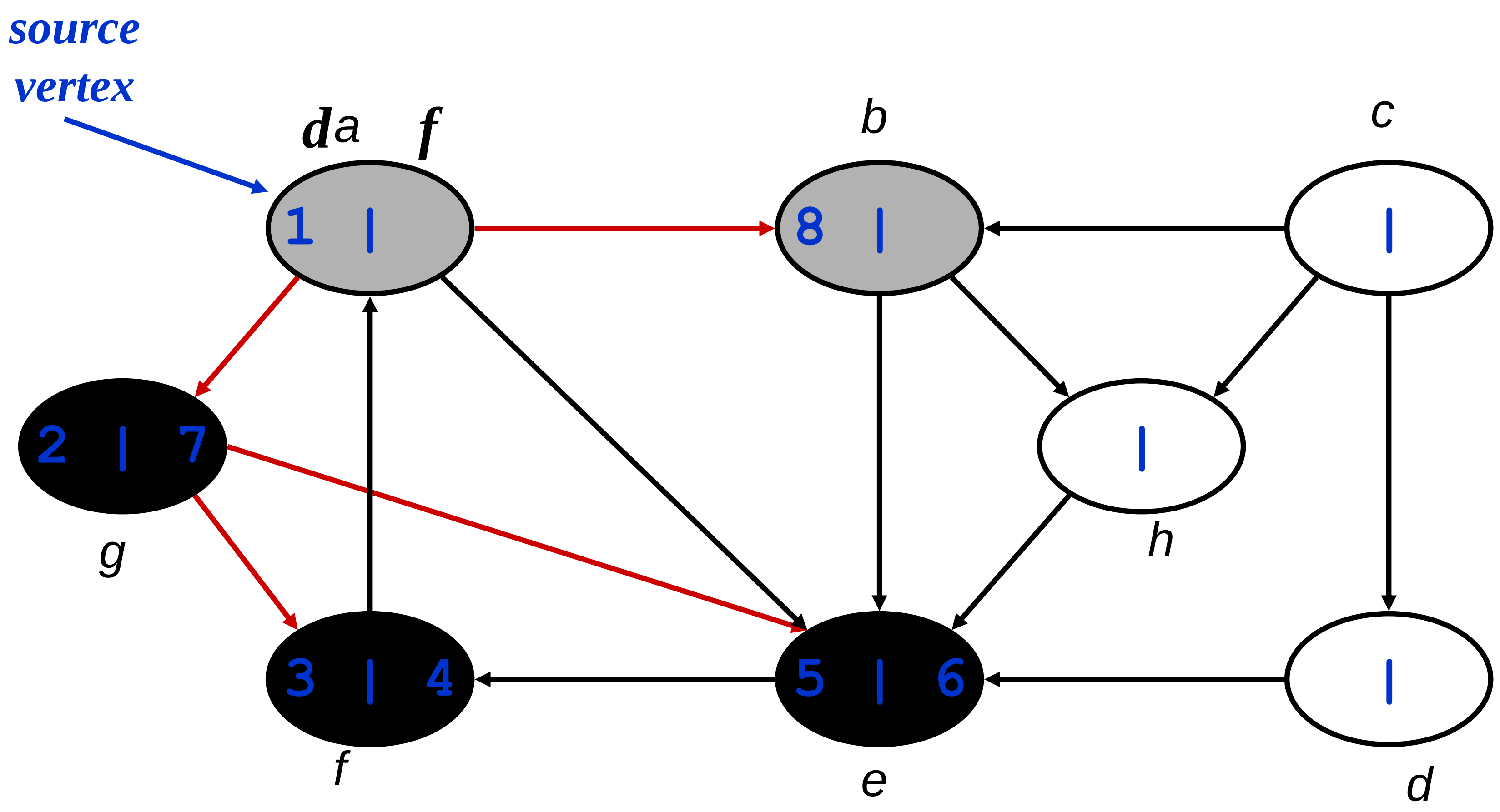

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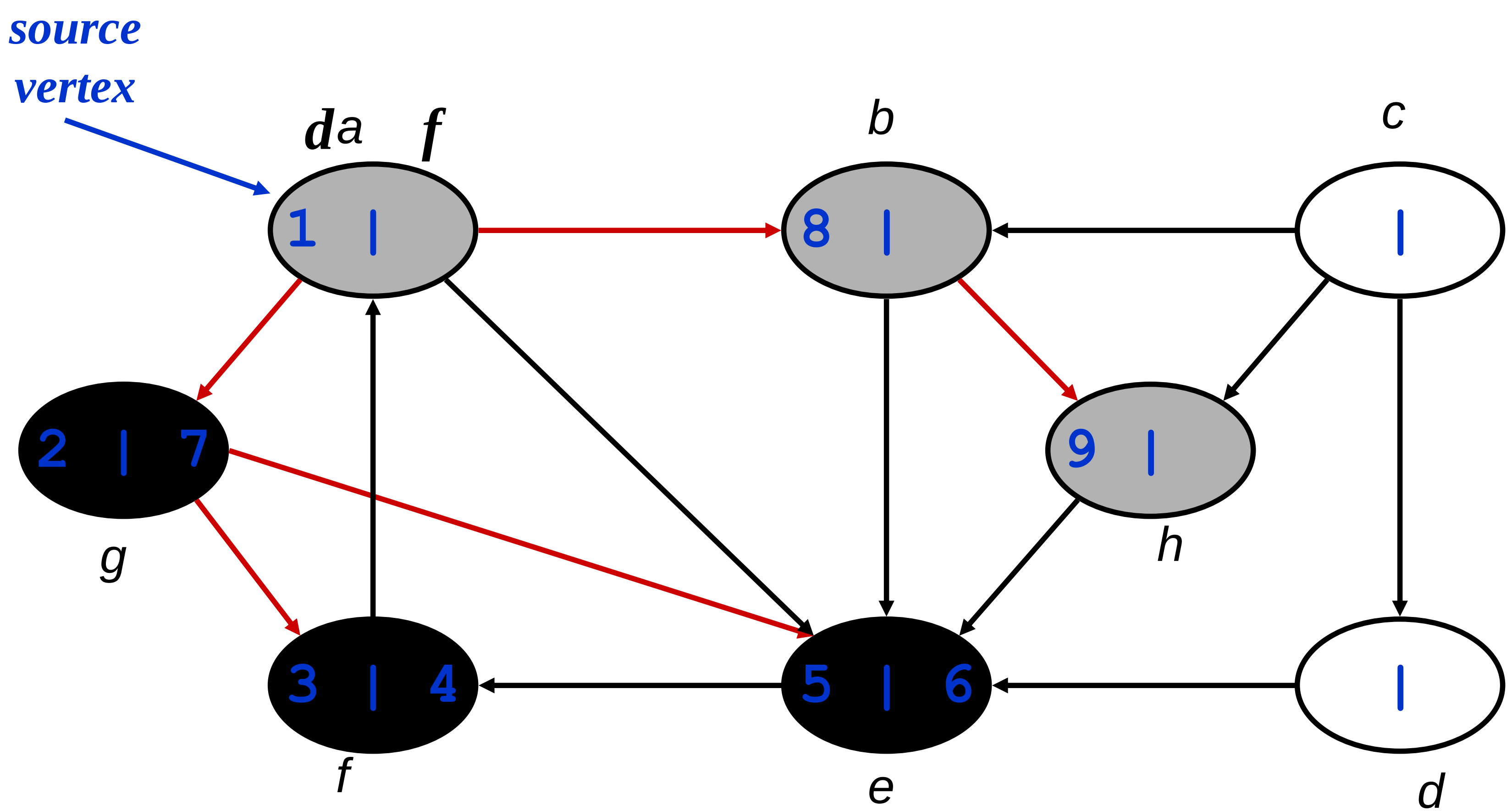
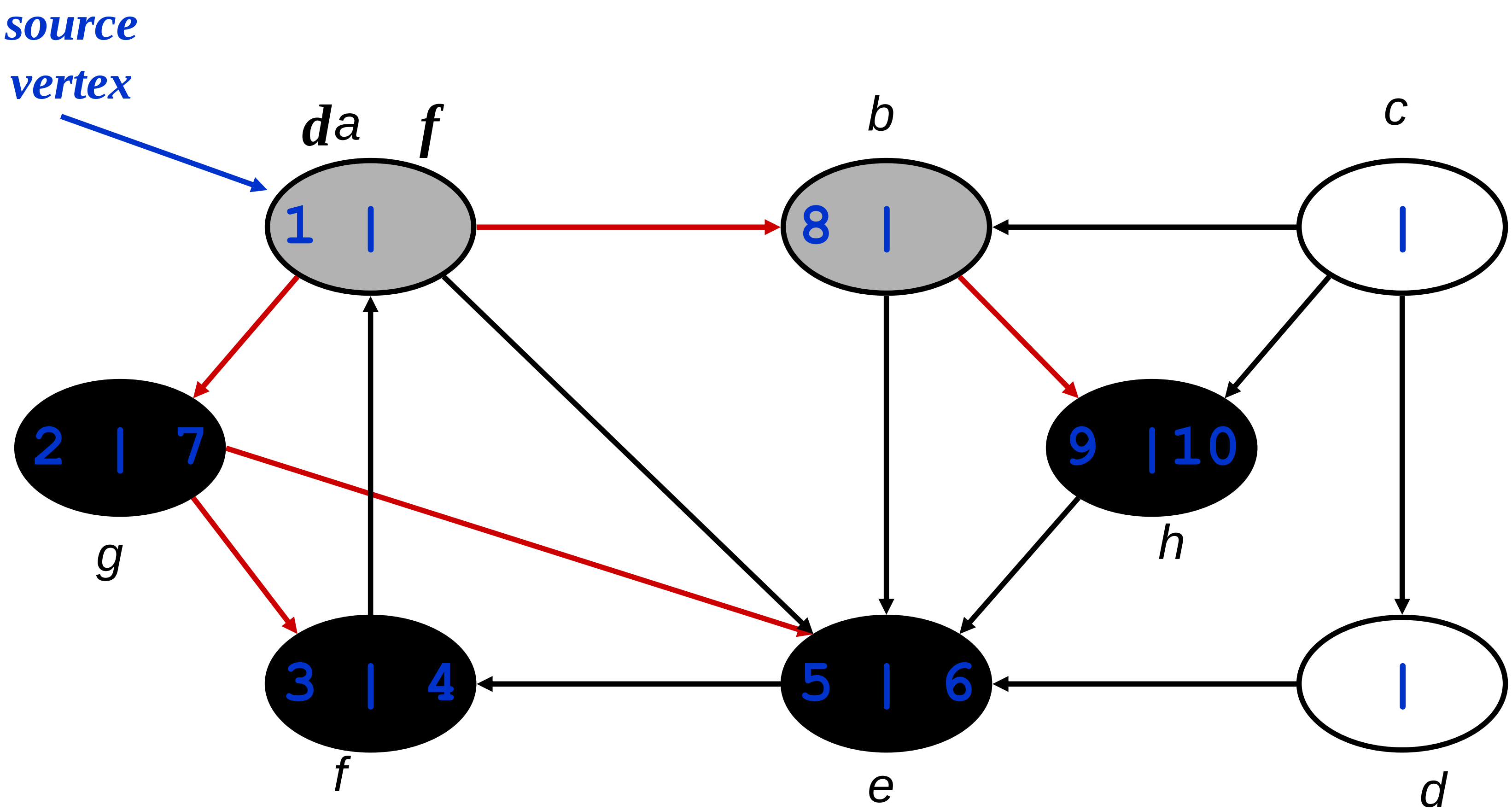



***What is the structure of the grey vertices?***
***What do they represent?***

# DFS Example



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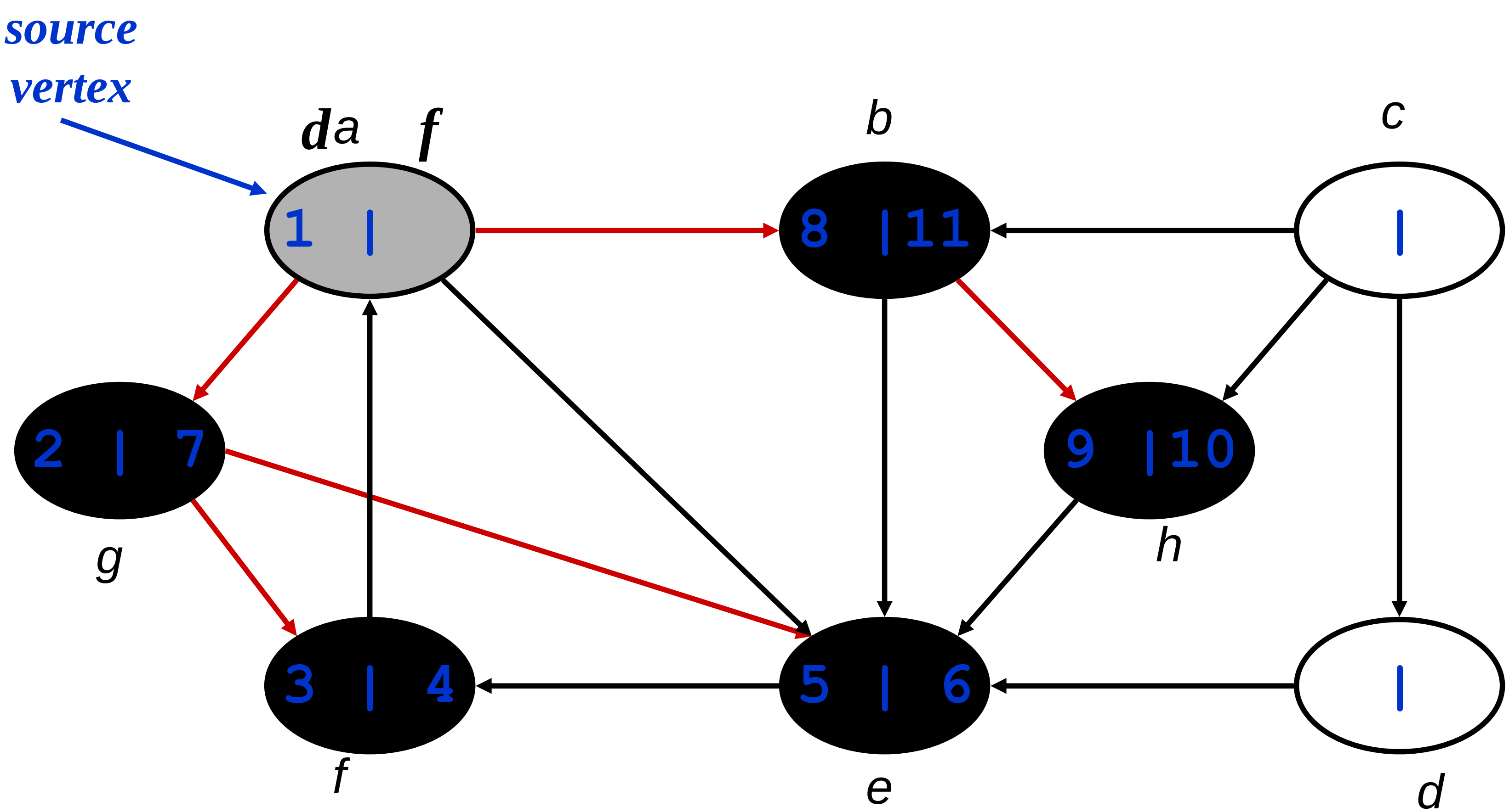

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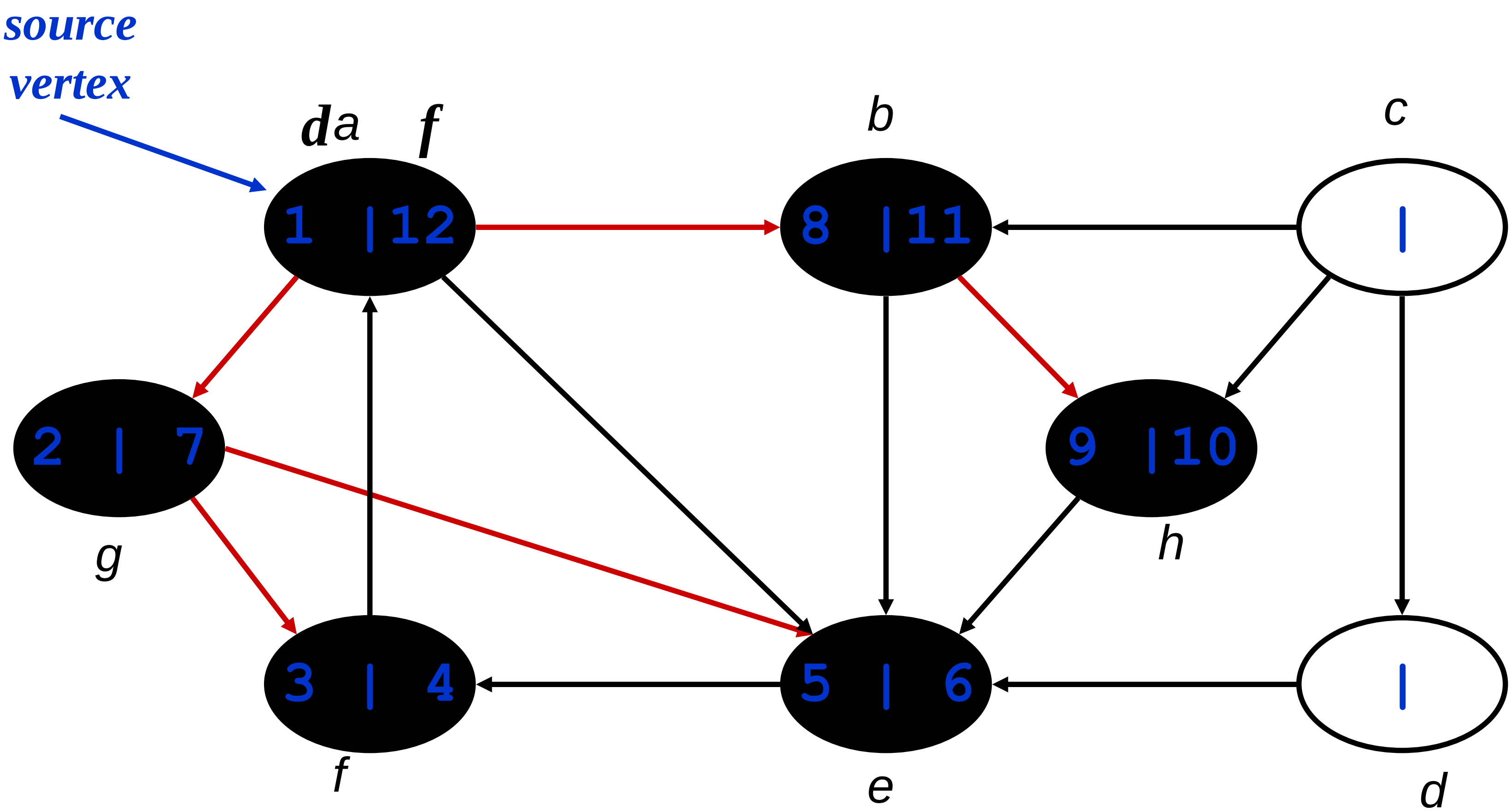

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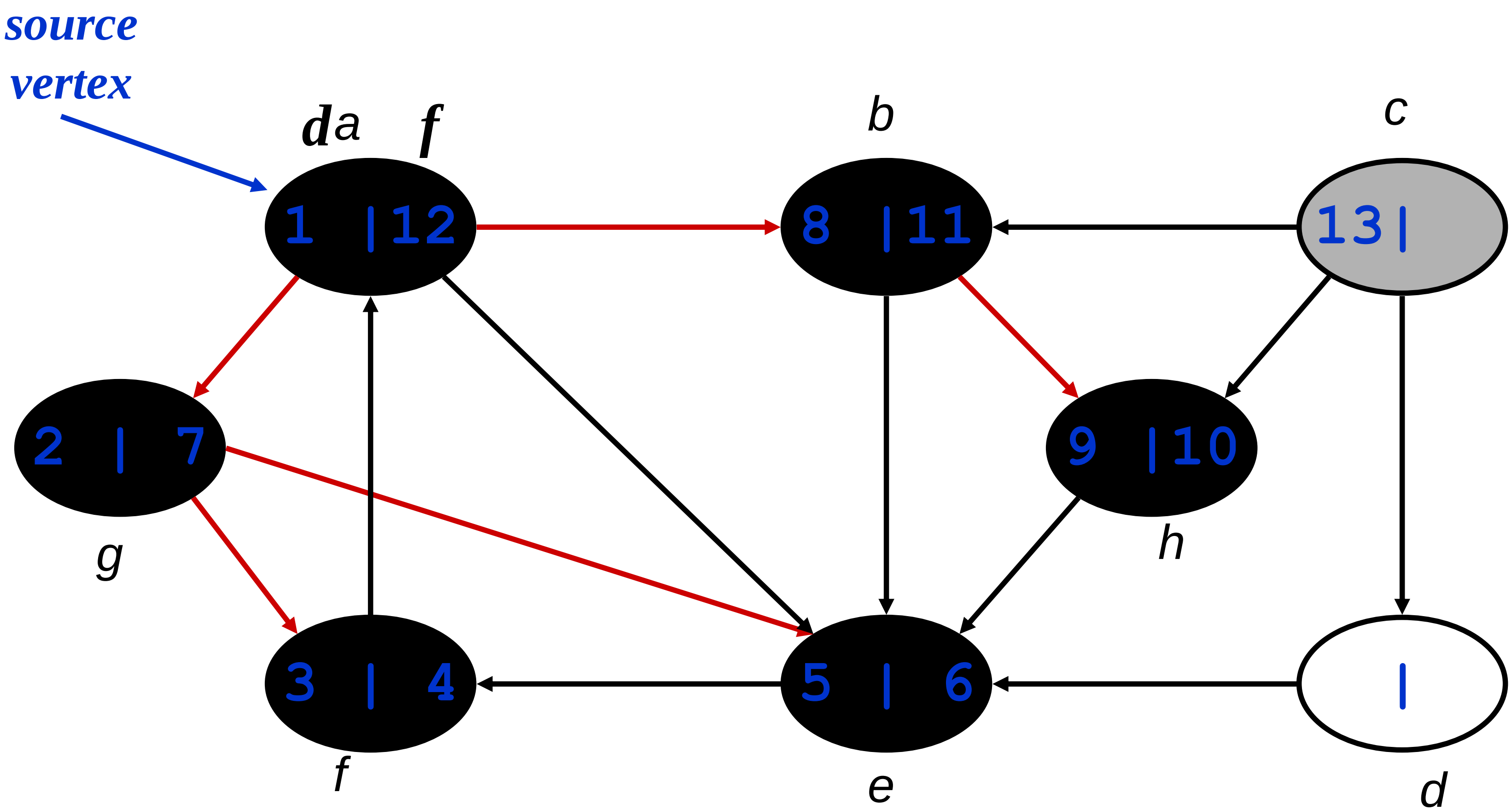

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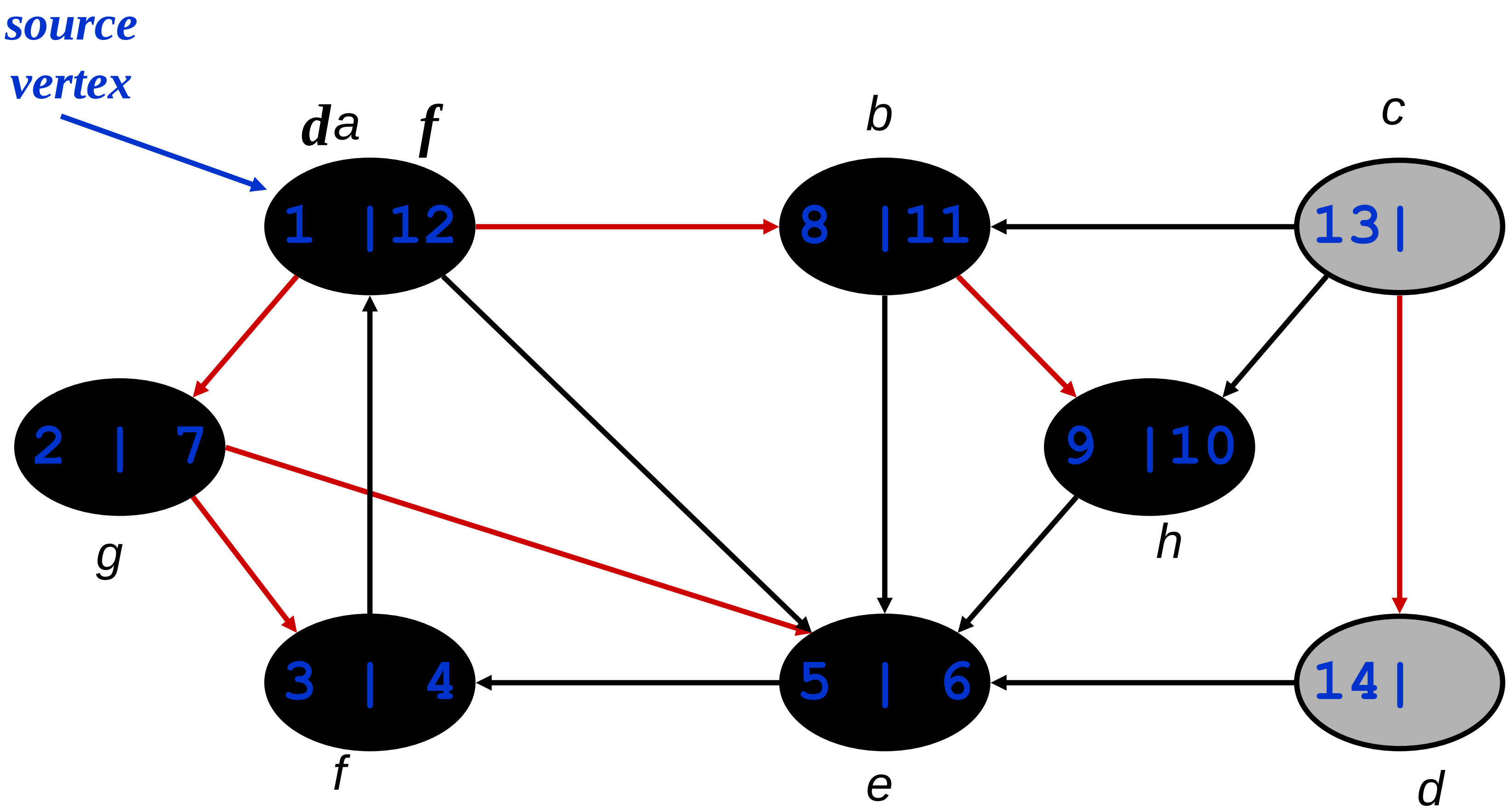

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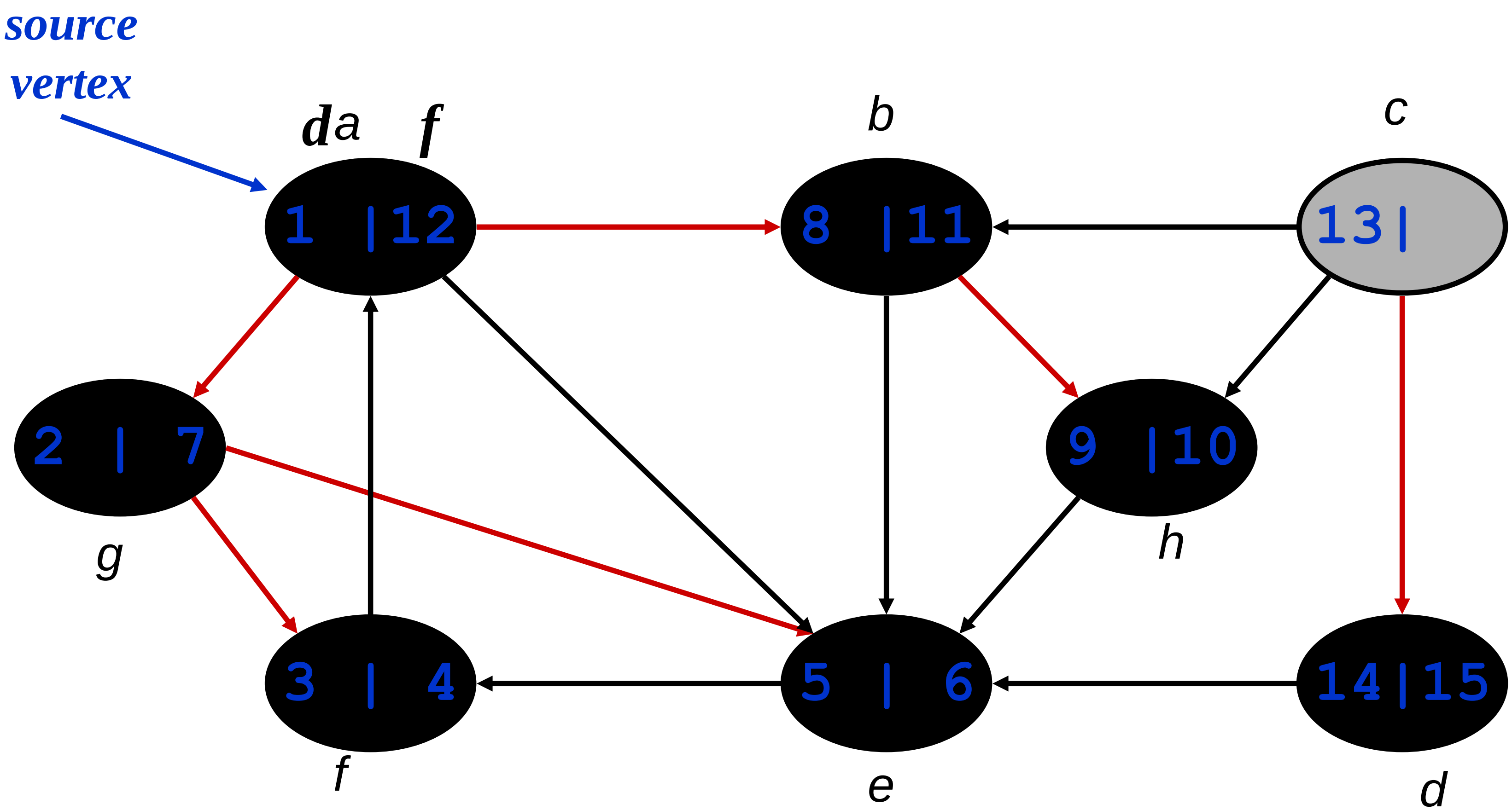

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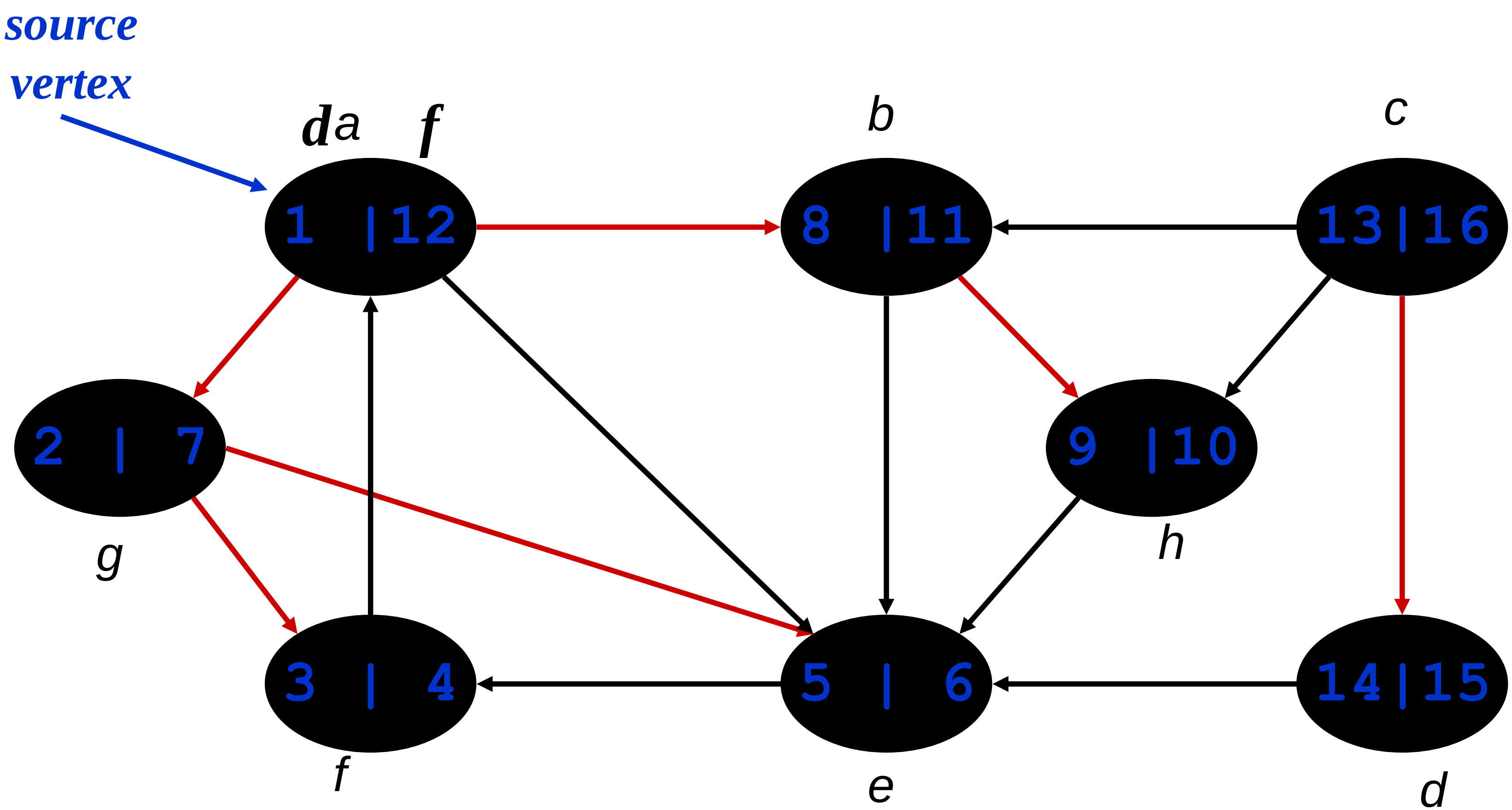
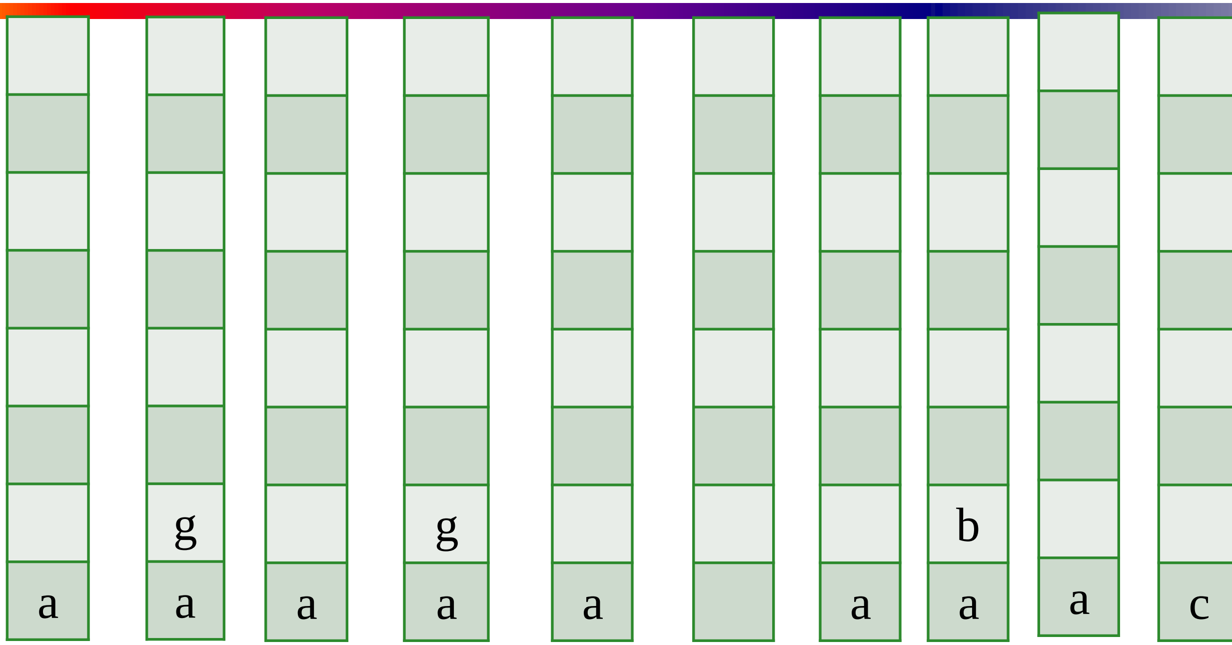
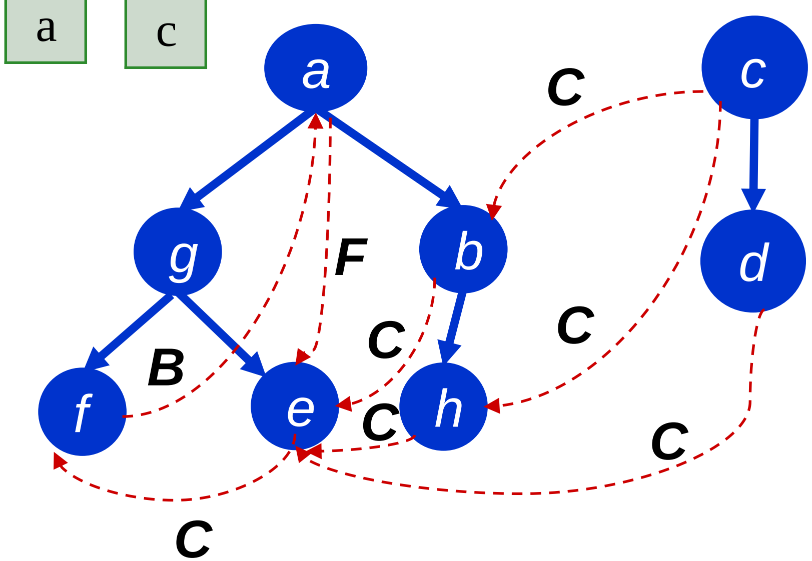





*a g f e b h c d*

# Pseudo-code

**DFS(*G*)**

1. **for** each vertex $u \in V[G]$
2. **do** *color*[*u*] ← white
3. $\pi[u]$ ← NIL
4. *time* ← 0
5. **for** each vertex $u \in V[G]$
6. **do if** *color*[*u*] = white
7. **then** DFS-Visit(*u*)

***DFS-Visit(u)***

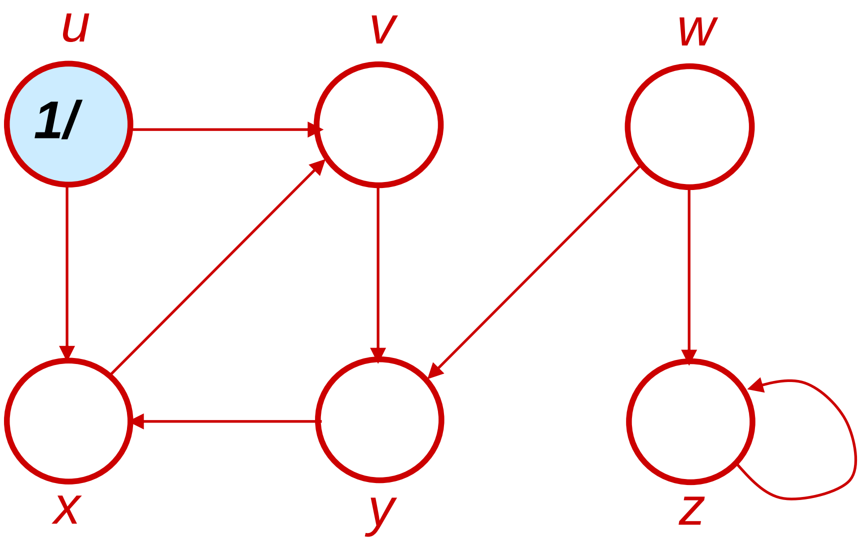
1. *color[u] ← GRAY ∇ White vertex u has been discovered*
2. *time ← time + 1*
3. *d[u] ← time*
4. ***for** each v ∈ Adj[u]*
5. ***do if** color[v] = WHITE*
6. ***then** π[v] ← u*
7. *DFS-Visit(v)*
8. *color[u] ← BLACK ∇ Blacken u; it is finished.*
9. *f[u] ← time ← time + 1*

*Uses a global timestamp **time**.*

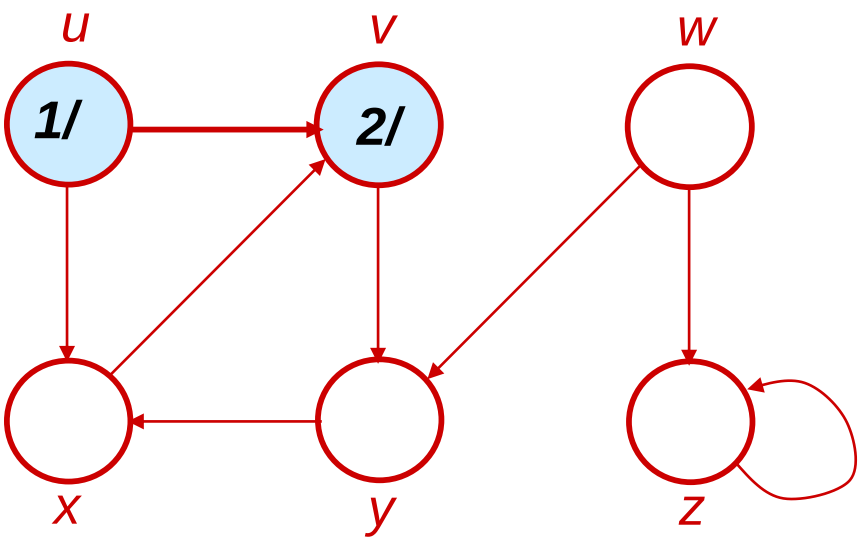
# Analysis of DFS

- Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.
- DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 4-7 of DFS-Visit is executed $|\mathrm{Adj}[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |\mathrm{Adj}[v]| = \Theta(E)$
- Total running time of DFS is $\Theta(V+E)$.

# Example 2



# Example (DFS)



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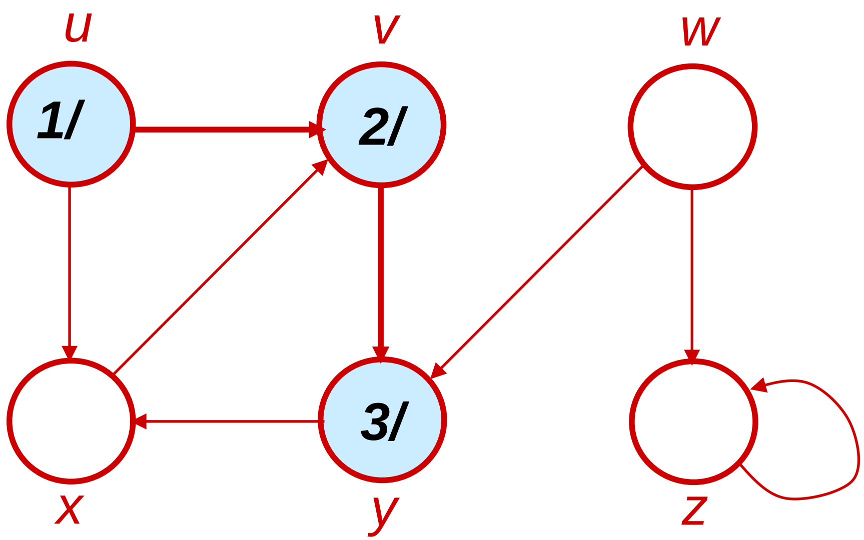

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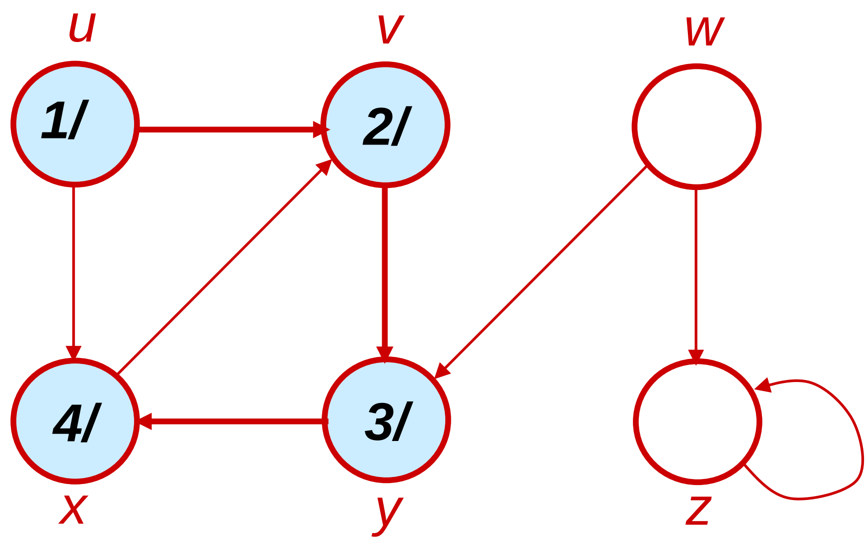

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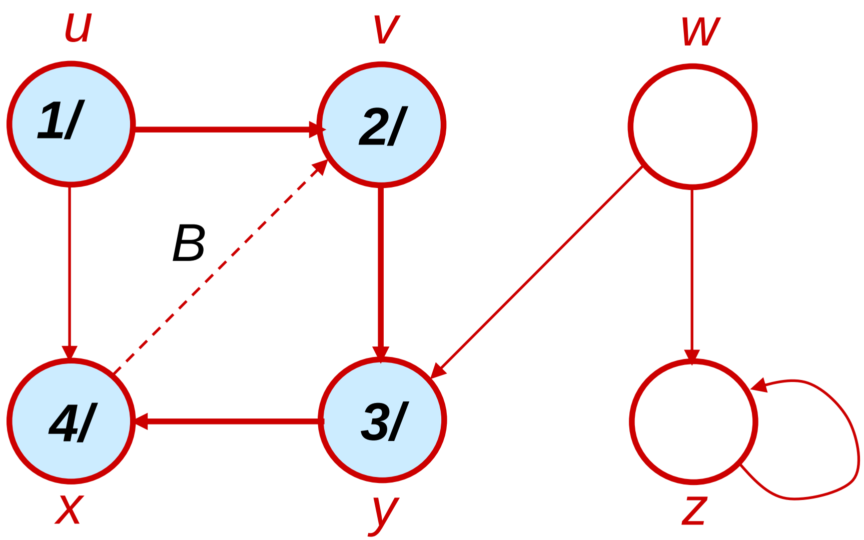

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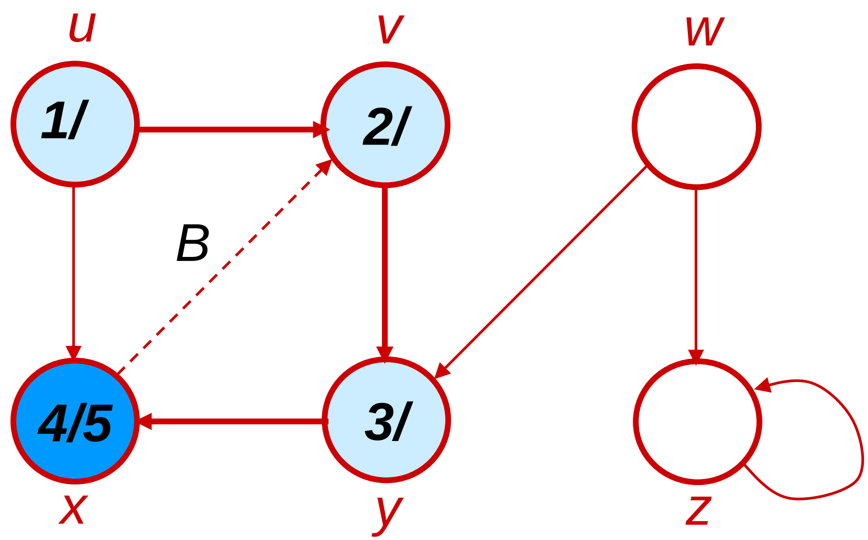

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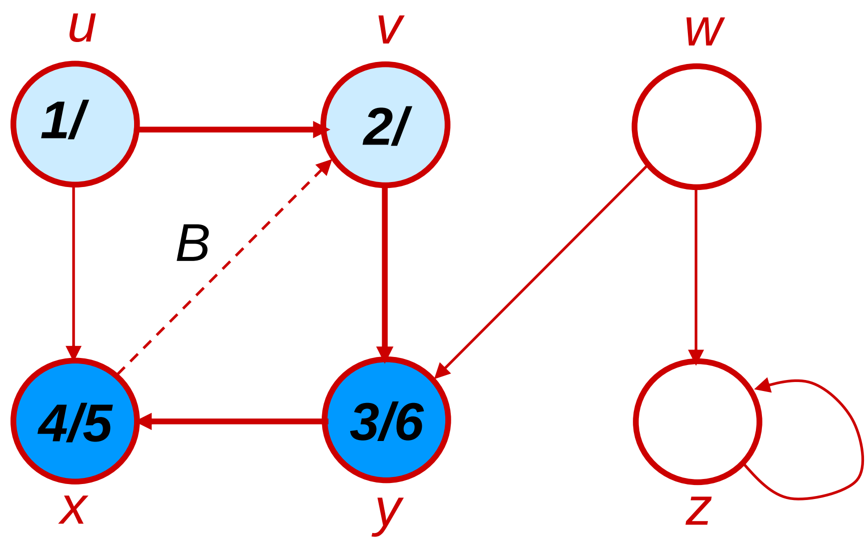
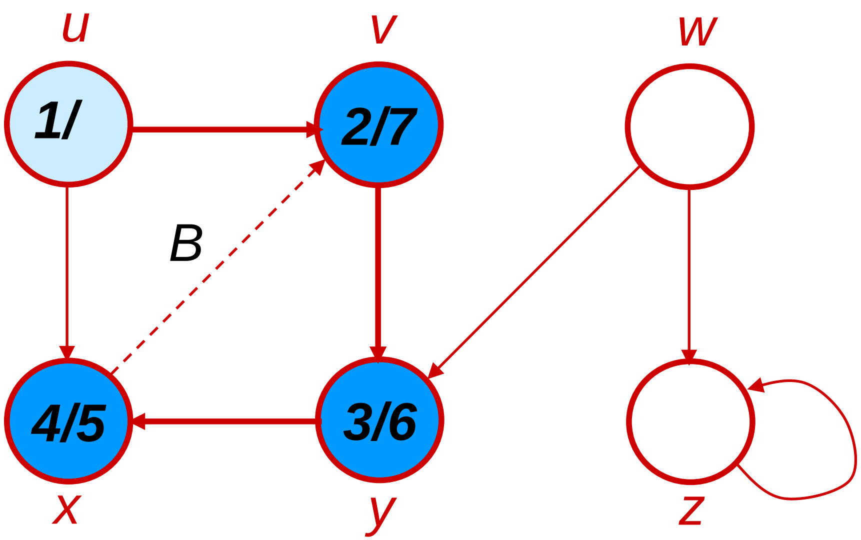
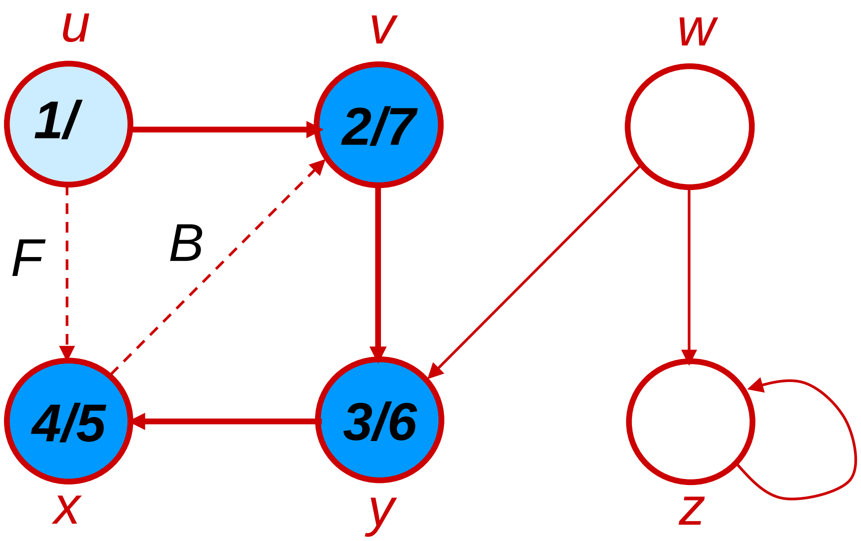


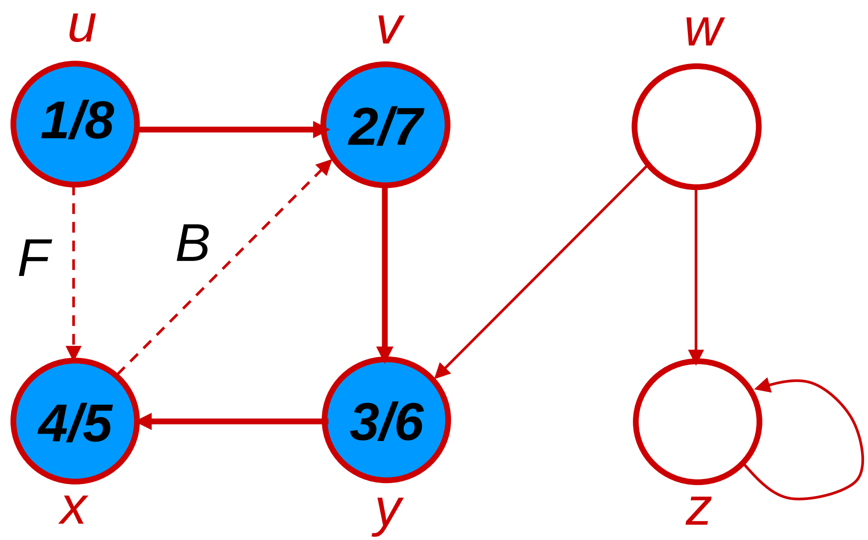
# Example (DFS)



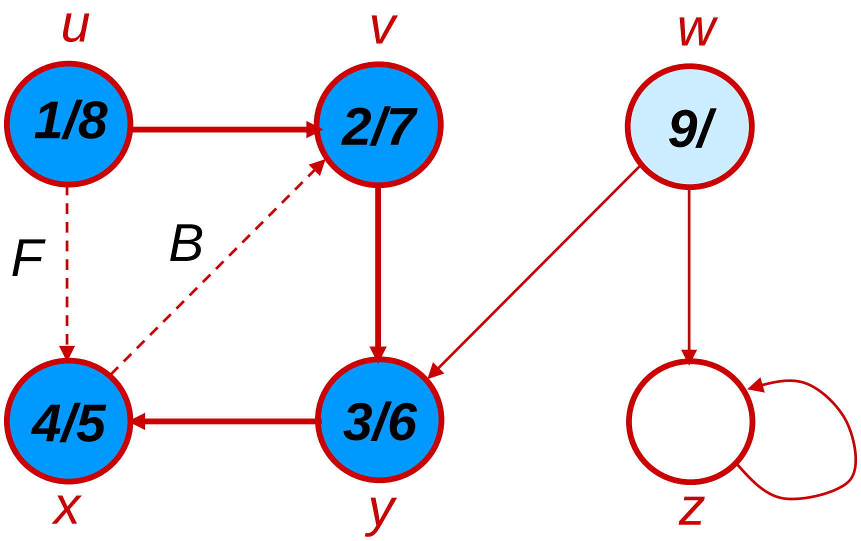
# Example (DFS)



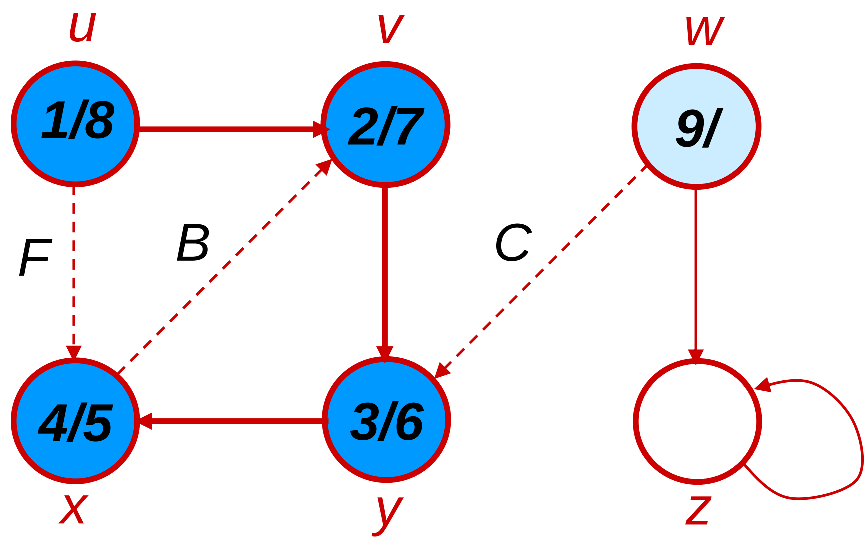
# Example (DFS)



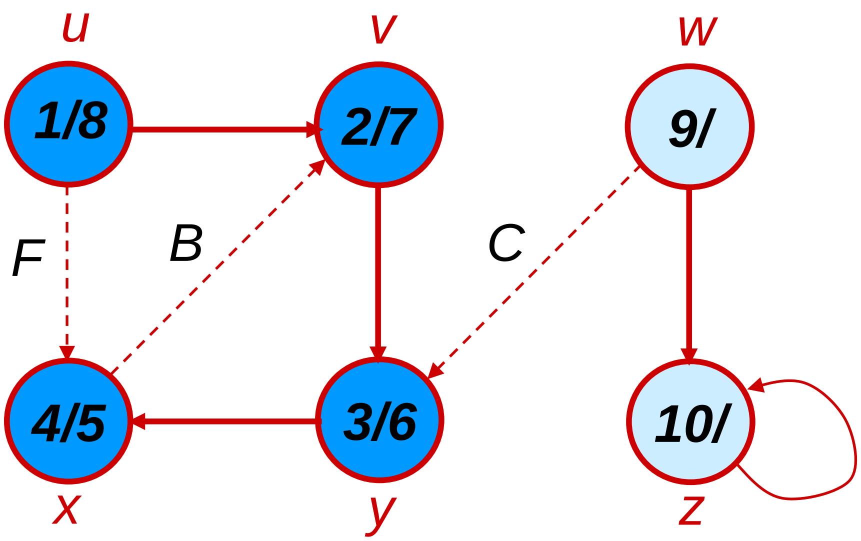
# Example (DFS)



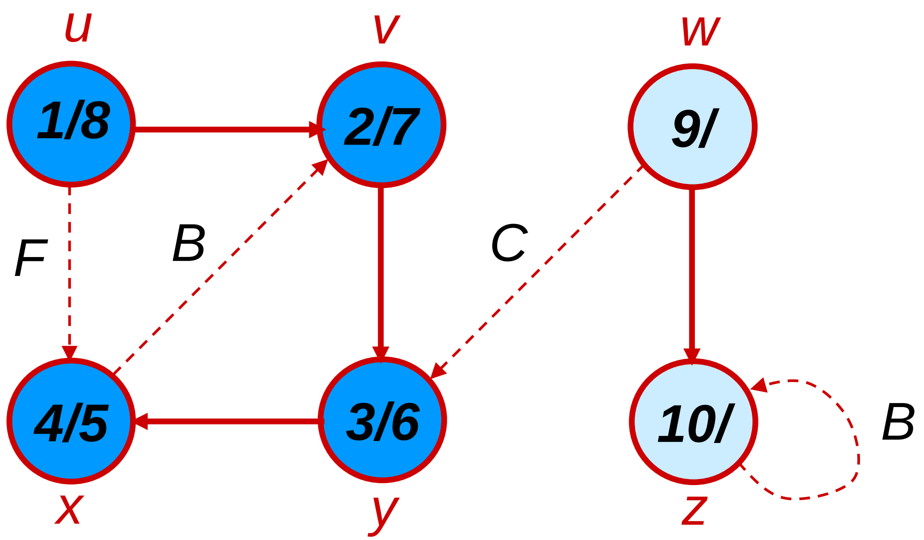
# Example (DFS)



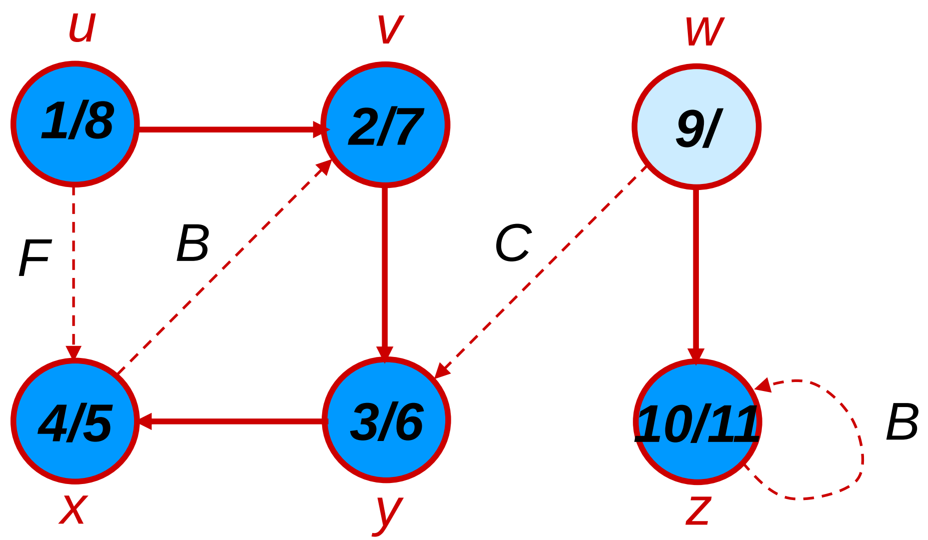
# Example (DFS)



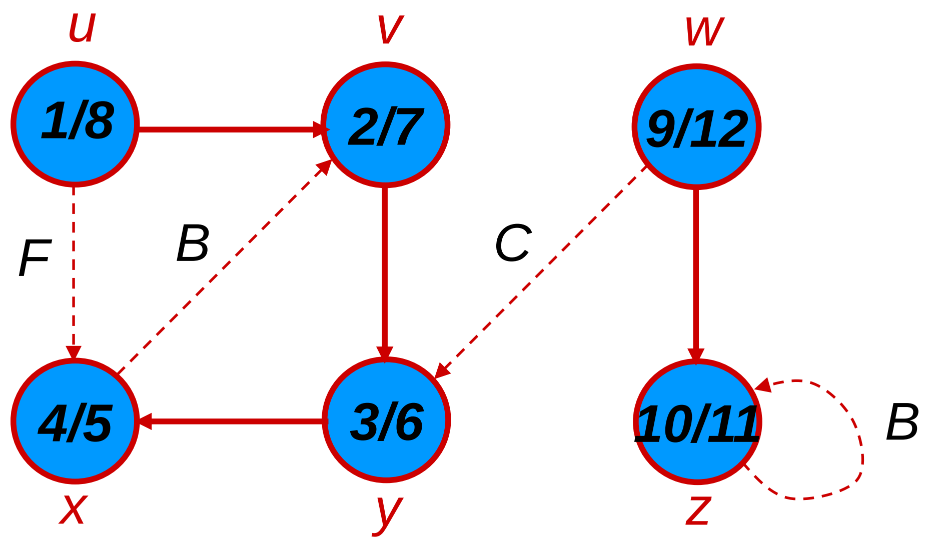
# Example (DFS)



# Example (DFS)



# Example (DFS)



# DFS-Properties

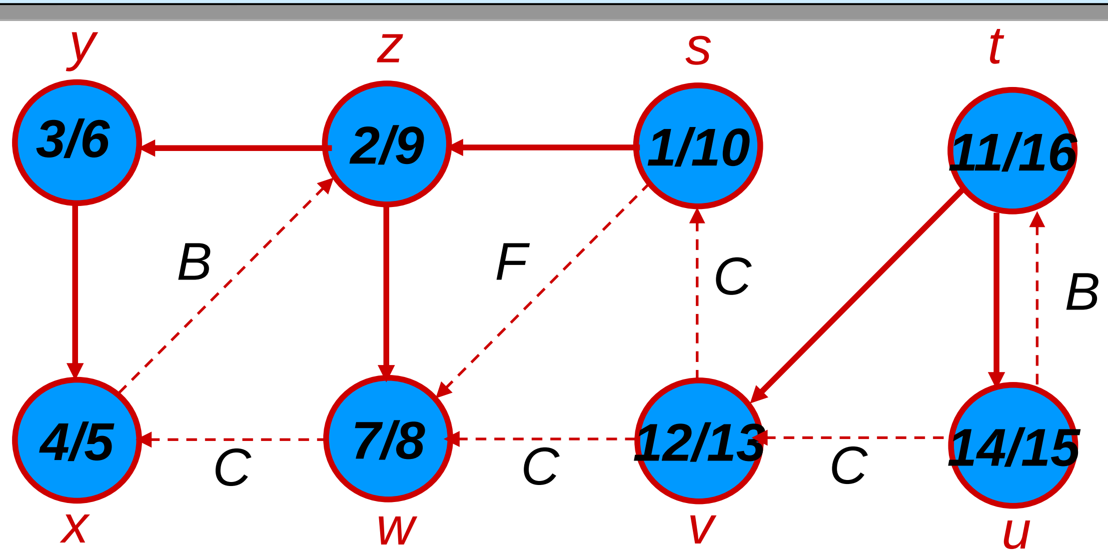
- u=v. π,iff DFS-Visit(G,v) was called during a search of u's adjacency list
- V is descendent of u in DFS iff v is discovered when u is grey.
- Paranthesis structure
  - If we represent the discovery of vertex *u with* a left parenthesis "*(u" and represent its finishing by a right parenthesis "u)", then* the history of discoveries and finishings makes a well-formed expression in the sense that the parentheses are properly nested.

# Parenthesis Theorem

**Theorem 22.7**

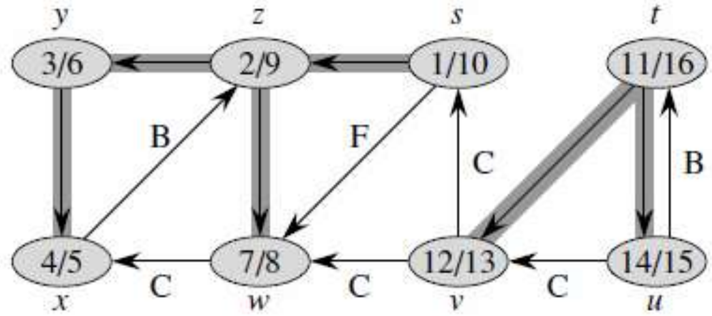
For all $u$, $v$, exactly one of the following holds:

1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither $u$ nor $v$ is a descendant of the other.
2. $d[u] < d[v] < f[v] < f[u]$ and $v$ is a descendant of $u$.
3. $d[v] < d[u] < f[u] < f[v]$ and $u$ is a descendant of $v$.

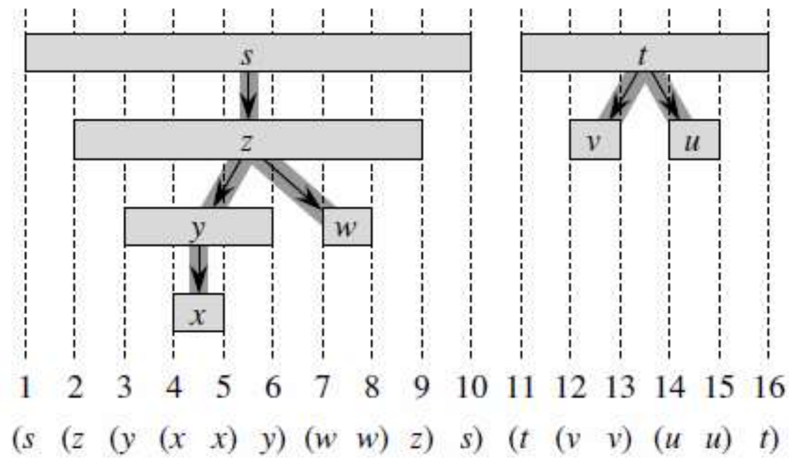



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

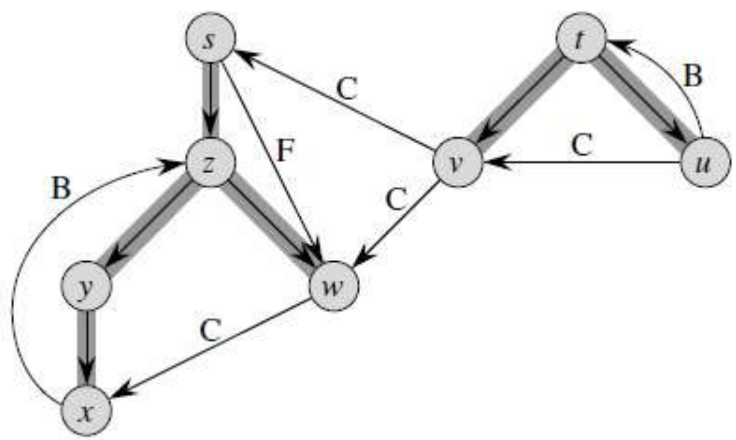
(a)


y
z
s
t
3/6
2/9
1/10
11/16
B
F
C
B
4/5
7/8
12/13
14/15
C
C
C
x
w
v
u

(b)


s
t
z
v
u
y
w
x
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

(c)

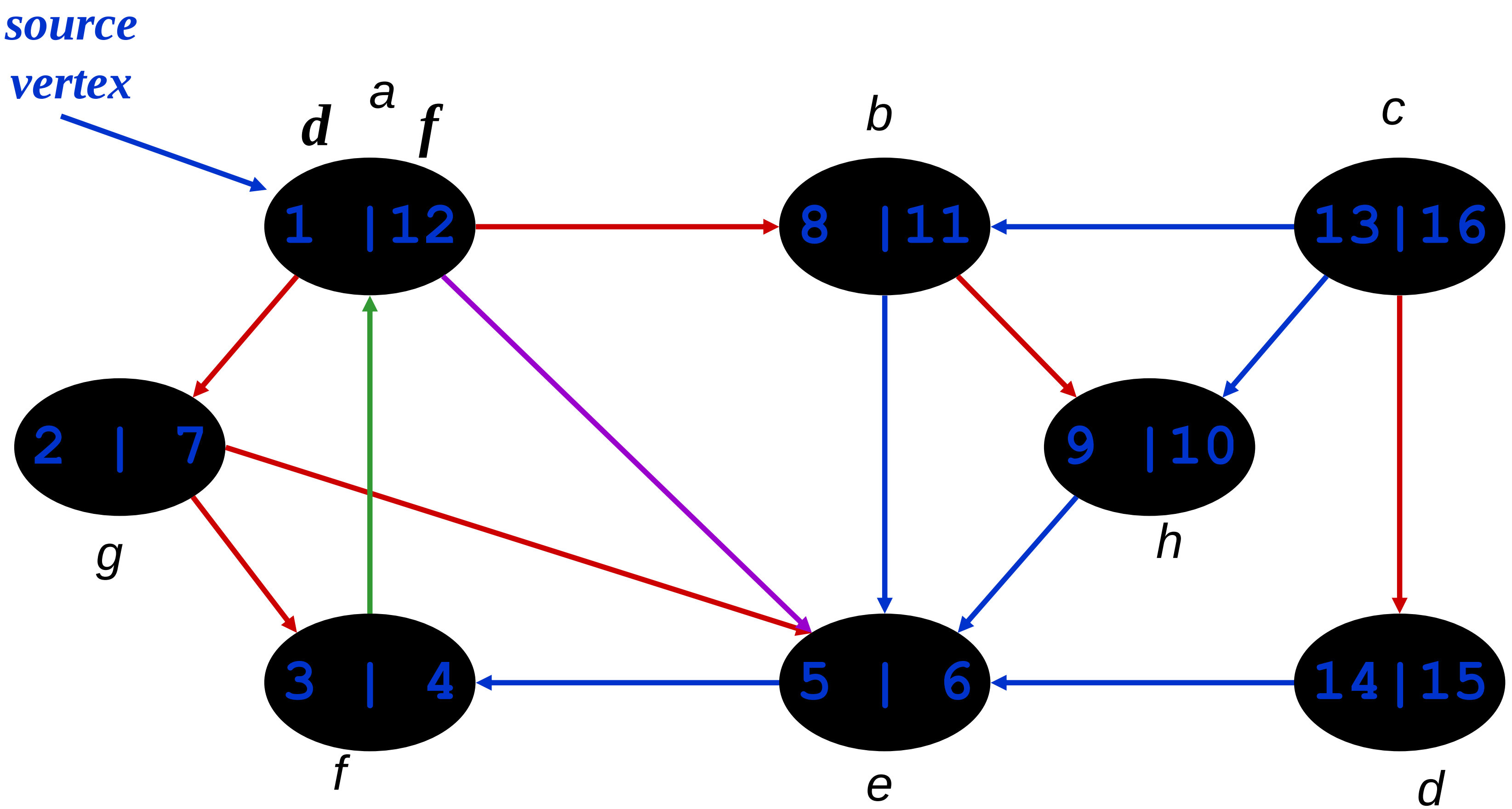

s
t
C
B
z
F
v
C
u
B
C
y
w
C
x

# Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring $(u, v)$.
- **Back edge:** $(u, v)$, where $u$ is a descendant of $v$ (in the depth-first tree).
- **Forward edge:** $(u, v)$, where $v$ is a descendant of $u$, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

***Theorem:***
*In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.*

# DFS Example

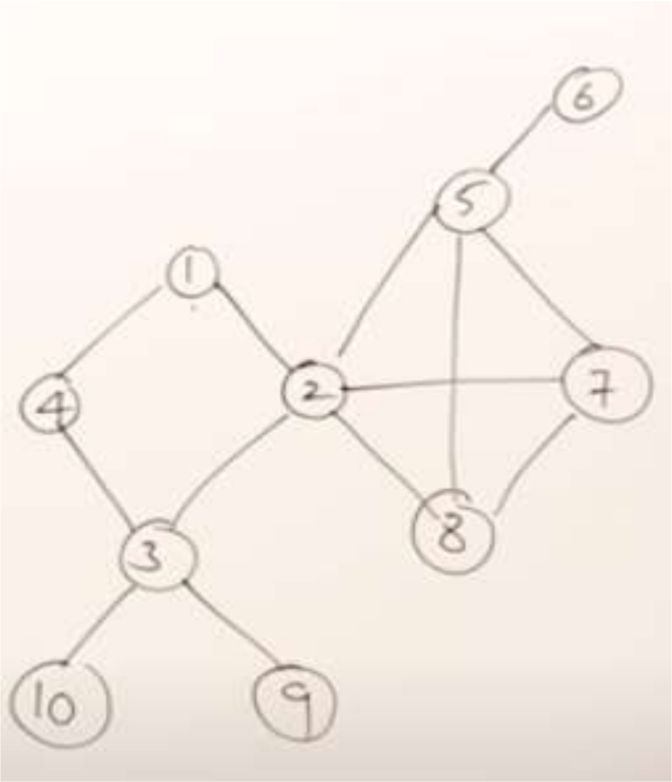


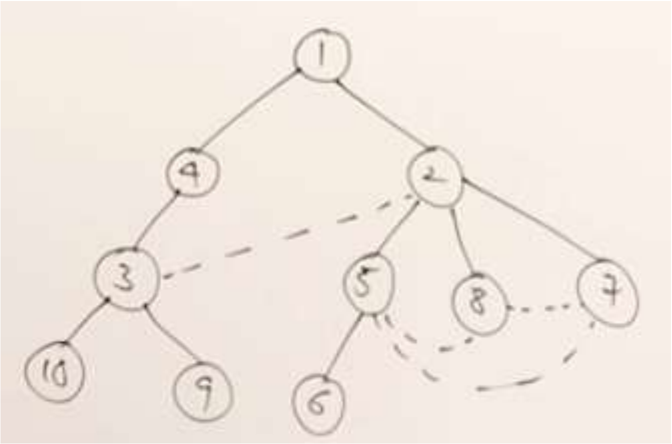

*Tree edges* *Back edges* *Forward edges* *Cross edges*

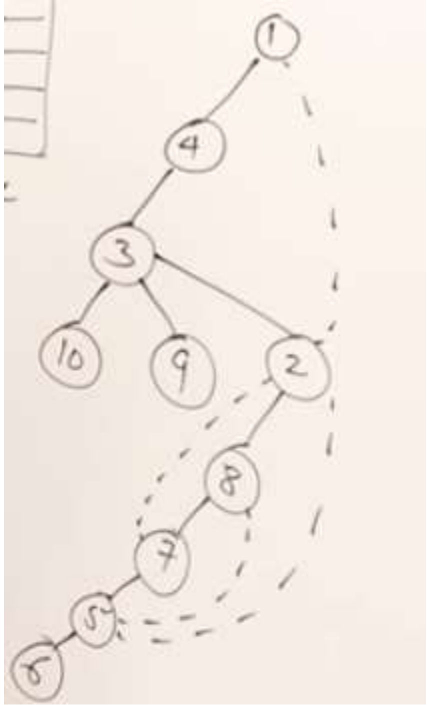
# DFS And Graph Cycles

- Theorem: An undirected graph is *acyclic* iff a DFS yields no back edges
  - If acyclic, no back edges (because a back edge implies a cycle
  - If no back edges, acyclic
    - No back edges implies only tree edges
    - Only tree edges implies we have a tree or a forest which by definition is acyclic
- Thus, can run DFS to find whether a graph has a cycle

# Problem




1
4
2
3
5
8
7
10
9
6


1
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