Linear Transformation in Linear Space

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1 Linear Transformation

Let V and W be an n dimensional vector space over a field $\mathbb{F}$. Let $T : V \to W$ be a function with V as its domain and its range contained in W.

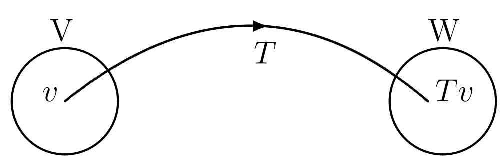
$$T(V) \subset W$$

We also assume T is linear in the sense that

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$T(\alpha v_1) = \alpha T(v_1)$$

$\forall\ v_1, v_2 \in V$ and $\alpha \in \mathbb{F}$.

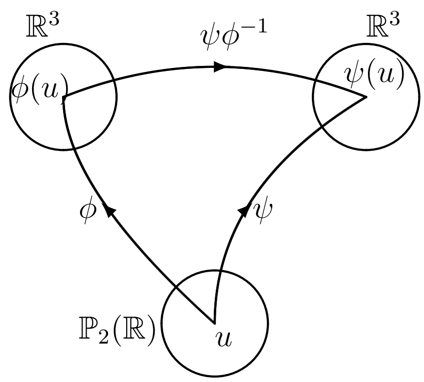


Let $L(V, W)$ denote the set of linear transformation from V to W. If $T \in L(V, W)$, T is defined if we prescribe the action of T on a basis of V.
Let $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ be a basis of V. Then $v \in V$ given by $v = x_1v_1 + x_2v_2 + ... + x_nv_n$, $\forall\ x_i \in \mathbb{F}$

$$T(v) = T(x_1v_1 + x_2v_2 + ... + x_nv_n)$$

$$= x_1T(v_1) + x_2T(v_2) + ... + x_nT(v_n)$$

If we know each $T(v_i)$ we will get $T(v)$.



2 Matrix Representation of Linear Transformation

Let $T : V \to V$ and $\mathfrak{B} = \{u_1, u_2, ..., u_n\}$ be the basis for the set V.
If $f(t)$ is a polinomial in $\mathbb{F}$ and T is represented by a diagonal matrix . $\Lambda = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$

then $P(T)$ is presented with respect to the same basis by $P_\lambda \begin{bmatrix} P_{\lambda_1} & 0 & ... & 0 \\ 0 & P_{\lambda_2} & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & P_{\lambda_n} \end{bmatrix}$

When $T \in L(V)$, does there exist an ordered basis for V with respect to T so that T has a diagonal representation?If such a diagonal representation exists,how to find the ordered basis?

Example : Let $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ is a linear transformation defined by $T(1) := 3.1 + 1.t + (-2).t^2$, $T(t) := 2.1 + 4.t + (-4).t^2$ and $T(t^2) := 2.1 + 1.t + (-1).t^2$. Show that $[T]_{\{1,t,t^2\}} = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$.Does this diagonizable?

Solution :T is diagonizable if $\exists$ a base for $\mathbb{P}_2(\mathbb{R})$ consisting of eigen vectors of T.

$$[T]-\lambda I = \begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{bmatrix}$$; λ is an eigen value of$[T]$ iff $[T]-\lambda I$ is singular.

$$det([T]-\lambda I) = 0$$

The characteristic polynomial is

$$det([T]-\lambda I) = -(\lambda-1)(\lambda-2)(\lambda-3)$$

Hence eigen values of $[T]$ are 1,2 and 3. So algebraic degree is 3. We need to find the eigen vectors corresponding to each eigen value.

When $\lambda = 1$, $[T]-1\lambda = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{bmatrix}$

-+ Rank $([T]-1\lambda) = 3$. All rows are linearly independent.

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + 3x_2 + x_3 = 0$$

$$\implies x_2 = 0, x_1 = -x_3$$

So the eigen vector corresponding to each eigen value 1 is (1 0 -1).

$$P_1(t) = 1 + (-1)t^2$$

$$\begin{aligned} T(P_1) &= T(1+(-1)t^2) \\ &= T(1) - T(t^2) \\ &= 3.1 + 1.t + (-2).t^2 - (2.1 + 1.t + (-1).t^2) \\ &= 1 + (-1)t^2 \\ &= 1.P_1 \end{aligned}$$

When $\lambda = 2$, $[T]-2\lambda = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{bmatrix}$

Rank $([T]-1\lambda) = 3$. All rows are linearly independent.

$$x_1 + 2x_2 + 2x_3 = 0$$

$$x_1 + 2x_2 + x_3 = 0$$

$$\implies x_3 = 0, x_1 = -2x_2$$

So the eigen vector corresponding to each eigen value 2 is (-2 1 0).

$$P_2(t) = (-2).1 + 1.t$$

$$\begin{aligned} T(P_2) &= T((-2).1 + 1.t) \\ &= (-2)T(1) + T(t) \\ &= -2(3.1 + 1.t + (-2).t^2) + 2.1 + 4.t + (-4).t^2 \\ &= -4 + 2t \\ &= 2.P_2 \end{aligned}$$

When $\lambda = 3$, $[T] - 3\lambda = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}$

Rank $([T] - 1\lambda) = 3$. All rows are linearly independent.

$$2x_2 + 2x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$\implies x_1 = 0, x_2 = -x_3$$

So the eigen vector corresponding to each eigen value 2 is (0 1 -1).

$$P_3(t) = 1.t + (-1).t^2$$

$$\begin{aligned} T(P_3) &= T(1.t + (-1).t^2) \\ &= T(t) - T(t^2) \\ &= 2.1 + 4.t + (-4).t^2 - (2.1 + 1.t + (-1).t^2) \\ &= 3t - 3t^2 \\ &= 3.P_3 \end{aligned}$$

For every eigen values $\exists$ an eigen vector which are the basis for the new transformation. Since we could find three basis the matrix is diagonizable.

$T_{P_1} = P_1$, $T_{P_2} = 2P_2$ and $T_{P_3} = 3P_3$

It also can represent as

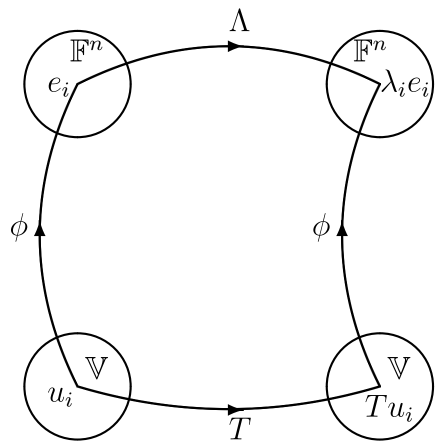
$$T_{P_1} = 1.P_1 + 0.P_2 + 0.P_3$$

$$T_{P_1} = 0.P_1 + 2.P_2 + 0.P_3$$

$$T_{P_1} = 0.P_1 + 0.P_2 + 3.P_3$$

So the diagonal matrix equilent to the given linear transformations is

$$[T]_{P_1,P_2,P_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$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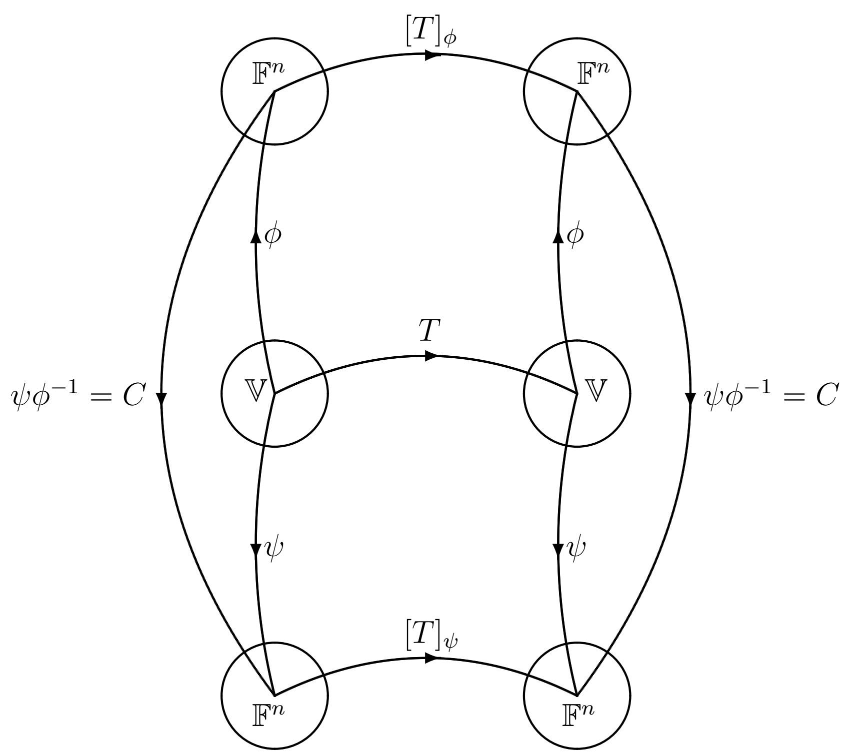
$[T]_\phi$
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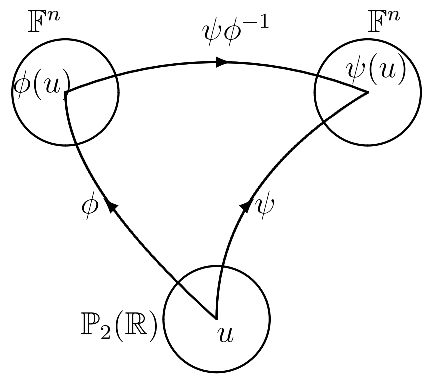

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