Linear Transformation on Linear Space

Internship Report

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1 Linear Transformation

Let V and W be an n dimensional vector space over a field $\mathbb{F}$. Let $T : V \to W$ be a function with V as its domain and its range contained in W.

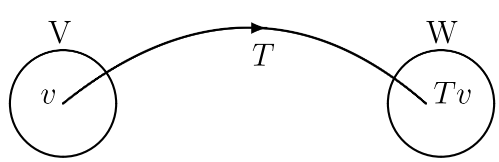
$$T(V) \subset W$$

T is linear in the sense that

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$T(\alpha v_1) = \alpha T(v_1)$$

$\forall\ v_1, v_2 \in V$ and $\alpha \in \mathbb{F}$.



Let $L(V, W)$ denote the set of linear transformation from V to W. If $T \in L(V, W)$, T is defined if we prescribe the action of T on a basis of V.

Let $\mathcal{B} = v_1, v_2, ..., v_n$ be a basis of V. Then $v \in V$ given by $v = x_1v_1 + x_2v_2 + ... + x_nv_n$, $\forall\ x_i \in \mathbb{F}$

$$T(v) = T(x_1 v_1 + x_2 v_2 + ... + x_n v_n)$$

$$= x_1 T(v_1) + x_2 T(v_2) + ... + x_n T(v_n)$$

If we know every $T(v_i)$ we will get T(v).

Let T be the linear transformation from $\mathbb{P}_2(\mathbb{R})$ to $\mathbb{R}^3$ defined by $\phi(p) := (p_0 \;\; p_1 \;\; p_2)$. Let $\mathcal{B}_1 = \{1, t, t^2\}$ be the basis for the vector space $\mathbb{P}_2(\mathbb{R})$. So the function can be represented as $p_0 + p_1 t + p_3 t^2 \rightarrow (p_0 \;, \; p_1 \;, \; p_2)$.

$$T_{\mathcal{B}_1} = \phi$$

Let $\mathcal{B}_2 = \{1 + t, 1 - t, t + t^2\}$ be another basis in $\mathbb{P}_2(\mathbb{R})$. Then

$$p = q_0(1 + t) + q_1(1 - t) + q_2(t + t^2)$$

$$= (q_0 + q_1) + (q_0 - q_1 + q_2)t + (q_2 t^2)$$

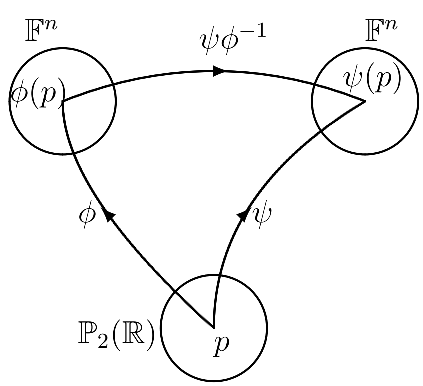
$$\implies q_0 + q_1 = p_0, q_0 - q_1 + q_2 = p_1, q_2 = p_2$$

$$q_0 = \frac{1}{2}(p_0 + p_1 - p_2), q_1 = \frac{1}{2}(p_0 + p_1 + p_2), q_2 = p_2$$

The transformation with respect to the basis $\mathcal{B}_2$ can be represented as

$$T_{\mathcal{B}_2} = \psi(p) = (q_0, q_1, q_2)$$

$$=(\tfrac{1}{2}(p_0 + p_1 - p_2) \;, \; \tfrac{1}{2}(p_0 1 p_1 + p_2) \;, \; p_2)$$



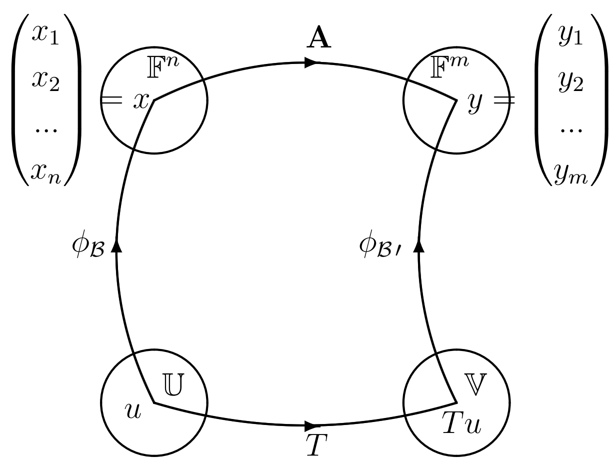
2 Matrix Representation of Linear Transformation

Let U be an n-dimensional vector space over the field $\mathbb{F}$ and V an m-dimensional vector space over $\mathbb{F}$. Let $\mathcal{B}$:=$\{u_1, u_2, ..., u_n\}$ be an ordered basis for U and $\mathcal{B}'$:=$\{v_1, v_2, ..., v_m\}$ an ordered basis for V. For each linear transformation T from U into V, there is an $m \times n$ matrix $\mathbf{A}$ with entries in $\mathbb{F}$.
Let T be given by

$$T(u_i) = a_{1i}v_1 + a_{2i}v_2 + ... + a_{mi}v_m$$

i = (1,2,...,n)
Let $u \in U$, then $u = x_1u_1 + x_2u_2 + ... + x_nu_n$
$T(u) = x_1T(u_1) + x_2T(u_2) + ... + x_nT(u_n)$
$\quad = x_1(a_{11}v_1 + ... + a_{1n}v_m) + ... + x_n(a_{m1}v_1 + ... + a_{mn}v_m)$
The matrix representation of T,that is $\mathbf{A}$ is given by a_{ij}.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \text{,an } m \times n \text{ matrix.}$$

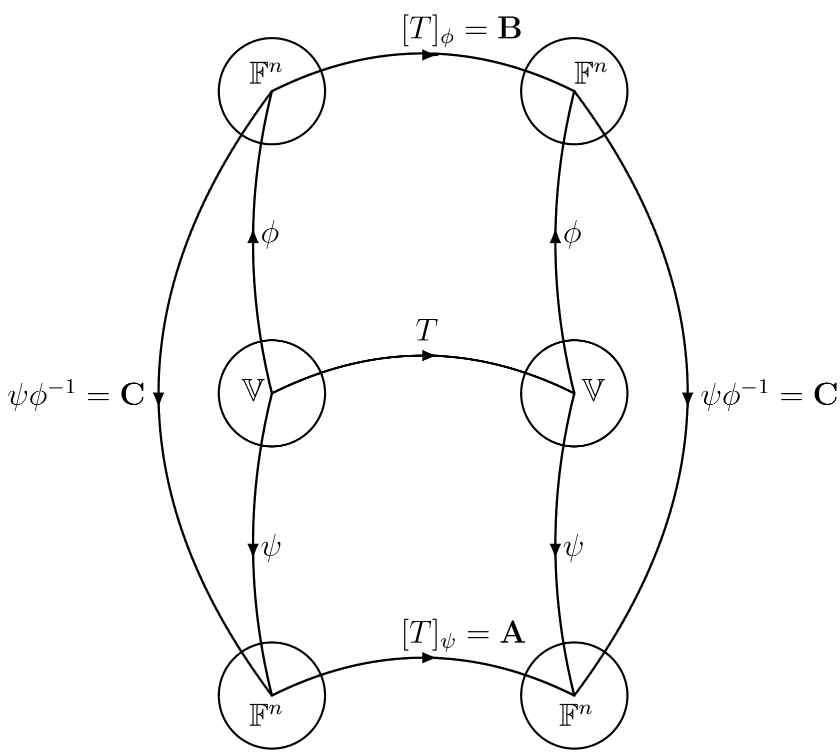


Let $u \in U$ and $T(u) = v$,where $v \in V$.
Let $\phi_{\mathcal{B}}(u) = x, x \in \mathbb{F}^n$,
$\quad \phi_{\mathcal{B}'}(v) = y, y \in \mathbb{F}^n$
$\implies \phi_{\mathcal{B}'}(Tu) = y$

$\implies \phi_{\mathcal{B}'} \circ T(u) = y$
$\implies \phi_{\mathcal{B}'} \circ T({\phi_{\mathcal{B}}}^{-1}x) = y$, $\phi_{\mathcal{B}}u = x$
$\implies \phi_{\mathcal{B}'} \circ T \circ {\phi_{\mathcal{B}}}^{-1}(x) = y$
Since $\mathbf{A}x = y$, we will get $\boxed{\mathbf{A} = \phi_{\mathcal{B}'} \circ T \circ {\phi_{\mathcal{B}}}^{-1}}$.

3 Similarity Transformation

Let V be an n dimenstional vector space. T is a linear transformation such that $T \in L(V)$.Let $\mathcal{B}$:=$\{v_1, v_2, ..., v_n\}$ and $\mathcal{B}'$:=$\{v'_1, v'_2, ..., v'_m\}$ be an ordered basis for V.
ϕ is a function under basis $\mathcal{B}$, $\phi = \phi_{\mathcal{B}}$.
ψ is a function under basis $\mathcal{B}'$, $\psi = \psi_{\mathcal{B}'}$.



$$[T]_\phi = \phi\psi^{-1}[T]_\psi(\phi\psi^{-1})^{-1}$$

$\implies$

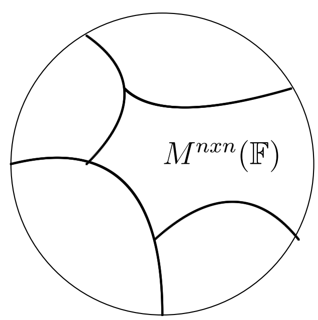
$$\mathbf{B} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$$

where C is an invertible $n \times n$ matrix, and $\mathbf{C} = \phi\psi^{-1}$
If $\mathbf{A}$ and $\mathbf{B}$ be n×n (square) matrices over field $\mathbb{F}$. We say that $\mathbf{B}$ is similar to $\mathbf{A}$ over $\mathbb{F}$, if there is an invertible n× n matrix $\mathbf{C}$ over $\mathbb{F}$ such that $\mathbf{B}$=$\mathbf{C^{-1}AC}$.There exist

an equivalence relation between the matrices which have this properties.
Proof : Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbf{M}_{n\times n}(\mathbb{R})$,
$\forall\ \mathbf{X} \in \mathbf{M}_{n\times n}(\mathbb{R})$, $\mathbf{X} = \mathbf{I}^{-1}\mathbf{X}\mathbf{I}$
So,$\mathbf{X} \sim \mathbf{X}$,(Reflexive relation)
Let $\mathbf{X} \sim \mathbf{Y}$, then $\mathbf{X} = \mathbf{C}^{-1}\mathbf{Y}\mathbf{C}$
$\implies \mathbf{Y} = \mathbf{C}\mathbf{X}\mathbf{C}^{-1}$, so $\mathbf{Y} \sim \mathbf{X}$,(Symmetric relation)
Let $\mathbf{X} \sim \mathbf{Y}, \mathbf{Y} \sim \mathbf{X}$ then,
$\mathbf{X} = \mathbf{C}^{-1}\mathbf{Y}\mathbf{C}$
$\mathbf{Y} = \mathbf{D}^{-1}\mathbf{Z}\mathbf{D}$,
$\mathbf{C}$ and $\mathbf{D}$ are invertible matrix.

$$\implies \mathbf{X} = \mathbf{C}^{-1}\mathbf{D}^{-1}\mathbf{Z}\mathbf{D}\mathbf{C}$$
$$= (\mathbf{D}\mathbf{C})^{-1}\mathbf{Z}\mathbf{D}\mathbf{C}$$

$\implies \mathbf{X} \sim \mathbf{Z}$(Transitive Relation)
Thus $\mathbf{B}$=$\mathbf{C^{-1}AC}$ an equivalence relation.
This relation $\sim$ induces a partition within the matrices



Equivalent matrix represents the same linear transformation. The matrices which are not in the same partition cannot involve in the same linear transformation.

Example :Zero matrix is the only element in its partition as well identity matrix is the only element in its partition.

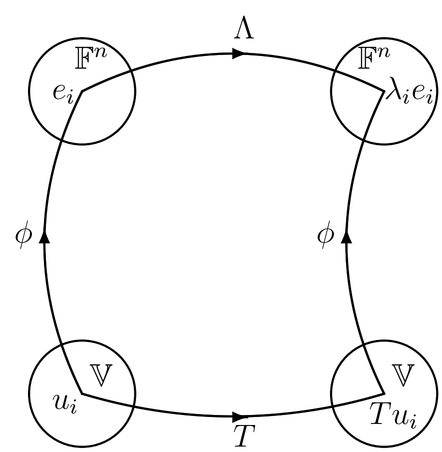
4 Diagonal Matrix

Does there exist a 'simple' matrix with as many zero entries representing a given linear transformation?
A simple non-trivial matrix will be diagonal matrix.

Let $T : V \to V$ and $\mathfrak{B} = \{u_1, u_2, ..., u_n\}$ be the basis for the set V.
If $f(t)$ is a polinomial in $\mathbb{F}$ and T is represented by a diagonal matrix . $\Lambda = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$

then $P(T)$ is presented with respect to the same basis by $P_\lambda = \begin{bmatrix} P_{\lambda_1} & 0 & ... & 0 \\ 0 & P_{\lambda_2} & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & P_{\lambda_n} \end{bmatrix}$

When $T \in L(V)$, does there exist an ordered basis for V with respect to T so that T has a diagonal representation? If such a diagonal representation exists,how to find the ordered basis?



Λ maps $\mathbb{F}^n$ to $\mathbb{F}^n$ and has the property that

$$\Lambda e_i = \lambda e_i$$

$$T(u_i) = \lambda e_i$$

where $u_i = \phi^{-1}(e_i)$, $(i = 1, 2, 3, ...n)$
$\implies$ $\lambda_1, \lambda_2, ...\lambda_n$ are the eigen values of T and $u_1, u_2, ..., u_n$ are the corresponding eigen vectors.

$$Tu_i = \lambda_i u_i$$

$$
\begin{aligned}
T(\phi^{-1}e_i) &= \lambda_i\phi^{-1}e_i \\
(\phi T\phi^{-1})(e_i) &= \Lambda e_i = \lambda_i e_i = \lambda_i\phi(u_i) \\
\phi^{-1}\phi T(\phi^{-1}e_i) &= \phi^{-1}\lambda_i\phi(u_i) \\
T(\phi^{-1}e_i) &= \lambda_i\phi^{-1}e_i \\
T(\phi^{-1}(e_i)) &= \lambda_i u_i \\
Tu_i &= \lambda_i u_i
\end{aligned}
$$

So the problem reduces to finding u_i's which are eigen vectors of T such that $\{u_1, u_2, ..., u_n\}$ forms an ordered basis for V. However, finding such eigen vectors which spans the V is not always possible. $\Longrightarrow$ Every linear transformation cannot be represented as diagonal matrices.
Let us see some examples.

Example : **1**
T is a linear transformation in $P_2(\mathbb{R})$
$T(1) = 5 + 1t + 3t^2$
$T(t) = -6 + 4t - 6t^2$
$T(t^2) = -6 + 2t + -4t^2$
Find a diagonal represention for T
Ans:
The matrix representation corresponding to the linear transformation T is $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & 4 \end{bmatrix}$.
If T is diagonizable, the $\det[T - \lambda I] = 0$ for λ is the eigen value.
$T - \lambda I = \begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & 4-\lambda \end{bmatrix}$.
$det \begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & 4-\lambda \end{bmatrix}$ = -$(\lambda^3 - 3\lambda^2 + 6\lambda - 4)$
Since $\det[T - \lambda I] = 0$,
$\lambda^3 + 3\lambda^2 - 6\lambda - 4 = 0$.
$(\lambda - 1)(\lambda - 2)^2 = 0$
So, the eigen values are 1,2,2.

When $\lambda = 1$,

$$[T-1I]=\begin{bmatrix}4 & -6 & -6\\ -1 & 3 & 2\\ 3 & -6 & 3\end{bmatrix}$$

$$\begin{bmatrix}4 & -6 & -6\\ -1 & 3 & 2\\ 3 & -6 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\implies 4x_1-6x_2-6x_3=0.$

$-x_1+3x_2+2x_3=0.$

$\implies x_1=x_3, x_1=-3x_2.$

So. $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-3x_2\\ x_2\\ -3x_2\end{bmatrix}$

When x_2 = -1, $\begin{bmatrix}-3x_2\\ x_2\\ -3x_2\end{bmatrix}=\begin{bmatrix}3\\ -1\\ 3\end{bmatrix}$

$W_1=Nullspace(T-I).$

When $\lambda=2$,

$$[T-2I]=\begin{bmatrix}3 & -6 & -6\\ -1 & 2 & 2\\ 3 & -6 & 2\end{bmatrix}$$

$$\begin{bmatrix}3 & -6 & -6\\ -1 & 2 & 2\\ 3 & -6 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\implies -x_1+2x_2+2x_3=0.$

$\implies x_1=0, x_2=1, x_3=-1.$

$x_1=2, x_2=0, x_3=-1.$

$W_2=Nullspace(T-2I).$

$\{(0,1,-1),(2,0,1)\}$ spans W_2.

The linear transformation T can be also expressed in terms of a diagonal matrix with the ordered basis $\mathcal{B}\prime=\{(1+2t+2t^2),(t-t^2),(2+t^2)\}$.

$p_1=1+2t+2t^2$

$p_2=t-t^2$

$p_3=2+t^2$

p_1, p_2 and p_3 are the eigen vectors of T.

$Tp_1=1p_1+0p_2+0p_3$

$Tp_1 = 0p_1 + 20p_2 + 0p_3$
$Tp_1 = 0p_1 + 0p_2 + 2p_3$
$$[T]_{\{p_1,p_2,p_3\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
So T is diagonizable.

Example(2)
T is a linear transformation in $P_2(\mathbb{R})$
. $T(1) = 3 + 2t + 2t^2$
$T(t) = 1 + 2t + 2t^2$
$T(t^2) = -1 + -1t$
Find a diagonal represention for T
Ans:
The matrix representation corresponding to the linear transformation T is $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$.
If T is diagonizable, the $\det[T - \lambda I] = 0$ for λ is the eigen value.
$$T - \lambda I = \begin{bmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{bmatrix}.$$
$$det \begin{bmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{bmatrix} = \text{-}(\lambda^3 - 3\lambda^2 + 6\lambda - 4)$$
Since $\det[T - \lambda I] = 0$,
$\lambda^3 + 3\lambda^2 - 6\lambda - 4 = 0.$
$(\lambda - 1)(\lambda - 2)^2 = 0$
So, the eigen values are 1,2,2.

When $\lambda = 1$,
$$[T - I] = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$\implies 2x_1 + 2x_2 - 1x_3 = 0.$

$\implies 2x_1 + 1x_2 - 1x_3 = 0.$
$\implies x_2 = 0$
$\implies 2x_1 = x_3.$
So. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 2x_1 \end{bmatrix}$
When $x_1 = 1$, $\begin{bmatrix} -x_1 \\ 0 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$
$W_1 = Nullspace(T - I).$
$\{(1, 0, 2)\}$ spans W_1.
When $\lambda = 2$,
$[T - 2I] = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{bmatrix}$
$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
$\implies 2x_1 - x_3 = 0.$
$\implies 2x_1 = x_3, x_2 = x_1.$
So. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 2x_1 \end{bmatrix}$
When $x_1 = 1$, $\begin{bmatrix} x_1 \\ x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$
$W_2 = Nullspace(T - 2I).$
$\{(1, 1, 2)\}$ does not span W_2.

So, T cannot be diagonized.

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5 Diagonizability

T is diagonizable if there exists an ordered basis for V consisting of eigenvectors of T.

5.1 Diagonizable Operators

Let $T \in L(V)$ be diagonizable.$\exists$ distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ each of algebric multiplicity $n_1, n_2, ..., n_k$ respectively, and eigen spaces $W_1, W_2, ..., W_k$ respectively with $\dim(W_i) = n_i$.

$$n_1 + n_2 + ... + n_k = n$$

$$W_i = N(T - \lambda_i I)$$

$V = W_1 \oplus W_2 \oplus ... \oplus W_k$, $(W_i \cap W_j = \{0\}$ for eigen spaces W_i and W_j belonging to the eigenvalues λ_i and λ_j whenever $\lambda_i \neq \lambda_j)$. That is, $v \in V$ has a unique representation .

$$v = w_1 + w_2 + ... + w_k, (w_i \in W_i)$$

When dim $W_i = n_i$, basis for W_i is $\{w_{i1}, w_{i2}, ..., w_{in_i}\}$.

$$Tw_{ij} = \lambda_i w_{ij}$$

Then $\{w_{11}, w_{12}, ..., w_{1n_1}, w_{21}, w_{22}, ..., w_{2n_2}, ..., w_{k1}, w_{k2}, ..., w_{kn_k}\}$ forms a basis for V.
Then the characteristic polynomial p is given by

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2}...(\lambda - \lambda_k)^{n_k}$$

n_i is the algebraic multiplicity of λ_i and dim W_i is called the algebraic multiplicity of λ_i.
T is diagonizable if and only if the algebraic multiplicity of each eigen values should be the geometric multiplicity.
Example(1) was diagonalized with the help of the eigen values 1,2,2. However the matrix in **Example(2)** had the same eigen value of **Example(1)**, **Example(2)** was not diagonizable because the geometrical multiplicity of the eigen value,2 $(= 1)$ is not equal to the algebraic multiplicity of the eigen value,2$(= 2)$.

5.2 Spectral Thoery

One can define, $P_i : V \to V$ by $P_i v = w_i$
Then P_i 's are linear and ${P_i}^2 = P_i$ (Idempotent). Then P_i is called projection on W_i along $W\prime_i$ where

$$W\prime_i = W_1 \oplus W_2 \oplus,,, \oplus W_{i-1} \oplus W_{i+1} \oplus,,, \oplus W_k$$

So, $V = W_i \oplus W\prime_i$
Now, $v = w_1 + w_2 + ... + w_k = P_1 v + P_2 v + ... + P_k v$
$\implies \boxed{I = P_1 + P_2 + ... + P_k}$ - (1)
$Tv = Tw_! + Tw_2 + ... + Tw_k$
. $=\lambda_1 w_1 + \lambda_2 w_2 + ...\lambda_k w_k$
. $=\lambda_1 P_1 + \lambda_2 P_2 + ...\lambda_k P_k$
$\implies \boxed{T = \lambda_1 P_1 + \lambda_2 P_2 + ...\lambda_k P_k}$ - (2)
1 and 2 constitute the celebrated **Spectral Theorom**.

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Q1. Let T be a linear transformation with respect to the ordered basis $\mathcal{B} := \{1, t, t^2\}$. $T(1) = 3 + 2t + 4t^2$
$T(t) = 2 + 2t * 2$
$T(t^2) = 4 + 2t + 3t^2$
Analyse this example and verify the spectral theory.
Ans:

The matrix representation corresponding to the linear transformation T is $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.
If T is diagonizable, the $\det[T - \lambda I] = 0$ for λ is the eigen value.
$T - \lambda I = \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$.
$det \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}$ = -$(\lambda^3 + 6\lambda^2 + 15\lambda + 8)$
Since $\det[T - \lambda I] = 0$,
$\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$.

$(\lambda - 8)(\lambda + 1)^2 = 0$
So, the eigen values are 8,-1,-1.

When λ = 8,

$$[T - 8I] = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 2x_1 + -8x_2 + 2x_3 = 0.$
$4x_1 + -2x_2 + -5x_3 = 0.$
$\implies x_1 = x_3, x_1 = 2x_2.$

So. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 2x_2 \end{bmatrix}$

When $x_2 = 1$, $\begin{bmatrix} 2x_2 \\ x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$W_1 = Nullspace(T - 8I).$
$\{(2, 1, 2)\}$ spans W_1.
When λ = -1,

$$[T - (-1)I] = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 2x_1 + 1x_2 + 2x_3 = 0.$
$x_1 = 0, x_2 = -2, x_3 = 1.$

$x_2 = 0, x_1 = 1.x_3 = -1.$ So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$W_2 = Nullspace(T - 8I).$
$\{(0, -2, 1), (1, 0, -1)\}$ spans W_1.

$(x_1, x_2, x_3) \in \mathbb{R}^3$ then, $p(t) = x_1 + x_2 t + x_3 t^2$

$(x_1, x_2, x_3) = y_1(2,1,2) + y_2(0,-2,1) + y_3(1,0,-1)$
$\quad = ((2y_1 + y_3), (y_1 - 2y_2), (2y_1 + y_2 - y_3))$
$2y_1 + y_3 = x_1,$
$y_1 - 2y_2 = x_2,$
$2y_1 + y_2 - y_3 = x_3$
$\implies y_1 = \frac{1}{9}(2x_1 + 1x_2 + 2x_3)$
$y_2 = \frac{1}{9}(1x_1 + -4x_2 + 1x_3)$
$y_3 = \frac{1}{9}(5x_1 + -2x_2 + -4x_3)$
$\therefore (x_1, x_2, x_3) = [\frac{1}{9}(2x_1 + 1x_2 + 2x_3)](2,1,2) + \frac{1}{9}(x_1 + -4x_2 + x_3)](0,-2,1) + [\frac{1}{9}(5x_1 + -2x_2 + -4x_3)](1,0,-1).$

$P_1(x_1, x_2, x_3) = [\frac{1}{9}(2x_1 + 1x_2 + 2x_3)](2,1,2)]$
. $\quad = [\frac{1}{9}(4x_1 + 2x_2 + 4x_3), \frac{1}{9}(2x_1 + 1x_2 + 2x_3), \frac{1}{9}(4x_1 + 2x_2 + 4x_3)]$

So $[P_1] = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix}$

$P_2(x_1, x_2, x_3) = \frac{1}{9}(x_1 + -4x_2 + x_3)](0,-2,1) + [\frac{1}{9}(5x_1 + -2x_2 + -4x_3)](1,0,-1)$
. $\quad = [\frac{1}{9}(5x_1 + -2x_2 + -4x_3), \frac{1}{9}(-2x_1 + 8x_2 + -2x_3), \frac{1}{9}(-4x_1 + -2x_2 + 5x_3)]$

So $[P_2] = \begin{bmatrix} \frac{5}{9} & \frac{-2}{9} & \frac{-4}{9} \\ \frac{-2}{9} & \frac{8}{9} & \frac{-2}{9} \\ \frac{-4}{9} & \frac{-2}{9} & \frac{5}{9} \end{bmatrix}$

$$[P_1] + [P_2] = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix} + \begin{bmatrix} \frac{5}{9} & \frac{-2}{9} & \frac{-4}{9} \\ \frac{-2}{9} & \frac{8}{9} & \frac{-2}{9} \\ \frac{-4}{9} & \frac{-2}{9} & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\lambda_1[P_1] + \lambda_2[P_2] = 8\begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix} + -1\begin{bmatrix} \frac{5}{9} & \frac{-2}{9} & \frac{-4}{9} \\ \frac{-2}{9} & \frac{8}{9} & \frac{-2}{9} \\ \frac{-4}{9} & \frac{-2}{9} & \frac{5}{9} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} = [T]$$

Thus the properties of spectral theory have been verified with the help of the example. Moreover,the linear transformation T can be also expressed in terms of a diagonal matrix with the ordered basis $\mathcal{B}\prime = \{(2 + t + 2t^2), (-2t + t^2), (1 - t^2)\}$.

$$[T]_{\{(2+t+2t^2),(-2t+t^2)(1-t^2)\}} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

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