Linear Transformation on Linear Space

Internship Report

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1 Linear Transformation

Let V and W be an n dimensional vector space over a field $\mathbb{F}$. Let $T : V \to W$ be a function with V as its domain and its range contained in W.

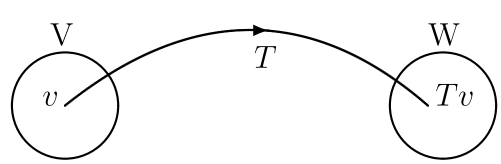
$$T(V) \subset W$$

We also assume T is linear in the sense that

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

$$T(\alpha v_1) = \alpha T(v_1)$$

$\forall\ v_1, v_2 \in V$ and $\alpha \in \mathbb{F}$.

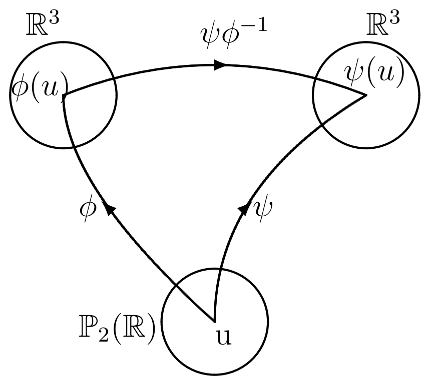


Let $L(V, W)$ denote the set of linear transformation from V to W. If $T \in L(V, W)$, T is defined if we prescribe the action of T on a basis of V.

Let $\mathcal{B} = v_1, v_2, ..., v_n$ be a basis of V. Then $v \in V$ given by $v = x_1v_1 + x_2v_2 + ... + x_nv_n$, $\forall\ x_i \in \mathbb{F}$

$$T(v) = T(x_1v_1 + x_2v_2 + ... + x_nv_n)$$

$= x_1T(v_1) + x_2T(v_2) + ... + x_nT(v_n)$
If we know every $T(v_i)$ we will get T(v).



2 Matrix Representation of Linear Transformation

Let V be an n-dimensional vector space over the field $\mathbb{F}$ and W an m-dimensional vector space over $\mathbb{F}$. Let $\mathcal{B}$:=$\{u_1, u_2, ..., u_n\}$ be an ordered basis for U and $\mathcal{B}\prime$:=$\{v_1, v_2, ..., v_n\}$ an ordered basis for V. For each linear transformation T from U into V, there is an $m \times n$ matrix **A** with entries in $\mathbb{F}$.
Let T be given by

$$T(u_i) = a_{1_i}v_1 + a_{2_i}v_2 + ... + a_{m_i}v_m$$

i = (1,2,...,n)

3 Similarity Transformation

Let **A** and **B** be n×n (square) matrices over the field $\mathbb{F}$. We say that **B** is similar to **A** over $\mathbb{F}$ if there is an invertible n× n matrix **P** over $\mathbb{F}$ such that $\mathbf{B}$=$\mathbf{P^{-1}AP}$.Therefore we will get an equivalence class between **A** and **B**.This relation $\sim$ induces a partition within the matrices.
Equivalent matrix represents the same linear transformation. The matrices which are not in the same partition cannot involve in the same linear transformation.

Example :Zero matrix is the only element in its partition so as identity matrix.

4 Diagonal Matrix

Does there exist a 'simple' matrix with as many zero entries representing a given linear transformation?
A simple non-trivial matrix will be diagonal matrix.
Let $T : V \to V$ and $\mathfrak{B} = \{u_1, u_2, ..., u_n\}$ be the basis for the set V.
If $f(t)$ is a polinomial in $\mathbb{F}$ and T is represented by a diagonal matrix . $\Lambda = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$

then $P(T)$ is presented with respect to the same basis by $P_\lambda \begin{bmatrix} P_{\lambda_1} & 0 & ... & 0 \\ 0 & P_{\lambda_2} & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & P_{\lambda_n} \end{bmatrix}$
When $T \in L(V)$, does there exist an ordered basis for V with respect to T so that T has a diagonal representation?If such a diagonal representation exists,how to find the ordered basis?

5 Diagonizability

T is diagonizable if there exists an ordered basis for V consisting of eigenvectors of T.

5.1 Diagonizable Operators

Let $T \in L(V)$ be diagonizable.$\exists$ distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ each of algebric multiplicity $n_1, n_2, ..., n_k$ respectively, and eigen spaces $W_1, W_2, ..., W_k$ respectively with $\dim(W_i) = n_i$.

$$n_1 + n_2 + ... + n_k = n$$

$V = W_1 \oplus W_2 \oplus ... \oplus W_k$, $(W_i \cap W_j = \{0\}$ for eigen spaces W_i and W_j belonging to the wigenvalues λ_i and λ_j whenever $\lambda_i \neq \lambda_j)$ That is, $v \in V$ has a unique representation .

$$v = w_1 + w_2 + ... + w_k, (w_i \in W_i)$$

$$Tw_i = \lambda_i w_i$$

One can define, $P_i : V \to V$ by $P_i v = w_i$
Then P_i 's are linear and $P_i^2 = P_i$ (Idempotent). Then P_i is called projection on W_i along $W\prime_i$ where

$$W\prime_i = W_1 \oplus W_2 \oplus , , , \oplus W_{i-1} \oplus W_{i+1} \oplus , , , \oplus W_k$$

So, $V = W_i \oplus W\prime_i$
Now, $v = w_1 + w_2 + ... + w_k = P_1 v + P_2 v + ... + P_k v$
$\implies \boxed{I = P_1 + P_2 + ... + P_k}$ - (1)
$Tv = Tw_! + Tw_2 + ... + Tw_k$
. $=\lambda_1 w_1 + \lambda_2 w_2 + ... \lambda_k w_k$
. $=\lambda_1 P_1 + \lambda_2 P_2 + ... \lambda_k P_k$
$\implies \boxed{T = \lambda_1 P_1 + \lambda_2 P_2 + ... \lambda_k P_k}$ - (2)
1 and 2 constitute the celebrated **Spectral Theorom**.

Example:1 $T(1) = 5 + 1t + 3t^2$
$T(t) = -6 + 4t - 6t^2$
$T(t^2) = -6 + 2t + -4t^2$
Ans:

The matrix representation corresponding to the linear transformation T is $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & 4 \end{bmatrix}$.

If T is diagonizable, the $\det[T - \lambda I] = 0$ for λ is the eigen value.

$T - \lambda I = \begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & 4-\lambda \end{bmatrix}$.

$det \begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & 4-\lambda \end{bmatrix}$ = -$(\lambda^3 - 3\lambda^2 + 6\lambda - 4)$

Since $\det[T - \lambda I] = 0$,
$\lambda^3 + 3\lambda^2 - 6\lambda - 4 = 0$.
$(\lambda - 1)(\lambda - 2)^2 = 0$

So, the eigen values are 1,2,2.

When $\lambda = 1$,

$$[T - 1I] = \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 4x_1 - 6x_2 - 6x_3 = 0.$

$-x_1 + 3x_2 + 2x_3 = 0.$

$\implies x_1 = x_3, x_1 = -3x_2.$

So. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \\ -3x_2 \end{bmatrix}$

When x_2 = -1, $\begin{bmatrix} -3x_2 \\ x_2 \\ -3x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

$W_1 = Nullspace(T - I).$

When $\lambda = 2$,

$$[T - 2I] = \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies -x_1 + 2x_2 + 2x_3 = 0.$

$\implies x_1 = 0, x_2 = 1, x_3 = -1.$

$x_1 = 2, x_2 = 0, x_3 = -1.$

$W_2 = Nullspace(T - 2I).$

$\{(0, 1, -1), (2, 0, 1)\}$ spans W_2.

The linear transformation T can be also expressed in terms of a diagonal matrix with the ordered basis $\mathcal{B}\prime = \{(1 + 2t + 2t^2), (t - t^2), (2 + t^2)\}$.

$p_1 = 1 + 2t + 2t^2$

$p_2 = t - t^2$

$p_3 = 2 + t^2$

p_1, p_2 and p_3 are the eigen vectors of T.

$Tp_1 = 1p_1 + 0p_2 + 0p_3$

$Tp_1 = 0p_1 + 20p_2 + 0p_3$

$Tp_1 = 0p_1 + 0p_2 + 2p_3$

$$[T]_{\{p_1,p_2,p_3\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example:1

$T(1) = 3 + 2t + 2t^2$

$T(t) = 1 + 2t + 2t^2$

$T(t^2) = -1 + -1t$

Ans:

The matrix representation corresponding to the linear transformation T is $\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$.

If T is diagonizable, the $\det[T - \lambda I] = 0$ for λ is the eigen value.

$$T - \lambda I = \begin{bmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{bmatrix}.$$

$$det \begin{bmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{bmatrix} = \text{-}(\lambda^3 - 3\lambda^2 + 6\lambda - 4)$$

Since $\det[T - \lambda I] = 0$,

$\lambda^3 + 3\lambda^2 - 6\lambda - 4 = 0.$

$(\lambda - 1)(\lambda - 2)^2 = 0$

So, the eigen values are 1,2,2.

When $\lambda = 2$,

$$[T - 2I] = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 2x_1 - x_3 = 0.$

$\implies 2x_1 = x_3, x_2 = x_1.$

So. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 2x_1 \end{bmatrix}$

When $x_1 = 1$, $\begin{bmatrix} x_1 \\ x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$W_1 = Nullspace(T - I).$

When $\lambda = 2$,

$[T - 2I] = \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & 2 \end{bmatrix}$

$\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies -x_1 + 2x_2 + 2x_3 = 0.$

$\implies x_1 = 0, x_2 = 1, x_3 = -1.$

$x_1 = 2, x_2 = 0, x_3 = -1.$

$W_2 = Nullspace(T - 2I).$

$\{(0, 1, -1), (2, 0, 1)\}$ spans W_2.

The linear transformation T can be also expressed in terms of a diagonal matrix with the ordered basis $\mathcal{B}\prime = \{(1 + 2t + 2t^2), (t - t^2), (2 + t^2)\}$.

$p_1 = 1 + 2t + 2t^2$

$p_2 = t - t^2$

$p_3 = 2 + t^2$

p_1, p_2 and p_3 are the eigen vectors of T.

$Tp_1 = 1p_1 + 0p_2 + 0p_3$

$Tp_1 = 0p_1 + 20p_2 + 0p_3$

$Tp_1 = 0p_1 + 0p_2 + 2p_3$

$[T]_{\{p_1,p_2,p_3\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

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Q1. Let T be a linear transformation with respect to the ordered basis $\mathcal{B} := \{1, t, t^2\}$. $T(1) = 3 + 2t + 4t^2$

$T(t) = 2 + 2t * 2$
$T(t^2) = 4 + 2t + 3t^2$
Analyse this example and verify the spectral theory.
Ans:
The matrix representation corresponding to the linear transformation T is $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.
If T is diagonizable, the $\det[T - \lambda I] = 0$ for λ is the eigen value.

$$T - \lambda I = \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix}.$$

$$det \begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix} = \text{-}(\lambda^3 + 6\lambda^2 + 15\lambda + 8)$$

Since $\det[T - \lambda I] = 0$,
$\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0.$
$(\lambda - 8)(\lambda + 1)^2 = 0$
So, the eigen values are 8,-1,-1.

When $\lambda = 8$,

$$[T - 8I] = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 2x_1 + -8x_2 + 2x_3 = 0.$
$4x_1 + -2x_2 + -5x_3 = 0.$
$\implies x_1 = x_3, x_1 = 2x_2.$

So. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 2x_2 \end{bmatrix}$

When $x_2 = 1$, $\begin{bmatrix} 2x_2 \\ x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$W_1 = Nullspace(T - 8I).$

When λ = -1,

$[T-(-1)I] = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$

$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies 2x_1 + 1x_2 + 2x_3 = 0.$

$x_1 = 0, x_2 = -2, x_3 = 1.$

$x_2 = 0, x_1 = 1.x_3 = -1.$ So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \, and \, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$(x_1, x_2, x_3) \in \mathbb{R}^3$ then, $p(t) = x_1 + x_2 t + x_3 t^2$

$(x_1, x_2, x_3) = y_1(2,1,2) + y_2(0,-2,1) + y_3(1,0,-1)$

$= ((2y_1 + y_3), (y_1 - 2y_2), (2y_1 + y_2 - y_3))$

$2y_1 + y_3 = x_1,$

$y_1 - 2y_2 = x_2,$

$2y_1 + y_2 - y_3 = x_3$

$\implies y_1 = \frac{1}{9}(2x_1 + 1x_2 + 2x_3)$

$y_2 = \frac{1}{9}(1x_1 + -4x_2 + 1x_3)$

$y_3 = \frac{1}{9}(5x_1 + -2x_2 + -4x_3)$

$\therefore (x_1, x_2, x_3) = [\frac{1}{9}(2x_1 + 1x_2 + 2x_3)](2,1,2) + \frac{1}{9}(x_1 + -4x_2 + x_3)](0,-2,1) + [\frac{1}{9}(5x_1 + -2x_2 + -4x_3)](1,0,-1).$

$P_1(x_1, x_2, x_3) = [\frac{1}{9}(2x_1 + 1x_2 + 2x_3)](2,1,2)]$

. $= [\frac{1}{9}(4x_1 + 2x_2 + 4x_3), \frac{1}{9}(2x_1 + 1x_2 + 2x_3), \frac{1}{9}(4x_1 + 2x_2 + 4x_3)]$

So $[P_1] = \begin{bmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix}$

$P_2(x_1, x_2, x_3) = \frac{1}{9}(x_1 + -4x_2 + x_3)](0,-2,1) + [\frac{1}{9}(5x_1 + -2x_2 + -4x_3)](1,0,-1)$

. $= [\frac{1}{9}(5x_1 + -2x_2 + -4x_3), \frac{1}{9}(-2x_1 + 8x_2 + -2x_3), \frac{1}{9}(-4x_1 + -2x_2 + 5x_3)]$

So $[P_2] = \begin{bmatrix} \frac{5}{9} & \frac{-2}{9} & \frac{-4}{9} \\ \frac{-2}{9} & \frac{8}{9} & \frac{-2}{9} \\ \frac{-4}{9} & \frac{-2}{9} & \frac{5}{9} \end{bmatrix}$

$$[P_1]+[P_2]=\begin{bmatrix}\frac{4}{9}&\frac{2}{9}&\frac{4}{9}\\\frac{2}{9}&\frac{1}{9}&\frac{2}{9}\\\frac{4}{9}&\frac{2}{9}&\frac{4}{9}\end{bmatrix}+\begin{bmatrix}\frac{5}{9}&\frac{-2}{9}&\frac{-4}{9}\\\frac{-2}{9}&\frac{8}{9}&\frac{-2}{9}\\\frac{-4}{9}&\frac{-2}{9}&\frac{5}{9}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

$$\lambda_1[P_1]+\lambda_2[P_2]=8\begin{bmatrix}\frac{4}{9}&\frac{2}{9}&\frac{4}{9}\\\frac{2}{9}&\frac{1}{9}&\frac{2}{9}\\\frac{4}{9}&\frac{2}{9}&\frac{4}{9}\end{bmatrix}+-1\begin{bmatrix}\frac{5}{9}&\frac{-2}{9}&\frac{-4}{9}\\\frac{-2}{9}&\frac{8}{9}&\frac{-2}{9}\\\frac{-4}{9}&\frac{-2}{9}&\frac{5}{9}\end{bmatrix}$$

$$=\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}=[T]$$

Thus the properties of spectral theory have been verified with the help of the example. Moreover,the linear transformation T can be also expressed in terms of a diagonal matrix with the ordered basis $\mathcal{B}\prime=\{(2+t+2t^2),(-2t+t^2),(1-t^2)\}$.

$$[T]_{\{(2+t+2t^2),(-2t+t^2)(1-t^2)\}}=\begin{bmatrix}8&0&0\\0&-1&0\\0&0&-1\end{bmatrix}.$$

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